Information and Computation 287 (2022) 104755

Contents lists available at ScienceDirect

Information and Computation

www.elsevier.com/locate/yinco

One-variable fragments of intermediate logics over linear frames ☆

Xavier Caicedo [a,1], George Metcalfe [b,*,2], Ricardo Rodríguez [c,3], Olim Tuyt [b,2]

[a] Departamento de Matemáticas, Universidad de los Andes, Bogotá, Colombia
[b] Mathematical Institute, University of Bern, Switzerland
[c] Departamento de Computación, Universidad de Buenos Aires, Argentina

ARTICLE INFO

Article history:
Available online 6 May 2021

Keywords:
Intermediate logics
One-variable fragments
Gödel modal logics

ABSTRACT

A correspondence is established between one-variable fragments of (first-order) intermediate logics defined over a fixed countable linear frame and Gödel modal logics defined over many-valued equivalence relations with values in a closed subset of the real unit interval. It is also shown that each of these logics can be interpreted in the one-variable fragment of the corresponding constant domain intermediate logic, which is equivalent to a Gödel modal logic defined over (crisp) equivalence relations. Although the latter modal logics in general lack the finite model property with respect to their frame semantics, an alternative semantics is defined that has this property and used to establish co-NP-completeness results for the one-variable fragments of the corresponding intermediate logics both with and without constant domains.

1. Introduction

One-variable fragments of first-order logics are often studied in the guise of propositional modal logics, where each unary predicate $P_i(x)$ is replaced with a propositional variable p_i and the quantifiers $(\forall x)$ and $(\exists x)$ are replaced with the modal operators $\Box$ and $\Diamond$, respectively. This shift in perspective can be useful in bringing algebraic methods to bear on these fragments to obtain axiomatization, finite model properties, and complexity results. Notably, the modal counterparts of the one-variable fragments of first-order classical logic CQC and intuitionistic logic IQC are the modal logic S5 and intuitionistic modal logic MIPC, respectively, both of which possess the finite model property and are decidable. One-variable fragments of (first-order) intermediate logics have been studied from an algebraic perspective in [2–5] and as fragments of classical bimodal logics in [6,7]. A key motivation for studying these fragments and their modal counterparts is that they provide some first-order expressivity (e.g., for modelling the knowledge and belief of agents), while typically remaining decidable; by

☆ Preliminary results from this work were reported in the proceedings of WoLLIC 2019 [1].

* Corresponding author.
E-mail addresses: xcaicedo@uniandes.edu.co (X. Caicedo), george.metcalfe@math.unibe.ch (G. Metcalfe), ricardo@dc.uba.ar (R. Rodríguez), olim.tuyt@math.unibe.ch (O. Tuyt).

[1] Supported by Universidad de los Andes Science Faculty Research Fund.
[2] Supported by Swiss National Science Foundation (SNF) grant 200021_184693.
[3] Partially supported by the Argentinean projects UBA-CyT 20020190100021BA and CONICET PIP 112-2015-0100412 CO (CONICET).

https://doi.org/10.1016/j.ic.2021.104755

contrast, the two-variable fragments of many of these logics are known to be undecidable, even with just a single monadic predicate symbol [8,9].[4]

It was proved in [10] that the constant domain intermediate logic defined over the frame $\langle \mathbb{Q}, \leq \rangle$ (or, equivalently, all linear frames) coincides with the first-order Gödel logic defined over the real unit interval $[0, 1]$. This first-order logic can be axiomatized by extending IQC with the prelinearity and constant domain axiom schemas $(\alpha \to \beta) \vee (\beta \to \alpha)$ and $(\forall x)(\alpha \vee \beta) \to ((\forall x)\alpha \vee \beta)$, respectively, where α and β are first-order formulas and, in the second schema, the variable x is not free in β [11]. The modal counterpart of its one-variable fragment is the Gödel modal logic $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, defined over equivalence relations, that can be axiomatized by extending MIPC with prelinearity and a modal analogue of the constant domain axiom schema [12]. More generally, it was proved in [13] that every constant domain intermediate logic defined over a countable linear frame coincides with a first-order Gödel logic defined over a closed subset A of $[0, 1]$ that contains 0 and 1 (called a Gödel set), and vice versa. The one-variable fragments of these logics correspond to Gödel modal logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$, defined over a Gödel set A, that form modal counterparts of one-variable fragments of first-order Gödel logics. These and other Gödel modal logics have been studied extensively in recent years in the framework of many-valued modal logics (see [14,15,12,16]) with the aim of modelling modal notions such as necessity, belief, and spatio-temporal relations in the presence of multiple degrees of truth, certainty, and possibility (see, e.g., [17–19]) and as a basis for fuzzy description logics (see, e.g., [20–22]).

In this paper, we investigate the analogous situation for one-variable fragments of intermediate logics defined over a countable linear frame without the constant domain assumption. More precisely, we prove that the modal counterpart of such a one-variable fragment is a Gödel modal logic $\mathsf{S5}(\mathbf{A})$ defined over A-valued equivalence relations for some Gödel set A (Section 4), and vice versa (Section 5). Notably, the one-variable fragment of *Corsi logic*, the intermediate logic defined over $\langle \mathbb{Q}, \leq \rangle$ (or, equivalently, all linear frames), axiomatized by extending IQC with prelinearity [23], corresponds to the Gödel modal logic $\mathsf{S5}(\mathbf{G})$, axiomatized by extending MIPC with prelinearity [12]. We also prove that the one-variable fragment of an intermediate logic defined over a linear frame can be interpreted in the one-variable fragment of the associated constant domain logic, obtaining an interpretation of $\mathsf{S5}(\mathbf{A})$ in $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ for any Gödel set A (Section 6).

In general, for an infinite Gödel set A, the Gödel modal logics $\mathsf{S5}(\mathbf{A})$ and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ do not have the finite model property with respect to their standard frame semantics. However, by restricting the values for box and diamond formulas to a subset of A, an alternative semantics is obtained for $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ (and implicitly, by the results of the previous section, $\mathsf{S5}(\mathbf{A})$) that has this property (Section 7).[5] This semantics is used to prove decidability and co-NP-completeness results for Gödel modal logics based on a large class of Gödel sets for which consistency can be checked with respect to certain finite structures (Section 8). The correspondences established in previous sections then yield decidability and co-NP-completeness results for one-variable fragments of a broad family of first-order intermediate logics defined over a countable linear frame. In particular, it follows that the one-variable fragment of Corsi logic (the intermediate logic of all linear frames) and its modal counterpart $\mathsf{S5}(\mathbf{G})$ are co-NP-complete.

2. The one-variable fragments

In this section, we introduce the one-variable fragments of intermediate logics over linear frames with and without constant domains that form the main focus of this paper. For convenience, we restrict our definitions to the set Fm_1 of one-variable first-order formulas $\alpha, \beta, \ldots$ built inductively from a countably infinite set of unary predicates $\{P_i\}_{i \in \mathbb{N}}$, propositional connectives $\wedge, \vee, \to, \bot, \top$, a fixed variable x, and quantifiers $\forall, \exists$.

A *frame* is a non-empty poset $\mathbf{K} = \langle K, \preceq \rangle$ and is said to be *linear* if $\preceq$ is a linear order. A *one-variable intuitionistic Kripke model* (or IK_1*-model* for short) based on $\mathbf{K}$ is a 4-tuple

$$\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k \in K}, \{I_k\}_{k \in K} \rangle,$$

such that for all $k \in K$, each D_k is a non-empty set (called the *domain* of k) and each I_k is a function mapping P_i to $I_k(P_i) \subseteq D_k$ for $i \in \mathbb{N}$, satisfying

$$k \preceq l \implies D_k \subseteq D_l \text{ and } I_k(P_i) \subseteq I_l(P_i).$$

Satisfaction in $\mathfrak{M}$ is defined inductively as follows for $k \in K$ and $a \in D_k$:

[4] Note that the results of [9] do not apply to the two-variable fragment of an intermediate logic defined over linear frames; indeed, the decidability of the two-variable fragment of first-order Gödel logic remains an intriguing open problem.

[5] A related alternative semantics for these logics was provided in a previous paper [16] by the first three authors and J. Rogger, but this account contained an error.

$$
\begin{array}{lcl}
\mathfrak{M}, k \models^a \bot & \iff & \text{never} \\
\mathfrak{M}, k \models^a \top & \iff & \text{always} \\
\mathfrak{M}, k \models^a P_i(x) & \iff & a \in I_k(P_i) \\
\mathfrak{M}, k \models^a \alpha \wedge \beta & \iff & \mathfrak{M}, k \models^a \alpha \ \text{ and } \ \mathfrak{M}, k \models^a \beta \\
\mathfrak{M}, k \models^a \alpha \vee \beta & \iff & \mathfrak{M}, k \models^a \alpha \ \text{ or } \ \mathfrak{M}, k \models^a \beta \\
\mathfrak{M}, k \models^a \alpha \to \beta & \iff & \mathfrak{M}, l \models^a \alpha \ \text{ implies } \ \mathfrak{M}, l \models^a \beta \ \text{ for all } l \succeq k \\
\mathfrak{M}, k \models^a (\forall x)\alpha & \iff & \mathfrak{M}, l \models^b \alpha \ \text{ for all } l \succeq k \text{ and } b \in D_l \\
\mathfrak{M}, k \models^a (\exists x)\alpha & \iff & \mathfrak{M}, k \models^b \alpha \ \text{ for some } b \in D_k.
\end{array}
$$

We call $\mathfrak{M}$ an IKL_1*-model* if $\mathbf{K}$ is linear, a CDIK_1*-model* if it has constant domains (i.e., $D_k = D_l$ for all $k, l \in K$), and a CDIKL_1*-model* if it satisfies both these conditions. A formula $\alpha \in \mathrm{Fm}_1$ is said to be *valid* in $\mathfrak{M}$ if $\mathfrak{M}, k \models^a \alpha$ for all $k \in K$ and $a \in D_k$. Given $\mathsf{L} \in \{\mathsf{IK}_1, \mathsf{IKL}_1, \mathsf{CDIK}_1, \mathsf{CDIKL}_1\}$, we say that $\alpha \in \mathrm{Fm}_1$ is L*-valid*, denoted by $\models_{\mathsf{L}} \alpha$, if it is valid in all L-models.

Let us briefly survey some known results for these one-variable fragments. Let $\mathcal{IQC}$ be any axiomatization for first-order intuitionistic logic and consider the following *prelinearity* (prl) and *constant domain* (cd) axiom schemas for all first-order formulas α and β, where x is not free in β for (cd):

$$
(\mathrm{prl}) \ \ (\alpha \to \beta) \vee (\beta \to \alpha) \quad \text{ and } \quad (\mathrm{cd}) \ \ (\forall x)(\alpha \vee \beta) \to ((\forall x)\alpha \vee \beta).
$$

As a direct consequence of completeness results established in the literature for the corresponding first-order logics (see [24], [25], [23], [10]), we obtain for any $\alpha \in \mathrm{Fm}_1$,

$$
\begin{array}{lcll}
\models_{\mathsf{IK}_1} \alpha & \iff & \vdash_{\mathcal{IQC}} \alpha & [24] \\
\models_{\mathsf{CDIK}_1} \alpha & \iff & \vdash_{\mathcal{IQC}+(\mathrm{cd})} \alpha & [25] \\
\models_{\mathsf{IKL}_1} \alpha & \iff & \vdash_{\mathcal{IQC}+(\mathrm{prl})} \alpha & [23] \\
\models_{\mathsf{CDIKL}_1} \alpha & \iff & \vdash_{\mathcal{IQC}+(\mathrm{cd})+(\mathrm{prl})} \alpha & [10].
\end{array}
$$

One-variable fragments of first-order logics may also be studied as propositional (modal) logics. Let $\mathrm{Fm}_{\Box\Diamond}$ be the set of propositional formulas $\varphi, \psi, \ldots$ built inductively over a set of propositional variables $\{p_i\}_{i \in \mathbb{N}}$, propositional connectives $\wedge, \vee, \to, \bot, \top$, and modal connectives $\Box, \Diamond$. The standard translation functions $(-)^*$ and $(-)^\circ$ between Fm_1 and $\mathrm{Fm}_{\Box\Diamond}$ are defined inductively as follows for $\star \in \{\wedge, \vee, \to\}$ and $\dagger \in \{\bot, \top\}$:

$$
\begin{array}{rlrl}
(P_i(x))^* &= p_i & p_i^\circ &= P_i(x) \\
\dagger^* &= \dagger & \dagger^\circ &= \dagger \\
(\alpha \star \beta)^* &= \alpha^* \star \beta^* & (\varphi \star \psi)^\circ &= \varphi^\circ \star \psi^\circ \\
((\forall x)\alpha)^* &= \Box\alpha^* & (\Box\varphi)^\circ &= (\forall x)\varphi^\circ \\
((\exists x)\alpha)^* &= \Diamond\alpha^* & (\Diamond\varphi)^\circ &= (\exists x)\varphi^\circ.
\end{array}
$$

Clearly $(\alpha^*)^\circ = \alpha$ for any $\alpha \in \mathrm{Fm}_1$ and $(\varphi^\circ)^* = \varphi$ for any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, so we may alternate between the first-order and modal notations as convenient.

Now let $\mathcal{MIPC}$ be any axiomatization of intuitionistic propositional logic extended with the necessitation rule $\varphi / \Box\varphi$ and the axiom schemas

$$
\begin{array}{ll}
\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) & \Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi) \\
\Box\varphi \to \varphi & \varphi \to \Diamond\varphi \\
\Diamond\varphi \to \Box\Diamond\varphi & \Diamond\Box\varphi \to \Box\varphi \\
\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi), &
\end{array}
$$

and consider the additional axiom schemas

$$
(\mathrm{prl}) \ \ (\varphi \to \psi) \vee (\psi \to \varphi) \quad \text{ and } \quad (\mathrm{cd})_\Box \ \ \Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi).
$$

The following completeness results may be found in the literature (see [26], [27], [12]):

$$
\begin{array}{lcll}
\models_{\mathsf{IK}_1} \alpha & \iff & \vdash_{\mathcal{MIPC}} \alpha^* & [26] \\
\models_{\mathsf{CDIK}_1} \alpha & \iff & \vdash_{\mathcal{MIPC}+(\mathrm{cd})_\Box} \alpha^* & [27] \\
\models_{\mathsf{CDIKL}_1} \alpha & \iff & \vdash_{\mathcal{MIPC}+(\mathrm{cd})_\Box+(\mathrm{prl})} \alpha^* & [12].
\end{array}
$$

In Section 4 of this paper, we establish the missing result:

$$\models_{\mathsf{IKL}_1} \alpha \iff \vdash_{\mathcal{MIPC}+(\mathrm{prl})} \alpha^*.$$

We also extend the correspondence with modal logics to one-variable fragments of intermediate logics defined over a single countable linear frame. Let us call an IKL_1-model based on a linear frame $\mathbf{K}$ an $\mathsf{IKL}_1(\mathbf{K})$-*model*. We say that $\alpha \in \mathrm{Fm}_1$ is $\mathsf{IKL}_1(\mathbf{K})$-*valid* and write $\models_{\mathsf{IKL}_1(\mathbf{K})} \alpha$ if it is valid in all $\mathsf{IKL}_1(\mathbf{K})$-models. Similarly, we say that α is $\mathsf{CDIKL}_1(\mathbf{K})$-*valid* and write $\models_{\mathsf{CDIKL}_1(\mathbf{K})} \alpha$ if it is valid in all constant domain $\mathsf{IKL}_1(\mathbf{K})$-models.

3. The many-valued modal logics

In this section, we define a family of many-valued modal logics that will be shown in Sections 4 and 5 to correspond to the one-variable fragments defined in the previous section. The propositional connectives of these logics are interpreted via the semantics of Gödel propositional logic, and they therefore belong to the family of Gödel modal logics studied in [28,5,14,15,12,16].

Following [13,29], let us call a closed subset A of the real unit interval $[0,1]$ containing 0 and 1 a *Gödel set*, and define the corresponding Heyting algebra

$$\mathbf{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle, \quad \text{where } x \to y := \begin{cases} 1 & \text{if } x \le y \\ y & \text{otherwise.} \end{cases}$$

Notable Gödel sets include $G := [0,1]$, yielding the *standard Gödel algebra* $\mathbf{G}$, $G_{\downarrow} := \{\frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{0\}$, $G_{\uparrow} := \{\frac{n-1}{n} \mid n \in \mathbb{N}^+\} \cup \{1\}$, and the finite Gödel sets $G_n := \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ for $n \in \mathbb{N}^+$.

An $\mathsf{S5}(\mathbf{A})$-*frame* is an A-valued equivalence relation: a pair $\langle W, R\rangle$ consisting of a non-empty set W and a map $R\colon W \times W \to A$ satisfying for all $u, v, w \in W$,

(i) $Rww = 1$ (reflexivity)
(ii) $Rvw = Rwv$ (symmetry)
(iii) $Ruv \wedge Rvw \le Ruw$ (transitivity).

It is called *crisp* if $Rvw \in \{0,1\}$ for all $v, w \in W$, in which case R determines an equivalence relation on W defined by $v \sim w :\Leftrightarrow Rvw = 1$.

An $\mathsf{S5}(\mathbf{A})$-*model* is a triple $\mathcal{M} = \langle W, R, V\rangle$ consisting of an $\mathsf{S5}(\mathbf{A})$-frame $\langle W, R\rangle$ and a map $V\colon \{p_i\}_{i\in\mathbb{N}} \times W \to A$. The map V is extended inductively to $V\colon \mathrm{Fm}_{\Box\Diamond} \times W \to A$ as follows, where $\star \in \{\wedge, \vee, \to\}$:

$$\begin{aligned} V(\bot, w) &= 0 \\ V(\top, w) &= 1 \\ V(\varphi \star \psi, w) &= V(\varphi, w) \star V(\psi, w) \\ V(\Box\varphi, w) &= \bigwedge\{Rwv \to V(\varphi, v) \mid v \in W\} \\ V(\Diamond\varphi, w) &= \bigvee\{Rwv \wedge V(\varphi, v) \mid v \in W\}. \end{aligned}$$

A formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ is said to be *valid* in $\mathcal{M}$ if $V(\varphi, w) = 1$ for all $w \in W$, and $\mathsf{S5}(\mathbf{A})$-*valid*, written $\models_{\mathsf{S5}(\mathbf{A})} \varphi$, if it is valid in all $\mathsf{S5}(\mathbf{A})$-models. We also say that $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ is $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-*valid*, written $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$, if it is valid in all $\mathsf{S5}(\mathbf{A})$-models based on a crisp $\mathsf{S5}(\mathbf{A})$-frame.

An $\mathsf{S5}(\mathbf{A})$-model $\mathcal{M} = \langle W, R, V\rangle$ is called *universal* if $Rwv = 1$ for all $w, v \in W$; we then write $\mathcal{M} = \langle W, V\rangle$, since the conditions for $\Box, \Diamond$ simplify to

$$V(\Box\varphi, w) = \bigwedge\{V(\varphi, v) \mid v \in W\} \quad \text{and} \quad V(\Diamond\varphi, w) = \bigvee\{V(\varphi, v) \mid v \in W\}.$$

It is easily proved that $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$ if and only if φ is valid in all universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-models, and hence that $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity corresponds to validity in a corresponding first-order Gödel logic. The fact, proved in [10], that validity in the first-order Gödel logic defined over $[0,1]$ coincides with validity in the first-order logic of linear intuitionistic Kripke models with constant domains then yields

$$\models_{\mathsf{CDIKL}_1} \alpha \iff \models_{\mathsf{S5}(\mathbf{G})^{\mathsf{C}}} \alpha^*.$$

A general correspondence, established in [13], between first-order Gödel logics and constant domain logics defined over a countable linear frame yields analogous results for their one-variable fragments and crisp many-valued modal logics. That is, the correspondence shows that for any countable linear frame $\mathbf{K}$, there exists a Gödel set A, and conversely, for any Gödel set A, there exists a countable linear frame $\mathbf{K}$ such that

$$\models_{\mathsf{CDIKL}_1(\mathbf{K})} \alpha \iff \models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \alpha^*.$$

In Sections 4 and 5, we extend these results to logics defined over a countable linear frame without the constant domain assumption, proving also that

$$\models_{\mathsf{IKL}_1} \alpha \iff \models_{\mathsf{S5}(\mathbf{G})} \alpha^*.$$

In [11] it was proved that the propositional Gödel logic defined over the Gödel set $G_\downarrow$ can be axiomatized by extending propositional intuitionistic logic with the prelinearity axiom schema. It is also not hard to see that this is the propositional logic of *any* infinite Gödel set A. On the other hand, it was proved in [29] that there are countably infinitely many such first-order Gödel logics (considered as sets of valid formulas) and that this bound in precise. Here we show that the same is true when we restrict attention to their one-variable fragments; that is, we prove that there are countably infinitely many logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ (and, similarly, countably infinitely many logics $\mathsf{S5}(\mathbf{A})$) where A ranges over infinite Gödel sets.

Let us say that an element a of a Gödel set A is a *right accumulation point of* A if $a < 1$ and for all $b \in A$ such that $a < b$, there exists $c \in A$ such that $a < c < b$. Left accumulation points of A are defined analogously. Let $\mathsf{R}(A)$ and $\mathsf{L}(A)$ denote the sets of right and left accumulation points of A, respectively. We use the following formula to detect right accumulation points of A:

$$\chi(p) := \Box((p \to \Box p) \to \Box p) \to \Box p.$$

Lemma 1. *Let A be any Gödel set and let $\langle W, R, V\rangle$ be an $\mathsf{S5}(\mathbf{A})$-model with $w \in W$. If $V(\chi(p), w) < 1$, then $V(\Box p, w)$ is a right accumulation point of A.*

Proof. Suppose that $V(\chi(p), w) < 1$. To prove that $V(\Box p, w)$ is a right accumulation point of A, it suffices to show that $V(\Box p, w) < Rwv \to V(p, v)$ for all $v \in W$. For a contradiction, suppose that there exists $v \in W$ such that $V(\Box p, w) = Rwv \to V(p, v)$. From $V(\Box p, w) = V(\chi(p), w) < 1$ it follows that $V(\Box p, w) = V(p, v) < Rwv$. By the symmetry and transitivity of R, we have $Rwv \wedge Rvu = Rwv \wedge Rwu$ for all $u \in W$, so

$$\begin{aligned}
Rwv \to V(\Box p, v) &= Rwv \to \bigwedge\{Rvu \to V(p, u) \mid u \in W\}\\
&= Rwv \to \bigwedge\{(Rwv \wedge Rvu) \to V(p, u) \mid u \in W\}\\
&= Rwv \to \bigwedge\{(Rwv \wedge Rwu) \to V(p, u) \mid u \in W\}\\
&= Rwv \to \bigwedge\{Rwu \to V(p, u) \mid u \in W\}\\
&= Rwv \to V(\Box p, w).
\end{aligned}$$

This gives $Rwv \to V(\Box p, v) = Rwv \to V(\Box p, w) = V(\Box p, w) < 1$ and hence $V(\Box p, v) = V(\Box p, w) = V(p, v) < Rwv$. Now note that from $V(\chi(p), w) < 1$, we obtain $V(\Box((p \to \Box p) \to \Box p, w) > V(\Box p, w)$ and so $V(\Box p, w) < Rwu \to V((p \to \Box p) \to \Box p, u)$ for all $u \in W$. We obtain a contradiction

$$\begin{aligned}
V(\Box p, w) &< Rwv \to V((p \to \Box p) \to \Box p, v)\\
&= Rwv \to V(\Box p, v)\\
&= V(\Box p, v)\\
&= V(\Box p, w). \quad \Box
\end{aligned}$$

Lemma 2. *The sets of logics $\mathsf{S5}(\mathbf{A})$ and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ (considered as sets of valid formulas), where A ranges over infinite Gödel sets, are both countably infinite.*

Proof. It was proved in [29] that there are countably infinitely many first-order Gödel logics. Hence, since each logic $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ corresponds to the one-variable fragment of a first-order Gödel logic, there are at most countably infinitely many logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$, where A ranges over infinite Gödel sets. Moreover, since $\mathsf{S5}(\mathbf{A})$ can be interpreted in $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ for any infinite Gödel set A, as will be proved in Theorem 18, there are at most countably infinitely many logics $\mathsf{S5}(\mathbf{A})$, where A again ranges over infinite Gödel sets.

It remains to show that infinitely many of the logics $\mathsf{S5}(\mathbf{A})$, and similarly the logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$, can be distinguished by formulas. For each $n \in \mathbb{N}^+$, let

$$\chi_n := \bigvee_{j=1}^{n} \chi(p_j) \vee \bigvee_{i=1}^{n-1} (\Box p_{i+1} \to \Box p_i).$$

We prove first that $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \chi_n$ implies $|\mathsf{R}(A)| < n$. Suppose that A has n distinct right accumulation points $a_1 < \dots < a_n$. Then for each $j \in \{1, \dots, n\}$, there exists a strictly descending sequence $(c_n^j)_{n\in\mathbb{N}} \subseteq (a_j, 1] \cap A$ converging to a_j. We define an $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M} = \langle \mathbb{N}, V\rangle$ with $V(p_j, m) = c_m^j$ for all $m \in \mathbb{N}$ and $j \in \{1, \dots, n\}$. Then $V(\Box p_j, m) = a_j < c_m^j = V(p_j, m)$ for all $j \in \{1, \dots, n\}$ and $m \in \mathbb{N}$, which implies $V(\chi(p_j), 0) = a_j$ for all $j \in \{1, \dots, n\}$. Moreover, $V(\Box p_{i+1} \to \Box p_i, 0) = V(\Box p_i, 0) = a_i$ for all $i \in \{1, \dots, n-1\}$. Hence $V(\chi_n, 0) = a_n < 1$ and $\not\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \chi_n$.

Next, we prove that $|\mathsf{R}(A)| < n$ implies $\models_{\mathsf{S5}(\mathbf{A})} \chi_n$. Suppose that $V(\chi_n, w) < 1$ for some $\mathsf{S5}(\mathbf{A})$-model $\langle W, R, V\rangle$ and $w \in W$. It follows that $V(\Box p_1, w) < \dots < V(\Box p_n, w)$ and $V(\chi(p_j), w) < 1$ for all $j \in \{1, \dots, n\}$. By Lemma 1, each of the $V(\Box p_j, w)$ is a right accumulation point of A and so $|\mathsf{R}(A)| \ge n$.

Since also $\models_{\mathsf{S5}(\mathbf{A})} \chi_n$ implies $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \chi_n$, we have that $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \chi_n$ if and only if $|\mathsf{R}(A)| < n$, and $\models_{\mathsf{S5}(\mathbf{A})} \chi_n$ if and only if $|\mathsf{R}(A)| < n$. Hence the sets of logics $\mathsf{S5}(\mathbf{A})$ and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$, where A ranges over infinite Gödel sets, are both countably infinite. $\square$

Similarly, we can detect left accumulation points. For each $n \in \mathbb{N}^+$, let

$$\theta_n := \bigvee_{i=1}^{n} (\Diamond(\Diamond p_i \to p_i)) \vee \bigvee_{i=1}^{n-1} (\Diamond p_{i+1} \to \Diamond p_i).$$

It is then easy to verify that for any Gödel set A and $n \in \mathbb{N}^+$,

$$\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \theta_n \iff \models_{\mathsf{S5}(\mathbf{A})} \theta_n \iff |\mathsf{L}(A) \setminus \{1\}| < n.$$

4. From linear frames to Gödel sets

In this section, we match the one-variable fragment of an intermediate logic defined over a single countable linear frame to a corresponding Gödel modal logic. In particular, we match the one-variable fragment of the intermediate logic defined over $\langle \mathbb{Q}, \le\rangle$ to the standard Gödel modal logic $\mathsf{S5}(\mathbf{G})$.

Let $\mathbf{K} = \langle K, \preceq\rangle$ be any countable linear frame. A subset $U \subseteq K$ is called an *upset* of $\mathbf{K}$ if whenever $k \in U$, $l \in K$, and $k \preceq l$, also $l \in U$. For each $k \in K$, we denote the upset $\{l \in K \mid k \preceq l\}$ by $[k)$. Now let $\mathrm{Up}(\mathbf{K})$ be the set of upsets of $\mathbf{K}$. Then $\langle \mathrm{Up}(\mathbf{K}), \subseteq\rangle$ is a complete linearly ordered set with greatest and least elements K and $\emptyset$, respectively. Moreover, since K is countable, there exists a complete (i.e., preserving all suprema and infima) order-embedding of $\langle \mathrm{Up}(\mathbf{K}), \subseteq\rangle$ into $\langle [0, 1], \le\rangle$ (see [30]). Hence we may identify $\mathrm{Up}(\mathbf{K})$ with a Gödel set and obtain an $\mathsf{S5}(\mathbf{Up}(\mathbf{K}))$-model based on the Heyting algebra

$$\mathbf{Up}(\mathbf{K}) = \langle \mathrm{Up}(\mathbf{K}), \cap, \cup, \to, \emptyset, K\rangle, \quad \text{where } X \to Y := \begin{cases} K & \text{if } X \subseteq Y \\ Y & \text{otherwise.} \end{cases}$$

Now let $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k\in K}, \{I_k\}_{k\in K}\rangle$ be any $\mathsf{IKL}_1(\mathbf{K})$-model. We define for all $a, b \in W$ and $i \in \mathbb{N}$,

$$W := \bigcup_{k\in K} D_k$$

$$U(a) := \{k \in K \mid a \in D_k\}$$

$$Rab := \begin{cases} K & a = b \\ U(a) \cap U(b) & a \ne b \end{cases}$$

$$V(p_i, a) := \{k \in K \mid a \in I_k(P_i)\}.$$

Note that each $V(p_i, a)$ is an upset of $\mathbf{K}$ since $k \preceq l$ implies $I_k(P_i) \subseteq I_l(P_i)$. Moreover, $Raa = K$, $Rab = Rba$, and $Rab \cap Rbc \subseteq Rac$ for all $a, b, c \in W$. Hence $\mathcal{M}_{\mathfrak{M}} := \langle W, R, V\rangle$ is an $\mathsf{S5}(\mathbf{Up}(\mathbf{K}))$-model. Moreover, if $\mathfrak{M}$ is a $\mathsf{CDIKL}_1(\mathbf{K})$-model, then $\mathcal{M}_{\mathfrak{M}}$ is universal.

Lemma 3. *Let $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k\in K}, \{I_k\}_{k\in K}\rangle$ be an $\mathsf{IKL}_1(\mathbf{K})$-model over a countable linear frame $\mathbf{K} = \langle K, \preceq\rangle$ with $\mathcal{M}_{\mathfrak{M}} = \langle W, R, V\rangle$. Then for any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, $k \in K$, and $a \in D_k$,*

$$\mathfrak{M}, k \models^a \varphi^\circ \iff k \in V(\varphi, a).$$

Proof. We begin with the following useful observation. If $a \in D_k$ and $b \in W$, then $b \in D_k$ if and only if $k \in Rab$. Just note that if $b = a$, this is trivial, and if $b \ne a$, then $k \in U(a) \cap U(b)$ if and only if $k \in U(b)$, i.e., $b \in D_k$.

We prove the claim by induction on the length of φ. The base cases for $\bot$, $\top$, and p_i are immediate from the definitions, and the cases for $\wedge$ and $\vee$ are straightforward, so we just consider the cases for $\to$, $\Box$, and $\Diamond$.

- Suppose that $\varphi = \psi_1 \to \psi_2$.

$$\begin{aligned}\mathfrak{M}, k \models^a (\psi_1 \to \psi_2)^\circ &\iff \mathfrak{M}, l \models^a \psi_1^\circ \text{ implies } \mathfrak{M}, l \models^a \psi_2^\circ \text{ for all } l \succeq k\\ &\iff l \in V(\psi_1, a) \text{ implies } l \in V(\psi_2, a) \text{ for all } l \succeq k\\ &\iff [k) \cap V(\psi_1, a) \subseteq V(\psi_2, a)\\ &\iff [k) \subseteq (V(\psi_1, a) \to V(\psi_2, a))\\ &\iff k \in V(\psi_1 \to \psi_2, a).\end{aligned}$$

- Suppose that $\varphi = \Box\psi$.

$$\begin{aligned}\mathfrak{M}, k \models^a (\Box\psi)^\circ &\iff \mathfrak{M}, l \models^b \psi^\circ \text{ for all } l \succeq k \text{ and } b \in D_l\\ &\iff l \in V(\psi, b) \text{ for all } l \succeq k \text{ and } b \in W \text{ such that } l \in Rab\\ &\iff [k) \cap Rab \subseteq V(\psi, b) \text{ for all } b \in W\\ &\iff [k) \subseteq (Rab \to V(\psi, b)) \text{ for all } b \in W\\ &\iff k \in \bigcap\{Rab \to V(\psi, b) \mid b \in W\}\\ &\iff k \in V(\Box\psi, a).\end{aligned}$$

- Suppose that $\varphi = \Diamond\psi$.

$$\begin{aligned}\mathfrak{M}, k \models^a (\Diamond\psi)^\circ &\iff \mathfrak{M}, k \models^b \psi^\circ \text{ for some } b \in D_k\\ &\iff k \in V(\psi, b) \text{ and } k \in Rab \text{ for some } b \in W\\ &\iff k \in \bigcup\{Rab \cap V(\psi, b) \mid b \in W\}\\ &\iff k \in V(\Diamond\psi, a). \quad \Box\end{aligned}$$

For the converse direction, suppose again that $\mathbf{K} = \langle K, \preceq\rangle$ is a countable linear frame. Let $\mathcal{M} = \langle W, R, V\rangle$ be any $\mathsf{S5}(\mathbf{Up}(\mathbf{K}))$-model and fix $w_0 \in W$. We define for each $k \in K$ and $i \in \mathbb{N}$,

$$D_k := \{v \in W \mid k \in Rw_0v\}$$

$$I_k(P_i) := \{v \in W \mid k \in V(p_i, v)\} \cap D_k.$$

It is easily checked that if $k \preceq l$, then $D_k \subseteq D_l$ and $I_k(P_i) \subseteq I_l(P_i)$ for each $i \in \mathbb{N}$. Hence $\mathfrak{M}_{\mathcal{M},w_0} := \langle K, \preceq, \{D_k\}_{k\in K}, \{I_k\}_{k\in K}\rangle$ is an $\mathsf{IKL}_1(\mathbf{K})$-model. Moreover, if $\mathcal{M}$ is universal, then $\mathfrak{M}_{\mathcal{M},w_0}$ is a $\mathsf{CDIKL}_1(\mathbf{K})$-model.

Lemma 4. *Let $\mathbf{K} = \langle K, \preceq\rangle$ be a countable linear frame and let $\mathcal{M} = \langle W, R, V\rangle$ be an $\mathsf{S5}(\mathbf{Up}(\mathbf{K}))$-model with $w_0 \in W$ and $\mathfrak{M}_{\mathcal{M},w_0} = \langle K, \preceq, \{D_k\}_{k\in K}, \{I_k\}_{k\in K}\rangle$. For any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, $k \in K$, and $v \in D_k$,*

$$\mathfrak{M}_{\mathcal{M},w_0}, k \models^v \varphi^\circ \iff k \in V(\varphi, v).$$

Proof. Note first that if $v \in D_k$, then $k \in Rw_0v$ and for any $l \succeq k$ and $u \in W$,

$$l \in Rw_0u \implies l \in Ruw_0 \cap Rw_0v \subseteq Ruv$$

$$l \in Ruv \implies l \in Rw_0v \cap Rvu \subseteq Rw_0u;$$

that is, $Rw_0u \cap [k) = Rvu \cap [k)$.

We prove the claim by induction on the length of φ. The base cases for $\bot$, $\top$, and p_i are immediate from the definitions and the cases for $\wedge$ and $\vee$ are straightforward, so we just consider the cases for $\to$, $\Box$, and $\Diamond$.

- Suppose that $\varphi = \psi_1 \to \psi_2$. Then $\mathfrak{M}_{\mathcal{M},w_0}, k \models^v (\psi_1 \to \psi_2)^\circ$

$$\iff \mathfrak{M}_{\mathcal{M},w_0}, l \models^v \psi_1^\circ \text{ implies } \mathfrak{M}_{\mathcal{M},w_0}, l \models^v \psi_2^\circ \text{ for all } l \succeq k$$
$$\iff l \in V(\psi_1, v) \text{ implies } l \in V(\psi_2, v) \text{ for all } l \succeq k$$
$$\iff [k) \cap V(\psi_1, v) \subseteq V(\psi_2, v)$$
$$\iff [k) \subseteq (V(\psi_1, v) \to V(\psi_2, v))$$
$$\iff k \in V(\psi_1 \to \psi_2, v).$$

- Suppose that $\varphi = \Box\psi$. Then $\mathfrak{M}_{\mathcal{M},w_0}, k \models^v (\Box\psi)^\circ$

$$\iff \mathfrak{M}_{\mathcal{M},w_0}, l \models^u \psi^\circ \text{ for all } l \succeq k \text{ and } u \in D_l$$
$$\iff l \in V(\psi, u) \text{ for all } l \succeq k \text{ and } u \in W \text{ such that } l \in Rw_0u$$
$$\iff l \in V(\psi, u) \text{ for all } l \succeq k \text{ and } u \in W \text{ such that } l \in Rvu$$
$$\iff [k) \cap Rvu \subseteq V(\psi, u) \text{ for all } u \in W$$
$$\iff [k) \subseteq (Rvu \to V(\psi, u)) \text{ for all } u \in W$$
$$\iff k \in (Rvu \to V(\psi, u)) \text{ for all } u \in W$$
$$\iff k \in \bigcap\{Rvu \to V(\psi, u) \mid u \in W\}$$
$$\iff k \in V(\Box\psi, v).$$

- Suppose that $\varphi = \Diamond\psi$. Then $\mathfrak{M}_{\mathcal{M},w_0}, k \models^v (\Diamond\psi)^\circ$

$$\iff \mathfrak{M}_{\mathcal{M},w_0}, k \models^u \psi^\circ \text{ for some } u \in D_k$$
$$\iff k \in V(\psi, u) \text{ for some } u \in W \text{ such that } k \in Rw_0u$$
$$\iff k \in V(\psi, u) \text{ for some } u \in W \text{ such that } k \in Rvu$$
$$\iff k \in V(\psi, u) \cap Rvu \text{ for some } u \in W$$
$$\iff k \in \bigcup\{V(\psi, u) \cap Rvu \mid u \in W\}$$
$$\iff k \in V(\Diamond\psi, v). \quad \Box$$

We now put these two lemmas together to obtain the desired correspondence, recalling that the result for the constant domain case is implicit in [13].

Theorem 5. *For any countable linear frame* $\mathbf{K} = \langle K, \preceq \rangle$ *and* $\alpha \in \mathrm{Fm}_1$,

$$\models_{\mathsf{IKL}_1(\mathbf{K})} \alpha \iff \models_{\mathsf{S5}(\mathbf{Up}(\mathbf{K}))} \alpha^*$$
$$\models_{\mathsf{CDIKL}_1(\mathbf{K})} \alpha \iff \models_{\mathsf{S5}(\mathbf{Up}(\mathbf{K}))^{\mathsf{C}}} \alpha^*.$$

Proof. Suppose first that $\not\models_{\mathsf{IKL}_1(\mathbf{K})} \alpha$. Then there exists an $\mathsf{IKL}_1(\mathbf{K})$-model $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k\in K}, \{I_k\}_{k\in K}\rangle$, $k \in K$, and $a \in D_k$ such that $\mathfrak{M}, k \not\models^a \alpha$. An application of Lemma 3 with $\mathcal{M}_{\mathfrak{M}} = \langle W, R, V\rangle$ yields $k \notin V(\alpha^*, a)$. Hence $V(\alpha^*, a) \neq K$ and $\not\models_{\mathsf{S5}(\mathbf{Up}(\mathbf{K}))} \alpha^*$.

Suppose now that $\not\models_{\mathsf{S5}(\mathbf{Up}(\mathbf{K}))} \alpha^*$. Then there exists an $\mathsf{S5}(\mathbf{Up}(\mathbf{K}))$-model $\mathcal{M} = \langle W, R, V\rangle$ and $w_0 \in W$ such that $V(\alpha^*, w_0) \neq K$. Let $k \in K \setminus V(\alpha^*, w_0)$. Then Lemma 4 yields $\mathfrak{M}_{\mathcal{M},w_0}, k \not\models^{w_0} \alpha$, so $\not\models_{\mathsf{IKL}_1(\mathbf{K})} \alpha$.

The second equivalence follows from the fact that if $\mathfrak{M}$ is a $\mathsf{CDIKL}_1(\mathbf{K})$-model, then $\mathcal{M}_{\mathfrak{M}}$ is universal, and, conversely, if $\mathcal{M}$ is universal, then $\mathfrak{M}_{\mathcal{M},w_0}$ is a $\mathsf{CDIKL}_1(\mathbf{K})$-model. $\Box$

By choosing suitable linear frames, we obtain the corresponding Gödel modal logics defined over certain notable Gödel sets.

Corollary 6. *For any formula $\alpha \in \mathrm{Fm}_1$ and $n \in \mathbb{N} \setminus \{0,1\}$,*

$$\models_{\mathsf{IKL}_1(\langle \mathbb{N}, \leq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_\downarrow)} \alpha^* \qquad \models_{\mathsf{CDIKL}_1(\langle \mathbb{N}, \leq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_\downarrow)^\mathsf{C}} \alpha^*$$

$$\models_{\mathsf{IKL}_1(\langle \mathbb{N}, \geq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_\uparrow)} \alpha^* \qquad \models_{\mathsf{CDIKL}_1(\langle \mathbb{N}, \geq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_\uparrow)^\mathsf{C}} \alpha^*$$

$$\models_{\mathsf{IKL}_1(\langle \{1,\ldots,n\}, \leq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_\mathsf{n})} \alpha^* \qquad \models_{\mathsf{CDIKL}_1(\langle \{1,\ldots,n\}, \leq \rangle)} \alpha \iff \models_{\mathsf{S5}(\mathbf{G}_\mathsf{n})^\mathsf{C}} \alpha^*.$$

For the logic $\mathsf{S5}(\mathbf{G})$, however, the obvious choice of a countable linear frame $\mathbf{Q} = \langle \mathbb{Q}, \leq \rangle$ produces a Gödel set $\mathrm{Up}(\mathbf{Q})$ that is not order-isomorphic to $[0,1]$.[6] In the next section, we will show that there exists a matching countable linear frame for every Gödel set A, but first we give here a construction that directly relates $\mathsf{S5}(\mathbf{G})$-validity to $\mathsf{IKL}_1(\mathbf{Q})$-validity.

For technical reasons, we begin by showing that we can restrict our attention to a particular class of $\mathsf{S5}(\mathbf{G})$-models. We say that an $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V \rangle$ is *irrational* if $V(\varphi, w)$ is irrational, 0, or 1 for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ and $w \in W$.

Lemma 7. *For any countable $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V \rangle$, there exists an irrational $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M}' = \langle W, R', V' \rangle$ such that for all $\varphi, \psi \in \mathrm{Fm}_{\Box\Diamond}$ and $w \in W$,*

$$V(\varphi, w) < V(\psi, w) \iff V'(\varphi, w) < V'(\psi, w).$$

Proof. By [30, Lemma 3.7], there exists a complete order-embedding f from the countable set $S = \{V(\varphi, w) \mid w \in W;\ \varphi \in \mathrm{Fm}_{\Box\Diamond}\} \cup R[W \times W]$ into $\mathbb{Q} \cap [0,1]$. For each $q \in \mathbb{Q} \cap [0,1]$, define

$$g(q) := \begin{cases} \frac{\pi}{3} q & q \leq \frac{1}{2} \\ \frac{\pi}{6} + (2 - \frac{\pi}{3})(q - \frac{1}{2}) & q > \frac{1}{2}. \end{cases}$$

Then g is a complete order-embedding from $\mathbb{Q} \cap [0,1]$ into $([0,1] \setminus \mathbb{Q}) \cup \{0,1\}$ with $g(0) = 0$, $g(1) = 1$. So $h = g \circ f$ is a complete order-embedding from S into $([0,1] \setminus \mathbb{Q}) \cup \{0,1\}$ with $h(0) = 0$, $h(1) = 1$. Now let $\mathcal{M}' = \langle W, R', V' \rangle$ where $R'wv = h(Rwv)$ and $V'(p_i, w) = h(V(p_i, w))$ for $w, v \in W$ and $i \in \mathbb{N}$. A straightforward induction on formula length yields $V'(\varphi, w) = h(V(\varphi, w))$ for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ and $w \in W$ and the claim follows immediately. $\square$

Now let $(0,1)_\mathbb{Q} := (0,1) \cap \mathbb{Q}$ and $(0,1)_\mathbf{Q} := \langle (0,1)_\mathbb{Q}, \geq \rangle$. Given any irrational $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V \rangle$ and $w_0 \in W$, we define for $q \in (0,1)_\mathbb{Q}$ and $i \in \mathbb{N}$,

$$D_q = \{v \in W \mid Rw_0v \geq q\} \quad \text{and} \quad I_q(P_i) = \{v \in W \mid V(p_i, v) \geq q\} \cap D_q.$$

It is easily checked that if $q \geq r$, then $D_q \subseteq D_r$ and $I_q(P_i) \subseteq I_r(P_i)$ for each $i \in \mathbb{N}$ and $q, r \in (0,1)_\mathbb{Q}$, so we obtain an $\mathsf{IKL}_1((0,1)_\mathbf{Q})$-model

$$\mathfrak{M}^i_{\mathcal{M}, w_0} := \langle (0,1)_\mathbb{Q}, \geq, \{D_q\}_{q \in (0,1)_\mathbb{Q}}, \{I_q\}_{q \in (0,1)_\mathbb{Q}} \rangle.$$

Moreover, if $\mathcal{M}$ is universal, then $\mathfrak{M}^i_{\mathcal{M}, w_0}$ is a $\mathsf{CDIKL}_1((0,1)_\mathbf{Q})$-model.

Lemma 8. *Let $\mathcal{M} = \langle W, R, V \rangle$ be an irrational $\mathsf{S5}(\mathbf{G})$-model with $w_0 \in W$ and $\mathfrak{M}^i_{\mathcal{M}, w_0} = \langle (0,1)_\mathbb{Q}, \geq, \{D_q\}_{q \in (0,1)_\mathbb{Q}}, \{I_q\}_{q \in (0,1)_\mathbb{Q}} \rangle$. For any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, $q \in (0,1)_\mathbb{Q}$, and $w \in D_q$,*

$$\mathfrak{M}^i_{\mathcal{M}, w_0}, q \models^w \varphi^\circ \iff V(\varphi, w) \geq q.$$

Proof. We prove the claim by induction on the length of φ. The base cases for $\bot$, $\top$, and p_i are immediate from the definitions and the cases for $\wedge$, $\vee$, and $\to$ are straightforward, so we just consider the cases for $\Box$, and $\Diamond$.

- For $\varphi = \Box\psi$, observe first that

$$\begin{aligned} \mathfrak{M}^i_{\mathcal{M}, w_0}, q \models^w (\forall x)\psi^\circ &\iff \mathfrak{M}^i_{\mathcal{M}, w_0}, r \models^v \psi^\circ \text{ for all } r \leq q \text{ and } v \in D_r \\ &\iff V(\psi, v) \geq r \text{ for all } r \leq q \text{ and } v \in D_r; \end{aligned}$$

$$\begin{aligned} V(\Box\psi, w) \geq q &\iff \bigwedge\{Rwv \to V(\psi, v) \mid v \in W\} \geq q \\ &\iff Rwv \to V(\psi, v) \geq q \text{ for all } v \in W \\ &\iff V(\psi, v) \geq q \wedge Rwv \text{ for all } v \in W. \end{aligned}$$

[6] Indeed, as explained in [13], the Gödel set $\mathrm{Up}(\mathbf{Q})$ is isomorphic to the Cantor set.

For the left-to-right direction suppose that $V(\psi, v) \geq r$ for all $r \leq q$ and $v \in D_r$. By assumption, $w \in D_q$, so $Rw_0w \geq q$. Let $v \in W$. If $q \leq Rwv$, then, by symmetry and transitivity, $Rw_0v \geq q$, i.e., $v \in D_q$, and hence $V(\psi, v) \geq q = q \wedge Rwv$ as required. Suppose now that $q > Rwv$. Then $Rw_0w \geq q > Rwv$ and, by transitivity, $Rwv = Rw_0w \wedge Rwv \leq Rw_0v$. But also, if $Rwv < Rw_0v$, then, by symmetry and transitivity, $Rwv < Rw_0v \wedge Rw_0w = Rww_0 \wedge Rw_0v \leq Rwv$, a contradiction. So $Rw_0v = Rwv$. It follows that for any $r \in (0,1)_{\mathbb{Q}}$ satisfying $r \leq Rw_0v$, we have $v \in D_r$ and hence $V(\psi, v) \geq r$. Finally, since $(0,1)_{\mathbb{Q}}$ is dense in $(0,1) \setminus \mathbb{Q}$, we have $\sup\{r \in (0,1)_{\mathbb{Q}} \mid Rw_0v \geq r\} = Rw_0v$, so $V(\psi, v) \geq Rw_0v = q \ \wedge Rwv$.

For the right-to-left direction, suppose that $V(\psi, v) \geq q \wedge Rwv$ for every $v \in W$. Let $r \leq q$ and $v \in D_r$. Then $Rw_0v \geq r$. Since $w \in D_q$, also $Rw_0w \geq q \geq r$, and by symmetry and transitivity, $Rwv \geq r$. So $V(\psi, v) \geq q \wedge Rwv \geq r$.

- For $\varphi = \Diamond\psi$, observe first that since $\mathfrak{M}$ is irrational and $q \in (0,1)_{\mathbb{Q}}$, $V(\varphi, w) \geq q$ if and only if $V(\varphi, w) > q$. Now observe that

$$
\begin{aligned}
\mathfrak{M}^i_{\mathcal{M},w_0}, q \models^w (\exists x)\psi^\circ &\iff \mathfrak{M}^i_{\mathcal{M},w_0}, q \models^v \psi^\circ \text{ for some } v \in D_q \\
&\iff V(\psi, v) \geq q \text{ for some } v \in D_q; \\
V(\Diamond\psi, w) \geq q &\iff \bigvee\{Rwv \wedge V(\psi, v) \mid v \in W\} \geq q \\
&\iff \bigvee\{Rwv \wedge V(\psi, v) \mid v \in W\} > q \\
&\iff Rwv \wedge V(\psi, v) \geq q \text{ for some } v \in W.
\end{aligned}
$$

For the left-to-right direction, suppose that $V(\psi, v) \geq q$ for some $v \in D_q$. Since $w, v \in D_q$, by transitivity, $Rwv \geq q$ and hence $Rwv \wedge V(\psi, v) \geq q$. For the right-to-left direction, suppose that there exists $v \in W$ such that $Rwv \wedge V(\psi, v) \geq q$, i.e., $Rwv \geq q$ and $V(\psi, v) \geq q$. Since $w \in D_q$, also $Rw_0v \geq q$, so $v \in D_q$ and $V(\psi, v) \geq q$. $\square$

We can now use this last lemma to prove the desired result, noting that the constant domain case was already proved in [10].

Theorem 9. *For any* $\alpha \in \mathrm{Fm}_1$,

$$
\begin{aligned}
\models_{\mathsf{S5}(\mathbf{G})} \alpha^* &\iff \models_{\mathsf{IKL}_1} \alpha &\iff \models_{\mathsf{IKL}_1(\mathbf{Q})} \alpha \\
\models_{\mathsf{S5}(\mathbf{G})^C} \alpha^* &\iff \models_{\mathsf{CDIKL}_1} \alpha &\iff \models_{\mathsf{CDIKL}_1(\mathbf{Q})} \alpha.
\end{aligned}
$$

Proof. Clearly, $\models_{\mathsf{IKL}_1} \alpha$ implies $\models_{\mathsf{IKL}_1(\mathbf{Q})} \alpha$. Suppose now that $\not\models_{\mathsf{IKL}_1} \alpha$. Then there exists a countable linear frame $\mathbf{K} = \langle K, \preceq \rangle$ and an $\mathsf{IKL}_1(\mathbf{K})$-model $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k \in K}, \{I_k\}_{k \in K} \rangle$, $k \in K$, and $a \in D_k$ such that $\mathfrak{M}, k \not\models^a \alpha$. An application of Lemma 3 with $\mathcal{M}_{\mathfrak{M}} = \langle W, R, V \rangle$ yields $k \notin V(\alpha^*, a)$. Hence $V(\alpha^*, a) \neq K$ and, since there exists a complete embedding of $\langle \mathrm{Up}(\mathbf{K}), \subseteq \rangle$ into $\langle [0,1], \leq \rangle$, we obtain $\not\models_{\mathsf{S5}(\mathbf{G})} \alpha^*$.

Now suppose that $\not\models_{\mathsf{S5}(\mathbf{G})} \alpha^*$. It follows that there exist a countable $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M} = \langle W, R, V \rangle$ and $w \in W$ such that $V(\alpha^*, w) < 1$. By Lemma 7, there exist an irrational $\mathsf{S5}(\mathbf{G})$-model $\mathcal{M}' = \langle W, R', V' \rangle$ and $r \in (0,1)_{\mathbb{Q}}$ such that $V'(\alpha^*, w) < r < 1$. But then Lemma 8 gives an $\mathsf{IKL}_1((0,1)_{\mathbf{Q}})$-model $\mathfrak{M}^i_{\mathcal{M}',w}$ such that $\mathfrak{M}^i_{\mathcal{M}',w}, r \not\models^w \alpha$. So $\not\models_{\mathsf{IKL}_1((0,1)_{\mathbf{Q}})} \alpha$ and since $(0,1)_{\mathbf{Q}}$ is order-isomorphic to $\mathbf{Q}$, also $\not\models_{\mathsf{IKL}_1(\mathbf{Q})} \alpha$.

Finally, for the second chain of equivalences, it suffices to recall that if $\mathfrak{M}$ is a $\mathsf{CDIKL}_1(\mathbf{K})$-model, then $\mathcal{M}_{\mathfrak{M}}$ is universal, and if $\mathcal{M}$ is universal, then $\mathfrak{M}^i_{\mathcal{M},w_0}$ is a $\mathsf{CDIKL}_1((0,1)_{\mathbf{Q}})$-model. $\square$

5. From Gödel sets to linear frames

In the previous section, we proved that for every countable linear frame $\mathbf{K}$, there exists a Gödel set A such that the validity of any $\alpha \in \mathrm{Fm}_1$ in $\mathsf{IKL}_1(\mathbf{K})$ corresponds to the validity of α^* in $\mathsf{S5}(\mathbf{A})$ (Theorem 5). In this section, we prove the converse: for any Gödel set A, there exists a countable linear frame $\mathbf{K}$ such that the validity of any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ in $\mathsf{S5}(\mathbf{A})$ corresponds to the validity of φ° in $\mathsf{IKL}_1(\mathbf{K})$ (Theorem 15).

We follow the strategy used in [13] to establish a correspondence between first-order Gödel logics and constant domain logics defined over a countable linear frame, making the necessary adjustments to accommodate many-valued relations and increasing domains. First we show that for any countable Gödel set A, the algebra $\mathbf{A}$ is isomorphic to $\mathbf{Up}(\mathbf{K})$ for some linear frame $\mathbf{K}$. Then, for the general case, we partition any Gödel set A into a countable part and an uncountable part. Using this partition and the result for countable Gödel sets, we show that $\models_{\mathsf{S5}(\mathbf{A})}$ coincides with $\models_{\mathsf{S5}(\mathbf{B})}$ for some Gödel set B such that $\mathbf{B}$ is isomorphic to some $\mathbf{Up}(\mathbf{K})$. Theorem 5 then gives the desired result.

Let us begin by recalling some topological terminology, referring to [31] for further details. A point $x \in \mathbb{R}$ is called a *limit point* if every open neighbourhood of x contains a point $y \neq x$. A subset $X \subseteq \mathbb{R}$ is called *perfect* if it is closed and all points in X are limit points in its relative topology. By a result of Cantor, every non-empty perfect set is uncountable. A proof of the following classical theorem may be found in [31].

Theorem 10 *(Cantor-Bendixson). If A is a closed subset of $\mathbb{R}$, then it can be (uniquely) written as $A = X \cup C$ for some perfect set X and countable set C such that $X \cap C = \emptyset$. The set X is called the* perfect kernel *of A and the set C is called the* scattered part *of A.*

We also recall a useful lemma proved in [32].

Lemma 11 *([32, Section 5.4.1]). Let $C \subseteq [0, 1]$ be a countable set and $X \subseteq [0, 1]$ a perfect set. Then there exists an order-embedding h from C into X preserving all existing suprema and infima, and satisfying $h(\inf C) = \inf X$ if $\inf C \in C$.*

We now consider the case of countable Gödel sets.

Lemma 12. *For any countable Gödel set A, there exists a countable linear frame $\mathbf{K}$ such that $\mathbf{Up}(\mathbf{K})$ and $\mathbf{A}$ are isomorphic.*

Proof. We call $x \in A$ *left isolated in A* if $\sup\{y \in A \mid y < x\} < x$ and define $K := \{x \in A \mid x \text{ left isolated in } A\}$. Note that K is non-empty, since otherwise A would be perfect and thus uncountable. Let $\mathbf{K} = \langle K, \geq \rangle$ and consider the map $h\colon \mathrm{Up}(\mathbf{K}) \to A$; $U \mapsto \sup U$. Since A is closed, h is well-defined. First we show that h is an order-embedding. Suppose that $U \subsetneq U'$ for some $U, U' \in \mathrm{Up}(\mathbf{K})$, and let $x \in U' \setminus U$. Since x is left isolated in A, we have $h(U) = \sup U < x \leq \sup U' = h(U')$. It remains to prove that h is surjective. Given $a \in A$, we consider the upset $U_a := \{x \in K \mid x \leq a\}$ of $\mathbf{K}$. Note that $h(U_a) \leq a$. Suppose for a contradiction that $h(U_a) < a$. Then $a \notin K$, since if $a \in K$, clearly $h(U_a) = a$. So a is not left isolated in A, i.e., $\sup\{y \in A \mid y < a\} = a$, and $[h(U_a), a] \cap A$ contains infinitely many points. Moreover, for any $c \in A$ such that $h(U_a) < c < a$, the set $[c, a] \cap A$ is again infinite and contains no left isolated points. But then $[c, a] \cap A$ is perfect and hence uncountable, a contradiction.

It follows that h is an order-isomorphism and since $h(\emptyset) = 0$ and $h(K) = 1$, h is an isomorphism between the Heyting algebras $\mathbf{Up}(\mathbf{K})$ and $\mathbf{A}$. $\square$

For an uncountable Gödel set A, we obtain a partition of A into a non-empty (uncountable) perfect kernel X and a countable set C, by Theorem 10. To deal with such uncountable Gödel sets, we prove the following lemma, noting that the case for $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ follows already from results in [13].

Lemma 13. *Let A be a Gödel set with a non-empty perfect kernel X, and let $B := A \cup [\inf X, 1]$. Then for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$,*

$$\models_{\mathsf{S5}(\mathbf{A})} \varphi \iff \models_{\mathsf{S5}(\mathbf{B})} \varphi \quad \text{and} \quad \models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi \iff \models_{\mathsf{S5}(\mathbf{B})^{\mathsf{C}}} \varphi.$$

Proof. The right-to-left-direction of both statements follows from the fact that $A \subseteq B$. For the other direction, suppose that $V_B(\varphi, w) < 1$ for some $\mathsf{S5}(\mathbf{B})$-model $\mathcal{M}_B = \langle W, R_B, V_B \rangle$ and $w \in W$. For each subformula $\Box\psi$ or $\Diamond\psi$ of φ, there exists a countable subset $W_{\Box\psi}$ or $W_{\Diamond\psi}$ of W such that, respectively, $V_B(\Box\psi, w) = \bigwedge\{Rwv \to V(\psi, v) \mid v \in W_{\Box\psi}\}$ or $V_B(\Diamond\psi, w) = \bigvee\{Rwv \wedge V(\psi, v) \mid v \in W_{\Diamond\psi}\}$. An easy induction yields $V'_B(\varphi, w) < 1$ when R'_B and V'_B are R_B and V_B restricted to $W' = \{w\} \cup \bigcup\{W_{\psi'} \mid \psi' \text{ is a subformula } \Box\psi \text{ or } \Diamond\psi \text{ of } \varphi\}$. We may therefore assume that W is countable and hence also that $C := \{V_B(\psi, v) \mid \psi \text{ a subformula of } \varphi;\ v \in W\}$ is countable. So, as B is uncountable, there exists $b \in B \setminus C$ such that $V_B(\varphi, w) < b < 1$. By Lemma 11, there exists an order-embedding h from $[\inf X, b] \cap (C \cup \{b\})$ into X. We define the following function $k_b\colon B \to A$ such that for every $a \in B$,

$$k_b(a) := \begin{cases} a & a < \inf X \\ h(a) & \inf X \leq a \leq b \\ 1 & \text{otherwise.} \end{cases}$$

Now let $\mathcal{M}_A = \langle W, R_A, V_A \rangle$ be the $\mathsf{S5}(\mathbf{A})$-model where $R_A vu = k_b(R_B vu)$ and $V_A(p_i, v) = k_b(V_B(p_i, v))$ for all $u, v \in W$ and each p_i that occurs in φ and $V_A(p_j, v) = 1$ for all other propositional variables p_j.[7]

We claim that this valuation extends to all subformulas of φ; that is, $V_A(\psi, v) = k_b(V_B(\psi, v))$ for every subformula ψ of φ and $v \in W$. It follows from this claim that $V_A(\varphi, w) < 1$, since

either $V_B(\varphi, w) < \inf X$ and $V_A(\varphi, w) = V_B(\varphi, w) < b < 1$
or $\inf X \leq V_B(\varphi, w) < b$ and $V_A(\varphi, w) = h(V_B(\varphi, w)) < h(b) \leq 1$.

So $\not\models_{\mathsf{S5}(\mathbf{A})} \varphi$. Moreover, if $\mathcal{M}_B$ is universal, then so is $\mathcal{M}_A$, so we also obtain that $\not\models_{\mathsf{S5}(\mathbf{B})^{\mathsf{C}}} \varphi$ implies $\not\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$.

We prove the claim by induction on the length of a subformula ψ of φ. The base cases follow by definition and the cases for the propositional connectives are straightforward, using the fact that $k_b(c \star d) = k_b(c) \star k_b(d)$ for all $c, d \in B$ and $\star \in \{\wedge, \vee, \to\}$. For a subformula $\Box\psi$, we have

[7] Note that this function differs slightly from the one used for the constant domain case in [29].

$$
\begin{aligned}
V_A(\Box\psi, v) &= \bigwedge\{R_A vu \to V_A(\psi, u) \mid u \in W\} \\
&= \bigwedge\{k_b(R_B vu) \to k_b(V_B(\psi, u)) \mid u \in W\} \\
&= \bigwedge\{k_b(R_B vu \to V_B(\psi, u)) \mid u \in W\} \\
&= k_b(\bigwedge\{R_B vu \to V_B(\psi, u) \mid u \in W\}) \qquad (1) \\
&= k_b(V_B(\Box\psi, v)).
\end{aligned}
$$

To prove (1), there are three cases to consider:

(i) $V_B(\Box\psi, v) < \inf X$. Then $k_b(V_B(\Box\psi, v)) = V_B(\Box\psi, v)$. Moreover, $U := \{u \in W \mid R_B vu_0 \to V_B(\psi, u_0) < \inf X\} \neq \emptyset$ and, by definition, $k_b(R_B vu \to V_B(\psi, u)) = R_B vu \to V_B(\psi, u)$ for all $u \in U$.
(ii) $\inf X \le V_B(\Box\psi, v) \le b$. By the choice of b, we have $V_B(\Box\psi, v) < b$. So $\inf X \le R_B vu \to V_B(\psi, u)$ for all $u \in W$ and $R_B vt \to V_B(\psi, t) < b$ for some $t \in W$. It follows that $\bigwedge\{k_b(R_B vu \to V_B(\psi, u)) \mid u \in W\} = \bigwedge\{h(R_B vu \to V_B(\psi, u)) \mid u \in W;\ R_B vu \to V_B(\psi, u) < b\}$ and the fact that h preserves infima concludes the case.
(iii) $b < V_B(\Box\psi, v)$. Then $b \le R_B vu \to V_B(\psi, u)$ for all $u \in W$ and hence $k_b(R_B vu \to V_B(\psi, u)) = 1 = k_b(V_B(\Box\psi, v))$ for all $u \in W$.

Next, for a subformula $\Diamond\psi$, we have

$$
\begin{aligned}
V_A(\Diamond\psi, v) &= \bigvee\{R_A vu \wedge V_A(\psi, u) \mid u \in W\} \\
&= \bigvee\{k_b(R_B vu) \wedge k_b(V_B(\psi, u)) \mid u \in W\} \\
&= \bigvee\{k_b(R_B vu \wedge V_B(\psi, u)) \mid u \in W\} \\
&= k_b(\bigvee\{R_B vu \wedge V_B(\psi, u) \mid u \in W\}) \qquad (2) \\
&= k_b(V_B(\Diamond\psi, v)).
\end{aligned}
$$

To prove (2), there are again three cases to consider:

(i) $V_B(\Diamond\psi, v) < \inf X$. Then $k_b(V_B(\Diamond\psi, v)) = V_B(\Diamond\psi, v)$ and, since $R_B vu \wedge V_B(\psi, u) < \inf X$ for all $u \in W$, also $k_b(R_B vu \wedge V_B(\psi, u)) = R_B vu \wedge V_B(\psi, u)$ for all $u \in W$, yielding (2).
(ii) $\inf X \le V_B(\Diamond\psi, v) \le b$. If $\inf X \le R_B vt \wedge V_B(\psi, t)$ for some $t \in W$, then (2) follows since h preserves existing suprema. Otherwise $R_B vu \wedge V_B(\psi, u) < \inf X$ for all $u \in W$, and so $V_B(\Diamond\psi, v) = \inf X$. But then $k_b(R_B vu \wedge V_B(\psi, u)) = R_B vu \wedge V_B(\psi, u)$ for all $u \in W$, and their join is $\inf X$. The equality (2) then follows from the fact that $h(\inf X) = \inf X$.
(iii) $b < V_B(\Diamond\psi, v)$. Then there exists $u \in W$ such that $b < R_B vu \wedge V_B(\psi, u)$, i.e., $k_b(R_B vu \wedge V_B(\psi, u)) = 1 = k_b(V_B(\Diamond\psi, v))$. □

We will also make use of the following lemma from [13] for composing Gödel sets and linear frames.

Lemma 14 *([13, Lemma 24]). Let A_1 and A_2 be Gödel sets and let $\mathbf{K}_1 = \langle K_1, \preceq_1\rangle$ and $\mathbf{K}_2 = \langle K_2, \preceq_2\rangle$ be linear frames with $K_1 \cap K_2 = \emptyset$ such that $\mathbf{Up}(\mathbf{K}_1) \cong \mathbf{A}_1$ and $\mathbf{Up}(\mathbf{K}_2) \cong \mathbf{A}_2$. Define $\mathbf{K} = \langle K, \preceq\rangle$, where $K := K_1 \cup K_2$ and*

$$\preceq := \preceq_1 \cup \preceq_2 \cup \{\langle k_2, k_1\rangle \mid k_2 \in K_2;\ k_1 \in K_1\},$$

and for any $\rho \in (0, 1)$, the Gödel set

$$A := \rho A_1 \cup ((1-\rho)A_2 + \rho).$$

Then $\mathbf{Up}(\mathbf{K}) \cong \mathbf{A}$.

We are now able to prove the main theorem of this section.

Theorem 15. *For each Gödel set A, there exists a countable linear frame $\mathbf{K}$ such that for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$,*

$$\models_{\mathsf{S5}(\mathbf{A})} \varphi \iff \models_{\mathsf{IKL}_1(\mathbf{K})} \varphi^\circ \quad \text{and} \quad \models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi \iff \models_{\mathsf{CDIKL}_1(\mathbf{K})} \varphi^\circ.$$

Proof. Let A be a Gödel set. By Theorem 10, there exists a partition of A into a countable set C and a perfect set X. If A is countable, then $X = \emptyset$ and so by Lemma 12 and Theorem 5, we are done. Now suppose that A is uncountable and so $X \neq \emptyset$. We define

$$A_1 := A \cup [\inf X, 1] \quad \text{and} \quad A_2 := (A \cap [0, \inf X]) \cup \mathcal{C}_{[\inf X, 1]},$$

where $\mathcal{C}_{[\inf X,1]}$ is the middle third Cantor set on the interval $[\inf X, 1]$. Note that the perfect kernel X_2 of A_2 is $\mathcal{C}_{[\inf X,1]}$ and so $A_2 \cup [\inf X_2, 1] = A_1$. By Lemma 13, for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$,

$$\models_{\mathsf{S5}(\mathbf{A})} \varphi \iff \models_{\mathsf{S5}(\mathbf{A}_1)} \varphi \iff \models_{\mathsf{S5}(\mathbf{A}_2)} \varphi$$
$$\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi \iff \models_{\mathsf{S5}(\mathbf{A}_1)^{\mathsf{C}}} \varphi \iff \models_{\mathsf{S5}(\mathbf{A}_2)^{\mathsf{C}}} \varphi.$$

If $\inf X = 0$, then $A_1 = [0, 1]$, in which case $\mathsf{S5}(\mathbf{A})$ coincides with $\mathsf{S5}(\mathbf{G})$ and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ coincides with $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. If $\inf X > 0$, we can write $A_2 = \rho B_1 \cup ((1-\rho)B_2 + \rho)$, where $\rho = \inf X$, $B_1 = (1/\rho)(A \cap [0, \rho])$, and $B_2 = \mathcal{C}_{[0,1]}$. Since $A \cap [0, \rho] \subseteq C \cup \{\inf X\}$, B_1 is countable. Therefore, by Lemma 12, $\mathbf{B}_1$ is isomorphic to $\mathbf{Up}(\mathbf{K}_1)$ for some linear frame $\mathbf{K}_1 = \langle K_1, \preceq_1\rangle$. Moreover, $\mathbf{B}_2$ is isomorphic to $\mathbf{Up}((0, 1)_{\mathbf{Q}})$. So by Lemma 14, we obtain a linear frame $\mathbf{K} = \langle K, \preceq\rangle$ such that $\mathbf{A}_2$ is isomorphic to $\mathbf{Up}(\mathbf{K})$. Theorem 5 then completes the proof. □

6. An interpretation theorem

In this section we provide an interpretation of the one-variable fragment of an intermediate logic defined over a linear frame in the one-variable fragment of the corresponding constant domain logic, thereby obtaining also an interpretation of $\mathsf{S5}(\mathbf{A})$ in $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ for any Gödel set A. The key idea is to describe the domains of an IKL_1-model using a distinguished unary predicate P_0 for the corresponding CDIKL_1-model. To this end, let $\mathrm{Fm}_1^r \subseteq \mathrm{Fm}_1$ denote the set of one-variable first-order formulas not containing P_0. An $\mathsf{IKL}_1^r(\mathbf{K})$-*model*, based on a linear frame $\mathbf{K} = \langle K, \preceq\rangle$, is an $\mathsf{IKL}_1(\mathbf{K})$-model $\mathfrak{M} = \langle K, \preceq, \{D_k\}_{k\in K}, \{I_k\}_{k\in K}\rangle$ such that the functions $\{I_k\}_{k\in K}$ are restricted to $\{P_i\}_{i\in\mathbb{N}^+}$.

Now let $\mathbf{K} = \langle K, \preceq\rangle$ be any linear frame and let $\mathfrak{M} = \langle K, \preceq, \{D\}, \{I_k\}_{k\in K}\rangle$ be a $\mathsf{CDIKL}_1(\mathbf{K})$-model satisfying

$$\bigcap_{k\in K} I_k(P_0) \neq \emptyset.$$

Define for each $k \in K$ and $i \in \mathbb{N}^+$,

$$D_k := I_k(P_0) \quad \text{and} \quad I_k^r(P_i) := I_k(P_i) \cap D_k.$$

Then $\mathfrak{M}^r = \langle K, \preceq, \{D_k\}_{k\in K}, \{I_k^r\}_{k\in K}\rangle$ is an $\mathsf{IKL}_1^r(\mathbf{K})$-model. Indeed, $\mathfrak{M} \mapsto \mathfrak{M}^r$ is a surjective map from $\mathsf{CDIKL}_1(\mathbf{K})$-models to $\mathsf{IKL}_1^r(\mathbf{K})$-models.

For each $\alpha \in \mathrm{Fm}_1^r$, we define $\alpha^c \in \mathrm{Fm}_1$ inductively by relativizing quantifiers to the unary predicate P_0. That is, $(P_i(x))^c := P_i(x)$ for each $i \in \mathbb{N}^+$, $\bot^c := \bot$, $\top^c := \top$, $(\alpha \star \beta)^c := \alpha^c \star \beta^c$ for $\star \in \{\wedge, \vee, \to\}$, and

$$((\forall x)\alpha)^c := (\forall x)(P_0(x) \to \alpha^c)$$
$$((\exists x)\alpha)^c := (\exists x)(P_0(x) \wedge \alpha^c).$$

Lemma 16. *Let $\mathbf{K} = \langle K, \preceq\rangle$ be a linear frame and let $\mathfrak{M} = \langle K, \preceq, \{D\}, \{I_k\}_{k\in K}\rangle$ be a $\mathsf{CDIKL}_1(\mathbf{K})$-model satisfying $\bigcap_{k\in K} I_k(P_0) \neq \emptyset$. Then for any $\alpha \in \mathrm{Fm}_1^r$, $k \in K$, and $a \in I_k(P_0)$,*

$$\mathfrak{M}^r, k \models^a \alpha \iff \mathfrak{M}, k \models^a \alpha^c.$$

Proof. We prove the claim by induction on the length of α. For the base case, using the assumption that $a \in I_k(P_0) = D_k$, we have for each $i \in \mathbb{N}^+$,

$$\mathfrak{M}^r, k \models^a P_i(x) \iff a \in I_k^r(P_i) \iff a \in I_k(P_i) \iff \mathfrak{M}, k \models^a P_i(x).$$

The cases for the propositional connectives follow easily using the induction hypothesis and the definition of α^c, so we just check the cases for the quantifiers:

$$\begin{aligned}
\mathfrak{M}^r, k \models^a (\forall x)\beta &\iff \mathfrak{M}^r, l \models^b \beta \ \text{ for all } l \succeq k \text{ and } b \in D_l \\
&\iff \mathfrak{M}, l \models^b \beta^c \ \text{ for all } l \succeq k \text{ and } b \in I_l(P_0) \\
&\iff (\mathfrak{M}, l \models^b P_0(x) \Rightarrow \mathfrak{M}, l \models^b \beta^c) \text{ for all } l \succeq k \text{ and } b \in D \\
&\iff \mathfrak{M}, l \models^b P_0(x) \to \beta^c \ \text{ for all } l \succeq k \text{ and } b \in D \\
&\iff \mathfrak{M}, k \models^a (\forall x)(P_0(x) \to \beta^c) \\
&\iff \mathfrak{M}, k \models^a ((\forall x)\beta)^c.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{M}^r, k \models^a (\exists x)\beta &\iff \mathfrak{M}^r, k \models^b \beta \text{ for some } b \in D_k \\
&\iff \mathfrak{M}, k \models^b \beta^c \text{ for some } b \in I_k(P_0) \\
&\iff (\mathfrak{M}, k \models^b P_0(x) \text{ and } \mathfrak{M}, k \models^b \beta^c) \text{ for some } b \in D \\
&\iff \mathfrak{M}, k \models^b P_0(x) \wedge \beta^c \text{ for some } b \in D \\
&\iff \mathfrak{M}, k \models^a (\exists x)(P_0(x) \wedge \beta^c) \\
&\iff \mathfrak{M}, k \models^a ((\exists x)\beta)^c. \quad \square
\end{aligned}$$

Theorem 17. *For any linear frame* $\mathbf{K} = \langle K, \preceq\rangle$ *and formula* $\alpha \in \mathrm{Fm}_1^r$,

$$\models_{\mathsf{IKL}_1(\mathbf{K})} (\forall x)\alpha \iff \models_{\mathsf{CDIKL}_1(\mathbf{K})} ((\forall x)\alpha)^c.$$

Proof. ($\Rightarrow$) Suppose that $\not\models_{\mathsf{CDIKL}_1(\mathbf{K})} ((\forall x)\alpha)^c$, i.e., $\mathfrak{M}_1, k_0 \not\models^a \alpha^c$ for some $\mathsf{CDIKL}_1(\mathbf{K})$-model $\mathfrak{M}_1 = \langle K, \preceq, \{D\}, \{I_k\}_{k\in K}\rangle$, $k_0 \in K$, and $a \in I_{k_0}(P_0)$. Let $\mathbf{K}_0 = \langle [k_0), \preceq\rangle$. Then also $\mathfrak{M}_2, k_0 \not\models^a \alpha^c$, where $\mathfrak{M}_2$ is the $\mathsf{CDIKL}_1(\mathbf{K}_0)$-model $\langle [k_0), \preceq, \{D\}, \{I_k\}_{k\in[k_0)}\rangle$ satisfying $\bigcap_{k\in[k_0)} I_k(P_0) \neq \emptyset$. An application of Lemma 16 yields $\mathfrak{M}_2^r, k_0 \not\models^a \alpha$. We can then extend $\mathfrak{M}_2^r$ to an $\mathsf{IKL}_1^r(\mathbf{K})$-model by defining $D_l := D_{k_0}$ and $I_l(P_i) := I_{k_0}(P_i)$ for all $l \prec k_0$, giving $\not\models_{\mathsf{IKL}_1(\mathbf{K})} (\forall x)\alpha$ as required.

($\Leftarrow$) Suppose that $\not\models_{\mathsf{IKL}_1(\mathbf{K})} (\forall x)\alpha$, i.e., $(\forall x)\alpha$ is not valid in some $\mathsf{IKL}_1(\mathbf{K})$-model $\mathfrak{M}$. Since α does not contain P_0, we can assume that $\mathfrak{M}$ is an $\mathsf{IKL}_1^r(\mathbf{K})$-model. Because the map $(-)^r$ is surjective, there exists a $\mathsf{CDIKL}_1(\mathbf{K})$-model $\mathfrak{N}$ such that $\mathfrak{M} = \mathfrak{N}^r$. By Lemma 16, the formula $((\forall x)\alpha)^c$ is not valid in $\mathfrak{N}$ and hence $\not\models_{\mathsf{CDIKL}_1(\mathbf{K})} ((\forall x)\alpha)^c$ as required. $\square$

Now let $\mathrm{Fm}_{\Box\Diamond}^r \subseteq \mathrm{Fm}_{\Box\Diamond}$ denote the set of modal formulas not containing p_0. For each $\varphi \in \mathrm{Fm}_{\Box\Diamond}^r$, we define $\varphi^c \in \mathrm{Fm}_{\Box\Diamond}$ inductively by relativizing modalities to p_0. That is, $(p_i)^c := p_i$ for each $i \in \mathbb{N}^+$, $\bot^c := \bot$, $\top^c := \top$, $(\varphi \star \psi)^c := \varphi^c \star \psi^c$ for $\star \in \{\wedge, \vee, \to\}$, $(\Box\varphi)^c := \Box(p_0 \to \varphi^c)$, and $(\Diamond\varphi)^c := \Diamond(p_0 \wedge \varphi^c)$.

Theorem 18. *For any formula* $\varphi \in \mathrm{Fm}_{\Box\Diamond}^r$ *and Gödel set* A,

$$\models_{\mathsf{S5}(\mathbf{A})} \varphi \iff \models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} (\Box\varphi)^c.$$

Proof. Consider any Gödel set A. By Theorem 15, there exists a countable linear frame $\mathbf{K}$ such that both $\models_{\mathsf{S5}(\mathbf{A})} \varphi$ if and only if $\models_{\mathsf{IKL}_1(\mathbf{K})} \varphi^\circ$, and $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$ if and only if $\models_{\mathsf{CDIKL}_1(\mathbf{K})} \varphi^\circ$ hold. Note that the translations $(-)^\circ$ and $(-)^c$ commute on formulas $\varphi \in \mathrm{Fm}_{\Box\Diamond}^r$. Combining this with Theorem 17 gives for every $\varphi \in \mathrm{Fm}_{\Box\Diamond}^r$,

$$\begin{aligned}
\models_{\mathsf{S5}(\mathbf{A})} \varphi &\iff \models_{\mathsf{S5}(\mathbf{A})} \Box\varphi \\
&\iff \models_{\mathsf{IKL}_1(\mathbf{K})} (\Box\varphi)^\circ \\
&\iff \models_{\mathsf{CDIKL}_1(\mathbf{K})} ((\Box\varphi)^\circ)^c \\
&\iff \models_{\mathsf{CDIKL}_1(\mathbf{K})} ((\Box\varphi)^c)^\circ \\
&\iff \models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} (\Box\varphi)^c. \quad \square
\end{aligned}$$

Let us remark finally that the predicate used in this interpretation corresponds exactly to the existence predicate considered in the context of Scott logics in [33] and is closely related also to the normalized probability distribution used for the possibilistic logic studied in [34].

7. A finite model property

In this section, we establish a finite model property for the logic $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ (and hence also $\mathsf{S5}(\mathbf{A})$) for any Gödel set A. Crucially, however, this property does not hold in general with respect to the "standard" $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-models defined in Section 3. Indeed, for any Gödel set A containing at least one right accumulation point c, the formula $\Diamond(p_1 \to \Box p_1)$ is valid in all finite $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-models, but not in any infinite universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\langle \mathbb{N}^+, V\rangle$ satisfying $V(p_1, n) \in A \cap (c, c + \frac{1}{n}]$ for each $n \in \mathbb{N}^+$. A previous paper [16] by three of the authors with J. Rogger, contains a flawed proof that these logics have the finite model property with respect to an alternative semantics.[8] Here, we introduce a further (related) alternative semantics that avoids the problem encountered in that paper.

Let $P \subseteq \{p_i\}_{i\in\mathbb{N}}$ be a set of propositional variables and let $\mathrm{Fm}_{\Box\Diamond}(P)$ denote the set of formulas in $\mathrm{Fm}_{\Box\Diamond}$ with variables in P. A *relativized universal* $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$*-model over* P (for short, $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$*-model over* P) based on a Gödel set A is a triple

[8] More precisely, Lemma 23 of [16] is false unless $T_\Box = T_\Diamond$; this restriction does not cause any problems for $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, but is not sufficient for other cases.

$\mathcal{M} = \langle W, V, T\rangle$ consisting of finite non-empty sets W and T satisfying $\{0,1\} \subseteq T \subseteq A$, and a map $V : P \times W \to A$. The map V is extended inductively to $V : \mathrm{Fm}_{\Box\Diamond}(P) \times W \to A$ as follows, where $\star \in \{\wedge, \vee, \to\}$:

$$V(\bot, w) = 0$$
$$V(\top, w) = 1$$
$$V(\varphi \star \psi, w) = V(\varphi, w) \star V(\psi, w)$$
$$V(\Box\varphi, w) = \bigvee\{r \in T \mid r \leq \bigwedge\{V(\varphi, v) \mid v \in W\}\}$$
$$V(\Diamond\varphi, w) = \bigwedge\{r \in T \mid r \geq \bigvee\{V(\varphi, v) \mid v \in W\}\}.$$

We say that $\varphi \in \mathrm{Fm}_{\Box\Diamond}(P)$ is *valid* in $\mathcal{M}$ if $V(\varphi, w) = 1$ for all $w \in W$.

Note that since W and T are finite, $V(\Box\varphi, w), V(\Diamond\varphi, w) \in T$ for all $\Box\varphi, \Diamond\varphi \in \mathrm{Fm}_{\Box\Diamond}(P)$ and $w \in W$, and these values are independent of w. Moreover, a simple induction on the length of $\varphi \in \mathrm{Fm}_{\Box\Diamond}(P)$ shows that always

$$V(\varphi, w) \in B_{\mathcal{M}} := \{V(p_i, v) \mid p_i \in P;\ v \in W\} \cup T.$$

Indeed, $\mathcal{M}$ may also be viewed as an $\mathsf{ruS5}(\mathbf{B}_{\mathcal{M}})^{\mathsf{C}}$-model over P; that is, we may assume that V is a function from $P \times W$ to $B_{\mathcal{M}}$. In particular, if P is finite, then $\mathcal{M}$ is a truly finite object.

Recall that $\mathsf{R}(A)$ and $\mathsf{L}(A)$ denote the sets of right and left accumulation points, respectively, of a Gödel set A. An $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M} = \langle W, V, T\rangle$ over $P \subseteq \{p_i\}_{i\in\mathbb{N}}$ is called Σ-*normal* for $\Sigma \subseteq \mathrm{Fm}_{\Box\Diamond}(P)$ if for all $\Box\varphi, \Diamond\psi \in \Sigma$ and $w \in W$,

$$V(\Box\varphi, w) \notin \mathsf{R}(A) \implies \text{there exists } v \in W \text{ such that } V(\Box\varphi, w) = V(\varphi, v)$$
$$V(\Diamond\psi, w) \notin \mathsf{L}(A) \implies \text{there exists } v \in W \text{ such that } V(\Diamond\psi, w) = V(\psi, v).$$

Let us also call $\Sigma \subseteq \mathrm{Fm}_{\Box\Diamond}$ a *fragment* if it is closed under subformulas. The next lemma shows that (roughly speaking) for a finite fragment, validity in a (possibly infinite) universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model can be matched to validity in a corresponding $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model that is normal for the fragment.

Lemma 19. *Let A be a Gödel set and let $\mathcal{M} = \langle W, V\rangle$ be a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model with $w \in W$. For any $P \subseteq \{p_i\}_{i\in\mathbb{N}}$ and finite fragment $\Sigma \subseteq \mathrm{Fm}_{\Box\Diamond}(P)$, there exists a Σ-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M}' = \langle W', V', T\rangle$ over P with $w \in W' \subseteq W$, $|W'| \leq |\Sigma|$, and $|B_{\mathcal{M}'}| \leq |\Sigma|^2$, satisfying $V'(\varphi, v) = V(\varphi, v)$ for all $\varphi \in \Sigma$ and $v \in W'$.*

Proof. We define

$$T := \{V(\Box\varphi, w) \mid \Box\varphi \in \Sigma\} \cup \{V(\Diamond\varphi, w) \mid \Diamond\varphi \in \Sigma\} \cup \{0,1\}$$

and write $T = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$. Then for each $\Box\varphi \in \Sigma$, we have $V(\Box\varphi, w) = t_i$ for some $0 \leq i \leq n$ and we choose a witness $v_{\Box\varphi} \in W$ satisfying

$$t_i \in \mathsf{R}(A) \implies V(\varphi, v_{\Box\varphi}) \in [t_i, t_{i+1}) \cap A$$
$$t_i \notin \mathsf{R}(A) \implies V(\Box\varphi, w) = V(\varphi, v_{\Box\varphi}) = t_i.$$

Similarly, for each $\Diamond\varphi \in \Sigma$, we have $V(\Diamond\varphi, w) = t_i$ for some $0 \leq i \leq n$, and we choose a witness $v_{\Diamond\varphi} \in W$ satisfying

$$t_i \in \mathsf{L}(A) \implies V(\varphi, v_{\Diamond\varphi}) \in (t_{i-1}, t_i] \cap A$$
$$t_i \notin \mathsf{L}(A) \implies V(\Diamond\varphi, w) = V(\varphi, v_{\Diamond\varphi}) = t_i.$$

We now define

$$W' := \{w\} \cup \{v_{\Box\varphi} \mid \Box\varphi \in \Sigma\} \cup \{v_{\Diamond\varphi} \mid \Diamond\varphi \in \Sigma\}$$

and $V'(p_i, v) := V(p_i, v)$ for all $p_i \in P$ and $v \in W'$. Then by construction, $\mathcal{M}' = \langle W', V', T\rangle$ is a Σ-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model over P and clearly $|W'| \leq |\Sigma|$. It follows by induction on formula length that $V'(\varphi, v) = V(\varphi, v)$ for all $\varphi \in \Sigma$, $v \in W'$. The base cases and the cases of the propositional connectives are straightforward. If $\varphi = \Box\psi$, then $V(\Box\psi, w) = t_i$ for some $0 \leq i \leq n$, and we have two cases. If $t_i \in \mathsf{R}(A)$, then

$$V(\Box\psi, w) = \bigwedge\{V(\psi, v) \mid v \in W\} \leq \bigwedge\{V(\psi, v) \mid v \in W'\} \leq V(\psi, v_{\Box\psi}) < t_{i+1},$$

and if $t_i \notin \mathsf{R}(A)$, then

$$V(\Box\psi, w) = \bigwedge\{V(\psi, v) \mid v \in W\} \leq \bigwedge\{V(\psi, v) \mid v \in W'\} \leq V(\psi, v_{\Box\psi}) = t_i.$$

Together with the induction hypothesis, this gives

$$\begin{aligned} V(\Box\psi, w) &= \bigvee\{r \in T \mid r \leq \bigwedge\{V(\psi, v) \mid v \in W'\}\} \\ &= \bigvee\{r \in T \mid r \leq \bigwedge\{V'(\psi, v) \mid v \in W'\}\} \\ &= V'(\Box\psi, w). \end{aligned}$$

The case $\varphi = \Diamond\psi$ is very similar. It easily follows also that $|B_{\mathcal{M}'}| \leq |\Sigma|^2$. $\square$

The second crucial lemma proceeds in the other direction; it shows that (roughly speaking) validity for a fragment in an $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model can be matched to validity in a corresponding universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model. The key idea here is to approximate values in the set T taken by formulas $\Box\varphi$ and $\Diamond\varphi$ by taking multiple copies of the set of worlds and choosing elements in A that get closer and closer to the values in T from the left or right as appropriate.

Lemma 20. *Let $\mathcal{M} = \langle W, V, T\rangle$ be a Σ-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model over a finite set $P \subseteq \{p_i\}_{i\in\mathbb{N}}$ for a fragment $\Sigma \subseteq \mathrm{Fm}_{\Box\Diamond}(P)$. Then there exists a (countable) universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M}' = \langle W', V'\rangle$ such that $W \subseteq W'$ and $V(\varphi, w) = V'(\varphi, w)$ for all $\varphi \in \Sigma$ and $w \in W$.*

Proof. Let $T = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$. For each $t_i \in \mathsf{R}(A)$, we fix a strictly descending sequence $(r^i_n)_{n\in\mathbb{N}^+} \subseteq A \cap (t_i, t_{i+1})$ such that $t_i < r^i_n < t_i + \frac{1}{n}$ for each $n \in \mathbb{N}^+$. Similarly, for each $t_i \in \mathsf{L}(A)$, we fix a strictly ascending sequence $(s^i_n)_{n\in\mathbb{N}^+} \subseteq A \cap (t_{i-1}, t_i)$ such that $t_i - \frac{1}{n} < s^i_n < t_i$ for each $n \in \mathbb{N}^+$. For each $0 \leq i < N$, we write

$$[t_i, t_{i+1}] \cap B_{\mathcal{M}} = \{t_i = b^i_0 < b^i_1 < \cdots < b^i_{k_i} < b^i_{k_{i+1}} = t_{i+1}\}.$$

Note that $B_{\mathcal{M}} = \bigcup_{0\leq i<N}([t_i, t_{i+1}] \cap B_{\mathcal{M}})$.

We now define a map $h_n\colon B_{\mathcal{M}} \to A$ for each $n \in \mathbb{N}$, where

(i) $h_0\colon B_{\mathcal{M}} \to A$ is the identity embedding;

(ii) if $n > 0$ is odd, then $h_n(t_N) := t_N$ and for each $i \in \{0, 1, \ldots, N-1\}$, we define $h_n(t_i) := t_i$ and for each $j \in \{1, \ldots, k_i\}$,

$$h_n(b^i_j) := \begin{cases} r^i_{n+k_i-j} & t_i \in \mathsf{R}(A) \\ b^i_j & t_i \notin \mathsf{R}(A) \end{cases}$$

(iii) if $n > 0$ is even, then $h_n(t_N) := t_N$ and for each $i \in \{0, 1, \ldots, N-1\}$, we define $h_n(t_i) := t_i$, and for each $j \in \{1, \ldots, k_i\}$,

$$h_n(b^i_j) := \begin{cases} s^{i+1}_{n+j} & t_{i+1} \in \mathsf{L}(A) \\ b^i_j & t_{i+1} \notin \mathsf{L}(A). \end{cases}$$

Note that each $h_n\colon B_{\mathcal{M}} \to A$ is a strictly order-preserving embedding that fixes T.

For each $n \in \mathbb{N}$, let W_n denote a disjoint copy of W with elements $w_n \in W_n$ corresponding to the element $w \in W$, with $W_0 = W$. Now for each $p_i \in P$, $w \in W$, and $n \in \mathbb{N}$, define

$$W' := \bigcup_{n\in\mathbb{N}} W_n \quad \text{and} \quad V'(p_i, w_n) := h_n(V(p_i, w)).$$

Defining also $V'(p_j, w_n) := 0$ for $j \in \mathbb{N}$ and $p_j \notin P$, we obtain a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M}' = \langle W', V'\rangle$.

We prove by induction on formula length that $V'(\varphi, w_n) = h_n(V(\varphi, w))$ for all $\varphi \in \Sigma$, $w \in W$, and $n \in \mathbb{N}$. The base cases follow by definition and the fact that each h_n fixes 0 and 1. The cases for propositional connectives follow from the fact that each h_n is a strictly order-preserving embedding fixing 0 and 1.

Now consider $\varphi = \Box\psi \in \Sigma$ with $V(\Box\psi, w) = t_i$. Then $V(\Box\psi, w) \leq V(\psi, v)$ and so $h_n(V(\Box\psi, w)) \leq h_n(V(\psi, v))$ for all $v \in W$. We consider two cases. If $t_i \notin \mathsf{R}(A)$, then since $\mathcal{M}$ is Σ-normal, there exists $v \in W$ such that $V(\Box\psi, w) = V(\psi, v)$ and so $h_n(V(\psi, v)) = t_i$ for all $n \in \mathbb{N}$. If $t_i \in \mathsf{R}(A)$, then $i < N$ and there exists $v \in W$ such that $V(\psi, v) \in [t_i, t_{i+1}) \cap B_{\mathcal{M}}$. Then by construction, $h_n(V(\psi, v)) \in [t_i, r^i_n] \subseteq [t_i, t_i + \frac{1}{n}]$ for each odd $n \in \mathbb{N}$. In both cases,

$$\begin{aligned} t_i &\leq \bigwedge\{h_n(V(\psi, v)) \mid v \in W;\ n \in \mathbb{N}\} \\ &\leq \bigwedge\{h_n(V(\psi, v)) \mid v \in W;\ n \in \mathbb{N} \text{ odd}\} \\ &= t_i = V(\Box\psi, w). \end{aligned}$$

Applying the induction hypothesis then gives

$$\begin{aligned} V'(\Box\psi, w) &= \bigwedge\{V'(\psi, w_n) \mid w \in W;\ n \in \mathbb{N}\} \\ &= \bigwedge\{h_n(V(\psi, w)) \mid w \in W;\ n \in \mathbb{N}\} \\ &= V(\Box\psi, w). \end{aligned}$$

The case for $\varphi = \Diamond\psi \in \Sigma$ is similar. So we have $V'(\varphi, w_n) = h_n(V(\varphi, w))$ for all $\varphi \in \Sigma$, $w \in W$, and $n \in \mathbb{N}$. Taking $n = 0$ then gives $V'(\varphi, w) = V(\varphi, w)$ for all $\varphi \in \Sigma$ and $w \in W$. $\Box$

Let P_φ denote the (finite) set of propositional variables occurring in a formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, and let Σ_φ denote the fragment of subformulas in φ. The following theorem expresses the desired finite model property $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$, recalling that an $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M}$ over a finite set of variables not only has a finite set of worlds, but may be considered a finite object if $\mathbf{A}$ is replaced by $\mathbf{B}_{\mathcal{M}}$.

Theorem 21. *Let A be a Gödel set. For any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$,*

$$\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi \iff \varphi \text{ is valid in all } \Sigma_\varphi\text{-normal } \mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}\text{-models over } P_\varphi.$$

Proof. If $\not\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$, then there exists a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M} = \langle W, V\rangle$ and $w \in W$ such that $V(\varphi, w) < 1$. By Lemma 19, there exists a Σ_φ-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M}' = \langle W', V', T\rangle$ over P_φ such that $V'(\varphi, w) = V(\varphi, w) < 1$.

Conversely, if $V(\varphi, w) < 1$ for some $w \in W$ in a Σ_φ-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model $\langle W, V, T\rangle$ over P_φ, then, by Lemma 20 there exists a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\langle W', V'\rangle$ such that $V'(\varphi, w) = V(\varphi, w) < 1$. $\Box$

Let us remark finally that for the Gödel set $[0, 1]$, the above reasoning yields also an algebraic finite model property for the logic $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. That is, the algebraic semantics for this logic is the variety of monadic Gödel algebras (a subvariety of monadic Heyting algebras) and it can be shown that Σ_φ-normal $\mathsf{ruS5}(\mathbf{G})^{\mathsf{C}}$-models over P_φ correspond to evaluations into finite members of this variety. A more straightforward proof of this result, avoiding the use of the machinery of an alternative frame semantics, is given in [1], but it is currently unclear if such an approach can be generalized to arbitrary Gödel sets.

8. Decidability and complexity

The finite model property established in Theorem 21 does not directly yield decidability of $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity for an arbitrary Gödel set A. In order to check the normality condition for an $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model, we require some representation of the sets $\mathsf{R}(A)$ and $\mathsf{L}(A)$, which in general, might not even be recursive. We resolve this issue here by specifying sufficient conditions on a Gödel set A that ensure the decidability and even co-NP-completeness of $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity, and hence also of $\mathsf{S5}(\mathbf{A})$-validity and the corresponding one-variable fragments of first-order Gödel logics and Corsi logics with or without constant domains.

Observe first that to determine the $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity of a formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ it suffices, by Lemmas 19 and 20, to check validity in Σ_φ-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-models $\mathcal{M} = \langle W, V, T\rangle$ over P_φ. Indeed, as remarked in the previous section, such an $\mathcal{M}$ may be viewed as an $\mathsf{ruS5}(\mathbf{B}_{\mathcal{M}})^{\mathsf{C}}$-model, where $B_{\mathcal{M}}$ is finite. Let us also note that the property of Σ_φ-normality of $\mathcal{M}$ is determined by the sets $T_r := T \cap \mathsf{R}(A)$ and $T_l := T \cap \mathsf{L}(A)$. It therefore follows that the $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity of a formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ of length n is determined by structures of the form

$$\langle W, V, B, \leq, 0, 1, T, T_r, T_l\rangle$$

satisfying the following conditions:

(i) $|W|, |T|, |T_r|, |T_l| \leq n$ and $|B|, |V| \leq n^2$;
(ii) $\{0, 1\}, T_r, T_l \subseteq T \subseteq B$ and $0 \notin T_l$, $1 \notin T_r$;
(iii) $\leq \subseteq B^2$ is a linear order with top and bottom elements 1 and 0, respectively;
(iv) $\langle W, V, T\rangle$ is an $\mathsf{ruS5}(\mathbf{B}_{\mathcal{M}})^{\mathsf{C}}$-model over P_φ such that for all $\Box\psi, \Diamond\psi \in \Sigma_\varphi$ and $w \in W$,

$$\begin{aligned} V(\Box\psi, w) \notin T_r &\implies \text{there exists } v \in W \text{ such that } V(\Box\psi, w) = V(\psi, v) \\ V(\Diamond\psi, w) \notin T_l &\implies \text{there exists } v \in W \text{ such that } V(\Diamond\psi, w) = V(\psi, v); \end{aligned}$$

(v) the finite structure $\langle B, \leq, 0, 1, T, T_r, T_l\rangle$ is *consistent with* A; that is, there exists an order-embedding $f\colon \langle B, \leq, 0, 1\rangle \to \langle A, \leq, 0, 1\rangle$ preserving 0 and 1 such that

$$f[T_r] = f[T] \cap \mathsf{R}(A) \quad \text{and} \quad f[T_l] = f[T] \cap \mathsf{L}(A).$$

Table 1
Examples of Gödel sets and corresponding consistent words (i.e. structures).

Gödel set A	Words consistent with A
$\{0, \frac{1}{n+1}, \dots, \frac{n+1}{n+1}\}$	$\{twt \mid w \in \{a,t\}^*$ of length at most $n\}$
$[0,1]$	$\{rwl \mid w \in \{a,d\}^*\}$
$G_\uparrow$	$\{twl \mid w \in \{a,t\}^*\}$
$G_\downarrow$	$\{rwt \mid w \in \{a,t\}^*\}$
$G_\uparrow \oplus G_\uparrow$	$\{twl \mid w \in \{a,t\}^*\} \cup \{twlw'l \mid w, w' \in \{a,t\}^*\}$
$G_\downarrow \oplus G_\uparrow$	$\{twt \mid w \in \{a,t\}^*\} \cup \{twdw't \mid w, w' \in \{a,t\}^*\}$
$G_\uparrow \times_{\text{lex}} G_\uparrow$	$\{twl \mid w \in \{a,t,l\}^*\}$
$G_\downarrow \times_{\text{lex}} G_\uparrow$	$\{rwl \mid w \in \{a,t\}^*\}$

Theorem 22. *Let A be a Gödel set. Then $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity and $\mathsf{S5}(\mathbf{A})$-validity are decidable (co-NP-complete) relative to the problem of checking the consistency of finite structures with A.*

Proof. Consider the following procedure to check the non-validity of a formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ of length n in $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$, where we may assume that all sets involved are subsets of $\{0, \dots, n^2\}$:

(1) Guess a structure $\langle W, V, B, \leq, 0, 1, T, T_r, T_l\rangle$ satisfying (i), (ii), (iii), and (iv);
(2) Check that $V(\varphi, i) < 1$ for some $i \in W$;
(3) Check that $\langle B, \leq, 0, 1, T, T_r, T_l\rangle$ is consistent with A.

It is easy to see that (1) and (2) are problems with complexity in NP, and hence that the complexity of the full procedure is decidable (in NP) relative to step (3). Co-NP-hardness follows from the fact that propositional classical logic can be interpreted in $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$. Finally, the same result for $\mathsf{S5}(\mathbf{A})$-validity follows from the interpretation of $\mathsf{S5}(\mathbf{A})$ in $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ provided by Theorem 18. □

Determining the consistency of a structure $\langle B, \leq, 0, 1, T, T_r, T_l\rangle$ with respect to a Gödel set A is trivial for some Gödel sets. For example, such a structure is consistent with $A = [0,1]$ if and only if $T_r = T \setminus \{0\}$ and $T_l = T \setminus \{1\}$, with $A = G_\uparrow$ if and only if $T_r = \emptyset$ and $T_l = \{1\}$, and with $A = G_\downarrow$ if and only if $T_r = \{0\}$ and $T_l = \emptyset$. Determining consistency with respect to other Gödel sets may be more complicated, however. The following observation simplifies the problem.

A finite structure $\langle B, \leq, 0, 1, T, T_r, T_l\rangle$ satisfying (ii) and (iii) may be coded via a finite word in the alphabet $\{a, t, r, l, d\}$, where each letter represents the "status" of an element of B with respect to their membership in T, T_r, and T_l:

a for an element of $B \setminus T$;	r for an element of $T_r \setminus T_l$;
t for an element of $T \setminus (T_r \cup T_l)$;	l for an element of $T_l \setminus T_r$;
d for an element of $T_r \cap T_l$.	

We say that a finite word in the alphabet $\{a, t, r, l, d\}$ is *consistent with a Gödel set A* if this is true of the corresponding finite structure. These words must start with t or r (the possible status of 0) and end with t or l (the possible status of 1). In Table 1 we state a number of examples; for Gödel sets A and B, we write $A \oplus B$ to denote the ordered sum identifying 1_A and 0_B and, if A is countable, we write $A \times_{\text{lex}} B$ for the lexicographic product. We assume harmlessly that the results of these operations are also Gödel sets.

All classes of words consistent with the respective Gödel sets in Table 1 form regular sets of words and are therefore decidable in linear time. It is not difficult to check that this property is preserved by a number of operations. That is, if the sets of words consistent with Gödel sets A and B are regular, then so are the sets of words consistent with A^*, $A \oplus B$, $\Sigma_\omega A = (A \oplus A \oplus \dots) \sqcup \{1\}$, and $A \times_{\text{lex}} B$ (if A is countable), where A^* denotes the Gödel set A with the ordering reversed. Note that $A \oplus G_2$ adds a new top element and $G_2 \oplus A$ adds a new bottom element to A. Hence, the disjoint ordered sum is recovered as $A \oplus^d B = A \oplus G_2 \oplus B$. Using Cantor's normal form, it is easy to see that any successor ordinal $2 \leq \alpha + 1 < \omega^\omega$ is a combination of $G_\uparrow$, which corresponds to the ordinal $\omega + 1$, and finite Gödel sets G_n using the above operations; for example,

$$\begin{aligned} \omega + n &= G_\uparrow \oplus G_n \quad (n \geq 2) \\ \omega^2 + 1 &= \Sigma_\omega G_\uparrow \\ \omega^2 + \omega + 1 &= G_\uparrow \times_{\text{lex}} G_\uparrow = (\Sigma_\omega G_\uparrow) \oplus G_\uparrow \\ \omega^3 + \omega 2 + 5 &= \Sigma_\omega(\Sigma_\omega G_\uparrow) \oplus G_\uparrow \oplus G_\uparrow \oplus G_5. \end{aligned}$$

This gives a large family of Gödel sets with a linearly decidable consistency problem.

Corollary 23. *$\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ and $\mathsf{S5}(\mathbf{A})$ are co-NP-complete for $A = [0,1]$, $A = G_{\uparrow}$, $A = G_{\downarrow}$, $A = G_n$ for any $n \geq 2$, and all finite combinations of these Gödel sets by $(-)^*$, $\oplus$, Σ_ω, and, if its first argument is countable, $\times_{\text{lex}}$.*

Note that $\text{Up}(\omega)$ and $\text{Up}(\omega^*)$ are isomorphic to $G_{\downarrow}$ and $G_{\uparrow}$, respectively. In general, for any ordinal α, $\text{Up}(\alpha)$ is isomorphic to $(\alpha+1)^*$ and $\text{Up}(\alpha^*)$ to $\alpha+1$. Moreover, for any pair of linear frames $\mathbf{K}$ and $\mathbf{L}$ we have $\mathbf{Up}(\mathbf{K}^*) = \mathbf{Up}(\mathbf{K})^*$, $\mathbf{Up}(\mathbf{K} \oplus^d \mathbf{L}) = \mathbf{Up}(\mathbf{L}) \oplus \mathbf{Up}(\mathbf{K})$, and if $\oplus^d_\omega \mathbf{K}$ denotes the disjoint ordered sum of $\mathbf{K}$ with itself ω times then $\mathbf{Up}(\oplus^d_\omega \mathbf{K}) = \Sigma_\omega \mathbf{Up}(\mathbf{K})$. Hence Theorem 5 yields the following decidability results.

Corollary 24. *$\mathsf{IKL}_1(\mathbf{K})$ and $\mathsf{CDIKL}_1(\mathbf{K})$ are co-NP-complete if $\mathbf{K}$ is any finite combination of countable ordinals below ω^ω and their reverses by $(-)^*$, $\oplus^d$, and $\oplus^d_\omega$.*

This notion of consistency can also be used to compare logics.

Theorem 25. *Let A and A' be two Gödel sets. Suppose that any finite structure $\langle B, \leq, 0, 1, T, T_r, T_l \rangle$ satisfying (ii) and (iii) is consistent with A if and only if it is consistent with A'. Then for all $\varphi \in \text{Fm}_{\Box\Diamond}$,*

$$\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi \iff \models_{\mathsf{S5}(\mathbf{A}')^{\mathsf{C}}} \varphi.$$

Proof. Suppose that $\not\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$ for some $\varphi \in \text{Fm}_{\Box\Diamond}$. Then by Lemma 19, there exists a Σ_φ-normal $\mathsf{ruS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathcal{M} = \langle W, V, T \rangle$ over P_φ such that $V(\varphi, w) < 1$ for some $w \in W$. Then the finite structure $\langle B_{\mathcal{M}}, \leq, 0, 1, T, T \cap \mathsf{R}(A), T \cap \mathsf{L}(A) \rangle$ is consistent with A, so by assumption it is also consistent with A'. We may therefore assume that $B_{\mathcal{M}} \subseteq A'$, $T \cap \mathsf{R}(A) = T \cap \mathsf{R}(A')$, and $T \cap \mathsf{L}(A) = T \cap \mathsf{L}(A')$. By Lemma 20, $\mathcal{M}$ can be extended to a universal $\mathsf{S5}(\mathbf{A}')^{\mathsf{C}}$-model $\mathcal{M}' = \langle W', V' \rangle$ such that $V'(\varphi, w) < 1$ and so $\not\models_{\mathsf{S5}(\mathbf{A}')^{\mathsf{C}}} \varphi$. The other direction follows by symmetry. $\Box$

Even undecidable Gödel sets can have a decidable consistency problem. For example, consider any countable limit ordinal $\alpha \geq \omega^2$. Then the words consistent with $\alpha + 1$ are all twl with $w \in \{a, t, l\}^*$. The same holds for $\omega^2 + 1$, so by Theorem 25, we obtain for any $\varphi \in \text{Fm}_{\Box\Diamond}$,

$$\models_{\mathsf{S5}(\alpha+1)^{\mathsf{C}}} \varphi \iff \models_{\mathsf{S5}(\omega^2+1)^{\mathsf{C}}} \varphi.$$

Since undecidable countable ordinals $\alpha \geq \omega^2$ exist, there are decidable logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ (and corresponding one-variable fragments) for which A is undecidable. In contrast, none of the full first-order Gödel logics determined by these ordinals are recursively enumerable [35].

A sequel to this paper will provide a full classification of all logics $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ and show that any logic $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ is equivalent to $\mathsf{S5}(\mathbf{B})^{\mathsf{C}}$ where B is a countable Gödel set obtained as in Corollary 23. From this result it follows that $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ and $\mathsf{S5}(\mathbf{A})$ are co-NP-complete for any Gödel set A and hence, in light of Theorems 5 and 15, that the one-variable fragments $\mathsf{IKL}_1(\mathbf{K})$ and $\mathsf{CDIKL}_1(\mathbf{K})$ are co-NP-complete for any countable linear frame $\mathbf{K}$. These complexity results also apply to the one-variable fragment of any first-order Gödel logic determined by a Gödel set A, extending results in [36] for weaker fragments.

In particular, it will be shown that for any countable ordinal α, there is an ordinal $\beta \leq \omega^2$ such that $\mathsf{S5}(\alpha+1)^{\mathsf{C}}$ is equivalent to $\mathsf{S5}(\beta+1)^{\mathsf{C}}$, and the same is true for the reversed ordinals $(\alpha+1)^*$. By Theorems 5 and 15 and the observations above, this reduction applies also to one-variable fragments of intermediate logics over countable frames α or α^*, yielding one-variable versions of results in [37] and [38].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We would like to thank the referees for their careful reading of the submitted version of this paper and their constructive suggestions for improvements.

References

[1] X. Caicedo, G. Metcalfe, R. Rodríguez, O. Tuyt, The one-variable fragment of Corsi logic, in: Proceedings of WoLLIC 2019, in: LNCS, vol. 11541, Springer, 2019, pp. 70–83.
[2] H. Ono, N.-Y. Suzuki, Relations between intuitionistic modal logics and intermediate predicate logics, Rep. Math. Log. 22 (1988) 65–87.
[3] N.-Y. Suzuki, Kripke bundles for intermediate predicate logics and Kripke frames for intuitionistic modal logics, Stud. Log. 49 (3) (1990) 289–306.
[4] G. Bezhanishvili, Varieties of monadic Heyting algebras - part I, Stud. Log. 61 (3) (1998) 367–402.
[5] P. Hájek, On fuzzy modal logics S5(C), Fuzzy Sets Syst. 161 (18) (2010) 2389–2396.

[6] F. Wolter, M. Zakharyaschev, Intuitionistic modal logics as fragments of classical bimodal logics, in: E. Orlowska (Ed.), Logic at Work, Springer, 1999, pp. 168–186.
[7] D. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev, Many-Dimensional Modal Logics, Elsevier, 2003.
[8] R. Kontchakov, A. Kurucz, M. Zakharyaschev, Undecidability of first-order intuitionistic and modal logics with two variables, Bull. Symb. Log. 11 (3) (2005) 428–438.
[9] M. Rybakov, D. Shkatov, Undecidability of first-order modal and intuitionistic logics with two variables and one monadic predicate letter, Stud. Log. 107 (2019) 695–717.
[10] M. Takano, Ordered sets R and Q as bases of Kripke models, Stud. Log. 46 (1987) 137–148.
[11] M. Dummett, A propositional calculus with denumerable matrix, J. Symb. Log. 24 (1959) 97–106.
[12] X. Caicedo, R. Rodríguez, Bi-modal Gödel logic over [0, 1]-valued Kripke frames, J. Log. Comput. 25 (1) (2015) 37–55.
[13] A. Beckmann, N. Preining, Linear Kripke frames and Gödel logics, J. Symb. Log. 72 (2007) 26–44.
[14] X. Caicedo, R. Rodríguez, Standard Gödel modal logics, Stud. Log. 94 (2) (2010) 189–214.
[15] G. Metcalfe, N. Olivetti, Towards a proof theory of Gödel modal logics, Log. Methods Comput. Sci. 7 (2) (2011) 1–27.
[16] X. Caicedo, G. Metcalfe, R. Rodríguez, J. Rogger, Decidability in order-based modal logics, J. Comput. Syst. Sci. 88 (2017) 53–74.
[17] L. Blandi, L. Godo, R. Rodríguez, A connection between similarity logic programming and Gödel modal logic, in: Proc. EUSFLAT 2005, Universidad Polytecnica de Catalunya, 2005, pp. 775–7080.
[18] S. Schockaert, M.D. Cock, E. Kerre, Spatial reasoning in a fuzzy region connection calculus, Artif. Intell. 173 (2) (2009) 258–298.
[19] P. Dellunde, L. Godo, E. Marchioni, Extending possibilistic logic over Gödel logic, Int. J. Approx. Reason. 52 (1) (2011) 63–75.
[20] P. Hájek, Making fuzzy description logic more general, Fuzzy Sets Syst. 154 (1) (2005) 1–15.
[21] F. Bobillo, M. Delgado, J. Gómez-Romero, U. Straccia, Fuzzy description logics under Gödel semantics, Int. J. Approx. Reason. 50 (3) (2009) 494–514.
[22] S. Borgwardt, F. Distel, R. Peñaloza, The limits of decidability in fuzzy description logics with general concept inclusions, Artif. Intell. 218 (2015) 23–55.
[23] G. Corsi, Completeness theorem for Dummett's LC quantified, Stud. Log. 51 (1992) 317–335.
[24] S. Kripke, Semantical analysis of intuitionistic logic I, in: J. Crossley, M. Dummett (Eds.), Formal Systems and Recursive Functions, in: Studies in Logic and the Foundations of Mathematics, vol. 40, Elsevier, 1965, pp. 92–130.
[25] S. Görnemann, A logic stronger than intuitionism, J. Symb. Log. 36 (2) (1971) 249–261.
[26] R. Bull, MIPC as formalisation of an intuitionist concept of modality, J. Symb. Log. 31 (1966) 609–616.
[27] H. Ono, On some intuitionistic modal logics, Publ. RIMS, Kyoto Univ. 13 (1977) 687–722.
[28] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer, Dordrecht, 1998.
[29] A. Beckmann, M. Goldstern, N. Preining, Continuous Fraissé conjecture, Order 25 (2008) 281–298.
[30] A. Horn, Logic with truth values in a linearly ordered Heyting algebra, J. Symb. Log. 34 (3) (1969) 395–409.
[31] Y. Moschovakis, Descriptive Set Theory, North-Holland, 1980.
[32] N. Preining, Characterization of Gödel Logics, PhD Thesis, TU Wien, 2003.
[33] R. Iemhoff, A note on linear Kripke models, J. Log. Comput. 15 (4) (2005) 489–506.
[34] F. Bou, F. Esteva, L. Godo, R. Rodríguez, Possibilistic Semantics for a Modal KD45 Extension of Gödel Fuzzy Logic, Proceedings of IPMU 2016, vol. 21, Springer, 2016, pp. 123–135.
[35] M. Baaz, N. Preining, R. Zach, First-order Gödel logics, Ann. Pure Appl. Log. 147 (2007) 23–47.
[36] M. Baaz, A. Ciabattoni, N. Preining, First-order satisfiability in Gödel logics: an NP-complete fragment, Theor. Comput. Sci. 412 (47) (2011) 6612–6623.
[37] P. Minari, M. Takano, H. Ono, Intermediate predicate logics determined by ordinals, J. Symb. Log. 55 (3) (1990) 1099–1124.
[38] A. Beckmann, N. Preining, Separating intermediate predicate logics of well-founded and dually well-founded structures by monadic sentences, J. Log. Comput. 25 (3) (2014) 527–547.