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Research Article

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Sobolev-Gaffney type inequalities for differential forms on sub-Riemannian contact manifolds with bounded geometry

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Abstract: In this article, we establish a Gaffney type inequality, in $W^{\ell,p}$-Sobolev spaces, for differential forms on sub-Riemannian contact manifolds without boundary, having bounded geometry (hence, in particular, we have in mind noncompact manifolds). Here, $p \in]1, \infty[$ and $\ell = 1, 2$ depending on the order of the differential form we are considering. The proof relies on the structure of the Rumin's complex of differential forms in contact manifolds, on a Sobolev-Gaffney inequality proved by Baldi-Franchi in the setting of the Heisenberg groups and on some geometric properties that can be proved for sub-Riemannian contact manifolds with bounded geometry.

Keywords: Heisenberg groups, differential forms, Gaffney inequality, contact manifolds

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1 Introduction

A well-known formulation of Gaffney's inequality in $\mathbb{R}^3$ is the following div-curl type estimate: There exists a geometric constant $C > 0$ such that for any vector field $\overrightarrow{F}$ in $W^{1,2}(\mathbb{R}^3, \mathbb{R}^3)$

$$\|\overrightarrow{F}\|_{W^{1,2}} \leq C\|\mathrm{curl}\overrightarrow{F}\|_{L^2} + \|\mathrm{div}\overrightarrow{F}\|_{L^2} + \|\overrightarrow{F}\|_{L^2}.$$

Such an estimate plays a fundamental role in mathematics, for example, in classical continuum and electromagnetic field theories. In the context of differential forms (by identifying 1-forms with vector fields), the previous inequality generalizes in $\mathbb{R}^n$ to the following. If $W^{1,2}(\mathbb{R}^n, \bigwedge^h\mathbb{R}^n)$ is the Sobolev space of differential forms on $\mathbb{R}^n$ of degree h and $\alpha \in W^{1,2}(\mathbb{R}^n, \bigwedge^h\mathbb{R}^n)$, then

$$\|\alpha\|_{W^{1,2}} \leq C\|\mathrm{d}\alpha\|_{L^2} + \|\delta\alpha\|_{L^2} + \|\alpha\|_{L^2}, \tag{1}$$

where d and δ denote, respectively, the differential and codifferential of the de Rham complex in $\mathbb{R}^n$. Gaffney's inequality is the key estimate for the Hodge decomposition theorem for differential forms and makes sense also in the more general framework of manifolds. Roughly speaking, the Hodge decomposition for a C^∞ differential form α of degree h says that $\alpha = \beta + (d\delta + \delta d)\gamma$, where $d\beta = \delta\beta = 0$ (for this problem on Riemannian manifolds M we refer, e.g., to [37], see also [13]).

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The proof of Gaffney's inequality for differential forms goes back to Gaffney [20] for manifolds without boundary, and to Friedrichs [19] and Morrey [33] for manifolds with boundary, where the differential forms satisfy additional conditions on the boundary (see also [37] and the references therein). The proof of the above inequality in a compact Riemannian manifold without boundary, replacing the Sobolev space $W^{1,2}$ with a more general Sobolev space $W^{1,p}$, is due to Scott [38] where, as a corollary, a L^p-Hodge decomposition result was obtained. Its counterpart, for compact Riemannian manifolds with boundary, is due to Iwaniec et al. [26]. The study of the L^p-Hodge theory for noncompact Riemannian manifolds was considered also in [2]. Further contributions presenting various generalizations of Gaffney's inequality can be found in the literature. For an exhaustive overview of such Gaffney-type inequalities in the Riemannian setting, we refer for example to [11].

In this article, we deal with sub-Riemannian contact manifolds, and since by the classical Darboux' theorem any contact manifold is locally diffeomorphic to the Heisenberg group of corresponding dimension via a contact map, it is worth mentioning the specific generalization of the Gaffney-Friedrichs inequality recently proved in Heisenberg groups [17].

By replacing the exterior differential d by a suitable differential d_c^M, which acts on differential forms "adapted" to the contact geometry (the so called Rumin complex, see subsection 1.1), we shall prove a $W^{\ell,p}$-Gaffney-type inequality for a complete, noncompact sub-Riemannian contact manifold M without boundary (here, $\ell = 1$ or $\ell = 2$ depending on the degree of the differential form we are dealing with). We shall also assume that M has bounded geometry, which means that, roughly speaking, there exist uniform bounds on the geometric invariants of the manifold (in the article, we adopt the definition of sub-Riemannian contact manifold with bounded geometry given in [6]). Manifolds with bounded geometry generalize the concept of compact manifolds and covering of compact manifolds. Examples of manifolds with bounded geometry are Lie groups or, more generally, homogeneous spaces. Examples of noncompact sub-Riemannian contact manifolds with bounded geometry are given in [6] (see, in particular, Remark 4.10 and Section 7 therein).

1.1 Sub-Riemannian contact manifolds

A contact manifold is given by the couple (M, H), where M is a smooth odd-dimensional (connected) manifold of dimension $2n+1$ and H is the so-called contact structure on M, that is, H is a smooth distribution of hyperplanes which is maximally nonintegrable: given θ^M a smooth 1-form defined on M such that $H = \ker(\theta^M)$, then $\mathrm{d}\theta^M$ restricts to a nondegenerate 2-form on H. Roughly speaking, to be maximally nonintegrable means that the contact subbundle H is as far as possible from being integrable. Indeed, in general, for a subbundle defined by a 1-form η to be integrable, it is necessary and sufficient that $\eta \wedge (\mathrm{d}\eta)^n \equiv 0$ (see [31], Section 3.4). Therefore, a measure on a contact manifold (M, H) can be defined through the nondegenerate top degree form $\theta^M \wedge (\mathrm{d}\theta^M)^n$. There exists, in addition, a unique vector field ξ^M transverse to $\ker\theta^M$ (the so-called Reeb vector field) such that $\theta^M(\xi^M) = 1$ and $\mathcal{L}_{\xi^M} = 0$.

Among stratified nilpotent Lie groups, the Heisenberg groups are the simplest example of a group endowed with a contact structure. We recall that the Heisenberg group $\mathbb{H}^n$ is the Lie group with stratified nilpotent Lie algebra $\mathfrak{h}$ of step 2

$$\mathfrak{h} = \operatorname{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\} \oplus \operatorname{span}\{T\} := \mathfrak{h}_1 \oplus \mathfrak{h}_2,$$

where the only nontrivial commutation rules are $[X_j, Y_j] = T$, $j = 1, \ldots, n$. We denote by $\theta^{\mathbb{H}}$ the 1-form on $\mathbb{H}^n$ such that $\ker\theta^{\mathbb{H}} = \exp(\mathfrak{h}_1)$ and $\theta^{\mathbb{H}}(T) = 1$. We recall also that $\mathbb{H}^n$ can be identified with $\mathbb{R}^{2n+1}$ through the exponential map. The stratification of the algebra induces a family of dilations δ_λ in the group via the exponential map,

$$\delta_\lambda = \lambda \text{ on } \mathfrak{h}_1, \quad \delta_\lambda = \lambda^2 \text{ on } \mathfrak{h}_2, \tag{2}$$

which are analogous to the Euclidean homotheties.

As already stressed earlier, the Heisenberg groups are the local model of all contact manifolds (see [31], p. 112).

Following Rumin (see [34], p. 288), we can assume that there is a metric g^M, which is globally adapted to the symplectic form $d\theta^M$. Indeed, there exists an endomorphism J of H such that $J^2 = -Id$, $d\theta^M(X, JY) = -d\theta^M(JX, Y)$ for all $X, Y \in H$, and $d\theta^M(X, JX) > 0$ for all $X \in H\backslash\{0\}$. Then, if $X, Y \in H$, we define $g_H(X, Y) := d\theta^M(X, JY)$. Finally, we extend J to TM by setting $J\xi^M = \xi^M$ and setting $g^M(X, Y) = \theta^M(X)\theta^M(Y) + d\theta^M(X, JY)$ for all $X, Y \in TM$.

The couple (M, H) equipped with the Riemannian metric g_H is called a *sub-Riemannian* contact manifold, and in the sequel, it will be denoted by (M, H, g^M), where g^M is obtained as mentioned earlier. In any sub-Riemannian contact manifold (M, H, g^M), we can define a sub-Riemannian distance d_M (see, e.g., [43]) inducing on M the same topology of M as a manifold. In particular, Heisenberg groups $\mathbb{H}^n$ can be viewed as sub-Riemannian contact manifolds. If we choose on the contact sub-bundle of $\mathbb{H}^n$ a left-invariant metric, it turns out that the associated sub-Riemannian metric is also left-invariant. It is customary to call this distance in $\mathbb{H}^n$ a Carnot-Carathéodory distance (note that all left-invariant sub-Riemannian metrics on Heisenberg groups are bi-Lipschitz equivalent).

A natural setting when dealing with differential forms in Heisenberg groups is provided by Rumin's complex $(E_0^\bullet, d_c^{\mathbb{H}})$ of differential forms in $\mathbb{H}^n$ (see, e.g., [34]), since de Rham's complex $(\Omega^\bullet, d)$ in $\mathbb{R}^{2n+1}$, endowed with the usual exterior differential d, does not fit the very structure of the group $\mathbb{H}^n$. Indeed, differential forms on $\mathfrak{h}$ split into two eigenspaces under δ_λ (see (2)); therefore, de Rham's complex lacks scale invariance under the anisotropic dilations δ_λ, basically since it mixes derivatives along all the layers of the stratification. Rumin's substitute for de Rham's exterior differential is a linear differential operator $d_c^{\mathbb{H}}$ from sections of E_0^h to sections of E_0^{h+1} $(0 \le h \le 2n + 1)$ such that $(d_c^{\mathbb{H}})^2 = 0$.

We note explicitly that Rumin's differential $d_c^{\mathbb{H}}$ may be a left-invariant differential operator of order higher than 1.

Rumin's construction of the intrinsic complex makes sense for arbitrary contact manifolds (M, H) (see [34]). The main features of Rumin's complex defined on M are the same as those already stated in $\mathbb{H}^n$. Assuming $E_0^\bullet = \oplus_{h=0}^{2n+1} E_0^h$ endowed with the exterior differential d_c^M, we have:

(i) $(d_c^M)^2 = 0$;

(ii) the complex $(E_0^\bullet, d_c^M)$ is homotopically equivalent to de Rham's complex $(\Omega^\bullet, d)$;

(iii) $d_c^M : E_0^h \to E_0^{h+1}$ is a homogeneous differential operator in the horizontal derivatives of order 1 if $h \neq n$, whereas $d_c^M : E_0^n \to E_0^{n+1}$ is a homogeneous differential operator in the horizontal derivatives of order 2.

Given (M, H, g^M), the scalar product on H determines a norm on the line bundle TM/H. Therefore, for any h, the vector spaces E_0^h are endowed with a scalar product. By using $\theta^M \wedge d\theta^M$ as a volume form, one obtains L^p-norms on spaces of smooth Rumin's differential forms on M (see Remark 3.4).

We denote by $*$ the Hodge duality associated with the inner product in $E_0^\bullet$ and the volume form (see also Section 4), and by δ_c^M, the formal adjoint in $L^2(M, E_0^\bullet)$ of the operator d_c^M; we have $\delta_c^M = (-1)^h * d_c^M *$ on E_0^h (see [34]).

1.2 Bounded geometry

In the sequel, we will assume that M is a noncompact manifold with bounded geometry, without boundary. But, of course, our approach covers also the case of a compact manifold without boundary.

Definition 1.1. Let $k \in \mathbb{N}$. Let $B(e, 1)$ denote the sub-Riemannian unit ball in $\mathbb{H}^n$. We say that a complete smooth sub-Riemannian contact manifold (M, H, g^M) has C^k-*bounded geometry* if there exist two constants $r > 0$ and $C_M = C(M) > 0$ such that, for every $x \in M$, if we denote by $B(x, r)$ the sub-Riemannian ball for

(M, H, g^M) centered at x and of radius r, there exists a contactomorphism (i.e., a diffeomorphism preserving the contact forms) $\phi_x : B(e,1) \to M$ that satisfies

(1) $B(x,r) \subset \phi_x(B(e,1))$.

(2) ϕ_x is C_M-bi-Lipschitz.

(3) Coordinate changes $\phi_y^{-1} \circ \phi_x$ and their derivatives up to order k with respect to unit left-invariant horizontal vector fields are bounded by C_M.

Examples of noncompact sub-Riemannian contact manifolds with bounded geometry are given in [6] (see, in particular, Remark 4.10 therein and Section 7).

The formulation of a Gaffney-type inequality in this setting requires different statements, depending on the degree of the differential forms we are considering. Indeed, the fact that the operator d_c^M has order 1 or 2, depending on the degree of the form on which it acts, will be a major issue in the proofs of our results and will change the class of Sobolev spaces in our inequalities.

To introduce the notion of Sobolev space in M, we will make use of the analogous notion in Heisenberg groups $\mathbb{H}^n$. This notion is associated with the stratification of their algebra and nowadays is quite classical: we refer, e.g., to Section 4 of [14] and Section 2.1.1 of this article Roughly speaking, if $\ell \in \mathbb{N}$ and $p \geq 1$, the Sobolev space $W^{\ell,p}(\mathbb{H}^n)$ (the so-called Folland-Stein Sobolev space) can be defined as follows. Fix an orthonormal basis $\{W_i,\ i = 1, \dots, 2n+1\}$ of $\mathfrak{h}$ such that $W_i \in \mathfrak{h}_1$ for $i = 1, \dots, 2n$ and $W_{2n+1} \in \mathfrak{h}_2$. We call *homogeneous order* of a monomial in $\{W_i\}$ its degree of homogeneity with respect to the group dilations δ_λ, $\lambda > 0$. We say that a differential form on $\mathbb{H}^n$ belongs to $W^{\ell,p}(\mathbb{H}^n)$ if all its derivatives of homogeneous order $\leq \ell$ along $\{W_i\}$ belong to $L^p(\mathbb{H}^n)$. Using C^k-bounded charts, this notion extends to C^k-bounded geometry sub-Riemannian contact manifolds M (we refer to Section 3 for precise definitions).

Our main result reads as follows.

Theorem 1.2. *Let (M, H, g^M) be a complete, smooth contact manifold with C^k-bounded geometry ($k \geq 3$), without boundary. We have:*

(i) if $1 \leq h \leq 2n$, and $1 < p < \infty$, then there exists $C > 0$ such that for all $\alpha \in W^{1,p}(M, E_0^h)$ with $h \neq n, n+1$

$$\|\alpha\|_{W^{1,p}(M,E_0^h)} \leq C(\|d_c^M \alpha\|_{L^p(M,E_0^{h+1})} + \|\delta_c^M \alpha\|_{L^p(M,E_0^{h-1})} + \|\alpha\|_{L^p(M,E_0^h)}).$$

Moreover,

(ii) if $h = n$, and $1 < p < \infty$, then there exists $C > 0$ such that for all $\alpha \in W^{2,p}(M, E_0^n)$

$$\|\alpha\|_{W^{2,p}(M,E_0^n)} \leq C(\|d_c^M \alpha\|_{L^p(M,E_0^{n+1})} + \|d_c^M \delta_c^M \alpha\|_{L^p(M,E_0^n)} + \|\alpha\|_{L^p(M,E_0^n)}).$$

On the other hand,

(iii) if $h = n+1$, and $1 < p < \infty$, then there exists $C > 0$ such that for all $\alpha \in W^{2,p}(M, E_0^n)$

$$\|\alpha\|_{W^{2,p}(M,E_0^{n+1})} \leq C(\|\delta_c^M d_c^M \alpha\|_{L^p(M,E_0^{n+1})} + \|\delta_c^M \alpha\|_{L^p(M,E_0^n)} + \|\alpha\|_{L^p(M,E_0^{n+1})}).$$

This article is organized as follows. Section 2 contains some definitions and properties of Heisenberg groups and a very short review of Rumin's complex. Section 3 contains a precise definition of manifold with bounded geometry and the definition of its associated Sobolev spaces. Section 3 also contains a fine result concerning the existence of a map that associates an orthonormal symplectic basis of $\ker \theta^M$ with the canonical orthonormal symplectic basis of Ker $\theta_e^{\mathbb{H}}$, where $\theta_e^{\mathbb{H}}$ denotes the form $\theta^{\mathbb{H}}$ evaluated at the point $e \in \mathbb{H}^n$. It should be noted that throughout the article, we will use both symbols α_p and $\alpha(p)$ to indicate when the form α is evaluated at the point p. In our setting, if $\psi : \mathbb{H}^n \to M$ is a contact diffeomorphism, we have $\psi^\sharp d_c^M = d_c^{\mathbb{H}} \psi^\sharp$, where $\psi^\sharp$ is the pullback of the map ψ. This is not the case when we replace the differential with the codifferential; in Section 4, we show that if M has bounded geometry we can locally control the difference $\psi^\sharp \delta_c^M - \delta_c^{\mathbb{H}} \psi^\sharp$ with constants depending only on the geometry of M. In Section 5, we prove at first a local Gaffney inequality, and then we pass from the local case to the global one using an ad hoc covering for M with balls of suitably small radius obtained in Section 3.

We now list a few difficulties we have to deal with in this note. To prove the Gaffney inequality on M, we pass from the corresponding result for the "flat" model $\mathbb{H}^n$ proved in [4]. A delicate point is to show that we can replace the contactomorphism ϕ_x appearing in Definition 1.1 with another contactomorphism ψ_x, which "sends" an orthonormal symplectic basis of $\ker\theta_x^M$ into the canonical orthonormal symplectic basis of $\ker\theta_e^{\mathbb{H}}$ and that still depends only on the bounded geometric constants and not on the point x. This is accomplished in Theorem 3.10, and it is a convenient step if we want to write the difference $\psi^\sharp\delta_c^M - \delta_c^{\mathbb{H}}\psi^\sharp$ in local coordinates. Another subtle issue comes from the fact that the order of the differentials d_c^M and $d_c^{\mathbb{H}}$ can be one or two depending on the degree of the form, and this problem is reflected in several proofs (e.g., in Section 4, when we estimate $\psi^\sharp\delta_c^M - \delta_c^{\mathbb{H}}\psi^\sharp$, or in Theorem 5.4 when we have to estimate the commutators between differentials and functions).

2 Basic properties of Rumin's complex $(E_0^\bullet, d_c)$ on Heisenberg groups and on general contact manifolds

2.1 Heisenberg groups

We denote by $\mathbb{H}^n$ the $(2n+1)$-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If we consider two points in $\mathbb{H}^n$, $p = (x, y, t)$ and $q = (\tilde{x}, \tilde{y}, \tilde{t})$, the (noncommutative) group operation is denoted by $p \cdot q := (x + \tilde{x}, y + \tilde{y}, t + \tilde{t} + \frac{1}{2}\sum_{j=1}^n (x_j\tilde{y}_j - y_j\tilde{x}_j))$. In this system of coordinates, the unit element of $\mathbb{H}^n$, which will be denote by e, is the zero of the vector space $\mathbb{R}^{2n+1}$, and $p^{-1} = -p$.

For any $q \in \mathbb{H}^n$, the (*left*) *translation* $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ is defined as follows:

$$p \mapsto \tau_q p := q \cdot p.$$

The Lebesgue measure in $\mathbb{R}^{2n+1}$ is a Haar measure in $\mathbb{H}^n$.

For a general review on Heisenberg groups and their properties, we refer to [24,39,42]. We limit ourselves to fix some notation, following [6].

First, we notice that Heisenberg groups are smooth manifolds (and therefore are Lie groups). In particular, the pullback of differential forms is well defined as follows (see, e.g., [21], Proposition 1.106).

Definition 2.1. If $\mathcal{U}$ and $\mathcal{V}$ are open subsets of $\mathbb{H}^n$, and $f : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism, then for any differential form α of degree h, we denote by $f^\sharp\alpha$ the pullback form in $\mathcal{U}$ defined by

$$(f^\sharp\alpha)(p)(v_1, \dots, v_h) := \alpha(f(p))(df(p)v_1, \dots, df(p)v_h)$$

for any h-tuple $(v_1, \dots, v_h)$ of tangent vectors at p.

The Heisenberg group $\mathbb{H}^n$ can be endowed with the homogeneous norm (known as Korányi norm)

$$\varrho(p) = ((|x|^2 + |y|^2)^2 + 16t^2)^{1/4},$$

and we define the gauge distance (a true distance, see [39], p. 638) as follows:

$$d(p, q) := \varrho(p^{-1} \cdot q). \tag{3}$$

The metric d behaves well with respect to left-translations, that is,

$$d(\tau_q p, \tau_q p') = d(p, p'),$$

for all $q, p, p' \in \mathbb{H}^n$. Finally, the balls for the metric d are the so-called Korányi balls

$$B(p,r) := \{q \in \mathbb{H}^n;\ d(p,q) < r\}. \tag{4}$$

We denote by $\mathfrak{h}$ the Lie algebra of the left-invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1,\dots,n$, by

$$X_i := \partial_{x_i} - \frac{1}{2}y_i\partial_t, \quad Y_i := \partial_{y_i} + \frac{1}{2}x_i\partial_t, \quad T := \partial_t.$$

The only nontrivial bracket relations are $[X_j, Y_j] = T$, for $j = 1,\dots,n$. The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$. Coherently, from now on, we refer to $X_1,\dots,X_n, Y_1,\dots,Y_n$ (identified with first-order differential operators) as the *horizontal derivatives*. Denoting by $\mathfrak{h}_2$ the linear span of T, the two-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The stratification of the Lie algebra $\mathfrak{h}$ induces a family of nonisotropic dilations $\{\delta_\lambda\}$, $\lambda > 0$, in $\mathbb{H}^n$ so that for any $p = (x,y,t) \in \mathbb{H}^n$,

$$\delta_\lambda(p) = (\lambda x, \lambda y, \lambda^2 t). \tag{5}$$

Notice that the gauge norm is positively δ_λ-homogenous (i.e., $d(\delta_\lambda(p), \lambda(p')) = \lambda d(p,p')$ for all $q, p, p' \in \mathbb{H}^n$ and $\lambda > 0$) so that the Lebesgue measure of the ball $B(x,r)$ is r^{2n+2} up to a geometric constant (the Lebesgue measure of $B(e,1)$). Thus, the *homogeneous dimension* of $\mathbb{H}^n$ with respect to δ_λ, $\lambda > 0$, equals

$$Q := 2n + 2.$$

It is well known that the topological dimension of $\mathbb{H}^n$ is $2n+1$, since as a smooth manifold it coincides with $\mathbb{R}^{2n+1}$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d)$ is Q.

The vector space $\mathfrak{h}$ can be endowed with an inner product, indicated by $\langle\cdot,\cdot\rangle$, making $X_1,\dots,X_n, Y_1,\dots,Y_n$ and T orthonormal.

Throughout this article, we write also

$$W_i^{\mathbb{H}} := X_i, \quad W_{i+n}^{\mathbb{H}} := Y_i, \quad W_{2n+1}^{\mathbb{H}} := T, \quad \text{for } i = 1,\dots,n. \tag{6}$$

As in [15], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_{2n+1})$ is a multi-index, we set

$$W^{\mathbb{H},I} = (W_1^{\mathbb{H}})^{i_1}\cdots(W_{2n}^{\mathbb{H}})^{i_{2n}}\,(W_{2n+1}^{\mathbb{H}})^{i_{2n+1}}. \tag{7}$$

By the Poincaré-Birkhoff-Witt theorem, the differential operators $W^{\mathbb{H},I}$ form a basis for the algebra of left invariant differential operators in $\mathbb{H}^n$. Furthermore, we set

$$|I| := i_1 + \cdots + i_{2n} + i_{2n+1}$$

to denote the order of the differential operator $W^{\mathbb{H},I}$, and

$$d(I) := i_1 + \cdots + i_{2n} + 2i_{2n+1}$$

to denote its degree of homogeneity with respect to group dilations.

2.1.1 Sobolev spaces in $\mathbb{H}^n$

Let $U \subset \mathbb{H}^n$ be an open set. We shall use the following classical notation: $\mathcal{E}(U)$ is the space of all smooth function on U, and $\mathcal{D}(U)$ is the space of all compactly supported smooth functions on U, endowed with the standard topologies (see, e.g., [40]).

We recall the notion of (integer order) Folland-Stein Sobolev space (for a general presentation, see, e.g., [14] and [15]).

Definition 2.2. If $U \subset \mathbb{H}^n$ is an open set, $1 \le p \le \infty$ and $k \in \mathbb{N}$, then the space $W^{k,p}(U)$ is the space of all $u \in L^p(U)$ such that, with the notation (7),

$$W^{\mathbb{H},I}u \in L^p(U) \quad \text{for all multi-indices} I \text{with } d(I) \le k,$$

endowed with the natural norm

$$\|u\|_{W^{k,p}(U)} := \sum_{d(I)\le k} \|W^{\mathbb{H},I}u\|_{L^p(U)}.$$

Folland-Stein Sobolev spaces enjoy the following properties akin to those of the usual Euclidean Sobolev spaces (see [14], and e.g., [18]).

Theorem 2.3. *If $U \subset \mathbb{H}^n$, $1 \le p \le \infty$, and $k \in \mathbb{N}$, then we have*:
(*i*) *$W^{k,p}(U)$ is a Banach space.*

In addition, if $p < \infty$, we have:
(*ii*) *$W^{k,p}(U) \cap \mathcal{E}(U)$ is dense in $W^{k,p}(U)$*;
(*iii*) *if $U = \mathbb{H}^n$, then $\mathcal{D}(\mathbb{H}^n)$ is dense in $W^{k,p}(\mathbb{H}^n)$*;
(*iv*) *if $1 < p < \infty$, then $W^{k,p}(U)$ is reflexive.*

Definition 2.4. If $U \subset \mathbb{H}^n$ is open and if $1 \le p < \infty$, we denote by $\mathring{W}^{k,p}$ the completion of $\mathcal{D}(U)$ in $W^{k,p}(U)$.

Remark 2.5. If $U \subset \mathbb{H}^n$ is bounded, by (iterated) Poincaré inequality (see, e.g., [27]), it follows that the norms

$$\|u\|_{W^{k,p}(U)} \quad \text{and} \quad \sum_{d(I)=k} \|W^{\mathbb{H},I}u\|_{L^p(U)}$$

are equivalent on $\mathring{W}^{k,p}$ when $1 \le p < \infty$.

We recall the following inequality (see, e.g., [37], Lemma 1.5.3 for a proof):

Lemma 2.6. (Ehrling's inequality). *Let $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$ be Banach spaces. Let $A : \mathbb{B}_1 \to \mathbb{B}_2$ be a compact map and $\mathbb{B}_1 \hookrightarrow \mathbb{B}_3$ be a continuous embedding. Then for any $\varepsilon > 0$, there exists a constant $c = c(\varepsilon)$ such that*

$$\|Ax\|_{\mathbb{B}_2} \le \varepsilon\|x\|_{\mathbb{B}_1} + c\|x\|_{\mathbb{B}_3} \quad \forall x \in \mathbb{B}_1.$$

Remark 2.7. Let $B(e,1)$ be the Korányi ball in $\mathbb{H}^n$. We want to show that taking $\mathbb{B}_1 = W^{2,p}(B(e,1))$, $\mathbb{B}_2 = W^{1,p}(B(e,1))$ and $\mathbb{B}_3 = L^p(B(e,1))$, we can apply the previous lemma in our content. We stress that imbedding theorems in Heisenberg groups (or in more general Carnot groups) are not trivial at all. Indeed, they require suitable condition on the domain and involve Poincaré inequalities and extensions results. In our setting, the crucial step we need to prove is that the inclusion $W^{2,p}(B(e,1)) \hookrightarrow W^{1,p}(B(e,1))$ is compact. Indeed, it is shown in [32] that $B(e,1)$ is a (ε,δ) domain, and hence, we have the compact embedding $W^{1,p}(B(e,1)) \hookrightarrow L^p(B(e,1))$ by Theorem 1.27 in [22] (see also [16,25,28–30]). By Poincaré inequality, we have also that the imbedding $W_0^{2,p}(B(e,2)) \hookrightarrow W^{1,p}(B(e,2))$ is compact. The delicate part, in order to pass from spaces of order one to spaces of order two, that we show below, is to obtain the extension from $W^{2,p}(B(e,1))$ to $W_0^{2,p}(B(e,2))$ and hence to have that $W^{2,p}(B(e,1)) \hookrightarrow W^{1,p}(B(e,1))$ is compact.

To this aim let $\{u_k\}$ be a bounded sequence in $W^{2,p}(B(e,1))$. By [30], Theorem B, there exists an extension operator Λ so that Λu_k is bounded in $W^{2,p}(\mathbb{H}^n)$. Let ψ be a smooth cut-off function such that $\psi \equiv 1$ in $B(e,1)$ and with support supp $\psi \subseteq B(e,2)$. Hence, the sequence $\{\psi\Lambda u_k\}$ is bounded in $W_0^{2,p}(B(e,2))$. Therefore, up a subsequence, $\psi\Lambda u_k \to u$ in $W^{1,p}(B(e,2))$, as $k \to \infty$. Hence,

$$\psi\Lambda u_{k|_{B(e,1)}} = u_k \to u_{|B(e,1)}$$

in $W^{1,p}(B(e,1))$. In conclusion, $W^{2,p}(B(e,1)) \hookrightarrow W^{1,p}(B(e,1)) \hookrightarrow L^p(B(e,1))$, and all the inclusions are compact. Hence, by Ehrling's inequality, if $v \in W^{2,p}(B(e,1))$ for any $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ such that

$$\|v\|_{W^{1,p}(B(e,1))} \le \varepsilon \|v\|_{W^{2,p}(B(e,1))} + c(\varepsilon)\|v\|_{L^p(B(e,1))}.$$

2.2 Multilinear algebra and Rumin's complex in Heisenberg groups

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1\mathfrak{h}$. The basis of $\bigwedge^1\mathfrak{h}$, dual to the basis $\{X_1, \dots, Y_n, T\}$, is the family of covectors $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta^{\mathbb{H}}\}$, where

$$\theta^{\mathbb{H}} := dt - \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$$

is called the *contact form* in $\mathbb{H}^n$.

We indicate again by $\langle\cdot,\cdot\rangle$ the inner product in $\bigwedge^1\mathfrak{h}$ that makes $(dx_1, \dots, dy_n, \theta^{\mathbb{H}})$ an orthonormal basis. Coherently with the previous notation (6), we set

$$\omega_i^{\mathbb{H}} := dx_i, \quad \omega_{i+n}^{\mathbb{H}} := dy_i, \quad \omega_{2n+1}^{\mathbb{H}} := \theta^{\mathbb{H}}, \quad \text{for } i = 1, \dots, n.$$

The volume $(2n+1)$-form $\omega_1^{\mathbb{H}} \wedge \cdots \wedge \omega_{2n+1}^{\mathbb{H}}$ will be also written as dV.

We denote by $\bigwedge_0\mathfrak{h} := \bigwedge^0\mathfrak{h} = \mathbb{R}$ and, for $1 \le h \le 2n+1$, we can define

$$\begin{aligned}\textstyle\bigwedge_h\mathfrak{h} &:= \operatorname{span}\left\{W_{i_1}^{\mathbb{H}} \wedge \cdots \wedge W_{i_h}^{\mathbb{H}} : 1 \le i_1 < \cdots < i_h \le 2n+1\right\},\\ \textstyle\bigwedge^h\mathfrak{h} &:= \operatorname{span}\left\{\omega_{i_1}^{\mathbb{H}} \wedge \cdots \wedge \omega_{i_h}^{\mathbb{H}} : 1 \le i_1 < \cdots < i_h \le 2n+1\right\}.\end{aligned}$$

In the sequel, we shall denote by Θ^h the basis of $\bigwedge^h\mathfrak{h}$ defined by

$$\Theta^h := \left\{\omega_{i_1}^{\mathbb{H}} \wedge \cdots \wedge \omega_{i_h}^{\mathbb{H}} : 1 \le i_1 < \cdots < i_h \le 2n+1\right\}.$$

The inner product $\langle\cdot,\cdot\rangle$ on $\bigwedge^1\mathfrak{h}$ yields naturally an inner product $\langle\cdot,\cdot\rangle$ on $\bigwedge^h\mathfrak{h}$ making Θ^h an orthonormal basis.

If $1 \le h \le 2n+1$, the Hodge isomorphisms

$$* : \textstyle\bigwedge_h\mathfrak{h} \leftrightarrow \bigwedge_{2n+1-h}\mathfrak{h} \quad \text{and} \quad * : \bigwedge^h\mathfrak{h} \leftrightarrow \bigwedge^{2n+1-h}\mathfrak{h},$$

are defined by setting

$$\begin{aligned}v \wedge * w &= \langle v, w\rangle W_1^{\mathbb{H}} \wedge \cdots \wedge W_{2n+1}^{\mathbb{H}} \quad \forall v, w \in \textstyle\bigwedge_h\mathfrak{h},\\ \varphi \wedge * \psi &= \langle \varphi, \psi\rangle \omega_1^{\mathbb{H}} \wedge \cdots \wedge \omega_{2n+1}^{\mathbb{H}} \quad \forall v, w \in \textstyle\bigwedge^h\mathfrak{h}.\end{aligned}$$

If $v \in \bigwedge_h\mathfrak{h}$, we define its dual $v^\natural \in \bigwedge^h\mathfrak{h}$ by the identity $\langle v^\natural | w\rangle := \langle v, w\rangle$, and analogously we define $\varphi^\natural \in \bigwedge_h\mathfrak{h}$ for $\varphi \in \bigwedge^h\mathfrak{h}$.

Throughout this article, the elements of $\bigwedge^h\mathfrak{h}$ are identified with *left invariant* differential forms of degree h on $\mathbb{H}^n$.

Definition 2.8. A h-form α on $\mathbb{H}^n$ is said left invariant if

$$\tau_q^\sharp \alpha = \alpha \quad \text{for any } q \in \mathbb{H}^n.$$

Here, $\tau_q^\sharp\alpha$ denotes the pull-back of α through the left translation τ_q.

The same construction given earlier can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$, obtaining *horizontal h-covectors* and *horizontal h-vectors*

$$\begin{aligned}\textstyle\bigwedge_h\mathfrak{h}_1 &:= \operatorname{span}\left\{W_{i_1}^{\mathbb{H}} \wedge \cdots \wedge W_{i_h}^{\mathbb{H}} : 1 \le i_1 < \cdots < i_h \le 2n\right\},\\ \textstyle\bigwedge^h\mathfrak{h}_1 &:= \operatorname{span}\left\{\omega_{i_1}^{\mathbb{H}} \wedge \cdots \wedge \omega_{i_h}^{\mathbb{H}} : 1 \le i_1 < \cdots < i_h \le 2n\right\}.\end{aligned}$$

Moreover,

$$\Theta_0^h := \Theta^h \cap \textstyle\bigwedge^h \mathfrak{h}_1$$

provides an orthonormal basis of $\bigwedge^h \mathfrak{h}_1$.

The *symplectic 2-form* $d\theta^{\mathbb{H}} \in \bigwedge^2 \mathfrak{h}_1$ is defined by

$$d\theta^{\mathbb{H}} = -\sum_{i=1}^{n} \omega_i^{\mathbb{H}} \wedge \omega_{i+n}^{\mathbb{H}}.$$

Keeping in mind that the Lie algebra $\mathfrak{h}$ can be identified with the tangent space to $\mathbb{H}^n$ at $x = e$ (see, e.g., [21], Proposition 1.72), starting from $\bigwedge^h \mathfrak{h}$ we can define by left translation a fiber bundle over $\mathbb{H}^n$, which is still denoted by $\bigwedge^h \mathfrak{h}$. We can think of h-forms as sections of $\bigwedge^h \mathfrak{h}$. We denote by Ω^h the vector space of all smooth h-forms. In addition, the symplectic 2-form $-d\theta^{\mathbb{H}}$ induces on $\mathfrak{h}_1$ a symplectic structure. Notice that $\{W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}}\}$ is a symplectic basis of $\ker \theta^{\mathbb{H}}$.

2.2.1 Rumin's complex

Unfortunately, when dealing with differential forms in $\mathbb{H}^n$, the de Rham complex lacks scale invariance under anisotropic dilations (see (2)). Thus, a substitute for de Rham's complex, which recovers scale invariance under δ_λ, has been defined by Rumin, [34].

Here, we give only a short introduction to Rumin's complex. For a more detailed presentation, we refer to Rumin's papers [36] or to the presentation in [7].

Let $L : \bigwedge^h \mathfrak{h} \to \bigwedge^{h+2} \mathfrak{h}$ be the Lefschetz operator defined by

$$L\xi = d\theta^{\mathbb{H}} \wedge \xi. \tag{8}$$

Then the spaces $E_0^\bullet \subset \bigwedge^\bullet \mathfrak{h}$ can be defined explicitly as follows.

Theorem 2.9. (See [34,35]). *We have*:

(*i*) $E_0^1 = \bigwedge^1 \mathfrak{h}_1$;

(*ii*) *if* $2 \le h \le n$, *then* $E_0^h = \bigwedge^h \mathfrak{h}_1 \cap (\bigwedge^{h-2} \mathfrak{h}_1 \wedge d\theta^{\mathbb{H}})^\perp$ (*i.e.*, E_0^h *is the space of the so-called primitive covectors of* $\bigwedge^h \mathfrak{h}_1$);

(*iii*) *if* $n < h \le 2n+1$, *then*

$$E_0^h = \{\alpha = \beta \wedge \theta^{\mathbb{H}},\ \beta \in \textstyle\bigwedge^{h-1} \mathfrak{h}_1,\ \beta \wedge d\theta^{\mathbb{H}} = 0\} = \theta^{\mathbb{H}} \wedge \ker L;$$

(*iv*) *if* $1 < h \le n$, *then* $\dim E_0^h = \binom{2n}{h} - \binom{2n}{h-2}$;

(*v*) *if* $*$ *denotes the Hodge duality associated with the inner product in* $\bigwedge^\bullet \mathfrak{h}$ *and the volume form, then* $*E_0^h = E_0^{2n+1-h}$.

For $h = 0, \dots, 2n+1$, the space of Rumin h-forms is the space of smooth sections of a left-invariant subbundle of $\bigwedge^h \mathfrak{h}$, which we still denote by E_0^h. Hence, it inherits the inner product and the norm of $\bigwedge^h \mathfrak{h}$.

The core of Rumin's theory consists in the construction of a suitable "exterior differential" $d_c^{\mathbb{H}} : E_0^h \to E_0^{h+1}$ making $\mathcal{E}_0 := (E_0^\bullet, d_c^{\mathbb{H}})$ a complex homotopic to the de Rham complex (i.e., $d_c^{\mathbb{H}} \circ d_c^{\mathbb{H}} = 0$).

More precisely, the exterior differential $d_c^{\mathbb{H}} : E_0^h \to E_0^{h+1}$ is a left-invariant, homogeneous operator with respect to group dilations. It is a first-order homogeneous operator in the horizontal derivatives in degree $\neq n$, whereas it is a second-order homogeneous horizontal operator in degree n. There exists a left invariant orthonormal basis of E_0^h. This basis is explicitly constructed by induction in [3]. Explicit computations of the classes E_0^h and of the differential $d_c^{\mathbb{H}} : E_0^h \to E_0^{h+1}$ in $\mathbb{H}^1$ and $\mathbb{H}^2$ are given in [4] (see Examples 3.11 and 3.12 therein).

The next remarkable property of Rumin's complex is its invariance under contact transformations.

Proposition 2.10. *If we write a form $\alpha = \sum_I \alpha_I \xi_I^{\mathbb{H}}$ in coordinates with respect to a left invariant basis $\{\xi_I^{\mathbb{H}}\}_I$ of E_0^h, we have*

$$\tau_q^\sharp \alpha = \sum_I (\alpha_I \circ \tau_q) \xi_I^{\mathbb{H}} \tag{9}$$

for all $q \in \mathbb{H}^n$. In addition, for $t > 0$,

$$\delta_t^\sharp \alpha = t^h \sum_I (\alpha_I \circ \delta_t) \xi_I^{\mathbb{H}}, \quad \text{if } 1 \le h \le n \tag{10}$$

and

$$\delta_t^\sharp \alpha = t^{h+1} \sum_I (\alpha_I \circ \delta_t) \xi_I^{\mathbb{H}}, \quad \text{if } n+1 \le h \le 2n+1. \tag{11}$$

We fix the notation for vector-valued function spaces (for the scalar case, we refer to Section 2.1.1).

Definition 2.11. If $U \subset \mathbb{H}^n$ is an open set, $0 \le h \le 2n+1$, $1 \le p \le \infty$ and $m \ge 0$, we denote by $L^p(U, \bigwedge^h \mathfrak{h})$, $\mathcal{E}(U, \bigwedge^h \mathfrak{h})$, $\mathcal{D}(U, \bigwedge^h \mathfrak{h})$, $W^{m,p}(U, \bigwedge^h \mathfrak{h})$ and by $\mathring{W}^{m,p}(U, \bigwedge^h \mathfrak{h})$ the space of all sections of $\bigwedge^h \mathfrak{h}$ such that their components with respect to a given left invariant frame belong to the corresponding scalar spaces.

The spaces $L^p(U, E_0^h)$, $\mathcal{E}(U, E_0^h)$, $\mathcal{D}(U, E_0^h)$, $W^{m,p}(U, E_0^h)$ and $\mathring{W}^{m,p}(U, E_0^h)$ are defined in the same way. Clearly, all these definitions are independent of the choice of frame.

In addition, Sobolev spaces of differential forms are invariant with respect to the pullback operator associated with contact diffeomorphisms (see [6], Lemma 4.8).

Proposition 2.12. *Denote by $\delta_c^{\mathbb{H}}$ the formal adjoint of $d_c^{\mathbb{H}}$ in $L^2(\mathbb{H}^n, E_0^h)$. Then $\delta_c^{\mathbb{H}} = (-1)^h * d_c^{\mathbb{H}} *$ on E_0^h.*

We remind the reader that $\delta_c^{\mathbb{H}}$ can be written in coordinates as a left-invariant homogeneous differential operator in the horizontal variables, of order 1 if $h \neq n+1$ and of order 2 if $h = n+1$ (see Examples 3.11 and 3.12 [4] for explicit expressions of the codifferential).

When $d_c^{\mathbb{H}}$ is second-order (i.e., when $d_c^{\mathbb{H}}$ acts on forms of degree n), the complex $(E_0^\bullet, d_c^{\mathbb{H}})$ stops behaving like a differential module. This is the source of many complications. In particular, the classical Leibniz formula for the de Rham complex $d(\alpha \wedge \beta) = d\alpha \wedge \beta \pm \alpha \wedge d\beta$ in general fails to hold (see [8], Proposition A.7). This causes several technical difficulties when we want to localize our estimates by means of cut-off functions.

If ζ is a smooth real function and $\alpha \in L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^h)$ we write $d_c^{\mathbb{H}}(\zeta\alpha) = \zeta d_c^{\mathbb{H}} \alpha + [d_c^{\mathbb{H}}, \zeta]\alpha$. The proof of the following Leibniz-type formula can be found in [5] (see Lemma 4.1) and basically is due to the fact that the exterior differential $d_c^{\mathbb{H}}$ on E_0^h can be written in coordinates as a left-invariant homogeneous differential operator in the horizontal variables, of order 1 if $h \neq n$ and of order 2 if $h = n$ (analogously, the codifferential $\delta_c^{\mathbb{H}}$ can be written in coordinates as a left-invariant homogeneous differential operator in the horizontal variables, of order 1 if $h \neq n+1$ and of order 2 if $h = n+1$).

Lemma 2.13. *If ζ is a smooth real function, then the following formulae hold:*

(*i*) *if $h \neq n$, then on E_0^h we have*

$$[d_c^{\mathbb{H}}, \zeta] = P_0^h(W\zeta),$$

where $P_0^h(W\zeta) : E_0^h \to E_0^{h+1}$ is a linear homogeneous differential operator of order 0 with coefficients depending only on the horizontal derivatives of ζ. If $h \neq n+1$, an analogous statement holds if we replace $d_c^{\mathbb{H}}$ in degree h with $\delta_c^{\mathbb{H}}$ in degree $h+1$;

(*ii*) *if* $h = n$, *then on* E_0^n *we have*

$$[d_c^{\mathbb{H}}, \zeta] = P_1^n(W\zeta) + P_0^n(W^2\zeta),$$

where $P_1^n(W\zeta) : E_0^n \to E_0^{n+1}$ *is a linear homogeneous differential operator of order* 1 (*and therefore horizontal*) *with coefficients depending only on the horizontal derivatives of* ζ, *and where* $P_0^h(W^2\zeta) : E_0^n \to E_0^{n+1}$ *is a linear homogeneous differential operator in the horizontal derivatives of order* 0 *with coefficients depending only on the second-order horizontal derivatives of* ζ. *If* $h = n + 1$, *an analogous statement holds if we replace* $d_c^{\mathbb{H}}$ *in degree n with* $\delta_c^{\mathbb{H}}$ *in degree* $n + 1$.

(*iii*) *if* $h \neq n + 1$, *then*

$$[d_c^{\mathbb{H}}\delta_c^{\mathbb{H}}, \zeta] = P_1^h(W\zeta) + P_0^h(W^2\zeta),$$

where $P_1^h(W\zeta) : E_0^h \to E_0^h$ *is a linear homogeneous differential operator of order* 1 (*and therefore horizontal*) *with coefficients depending only on the horizontal derivatives of* ζ, *and where* $P_0^h(W^2\zeta) : E_0^h \to E_0^h$ *is a linear homogeneous differential operator in the horizontal derivatives of order* 0 *with coefficients depending only on the second-order horizontal derivatives of* ζ.

(*iv*) *if* $h \neq n$, *then*

$$[\delta_c^{\mathbb{H}}d_c^{\mathbb{H}}, \zeta] = P_1^h(W\zeta) + P_0^h(W^2\zeta),$$

where $P_1^h(W\zeta) : E_0^h \to E_0^h$ *is a linear homogeneous differential operator of order* 1 (*and therefore horizontal*) *with coefficients depending only on the horizontal derivatives of* ζ, *and where* $P_0^h(W^2\zeta) : E_0^h \to E_0^h$ *is a linear homogeneous differential operator in the horizontal derivatives of order* 0 *with coefficients depending only on the second-order horizontal derivatives of* ζ.

Remark 2.14. On forms of degree $h > n$, Lemma 2.13 (i) takes the following simpler form. If $\alpha \in L^1_{\mathrm{loc}}(\mathbb{H}^n, E_0^h)$ with $h > n$ and $\psi \in \mathcal{E}(\mathbb{H}^n)$, then

$$d_c^{\mathbb{H}}(\psi\alpha) = d(\psi\alpha) = \mathrm{d}\psi \wedge \alpha + \psi\mathrm{d}\alpha = d_c^{\mathbb{H}}\psi \wedge \alpha + \psi d_c^{\mathbb{H}}\alpha$$

(in the sense of distributions). This follows from (iii) of Theorem 2.9, since α is a multiple of θ.

2.2.2 Rumin's complex in contact manifolds

The notion of Rumin's complex makes sense for arbitrary contact manifolds.

Let us start with the following definition (see [31], Section I-3).

Definition 2.15. If (M_1, H_1) and (M_2, H_2) are contact manifolds with $H_i = \ker\theta^{M_i}$ (i.e., θ^{M_i} are contact forms for $i = 1, 2$), $\mathcal{U}_1 \subset M_1$ and $\mathcal{U}_2 \subset M_2$ are open sets and f is a diffeomorphism from $\mathcal{U}_1$ onto $\mathcal{U}_2$, then f is said a contact diffeomorphism if there exists a nonvanishing real function τ defined in $\mathcal{U}_1$ such that

$$f^\sharp\theta^{M_2} = \tau\theta^{M_1}.$$

As already pointed out, by the classical Darboux theorem, any contact manifold (M, H) of dimension $2n + 1$ is locally contact diffeomorphic to the Heisenberg group $\mathbb{H}^n$. It turns out that Rumin's intrinsic complex is invariant under contactomorphisms; hence, it is invariantly defined for general contact manifolds (M, H). A detailed construction of this complex can be found in [34] (see also [6] Section 3.3 for details). Alternative contact invariant definitions of Rumin's complex can be found in [9] and [10].

In this article, we do not enter into the details of Rumin's construction. Instead, we just recall the basic properties enjoyed by the complex, exactly analogous to the ones in $\mathbb{H}^n$:

(i) $d_c^M \circ d_c^M = 0$;

(ii) the complex $\mathcal{E}_0 := (E_0^\bullet, d_c^M)$ is homotopically equivalent to the de Rham complex $(\Omega^\bullet, d)$;

(iii) $d_c^M : E_0^h \to E_0^{h+1}$ is a differential operator of order 1 if $h \neq n$, whereas $d_c^M : E_0^n \to E_0^{n+1}$ is a differential operator of order 2.

For later purposes, we recall the following statement, proved in [6] (see Proposition 2.14), which expresses the fact that Rumin's complex is invariant under contactomorphisms.

Proposition 2.16. *If ϕ is a contactomorphism from an open set $\mathcal{U} \subset \mathbb{H}^n$ to M, and we denote by $\mathcal{V}$ the open set $\mathcal{V} := \phi(\mathcal{U})$, the pullback operator $\phi^\sharp$ satisfies:*
(i) $\phi^\sharp E_0^\bullet(\mathcal{V}) = E_0^\bullet(\mathcal{U})$;
(ii) $d_c^{\mathbb{H}} \phi^\sharp = \phi^\sharp d_c^M$.

3 Sobolev spaces on contact sub-Riemannian manifolds with bounded geometry

There are several possibilities for defining and investigating Sobolev spaces over complete Riemannian manifolds. Here, we define Sobolev spaces (involving a positive number of derivatives) on contact sub-Riemannian manifolds with bounded geometry, following the approach already used in [6] (see [41] or [12]).

We make more precise the definition of contact manifold of bounded geometry already given in Definition 1.1.

Definition 3.1. Let k be a positive integer and let $B(e, 1)$ denote the unit sub-Riemannian ball in $\mathbb{H}^n$. We say that a sub-Riemannian contact manifold (M, H, g^M) has *bounded C^k-geometry* if there exist constants $r, C_M > 0$ such that, for every $x \in M$, there exists a contactomorphism (i.e., a diffeomorphism preserving the contact forms) $\phi_x : B(e, 1) \to M$ that satisfies,
(1) $B(x, r) \subset \phi_x(B(e, 1))$;
(2) ϕ_x is C_M-bi-Lipschitz, i.e.,

$$\frac{1}{C_M} d(p, q) \leq d_M(\phi_x(p), \phi_x(q)) \leq C_M d(p, q) \quad \text{for all } p, q \in B(e, 1); \tag{12}$$

(3) coordinate changes $\phi_y^{-1} \circ \phi_x$ and their first k derivatives with respect to unit left-invariant horizontal vector fields are bounded by C_M.

We recall the following covering lemma and definition from [6].

Lemma 3.2. (See [6], Lemma 4.11). *Let (M, H, g^M) be a C^k-bounded geometry sub-Riemannian contact manifold, where k is a positive integer. Then there exists $\rho > 0$ (depending only on the radius r of* Definition 3.1) *and an at most countable covering $\{B(x_j, \rho)\}$ of M such that:*
(i) each ball $B(x_j, \rho)$ is contained in the image of one of the contact charts of Definition 3.1;
(ii) $B(x_j, \frac{1}{5}\rho) \cap B(x_i, \frac{1}{5}\rho) = \emptyset$ *if $i \neq j$*;
(iii) the covering is uniformly locally finite. Even more, there exists $N = N(M) \in \mathbb{N}$ such that for each ball $B(x, \rho)$

$$\#\{k \in \mathbb{N} \text{ such that } B(x_k, \rho) \cap B(x, \rho) \neq \emptyset\} \leq N.$$

In addition, if $B(x_k, \rho) \cap B(x, \rho) \neq \emptyset$, then $B(x_k, \rho) \subset B(x, r)$; see Definition 3.1.

We are in position to define Sobolev spaces on M on bounded geometry contact sub-Riemannian manifolds.

Definition 3.3. Let (M, H, g^M) be a smooth sub-Riemannian contact manifold with C^k-bounded geometry ($k \in \mathbb{N}$), and let $\{\chi_j\}$ be a partition of the unity subordinated to the atlas $\mathcal{U} := \{B(x_j, \rho), \phi_{x_j}\}$ of Lemma 3.2. We stress explicitly that $\phi_{x_j}^{-1}(\operatorname{supp} \chi_j) \subset B(e, 1)$. If α is a Rumin's differential form on M, we say that $\alpha \in W^{\ell,p}_{\mathcal{U}}(M, E_0^\bullet)$ for $\ell = 0, 1, \dots, k-1$, and $p \geq 1$, if

$$\phi_{x_j}^\sharp(\chi_j \alpha) \in W^{\ell,p}(\mathbb{H}^n, E_0^\bullet) \quad \text{for } j \in \mathbb{N}$$

(notice that $\phi_{x_j}^\sharp(\chi_j \alpha)$ is compactly supported in $B(e, 1)$ and therefore can be continued by zero on the whole $\mathbb{H}^n$). Then, we set

$$\|\alpha\|_{W^{\ell,p}_{\mathcal{U}}(M,E_0^\bullet)} := \left(\sum_j \|\phi_{x_j}^\sharp(\chi_j \alpha)\|^p_{W^{\ell,p}(\mathbb{H}^n, E_0^\bullet)} \right)^{1/p}. \tag{13}$$

A different uniform covering and other choices of controlled charts lead to an equivalent norm. The definition of the Sobolev spaces $W^{\ell,p}_{\mathcal{U}}(M, E_0^\bullet)$, indeed, does not depend on the atlas $\mathcal{U}$, as shown in [6], Proposition 4.13. Therefore, from now on, we drop the index $\mathcal{U}$ from the notation of Sobolev norms and we shall write simply $W^{\ell,p}(M, E_0^\bullet)$.

Remark 3.4. Setting $\ell = 0$ in (13), we found that the norm in $W^{0,p}(M, E_0^\bullet)$ is equivalent to the norm $L^p(M, E_0^\bullet)$ associated with the volume form $\mu := \theta^M \wedge (d\theta^M)^n$ defined in the introduction.

Proof. Let us denote the norm in $W^{0,p}(M, E_0^\bullet)$ by

$$|||\alpha|||_{L^p(M,E_0^\bullet)} := \|\alpha\|_{W^{0,p}(M,E_0^\bullet)} = \left(\sum_j \|\phi_{x_j}^\sharp(\chi_j \alpha)\|^p_{L^p(\mathbb{H}^n, E_0^\bullet)} \right)^{1/p},$$

and by

$$\|\alpha\|_{L^p_\mu(M,E_0^\bullet)}$$

the L^p-norm associated with μ.

First, since $\chi_j \alpha$ is compactly supported and

$$\phi_{x_j}^\sharp(\theta^M \wedge (d\theta^M)^n) = \phi_{x_j}^\sharp(\theta^M) \wedge \phi_{x_j}^\sharp((d\theta^M)^n) = \theta^{\mathbb{H}} \wedge (d\theta^{\mathbb{H}})^n,$$

we have $\|\phi_{x_j}^\sharp(\chi_j \alpha)\|^p_{L^p(\mathbb{H}^n, E_0^\bullet)} \approx \|\chi_j \alpha\|^p_{L^p_\mu(M,E_0^\bullet)}$. Hence,

$$|||\alpha|||_{L^p(M,E_0^\bullet)} = \left(\sum_j \|\phi_{x_j}^\sharp(\chi_j \alpha)\|^p_{L^p(\mathbb{H}^n, E_0^\bullet)} \right)^{1/p} \approx \left(\sum_j \|\chi_j \alpha\|^p_{L^p_\mu(M,E_0^\bullet)} \right)^{1/p}.$$

We are left to show that

$$\left(\sum_j \|\chi_j \alpha\|^p_{L^p_\mu(M,E_0^\bullet)} \right)^{1/p} \approx \|\alpha\|_{L^p_\mu(M,E_0^\bullet)}.$$

To this aim, note that since $\{\chi_j\}$ is a partition of unity $|\chi_j| \leq 1$, $\sum_j |\chi_j|^p \leq \sum_j |\chi_j| \leq 1$ and therefore

$$\sum_j \int_M |\chi_j \alpha|^p \, d\mu = \int_M \sum_j |\chi_j \alpha|^p \, d\mu = \int_M \sum_j |\chi_j|^p |\alpha|^p \leq \|\alpha\|^p_{L^p_\mu(M,E_0^\bullet)}.$$

On the other hand, since $\alpha = \sum_j \chi_j \alpha$,

$$\|\alpha\|^p_{L^p_\mu(M,E_0^\bullet)} \leq \|\sum_j \chi_j \alpha\|^p_{L^p_\mu(M,E_0^\bullet)} = \int_M |\sum_j \chi_j|^p |\alpha|^p \, d\mu.$$

Now, for any $x \in M$, $\sum_j \chi_j(x)$ is a finite sum with a number of terms less than or equal to N (see (iii) in Lemma 3.2), and hence, there exists a constant c_N so that $|\sum_j \chi_j|^p \le c_N \sum_j |\chi_j|^p$, and in the inequality given earlier, we obtain

$$\|\alpha\|^p_{L^p_\mu(M,E_0^\bullet)} \le c_N \sum_j \int_M |\chi_j|^p |\alpha|^p d\mu = c_N \sum_j \|\chi_j \alpha\|^p_{L^p_\mu(M,E_0^\bullet)}. \qquad \square$$

Thanks to the previous remark, from now on, we shall denote with the same symbol $\|\cdot\|_{L^p(M,E_0^\bullet)}$ the two equivalent norms $\|\cdot\|_{L^p_\mu(M,E_0^\bullet)}$ and $|||\cdot|||_{L^p(M,E_0^\bullet)}$.

In the sequel of the article, we will need a covering of M with balls of suitable, fixed radius, which has the same properties as the covering of the Lemma 3.2. In analogy with what happens in the Riemannian setting (see, e.g., [41], 7.2.1) we have:

Remark 3.5. If $\eta > 0$ is small, then there exists an at most countable covering of M with balls $\{B(a_\ell, \eta)\}$, which satisfy the same properties as the covering given in Lemma 3.2. In particular, the covering is uniformly locally finite, i.e., there exists $N = N(\eta) \in \mathbb{N}$ such that any point of M has an open neighborhood of radius η that is covered by at most $N(\eta)$ balls of the covering. Moreover, the functions of the atlas $\{B(a_\ell, \eta), \phi_\ell\}$, with $\phi_\ell : B(e, \eta/C_M) \to M$, satisfy conditions (2) and (3) of Definition 3.1 with constants depending only on C_M but independent of a_ℓ. In particular, we notice that, if α is supported in $\phi_\ell(B(e, \eta/C_M))$, then by Definition 3.1, the norms

$$\|\alpha\|_{W^{m,p}(M,E_0^\bullet)} \quad \text{and} \quad \|\phi_\ell^\sharp \alpha\|_{W^{m,p}(\mathbb{H}^n,E_0^\bullet)}$$

are equivalent, with equivalence constants independent of ℓ.

Proof. For any $x \in M$, let $\phi_x : B(e, 1) \subset \mathbb{H}^n \to M$ be a map satisfying the conditions contained in Theorem 3.10.

Let $\{B(x_j, \rho)\}$ be a countable locally finite subcovering of $\{\phi_x(B(e, 1)), x \in M\}$ as in Lemma 3.2, and let ϕ_{x_j} be the corresponding bounded contact charts from the unit Heisenberg ball, i.e., $\phi_{x_j} : B(e, 1) \to B(x_j, \rho)$. We show now that any ball $B(x_j, \rho)$ has a finite subcover of balls of radii η.

Without loss of generality, we can assume that the ϕ_{x_j} are defined on a larger ball $B(e, \lambda)$, where $\lambda > 1$ is fixed and they still satisfy conditions (2) and (3) of Definition 3.1.

Let $\eta > 0$. We can cover the ball $B(e, 1)$ by a finite number $k = k(\eta)$ of balls of radii $\frac{\eta}{C_M}$, that is,

$$B(e, 1) \subset \bigcup_{i=1}^{k} B\left(z_i, \frac{\eta}{C_M}\right) \subset B(e, \lambda). \tag{14}$$

For any z_i as in (14), we define the map $\phi^i_{x_j} : B\left(e, \frac{\eta}{C_M}\right) \to M$ as follows:

$$\phi^i_{x_j}(z) := \phi_{x_j} \circ \tau_{z_i}(z) = \phi_{x_j}(z_i \cdot z).$$

Notice that $\phi^i_{x_j}(e) = \phi_{x_j}(z_i \cdot e) =: a^i_{x_j} \in \phi_{x_j}(B(e, 1))$.

Since the map ϕ_{x_j} is a contact map, and we have composed with a translation, the map $\phi^i_{x_j}$ is a smooth contact map. Indeed, since ϕ_{x_j} is a contact map, we have $(\phi_{x_j})^\sharp \theta^M = \theta^{\mathbb{H}}$, and hence,

$$\left(\phi^i_{x_j}\right)^\sharp \theta^M = \left(\phi_{x_j} \circ \tau_{z_i}\right)^\sharp \theta^M = \tau^\sharp_{z_i} \circ \left(\phi_{x_j}\right)^\sharp \theta^M = \tau^\sharp_{z_i} \theta^{\mathbb{H}} = \theta^{\mathbb{H}}.$$

Moreover, the maps $\phi^i_{x_j}$ satisfy conditions (2) and (3) of Definition 3.1 with constants depending only on C_M but independent of x_j. We can see that (2) still holds for $\phi^i_{x_j}$, as a consequence of the left-invariance of the Korány distance (3). Indeed, for all $p, q \in B(e, 1)$, we have

$$d_M\left(\phi^i_{x_j}(p), \phi^i_{x_j}(q)\right) = d_M\left(\phi_{x_j}(z_i \cdot p), \phi_{x_j}(z_i \cdot q)\right) \approx d(z_i \cdot p, z_i \cdot q) = d(p, q), \tag{15}$$

where the symbol ≈ means that we have used the same constant C_M given in (12).

Now, reasoning as in (15), for any $i = 1, \ldots, k = k(\eta)$, we have $B(a^i_{x_j}, \eta) \supset \phi^i_{x_j}\left(B\left(e, \frac{\eta}{C_M}\right)\right)$. Hence,

$$\bigcup_{i=1}^{k(\eta)} B\left(a^i_{x_j}, \eta\right)$$

is a finite cover of the set $B(x_j, \rho)$.

Therefore, any ball $B(x_j, \rho)$ of the countable covering $\{B(x_j, \rho)\}$ of M has a finite subcover of balls of radii η, and eventually $\{B(a^i_{x_j}, \eta)\}$ is an at most countable covering of M uniformly locally finite.

From now on, we shall denote the covering $\{B(a^i_{x_j}, \eta), \phi^i_{x_j}\}$ simply by $\{B(a_\ell, \eta), \phi_\ell\}$. By construction, any point of M has an open neighborhood of radius η that is covered by at most $N(\eta)$ balls of the covering. The number $N(\eta)$ satisfies the relation $N(\eta) \approx k(\eta)N(M)$, where the symbol ≈ means that there are constants depending only on the geometry of M (i.e., on the C_M and r that were introduced in Definition 3.1), and where $N(M)$ is the number appearing in Lemma 3.2. Indeed, the constants appearing in (15) are only C_M and $1/C_M$, just like in (12). Moreover, in the proof of Lemma 3.2 (see Lemma 4.11 in [6]), it is explicitly shown that the constant $N(M)$ depends on the geometry of the underlying manifold, as $N(M) = N(C_M, r)$. □

Remark 3.6. We notice that, if we use a covering of M with balls of radius η small as mentioned earlier, the constant c_N, which gives the equivalence between the two norms in Remark 3.4, will depend also on η.

3.1 Symplectic basis and orthogonal linear transformations

In the sequel of the article, we need to cover M with atlases that enjoy further properties besides those contained in Definition 3.1. First, in the next theorem, we observe that we can replace the contactomorphism ϕ_x appearing in Definition 3.1 with another contactomorphism, which "sends" an orthonormal symplectic basis of $\ker\theta^M_x$ into the canonical orthonormal symplectic basis of $\ker\theta^{\mathbb{H}}_e$ (see (6)) and still depending only on the bounded geometric constants and not on the point x. We begin with the following remark.

Remark 3.7. Given a contact manifold M, for any $x \in M$, there exists an orthonormal basis of $\ker\theta^M_x$.

Proof. This can be shown by simply considering the endomorphism $J : \ker\theta^M \to \ker\theta^M$, with $J^2 = -Id$. The metric g^M on M was already defined in Subsection 1.1, and it is globally adapted to the symplectic form $\mathrm{d}\theta^M$. If we follow the steps of the proof of Theorem 2.1.3 in [31], and choose $Z_1 \in \ker\theta^M$ to be a unit vector field, i.e.,

$$1 = g^M(Z_1, Z_1) = \mathrm{d}\theta^M(Z_1, JZ_1),$$

then also JZ_1 is a unit vector field, as follows:

$$g^M(JZ_1, JZ_1) = \mathrm{d}\theta^M(JZ_1, J^2Z_1) = -\mathrm{d}\theta^M(J^2Z_1, JZ_1) = -\mathrm{d}\theta^M(-Z_1, JZ_1) = g^M(Z_1, Z_1) = 1.$$

Notice also that $g^M(Z_1, JZ_1) = \mathrm{d}\theta^M(Z_1, J^2Z_1) = -\mathrm{d}\theta^M(Z_1, Z_1) = 0$, so Z_1 and JZ_1 are orthonormal.

To extend $\{Z_1, JZ_1\}$ to an orthonormal basis of $\ker\theta^M$, let us consider a unitary vector field $Z_2 \in \operatorname{span}\{Z_1, JZ_1\}^\perp \cap \ker\theta^M$. Arguing as earlier, $g^M(Z_2, JZ_2) = 0$. To prove that the vectors fields Z_1, JZ_1, Z_2, JZ_2 are all orthonormal, we are left to show that $g^M(Z_1, JZ_2) = 0$ and $g^M(JZ_1, JZ_2) = 0$. Indeed,

$$g^M(Z_1, JZ_2) = \mathrm{d}\theta^M(Z_1, J^2Z_2) = -\mathrm{d}\theta^M(JZ_1, JZ_2) = -g^M(JZ_1, Z_2) = 0$$

by construction, and analogously

$$g^M(JZ_1, JZ_2) = \mathrm{d}\theta^M(JZ_1, J^2Z_2) = \mathrm{d}\theta^M(Z_1, JZ_2) = g^M(Z_1, Z_2) = 0$$

by construction.

Likewise, one can repeat the same reasoning n times and construct an orthonormal symplectic basis for $\ker\theta^M$. ◻

From now on, the basis $\{Z_1, \dots, Z_n, JZ_1, \dots, JZ_n\}$ will be denoted by

$$\{W_1^M, \dots, W_{2n}^M\},$$

and we will refer to it as an orthonormal symplectic basis of $\ker\theta^M$.

We recall now the following definition.

Definition 3.8. Let V and W be real vector spaces of dimension N, both endowed with scalar products $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$, respectively. We say that the linear map

$$T : V \to W$$

is an orthogonal linear transformation if, given $\{e_1, \dots, e_N\}$ an orthonormal basis of V, then $\{Te_1, \dots, Te_N\} = \{\varepsilon_1, \dots, \varepsilon_N\}$ is an orthonormal basis of W.

The following proposition follows easily from a result of [1].

Proposition 3.9. *Let $a \in M$ be a fixed point, and let $\{W_1^M, \dots, W_{2n}^M\}$ be an orthonormal symplectic basis of* $\ker\theta^M$ *in a neighbourhood of a. Then there exist $\varepsilon = \varepsilon(a) > 0$, and a smooth family of horizontal curves $\gamma^\ell : [0, \varepsilon] \to M$, for $\ell = 1, \dots, 2n$, such that*

(i) $\gamma^\ell(0) = a$;

(ii) $(\gamma^\ell)'(0) = W_\ell^M(a)$;

(iii) $d_M(\gamma^\ell(t), a) = \int_0^t g^M((\gamma^\ell)'(s), (\gamma^\ell)'(s))^{1/2}_{\gamma^\ell(s)}\mathrm{d}s$, *where g^M is the metric defined in Subsection* 1.1.

Proof. Fix $\ell \in \{1, \dots, 2n\}$. Following [1], Section 4.3.1, denote by $H : T^*M \to \mathbb{R}$ the sub-Riemannian Hamiltonian associated with $(M, \ker\theta^M, g^M)$, and let $\lambda : [0, 1] \to T^*M$ be the normal extremal, i.e., the solution of

$$\lambda'(t) = \overrightarrow{H}(\lambda(t)),$$

with $\lambda(0) = (W_\ell^M(a))^\natural$ (recall that the Hamiltonian vector field $\overrightarrow{H}$ is defined by $\mathrm{d}\theta^M(\overrightarrow{H}, \ \cdot) = -\mathrm{d}H$) , so that

$$\langle\lambda(0), W_j^M(a)\rangle = \delta_{\ell j}.$$

Then the assertion follows from Remark 4.28 and Theorem 4.65 in [1]. ◻

The main result of this section is the following theorem.

Theorem 3.10. *There exist $0 < r' < r$, and $0 < \mu \le 1$ (all depending only on the bounded geometry constants) such that for any $a \in M$, if we denote by $\{W_{1,a}^M, \dots, W_{2n,a}^M\}$ the orthonormal symplectic basis of $\ker\theta_a^M$ (and, as in* (6)*, $\{W_{1,e}^{\mathbb{H}}, \dots, W_{2n,e}^{\mathbb{H}}\}$ is the orthonormal symplectic basis of $\ker\theta_e^{\mathbb{H}}$), and there exists a contact map ψ_a (that is $\psi_a^\sharp(\theta^M) = \theta^{\mathbb{H}}$),*

$$\psi_a : B(e, \mu) \subset \mathbb{H}^n \to M$$

satisfying $B(a, r') \subset \psi_a(B(e, \mu))$ and conditions (2) *and* (3) *given in Definition* 3.1 *and such that:*

(i) $\psi_a(e) = a$;

(ii) $(\mathrm{d}\psi_a)_e W_{j,e}^{\mathbb{H}} = W_{j,a}^M$ *for $j = 1, \dots, 2n$, and $(\mathrm{d}\psi_a)_e\xi_e^{\mathbb{H}} = \xi_a^M$. In particular, the map*

$$(\mathrm{d}\psi_a)_e : T_e\mathbb{H}^n \to T_aM, \quad v \mapsto (\mathrm{d}\psi_a)_e(v),$$

is an orthogonal linear map.

Proof. Let $a \in M$ and let $\phi_a : B(e,1) \to M$ be a contactomorphism as in Definition 3.1 satisfying
(1) $B(a,r) \subset \phi_a(B(e,1))$;
(2)

$$\frac{1}{C_M} d(p,q) \le d_M(\phi_a(p), \phi_a(q)) \le C_M d(p,q) \quad \text{for all } p, q \in B(e,1); \tag{16}$$

(3) coordinate changes $\phi_b^{-1} \circ \phi_a$ and their first k derivatives with respect to unit left-invariant horizontal vector fields are bounded by C_M.

We can also assume that $\phi_a(e) = a$.

We consider the map

$$(\mathrm{d}\phi_a)_e : T_e\mathbb{H}^n \to T_a M.$$

The map $\phi_a^{-1} : \phi_a(B(e,1)) \to B(e,1)$ defines, by pushforward, the vector fields

$$\{\hat{W}_1, \dots, \hat{W}_{2n}\} := \{\mathrm{d}\phi_a^{-1}(W_1^M), \dots \mathrm{d}\phi_a^{-1}(W_{2n}^M)\},$$

which are a symplectic basis of $\ker \theta^{\mathbb{H}}$. Indeed, since ϕ_a is a contact map, i.e.,

$$\phi_a^\sharp(\theta^M) = \theta^{\mathbb{H}}.$$

For example, if $j = i + n$, we have

$$\begin{aligned}
\mathrm{d}\theta^{\mathbb{H}}(\hat{W}_i, \hat{W}_j) &= \mathrm{d}\theta^{\mathbb{H}}(\mathrm{d}\phi_a^{-1}(W_i^M), \mathrm{d}\phi_a^{-1}(W_j^M)) \\
&= d(\phi_a^\sharp\theta^M)(\mathrm{d}\phi_a^{-1}(W_i^M), \mathrm{d}\phi_a^{-1}(W_j^M)) \\
&= \phi_a^\sharp(\mathrm{d}\theta^M)(\mathrm{d}\phi_a^{-1}(W_i^M), \mathrm{d}\phi_a^{-1}(W_j^M)) \\
&= \langle \mathrm{d}\theta^M | \mathrm{d}\phi_a \mathrm{d}\phi_a^{-1}(W_i^M), \mathrm{d}\phi_a \mathrm{d}\phi_a^{-1}(W_j^M) \rangle = \mathrm{d}\theta^M(W_i^M, W_j^M) = \delta_{ij}.
\end{aligned}$$

In particular, $(\hat{W}_1(e), \dots, \hat{W}_{2n}(e))$ can be identified with a symplectic basis of $\mathbb{R}^{2n}$. Hence, if we denote by $\{e_1, \dots, e_{2n}\}$ the canonical basis of $\mathbb{R}^{2n}$, there exists a matrix $\tilde{A} = \tilde{A}_{\phi_a} \in Sp(2n)$ such that

$$\tilde{A} W_{i,e}^{\mathbb{H}} = \tilde{A} e_i = \hat{W}_i(e) \quad i = 1, \dots, 2n$$

(we stress that the matrix $\tilde{A}$ depends on $\mathrm{d}\phi_a^{-1}$). It is well defined the (Euclidean) linear contact map $L : \mathbb{H}^n \to \mathbb{H}^n$ associated with the matrix

$$A := \begin{pmatrix} \tilde{A} & 0_{2n\times 1} \\ 0_{1\times 2n} & 1 \end{pmatrix},$$

with $A \in GL(\mathbb{R}^{2n+1}, \mathbb{R}^{2n+1})$. In particular, A induces an automorphism of the group.

We are now ready to show that there exist $0 < r' < r$ and $0 < \mu \le 1$ (depending only on the constant C_M appearing in (16)) and a contact map ψ_a that satisfies the property of Definition 3.1, with constants depending only on the bounded geometry, but independent of a.

Claim: We claim that the norm of $\hat{W}_i(e)$ can be bounded from above and below by constants depending only on the constant C_M appearing in Definition 3.1 and not on the point a. We can then write $\|\hat{W}_i(e)\| \approx 1$ (independently of the point $a \in M$), where the symbol $\approx$ means that the constants appearing above depend only on C_M.

Let us assume for a while that the claim is true. It follows that also the norm of the matrix $\tilde{A}$ is controlled from below and above by a constants depending only on the constant C_M appearing in Definition 3.1, i.e., $1/C_M \le \|\tilde{A}\| \le C_M$.

First, we notice that if $p = (p_1, \dots, p_{2n}, p_{2n+1}) =: (p', p_{2n+1}) \in \mathbb{H}^n$ and $L(p) = A(p', p_{2n+1}) = (\tilde{A}p', p_{2n+1})$, then ($C_M > 1$)

$$C_M^{-4} d(p,e)^4 \le C_M^{-4}(|\tilde{A}^{-1}\tilde{A}p'|^4 + 16p_{2n+1}^2) \le (|\tilde{A}p'|^4 + 16p_{2n+1}^2) = d(L(p), e)^4 \le C_M^4 d(p,e)^4. \tag{17}$$

Therefore, if $0 < \mu \le 1$, we have

$$L(B(e,\mu)) \subset B(e,1) \tag{18}$$

if we take $\mu \le 1/C_M$.

Hence, we can define

$$\psi_a := \phi_a \circ L_{|B(e,\mu)} : B(e,\mu) \to M.$$

The map ψ_a is a contactomorphism and by construction satisfies

$$W^M_{j,a} = (\mathrm{d}\psi_a)_e W^{\mathbb{H}}_{j,e}.$$

We show now that if $r' < \frac{\mu}{C_M}$ we have

$$B(a,r') \subset \psi_a(B(e,\mu)). \tag{19}$$

Indeed, let $b \in B(a,r')$. There exists $p \in B(e,1)$ such that $b = \phi_a(p)$ (if $r' < r$). By (16), if we take $r' < \frac{\mu}{C_M^2}$, we have

$$d(p,e) \le C_M d_M(\phi_a(p),a) = C_M d_M(b,a) < \frac{\mu}{C_M},$$

that is, $p \in B\Big(e, \frac{\mu}{C_M}\Big)$. Since L is a linear isomorphism there exists $q \in \mathbb{H}^n$ such that $p = L(q)$. We show that $q \in B(e,\mu)$. Indeed, by (17),

$$d(q,e) \le C_M d(L(q),e) = C_M d(p,e) < C_M \frac{\mu}{C_M} = \mu.$$

Hence, $b = \phi_a(p) = (\phi_a \circ L)(q) = \psi_a(q) \in \psi_a(B(e,\mu))$ and (19) holds.

We need to show now that if $p, q \in B(e,\mu)$, then the condition (12) in Definition 3.1 is satisfied, that is,

$$d_M(\psi_a(p), \psi_a(q)) \approx d(p,q),$$

where, here, the symbol $\approx$ means that the constants appearing above depend only on C_M. Notice that if $p, q \in B(e,\mu)$, then $L(p), L(q) \in B(e,1)$, by (18), and

$$\begin{aligned} d_M(\psi_a(p), \psi_a(q)) = d_M(\phi_a(L(p)), \phi_a(L(q))) &\overbrace{\approx}^{\text{by (16)}} d(L(p), L(q)) \\ &= \rho(L(p)^{-1} \cdot L(q)) = \rho(L(p^{-1} \cdot q)) = d(L(p^{-1} \cdot q), e). \end{aligned}$$

As in (17),

$$d(L(p^{-1} \cdot q), e) \approx d(p,q),$$

hence,

$$d_M(\psi_a(p), \psi_a(q)) \approx d(p,q),$$

where, again, the equivalence constants depend only on the constant C_M appearing in Definition 3.1 and hence independent of $a \in M$.

We have now to check that the map ψ_a satisfies also the third requirement of Definition 3.1. Let $a, b \in M$, and consider the maps $\psi_a = \phi_a \circ L$ and $\psi_b = \phi_b \circ \hat{L}$ constructed as earlier. We recall that the maps L and $\hat{L}$, being linear contact maps, preserve horizontal derivatives. Since, by (3) in Definition 3.1, coordinate changes $\phi_b^{-1} \circ \phi_a$ and all their first k derivatives with respect to unit left-invariant horizontal vector fields are bounded by C_M, and it follows that $\psi_b^{-1} \circ \psi_a = \hat{L}^{-1} \circ \phi_b^{-1} \circ \phi_a \circ L$ enjoy the same property.

We are left with the proof of the Claim.

Let $W_1^M, \dots, W_{2n}^M$ be an orthonormal symplectic basis of $\ker \theta^M$ in a neighborhood of a. By Proposition 3.9, for any $j = 1, \dots, 2n$, there exists a curve $\gamma^j(t)$ for t small, such that

$$d_M(\gamma^j(t), a) = \int_0^t g^M((\gamma^j)'(s), (\gamma^j)'(s))^{1/2}_{\gamma^j(s)} \mathrm{d}s.$$

Notice that the basis is orthonormal, hence for any j, $g^M((\gamma^j)'(0), (\gamma^j)'(0))^{1/2}_a = g^M(W^M_{j,a}, W^M_{j,a})^{1/2}_a = 1$.

Let us take the map ϕ_a considered at the very beginning of the proof. Hence, again by condition (16), for $t \neq 0$ small, we have

$$\frac{1}{t}\int_0^t g^M((\gamma^j)'(s), (\gamma^j)'(s))^{1/2}_{\gamma^j(s)} \mathrm{d}s = \frac{d_M(\gamma^j(t), a)}{t} \approx \frac{d(\phi_a^{-1}\gamma^j(t), e)}{t}, \tag{20}$$

where the symbol $\approx$ means that the constants appearing depend only on C_M.

If we set $\sigma^j(t) := \phi_a^{-1}\gamma^j(t)$, then $\sigma^j(0) = \phi_a^{-1}\gamma^j(0) = e$. Since $\hat{W}_j = \mathrm{d}\phi_a^{-1}(W^M_j)$, we have $(\sigma^j)'(0) = (\mathrm{d}\phi_a^{-1})_e(\gamma^j)'(0) = (\mathrm{d}\phi_a^{-1})_e W^M_{j,a} = \hat{W}_j(e)$.

The map ϕ_a is a contact map, then the vector fields $\hat{W}_j$ are horizontal vector fields.

From now on, we argue with a fixed vector field $\hat{W}_j$. Hence, for the sake of simplicity, we shall drop the index j writing $\hat{W}$ instead of $\hat{W}_j$ and $\sigma(t)$ instead of $\sigma^j(t)$. Hence, we have

$$\hat{W}(p) = \sum_{k=1}^{2n} \lambda_k(p) W^{\mathbb{H}}_{k,p} \tag{21}$$

and $\sigma(t) = (\sigma_1(t), \dots, \sigma_{2n}(t), \sigma_{2n+1}(t))$ satisfies:
- $\sigma_k'(t) = \lambda_k(\sigma(t))$ if $1 \le k \le 2n$, and
- $\sigma_{2n+1}'(t) = \frac{1}{2}\sum_{k=1}^n (\sigma_k'(t)\sigma_{k+n}(t) - \sigma_{k+n}'(t)\sigma_k(t))$.

By Taylor's formula we have, for $t \to 0$:

$\sigma_k(t) = t\lambda_k(\sigma(0)) + O(t^2) = t\lambda_k(e) + O(t^2)$ if $1 \le k \le 2n$, and hence also $\sigma'{}_k(t) = \lambda_k(e) + O(t)$ if $1 \le k \le 2n$. Replacing these expressions in $\sigma_{2n+1}'(t)$, we obtain

$$\sigma_{2n+1}'(t) = \frac{1}{2}\sum_{k=1}^n ((\lambda_k(e) + O(t))(t\lambda_{k+n}(e) + O(t^2)) - (\lambda_{k+n}(e) + O(t))(t\lambda_k(e) + O(t^2))) = O(t^2) = o(t^3).$$

Therefore, for $t \to 0$,

$$\sigma(t) = (t(\lambda_1(e) + o(1)), \dots, t(\lambda_{2n}(e) + o(1)), t^2 \cdot o(1)) = \delta_t(\lambda_1(e) + o(1), \dots, \lambda_{2n}(e) + o(1), o(1)),$$

where δ_t is the dilation defined in (5). If we take $p = e$ in (21), $\hat{W}(e) = (\lambda_1(e), \dots, \lambda_{2n}(e), 0)$, hence

$$\sigma(t) = \delta_t(\hat{W}(e) + o(1)).$$

Thus,

$$\frac{d(\sigma(t), e)}{t} = \frac{\rho(\sigma(t))}{t} = \|\hat{W}_j(e) + o(1)\| \text{ as } t \to 0.$$

Therefore, by (20), it holds

$$\frac{1}{C_M}\|\hat{W}_j(e) + o(1)\| \le \frac{1}{t}\int_0^t g^M((\gamma^j)'(s), (\gamma^j)'(s))^{1/2}_{\gamma^j(s)} \le C_M\|\hat{W}_j(e) + o(1)\|,$$

as $t \to 0$. Passing now to the limit for $t \to 0$, and keeping in mind that $g^M(\gamma'(0), \gamma'(0))^{1/2}_a = 1$, we obtain

$$\frac{1}{C_M}\|\hat{W}_j(e)\| \le 1 \le C_M\|\hat{W}_j(e)\|,$$

and the claim is proved. □

4 Some results about the lack of commutation between the pull-back and the co-differential

In this section, we take the map $\psi : B(e,1) \to M$ as in Theorem 3.10.

Let α be a smooth differential form on M and set $\beta := \psi^\sharp \alpha$ the pullback of α. As already pointed out in Proposition 2.16, pulling back $d_c^M \alpha$ gives $d_c^{\mathbb{H}} \beta$. On the contrary, since $\delta_c^M \alpha$ involves the Hodge $*$-operator, it turns out that its pullback is not $\delta_c^{\mathbb{H}} \beta$, i.e., $\psi^\sharp \delta_c^M \neq \delta_c^{\mathbb{H}} \psi^\sharp$. In this section, we shall examine the relation between $\psi^\sharp \delta_c^M$ and $\delta_c^{\mathbb{H}} \psi^\sharp$. Remember that both $\delta_c^{\mathbb{H}}$ and δ_c^M are equal to $\pm * d_c^{\mathbb{H}} *$ and $\pm * d_c^M *$, respectively. In the sequel, we shall always drop the sign since we are only interested in estimates of norms.

In Definition 4.1 and Proposition 4.2, we recall some preliminary notations and results (see [23], Section 2.1).

Let V and W be real vector spaces of dimension N, both endowed with scalar products $\langle \cdot,\cdot \rangle_V$ and $\langle \cdot,\cdot \rangle_W$, respectively.

If we denote by $*$ the Hodge $*$-operator in V, the following equality holds

$$u \wedge * v := \langle u, v \rangle_V e_1 \wedge \cdots \wedge e_N,$$

where $\{e_1, \dots, e_N\}$ is an orthonormal basis of V. The Hodge $*$-operator on W is defined analogously.

To fix our notations, we recall the following definition (see [23], Section 2.1).

Definition 4.1. If V, W are finite dimensional linear vector spaces and $L : V \to W$ is a linear map, we define

$$\Lambda_h L : {\textstyle\bigwedge_h} V \to {\textstyle\bigwedge_h} W$$

as the linear map given by

$$(\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) = L(v_1) \wedge \cdots \wedge L(v_h)$$

for any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$, and

$$\Lambda^h L : {\textstyle\bigwedge^h} W \to {\textstyle\bigwedge^h} V$$

as the linear map defined by

$$\langle (\Lambda^h L)(\alpha) | v_1 \wedge \cdots \wedge v_h \rangle = \langle \alpha | (\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) \rangle$$

for any $\alpha \in \bigwedge^h W$ and any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$.

Proposition 4.2. *If V and W are N-dimensional vector spaces and $L : V \to W$ is an orthogonal linear transformation, we have*

$$*(\Lambda_h L) = (\Lambda_{N-h} L) * .$$

Let $\{W_1^M, \dots, W_{2n}^M\}$ be an orthonormal symplectic basis of $\ker \theta^M$. If we denote by ξ^M the Reeb vector field on M, then the metric can be extended to a Riemannian metric on TM (still denoted by g^M), so that $\{W_1^M, \dots, W_{2n}^M, \xi^M\}$ is an orthonormal frame of TM. Let $\{W_1^{\mathbb{H}}, \dots, W_{2n}^{\mathbb{H}}\}$ be the standard orthonormal symplectic basis of $\ker \theta^{\mathbb{H}}$ (see (6)).

If $m \in M$, we denote by

$$k_m : T_m M \to \mathbb{R}^{2n+1}$$

the map which associates with a vector $v \in T_m M$ its coordinates with respect to the basis $\{W_{1,m}^M, \dots, W_{2n,m}^M, \xi_m^M\}$, and by

$$f_x : T_x \mathbb{H}^n \to \mathbb{R}^{2n+1}$$

the analogous map which associates with a vector $v \in T_x \mathbb{H}^n$ its coordinates with respect to the basis $\{W_{1,x}^{\mathbb{H}}, \dots, W_{2n,x}^{\mathbb{H}}, W_{2n+1,x}^{\mathbb{H}}\}$.

We have the following property.

Lemma 4.3. *If $\psi : B(e,1) \to M$ is a map as in* Theorem 3.10 *and f and k are defined as earlier, we set*

$$\Psi_x := k_{\psi(x)} \circ (\mathrm{d}\psi)_x \circ f_x^{-1} : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1},$$

and we set

$$L_x := k_{\psi(x)}^{-1} \circ \Psi_e \circ f_x \quad \textit{and} \quad R_x := k_{\psi(x)}^{-1} \circ [\Psi_x - \Psi_e] \circ f_x.$$

Then

$$(\mathrm{d}\psi)_x = L_x + R_x,$$

where the map L_x is an orthogonal transformation for any $x \in \mathbb{H}^n$, and the linear map R_x is a smooth map vanishing at $x = e$. Moreover,

$$*\Lambda_h(\mathrm{d}\psi)_x = \Lambda_{2n+1-h}(\mathrm{d}\psi)_x * - \Lambda_{2n+1-h}R_x * + * \Lambda_h R_x. \tag{22}$$

Proof. By Theorem 3.10 $(\mathrm{d}\psi)_e$ is an orthogonal linear map, thus

$$k_{\psi(e)} \circ (\mathrm{d}\psi)_e \circ f_e^{-1} : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$$

is an orthonormal map, since by construction both f_x and k_m are orthonormal maps for any $x \in \mathbb{H}^n$ and for any $m \in M$, respectively, that is, Ψ_e is an orthogonal linear transformation.

Moreover, if we express

$$\Psi_x = \Psi_e + \Psi_x - \Psi_e,$$

we have

$$(\mathrm{d}\psi)_x = k_{\psi(x)}^{-1} \circ \Psi_x \circ f_x = k_{\psi(x)}^{-1} \circ \Psi_e \circ f_x + k_{\psi(x)}^{-1} \circ [\Psi_x - \Psi_e] \circ f_x.$$

Thus, if we set

$$L_x := k_{\psi(x)}^{-1} \circ \Psi_e \circ f_x \quad \text{and} \quad R_x := k_{\psi(x)}^{-1} \circ [\Psi_x - \Psi_e] \circ f_x,$$

we obtain

$$(\mathrm{d}\psi)_x = L_x + R_x \quad \text{and} \quad \Lambda_h(\mathrm{d}\psi)_x = \Lambda_h L_x + \Lambda_h R_x.$$

In particular, we notice that $R_x = (\mathrm{d}\psi)_x - k_{\psi(x)}^{-1} \circ \Psi_e \circ f_x$, which can be seen as a matrix-valued smooth map vanishing at $x = e$.

Moreover, the map $k_{\psi(x)}^{-1} \circ \Psi_e \circ f_x$ is an orthogonal transformation for any $x \in \mathbb{H}^n$, and by Proposition 4.2,

$$*\Lambda_h L_x = \Lambda_{2n+1-h} L_x * .$$

Therefore, we can then write

$$\begin{aligned} *\Lambda_h(\mathrm{d}\psi)_x &= *\Lambda_h L_x + * \Lambda_h R_x = \Lambda_{2n+1-h} L_x * + * \Lambda_h R_x \\ &= \Lambda_{2n+1-h}(\mathrm{d}\psi)_x * - \Lambda_{2n+1-h} R_x * + * \Lambda_h R_x. \end{aligned}$$

□

In the following lemma, we discuss the interplay between the Hodge $*$-operator and the pullback.

Lemma 4.4. *Let α be a smooth form on M of degree h, and let $\psi : B(e,1) \to M$ be as in* Theorem 3.10, *then*

$$\psi^\sharp(*\alpha) - * \psi^\sharp\alpha = \sum_I b_I \xi_I^{\mathbb{H}}, \tag{23}$$

where $\{\xi_I^{\mathbb{H}}\}_I$ is a left-invariant basis of the space of forms in $\mathbb{H}^n$ of degree $2n+1-h$, and $b_I \in C^\infty(\mathbb{H}^n)$ are smooth coefficients that vanish when evaluated at the point $x = e$.

Proof. Let α be a form on M of degree h, and take $v_1 \wedge \cdots \wedge v_{2n+1-h} \in \Lambda_{2n+1-h}\mathfrak{h}$ an arbitrary simple $(2n+1-h)$-vector of norm ≤ 1. If $x \in B(e,1)$, using (22), we can write

$$\begin{aligned}\langle(\psi^\sharp(*\alpha))_x | v_1 \wedge \cdots \wedge v_{2n+1-h}\rangle &= \langle *\alpha_{\psi(x)} | \Lambda_{2n+1-h}(\mathrm{d}\psi)_x(v_1 \wedge \cdots \wedge v_{2n+1-h})\rangle \\ &= \langle \alpha_{\psi(x)} | * [\Lambda_{2n+1-h}(\mathrm{d}\psi)_x(v_1 \wedge \cdots \wedge v_{2n+1-h})]\rangle \\ &= \langle \alpha_{\psi(x)} | \Lambda_h(\mathrm{d}\psi)_x * (v_1 \wedge \cdots \wedge v_{2n+1-h})\rangle \\ &\quad - \langle \alpha_{\psi(x)} | \Lambda_h R_x * (v_1 \wedge \cdots \wedge v_{2n+1-h})\rangle + \langle \alpha_{\psi(x)} | * \Lambda_{2n+1-h} R_x (v_1 \wedge \cdots \wedge v_{2n+1-h})\rangle \\ &= \langle (*\psi^\sharp\alpha)_x | (v_1 \wedge \cdots \wedge v_{2n+1-h})\rangle - \langle \alpha_{\psi(x)} | \Lambda_h R_x * (v_1 \wedge \cdots \wedge v_{2n+1-h})\rangle \\ &\quad + \langle \alpha_{\psi(x)} | * \Lambda_{2n+1-h} R_x (v_1 \wedge \cdots \wedge v_{2n+1-h})\rangle.\end{aligned}$$

Hence,

$$\psi^\sharp(*\alpha) - * \psi^\sharp\alpha =: \sum_I b_I \xi_I^{\mathbb{H}},$$

where the $\xi_I^{\mathbb{H}}$ belong to a basis of the space of the forms of degree $2n+1-h$ in $\mathbb{H}^n$.

Let $W_I^{\mathbb{H}}$ be the dual of the $(2n+1-h)$-covector $\xi_I^{\mathbb{H}}$. Then

$$b_I(x) = \langle (\psi^\sharp(*\alpha))_x - (*\psi^\sharp\alpha)_x | W_I^{\mathbb{H}}\rangle = \langle \alpha_{\psi(x)} | \Lambda_h(R_x)(* W_I^{\mathbb{H}})\rangle - \langle \alpha_{\psi(x)} | * \Lambda_{2n+1-h}(R_x) W_I^{\mathbb{H}}\rangle. \tag{24}$$

Since R vanishes at $x = e$ (see Lemma 4.3), then $b_I(e) = 0$. □

By using $\delta_c^M \alpha$ instead of α in (23), we obtain:

Corollary 4.5. *With the same hypotheses of* Lemma 4.4, *we have*

$$\psi^\sharp(*\delta_c^M\alpha) - * \psi^\sharp(\delta_c^M\alpha) =: \sum_J B_J \xi_J^{\mathbb{H}}, \tag{25}$$

where $\{\xi_J^{\mathbb{H}}\}_J$ *is a left-invariant basis of the space of forms in* $\mathbb{H}^n$ *of degree* $2n+2-h$, *and* $B_I \in C^\infty(\mathbb{H}^n)$ *are smooth coefficients defined by*

$$\begin{aligned}B_J(x) &= \langle (\psi^\sharp(*\delta_c^M\alpha))_x - (*\psi^\sharp(\delta_c^M\alpha))_x | W_J^{\mathbb{H}}\rangle \\ &= \langle (\delta_c^M\alpha)_{\psi(x)} | \Lambda_{h-1}(R_x)(* W_J^{\mathbb{H}})\rangle - \langle (\delta_c^M\alpha)_{\psi(x)} | * \Lambda_{2n+2-h}(R_x) W_J^{\mathbb{H}}\rangle\end{aligned} \tag{26}$$

and that vanish when evaluated at the point $x = e$.

Remark 4.6. Let us denote by $r_{j,x}$ the coefficients of $\Lambda_h(R_x)W_I^{\mathbb{H}}$. The $r_{j,x}$ are smooth maps that vanish at $x = e$. Let us notice that, even though the coefficients b_I are functions on $\mathbb{H}^n$, by the second equality of (24) in the proof mentioned earlier, they can be expressed as a linear combination of terms of the form

$$\alpha_{i,\psi(x)}\ r_{j,x} = (\alpha_i \circ \psi)(x)\ r_j(x), \tag{27}$$

where we used the subscript ψ to highlight the dependence of b_I on the map ψ.

Remark 4.7. Let us assume α to be a form on M of degree h, then by Lemma 4.4, we know that

$$d_c^{\mathbb{H}}\psi^\sharp(*\alpha) - d_c^{\mathbb{H}} * \psi^\sharp(\alpha) = d_c^{\mathbb{H}} \sum_I b_I \xi_I^{\mathbb{H}}. \tag{28}$$

Applying the Hodge $*$-operator, then we obtain

$$*d_c^{\mathbb{H}}\psi^\sharp(*\alpha) - * d_c^{\mathbb{H}} * \psi^\sharp(\alpha) = * d_c^{\mathbb{H}} \sum_I b_I \xi_I^{\mathbb{H}}.$$

Keeping in mind that $d_c^{\mathbb{H}}\psi^\sharp = \psi^\sharp d_c^M$, the expression mentioned earlier becomes $*\psi^\sharp(d_c^M * \alpha) - \delta_c^{\mathbb{H}}\psi^\sharp(\alpha) = *d_c^{\mathbb{H}}\sum_I b_I \xi_I^{\mathbb{H}}$. Therefore, writing, up to a sign, $1 = **$, from the last equality, we have $*\psi^\sharp(** d_c^M * \alpha) - \delta_c^{\mathbb{H}}\psi^\sharp(\alpha) = * d_c^{\mathbb{H}}\sum_I b_I \xi_I^{\mathbb{H}}$, and, again up to a sign, we eventually obtain

$$\delta_c^{\mathbb{H}}\psi^\sharp(\alpha) = *\ \psi^\sharp(*\delta_c^M\alpha) + *\ d_c^{\mathbb{H}}\sum_I b_I\xi_I^{\mathbb{H}}. \tag{29}$$

If $2n+1-h \neq n$ (i.e., $h \neq n+1$), the differential $d_c^{\mathbb{H}}$ has order 1, then by left-invariance, we have

$$*d_c^{\mathbb{H}}\sum_I b_I\xi_I^{\mathbb{H}} = \sum_{\ell,I}(W_\ell^{\mathbb{H}}b_I) * (\omega_\ell^{\mathbb{H}} \wedge \xi_I).$$

Then, $*d_c^{\mathbb{H}}\sum_I b_I\xi_I^{\mathbb{H}}$ is a form of degree $h-1$ whose coefficients are of the type

$$(W_\ell^{\mathbb{H}}\alpha_{i,\psi(x)})\cdot r_{j,x} \ \text{ or } \ \alpha_{i,\psi(x)}\cdot(W_\ell^{\mathbb{H}}r_{j,x}).$$

When $h = n+1$ the differential $d_c^{\mathbb{H}}$ has order two. Then $*d_c^{\mathbb{H}}\sum_I b_I\xi_I^{\mathbb{H}}$ is a form of degree n whose coefficients are of type

$$(W_\ell^{\mathbb{H}}W_\lambda^{\mathbb{H}}\alpha_{i,\psi(x)})\cdot r_{j,x},\ (W_\ell^{\mathbb{H}}\alpha_{i,\psi(x)})\cdot W_\lambda^{\mathbb{H}}r_{j,x} \ \text{ or } \ \alpha_{i,\psi(x)}\cdot(W_\ell^{\mathbb{H}}W_\lambda^{\mathbb{H}}r_{j,x}).$$

Moreover, by (26), the coefficients B_J are of the type

$$(W_\ell^{\mathbb{H}}\alpha_{i,\psi(x)})\cdot r_{j,x} \ \text{ if } h \neq n+1, \quad \text{or} \quad (W_\ell^{\mathbb{H}}W_\lambda^{\mathbb{H}}\alpha_{i,\psi(x)})\cdot r_{j,x} \ \text{ if } h = n+1.$$

We are now in position to examine the interplay between $\psi^\sharp\delta_c^M$ and $\delta_c^{\mathbb{H}}\psi^\sharp$.

Proposition 4.8. *With the same hypotheses of* Lemma 4.4, *we have*

$$\delta_c^{\mathbb{H}}\psi^\sharp(\alpha) = \psi^\sharp(\delta_c^M\alpha) + *\ d_c^{\mathbb{H}}\sum_I b_I\xi_I^{\mathbb{H}} + *\ \sum_J B_J\xi_J^{\mathbb{H}}, \tag{30}$$

where the coefficient b_I and B_J are defined in (24) *and* (26), *respectively*.

Proof. We start from (29) and combine with (25) to obtain, up to a sign,

$$\delta_c^{\mathbb{H}}\psi^\sharp(\alpha) = *\ *\ \psi^\sharp(\delta_c^M\alpha) + *\ d_c^{\mathbb{H}}\sum_I b_I\xi_I^{\mathbb{H}} + *\ \sum_J B_J\xi_J^{\mathbb{H}}.$$ □

Given a smooth h-form α on M, we show below that the L^p-norm of

$$\delta_c^{\mathbb{H}}\psi^\sharp\alpha - \psi^\sharp\delta_c^M\alpha$$

is small if α is supported in a suitably chosen small ball. To show this, we have to handle terms of the form $W_\ell^{\mathbb{H}}(\alpha_{i,\psi(x)})\cdot r_{j,x}$ and terms of the form $\alpha_{i,\psi(x)}\cdot(W_\ell^{\mathbb{H}}r_{j,x})$ (and with a little difference in the case $h = n+1$). The proof relies on two different approaches: for terms like $\alpha_{i,\psi(x)}\cdot(W_\ell^{\mathbb{H}}r_{j,x})$, we use a Sobolev inequality and the fact that the support of α is small; on the contrary, when we want to handle terms of the form $W_\ell^{\mathbb{H}}(\alpha_{i,\psi(x)})\cdot r_{j,x}$, we use the fact that the $r_{j,x}$ tend to zero as $x \to e$. Remember that the $r_{j,x}$ are the coefficients of $\Lambda_h(R_x)W_I^{\mathbb{H}}$, which depend only on the map ψ (see Lemma 4.3), and the map ψ can be controlled with constants depending only on the geometry of M, i.e., on r and C_M (by Theorem 3.10). In conclusion, we find that the radius of the support can be chosen independently of α and depends only on the geometry of M.

Proposition 4.9. *For any $a \in M$, we consider the map $\psi = \psi_a$ as in* Theorem 3.10. *With the notation of* Proposition 4.8, *we define the operator*

$$\mathcal{P}(x, W^{\mathbb{H}})\psi^\sharp\alpha := \delta_c^{\mathbb{H}}(\psi^\sharp\alpha) - \psi^\sharp(\delta_c^M\alpha) = *\ d_c^{\mathbb{H}}\sum_I b_I\xi_I^{\mathbb{H}} + *\ \sum_J B_J\xi_J^{\mathbb{H}}. \tag{31}$$

The operator $\mathcal{P}$ is a linear differential operator on $\mathbb{H}^n$, which is of second-order if $h = n+1$ and of first-order otherwise.

Let $1 < p < \infty$ and let $\varepsilon > 0$. Then there exists $\bar\eta = \bar\eta(C_M, r, \varepsilon) > 0$ such that if $\eta < \bar\eta(C_M, \varepsilon)$ and α is a smooth h-form on M supported in $B(a, \eta)$, we have

$$\|\mathcal{P}(\psi^\sharp\alpha)\|_{L^p(\psi^{-1}(B(a,\eta)),E_0^{h-1})} \le \varepsilon\|\psi^\sharp\alpha\|_{W^{1,p}(\psi^{-1}(B(a,\eta)),E_0^h)}, \tag{32}$$

if $h \neq n+1$, *or*

$$\|\mathcal{P}(\psi^\sharp\alpha)\|_{L^p(\psi^{-1}(B(a,\eta)),E_0^{n})} \le \varepsilon\|\psi^\sharp\alpha\|_{W^{2,p}(\psi^{-1}(B(a,\eta)),E_0^{n+1})}. \tag{33}$$

Proof. By Proposition 4.8, the operator $\mathcal{P}$ acts on $\psi^\sharp\alpha$, since we can write

$$\mathcal{P}(x, W^{\mathbb{H}})\psi^\sharp\alpha = \delta_c^{\mathbb{H}}(\psi^\sharp\alpha) - \psi^\sharp\delta_c^M(\psi^\sharp)^{-1}(\psi^\sharp\alpha).$$

As pointed out in Lemma 4.4, the smooth maps $r_{j,x}$ vanish at $x = e$, and since they are the coefficients of $\Lambda_{2n+1-h}(R_x)W_I^{\mathbb{H}}$, they do not depend on α but only on ψ.

Keeping into account (27), which expresses b_I, we obtain estimates of the type

$$|b_I(x)| \le |\alpha_{i,\psi(x)}|\Omega(x),$$

where $\Omega(x) = O(\rho(x))$ for $x \to e$ (with a slight abuse of notation here and in the sequel we avoid to take the sum over the index i).

Moreover, if $h \neq n+1$, from (29), by the triangular inequality, we also obtain estimates of the type

$$|\delta_c^{\mathbb{H}}\psi^\sharp(\alpha) - \psi^\sharp(\delta_c^M\alpha)| \le \sum_\ell |W_\ell^{\mathbb{H}}\alpha_{i,\psi(x)}|\Omega(x) + |\alpha_{i,\psi(x)}\cdot\sum_\ell W_\ell^{\mathbb{H}} r_{j,x}|, \tag{34}$$

where $\Omega(x) = O(\rho(x))$ for $x \to e$.

Similarly, if $h = n+1$, remembering also that d_c^M is a second-order differential operator, we have estimates of the type

$$|\delta_c^{\mathbb{H}}\psi^\sharp(\alpha) - \psi^\sharp(\delta_c^M\alpha)| \le \sum_{\ell,\lambda}|W_\ell^{\mathbb{H}}W_\lambda^{\mathbb{H}}\alpha_{i,\psi(x)}|\Omega(x) + \sum_\ell |W_\ell^{\mathbb{H}}\alpha_{i,\psi(x)}\cdot\sum_\lambda W_\lambda^{\mathbb{H}} r_{j,x}| + |\alpha_{i,\psi(x)}\cdot\sum_{\ell,\lambda}(W_\ell^{\mathbb{H}}W_\lambda^{\mathbb{H}})r_{j,x}|, \tag{35}$$

where $\Omega(x) = O(\rho(x))$ for $x \to e$.

- Case $h \neq n+1$.

Let us estimate the L^p-norm of the first term in the right-hand side of (34). If $x \to e$ then $\Omega(x) \to 0$, then the L^p-norm of term $\sum_\ell |W_\ell^{\mathbb{H}}\alpha_{i,\psi(x)}|\Omega(x)$ is controlled by $\varepsilon\sum_\ell \|W_\ell^{\mathbb{H}}\psi^\sharp\alpha\|_{L^p}$ provided x is sufficiently close to e. Thus, now we need to estimate the second term of the right-hand side of (34).

Hence, we need only to handle carefully the terms that can be expressed as a linear combination of terms of the form:

$$\alpha_{i,\psi(x)}\cdot(W_\ell^{\mathbb{H}} r_{j,x}) = \alpha_i \circ \psi(x)\cdot(W_\ell^{\mathbb{H}} r_{j,x}).$$

The functions $W_\ell^{\mathbb{H}} r_{j,x}$ are bounded in $\psi^{-1}(B(a,\eta))$. But $B\left(e, \frac{\eta}{C_M}\right) \subseteq \psi^{-1}(B(a,\eta)) \subseteq B(e, C_M\eta)$, and by the Sobolev inequality,

$$\left(\int_{B(e,C_M\eta)} (\alpha_i \circ \psi(x))^p \mathrm{d}x\right)^{1/p} \le c_p C_M \eta \left(\int_{B(e,C_M\eta)} \sum_\ell |W_\ell^{\mathbb{H}}(\alpha_i \circ \psi)(x)|^p \mathrm{d}x\right)^{1/p},$$

where c_p denotes the Sobolev constant (depending only on p and n).

If we chose η so that $c_p C_M \eta < \varepsilon$, i.e., $\eta < \bar\eta$, where

$$\bar\eta = \frac{\varepsilon}{c_p C_M}, \tag{36}$$

and finally, we obtain,

$$\|\alpha_{i,\psi(x)}\cdot(W_\ell^{\mathbb{H}} r_{j,x})\|_{L^p(\psi^{-1}(B(a,\eta)))} \le \varepsilon\sum_\ell \|W_\ell^{\mathbb{H}}\alpha_{i,\psi}\|_{L^p(\psi^{-1}(B(a,\eta)))}.$$

Therefore, reasoning on the differential form α, and possibly relabeling ε, we obtain (32).

- Case $h = n + 1$. Arguing again as mentioned earlier, we notice that the first term on the right-hand side of (35) is of the form $(W_\ell^{\mathbb{H}} W_\lambda^{\mathbb{H}}(\alpha_{i,\psi(x)}))\cdot r_{j,x}$ and can be estimated by $\varepsilon\sum_{\ell,\lambda}\|W_\ell^{\mathbb{H}} W_\lambda^{\mathbb{H}}(\psi^\sharp\alpha)\|_{L^p}$ since $r_{j,x} \to 0$ if $x \to e$.

Hence, we need only to handle carefully the terms that can be expressed as a linear combination of terms of the form

$$(W_\ell^{\mathbb{H}}\alpha_{i,\psi(x)} \cdot W_\lambda^{\mathbb{H}})r_{j,x} \quad \text{and} \quad \alpha_{i,\psi(x)} \cdot (W_\ell^{\mathbb{H}} W_\lambda^{\mathbb{H}})r_{j,x}.$$

They can be handled again by using the Sobolev inequality, since both first and second derivatives of $r_{j,x}$ are bounded in $B(e, C_M\eta)$. Indeed, when we apply Sobolev inequality to terms of the form $W_\ell^{\mathbb{H}}\alpha_{i,\psi(x)}$, we obtain an estimate with terms of the form $c_pC_M\eta\sum_{\ell,\lambda}\|W_\ell^{\mathbb{H}} W_\lambda^{\mathbb{H}}(\psi^\sharp\alpha)\|_{L^p(\psi^{-1}(B(a,\eta)))}$. Similarly, the terms of the type $\alpha_{i,\psi(x)} \cdot (W_\ell^{\mathbb{H}} W_\lambda^{\mathbb{H}})r_{j,x}$ can be estimated by $c_pC_M\eta\sum_\ell\|W_\ell^{\mathbb{H}}(\psi^\sharp\alpha)\|_{L^p(\psi^{-1}(B(a,\eta)))}$.

Therefore, again choosing $c_pC_M\eta \le \varepsilon$, eventually we obtain (33). □

Notice that $\bar\eta$ depends also on p, n, but this dependence is not explicit in the statement given earlier since it is well known and what is relevant to us is to show the dependence on the geometry of M.

Remark 4.10. If the differential form α is of degree n, to prove Theorem 5.4, we shall also need to know the interplay between $\psi^\sharp(d_c^M\delta_c^M)$ and $d_c^{\mathbb{H}}\delta_c^{\mathbb{H}}\psi^\sharp$.

We set

$$Q(x, W^{\mathbb{H}})(\psi^\sharp\alpha) := d_c^{\mathbb{H}}\delta_c^{\mathbb{H}}(\psi^\sharp\alpha) - \psi^\sharp(d_c^M\delta_c^M\alpha). \tag{37}$$

Now, $Q(\psi^\sharp\alpha) := d_c^{\mathbb{H}}\delta_c^{\mathbb{H}}(\psi^\sharp\alpha) - \psi^\sharp(d_c^M\delta_c^M\alpha) = d_c^{\mathbb{H}}\delta_c^{\mathbb{H}}(\psi^\sharp\alpha) - d_c^{\mathbb{H}}\psi^\sharp(\delta_c^M\alpha)$. Hence, keeping in mind (31), we easily see that

$$Q(x, W^{\mathbb{H}}) = d_c^{\mathbb{H}}\mathcal{P}(x, W^{\mathbb{H}})$$

is a second-order differential operator (since $\mathcal{P}\psi^\sharp\alpha$ is a form of degree $n - 1$ and hence $d^{\mathbb{H}}$ is a differential operator of degree 1). To estimate the L^p norm of $Q(x, W^{\mathbb{H}})(\psi^\sharp\alpha)$, we have to estimate terms of type

$$(W_\ell^{\mathbb{H}} W_\lambda^{\mathbb{H}}(\alpha_{i,\psi(x)}))\cdot r_{j,x},\ (W_\ell^{\mathbb{H}}\alpha_{i,\psi(x)})\cdot W_\lambda^{\mathbb{H}}r_{j,x} \quad \text{or} \quad \alpha_{i,\psi(x)} \cdot (W_\ell^{\mathbb{H}} W_\lambda^{\mathbb{H}})r_{j,x}.$$

Hence, with the notation of Proposition 4.9, given $\varepsilon > 0$ if $\eta < \bar\eta(C_M, \varepsilon)$ and α is supported in $B(a, \eta)$, then

$$\|Q(\psi^\sharp\alpha)\|_{L^p} \le \varepsilon\|\psi^\sharp\alpha\|_{W^{2,p}}. \tag{38}$$

Likewise, if the differential form α is of degree $n + 1$, in proving Theorem 5.4, we shall need also to evaluate the difference $\delta_c^{\mathbb{H}}d_c^{\mathbb{H}}(\psi^\sharp\alpha) - \psi^\sharp(\delta_c^M d_c^M\alpha)$. Let us set

$$\mathcal{T}(x, W^{\mathbb{H}})(\psi^\sharp\alpha) := \delta_c^{\mathbb{H}}d_c^{\mathbb{H}}(\psi^\sharp\alpha) - \psi^\sharp(\delta_c^M d_c^M\alpha).$$

Hence,

$$\mathcal{T}(x, W^{\mathbb{H}}) = \mathcal{P}(x, W^{\mathbb{H}})d_c^{\mathbb{H}}$$

is again a second-order differential operator (since $d_c^{\mathbb{H}}\psi^\sharp\alpha$ is a form of degree $n + 2$ and hence $\mathcal{P}$ is a differential operator of degree 1). As for the operator (38), we have

$$\|\mathcal{T}(\psi^\sharp\alpha)\|_{L^p} \le \varepsilon\|\psi^\sharp\alpha\|_{W^{2,p}}. \tag{39}$$

The estimates (38) and (39) obtained in Proposition 4.9 will be used in the proof of Theorem 5.2.

In the sequel, to prove a Gaffney estimate on M, once we have chosen an atlas on M and a partition of the unity subordinated to the chosen covering of M, we will need also some L^p estimates of the commutator between the operator d_c^M (or δ_c^M) with a smooth function. A key step to obtain such L^p estimates is given by the following lemma (since we are only interested in estimates of norms, the following inequalities are true up to signs).

Lemma 4.11. *Let ψ be a contactomorphism from an open set $\mathcal{U} \subset \mathbb{H}^n$ to M, and denote by $\mathcal{V}$ the open set $\mathcal{V} = \phi(\mathcal{U})$. If χ is a smooth function in M and $\alpha \in E_0^h(\mathcal{V})$, we have*

$$\psi^\sharp([d_c^M, \chi]\alpha) = [d_c^{\mathbb{H}}, \chi \circ \psi]\psi^\sharp\alpha \quad \text{for any } h, \tag{40}$$

$$\psi^\sharp([\delta_c^M, \chi]\alpha) = [\delta_c^{\mathbb{H}}, \chi \circ \psi]\psi^\sharp\alpha + [\mathcal{P}, \chi \circ \psi]\psi^\sharp(*\alpha) \quad \text{if } h \neq n+1, \tag{41}$$

$$\psi^\sharp([d_c^M\delta_c^M, \chi]\alpha) = [d_c^{\mathbb{H}}\delta_c^{\mathbb{H}}\chi \circ \psi]\psi^\sharp\alpha + [d_c^{\mathbb{H}}\mathcal{P}, \chi \circ \psi]\psi^\sharp\alpha \quad \text{if } h = n, \tag{42}$$

$$\psi^\sharp([\delta_c^M d_c^M, \chi]\alpha) = [\delta_c^{\mathbb{H}} d_c^{\mathbb{H}}, \chi \circ \psi]\psi^\sharp\alpha + [\mathcal{P} d_c^{\mathbb{H}}, \chi \circ \psi^\sharp]\psi^\sharp\alpha \quad \text{if } h = n+1. \tag{43}$$

Proof. The first equality follows directly from the fact that the Rumin differential and the pullback of a contact map commute (see also Proposition 2.16):

$$\begin{aligned}\psi^\sharp([d_c^M, \chi]\alpha) &= \psi^\sharp d_c^M(\chi\alpha) - \psi^\sharp(\chi d_c^M\alpha) = d_c^{\mathbb{H}}\psi^\sharp(\chi\alpha) - \chi \circ \psi \cdot \psi^\sharp(d_c^M\alpha) = d_c^{\mathbb{H}}(\chi \circ \psi \cdot \psi^\sharp\alpha) - \chi \circ \psi \cdot d_c^{\mathbb{H}}\psi^\sharp\alpha \\ &= [d_c^{\mathbb{H}}, \chi \circ \psi]\psi^\sharp\alpha.\end{aligned}$$

For the second formula, the codifferential and the pullback map do not commute; however, we can use (31):

$$\begin{aligned}\psi^\sharp([\delta_c^M, \chi]\alpha) &= \psi^\sharp\delta_c^M(\chi\alpha) - \psi^\sharp(\chi\delta_c^M\alpha) \\ &= \delta_c^{\mathbb{H}}\psi^\sharp(\chi\alpha) + \mathcal{P}\psi^\sharp(\chi\alpha) - \chi \circ \psi(\psi^\sharp\delta_c^M\alpha) \\ &= \delta_c^{\mathbb{H}}(\chi \circ \psi \cdot \psi^\sharp\alpha) + \mathcal{P}(\chi \circ \psi \cdot \psi^\sharp\alpha) - \chi \circ \psi(\delta_c^{\mathbb{H}}\psi^\sharp\alpha + \mathcal{P}\psi^\sharp\alpha) \\ &= [\delta_c^{\mathbb{H}}, \chi_j \circ \psi_j]\psi_j^\sharp\alpha + [\mathcal{P}, \chi \circ \psi]\psi^\sharp\alpha.\end{aligned}$$

The third and fourth formulae will follow by using a similar reasoning as earlier:

$$\begin{aligned}\psi^\sharp([d_c^M\delta_c^M, \chi]\alpha) &= \psi^\sharp(d_c^M\delta_c^M(\chi\alpha)) - \psi^\sharp(\chi d_c^M\delta_c^M\alpha) \\ &= d_c^{\mathbb{H}}\psi^\sharp(\delta_c^M(\chi\alpha)) - \chi \circ \psi d_c^{\mathbb{H}}\psi^\sharp(\delta_c^M\alpha) \\ &= d_c^{\mathbb{H}}(\delta_c^{\mathbb{H}}\psi^\sharp(\chi\alpha) + \mathcal{P}\psi^\sharp(\chi\alpha)) - \chi \circ \psi d_c^{\mathbb{H}}(\delta_c^{\mathbb{H}}\psi^\sharp\alpha + \mathcal{P}\psi^\sharp\alpha) \\ &= d_c^{\mathbb{H}}(\delta_c^{\mathbb{H}}(\chi \circ \psi \cdot \psi^\sharp\alpha) + \mathcal{P}(\chi \circ \psi \cdot \psi^\sharp\alpha)) - \chi \circ \psi d_c^{\mathbb{H}}(\delta_c^{\mathbb{H}}\psi^\sharp\alpha + \mathcal{P}\psi^\sharp\alpha) \\ &= [d_c^{\mathbb{H}}\delta_c^{\mathbb{H}}, \chi \circ \psi]\psi^\sharp\alpha + [d_c^{\mathbb{H}}\mathcal{P}, \chi \circ \psi]\psi^\sharp\alpha\end{aligned}$$

and as for (43), using again (31), we obtain

$$\begin{aligned}\psi^\sharp([\delta_c^M d_c^M, \chi]\alpha) &= \psi^\sharp(\delta_c^M d_c^M(\chi\alpha)) - \psi^\sharp(\chi\delta_c^M d_c^M\alpha) \\ &= \delta_c^{\mathbb{H}}\psi^\sharp d_c^M(\chi\alpha) + \mathcal{P}\psi^\sharp d_c^M(\chi\alpha) - \chi \circ \psi(\delta_c^{\mathbb{H}}\psi^\sharp d_c^M\alpha + \mathcal{P}\psi^\sharp d_c^M\alpha) \\ &= \delta_c^{\mathbb{H}} d_c^{\mathbb{H}}\psi^\sharp(\chi\alpha) + \mathcal{P} d_c^{\mathbb{H}}\psi^\sharp(\chi\alpha) - \chi \circ \psi(\delta_c^{\mathbb{H}} d_c^{\mathbb{H}}\psi^\sharp\alpha + \mathcal{P} d_c^{\mathbb{H}}\psi^\sharp\alpha) \\ &= [\delta_c^{\mathbb{H}} d_c^{\mathbb{H}}, \chi \circ \psi]\psi^\sharp\alpha + [\mathcal{P} d_c^{\mathbb{H}}, \chi \circ \psi]\psi^\sharp\alpha.\end{aligned}$$

□

5 Sobolev-Gaffney type inequalities on contact manifolds

We recall the following Sobolev-Gaffney type inequalities proved in the setting of Heisenberg groups for differential forms in $\mathcal{D}(\mathbb{H}^n, E_0^h)$ (see [4], Remark 5.3, (i), (iii), (vi) therein). By density of $\mathcal{D}(\mathbb{H}^n, E_0^h)$ in $W^{k,p}(\mathbb{H}^n, E_0^h)$ (see Theorem 2.3), this result can be rephrased as follows.

Lemma 5.1. *Let $1 \leq h \leq 2n$, and $1 < p < \infty$. Then there exists a constant $C_G = C_G(p, n, h) > 0$ such that for all $u \in W^{1,p}(\mathbb{H}^n, E_0^h)$, we have:*

(i) if $h \neq n, n+1$,

$$\|u\|_{W^{1,p}(\mathbb{H}^n, E_0^h)} \leq C_G(\|d_c^{\mathbb{H}} u\|_{L^p(\mathbb{H}^n, E_0^{h+1})} + \|\delta_c^{\mathbb{H}} u\|_{L^p(\mathbb{H}^n, E_0^{h-1})} + \|u\|_{L^p(\mathbb{H}^n, E_0^h)}), \tag{44}$$

(ii) if $h = n$,

$$\|u\|_{W^{2,p}(\mathbb{H}^n,E_0^n)} \le C_G(\|d_c^{\mathbb{H}}u\|_{L^p(\mathbb{H}^n,E_0^{n+1})} + \|d_c^{\mathbb{H}}\delta_c^{\mathbb{H}}u\|_{L^p(\mathbb{H}^n,E_0^n)} + \|u\|_{L^p(\mathbb{H}^n,E_0^n)}), \tag{45}$$

and

(*iii*) *if* $h = n + 1$,

$$\|u\|_{W^{2,p}(\mathbb{H}^n,E_0^{n+1})} \le C_G(\|\delta_c^{\mathbb{H}}d_c^{\mathbb{H}}u\|_{L^p(\mathbb{H}^n,E_0^{n+1})} + \|\delta_c^{\mathbb{H}}u\|_{L^p(\mathbb{H}^n,E_0^n)} + \|u\|_{L^p(\mathbb{H}^n,E_0^{n+1})}). \tag{46}$$

Before stating the global result, we prove the following local one where we can use the groundwork just developed, together with the Gaffney-Sobolev inequality stated in the previous result.

Theorem 5.2. *If* $1 < p < \infty$, *there exists a positive constant* $\tilde\eta = \tilde\eta(C_M, C_G)$ *such that, if* $\eta < \tilde\eta$, $(B(a_\ell,\eta), \psi_{a_\ell})$ *is a chart of the atlas given in Remark* 3.5, *and* α *is a smooth form in* M *with support contained in* $B(a_\ell,\eta)$, *then there exists a constant* $C = C(C_M, C_G)$ *such that, if* $h \neq n, n+1$, *then*

$$\begin{aligned}\|\psi_{a_\ell}^\sharp\alpha\|_{W^{1,p}\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^h\right)} \le C(&\|\psi_{a_\ell}^\sharp\alpha\|_{L^p\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^h\right)} + \|\psi_{a_\ell}^\sharp(d_c^M\alpha)\|_{L^p\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^{h+1}\right)}\\ &+ \|\psi_{a_\ell}^\sharp(\delta_c^M\alpha)\|_{L^p\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^{h-1}\right)}).\end{aligned} \tag{47}$$

Whereas, if $h = n$, *we obtain*

$$\begin{aligned}\|\psi_{a_\ell}^\sharp\alpha\|_{W^{2,p}\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^n\right)} \le C(&\|\psi_{a_\ell}^\sharp\alpha\|_{L^p\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^n\right)} + \|\psi_{a_\ell}^\sharp(d_c^M\alpha)\|_{L^p\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^{n+1}\right)}\\ &+ \|\psi_{a_\ell}^\sharp(d_c^M\delta_c^M\alpha)\|_{L^p\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^n\right)}).\end{aligned} \tag{48}$$

and if $h = n + 1$, *we obtain*

$$\begin{aligned}\|\psi_{a_\ell}^\sharp\alpha\|_{W^{2,p}\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^{n+1}\right)} \le C(&\|\psi_{a_\ell}^\sharp\alpha\|_{L^p\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^{n+1}\right)} + \|\psi_{a_\ell}^\sharp(\delta_c^M d_c^M\alpha)\|_{L^p\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^{n+1}\right)}\\ &+ \|\psi_{a_\ell}^\sharp(\delta_c^M\alpha)\|_{L^p\left(\psi_{a_\ell}^{-1}(B(a_\ell,\eta)),E_0^n\right)}).\end{aligned} \tag{49}$$

Proof. We consider a chart $(B(a_\ell,\eta), \psi_{a_\ell})$ and assume that ψ_{a_ℓ} is taken as in Theorem 3.10.

To avoid cumbersome notation, in the sequel, we omit the subscripts, and we write $B(a,\eta)$ and ψ. Moreover, we set

$$\tilde B_\eta := \psi^{-1}(B(a,\eta))$$

and write $L^p(\tilde B_\eta)$ instead of $L^p(\psi^{-1}(B(a,\eta)), E_0^h)$ and similarly for the notation on Sobolev spaces. If α is supported in $B(a,\eta)$ then, without loss of generality, we may assume that $\psi^\sharp\alpha$ is compactly supported in $B(e, \eta/C_M)$, since $B\left(e, \frac{\eta}{C_M}\right) \subseteq \psi^{-1}(B(a,\eta)) \subseteq B(e, C_M\eta)$.

To prove (47), we use first (44) with (31) and then (32): given $\varepsilon > 0$, by Proposition 4.9, there exists $\overline{\eta}$ (see (36)) so that, if $\eta < \overline{\eta}$, we obtain

$$\begin{aligned}\|\psi^\sharp\alpha\|_{W^{1,p}(\tilde B_\eta)} &\le C_G(\|\psi^\sharp\alpha\|_{L^p(\mathbb{H}^n)} + \|d_c^{\mathbb{H}}(\psi^\sharp\alpha)\|_{L^p(\mathbb{H}^n)} + \|\delta_c^{\mathbb{H}}(\psi^\sharp\alpha)\|_{L^p(\mathbb{H}^n)})\\ &\le C_G(\|\psi^\sharp\alpha\|_{L^p(\tilde B_\eta)} + \|\psi^\sharp(d_c^M\alpha)\|_{L^p(\tilde B_\eta)} + \|\psi^\sharp(\delta_c^M\alpha)\|_{L^p(\tilde B_\eta)} + \|\mathcal{P}(\psi^\sharp\alpha)\|_{L^p(\tilde B_\eta)})\\ &\le C_G(\|\psi^\sharp\alpha\|_{L^p(\tilde B_\eta)} + \|\psi^\sharp(d_c^M\alpha)\|_{L^p(\tilde B_\eta)} + \|\psi^\sharp(\delta_c^M\alpha)\|_{L^p(\tilde B_\eta)} + \varepsilon\|\psi^\sharp\alpha\|_{W^{1,p}(\tilde B_\eta)}).\end{aligned}$$

Choosing $\varepsilon \le 1/(2C_G)$, we have proved (47) for $\eta < \tilde\eta := \frac{1}{2c_pC_MC_G}$ (notice that in the statement, again, the dependence on c_p was omitted).

Let now $h = n$. The argument mentioned earlier needs to be only slightly modified. Indeed, we will apply the Gaffney inequality (45), where both $d_c^{\mathbb{H}}$ and $d_c^{\mathbb{H}}\delta_c^{\mathbb{H}}$ appearing on the right-hand side are differential operators of order 2. Therefore, keeping also in mind (38), we obtain

$$\begin{aligned}\|\psi^\sharp\alpha\|_{W^{2,p}(\tilde B_\eta)} &\le C_G\{\|\psi^\sharp\alpha\|_{L^p(\mathbb{H}^n)} + \|d_c^{\mathbb{H}}(\psi^\sharp\alpha)\|_{L^p(\mathbb{H}^n)} + \|d_c^{\mathbb{H}}\delta_c^{\mathbb{H}}(\psi^\sharp\alpha)\|_{L^p(\mathbb{H}^n)}\}\\ &\le C_G\{\|\psi^\sharp\alpha\|_{L^p(\tilde B_\eta)} + \|\psi^\sharp(d_c^M\alpha)\|_{L^p(\tilde B_\eta)} + \|\psi^\sharp(d_c^M\delta_c^M\alpha)\|_{L^p(\tilde B_\eta)} + \|Q(\psi^\sharp\alpha)\|_{L^p(\tilde B_\eta)}\}\\ &\le C_G\{\|\psi^\sharp\alpha\|_{L^p(\tilde B_\eta)} + \|\psi^\sharp(d_c^M\alpha)\|_{L^p(\tilde B_\eta)} + \|\psi^\sharp(d_c^M\delta_c^M\alpha)\|_{L^p(\tilde B_\eta)} + \varepsilon\|\psi^\sharp\alpha\|_{W^{2,p}(\tilde B_\eta)}\}.\end{aligned}$$

Choosing $\varepsilon < 1/(2C_G)$, we can absorb the term $\varepsilon\|\psi^\sharp\alpha\|_{W^{2,p}(\tilde B_\eta)}$ in the left-hand side and eventually obtain (48).

The case $h = n+1$ can be handled similarly, taking into account (39) and proving therefore (49). □

As noticed in Remark 3.5, if α is supported in $B(a,\eta)$, then the norms

$$\|\alpha\|_{W^{\ell,p}(M,E_0^\bullet)} \quad \text{and} \quad \|\psi^\sharp\alpha\|_{W^{\ell,p}(\mathbb{H}^n,E_0^\bullet)}$$

are equivalent, with equivalence constants independent of ψ. From the previous theorem, we immediately obtain the following local result in M.

Remark 5.3. Under the notation of Theorem 5.2, let $B(a,\eta)$ be a ball satisfying Remark 3.5. Let α be a smooth form supported in $B(a,\eta)$. If $\eta < \tilde\eta(C_M, C_G)$, then there exists a constant $C > 0$ depending only on C_M and C_G, so that, if $h \neq n, n+1$, we have

$$\|\alpha\|_{W^{1,p}(B(a,\eta),E_0^h)} \le C(\|\alpha\|_{L^p(B(a,\eta),E_0^h)} + \|d_c^M\alpha\|_{L^p(B(a,\eta),E_0^{h+1})} + \|\delta_c^M\alpha\|_{L^p(B(a,\eta),E_0^{h-1})}). \tag{50}$$

Whereas, if $h = n$, we obtain

$$\|\alpha\|_{W^{2,p}(B(a,\eta),E_0^n)} \le C(\|\alpha\|_{L^p(B(a,\eta),E_0^n)} + \|d_c^M\alpha\|_{L^p(B(a,\eta),E_0^{n+1})} + \|d_c^M\delta_c^M\alpha\|_{L^p(B(a,\eta),E_0^n)}), \tag{51}$$

and if $h = n+1$, we obtain

$$\|\alpha\|_{W^{2,p}(B(a,\eta),E_0^{n+1})} \le C(\|\alpha\|_{L^p(B(a,\eta),E_0^{n+1})} + \|\delta_c^M d_c^M\alpha\|_{L^p(B(a,\eta),E_0^{n+1})} + \|\delta_c^M\alpha\|_{L^p(B(a,\eta),E_0^n)}). \tag{52}$$

We are now in position to prove the following Sobolev-Gaffney type inequalities on M if we assume M to be a smooth sub-Riemannian contact manifold without boundary with bounded geometry.

Theorem 5.4. *Let (M,H,g) be a smooth contact manifold with bounded geometry, without boundary. Let $1 \le h \le 2n$, and $1 < p < \infty$. There exists a positive constant $C = C(C_M, C_G)$ such that for all $\alpha \in W^{1,p}(M,E_0^h)$, we have*:

(*i*) *for $h \neq n, n+1$,*

$$\|\alpha\|_{W^{1,p}(M,E_0^h)} \le C(\|d_c^M\alpha\|_{L^p(M,E_0^{h+1})} + \|\delta_c^M\alpha\|_{L^p(M,E_0^{h-1})} + \|\alpha\|_{L^p(M,E_0^h)}); \tag{53}$$

(*ii*) *for $h = n$,*

$$\|\alpha\|_{W^{2,p}(M,E_0^n)} \le C(\|d_c^M\alpha\|_{L^p(M,E_0^{n+1})} + \|d_c^M\delta_c^M\alpha\|_{L^p(M,E_0^n)} + \|\alpha\|_{L^p(M,E_0^n)}); \tag{54}$$

(*iii*) *for $h = n+1$,*

$$\|\alpha\|_{W^{2,p}(M,E_0^{n+1})} \le C(\|\delta_c^M d_c^M\alpha\|_{L^p(M,E_0^{n+1})} + \|\delta_c^M\alpha\|_{L^p(M,E_0^n)} + \|\alpha\|_{L^p(M,E_0^{n+1})}). \tag{55}$$

Proof. Let $\frac{1}{2}\tilde\eta(C_M,C_G) < \eta < \tilde\eta(C_M,C_G)$ (where $\tilde\eta$ is taken as in Theorem 5.2) and consider the countable, locally finite, atlas $\mathcal{U} := \{B(a_j,\eta),\psi_j\}$ of Remark 3.5, where $\psi_j : B(e,\eta/C_M) \to M$. As in Definition 3.3, let now $\{\chi_j\}$ be a partition of unity subordinate to the atlas. Without loss of generality, we can assume $\psi_j^{-1}(\operatorname{supp}\chi_j) \subset B(e,\eta/C_M)$.

We have

$$\alpha = \sum_j \chi_j\alpha.$$

Notice that $\chi_j\alpha$ is supported in $\psi_j(B(e,\eta/C_M))$. By definition, if $\ell = 1, 2$, we have

$$\|\alpha\|_{W^{\ell,p}(M,E_0^\bullet)} := \left(\sum_j \|\psi_j^\sharp(\chi_j\alpha)\|^p_{W^{\ell,p}(\mathbb{H}^n,E_0^\bullet)}\right)^{1/p}.$$

In the sequel c will denote a geometric constant that may vary from line to line, depending in principle on C_M, C_G, η (and on p, h, n). Once we have chosen η such that $\frac{1}{2}\tilde\eta(C_M,C_G) < \eta < \tilde\eta(C_M,C_G)$, the dependence of c is only on C_M, C_G (and on p, h, n).

- Suppose first $h \neq n, n+1$. We divide the proof in three steps.

Step 1. Let $j \in \mathbb{N}$ be fixed, and let $(B(a_j,\eta),\psi_j)$ be a chart of $\mathcal{U}$. We apply Theorem 5.2 to $\psi_j^\sharp(\chi_j\alpha)$ (see (47)), and hence,

$$\|\psi_j^\sharp(\chi_j\alpha)\|_{W^{1,p}\left(B\left(e,\frac{\eta}{C_M}\right),E_0^h\right)} \le c\left\{\|\psi_j^\sharp(d_c^M\chi_j\alpha)\|_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h+1}\right)} + \|\psi_j^\sharp(\delta_c^M\chi_j\alpha)\|_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h-1}\right)} + \|\psi_j^\sharp(\chi_j\alpha)\|_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^h\right)}\right\}.$$

Now, since

$$d_c^M(\chi_j\alpha) = \chi_j d_c^M\alpha + [d_c^M,\chi_j]\alpha, \quad \delta_c^M(\chi_j\alpha) = \chi_j\delta_c^M\alpha + [\delta_c^M,\chi_j]\alpha,$$

from the previous inequality, we obtain

$$\begin{aligned}
&\|\psi_j^\sharp(\chi_j\alpha)\|^p_{W^{1,p}\left(B\left(e,\frac{\eta}{C_M}\right),E_0^h\right)}\\
&\quad\le c\left\{\|\psi_j^\sharp(\chi_j d_c^M\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h+1}\right)} + \|\psi_j^\sharp(\chi_j\delta_c^M\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h-1}\right)}\right.\\
&\qquad\left. + \|\psi_j^\sharp(\chi_j\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^h\right)} + \|\psi_j^\sharp([d_c^M,\chi_j]\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h+1}\right)} + \|\psi_j^\sharp([\delta_c^M,\chi_j]\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h-1}\right)}\right\}.
\end{aligned}\tag{56}$$

Step 2. We show now that we can control the sum with respect to j, of the last two terms in (56), with the norm $\|\alpha\|^p_{L^p(M,E_0^h)}$.

First, by Lemma 4.11, (40) and (41),

$$\psi_j^\sharp[d_c^M,\chi_j]\alpha = [d_c^{\mathbb{H}},\chi_j\circ\psi_j]\psi_j^\sharp\alpha$$

and

$$\psi_j^\sharp[\delta_c^M,\chi_j]\alpha = [\delta_c^{\mathbb{H}},\chi_j\circ\psi_j]\psi_j^\sharp\alpha + * [d_c^{\mathbb{H}},\chi_j\circ\psi_j]\psi_j^\sharp * \alpha.$$

On the other hand, by Lemma 2.13, the differential operators $[d_c^{\mathbb{H}},\chi_j\circ\psi_j]$ and $[\delta_c^{\mathbb{H}},\chi_j\circ\psi_j]$ in $\mathbb{H}^n$ have order 0 if $h \neq n, n+1$. Keeping in mind this fact, we start from the estimate of the L^p-norm of $\psi_j^\sharp[d_c^M,\chi_j]\alpha$. We have:

$$\begin{aligned}
\|[d_c^{\mathbb{H}},\chi_j\circ\psi_j]\psi_j^\sharp\alpha\|_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h+1}\right)} &\le c\sum_{k\in I_j}\|[d_c^{\mathbb{H}},\chi_j\circ\psi_j]\psi_j^\sharp(\chi_k\alpha)\|_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h+1}\right)}\\
&\le c\sum_{k\in I_j}\|\psi_j^\sharp(\chi_k\alpha)\|_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^h\right)}\\
&\le c\sum_{k\in I_j}\|\psi_j^\sharp(\psi_k^\sharp)^{-1}\psi_k^\sharp(\chi_k\alpha)\|_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^h\right)}\\
&\le c\sum_{k\in I_j}\|\psi_k^\sharp(\chi_k\alpha)\|_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^h\right)},
\end{aligned}\tag{57}$$

where $\#I_j \le N(\eta)$ since, by Remark 3.5, $B(a_j,\eta)$ intersect at most $N(\eta)$ other balls of the covering (and where the constant also depends on the uniform bound by C_M, of the horizontal derivatives of $\psi_k^{-1}\circ\psi_j$). However, since η is bounded above and below by a quantity that depends only on C_M and C_G, also the $\#I_j$ can be controlled only by a geometric constant. Hence, also

$$\|[d_c^{\mathbb{H}}, \chi_j \circ \psi_j]\psi_j^{\sharp}\alpha\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^{h+1})} \le c\sum_{k\in I_j}\|\psi_k^{\sharp}(\chi_k\alpha)\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^h)}.$$

Finally, an analogous estimate of the L^p-norm of $[\delta_c^{\mathbb{H}}, \chi_j \circ \psi_j]\psi_j^{\sharp}\alpha$, gives an estimate of the L^p-norm of $\psi_j^{\sharp}[\delta_c^M, \chi_j]\alpha$. Indeed,

$$\|[\delta_c^{\mathbb{H}}, \chi_j \circ \psi_j]\psi_j^{\sharp}\alpha\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h-1}\right)} \le c\sum_{k\in I_j}\|\psi_k^{\sharp}(\chi_k\alpha)\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^h)}.$$

Using once again the fact that the cover is uniformly locally finite and that $\sum_j\sum_{k\in I_j} = \sum_k\sum_{j\in I_k}$, summing up over j in the inequalities mentioned earlier and keeping in mind Remark 3.4, we obtain

$$\begin{aligned}&\sum_j\|\psi_j^{\sharp}([d_c^M, \chi_j]\alpha)\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^{h+1})} + \sum_j\|\psi_j^{\sharp}([\delta_c^M, \chi_j]\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h-1}\right)}\\ &\le c\sum_k\|\psi_k^{\sharp}(\chi_k\alpha)\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^h)} \le c\|\alpha\|^p_{L^p(M,E_0^h)}.\end{aligned}$$

Step 3. Finally, summing up over j in (56), and using the last estimates, we obtain

$$\begin{aligned}&\left(\sum_j\|\psi_j^{\sharp}(\chi_j\alpha)\|^p_{W^{1,p}\left(B\left(e,\frac{\eta}{C_M}\right),E_0^h\right)}\right)^{1/p}\\ &\le c\left\{\left(\sum_j\|\psi_j^{\sharp}(\chi_j d_c^M\alpha)\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^{h+1})}\right)^{1/p} + \left(\sum_j\|\psi_j^{\sharp}(\chi_j\delta_c^M\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{h-1}\right)}\right)^{1/p} + \left(\sum_j\|\psi_j^{\sharp}(\chi_j\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^h\right)}\right)^{1/p}\right\}\\ &\quad + \|\alpha\|_{L^p(M,E_0^h)},\end{aligned}$$

which, keeping again in mind Remark 3.4, gives eventually (53).

- Let now $h = n$.

The argument mentioned earlier needs to be slightly modified. For $j \in \mathbb{N}$ fixed and keeping in mind again that $\chi_j\alpha$ is supported in $\psi_j\left(B\left(e, \frac{\eta}{C_M}\right)\right)$, we apply now (48) to obtain

$$\begin{aligned}&\|\psi_j^{\sharp}(\chi_j\alpha)\|^p_{W^{2,p}(B(e,\frac{\eta}{C_M}),E_0^n)}\\ &\le c\left\{\|\psi_j^{\sharp}(\chi_j d_c^M\alpha)\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^{n+1})} + \|\psi_j^{\sharp}(\chi_j d_c^M\delta_c^M\alpha)\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^n)}\right.\\ &\quad \left. + \|\psi_j^{\sharp}(\chi_j\alpha)\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^n)} + \|\psi_j^{\sharp}([d_c^M, \chi_j]\alpha)\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^{n+1})} + \|\psi_j^{\sharp}([d_c^M\delta_c^M, \chi_j]\alpha)\|^p_{L^p(B(e,\frac{\eta}{C_M}),E_0^n)}\right\}.\end{aligned} \tag{58}$$

Since $h = n$, by Lemmas 2.13 and 4.11, $\psi_j^{\sharp}[d_c^M, \chi_j]$ and $\psi_j^{\sharp}[d_c^M\delta_c^M, \chi_j]$ are now operators of order 1. Reasoning as in Step 2, we can write

$$\|\psi_j^{\sharp}([d_c^M, \chi_j]\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{n+1}\right)} \le \sum_{k\in I_j}\|\psi_k^{\sharp}(\chi_k\alpha)\|^p_{W^{1,p}\left(B\left(e,\frac{\eta}{C_M}\right),E_0^n\right)}.$$

If we combine this with Remark 2.7, then for any $0 < \varepsilon < 1$, there exists a constant $c(\varepsilon)$ such that

$$\begin{aligned}
&\sum_j \|\psi_j^\sharp([d_c^M, \chi_j]\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^{n+1}\right)} \\
&\quad\leq \sum_k \|\psi_k^\sharp(\chi_k\alpha)\|^p_{W^{1,p}\left(B\left(e,\frac{\eta}{C_M}\right),E_0^n\right)} \\
&\quad\leq \varepsilon\sum_k \|\psi_k^\sharp(\chi_k\alpha)\|^p_{W^{2,p}\left(B\left(e,\frac{\eta}{C_M}\right),E_0^n\right)} + c(\varepsilon)\|\sum_k \phi_k^\sharp(\chi_k\alpha)\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^n\right)} \\
&\quad\leq \varepsilon\|\alpha\|^p_{W^{2,p}(M,E_0^n)} + c(\varepsilon)\|\alpha\|^p_{L^p(M,E_0^n)}.
\end{aligned} \tag{59}$$

A similar argument shows that

$$\sum_j \|\psi_j^\sharp([d_c^M\delta_c^M, \chi_j](\chi_j\alpha))\|^p_{L^p\left(B\left(e,\frac{\eta}{C_M}\right),E_0^n\right)} \leq \varepsilon\|\alpha\|^p_{W^{2,p}(M,E_0^n)} + c(\varepsilon)\|\alpha\|^p_{L^p(M,E_0^n)}. \tag{60}$$

Going back to (58), summing over j and taking the power $1/p$ of all the addends, we obtain

$$\|\alpha\|_{W^{2,p}(M,E_0^n)} \leq c(\|d_c^M\alpha\|_{L^p(M,E_0^{n+1})} + \|d_c^M\delta_c^M\alpha\|_{L^p(M,E_0^n)} + \|\alpha\|_{L^p(M,E_0^n)} + \varepsilon\|\alpha\|_{W^{2,p}(M,E_0^n)} + c(\varepsilon)\|\alpha\|_{L^p(M,E_0^n)}).$$

Therefore, absorbing $\varepsilon\|\alpha\|_{W^{2,p}(M,E_0^n)}$ in the left-hand side, up to changing the constants from line to line, we obtain (54).

- The case $h = n + 1$.

We fix again $j \in \mathbb{N}$ and we start from the local estimate (49). The case $h = n + 1$ can be dealt with a similar argument to the case $h = n$. Indeed, we have to use the fact that the differential operators $\psi_j^\sharp[\delta_c^M d_c^M, \chi_j]$ and $\psi_j^\sharp[\delta_c^M, \chi_j]$, again by Lemmas 4.11 and 2.13, are operators in $\mathbb{H}^n$ of order 1. Then we can produce estimates for these operators analogous to the ones in (59) and (60). Finally, to conclude we need, once again, Remark 2.7. □

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