GEOMETRY OF CERTAIN FOLIATIONS ON THE COMPLEX PROJECTIVE PLANE

by

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Abstract. — Let $d \geq 2$ be an integer. The set $\mathbf{F}(d)$ of foliations of degree d on the complex projective plane can be identified with a ZARISKI's open set of a projective space of dimension d^2+4d+2 on which $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ acts. We show that there are exactly two orbits $O(\mathcal{F}_1^d)$ and $O(\mathcal{F}_2^d)$ of minimal dimension 6, necessarily closed in $\mathbf{F}(d)$. This generalizes known results in degrees 2 and 3. We deduce that an orbit $O(\mathcal{F})$ of an element $\mathcal{F} \in \mathbf{F}(d)$ of dimension 7 is closed in $\mathbf{F}(d)$ if and only if $\mathcal{F}_i^d \notin \overline{O(\mathcal{F})}$ for $i=1,2$. This allows us to show that in any degree $d \geq 3$ there are closed orbits in $\mathbf{F}(d)$ other than the orbits $O(\mathcal{F}_1^d)$ and $O(\mathcal{F}_2^d)$, unlike the situation in degree 2. On the other hand, we introduce the notion of the basin of attraction $\mathbf{B}(\mathcal{F})$ of a foliation $\mathcal{F} \in \mathbf{F}(d)$ as the set of $\mathcal{G} \in \mathbf{F}(d)$ such that $\mathcal{F} \in \overline{O(\mathcal{G})}$. We show that the basin of attraction $\mathbf{B}(\mathcal{F}_1^d)$, resp. $\mathbf{B}(\mathcal{F}_2^d)$, contains a quasi-projective subvariety of $\mathbf{F}(d)$ of dimension greater than or equal to $\dim \mathbf{F}(d)-(d-1)$, resp. $\dim \mathbf{F}(d)-(d-3)$. In particular, we obtain that the basin $\mathbf{B}(\mathcal{F}_2^3)$ contains a non-empty ZARISKI open subset of $\mathbf{F}(3)$. This is an analog in degree 3 of a result on foliations of degree 2 due to CERVEAU, DÉSERTI, GARBA BELKO and MEZIANI.
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Introduction

The set $\mathbf{F}(d)$ of holomorphic foliations of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is identified with a ZARISKI open subset of the projective space $\mathbb{P}^{d^2+4d+2}_{\mathbb{C}}$. We are interested here in the action of the group $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) = \mathrm{PGL}_3(\mathbb{C})$ on $\mathbf{F}(d)$. We generalize to arbitrary degree some results known in small degrees [9, 1, 5] on this action.
For $\mathcal{F} \in \mathbf{F}(d)$, we will respectively denote by $O(\mathcal{F})$ and $\mathrm{Iso}(\mathcal{F})$ the orbit and the isotropy group of $\mathcal{F}$ under the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$, *i.e.*

$$O(\mathcal{F}) := \{\varphi^*\mathcal{F} \in \mathbf{F}(d) \mid \varphi \in \mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})\} \qquad \text{and} \qquad \mathrm{Iso}(\mathcal{F}) := \{\varphi \in \mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) \mid \varphi^*\mathcal{F} = \mathcal{F}\}.$$

$O(\mathcal{F})$ is a ZARISKI irreducible subset of $\mathbf{F}(d)$ and $\mathrm{Iso}(\mathcal{F})$ is an algebraic subgroup of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$.

Key words and phrases. — foliation, singularity, inflection point, orbit, isotropy group.

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Following [14] we will say that a foliation of $\mathbf{F}(d)$ is convex if its leaves other than straight lines have no inflection points. We will denote by $\mathbf{FC}(d)$ the subset of $\mathbf{F}(d)$ consisting of convex foliations, which is a ZARISKI closed subset of $\mathbf{F}(d)$.
According to [7, Proposition 2.2] every foliation of degree 0 or 1 is convex. For $d \geq 2$, $\mathbf{FC}(d)$ is a proper closed subset of $\mathbf{F}(d)$ and it contains the foliation $\mathcal{F}_1^d$ defined in the affine chart (x,y) by the 1-form (see [3, page 75])

$$\omega_1^d = y^d \mathrm{d}x + x^d(x\mathrm{d}y - y\mathrm{d}x).$$

We know from [9, Proposition 2.3] that if $\mathcal{F}$ is an element of $\mathbf{F}(d)$ with $d \geq 2$, then the dimension of $O(\mathcal{F})$ is at least 6, or equivalently, the dimension of $\mathrm{Iso}(\mathcal{F})$ is at most 2. In addition these bounds are attained by the convex foliation $\mathcal{F}_1^d$ and the non convex foliation $\mathcal{F}_2^d$ defined by the 1-form (see [3])

$$\omega_2^d = x^d \mathrm{d}x + y^d(x\mathrm{d}y - y\mathrm{d}x).$$

The main result of this paper is the following.

Theorem A. — *Let d be an integer greater than or equal to 2 and let $\mathcal{F}$ be an element of $\mathbf{F}(d)$. Assume that the isotropy group $\mathrm{Iso}(\mathcal{F})$ of $\mathcal{F}$ has dimension 2. Then $\mathcal{F}$ is linearly conjugated to one of the two foliations $\mathcal{F}_1^d$ and $\mathcal{F}_2^d$ defined respectively by the 1-forms*

1. $\omega_1^d = y^d \mathrm{d}x + x^d(x\mathrm{d}y - y\mathrm{d}x)$;
2. $\omega_2^d = x^d \mathrm{d}x + y^d(x\mathrm{d}y - y\mathrm{d}x)$.

In other words, $O(\mathcal{F}_1^d)$ and $O(\mathcal{F}_2^d)$ are the only orbits of dimension 6. They are closed in $\mathbf{F}(d)$. Moreover we have

$$\mathrm{Iso}(\mathcal{F}_1^d) = \left\{ \left(\frac{\alpha^{d-1}x}{1+\beta x}, \frac{\alpha^d y}{1+\beta x} \right) \,\middle|\, \alpha \in \mathbb{C}^*,\ \beta \in \mathbb{C} \right\},$$
$$\mathrm{Iso}(\mathcal{F}_2^d) = \left\{ \left(\frac{\alpha^{d+1}x}{1+\beta x}, \frac{\alpha^d y}{1+\beta x} \right) \,\middle|\, \alpha \in \mathbb{C}^*,\ \beta \in \mathbb{C} \right\};$$

these two groups are not conjugated.

This theorem is a generalization in arbitrary degree of previous results on foliations of degrees $d = 2$ ([9, Proposition 2.7]) and $d = 3$ ([1, Theorem 10], [5, Corollary B]).
We also obtain the following corollary, which generalizes [5, Corollary 3.9]:

Corollary B. — *Let d be an integer greater than or equal to 2 and let $\mathcal{F}$ be an element of $\mathbf{F}(d)$. If $\dim O(\mathcal{F}) \leq 7$, then*

$$\overline{O(\mathcal{F})} \subset O(\mathcal{F}) \cup O(\mathcal{F}_1^d) \cup O(\mathcal{F}_2^d).$$

In particular, when $\dim O(\mathcal{F}) = 7$, the orbit $O(\mathcal{F})$ of $\mathcal{F}$ is closed in $\mathbf{F}(d)$ if and only if $\mathcal{F}_i^d \notin \overline{O(\mathcal{F})}$ for $i = 1, 2$.

In the spirit of Corollary B we can ask under what condition the closure in $\mathbf{F}(d)$ of the orbit $O(\mathcal{F})$ of an element $\mathcal{F}$ of $\mathbf{F}(d)$ contains the foliations $\mathcal{F}_1^d$ and $\mathcal{F}_2^d$, a question that we have already asked and studied in degree 3 in [5, Section 3]. In Section §3, we extend (Propositions 3.4 and 3.11) in arbitrary degree d our previous results in [5, Propositions 3.10, 3.12, 3.15, 3.17] concerning this question.
For $\mathcal{F} \in \mathbf{F}(d)$, we call *basin of attraction* of $\mathcal{F}$ the subset $\mathbf{B}(\mathcal{F})$ of $\mathbf{F}(d)$ defined by

$$\mathbf{B}(\mathcal{F}) := \{\mathcal{G} \in \mathbf{F}(d) \mid \mathcal{F} \in \overline{O(\mathcal{G})}\}.$$

It follows from [9, Theorem 2.15] that in degree 2 the basin $\mathbf{B}(\mathcal{F}_1^2)$ contains a quasi-projective subvariety of $\mathbf{F}(2)$ of dimension greater than or equal to $\dim\mathbf{F}(2)-1$. In Section §3, we establish an analogous result in any degree greater than 2.

Theorem C (Theorem 3.10). — *For any integer $d\geq 2$, the basin of attraction $\mathbf{B}(\mathcal{F}_1^d)$ of $\mathcal{F}_1^d$ contains a quasi-projective subvariety of $\mathbf{F}(d)$ of dimension greater than or equal to $\dim\mathbf{F}(d)-(d-1)$.*

Notice that the non-convexity of $\mathcal{F}_2^d$ and the fact that $\mathbf{FC}(d)$ is closed in $\mathbf{F}(d)$ imply that

$$\mathbf{B}(\mathcal{F}_2^d)\subset\mathbf{F}(d)\setminus\mathbf{FC}(d). \tag{0.1}$$

In degree 2, according to [9, Theorem 3], inclusion (0.1) is an equality:

$$\mathbf{B}(\mathcal{F}_2^2)=\mathbf{F}(2)\setminus\mathbf{FC}(2). \tag{0.2}$$

It follows in particular from equality (0.2) that the basin $\mathbf{B}(\mathcal{F}_2^2)$ is a ZARISKI open subset of $\mathbf{F}(2)$. For $d\geq 3$ we show the following result.

Theorem D (Theorem 3.18). — *In any degree $d\geq 3$, the basin of attraction $\mathbf{B}(\mathcal{F}_2^d)$ of $\mathcal{F}_2^d$ contains a quasi-projective subvariety of $\mathbf{F}(d)$ of dimension greater than or equal to $\dim\mathbf{F}(d)-(d-3)$. In particular, the basin $\mathbf{B}(\mathcal{F}_2^3)$ contains a non-empty ZARISKI open subset of $\mathbf{F}(3)$.*

Along the same order of ideas, we prove the following result.

Theorem E (Theorem 3.21). — *For any integer $d\geq 2$, the intersection $\mathbf{B}(\mathcal{F}_1^d)\cap\mathbf{B}(\mathcal{F}_2^d)$ is non-empty and it contains a quasi-projective subvariety of $\mathbf{F}(d)$ of dimension equal to $\dim\mathbf{F}(d)-3d$.*

By combining equality (0.2) with the classification of C. FAVRE and J. V. PEREIRA of convex foliations of degree two (*cf.* [10, Proposition 7.4] or [6, Theorem A]), we see that the only closed orbits in $\mathbf{F}(2)$ under the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ are those of $\mathcal{F}_1^2$ and $\mathcal{F}_2^2$. We show in Section §4 that in any degree $d\geq 3$ there are closed orbits in $\mathbf{F}(d)$ other than the orbits $\mathcal{O}(\mathcal{F}_1^d)$ and $\mathcal{O}(\mathcal{F}_2^d)$, unlike the situation in degree 2. More precisely, we will consider a family of elements of $\mathbf{F}(d)$ which has been already studied in degree $d=2$ in [9, page 189], namely the family $(\mathcal{F}_0^d(\lambda))_{\lambda\in\mathbb{C}^*}$ of foliations of degree d on $\mathbb{P}^2_{\mathbb{C}}$ defined by the 1-form

$$\omega_0^d(\lambda)=x\mathrm{d}y-\lambda y\mathrm{d}x+y^d\mathrm{d}y.$$

We will see that, for $\lambda=1$, $\mathcal{F}_0^d(1)$ is linearly conjugated to the foliation $\mathcal{F}_1^d$ and that, for any $\lambda\neq 1$, $\dim\mathcal{O}\big(\mathcal{F}_0^d(\lambda)\big)=7$. Moreover, we will show (Proposition 4.2) that the orbit $\mathcal{O}\big(\mathcal{F}_0^d(\lambda)\big)$ is closed for any $d\geq 3$ and $\lambda=-\frac{1}{d-1}$, resp. for any $d\in\{3,4,5\}$ and any $\lambda\in\mathbb{C}^*$, and we conjecture that it is so for any $d\geq 6$ and any $\lambda\in\mathbb{C}^*$ (see Conjectures 1 and 2).

1. Some definitions and notations

1.1. Singularities and local invariants. — A degree d holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ is defined in homogeneous coordinates $[x:y:z]$ by a 1-form

$$\omega=a(x,y,z)\mathrm{d}x+b(x,y,z)\mathrm{d}y+c(x,y,z)\mathrm{d}z,$$

where a, b and c are homogeneous polynomials of degree $d+1$ without common factor and satisfying the EULER condition $i_{\mathrm{R}}\omega=0$, where $\mathrm{R}=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$ denotes the radial vector field and i_{R} is the interior product by R.

Dually the foliation $\mathcal{F}$ can also be defined by a homogeneous vector field
$$\mathrm{Z}=U(x,y,z)\frac{\partial}{\partial x}+V(x,y,z)\frac{\partial}{\partial y}+W(x,y,z)\frac{\partial}{\partial z},$$
the coefficients U,V and W are homogeneous polynomials of degree d without common factor. The relation between Z and ω is given by
$$\omega=i_{\mathrm{R}}i_{\mathrm{Z}}(\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z).$$
The *singular locus* $\mathrm{Sing}\mathcal{F}$ of $\mathcal{F}$ is the projectivization of the singular locus of ω
$$\mathrm{Sing}\,\omega=\{(x,y,z)\in\mathbb{C}^3\,|\,a(x,y,z)=b(x,y,z)=c(x,y,z)=0\}.$$
Let $\mathcal{C}\subset\mathbb{P}^2_{\mathbb{C}}$ be an algebraic curve with homogeneous equation $F(x,y,z)=0$. We say that $\mathcal{C}$ is an *invariant curve* by $\mathcal{F}$ if $\mathcal{C}\smallsetminus\mathrm{Sing}\mathcal{F}$ is a union of (ordinary) leaves of the regular foliation $\mathcal{F}|_{\mathbb{P}^2_{\mathbb{C}}\smallsetminus\mathrm{Sing}\mathcal{F}}$. In algebraic terms, this is equivalent to require that the 2-form $\omega\wedge\mathrm{d}F$ is divisible by F, *i.e.* it vanishes along each irreducible component of $\mathcal{C}$.

Let p be an arbitrary point of $\mathcal{C}$. When each irreducible component of $\mathcal{C}$ passing through p is not $\mathcal{F}$-invariant, we define the *tangency order* $\mathrm{Tang}(\mathcal{F},\mathcal{C},p)$ of $\mathcal{F}$ with $\mathcal{C}$ at p as follows. We fix a local chart (u,v) such that $p=(0,0)$; let $f(\mathrm{u},\mathrm{v})=0$ be a reduced local equation of $\mathcal{C}$ in a neighborhood of p and let X be a vector field defining the germ of $\mathcal{F}$ at p. We denote by $\mathrm{X}(f)$ the LIE derivative of f along X and by $\langle f,\mathrm{X}(f)\rangle$ the ideal of $\mathbb{C}\{\mathrm{u},\mathrm{v}\}$ generated by f and $\mathrm{X}(f)$. Then
$$\mathrm{Tang}(\mathcal{F},\mathcal{C},p)=\dim_{\mathbb{C}}\frac{\mathbb{C}\{\mathrm{u},\mathrm{v}\}}{\langle f,\mathrm{X}(f)\rangle}.$$
Notice that $\mathrm{Tang}(\mathcal{F},\mathcal{C},p)$ coincides with the intersection multiplicity $(\mathcal{C}.\mathcal{C}')_p$ at p of the two algebraic curves $\mathcal{C}=\{F=0\}$ and $\mathcal{C}'=\{\mathrm{Z}(F)=0\}$. Moreover, $\mathrm{Tang}(\mathcal{F},\mathcal{C},p)<+\infty$ by the non-invariance of the irreducible components of $\mathcal{C}$ passing through p. By convention, we put $\mathrm{Tang}(\mathcal{F},\mathcal{C},p)=+\infty$ if there is at least one irreducible component of $\mathcal{C}$ invariant by $\mathcal{F}$ and passing through p.

Let us recall some local notions attached to the pair $(\mathcal{F},s)$, where $s\in\mathrm{Sing}\mathcal{F}$. The germ of $\mathcal{F}$ at s is defined, up to multiplication by a unity in the local ring $\mathcal{O}_s$ at s, by a vector field $\mathrm{X}=A(\mathrm{u},\mathrm{v})\frac{\partial}{\partial\mathrm{u}}+B(\mathrm{u},\mathrm{v})\frac{\partial}{\partial\mathrm{v}}$. The *algebraic multiplicity* $\nu(\mathcal{F},s)$ of $\mathcal{F}$ at s is given by
$$\nu(\mathcal{F},s)=\min\{\nu(A,s),\nu(B,s)\},$$
where $\nu(g,s)$ denotes the algebraic multiplicity of the function g at s. Let us denote by $\mathfrak{L}_s(\mathcal{F})$ the family of straight lines through s which are not invariant by $\mathcal{F}$. For any line ℓ of $\mathfrak{L}_s(\mathcal{F})$, we have the inequalities $1\leq\mathrm{Tang}(\mathcal{F},\ell,s)\leq d$. This allows us to associate to the pair $(\mathcal{F},s)$ the following (invariant) integers
$$\tau(\mathcal{F},s)=\min\{\mathrm{Tang}(\mathcal{F},\ell,s)\mid\ell\in\mathfrak{L}_s(\mathcal{F})\},\qquad\kappa(\mathcal{F},s)=\max\{\mathrm{Tang}(\mathcal{F},\ell,s)\mid\ell\in\mathfrak{L}_s(\mathcal{F})\}.$$
The invariant $\tau(\mathcal{F},s)$ represents the tangency order of $\mathcal{F}$ with a generic line passing through s. It is easy to see that
$$\tau(\mathcal{F},s)=\min\{k\geq 1\mid\det(J^k_s\mathrm{X},\mathrm{R}_s)\not\equiv 0\}\geq\nu(\mathcal{F},s),$$
where $J^k_s\mathrm{X}$ denotes the k-jet of X at s and R_s is the radial vector field centered at s. The MILNOR *number* of $\mathcal{F}$ at s is the integer
$$\mu(\mathcal{F},s)=\dim_{\mathbb{C}}\mathcal{O}_s/\langle A,B\rangle,$$
where $\langle A,B\rangle$ denotes the ideal of $\mathcal{O}_s$ generated by A and B.

The singularity s is called *radial of order* $n-1$, with $n\in\{2,\ldots,d\}$, if $\nu(\mathcal{F},s)=1$ and $\tau(\mathcal{F},s)=n$.

The singularity s is called *non-degenerate* if $\mu(\mathcal{F},s)=1$, or equivalently if the Jacobian matrix of X at s, denoted by $\mathrm{JacX}(s)$, possesses two nonzero eigenvalues λ,μ. In this case, the quantity

$$\mathrm{BB}(\mathcal{F},s)=\frac{\mathrm{tr}^2(\mathrm{JacX}(s))}{\det(\mathrm{JacX}(s))}=\frac{\lambda}{\mu}+\frac{\mu}{\lambda}+2$$

is called the BAUM-BOTT *index* of $\mathcal{F}$ at s, see [2].

We will say that the singularity s is *quasi-radial of order* $n-1$ if $\mu(\mathcal{F},s)=1$, $\mathrm{BB}(\mathcal{F},s)=4$ and $\kappa(\mathcal{F},s)=n$. In the sequel we will denote by $\mathrm{QRad}(\mathcal{F},n-1)$ the set of quasi-radial singularities of $\mathcal{F}$ of order $n-1$ and by $\widehat{\mathrm{QRad}}(\mathcal{F},n-1)$ the subset of $\mathrm{Sing}(\mathcal{F})\times\mathfrak{L}_s(\mathcal{F})$ defined by

$$\widehat{\mathrm{QRad}}(\mathcal{F},n-1):=\Big\{(s,\ell)\in\mathrm{Sing}(\mathcal{F})\times\mathfrak{L}_s(\mathcal{F})\mid\mu(\mathcal{F},s)=1,\ \mathrm{BB}(\mathcal{F},s)=4,\ \mathrm{Tang}(\mathcal{F},\ell,s)=n\Big\}.$$

Remark 1.1. — Every radial singularity s of order $n-1$ of a foliation $\mathcal{F}$ of degree $d\geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$ is quasi-radial of order $\geq n-1$, because $\kappa(\mathcal{F},s)\geq\tau(\mathcal{F},s)$. The converse is false: for instance, for the foliation defined in the affine chart $z=1$ by the 1-form $(x+y)\mathrm{d}y-y\mathrm{d}x+(x^n+y^d)\mathrm{d}x$, with $n\in\{2,3,\ldots,d\}$, the point $[0:0:1]$ is a quasi-radial singularity of order $n-1$, but it is not radial.

1.2. Inflection points. — Let us consider a foliation $\mathcal{F}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ and let p be a regular point of $\mathcal{F}$. Let us denote by $\mathrm{T}^{\mathbb{P}}_p\mathcal{F}$ the tangent line to the leaf of $\mathcal{F}$ passing through p; it is the straight line of $\mathbb{P}^2_{\mathbb{C}}$ passing through p with direction $\mathrm{T}_p\mathcal{F}$. If $k\in\{2,\ldots,d\}$, we will say that p is a *(transverse) inflection point of order* $k-1$ of $\mathcal{F}$ if $\mathrm{Tang}(\mathcal{F},\mathrm{T}^{\mathbb{P}}_p\mathcal{F},p)=k$, in which case the line $\mathrm{T}^{\mathbb{P}}_p\mathcal{F}$ is not invariant by $\mathcal{F}$. When $\mathrm{T}^{\mathbb{P}}_p\mathcal{F}$ is $\mathcal{F}$-invariant, the point p will be called a *trivial inflection point* of $\mathcal{F}$. If we denote by $\mathrm{Inv}(\mathcal{F})$ the set of invariant lines of $\mathcal{F}$, then the set of trivial inflection points of $\mathcal{F}$ is precisely $\mathrm{Inv}(\mathcal{F})\setminus\mathrm{Sing}(\mathcal{F})$. In the sequel, we will denote by $\mathrm{Flex}(\mathcal{F})$ the set of inflection points of $\mathcal{F}$ and by $\mathrm{Flex}(\mathcal{F},k-1)$ the subset of $\mathrm{Flex}(\mathcal{F})$ consisting of transverse inflection points of $\mathcal{F}$ of order $k-1$, *i.e.*

$$\mathrm{Flex}(\mathcal{F},k-1):=\Big\{p\in\mathrm{Flex}(\mathcal{F})\mid p\notin\mathrm{Sing}(\mathcal{F}),\ \mathrm{Tang}(\mathcal{F},\mathrm{T}^{\mathbb{P}}_p\mathcal{F},p)=k\Big\}.$$

Let us recall the notion of inflection divisor of $\mathcal{F}$, introduced by PEREIRA [16], which allows to determine the set $\mathrm{Flex}(\mathcal{F})$. Let Z be a homogeneous vector field of degree d on $\mathbb{C}^3$ defining $\mathcal{F}$. The *inflection divisor* of $\mathcal{F}$, denoted by $\mathrm{I}_{\mathcal{F}}$, is the divisor of $\mathbb{P}^2_{\mathbb{C}}$ defined by the homogeneous equation

$$\begin{vmatrix} x & \mathrm{Z}(x) & \mathrm{Z}^2(x)\\ y & \mathrm{Z}(y) & \mathrm{Z}^2(y)\\ z & \mathrm{Z}(z) & \mathrm{Z}^2(z)\end{vmatrix}=0. \tag{1.1}$$

According to [16], $\mathrm{I}_{\mathcal{F}}$ satisfies the following properties:

1. The support of $\mathrm{I}_{\mathcal{F}}$ is exactly the closure of the set $\mathrm{Flex}(\mathcal{F})$ of inflection points of $\mathcal{F}$. More precisely, $\mathrm{I}_{\mathcal{F}}$ can be decomposed as $\mathrm{I}_{\mathcal{F}}=\mathrm{I}^{\mathrm{inv}}_{\mathcal{F}}+\mathrm{I}^{\mathrm{tr}}_{\mathcal{F}}$, where the support of $\mathrm{I}^{\mathrm{inv}}_{\mathcal{F}}$ is the set $\mathrm{Inv}(\mathcal{F})$ of $\mathcal{F}$-invariant lines and the support of $\mathrm{I}^{\mathrm{tr}}_{\mathcal{F}}$ is the closure of the set of transverse inflection points of $\mathcal{F}$.
2. If $\mathcal{C}$ is an algebraic curve invariant by $\mathcal{F}$, then $\mathcal{C}\subset\mathrm{I}_{\mathcal{F}}$ if and only if $\mathcal{C}\subset\mathrm{Inv}(\mathcal{F})$.
3. The degree of the divisor $\mathrm{I}_{\mathcal{F}}$ is $3d$.

The foliation $\mathcal{F}$ will be called *convex* if its inflection divisor $\mathrm{I}_{\mathcal{F}}$ is totally invariant by $\mathcal{F}$, *i.e.* if $\mathrm{I}_{\mathcal{F}}$ is a product of invariant lines.

2. Description of the foliations $\mathcal{F}$ of degree greater than or equal to 2 such that $\dim O(\mathcal{F})=6$

Recall that the foliations $\mathcal{F}_1^d$ and $\mathcal{F}_2^d$ are respectively defined in the affine chart $z=1$ by the 1-forms

$$\omega_1^d = y^d \mathrm{d}x + x^d(x\mathrm{d}y - y\mathrm{d}x) \qquad \text{and} \qquad \omega_2^d = x^d\mathrm{d}x + y^d(x\mathrm{d}y - y\mathrm{d}x).$$

The foliation $\mathcal{F}_1^d$ is convex with inflection divisor $\mathrm{I}_{\mathcal{F}_1^d} = \mathrm{I}_{\mathcal{F}_1^d}^{\mathrm{inv}} = x^{d+1}y^{2d-1}$ and it has two singular points $s_1=[0:0:1]$ and $s_2=[0:1:0]$; the singularity s_1 has maximal algebraic multiplicity d and s_2 is radial of maximal order $d-1$. The foliation $\mathcal{F}_2^d$ is not convex with invariant inflection divisor $\mathrm{I}_{\mathcal{F}_2^d}^{\mathrm{inv}} = x^{2d+1}$ and transverse inflection divisor $\mathrm{I}_{\mathcal{F}_2^d}^{\mathrm{tr}} = y^{d-1}$. The singular locus $\mathrm{Sing}(\mathcal{F}_2^d)$ is reduced to the point $s_1=[0:0:1]$; moreover $\nu(\mathcal{F}_2^d, s_1) = d$. We note that the 1-forms $\dfrac{\omega_1^d}{x^2y^d}$ and $\dfrac{\omega_2^d}{x^{d+2}}$ are closed and they respectively admit as first integrals

$$\frac{1}{d-1}\left(\frac{x}{y}\right)^{d-1} + \frac{1}{x} \qquad \text{and} \qquad \frac{1}{d+1}\left(\frac{y}{x}\right)^{d+1} - \frac{1}{x};$$

this allows to check that

$$\mathrm{Iso}(\mathcal{F}_1^d) = \left\{\left(\frac{\alpha^{d-1}x}{1+\beta x}, \frac{\alpha^d y}{1+\beta x}\right) \,\Big|\, \alpha\in\mathbb{C}^*,\ \beta\in\mathbb{C}\right\} \quad \text{and} \quad \mathrm{Iso}(\mathcal{F}_2^d) = \left\{\left(\frac{\alpha^{d+1}x}{1+\beta x}, \frac{\alpha^d y}{1+\beta x}\right) \,\Big|\, \alpha\in\mathbb{C}^*,\ \beta\in\mathbb{C}\right\}.$$

In particular, $\dim \mathrm{Iso}(\mathcal{F}_i^d)=2$ for $i=1,2$. Thus the orbits $O(\mathcal{F}_1^d)$ and $O(\mathcal{F}_2^d)$ are both of dimension 6, which is the minimal dimension possible in any degree $d\geq 2$ ([9, Proposition 2.3]). Theorem A announced in the Introduction shows that the orbits $O(\mathcal{F}_1^d)$ and $O(\mathcal{F}_2^d)$ are the only orbits having minimal dimension 6. The goal of this section is to prove this theorem.

Let us denote by $\chi(\mathbb{P}^2_{\mathbb{C}})$ the LIE algebra of holomorphic vector fields on $\mathbb{P}^2_{\mathbb{C}}$: $\chi(\mathbb{P}^2_{\mathbb{C}})$ is of course the LIE algebra of the automorphism group of $\mathbb{P}^2_{\mathbb{C}}$. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$ and let X be an element of $\chi(\mathbb{P}^2_{\mathbb{C}})$. Following [9] we will say that X is a *symmetry* of the foliation $\mathcal{F}$ if the flow $\exp(t\mathrm{X})$ is, for each t, in the isotropy group $\mathrm{Iso}(\mathcal{F})$ of $\mathcal{F}$. If ω defines $\mathcal{F}$ in an affine chart, X is a symmetry of $\mathcal{F}$ if and only if $\mathrm{L_X}\omega\wedge\omega=0$, where $\mathrm{L_X}\omega$ denotes the LIE derivative of ω along X.

Lemma 2.1. — *Let $\mathcal{F}$ be a foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ and let X be a symmetry of $\mathcal{F}$. Assume that there is an affine chart $\mathbb{C}^2\subset\mathbb{P}^2_{\mathbb{C}}$ such that the vector field X is affine (i.e. $\deg X\leq 1$) and let ω be a 1-form defining $\mathcal{F}$ in this chart. Then there is a constant $\lambda\in\mathbb{C}$ such that $\mathrm{L_X}\omega=\lambda\omega$.*

Proof. — We will use an argument similar to one in [9, Proposition 2.5]. Since $\mathrm{L_X}\omega\wedge\omega=0$ and ω has isolated singularities, the DE RHAM-SAITO division theorem (*cf.* [17] or [8, Proposition 1.14]) ensures the existence of a holomorphic function g on $\mathbb{C}^2$ such that $\mathrm{L_X}\omega=g\omega$. The 1-form ω and the vector field X being polynomials, $\mathrm{L_X}\omega$ is also polynomial; therefore g is rational and holomorphic on $\mathbb{C}^2$ hence polynomial. The vector field X being affine we have $\deg \mathrm{L_X}\omega\leq\deg\omega$; the equality $\mathrm{L_X}\omega=g\omega$ implies that g is constant. □

If $\mathcal{F}$ is a foliation on $\mathbb{P}^2_{\mathbb{C}}$, we will denote by $\mathfrak{iso}(\mathcal{F})$ the LIE algebra of the algebraic group $\mathrm{Iso}(\mathcal{F})$; $\mathfrak{iso}(\mathcal{F})$ is a LIE subalgebra of $\chi(\mathbb{P}^2_{\mathbb{C}})$ and it consists of symmetries of $\mathcal{F}$. We know from [9, Proposition 2.5] that if $\dim\mathfrak{iso}(\mathcal{F})=2$ then $\mathfrak{iso}(\mathcal{F})$ is *affine*, *i.e.* generated by two vector fields X and Y such that $[\mathrm{X},\mathrm{Y}]=\mathrm{Y}$. The following lemma classifies the affine LIE subalgebras of $\chi(\mathbb{P}^2_{\mathbb{C}})$ and it will be used to prove Theorem A.

Lemma 2.2. — *Every affine LIE subalgebra of $\chi(\mathbb{P}^2_{\mathbb{C}})$ is linearly conjugated to one of the following models*

(a) $\langle \gamma x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\rangle$ *with* $\gamma\in\mathbb{C}^*$;

(b) $\left\langle y\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle$;
(c) $\left\langle \frac{\partial}{\partial x}+y\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle$;
(d) $\left\langle x\frac{\partial}{\partial x}+(x+y)\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle$;
(e) $\left\langle x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}, \frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right\rangle$.

Proof. — Let $\mathfrak{g}$ be an affine LIE subalgebra of $\chi(\mathbb{P}^2_{\mathbb{C}})$. Then there exist X and Y in $\chi(\mathbb{P}^2_{\mathbb{C}})$ such that $\mathfrak{g}=\langle \mathrm{X},\mathrm{Y}\rangle$ and $[\mathrm{X},\mathrm{Y}]=\mathrm{Y}$. Fixing homogeneous coordinates $[x:y:z]$ in $\mathbb{P}^2_{\mathbb{C}}$ we have an isomorphism of LIE algebras $\tau:\mathfrak{sl}_3(\mathbb{C})\to\chi(\mathbb{P}^2_{\mathbb{C}})$ defined, for $A\in\mathfrak{sl}_3(\mathbb{C})$, by

$$\tau(A)=(x\ \ y\ \ z)A\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}.$$

Notice that if $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\ a_{21}&a_{22}&a_{23}\\ a_{31}&a_{32}&a_{33}\end{pmatrix}\in\mathfrak{sl}_3(\mathbb{C})$, then in the affine chart $z=1$ the vector field $\tau(A)\in\chi(\mathbb{P}^2_{\mathbb{C}})$ writes as

$$\big(a_{31}+(a_{11}-a_{33})x+a_{21}y-a_{13}x^2-a_{23}xy\big)\frac{\partial}{\partial x}+\big(a_{32}+a_{12}x+(a_{22}-a_{33})y-a_{13}xy-a_{23}y^2\big)\frac{\partial}{\partial y}.$$

Let M and N be the matrices of $\mathfrak{sl}_3(\mathbb{C})$ associated to the vector fields X and Y respectively, *i.e.* $M=\tau^{-1}(\mathrm{X})$ and $N=\tau^{-1}(\mathrm{Y})$. Then the LIE bracket $[\mathrm{X},\mathrm{Y}]$ corresponds to $[M,N]:=MN-NM$ and therefore $[M,N]=N$. Let us write $M=\begin{pmatrix}-m_{22}-m_{33}&m_{12}&m_{13}\\ m_{21}&m_{22}&m_{23}\\ m_{31}&m_{32}&m_{33}\end{pmatrix}$. Taking into account the possible JORDAN forms of a matrix of $\mathfrak{sl}_3(\mathbb{C})$, it suffices us to treat the following possibilities

$$N=\begin{pmatrix}-a-b&0&0\\ 0&a&0\\ 0&0&b\end{pmatrix},\quad N=\begin{pmatrix}-2c&0&0\\ 0&c&0\\ 0&1&c\end{pmatrix},\quad N=\begin{pmatrix}0&0&0\\ 0&0&0\\ 0&1&0\end{pmatrix},\quad N=\begin{pmatrix}0&0&0\\ 1&0&0\\ 0&1&0\end{pmatrix}.$$

where $a,b\in\mathbb{C},c\in\mathbb{C}^*$, with $(a,b)\neq(0,0)$.

1. If $N=\begin{pmatrix}-a-b&0&0\\ 0&a&0\\ 0&0&b\end{pmatrix}$ then the equality $[M,N]=N$ implies that $a=b=0$: contradiction.

2. If $N=\begin{pmatrix}-2c&0&0\\ 0&c&0\\ 0&1&c\end{pmatrix}$ then the $(1,1)$ coefficient of the matrix $[M,N]-N$ is equal to $2c$ and is therefore nonzero: contradiction.

3. Assume that $N=\begin{pmatrix}0&0&0\\ 0&0&0\\ 0&1&0\end{pmatrix}$; the equality $[M,N]=N$ then leads to $M=\begin{pmatrix}1-2m_{33}&m_{12}&0\\ 0&m_{33}-1&0\\ m_{31}&m_{32}&m_{33}\end{pmatrix}$. Up to replacing M by $M-m_{32}N$ we can assume that $m_{32}=0$. Now we will distinguish several eventualities:

3.1. When $(3m_{33}-1)(3m_{33}-2)\neq 0$ the matrix $P=\begin{pmatrix}3m_{33}-1 & m_{12} & 0\\ 0 & 3m_{33}-2 & 0\\ -m_{31} & -m_{31}m_{12} & 3m_{33}-2\end{pmatrix}$ commutes with N and $P^{-1}MP=\begin{pmatrix}1-2m_{33} & 0 & 0\\ 0 & m_{33}-1 & 0\\ 0 & 0 & m_{33}\end{pmatrix}$. Thus $\mathfrak{g}$ is linearly conjugated to

$$\big\langle \tau(P^{-1}MP),\tau(N)\big\rangle=\big\langle (1-3m_{33})x\tfrac{\partial}{\partial x}-y\tfrac{\partial}{\partial y},\tfrac{\partial}{\partial y}\big\rangle=\big\langle \gamma x\tfrac{\partial}{\partial x}+y\tfrac{\partial}{\partial y},\tfrac{\partial}{\partial y}\big\rangle,\qquad \text{where } \gamma=3m_{33}-1\in\mathbb{C}^*.$$

3.2. Assume that $m_{33}=\frac{1}{3}$. If $\delta\in\mathbb{C}^*$ then the matrix $P=\begin{pmatrix}\frac{1}{\delta} & -m_{12} & 0\\ 0 & 1 & 0\\ 0 & m_{12}m_{31} & 1\end{pmatrix}$ commutes with N and $P^{-1}MP=\begin{pmatrix}\frac{1}{3} & 0 & 0\\ 0 & -\frac{2}{3} & 0\\ \frac{m_{31}}{\delta} & 0 & \frac{1}{3}\end{pmatrix}$. As a result $\mathfrak{g}$ is linearly conjugated to

$$\big\langle \tau(P^{-1}MP),\tau(N)\big\rangle=\big\langle \tfrac{m_{31}}{\delta}\tfrac{\partial}{\partial x}-y\tfrac{\partial}{\partial y},\tfrac{\partial}{\partial y}\big\rangle=\big\langle -\tfrac{m_{31}}{\delta}\tfrac{\partial}{\partial x}+y\tfrac{\partial}{\partial y},\tfrac{\partial}{\partial y}\big\rangle.$$

The case where $m_{31}=0$ leads to the model $(\mathfrak{b})$. If $m_{31}\neq 0$ then by taking $\delta=-m_{31}$ we get the model $(\mathfrak{c})$.

3.3. Assume that $m_{33}=\frac{2}{3}$. If $\delta\in\mathbb{C}^*$ then the matrix $P=\begin{pmatrix}\delta & 0 & 0\\ 0 & 1 & 0\\ -\delta m_{31} & -m_{12}m_{31} & 1\end{pmatrix}$ commutes with N and $P^{-1}MP=\begin{pmatrix}-\frac{1}{3} & \frac{m_{12}}{\delta} & 0\\ 0 & -\frac{1}{3} & 0\\ 0 & 0 & \frac{2}{3}\end{pmatrix}$. As a consequence $\mathfrak{g}$ is linearly conjugated to

$$\big\langle \tau(P^{-1}MP),\tau(N)\big\rangle=\big\langle -x\tfrac{\partial}{\partial x}+(\tfrac{m_{12}}{\delta}x-y)\tfrac{\partial}{\partial y},\tfrac{\partial}{\partial y}\big\rangle=\big\langle x\tfrac{\partial}{\partial x}+(y-\tfrac{m_{12}}{\delta}x)\tfrac{\partial}{\partial y},\tfrac{\partial}{\partial y}\big\rangle.$$

The case $m_{12}=0$ leads to the model $(\mathfrak{a})$ with $\gamma=1$. If $m_{12}\neq 0$ then by taking $\delta=-m_{12}$ we obtain the model $(\mathfrak{d})$.

4. Assume that $N=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$; then the equality $[M,N]=N$ implies that $M=\begin{pmatrix}-1&0&0\\m_{32}&0&0\\m_{31}&m_{32}&1\end{pmatrix}$. Up to replacing M by $M-m_{32}N$ we can assume that $m_{32}=0$. The matrix $P=\begin{pmatrix}1&0&0\\0&1&0\\-\frac{m_{31}}{2}&0&1\end{pmatrix}$ commutes with N and $P^{-1}MP=\begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix}$. Therefore $\mathfrak{g}$ is linearly conjugated to

$$\big\langle \tau(P^{-1}MP),\tau(N)\big\rangle=\big\langle -2x\tfrac{\partial}{\partial x}-y\tfrac{\partial}{\partial y},y\tfrac{\partial}{\partial x}+\tfrac{\partial}{\partial y}\big\rangle=\big\langle 2x\tfrac{\partial}{\partial x}+y\tfrac{\partial}{\partial y},y\tfrac{\partial}{\partial x}+\tfrac{\partial}{\partial y}\big\rangle.$$

By permuting the coordinates x and y we obtain the model $(\mathfrak{e})$. □

Proof of Theorem A. — Since $\dim\mathfrak{iso}(\mathcal{F})=\dim\mathrm{Iso}(\mathcal{F})=2$, [9, Proposition 2.5] implies that $\mathfrak{iso}(\mathcal{F})$ is affine. Therefore, up to linear conjugation, $\mathfrak{iso}(\mathcal{F})$ is one of the models $(\mathfrak{a})$–$(\mathfrak{e})$ of Lemma 2.2.
Let ω be a 1-form defining $\mathcal{F}$ in the affine chart $z=1$

$$\omega=A(x,y)\mathrm{d}x+B(x,y)\mathrm{d}y,\quad A,B\in\mathbb{C}[x,y],\ \ \gcd(A,B)=1.$$

We will study the five possible models $(\mathfrak{a})$–$(\mathfrak{e})$ of the LIE algebra $\mathfrak{iso}(\mathcal{F})$ and show that ω is linearly conjugated to one of the two 1-forms ω_1^d or ω_2^d.

1. Assume that $\mathfrak{iso}(\mathcal{F})$ is of one of the types $(\mathfrak{a})$–$(\mathfrak{d})$, *i.e.* that $\mathfrak{iso}(\mathcal{F})=\langle \mathrm{X},\mathrm{Y}\rangle$ where $\mathrm{X}\in\{\gamma x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\varepsilon\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},x\frac{\partial}{\partial x}+(x+y)\frac{\partial}{\partial y}\},\mathrm{Y}=\frac{\partial}{\partial y}$ with $\varepsilon\in\{0,1\}$ and $\gamma\in\mathbb{C}^*$. By Lemma 2.1 there exist $\lambda,\mu\in\mathbb{C}$ such that $\mathrm{L_X}\omega=\lambda\omega$ and $\mathrm{L_Y}\omega=\mu\omega$. Since $\mathrm{L_Y}\mathrm{d}x=\mathrm{dL_Y}x=0$ and $\mathrm{L_Y}\mathrm{d}y=\mathrm{dL_Y}y=0$, we have $\mathrm{L_Y}\omega=\mathrm{Y}(A)\mathrm{d}x+\mathrm{Y}(B)\mathrm{d}y=\frac{\partial A}{\partial y}\mathrm{d}x+\frac{\partial B}{\partial y}\mathrm{d}y$. Therefore $\mathrm{L_Y}\omega=\mu\omega$ if and only if $\frac{\partial A}{\partial y}=\mu A$ and $\frac{\partial B}{\partial y}=\mu B$. Since $A,B\in\mathbb{C}[x,y]$ and $\mu\in\mathbb{C}$, it follows that $\mu=0$, $A(x,y)=A(x)$ and $B(x,y)=B(x)$. Thus

$$\omega=A(x)\mathrm{d}x+B(x)\mathrm{d}y,\quad A,B\in\mathbb{C}[x],\ \ \gcd(A,B)=1.$$

1.1. Let us consider the case where $\mathrm{X}=\gamma x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ with $\gamma\in\mathbb{C}^*$. We have

$$\mathrm{L_X}\omega=\mathrm{X}(A)\mathrm{d}x+A\mathrm{dX}(x)+\mathrm{X}(B)\mathrm{d}y+B\mathrm{dX}(y)=\big(\gamma xA'+\gamma A\big)\mathrm{d}x+\big(\gamma xB'+B\big)\mathrm{d}y,$$

so that $\mathrm{L_X}\omega=\lambda\omega$ if and only if $\gamma xA'=(\lambda-\gamma)A$ and $\gamma xB'=(\lambda-1)B$. By putting $\kappa=\frac{\lambda-\gamma}{\gamma}$ and $\nu=\frac{\lambda-1}{\gamma}$, the last two equations can be rewritten as $xA'=\kappa A$ and $xB'=\nu B$ and can be immediately integrated to give $A(x)=\alpha x^{\kappa}$ and $B(x)=\beta x^{\nu}$, where $\alpha,\beta\in\mathbb{C}$. Since $A,B\in\mathbb{C}[x]$ and $\gcd(A,B)=1$, we deduce that $\alpha,\beta\in\mathbb{C}^*$, $\kappa,\nu\in\mathbb{N}$ and $\kappa\nu=0$. The equality $\deg\mathcal{F}=d$ then implies that

- either $\kappa=0$ and $\nu=d$, in which case $\omega=\alpha\mathrm{d}x+\beta x^d\mathrm{d}y$;
- or $\nu=0$ and $\kappa=d$, in which case $\omega=\alpha x^d\mathrm{d}x+\beta\mathrm{d}y$.

If $\omega=\alpha\mathrm{d}x+\beta x^d\mathrm{d}y$, resp. $\omega=\alpha x^d\mathrm{d}x+\beta\mathrm{d}y$, by making the change of coordinates $(x,y)\mapsto\left(\frac{y}{x},-\frac{\alpha}{\beta x}\right)$, we reduce ourselves to $\omega=\omega_1^d=y^d\mathrm{d}x+x^d(x\mathrm{d}y-y\mathrm{d}x)$, resp. $\omega=\omega_2^d=x^d\mathrm{d}x+y^d(x\mathrm{d}y-y\mathrm{d}x)$.

1.2. Let us examine the case where $X=\varepsilon\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ with $\varepsilon\in\{0,1\}$. Since $\mathrm{L_X}\mathrm{d}x=\mathrm{dL_X}x=0$ and $\mathrm{L_X}\mathrm{d}y=\mathrm{dL_X}y=\mathrm{d}y$, we have $\mathrm{L_X}\omega=\mathrm{X}(A)\mathrm{d}x+\mathrm{X}(B)\mathrm{d}y+B\mathrm{d}y=\varepsilon A'\mathrm{d}x+(\varepsilon B'+B)\mathrm{d}y$. Therefore $\mathrm{L_X}\omega=\lambda\omega$ if and only if $\varepsilon A'=\lambda A$ and $\varepsilon B'=(\lambda-1)B$. Since $A,B\in\mathbb{C}[x]$ and $\lambda\in\mathbb{C}$, it follows that $AB=0$: contradiction with $\gcd(A,B)=1$.

1.3. Let us study the eventuality: $\mathrm{X}=x\frac{\partial}{\partial x}+(x+y)\frac{\partial}{\partial y}$. We have $\mathrm{dX}(x)=\mathrm{d}x$ and $\mathrm{dX}(y)=\mathrm{d}x+\mathrm{d}y$, so that

$$\mathrm{L_X}\omega=\mathrm{X}(A)\mathrm{d}x+A\mathrm{d}x+\mathrm{X}(B)\mathrm{d}y+B(\mathrm{d}x+\mathrm{d}y)=(xA'+A+B)\mathrm{d}x+(xB'+B)\mathrm{d}y.$$

The condition $\mathrm{L_X}\omega=\lambda\omega$ is then equivalent to the system of differential equations $xA'+B=(\lambda-1)A$ and $xB'=(\lambda-1)B$, which can be easily integrated to yield $A(x)=(a-b\ln x)x^{\lambda-1}$ and $B(x)=bx^{\lambda-1}$, where $a,b\in\mathbb{C}$. Since $A\in\mathbb{C}[x]$, we deduce that $b=0$ and therefore $B\equiv 0$: contradiction with $\gcd(A,B)=1$.

2. Assume that $\mathfrak{iso}(\mathcal{F})$ is of type $(\mathfrak{e})$, *i.e.* $\mathfrak{iso}(\mathcal{F})=\langle\mathrm{X},\mathrm{Y}\rangle$ where $\mathrm{X}=x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y},\mathrm{Y}=\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$. As before by writing explicitly that $\mathrm{L_X}\omega=\lambda\omega$ and $\mathrm{L_Y}\omega=\mu\omega$, with $\lambda,\mu\in\mathbb{C}$ (Lemma 2.1), we obtain the system of partial differential equations

$$x\tfrac{\partial A}{\partial x}+2y\tfrac{\partial A}{\partial y}=(\lambda-1)A,\qquad x\tfrac{\partial B}{\partial x}+2y\tfrac{\partial B}{\partial y}=(\lambda-2)B,\qquad \tfrac{\partial A}{\partial x}+x\tfrac{\partial A}{\partial y}=\mu A-B,\qquad \tfrac{\partial B}{\partial x}+x\tfrac{\partial B}{\partial y}=\mu B.$$

It follows in particular that

$$(x^2-2y)\tfrac{\partial B}{\partial y}=(\mu x+2-\lambda)B\qquad\text{and}\qquad(x^2-2y)\tfrac{\partial A}{\partial y}=(\mu x+1-\lambda)A-xB.$$

Elementary integrations then lead to

$$B(x,y)=b(x)(x^2-2y)^{\frac{\lambda-2-\mu x}{2}}\qquad\text{and}\qquad A(x,y)=\Big(a(x)\sqrt{x^2-2y}-xb(x)\Big)\,(x^2-2y)^{\frac{\lambda-2-\mu x}{2}},$$

for some functions a and b of the coordinate x. Since $A,B \in \mathbb{C}[x,y]$ and $\gcd(A,B)=1$, we deduce that $\lambda-2-\mu x=0$ and $a(x)=0$ for any $x\in\mathbb{C}$, hence $\lambda=2,\mu=0$ and $a\equiv 0$. Therefore $B(x,y)=b(x)$ and $A(x,y)=-xb(x)=-xB(x,y)$: contradiction with $\gcd(A,B)=1$. $\square$

3. Foliations of $\mathbf{F}(d)$ degenerating onto $\mathcal{F}_1^d$ and $\mathcal{F}_2^d$

In this section we will study the problem of knowing whether the closure of the orbit of a foliation of $\mathbf{F}(d)$ contains the foliations $\mathcal{F}_1^d$ and $\mathcal{F}_2^d$. The following definition will be useful.

Definition 3.1 ([9]). — Let $\mathcal{F}$ and $\mathcal{F}'$ be two foliations of $\mathbf{F}(d)$. We say that $\mathcal{F}$ *degenerates* onto $\mathcal{F}'$ if the closure $\overline{\mathcal{O}(\mathcal{F})}$ (in $\mathbf{F}(d)$) of the orbit $\mathcal{O}(\mathcal{F})$ contains $\mathcal{O}(\mathcal{F}')$ and $\mathcal{O}(\mathcal{F})\neq\mathcal{O}(\mathcal{F}')$.

Remarks 3.2. — Let $\mathcal{F}$ and $\mathcal{F}'$ be two foliations such that $\mathcal{F}$ degenerates onto $\mathcal{F}'$. Then:

(i) $\dim\mathcal{O}(\mathcal{F}')<\dim\mathcal{O}(\mathcal{F})$;

(ii) $\deg \mathrm{I}^{\mathrm{inv}}_{\mathcal{F}}\leq\deg \mathrm{I}^{\mathrm{inv}}_{\mathcal{F}'}$, which is equivalent to $\deg \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}}\geq\deg \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}'}$. In particular, if $\mathcal{F}$ is convex, $\mathcal{F}'$ is also convex.

Corollary B is an immediate consequence of Theorem A and Remark 3.2 (i).

Remark 3.3. — Corollary B applies particularly to any foliation of $\mathbf{F}(d)$ which is *homogeneous*, *i.e.* which is given, for a suitable choice of affine coordinates (x,y), by a homogeneous 1-form $\omega=A(x,y)\mathrm{d}x+B(x,y)\mathrm{d}y$, where $A,B\in\mathbb{C}[x,y]_d$ and $\gcd(A,B)=1$. Indeed, for such a foliation $\mathcal{H}\in\mathbf{F}(d)$, we have (*cf.* [4])

$$\mathrm{Iso}(\mathcal{H})=\{(\alpha x,\alpha y)\mid\alpha\in\mathbb{C}^*\};$$

in particular, $\dim\mathcal{O}(\mathcal{H})=7$ and consequently

$$\overline{\mathcal{O}(\mathcal{H})}\subset\mathcal{O}(\mathcal{H})\cup\mathcal{O}(\mathcal{F}_1^d)\cup\mathcal{O}(\mathcal{F}_2^d).$$

Assertion ***1.*** (resp. ***2.***) of the following proposition gives a necessary (resp. sufficient) condition for a foliation of $\mathbf{F}(d)$ to degenerate onto the foliation $\mathcal{F}_1^d$.

Proposition 3.4. — *Let $\mathcal{F}$ be an element of $\mathbf{F}(d)$ such that $\mathcal{F}_1^d\notin\mathcal{O}(\mathcal{F})$. The following assertions hold:*
1. *If $\mathcal{F}$ degenerates onto $\mathcal{F}_1^d$, then $\mathcal{F}$ possesses a non-degenerate singularity m satisfying $\mathrm{BB}(\mathcal{F},m)=4$.*
2. *If $\mathcal{F}$ possesses a quasi-radial singularity of maximal order $d-1$, i.e. if $\mathrm{QRad}(\mathcal{F},d-1)\neq\emptyset$, then $\mathcal{F}$ degenerates onto $\mathcal{F}_1^d$.*

Proof. — ***1.*** Assume that $\mathcal{F}$ degenerates onto $\mathcal{F}_1^d$. Then there is an analytic family of foliations $(\mathcal{F}_\varepsilon)$ defined by a family of 1-forms (ω_ε) such that $\mathcal{F}_\varepsilon$ belongs to $\mathcal{O}(\mathcal{F})$ for $\varepsilon\neq0$ and $\mathcal{F}_{\varepsilon=0}=\mathcal{F}_1^d$. The non-degenerate singular point of $\mathcal{F}_1^d$, denoted by m_0, is "stable" in the sense that there is an analytic family (m_ε) of non-degenerate singular points of $\mathcal{F}_\varepsilon$ such that $m_{\varepsilon=0}=m_0$. The $\mathcal{F}_\varepsilon$'s being conjugated to $\mathcal{F}$ for $\varepsilon\neq0$, the foliation $\mathcal{F}$ admits a non-degenerate singular point m such that

$$\forall\,\varepsilon\in\mathbb{C}^*,\ \mathrm{BB}(\mathcal{F}_\varepsilon,m_\varepsilon)=\mathrm{BB}(\mathcal{F},m).$$

Since $\mu(\mathcal{F}_\varepsilon,m_\varepsilon)=1$ for any $\varepsilon\in\mathbb{C}$, the function $\varepsilon\mapsto\mathrm{BB}(\mathcal{F}_\varepsilon,m_\varepsilon)$ is continuous, hence constant on $\mathbb{C}$. As a consequence

$$\mathrm{BB}(\mathcal{F},m)=\mathrm{BB}(\mathcal{F}_{\varepsilon=0},m_{\varepsilon=0})=\mathrm{BB}(\mathcal{F}_1^d,m_0)=4.$$

2. Assume that $\mathcal{F}$ has a quasi-radial singularity m of order $d-1$. Then $\mu(\mathcal{F},m)=1, \mathrm{BB}(\mathcal{F},m)=4$ and $\kappa(\mathcal{F},m)=d$. This last equality ensures the existence of a line ℓ_m passing through m, not invariant by $\mathcal{F}$ and such that $\mathrm{Tang}(\mathcal{F},\ell_m,m)=d$. Let us choose an affine coordinate system (x,y) such that $m=(0,0)$ and $\ell_m=\{x=0\}$. The foliation $\mathcal{F}$ is defined in these coordinates by a 1-form ω of type

$$\omega=C_d(x,y)(x\mathrm{d}y-y\mathrm{d}x)+\sum_{i=1}^{d}\big(A_i(x,y)\mathrm{d}x+B_i(x,y)\mathrm{d}y\big),\qquad \text{where } A_i,\,B_i\in\mathbb{C}[x,y]_i,\; C_d\in\mathbb{C}[x,y]_d.$$

We have

$$\omega\wedge\mathrm{d}x\Big|_{x=0}=\sum_{i=1}^{d}B_i(0,y)\mathrm{d}y\wedge\mathrm{d}x=\sum_{i=1}^{d}B_i(0,1)y^i\mathrm{d}y\wedge\mathrm{d}x.$$

Then the equality $\mathrm{Tang}(\mathcal{F},\ell_m,m)=d$ translates into $B_i(0,1)=0$ for $i\in\{1,2,\ldots,d-1\}$ and $B_d(0,1)\neq0$. This allows to write

$$B_1(x,y)=\alpha x,\qquad B_d(x,y)=x\widehat{B}_{d-1}(x,y)+\gamma y^d,\qquad B_i(x,y)=x\widetilde{B}_{i-1}(x,y)\text{ for } i\in\{2,3,\ldots,d-1\},$$

where $\widetilde{B}_{i-1}\in\mathbb{C}[x,y]_{i-1},\;\widehat{B}_{d-1}\in\mathbb{C}[x,y]_{d-1},\;\gamma\in\mathbb{C}^*,\;\alpha\in\mathbb{C}$. The equalities $\mu(\mathcal{F},m)=1$ and $\mathrm{BB}(\mathcal{F},m)=4$ imply that $\alpha\neq0$ and $A_1(x,y)=\delta x-\alpha y$ for some $\delta\in\mathbb{C}$. Thus ω is of type

$$\omega=\delta x\mathrm{d}x+\big(x\widehat{B}_{d-1}(x,y)+\gamma y^d\big)\mathrm{d}y+\big(C_d(x,y)+\alpha\big)\big(x\mathrm{d}y-y\mathrm{d}x\big)+\sum_{i=2}^{d}A_i(x,y)\mathrm{d}x+x\sum_{i=2}^{d-1}\widetilde{B}_{i-1}(x,y)\mathrm{d}y,$$

where $A_i\in\mathbb{C}[x,y]_i,\;\widetilde{B}_{i-1}\in\mathbb{C}[x,y]_{i-1},\;\widehat{B}_{d-1}\in\mathbb{C}[x,y]_{d-1},\;\delta\in\mathbb{C},\;\alpha,\gamma\in\mathbb{C}^*$.

By putting $\varphi=\big(\varepsilon^dx,\varepsilon y\big)$ and $\theta=\alpha(x\mathrm{d}y-y\mathrm{d}x)+\gamma y^d\mathrm{d}y$, we obtain

$$\frac{1}{\varepsilon^{d+1}}\varphi^*\omega=\theta+\varepsilon^{d-1}\Big(\delta x\mathrm{d}x+x\widehat{B}_{d-1}(\varepsilon^{d-1}x,y)\mathrm{d}y\Big)+\varepsilon^dC_d(\varepsilon^{d-1}x,y)\,(x\mathrm{d}y-y\mathrm{d}x)+\sum_{i=2}^{d}\varepsilon^{i-1}A_i(\varepsilon^{d-1}x,y)\mathrm{d}x+x\sum_{i=2}^{d-1}\varepsilon^{i-1}\widetilde{B}_{i-1}(\varepsilon^{d-1}x,y)\mathrm{d}y$$

which tends to θ as ε tends to 0. By making the change of coordinates $(x,y)\mapsto\Big(\frac{x}{y}-\frac{\gamma}{\alpha y},\frac{x}{y}\Big)$, we reduce ourselves to $\theta=\omega_1^d=y^d\mathrm{d}x+x^d(x\mathrm{d}y-y\mathrm{d}x)$. As a result $\mathcal{F}$ degenerates onto $\mathcal{F}_1^d$. □

Example 3.5. — Let us consider the homogeneous foliation $\mathcal{H}_1^d$ defined in the affine chart $z=1$ by the 1-form

$$\overline{\omega}_1^d=y^d\mathrm{d}x-x^d\mathrm{d}y.$$

We know from [4, Proposition 4.1] that $\mathcal{H}_1^d$ is convex and admits the points $[1:0:0]$ and $[0:1:0]$ as radial singularities of maximal order $d-1$. Therefore $\mathcal{H}_1^d$ degenerates onto $\mathcal{F}_1^d$ (Proposition 3.4) and it does not degenerate onto $\mathcal{F}_2^d$, because $\mathcal{F}_2^d$ is not convex. Thus, according to Remark 3.3, we have

$$\overline{O(\mathcal{H}_1^d)}=O(\mathcal{H}_1^d)\cup O(\mathcal{F}_1^d).$$

Example 3.6. — Let us consider the family $(\mathcal{G}^d(\gamma))_{\gamma\in\mathbb{C}}$ of foliations of degree d on $\mathbb{P}^2_{\mathbb{C}}$ defined in the affine chart $z=1$ by

$$\eta^d(\gamma)=(x-\gamma y)\mathrm{d}y-y\mathrm{d}x+x^d\mathrm{d}x-y^d\mathrm{d}y.$$

We remark that the point $m=[0:0:1]$ is a non-degenerate singularity of $\mathcal{G}^d(\gamma)$ with BAUM-BOTT index 4. Moreover, along the line $\ell=\{y=0\}$ we have $\eta^d(\gamma)\wedge\mathrm{d}y\Big|_{y=0}=x^d\mathrm{d}x\wedge\mathrm{d}y$, so that $\mathrm{Tang}(\mathcal{G}^d(\gamma),\ell,m)=d$. It follows that the singularity m of $\mathcal{G}^d(\gamma)$ is quasi-radial of maximal order $d-1$. As a consequence $\mathcal{G}^d(\gamma)$ degenerates onto $\mathcal{F}_1^d$ (Proposition 3.4).

The converse of assertion ***2.*** of Proposition 3.4 is false as the following example shows.

Example 3.7. — Let $\mathcal{F}$ be the foliation of degree $d\geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$ defined in the affine chart $z=1$ by

$$\omega = x\mathrm{d}y - y\mathrm{d}x + P(y)\mathrm{d}y,$$

where P is a polynomial of $\mathbb{C}[y]$ of degree d admitting 0 as a root of multiplicity $\leq d-1$, *i.e.* P is of the form

$$P(y) = y^{\nu}(a_0 + a_1 y + \cdots + a_{d-\nu}y^{d-\nu}), \qquad \text{where } \nu\in\{1,2,\ldots,d-1\},\ a_i\in\mathbb{C},\ a_0 a_{d-\nu}\neq 0.$$

The singular locus of $\mathcal{F}$ consists of the two points $m=[0:0:1]$ and $m'=[1:0:0]$; moreover

$$\mu(\mathcal{F},m)=1, \qquad \mathrm{BB}(\mathcal{F},m)=4, \qquad \kappa(\mathcal{F},m)=\nu<d, \qquad \mu(\mathcal{F},m')>1.$$

It follows that $\mathcal{F}$ has no quasi-radial singularity of maximal order $d-1$, *i.e.* $\mathrm{QRad}(\mathcal{F},d-1)=\emptyset$. However, $\mathcal{F}$ degenerates onto $\mathcal{F}_1^d$. Indeed, by putting $\varphi=\left(\frac{a_{d-\nu}}{\varepsilon^d}x, \frac{1}{\varepsilon}y\right)$, we see that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{d+1}}{a_{d-\nu}}\varphi^*\omega = x\mathrm{d}y - y\mathrm{d}x + y^d\mathrm{d}y.$$

Question 1. — *Let $\mathcal{F}$ be a foliation of degree $d\geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$. Is it true that if $\mathcal{F}$ degenerates onto $\mathcal{F}_1^d$ then*
- *either $\mathcal{F}$ admits a quasi-radial singularity of maximal order $d-1$,*
- *or $\mathcal{F}$ is conjugated to Example 3.7, i.e. up to linear conjugation $\mathcal{F}$ is given by a 1-form of type $x\mathrm{d}y - y\mathrm{d}x + P(y)\mathrm{d}y$ with $P\in\mathbb{C}[y]$, $\deg P = d$ and $P(0)=0$?*

Proposition 3.8. — *Let d be an integer greater than or equal to 2. Let us denote by $U_1(d)$ the subset of $\mathbf{F}(d)$ defined by*

$$U_1(d) := \Big\{\mathcal{F}\in\mathbf{F}(d) \mid \forall s\in\mathrm{Sing}(\mathcal{F}),\ \mu(\mathcal{F},s)=1, \tau(\mathcal{F},s)=1\Big\}.$$

Then:

(i) $U_1(d)$ is a non-empty ZARISKI *open subset of $\mathbf{F}(d)$; in particular, for any $\gamma\in\mathbb{C}$, $\mathcal{G}^d(\gamma)\in U_1(d)$ if and only if $\gamma\left(\gamma^{d+1}+\frac{(d+1)^{d+1}}{d^d}\right)\neq 0$.*

(ii) Let $\mathcal{F}$ be an element of $U_1(d)$. For any singular point $s\in\mathrm{Sing}(\mathcal{F})$, the set

$$\Lambda(\mathcal{F},s) := \Big\{\ell_s\in\mathfrak{L}_s(\mathcal{F}) \mid \mathrm{Tang}(\mathcal{F},\ell_s,s)>1\Big\}$$

has at most 2 elements. In particular, the set $\bigcup_{n=2}^{d}\widehat{\mathrm{QRad}}(\mathcal{F},n-1)$ is finite.

To prove this proposition, we need the following lemma.

Lemma 3.9. — *Let $\mathcal{F}$ be a foliation of degree $d\geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$, s a singular point of $\mathcal{F}$, ℓ_s a line passing through s and not invariant by $\mathcal{F}$ and $\mathrm{X}=A(x,y)\frac{\partial}{\partial x}+B(x,y)\frac{\partial}{\partial y}$ a polynomial vector field defining $\mathcal{F}$ in an affine chart (x,y) containing s. Let us denote by (x_0,y_0) the coordinates of s and let $a(x-x_0)+b(y-y_0)=0$ be an equation of the line ℓ_s. Then, for any $n\in\{2,3,\ldots,d\}$, $\mathrm{Tang}(\mathcal{F},\ell_s,s)\geq n$ if and only if*

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}\Big(aA(x_0+bt,y_0-at)+bB(x_0+bt,y_0-at)\Big)\Big|_{t=0}=0, \quad \forall j\in\{1,2,\ldots,n-1\}.$$

In particular, the set $\Lambda(\mathcal{F},s) := \Big\{\ell_s\in\mathfrak{L}_s(\mathcal{F}) \mid \mathrm{Tang}(\mathcal{F},\ell_s,s)>\tau(\mathcal{F},s)\Big\}$ is finite and its cardinality is at most $\tau(\mathcal{F},s)+1$.

Proof. — The 1-form $\omega = A(x,y)\mathrm{d}y - B(x,y)\mathrm{d}x$ also defines the foliation $\mathcal{F}$ because $i_{\mathrm{X}}\omega = 0$. We have

$$\omega \wedge \mathrm{d}\big(a(x-x_0)+b(y-y_0)\big)\Big|_{(x,y)=(x_0+bt,y_0-at)} = P(t)\mathrm{d}y\wedge\mathrm{d}x,$$

where $P(t) = aA(x_0+bt,y_0-at)+bB(x_0+bt,y_0-at)$. Thus $\mathrm{Tang}(\mathcal{F},\ell_s,s) = \nu(P(t),0)$. Notice that $P(0)=0$ because the point s being singular for $\mathcal{F}$, we have $A(x_0,y_0)=B(x_0,y_0)=0$. Then $\mathrm{Tang}(\mathcal{F},\ell_s,s)\geq n$ if and only if the root $t=0$ of the polynomial P has multiplicity at least n, that is if and only if $P'(0)=P''(0)=\cdots=P^{(n-1)}(0)=0$, hence the announced equivalence holds.

By conjugating ω by the translation $(x+x_0,y+y_0)$, we can assume that $s=(0,0)$. Let us denote $\tau(\mathcal{F},s)$ simply by τ. Then the vector field X decomposes in the form

$$\mathrm{X} = C_{\tau-2}(x,y)\mathrm{R} + \sum_{i=\tau}^{d+1}\mathrm{X}_i,$$

where $\mathrm{R} = x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$, $C_{\tau-2}$ is a polynomial of degree $\leq \tau-2$, $\mathrm{X}_i = A_i(x,y)\frac{\partial}{\partial x}+B_i(x,y)\frac{\partial}{\partial y}$ is a homogeneous vector field of degree i, with $\det(\mathrm{X}_\tau,\mathrm{R})\not\equiv 0$. Thus, we have

$$\begin{aligned} aA(bt,-at)+bB(bt,-at) &= \left(a\Big(xC_{\tau-2}(x,y)+\sum_{i=\tau}^{d+1}A_i(x,y)\Big)+b\Big(yC_{\tau-2}(x,y)+\sum_{i=\tau}^{d+1}B_i(x,y)\Big)\right)\Bigg|_{(x,y)=(bt,-at)}\\ &= \sum_{i=\tau}^{d+1}\Big(aA_i(bt,-at)+bB_i(bt,-at)\Big)\\ &= \sum_{i=\tau}^{d+1}t^iQ_{i+1}(a,b),\end{aligned}$$

where $Q_{i+1}(a,b) := aA_i(b,-a)+bB_i(b,-a)$ is a homogeneous polynomial of degree $i+1$ in (a,b). From this, we deduce that $\mathrm{Tang}(\mathcal{F},\ell_s,s)>\tau$ if and only if $Q_{\tau+1}(a,b)=0$. As a result

$$\Lambda(\mathcal{F},s) = \Big\{\ell_s = \{ax+by=0\}\in\mathfrak{L}_s(\mathcal{F}) \mid Q_{\tau+1}(a,b)=0\Big\}.$$

Now, the polynomial $Q_{\tau+1}$ is not identically zero because $Q_{\tau+1}(a,b) = -\det(\mathrm{X}_\tau,\mathrm{R})\big|_{(x,y)=(b,-a)}\not\equiv 0$. It follows that $\Lambda(\mathcal{F},s)$ has cardinality at most $\tau+1$. □

Proof of Proposition 3.8. — We have

$$U_1(d) = \Big\{\mathcal{F}\in\mathbf{F}(d) \mid \forall s\in\mathrm{Sing}(\mathcal{F}),\ \det(\mathrm{Jac}\mathrm{X}(s))\neq 0,\ \det(J^1_s\mathrm{X},\mathrm{R}_s)\not\equiv 0\Big\},$$

where X denotes a polynomial vector field defining $\mathcal{F}$ in an affine chart containing s and R_s is the radial vector field centered at s. It follows that $U_1(d)$ is a ZARISKI open subset of $\mathbf{F}(d)$. To establish assertion *(i)*, it remains to show that for any $\gamma\in\mathbb{C}$, $\mathcal{G}^d(\gamma)\in U_1(d)$ if and only if $\gamma\Big(\gamma^{d+1}+\frac{(d+1)^{d+1}}{d^d}\Big)\neq 0$. In homogeneous coordinates, the foliation $\mathcal{G}^d(\gamma)$ is defined by the 1-form

$$\Omega^d(\gamma) = z\big(x^d-yz^{d-1}\big)\mathrm{d}x - z\big(y^d+\gamma yz^{d-1}-xz^{d-1}\big)\mathrm{d}y + \big(y^{d+1}-x^{d+1}+\gamma y^2z^{d-1}\big)\mathrm{d}z.$$

The singular locus $\mathrm{Sing}\big(\mathcal{G}^d(\gamma)\big)$ consists of the points

$$s_0=[0:0:1],\qquad s_k=[x_k:x_k^d:1],\qquad s'_l=[1:\xi^l:0],\qquad k\in\{1,2,\ldots,d^2-1\}, l\in\{0,1,\ldots,d\},$$

where $\xi=\exp\big(\frac{2i\pi}{d+1}\big)$ and the x_k's are the roots of the polynomial $P(x)=x^{d^2-1}+\gamma x^{d-1}-1$.

In the affine chart $z=1$, resp. $x=1$, $\mathcal{G}^d(\gamma)$ is given by the vector field

$$\mathrm{Y}=\big(y^d+\gamma y-x\big)\frac{\partial}{\partial x}+\big(x^d-y\big)\frac{\partial}{\partial y},\qquad \text{resp. } \mathrm{Z}=\big(y^{d+1}+\gamma y^2z^{d-1}-1\big)\frac{\partial}{\partial y}+z\big(y^d+\gamma yz^{d-1}-z^{d-1}\big)\frac{\partial}{\partial z}.$$

A direct computation show that $\det(\mathrm{Jac}\,\mathrm{Y}(s_0))=1\neq 0$, $\det(J^1_{s_0}\mathrm{Y},\mathrm{R}_{s_0})=\gamma y^2$ and

$$\det(\mathrm{Jac}\,\mathrm{Z}(s'_l))=(d+1)\xi^{-2l}\neq 0,\qquad \det(\mathrm{Jac}\,\mathrm{Y}(s_k))=1-d\gamma x_k^{d-1}-d^2x_k^{d^2-1}=(d-1)\big(d\gamma x_k^{d-1}-d-1\big),\ \text{because } P(x_k)=0,$$
$$\det(J^1_{s'_l}\mathrm{Z},\mathrm{R}_{s'_l})=d\xi^{-l}\big(y-\xi^l\big)z\not\equiv 0,\quad \det(J^1_{s_k}\mathrm{Y},\mathrm{R}_{s_k})=\big(dx_k^{d^2-d}+\gamma\big)\big(y-x_k^d\big)^2-dx_k^{d-1}\big(x-x_k\big)^2\not\equiv 0,\ \text{because } x_k\neq 0.$$

From these we deduce that $\mathcal{G}^d(\gamma)\in U_1(d)$ if and only if $\gamma\neq 0$ and $d\gamma x_k^{d-1}-d-1\neq 0$, *i.e.* if and only if $\gamma\neq 0$ and $x_k^{d-1}\neq\frac{d+1}{d\gamma}$. Now, by putting $Q(t)=t^{d+1}+\gamma t-1$, we have $P(x)=Q(x^{d-1})$ so that $t_0\in\mathbb{C}$ is a root of the polynomial $Q(t)$ if and only if there exists $k\in\{1,2,\ldots,d^2-1\}$ such that $t_0=x_k^{d-1}$. It follows that

$$\mathcal{G}^d(\gamma)\in U_1(d)\Longleftrightarrow \gamma Q\left(\frac{d+1}{d\gamma}\right)\neq 0\Longleftrightarrow \gamma\left(\gamma^{d+1}+\frac{(d+1)^{d+1}}{d^d}\right)\neq 0.$$

Assertion *(ii)* is an immediate consequence of Lemma 3.9. □

Theorem 3.10. — *Let d be an integer greater than or equal to 2. Let us denote by $\Sigma_1(d)$ the subset of $\mathbf{F}(d)$ defined by*

$$\Sigma_1(d):=\Big\{\mathcal{F}\in\mathbf{F}(d)\mid \mathrm{QRad}(\mathcal{F},d-1)\neq\emptyset\Big\}.$$

Then

(a) $\emptyset\neq\Sigma_1(d)\subsetneq\mathbf{B}(\mathcal{F}_1^d)$;

(b) $\Sigma_1(d)$ *is a constructible subset of* $\mathbf{F}(d)$ *of dimension greater than or equal to* $\dim\mathbf{F}(d)-(d-1)$.

Proof. — *(a)* $\Sigma_1(d)$ contains the foliations $\mathcal{H}_1^d$ and $\mathcal{G}^d(\gamma),\gamma\in\mathbb{C}$ (Examples 3.5 and 3.6) and is therefore non-empty. Assertion **2.** of Proposition 3.4 ensures that $\Sigma_1(d)\subset\mathbf{B}(\mathcal{F}_1^d)$; this inclusion is strict as Example 3.7 shows.

(b) Let us denote by $\check{\mathbb{P}}^2_{\mathbb{C}}$ the dual projective plane of $\mathbb{P}^2_{\mathbb{C}}$. Let $\pi:\mathbf{F}(d)\times\mathbb{P}^2_{\mathbb{C}}\times\check{\mathbb{P}}^2_{\mathbb{C}}\to\mathbf{F}(d)$ be the projection onto the first factor; we have $\Sigma_1(d)=\pi(W_1(d))$, where

$$\begin{aligned}W_1(d):&=\bigcup_{\mathcal{F}\in\Sigma_1(d)}\{\mathcal{F}\}\times\widehat{\mathrm{QRad}}(\mathcal{F},d-1)\\&=\Big\{(\mathcal{F},s,\ell)\in\mathbf{F}(d)\times\mathbb{P}^2_{\mathbb{C}}\times\check{\mathbb{P}}^2_{\mathbb{C}}\mid s\in\mathrm{Sing}(\mathcal{F}),\ \ell\in\mathfrak{L}_s(\mathcal{F}),\ \mu(\mathcal{F},s)=1,\ \mathrm{BB}(\mathcal{F},s)=4,\ \mathrm{Tang}(\mathcal{F},\ell,s)=d\Big\}.\end{aligned}$$

According to Lemma 3.9, $W_1(d)$ can be rewritten as

$$W_1(d)=\left\{(\mathcal{F},s,\ell)\in\mathbf{F}(d)\times\mathbb{P}^2_{\mathbb{C}}\times\check{\mathbb{P}}^2_{\mathbb{C}}\;\middle|\;\begin{array}{l}s=(x_0,y_0)\in\ell=\{a(x-x_0)+b(y-y_0)=0\}\\[4pt] A(x_0,y_0)=0,\ B(x_0,y_0)=0,\ \det(\mathrm{Jac}\,\mathrm{X}(s))\neq 0,\ \dfrac{\mathrm{tr}^2(\mathrm{Jac}\,\mathrm{X}(s))}{\det(\mathrm{Jac}\,\mathrm{X}(s))}=4\\[4pt] aA(x_0+bt,y_0-at)+bB(x_0+bt,y_0-at)\not\equiv 0\\[4pt] \dfrac{\mathrm{d}^j}{\mathrm{d}t^j}\Big(aA(x_0+bt,y_0-at)+bB(x_0+bt,y_0-at)\Big)\Big|_{t=0}=0,\ j=1,\ldots,d-1\end{array}\right\},\tag{3.1}$$

where $\mathrm{X}=A(x,y)\frac{\partial}{\partial x}+B(x,y)\frac{\partial}{\partial y}$ is a polynomial vector field defining $\mathcal{F}$ in an affine chart (x,y) containing s. It follows that $W_1(d)$ is a quasi-projective subvariety of $\mathbf{F}(d)\times\mathbb{P}^2_{\mathbb{C}}\times\check{\mathbb{P}}^2_{\mathbb{C}}$. Thus, by CHEVALLEY's Theorem [11, Exercise II.3.19], the set $\Sigma_1(d)=\pi(W_1(d))$ is constructible.

According to the above discussion and Proposition 3.8 *(i)*, the intersection $U_1(d)\cap\Sigma_1(d)$ contains the foliations $\mathcal{G}^d(\gamma)$, with $\gamma\left(\gamma^{d+1}+\frac{(d+1)^{d+1}}{d^d}\right)\neq 0$, and is therefore non-empty ($U_1(d)$ being the set of $\mathcal{F}\in\mathbf{F}(d)$ such that for any $s\in\operatorname{Sing}\mathcal{F}$, $\mu(\mathcal{F},s)=1$ and $\tau(\mathcal{F},s)=1$). Then there exists an irreducible component $\Sigma_1^0(d)$ of $\Sigma_1(d)$ such that $U_1(d)\cap\Sigma_1^0(d)\neq\emptyset$. Let $W_1(d)=\bigcup\limits_{i=1}^{k}W_1^i(d)$ be the decomposition of $W_1(d)$ into its irreducible components. Let us denote by $\pi_0: W_1(d)\to\mathbf{F}(d)$ the restriction of π to $W_1(d)$. Then, there is $n\in\{1,\ldots,k\}$ such that $\overline{\pi_0(W_1^n(d))}=\overline{\Sigma_1^0(d)}$. Indeed, since $\Sigma_1(d)=\pi_0(W_1(d))$, we have $\overline{\Sigma_1^0(d)}\subset\overline{\Sigma_1(d)}=\bigcup\limits_{i=1}^{k}\overline{\pi_0(W_1^i(d))}$. The irreducibility of $\Sigma_1^0(d)$ therefore ensures the existence of $n\in\{1,\ldots,k\}$ such that $\overline{\Sigma_1^0(d)}\subset\overline{\pi_0(W_1^n(d))}\subset\overline{\Sigma_1(d)}$. Since $\overline{\Sigma_1^0(d)}$ is an irreducible component of $\overline{\Sigma_1(d)}$ and since $\overline{\pi_0(W_1^n(d))}$ is irreducible by continuity of π_0, we deduce that $\overline{\pi_0(W_1^n(d))}=\overline{\Sigma_1^0(d)}$.
Thus, since $U_1(d)$ is a ZARISKI open subset of $\mathbf{F}(d)$ (Proposition 3.8 *(i)*), the morphism π_0 induces by restriction a dominant morphism of quasi-projective varieties $\pi_0^n: W_1^n(d)\cap\pi_0^{-1}(U_1(d))\to\overline{\Sigma_1^0(d)}\cap U_1(d)$.
Notice that all the fibers of π_0 over the elements of $U_1(d)\cap\Sigma_1(d)$ are finite and non-empty. Indeed, if $\mathcal{F}\in U_1(d)\cap\Sigma_1(d)$ then, by Proposition 3.8 *(ii)*, the set $\widehat{\operatorname{QRad}}(\mathcal{F},d-1)$ is finite and non-empty; therefore so is $\pi_0^{-1}(\mathcal{F})=\{\mathcal{F}\}\times\widehat{\operatorname{QRad}}(\mathcal{F},d-1)$. Since $\pi_0(W_1^n(d)\cap\pi_0^{-1}(U_1(d)))\subset U_1(d)\cap\Sigma_1(d)$, it follows that all the non-empty fibers of π_0^n are finite and therefore zero-dimensional. The fiber dimension theorem (*cf.* [15, Theorem 3, page 49]) then implies that $\dim(W_1^n(d)\cap\pi_0^{-1}(U_1(d)))=\dim(\overline{\Sigma_1^0(d)}\cap U_1(d))$; since $W_1^n(d)\cap\pi_0^{-1}(U_1(d))$ and $\overline{\Sigma_1^0(d)}\cap U_1(d)$ are non-empty open subsets of the irreducible varieties $W_1^n(d)$ and $\overline{\Sigma_1^0(d)}$ respectively, we have

$$\dim\overline{\Sigma_1^0(d)}=\dim(\overline{\Sigma_1^0(d)}\cap U_1(d))=\dim(W_1^n(d)\cap\pi_0^{-1}(U_1(d)))=\dim W_1^n(d).$$

Now, from (3.1) we deduce that each irreducible component $W_1^i(d)$ of $W_1(d)$ has dimension

$$\dim W_1^i(d)\geq\dim(\mathbf{F}(d)\times\mathbb{P}^2_{\mathbb{C}}\times\check{\mathbb{P}}^2_{\mathbb{C}})-4-(d-1)=\dim\mathbf{F}(d)-(d-1),$$

hence

$$\dim\Sigma_1(d)=\dim\overline{\Sigma_1(d)}\geq\dim\overline{\Sigma_1^0(d)}=\dim W_1^n(d)\geq\dim\mathbf{F}(d)-(d-1).$$

□

Assertion ***1.*** (resp. ***2.***) of the following proposition gives a necessary (resp. sufficient) condition for a foliation of $\mathbf{F}(d)$ to degenerate onto the foliation $\mathcal{F}_2^d$.

Proposition 3.11. — *Let $\mathcal{F}$ be an element of $\mathbf{F}(d)$ such that $\mathcal{F}_2^d\notin O(\mathcal{F})$. The following assertions hold:*
1. *If $\mathcal{F}$ degenerates onto $\mathcal{F}_2^d$, then $\deg\mathrm{I}_{\mathcal{F}}^{\mathrm{tr}}\geq d-1$.*
2. *If $\mathcal{F}$ admits an inflection point of maximal order $d-1$, i.e. if $\operatorname{Flex}(\mathcal{F},d-1)\neq\emptyset$, then $\mathcal{F}$ degenerates onto $\mathcal{F}_2^d$.*

Proof. — ***1.*** If $\mathcal{F}$ degenerates onto $\mathcal{F}_2^d$, then $\deg\mathrm{I}_{\mathcal{F}}^{\mathrm{tr}}\geq\deg\mathrm{I}_{\mathcal{F}_2^d}^{\mathrm{tr}}$. An immediate computation shows that $\mathrm{I}_{\mathcal{F}_2^d}^{\mathrm{tr}}=y^{d-1}$ so that $\deg\mathrm{I}_{\mathcal{F}_2^d}^{\mathrm{tr}}=d-1$, hence the announced inequality holds.
2. Assume that $\mathcal{F}$ possesses such a point. We choose an affine coordinate system (x,y) such that $p=(0,0)$ is an inflection point of order $d-1$ of $\mathcal{F}$ and $x=0$ is the tangent line to the leaf of $\mathcal{F}$ passing through p.

Let ω be a 1-form defining $\mathcal{F}$ in these coordinates. Since $\mathrm{T}_p^{\mathbb{P}}\mathcal{F} = \{x=0\}$, ω is of type

$$\omega = C_d(x,y)(x\mathrm{d}y - y\mathrm{d}x) + \alpha\mathrm{d}x + \sum_{i=1}^{d}\big(A_i(x,y)\mathrm{d}x + B_i(x,y)\mathrm{d}y\big), \quad \text{where } A_i, B_i \in \mathbb{C}[x,y]_i,\ C_d \in \mathbb{C}[x,y]_d,\ \alpha \in \mathbb{C}^*.$$

We have

$$\omega \wedge \mathrm{d}x\Big|_{x=0} = \sum_{i=1}^{d} B_i(0,y)\mathrm{d}y \wedge \mathrm{d}x = \sum_{i=1}^{d} B_i(0,1)y^i\mathrm{d}y \wedge \mathrm{d}x.$$

Therefore the hypothesis that $(0,0)$ is an inflection point of order $d-1$ of $\mathcal{F}$ translates into $B_i(0,1) = 0$ for $i \in \{1,2,\ldots,d-1\}$ and $B_d(0,1) \neq 0$. Then we can write

$$B_d(x,y) = x\widehat{B}_{d-1}(x,y) + \beta y^d, \qquad B_i(x,y) = x\widetilde{B}_{i-1}(x,y) \text{ for } i \in \{1,2,\ldots,d-1\},$$

where $\widetilde{B}_{i-1} \in \mathbb{C}[x,y]_{i-1}$, $\widehat{B}_{d-1} \in \mathbb{C}[x,y]_{d-1}$, $\beta \in \mathbb{C}^*$. Thus ω is of type

$$\omega = \alpha\mathrm{d}x + \big(x\widehat{B}_{d-1}(x,y) + \beta y^d\big)\mathrm{d}y + C_d(x,y)\big(x\mathrm{d}y - y\mathrm{d}x\big) + \sum_{i=1}^{d} A_i(x,y)\mathrm{d}x + x\sum_{i=1}^{d-1}\widetilde{B}_{i-1}(x,y)\mathrm{d}y,$$

where $A_i \in \mathbb{C}[x,y]_i$, $\widetilde{B}_{i-1} \in \mathbb{C}[x,y]_{i-1}$, $\widehat{B}_{d-1} \in \mathbb{C}[x,y]_{d-1}$, $\alpha,\beta \in \mathbb{C}^*$.

Let us consider the family of automorphisms $\varphi = \varphi_\varepsilon = (\varepsilon^{d+1}x, \varepsilon y)$. We have

$$\frac{1}{\varepsilon^{d+1}}\varphi^*\omega = \alpha\mathrm{d}x + \Big(\varepsilon^d x\widehat{B}_{d-1}(\varepsilon^d x, y) + \beta y^d\Big)\mathrm{d}y + \varepsilon^{d+1}C_d(\varepsilon^d x, y)\,(x\mathrm{d}y - y\mathrm{d}x) + \sum_{i=1}^{d}\varepsilon^i A_i(\varepsilon^d x, y)\mathrm{d}x + x\sum_{i=1}^{d-1}\varepsilon^i\widetilde{B}_{i-1}(\varepsilon^d x, y)\mathrm{d}y$$

which tends to $\alpha\mathrm{d}x + \beta y^d\mathrm{d}y$ as ε tends to 0. Clearly $\alpha\mathrm{d}x + \beta y^d\mathrm{d}y$ defines a foliation conjugated to $\mathcal{F}_2^d$; as a result $\mathcal{F}$ degenerates onto $\mathcal{F}_2^d$. □

Example 3.12. — Let us consider the homogeneous foliation $\mathcal{H}_2^d$ defined in the affine chart $z=1$ by the 1-form

$$\overline{\omega}_2^d = x^d\mathrm{d}x - y^d\mathrm{d}y.$$

We know from [4, Proposition 4.1] that $\mathcal{H}_2^d$ has no non-degenerate singularity with BAUM-BOTT index 4 and that

$$\mathrm{Flex}(\mathcal{H}_2^d, d-1) = \{xy=0\} \setminus \{[0:0:1]\} \neq \emptyset.$$

Thus $\mathcal{H}_2^d$ degenerates onto $\mathcal{F}_2^d$ (Proposition 3.11) and it does not degenerate onto $\mathcal{F}_1^d$ (Proposition 3.4). Consequently, according to Remark 3.3, we have

$$\overline{O(\mathcal{H}_2^d)} = O(\mathcal{H}_2^d) \cup O(\mathcal{F}_2^d).$$

***Example 3.13* (JOUANOLOU's foliation)**. — Let us consider the foliation $\mathcal{F}_J^d$ of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$ defined, in the affine chart $z=1$, by

$$\omega_J^d = (x^d y - 1)\mathrm{d}x + (y^d - x^{d+1})\mathrm{d}y.$$

This example is due to JOUANOLOU and is historically the first explicit example of foliation without invariant algebraic curve ([12]). The point $p=(0,0)$ is an inflection point of maximal order $d-1$ of $\mathcal{F}_J^d$ because $\mathrm{T}_p^{\mathbb{P}}\mathcal{F}_J^d = \{x=0\}$ and $\omega_J^d \wedge \mathrm{d}x\Big|_{x=0} = y^d\mathrm{d}y \wedge \mathrm{d}x$. As a result $\mathcal{F}_J^d$ degenerates onto $\mathcal{F}_2^d$ (Proposition 3.11). However, we know from [13, Section 3] that every singularity s of $\mathcal{F}_J^d$ is non-degenerate with BAUM-BOTT index

$$\mathrm{BB}(\mathcal{F}_J^d, s) = \frac{(d+2)^2}{d^2+d+1} \neq 4,$$

so that $\mathcal{F}_J^d$ does not degenerate onto $\mathcal{F}_1^d$ (Proposition 3.4).

The converse of assertion **2.** of Proposition 3.11 is false as the following example shows.

Example 3.14. — Let $\mathcal{F}$ be the foliation of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$ defined in the affine chart $z = 1$ by

$$\omega = \mathrm{d}x + P(y)\mathrm{d}y, \qquad\qquad \text{where } P \in \mathbb{C}[y],\ \deg P = d.$$

It is easy to check that $\mathrm{Sing}(\mathcal{F}) = \big\{[1:0:0]\big\}$ and $\mathrm{I}^{\mathrm{tr}}_{\mathcal{F}} = P'(y)$. If the derivative P' has a single root, *i.e* if P is of the form $P(y) = a(y-\alpha)^d + b$, where $\alpha, a, b \in \mathbb{C}, a \neq 0$, then $\mathcal{F}$ is conjugated to $\mathcal{F}_2^d$; indeed, we have

$$\frac{1}{a}\varphi^*\omega = \mathrm{d}x + y^d\mathrm{d}y, \quad \text{where } \varphi = (ax - by, y + \alpha).$$

We assume that the derivative P' has at least two distinct roots; this implies that $d \geq 3$. A straightforward computation shows that $\mathcal{F}$ has no inflection point of maximal order $d-1$, *i.e.* $\mathrm{Flex}(\mathcal{F}, d-1) = \emptyset$. However, $\mathcal{F}$ degenerates onto $\mathcal{F}_2^d$. Indeed, by writing $P(y) = a_0 + a_1 y + \cdots + a_d y^d$, $a_i \in \mathbb{C}$, $a_d \neq 0$, and by putting $\psi = \left(\dfrac{a_d}{\varepsilon^{d+1}}x, \dfrac{1}{\varepsilon}y\right)$, we obtain that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{d+1}}{a_d}\psi^*\omega = \mathrm{d}x + y^d\mathrm{d}y.$$

Question 2. — *Let $\mathcal{F}$ be a foliation of degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$. Is it true that if $\mathcal{F}$ degenerates onto $\mathcal{F}_2^d$ then*
- *either $\mathcal{F}$ possesses an inflection point of maximal order $d-1$,*
- *or $\mathcal{F}$ is conjugated to Example 3.14, i.e. up to linear conjugation $\mathcal{F}$ is given by a 1-form of type $\mathrm{d}x + P(y)\mathrm{d}y$ with $P \in \mathbb{C}[y]$, $\deg P = d$?*

Proposition 3.15. — *Let d be an integer greater than or equal to 2. Let us denote by $U_2(d)$ the set of foliations $\mathcal{F} \in \mathbf{F}(d)$ whose inflection divisor $\mathrm{I}_{\mathcal{F}}$ is transverse (i.e. $\mathrm{I}_{\mathcal{F}} = \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}}$) and reduced. Then*

(i) $U_2(d)$ contains the JOUANOLOU*'s foliation $\mathcal{F}_J^d$ and it is a (non-empty)* ZARISKI *open subset of $\mathbf{F}(d)$;*

(ii) for any $d \geq 3$, every foliation $\mathcal{F} \in U_2(d)$ has a finite number (possibly zero) of transverse inflection points of order greater than or equal to 2; in other words, the set $\bigcup\limits_{k=3}^{d} \mathrm{Flex}(\mathcal{F}, k-1)$ is finite.

To establish this proposition, let us first prove the following lemma.

Lemma 3.16. — *Let $\mathcal{F}$ be a foliation of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$, p a regular point of $\mathcal{F}$ and X a polynomial vector field defining $\mathcal{F}$ in an affine chart (x, y) containing p. Then, for any $k \in \{2, 3, \ldots, d\}$, $\mathrm{Tang}(\mathcal{F}, \mathrm{T}^{\mathbb{P}}_p\mathcal{F}, p) \geq k$ if and only if the matrix $\left.\begin{pmatrix} \mathrm{X}(x) & \mathrm{X}^2(x) & \cdots & \mathrm{X}^k(x) \\ \mathrm{X}(y) & \mathrm{X}^2(y) & \cdots & \mathrm{X}^k(y) \end{pmatrix}\right|_p$ has rank 1.*

Remark 3.17. — If $\mathrm{X} = \sum\limits_{i=1}^{n} \mathrm{X}_i(z_1, \ldots, z_n)\dfrac{\partial}{\partial z_i}$ is a holomorphic vector field on $\mathbb{C}^n$ and if $t \mapsto \alpha(t)$ is an integral curve of X, then we have the following formula which can be easily proved by induction on j :

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}\alpha(t) = (\mathrm{X}^j(z_1), \ldots, \mathrm{X}^j(z_n)) \circ \alpha(t). \tag{3.2}$$

Proof. — Let $t \mapsto \alpha(t)$ be the integral curve of X passing through p at $t = 0$. The point p being regular for $\mathcal{F}$, we have $\mathrm{T}_p\mathcal{F} \ni \alpha'(0) = \mathrm{X}(p) \neq 0$. Up to linear conjugation, we can assume that $p = (0,0)$ and $\mathrm{T}^{\mathbb{P}}_p\mathcal{F} = \{y = 0\}$. We can then write $\alpha(t) = \left(\sum\limits_{i \geq 1} x_i \frac{t^i}{i!}, \sum\limits_{i \geq 1} y_i \frac{t^i}{i!}\right)$ with $y_1 = 0$ and $x_1 \neq 0$.

Thus, $\mathrm{Tang}(\mathcal{F},\mathrm{T}_p^{\mathbb{P}}\mathcal{F},p)=\nu(g(t),0)$, where $g(t)=\sum\limits_{i\geq 2}y_i\frac{t^i}{i!}$. As a result, $\mathrm{Tang}(\mathcal{F},\mathrm{T}_p^{\mathbb{P}}\mathcal{F},p)\geq k$ if and only if $y_2=y_3=\cdots=y_k=0$, or equivalently if and only if the matrix $\begin{pmatrix} x_1 & x_2 & \cdots & x_k\\ 0 & y_2 & \cdots & y_k\end{pmatrix}$ has rank 1. Now, by using formula (3.2), we see that

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_k\\ 0 & y_2 & \cdots & y_k\end{pmatrix}=\begin{pmatrix} \mathrm{X}(x) & \mathrm{X}^2(x) & \cdots & \mathrm{X}^k(x)\\ \mathrm{X}(y) & \mathrm{X}^2(y) & \cdots & \mathrm{X}^k(y)\end{pmatrix}\Big|_{(x,y)=(0,0)},$$

hence the lemma follows. □

Proof of Proposition 3.15. — *(i)* For $\mathcal{F}\in\mathbf{F}(d)$, to say that $\mathrm{I}_{\mathcal{F}}$ is transverse and reduced means that $\mathcal{F}$ has no invariant line and that $\mathrm{I}_{\mathcal{F}}$ has no multiple component, which shows that $U_2(d)$ is a ZARISKI open subset of $\mathbf{F}(d)$.
As we have already mentioned in Example 3.13, the JOUANOLOU's foliation $\mathcal{F}_J^d$ has no invariant algebraic curve [12]; in particular, it has no invariant line and consequently $\mathrm{I}_{\mathcal{F}_J^d}=\mathrm{I}^{\mathrm{tr}}_{\mathcal{F}_J^d}$. To establish the first announced assertion, it remains to prove that $\mathrm{I}_{\mathcal{F}_J^d}$ is reduced. In homogeneous coordinates, the foliation $\mathcal{F}_J^d$ is defined by the vector field $y^d\frac{\partial}{\partial x}+z^d\frac{\partial}{\partial y}+x^d\frac{\partial}{\partial z}$; an immediate computation, using formula (1.1), shows that $\mathrm{I}_{\mathcal{F}_J^d}$ has equation $F(x,y,z)=0$, where

$$F(x,y,z)=x^{2d+1}z^{d-1}+y^{2d+1}x^{d-1}+z^{2d+1}y^{d-1}-3x^dy^dz^d.$$

We must show that F has no multiple factor in $\mathbb{C}[x,y,z]$. Since $F\in\mathbb{Z}[x,y,z]$, it suffices to show that F has no multiple factor in $\mathbb{F}_2[x,y,z]$. Indeed, if F had a multiple factor in $\mathbb{C}[x,y,z]$, then one of the resultants $\mathrm{Res}_x(F,\frac{\partial F}{\partial x})\in\mathbb{Z}[y,z]$ or $\mathrm{Res}_y(F,\frac{\partial F}{\partial y})\in\mathbb{Z}[x,z]$ or $\mathrm{Res}_z(F,\frac{\partial F}{\partial z})\in\mathbb{Z}[x,y]$ would be identically zero and therefore so would be its reduction modulo 2; so that F would also have a multiple factor in $\mathbb{F}_2[x,y,z]$. We have to show that $\gcd(F,\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z})=1$ in $\mathbb{F}_2[x,y,z]$, or equivalently that

$$\gcd(F,\tfrac{\partial F}{\partial x})=1 \text{ in } \mathbb{F}_2(y,z)[x],\qquad \gcd(F,\tfrac{\partial F}{\partial y})=1 \text{ in } \mathbb{F}_2(x,z)[y],\qquad \gcd(F,\tfrac{\partial F}{\partial z})=1 \text{ in } \mathbb{F}_2(x,y)[z].$$

The coordinates x,y,z playing a symmetric role, it suffices again to show that $\gcd(F,\frac{\partial F}{\partial x})=1$ in $\mathbb{F}_2(y,z)[x]$. In $\mathbb{F}_2[x,y,z]$ we have

$$F=x^{2d+1}z^{d-1}+y^{2d+1}x^{d-1}+z^{2d+1}y^{d-1}+x^dy^dz^d\qquad\text{and}\qquad \tfrac{\partial F}{\partial x}=x^{d-2}\left(x^{d+2}z^{d-1}+dxy^dz^d+(d+1)y^{2d+1}\right).$$

Then $x=0$ is not a root of $F\in\mathbb{F}_2(y,z)[x]$ and consequently

$$\mathbb{F}_2(y,z)[x]\ni\gcd(F,\tfrac{\partial F}{\partial x})=\gcd(F,\varphi),\quad\text{where}\quad \varphi=x^{d+2}+dxzy^d+(d+1)\frac{y^{2d+1}}{z^{d-1}}.$$

Moreover, a straightforward computation shows that

$$x^3F=\left(x^{d+2}z^{d-1}-(d-1)xy^dz^d-dy^{2d+1}\right)\varphi+y^{d-1}z^{2d+1}\left(x+\frac{y^{d+1}}{z^d}\right)\left(x^2+(d^2-d-1)\frac{y^{d+1}}{z^d}x+d(d+1)\frac{y^{2d+2}}{z^{2d}}\right),$$

so that

$$\begin{aligned}\mathbb{F}_2(y,z)[x] \ni \gcd(F,\varphi) &= \gcd\left(\left(x+\frac{y^{d+1}}{z^d}\right)\left(x-\frac{y^{d+1}}{z^d}\right),\varphi\right), \text{ because } d^2-d \equiv d(d+1) \equiv 0 \bmod 2\\ &= \gcd\left(x-\frac{y^{d+1}}{z^d},\, x^{d+2}+dxzy^d+(d+1)\frac{y^{2d+1}}{z^{d-1}}\right)\\ &= \gcd\left(x-\frac{y^{d+1}}{z^d},\, x^{d+2}-\frac{y^{2d+1}}{z^{d-1}}\right)\\ &= 1,\end{aligned}$$

because $\left(\frac{y^{d+1}}{z^d}\right)^{d+2} \neq \frac{y^{2d+1}}{z^{d-1}}$ in the field $\mathbb{F}_2(y,z)$. As a result $\mathbb{F}_2(y,z)[x] \ni \gcd(F,\frac{\partial F}{\partial x}) = 1$.

(ii) Let $\mathcal{F}$ be a foliation of degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$ with reduced and transverse inflection divisor $\mathrm{I}_{\mathcal{F}}$, *i.e.* $\mathcal{F} \in U_2(d)$. We want to show that the set $\Gamma(\mathcal{F}) := \bigcup_{k=3}^{d} \mathrm{Flex}(\mathcal{F},k-1)$ is finite. By definition of $\Gamma(\mathcal{F})$ we have

$$\Gamma(\mathcal{F}) \subset \left\{p \in \mathbb{P}^2_{\mathbb{C}} \mid p \notin \mathrm{Sing}(\mathcal{F}),\ \mathrm{Tang}(\mathcal{F},\mathrm{T}^{\mathbb{P}}_p\mathcal{F},p) \geq 3\right\}. \tag{3.3}$$

Let X be a vector field defining $\mathcal{F}$ in an affine chart $\mathbb{C}^2 = \{(x,y)\} \subset \mathbb{P}^2_{\mathbb{C}}$. Lemma 3.16 and inclusion (3.3) imply that $\Gamma(\mathcal{F}) \cap \mathbb{C}^2$ is contained in the set of points $p \in \mathbb{C}^2$ such that

$$\begin{pmatrix}\mathrm{X}(x)\\ \mathrm{X}(y)\end{pmatrix}(p) \neq \begin{pmatrix}0\\0\end{pmatrix}, \quad \mathrm{I}_{\mathrm{X}}(p) := \begin{vmatrix}\mathrm{X}(x) & \mathrm{X}^2(x)\\ \mathrm{X}(y) & \mathrm{X}^2(y)\end{vmatrix}(p) = 0, \quad \mathrm{X}(\mathrm{I}_{\mathrm{X}})(p) = \begin{vmatrix}\mathrm{X}(x) & \mathrm{X}^3(x)\\ \mathrm{X}(y) & \mathrm{X}^3(y)\end{vmatrix}(p) = 0.$$

Now, the affine chart $\mathbb{C}^2 = \{(x,y)\} \subset \mathbb{P}^2_{\mathbb{C}}$ being arbitrary, $\Gamma(\mathcal{F})$ is finite if and only if $\Gamma(\mathcal{F}) \cap \mathbb{C}^2$ is finite. It suffices therefore to show that the algebraic curves $\mathrm{I}_{\mathcal{F}} \cap \mathbb{C}^2 = \{\mathrm{I}_{\mathrm{X}}(x,y) = 0\}$ and $\mathcal{C} := \{\mathrm{X}(\mathrm{I}_{\mathrm{X}})(x,y) = 0\}$ intersect at a finite number of points, *i.e.* that they have no common component. Let us argue by contradiction and assume that there exist $K, L, L' \in \mathbb{C}[x,y]$, with $\deg K > 0$, such that $\mathrm{I}_{\mathrm{X}} = KL$ and $\mathrm{X}(\mathrm{I}_{\mathrm{X}}) = KL'$. Then $KL' = \mathrm{X}(KL) = \mathrm{X}(K)L + K\mathrm{X}(L)$ and therefore $\mathrm{X}(K)L = K(L' - \mathrm{X}(L))$. Moreover, the hypothesis that $\mathrm{I}_{\mathcal{F}}$ is reduced implies that $\gcd(K,L) = 1$. It follows that there is $L'' \in \mathbb{C}[x,y]$ such that $\mathrm{X}(K) = KL''$, which means that the algebraic curve $\mathcal{C}' := \{K(x,y) = 0\}$, contained in $\mathrm{I}_{\mathcal{F}}$, is invariant by $\mathcal{F}$, contradicting the hypothesis that $\mathrm{I}_{\mathcal{F}}$ is transverse. □

Theorem 3.18. — *Let d be an integer greater than or equal to 2. Let us denote by $\Sigma_2(d)$ the subset of $\mathbf{F}(d)$ defined by*

$$\Sigma_2(d) := \left\{\mathcal{F} \in \mathbf{F}(d) \mid \mathrm{Flex}(\mathcal{F},d-1) \neq \emptyset\right\}.$$

Then

(a) $\mathbf{B}(\mathcal{F}_2^2) = \mathbf{F}(2) \setminus \mathbf{FC}(2) = \Sigma_2(2)$ *and, for any $d \geq 3$, we have* $\emptyset \neq \Sigma_2(d) \subsetneq \mathbf{B}(\mathcal{F}_2^d)$;

(b) $\Sigma_2(d)$ *is a constructible subset of* $\mathbf{F}(d)$;

(c) for any $d \geq 3$, we have $\dim \Sigma_2(d) \geq \dim \mathbf{F}(d) - (d-3)$.

In particular, the set $\Sigma_2(3)$, and therefore $\mathbf{B}(\mathcal{F}_2^3)$, contains a non-empty ZARISKI *open subset of* $\mathbf{F}(3)$.

Proof. — *(a)* As we have already said in Introduction, the first equality $\mathbf{B}(\mathcal{F}_2^2) = \mathbf{F}(2) \setminus \mathbf{FC}(2)$ follows from [9, Theorem 3]. The second equality $\mathbf{F}(2) \setminus \mathbf{FC}(2) = \Sigma_2(2)$ is a consequence of the following obvious remark: if $\mathcal{F} \in \mathbf{F}(2) \setminus \mathbf{FC}(2)$ then every transverse inflection point of $\mathcal{F}$ is of order 1.

The set $\Sigma_2(d)$ contains the foliations $\mathcal{H}_2^d$ and $\mathcal{F}_J^d$ (Examples 3.12 and 3.13) and is therefore non-empty. According to assertion **2.** of Proposition 3.11, we have $\Sigma_2(d) \subset \mathbf{B}(\mathcal{F}_2^d)$; this inclusion is strict for any $d \geq 3$ as Example 3.14 shows.

(b) Let $\pi : \mathbf{F}(d) \times \mathbb{P}^2_{\mathbb{C}} \to \mathbf{F}(d)$ be the projection onto the first factor; notice that $\Sigma_2(d) = \pi(W_2(d))$, where

$$\begin{aligned} W_2(d) :&= \bigcup_{\mathcal{F} \in \Sigma_2(d)} \{\mathcal{F}\} \times \mathrm{Flex}(\mathcal{F}, d-1) \\ &= \Big\{(\mathcal{F}, p) \in \mathbf{F}(d) \times \mathbb{P}^2_{\mathbb{C}} \mid p \notin \mathrm{Sing}(\mathcal{F}),\ \mathrm{Tang}(\mathcal{F}, \mathrm{T}_p^{\mathbb{P}}\mathcal{F}, p) = d\Big\}. \end{aligned}$$

By Lemma 3.16, $W_2(d)$ can be rewritten as

$$W_2(d) = \left\{(\mathcal{F}, p) \in \mathbf{F}(d) \times \mathbb{P}^2_{\mathbb{C}} \mid \begin{pmatrix} \mathrm{X}(x) \\ \mathrm{X}(y) \end{pmatrix}(p) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{vmatrix} \mathrm{X}(x) & \mathrm{X}^j(x) \\ \mathrm{X}(y) & \mathrm{X}^j(y) \end{vmatrix}(p) = 0, j = 2, \ldots, d\right\}, \tag{3.4}$$

where X denotes a polynomial vector field defining $\mathcal{F}$ in an affine chart (x,y) containing p. It follows that $W_2(d)$ is a quasi-projective subvariety of $\mathbf{F}(d) \times \mathbb{P}^2_{\mathbb{C}}$. Therefore, by CHEVALLEY's theorem [11, Exercise II.3.19], the set $\Sigma_2(d) = \pi(W_2(d))$ is constructible.

(c) From the above discussion and Proposition 3.15 *(i)*, we have $\mathcal{F}_J^d \in U_2(d) \cap \Sigma_2(d) \neq \emptyset$ ($U_2(d)$ being the set of foliations of $\mathbf{F}(d)$ with reduced and transverse inflection divisor). Therefore there exists an irreducible component $\Sigma_2^0(d)$ of $\Sigma_2(d)$ such that $U_2(d) \cap \Sigma_2^0(d) \neq \emptyset$. We denote by $\pi_0 : W_2(d) \to \mathbf{F}(d)$ the restriction of π to $W_2(d)$. Let $W_2(d) = \bigcup_{i=1}^{n} W_2^i(d)$ be the decomposition of $W_2(d)$ into its irreducible components. Then, by arguing as in the proof of Theorem 3.10, we see that there is $k \in \{1, \ldots, n\}$ such that $\overline{\pi_0(W_2^k(d))} = \overline{\Sigma_2^0(d)}$. Since $U_2(d)$ is a ZARISKI open subset of $\mathbf{F}(d)$ (Proposition 3.15 *(i)*), the morphism π_0 therefore induces by restriction a dominant morphism of quasi-projective varieties $\pi_0^k : W_2^k(d) \cap \pi_0^{-1}(U_2(d)) \to \overline{\Sigma_2^0(d)} \cap U_2(d)$. Notice that, for any $\mathcal{F} \in U_2(d) \cap \Sigma_2(d)$, the fiber $\pi_0^{-1}(\mathcal{F})$ is finite and non-empty, because $\pi_0^{-1}(\mathcal{F}) = \{\mathcal{F}\} \times \mathrm{Flex}(\mathcal{F}, d-1)$ and $\mathrm{Flex}(\mathcal{F}, d-1)$ is finite and non-empty by assertion *(ii)* of Proposition 3.15. Since $\pi_0(W_2^k(d) \cap \pi_0^{-1}(U_2(d))) \subset U_2(d) \cap \Sigma_2(d)$, we deduce that all the non-empty fibers of π_0^k are finite and therefore zero-dimensional. The fiber dimension theorem (*cf.* [15, Theorem 3, page 49]) then ensures that $\dim(W_2^k(d) \cap \pi_0^{-1}(U_2(d))) = \dim(\overline{\Sigma_2^0(d)} \cap U_2(d))$; since $W_2^k(d) \cap \pi_0^{-1}(U_2(d))$ and $\overline{\Sigma_2^0(d)} \cap U_2(d)$ are non-empty open subsets of the irreducible varieties $W_2^k(d)$ and $\overline{\Sigma_2^0(d)}$ respectively, we have

$$\dim \overline{\Sigma_2^0(d)} = \dim(\overline{\Sigma_2^0(d)} \cap U_2(d)) = \dim(W_2^k(d) \cap \pi_0^{-1}(U_2(d))) = \dim W_2^k(d).$$

Now, it follows from (3.4) that each irreducible component $W_2^i(d)$ of $W_2(d)$ has dimension

$$\dim W_2^i(d) \geq \dim(\mathbf{F}(d) \times \mathbb{P}^2_{\mathbb{C}}) - (d-1) = \dim \mathbf{F}(d) - (d-3),$$

hence

$$\dim \Sigma_2(d) = \dim \overline{\Sigma_2(d)} \geq \dim \overline{\Sigma_2^0(d)} = \dim W_2^k(d) \geq \dim \mathbf{F}(d) - (d-3).$$

The subset $\Sigma_2(d) \subset \mathbf{F}(d)$ being constructible, it contains a dense open subset of its closure $\overline{\Sigma_2(d)}$. In degree $d = 3$ we have $\dim \overline{\Sigma_2(3)} \geq \dim \mathbf{F}(3)$ and therefore $\dim \overline{\Sigma_2(3)} = \dim \mathbf{F}(3)$, so that $\overline{\Sigma_2(3)} = \mathbf{F}(3)$ because $\mathbf{F}(3)$ is irreducible. It follows that $\Sigma_2(3)$ contains a dense open subset of $\mathbf{F}(3)$. This ends the proof of the theorem. □

Remark 3.19. — The set $\mathbf{F}(d)$ contains elements which degenerate onto both $\mathcal{F}_1^d$ and $\mathcal{F}_2^d$, *e.g.* the family of foliations $\mathcal{G}^d(\gamma)$, $\gamma\in\mathbb{C}$. Indeed, on the one hand, we have seen (Example 3.6) that $\mathcal{G}^d(\gamma)$ degenerates onto $\mathcal{F}_1^d$. On the other hand, by putting $\varphi=(\frac{x}{\varepsilon},\frac{y}{\varepsilon})$ we obtain that $\lim\limits_{\varepsilon\to0}\varepsilon^{d+1}\varphi^*\eta^d(\gamma)=\bar{\omega}_2^d$, which shows that $\mathcal{G}^d(\gamma)$ degenerates onto the homogeneous foliation $\mathcal{H}_2^d$ (Example 3.12) and therefore, by transitivity, onto $\mathcal{F}_2^d$.

Example 3.20. — Let us consider the homogeneous foliation $\mathcal{H}_{1,2}^d$ defined in the affine chart $z=1$ by the 1-form

$$\bar{\omega}_{1,2}^d=(x^d+y^d)\mathrm{d}x+x^d\mathrm{d}y.$$

This foliation degenerates onto both $\mathcal{F}_1^d$ and $\mathcal{F}_2^d$. Indeed, on the one hand, $\mathcal{H}_{1,2}^d$ is given in the affine chart $y=1$ by

$$\bar{\theta}_{1,2}^d=x\mathrm{d}z-z\mathrm{d}x+x^d\mathrm{d}z+x^d(x\mathrm{d}z-z\mathrm{d}x);$$

we see that the point $[0:1:0]$ is a radial singularity of maximal order $d-1$ of $\mathcal{H}_{1,2}^d$. Thus, by Proposition 3.4, $\mathcal{H}_{1,2}^d$ degenerates onto $\mathcal{F}_1^d$. On the other hand, a straightforward computation shows that

$$\mathrm{Flex}(\mathcal{H}_{1,2}^d,d-1)=\{y=0\}\setminus\{[0:0:1]\}\neq\emptyset;$$

consequently, $\mathcal{H}_{1,2}^d$ also degenerates onto $\mathcal{F}_2^d$ (Proposition 3.11).
Since $\overline{\mathcal{O}(\mathcal{H}_{1,2}^d)}\subset\mathcal{O}(\mathcal{H}_{1,2}^d)\cup\mathcal{O}(\mathcal{F}_1^d)\cup\mathcal{O}(\mathcal{F}_2^d)$ (Remark 3.3), we deduce that in fact

$$\overline{\mathcal{O}(\mathcal{H}_{1,2}^d)}=\mathcal{O}(\mathcal{H}_{1,2}^d)\cup\mathcal{O}(\mathcal{F}_1^d)\cup\mathcal{O}(\mathcal{F}_2^d).$$

Theorem 3.21. — *Let d be an integer greater than or equal to 2. Then*
(a) $\emptyset\neq\Sigma_1(d)\cap\Sigma_2(d)\subset\mathbf{B}(\mathcal{F}_1^d)\cap\mathbf{B}(\mathcal{F}_2^d)\supset\mathbf{B}(\mathcal{H}_{1,2}^d)$;
(b) $\mathbf{B}(\mathcal{H}_{1,2}^d)$ *contains a quasi-projective subvariety of* $\mathbf{F}(d)$ *of dimension equal to* $\dim\mathbf{F}(d)-3d$.

Proof. — *(a)* The intersection $\Sigma_1(d)\cap\Sigma_2(d)$ contains the homogeneous foliation $\mathcal{H}_{1,2}^d$ (Example 3.20) and is therefore non-empty. The inclusion $\Sigma_1(d)\cap\Sigma_2(d)\subset\mathbf{B}(\mathcal{F}_1^d)\cap\mathbf{B}(\mathcal{F}_2^d)$ follows from Theorems 3.10 and 3.18. Let us show the inclusion $\mathbf{B}(\mathcal{H}_{1,2}^d)\subset\mathbf{B}(\mathcal{F}_1^d)\cap\mathbf{B}(\mathcal{F}_2^d)$. Let $\mathcal{F}\in\mathbf{B}(\mathcal{H}_{1,2}^d)$, *i.e.* $\mathcal{F}\in\mathbf{F}(d)$ such that $\mathcal{H}_{1,2}^d\in\overline{\mathcal{O}(\mathcal{F})}$. Since $\mathcal{H}_{1,2}^d$ degenerates onto $\mathcal{F}_i^d$, $i=1,2$, it follows that $\mathcal{F}_i^d\in\overline{\mathcal{O}(\mathcal{H}_{1,2}^d)}\subset\overline{\mathcal{O}(\mathcal{F})}$, hence $\mathcal{F}\in\mathbf{B}(\mathcal{F}_1^d)\cap\mathbf{B}(\mathcal{F}_2^d)$.

(b) Let us denote by $\Sigma(\mathcal{H}_{1,2}^d)$ the subset of $\mathbf{F}(d)$ defined as follows: an element $\mathcal{F}$ of $\mathbf{F}(d)$ belongs to $\Sigma(\mathcal{H}_{1,2}^d)$ if and only if
(1) $\mathcal{F}$ admits an invariant line ℓ;
(2) there is a system of homogeneous coordinates $[x:y:z]\in\mathbb{P}^2_{\mathbb{C}}$ in which $\ell=\{z=0\}$ and $\mathcal{F}$ is defined in the affine chart $z=1$ by a 1-form ω of type

$$\omega=\sum_{i=0}^{d-1}\omega_i+\lambda\bar{\omega}_{1,2}^d=\sum_{i=0}^{d-1}\omega_i+\lambda\left((x^d+y^d)\mathrm{d}x+x^d\mathrm{d}y\right),$$

where $\lambda\in\mathbb{C}^*$ and the ω_i's are homogeneous 1-forms of degree i.

Notice that $\Sigma(\mathcal{H}_{1,2}^d)\subset\mathbf{B}(\mathcal{H}_{1,2}^d)$. Indeed, by putting $\varphi=(\frac{x}{\varepsilon},\frac{y}{\varepsilon})$ and by writing $\omega_i=P_i(x,y)\mathrm{d}x+Q_i(x,y)\mathrm{d}y$, where $P_i,Q_i\in\mathbb{C}[x,y]_i$, we obtain

$$\varepsilon^{d+1}\varphi^*\omega=\sum_{i=0}^{d-1}(\varepsilon^{d-i}P_i(x,y)\mathrm{d}x+\varepsilon^{d-i}Q_i(x,y)\mathrm{d}y)+\lambda\bar{\omega}_{1,2}^d$$

which tends to $\lambda\overline{\omega}_{1,2}^{d}$ as ε tends to 0. It follows that $\mathcal{H}_{1,2}^{d}\in\overline{O(\mathcal{F})}$ for any $\mathcal{F}\in\Sigma(\mathcal{H}_{1,2}^{d})$, hence the inclusion $\Sigma(\mathcal{H}_{1,2}^{d})\subset\mathbf{B}(\mathcal{H}_{1,2}^{d})$ holds.
Moreover, every foliation $\mathcal{F}\in\mathbf{F}(d)$ is given in the affine chart $z=1$ by a 1-form of type

$$\sum_{i=0}^{d}\big(A_i(x,y)\mathrm{d}x+B_i(x,y)\mathrm{d}y\big)+C_d(x,y)(x\mathrm{d}y-y\mathrm{d}x),$$

where $A_i,B_i\in\mathbb{C}[x,y]_i, C_d\in\mathbb{C}[x,y]_d$ with $\gcd\big(yC_d-\sum_{i=0}^{d}A_i, xC_d+\sum_{i=0}^{d}B_i\big)=1$. Condition *(2)* is then equivalent to taking $C_d\equiv 0, A_d(x,y)=\lambda(x^d+y^d), B_d(x,y)=\lambda x^d$. Since the set of foliations of $\mathbf{F}(d)$ admitting an invariant line is a ZARISKI closed subset of $\mathbf{F}(d)$, we deduce that $\Sigma(\mathcal{H}_{1,2}^{d})$ is a quasi-projective subvariety of $\mathbf{F}(d)$. Since ω and $\mu\omega$ define the same foliation if $\mu\neq 0$, and the choice of a line $\ell\subset\mathbb{P}^2_{\mathbb{C}}$ is equivalent to the choice of a point in $\check{\mathbb{P}}^2_{\mathbb{C}}$, conditions *(1)* and *(2)* imply that

$$\dim\Sigma(\mathcal{H}_{1,2}^{d})=2+2\sum_{i=0}^{d-1}(i+1)=d^2+d+2=\dim\mathbf{F}(d)-3d.$$

□

4. A family of foliations of $\mathbf{F}(d)$ with orbits of dimension less than or equal to 7

In this section we will establish some properties of the family $(\mathcal{F}_0^d(\lambda))_{\lambda\in\mathbb{C}^*}$ of foliations of degree d on $\mathbb{P}^2_{\mathbb{C}}$ defined in the affine chart $z=1$ by

$$\omega_0^d(\lambda)=x\mathrm{d}y-\lambda y\mathrm{d}x+y^d\mathrm{d}y.$$

In homogeneous coordinates, $\mathcal{F}_0^d(\lambda)$ is given by

$$\Omega_0^d(\lambda)=-\lambda yz^d\mathrm{d}x+z\left(xz^{d-1}+y^d\right)\mathrm{d}y+y\left((\lambda-1)xz^{d-1}-y^d\right)\mathrm{d}z.$$

Thus, the singular locus of $\mathcal{F}_0^d(\lambda)$ consists of the two points $s_1=[0:0:1]$ and $s_2=[1:0:0]$. The singularity s_1 is non-degenerate with BAUM-BOTT index $\mathrm{BB}(\mathcal{F}_0^d(\lambda),s_1)=2+\lambda+\frac{1}{\lambda}$ and the singular point s_2 has maximal algebraic multiplicity d. We see that for $\lambda=1$ the 1-form $\Omega_0^d(1)$ writes in the affine chart $x=1$ as

$$z^d\mathrm{d}y+y^d(z\mathrm{d}y-y\mathrm{d}z);$$

we deduce that $\mathcal{F}_0^d(1)$ is conjugated to the foliation $\mathcal{F}_1^d$ and is therefore convex.
In the sequel we assume that $\lambda\in\mathbb{C}\setminus\{0,1\}$. A direct computation, using formula (1.1), leads to

$$\mathrm{I}^{\mathrm{inv}}_{\mathcal{F}_0^d(\lambda)}=yz^{2d-1}\qquad\text{and}\qquad\mathrm{I}^{\mathrm{tr}}_{\mathcal{F}_0^d(\lambda)}=(\lambda-1)x-\big((d-1)\lambda+1\big)y^d;\tag{4.1}$$

it follows that, for any $\lambda\in\mathbb{C}\setminus\{0,1\}$, $\mathcal{F}_0^d(\lambda)$ is not convex.
A straightforward computation shows that the algebraic curve $(1-\lambda d)x+y^d=0$ is invariant by $\mathcal{F}_0^d(\lambda)$. What is more, the rational 1-form $\eta_0^d(\lambda)=\dfrac{\omega_0^d(\lambda)}{y\left((1-\lambda d)x+y^d\right)}$ is closed. For $\lambda=\frac{1}{d}$ we note that $\eta_0^d(\frac{1}{d})=\dfrac{\omega_0^d(\lambda)}{y^{d+1}}$ has as first integral $\dfrac{x}{dy^d}-\ln y$; this allows to see that $\mathrm{Iso}(\mathcal{F}_0^d(\frac{1}{d}))$ is the group $\{(\alpha^d x,\alpha y)\mid\alpha\in\mathbb{C}^*\}$.
When $\lambda\in\mathbb{C}\setminus\{0,1,\frac{1}{d}\}$ a straightforward computation shows that $\eta_0^d(\lambda)$ integrates into

$$\lambda\ln\left((1-\lambda d)x+y^d\right)-\ln y,$$

which allows to verify that the isotropy group is here again
$$\mathrm{Iso}(\mathcal{F}_0^d(\lambda)) = \{(\alpha^d x, \alpha y) \mid \alpha \in \mathbb{C}^*\}.$$
It follows in particular that, for any $\lambda \in \mathbb{C} \setminus \{0,1\}$, $\mathcal{O}(\mathcal{F}_0^d(\lambda))$ has dimension 7.

Notice that two foliations $\mathcal{F}_0^d(\lambda)$ and $\mathcal{F}_0^d(\lambda')$ are conjugated if and only if $\lambda = \lambda'$.

Proposition 4.1. — *Let λ be a nonzero complex number. Let $\mathcal{F}$ be an element of $\mathbf{F}(d)$ such that $\mathcal{F}_0^d(\lambda) \notin \mathcal{O}(\mathcal{F})$.*
1. *If $\mathcal{F}$ degenerates onto $\mathcal{F}_0^d(\lambda)$, then $\mathcal{F}$ admits a non-degenerate singular point m satisfying $\mathrm{BB}(\mathcal{F}, m) = 2 + \lambda + \frac{1}{\lambda}$.*
2. *If $\mathcal{F}$ possesses a non-degenerate singular point m such that*
$$\mathrm{BB}(\mathcal{F}, m) = 2 + \lambda + \frac{1}{\lambda} \qquad \textit{and} \qquad \kappa(\mathcal{F}, m) = d,$$
then $\mathcal{F}$ degenerates onto $\mathcal{F}_0^d(\lambda)$.

Proof. — It suffices to argue as in the proof of Proposition 3.4, replacing the foliation $\mathcal{F}_1^d$ by $\mathcal{F}_0^d(\lambda)$ and the equality $\mathrm{BB}(\mathcal{F}, m) = 4$ by $\mathrm{BB}(\mathcal{F}, m) = 2 + \lambda + \frac{1}{\lambda}$. □

Proposition 4.2. — *The orbit $\mathcal{O}\big(\mathcal{F}_0^d(\lambda)\big)$ is closed in $\mathbf{F}(d)$ in the following two cases:*

(i) $d \geq 3$ and $\lambda = -\dfrac{1}{d-1}$;

(ii) $d \in \{3,4,5\}$ and $\lambda \in \mathbb{C}^*$.

The proof of this proposition uses the following lemma.

Lemma 4.3. — *Let λ be a nonzero complex number. Then, the orbit $\mathcal{O}\big(\mathcal{F}_0^d(\lambda)\big)$ is closed in $\mathbf{F}(d)$ if and only if $\mathcal{F}_0^d(\lambda)$ does not degenerate onto $\mathcal{F}_2^d$.*

Proof. — The direct implication is obvious. Let us prove the converse. From the above discussion, $\mathcal{F}_0^d(1)$ is conjugated to the convex foliation $\mathcal{F}_1^d$; therefore its orbit $\mathcal{O}\big(\mathcal{F}_0^d(1)\big)$ is closed in $\mathbf{F}(d)$. For any $\lambda \in \mathbb{C} \setminus \{0,1\}$, the unique non-degenerate singular point $s_1 = [0:0:1]$ of $\mathcal{F}_0^d(\lambda)$ has BAUM-BOTT index $\mathrm{BB}(\mathcal{F}_0^d(\lambda), s_1) = 2 + \lambda + \frac{1}{\lambda} \neq 4$; this implies, according to assertion ***1.*** of Proposition 3.4, that $\mathcal{F}_0^d(\lambda)$ does not degenerate onto $\mathcal{F}_1^d$. Moreover, for any $\lambda \in \mathbb{C} \setminus \{0,1\}$, $\mathcal{O}(\mathcal{F}_0^d(\lambda))$ has dimension 7. The converse implication then follows immediately from Corollary B. □

Proof of Proposition 4.2. — *(i)* Let us put $\lambda_0 = -\frac{1}{d-1}$; according to (4.1) we have $\mathrm{I}^{\mathrm{tr}}_{\mathcal{F}_0^d(\lambda_0)} = (\lambda_0 - 1)x$, hence $\deg \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}_0^d(\lambda_0)} = 1 < d - 1$ for any $d \geq 3$. According to the first assertion of Proposition 3.11, it follows that, for any $d \geq 3$, the foliation $\mathcal{F}_0^d(\lambda_0)$ does not degenerate onto $\mathcal{F}_2^d$, so that its orbit $\mathcal{O}\big(\mathcal{F}_0^d(\lambda_0)\big)$ is closed in $\mathbf{F}(d)$ (Lemma 4.3).

(ii) Let $[x:y:z]$ be homogeneous coordinates in $\mathbb{P}^2_{\mathbb{C}}$. For $n \in \mathbb{N}$, let us denote by Λ^1_n the $\mathbb{C}$-vector space of 1-forms in the variables x, y, z, whose coefficients are homogeneous polynomials of degree n. Let us put $\alpha = y\mathrm{d}z - z\mathrm{d}y$, $\beta = z\mathrm{d}x - x\mathrm{d}z$ and $\gamma = x\mathrm{d}y - y\mathrm{d}x$. We have the identification
$$\begin{aligned}\mathbf{F}(d) &= \big\{[\Omega] \in \mathbb{P}(\Lambda^1_{d+1}) \mid \Omega = p\mathrm{d}x + q\mathrm{d}y + r\mathrm{d}z,\ p,q,r \in \mathbb{C}[x,y,z]_{d+1},\ xp + yq + zr = 0, \gcd(p,q,r) = 1\big\} \\ &= \big\{[\Omega] \in \mathbb{P}(\Lambda^1_{d+1}) \mid \Omega = A\alpha + B\beta + C\gamma,\ A, B \in \mathbb{C}[x,y,z]_d,\ C \in \mathbb{C}[x,y]_d,\ \gcd\big(yA - xB, zB - yC, xC - zA\big) = 1\big\}.\end{aligned}$$

By writting

$$\begin{aligned}
A &= \xi_1 x^d + \xi_3 x^{d-1}y + \cdots + \xi_{2d+1}y^d + \left(\xi_{2d+3}x^{d-1} + \xi_{2d+5}x^{d-2}y + \cdots + \xi_{4d+1}y^{d-1}\right)z + \left(\xi_{4d+3}x^{d-2} + \xi_{4d+5}x^{d-3}y + \cdots + \xi_{6d-1}y^{d-2}\right)z^2 + \cdots + \xi_{d^2+3d+1}z^d,\\
B &= \xi_2 x^d + \xi_4 x^{d-1}y + \cdots + \xi_{2d+2}y^d + \left(\xi_{2d+4}x^{d-1} + \xi_{2d+6}x^{d-2}y + \cdots + \xi_{4d+2}y^{d-1}\right)z + \left(\xi_{4d+4}x^{d-2} + \xi_{4d+6}x^{d-3}y + \cdots + \xi_{6d}y^{d-2}\right)z^2 + \cdots + \xi_{d^2+3d+2}z^d,\\
C &= \xi_{d^2+3d+3}x^d + \xi_{d^2+3d+4}x^{d-1}y + \xi_{d^2+3d+5}x^{d-2}y^2 + \cdots + \xi_{d^2+4d+2}xy^{d-1} + \xi_{d^2+4d+3}y^d,
\end{aligned}$$

we can identify the class $[\Omega]$ of $\Omega = A\alpha + B\beta + C\gamma$ to the element $[\xi_1 : \xi_2 : \cdots : \xi_{d^2+4d+3}] \in \mathbb{P}_{\mathbb{C}}^{d^2+4d+2}$. Thus, we can identify $\mathbf{F}(d)$ with the ZARISKI open set:

$$\left\{ [\xi_1 : \xi_2 : \cdots : \xi_{d^2+4d+3}] \in \mathbb{P}_{\mathbb{C}}^{d^2+4d+2} \;\middle|\; \begin{array}{l} A = \xi_1 x^d + \xi_3 x^{d-1}y + \cdots + \xi_{2d+1}y^d + (\xi_{2d+3}x^{d-1} + \xi_{2d+5}x^{d-2}y + \cdots + \xi_{4d+1}y^{d-1})z + \cdots + \xi_{d^2+3d+1}z^d \\ B = \xi_2 x^d + \xi_4 x^{d-1}y + \cdots + \xi_{2d+2}y^d + (\xi_{2d+4}x^{d-1} + \xi_{2d+6}x^{d-2}y + \cdots + \xi_{4d+2}y^{d-1})z + \cdots + \xi_{d^2+3d+2}z^d \\ C = \xi_{d^2+3d+3}x^d + \xi_{d^2+3d+4}x^{d-1}y + \xi_{d^2+3d+5}x^{d-2}y^2 + \cdots + \xi_{d^2+4d+2}xy^{d-1} + \xi_{d^2+4d+3}y^d \\ \gcd\left(yA - xB, zB - yC, xC - zA\right) = 1 \end{array} \right\}.$$

Then, via this identification, we have

$$\mathcal{F}_2^d = \left[\Omega_2^d\right] = \left[x^d\beta + y^d\gamma\right] = \left[0 : 1 : 0 : 0 : \cdots : 0 : 0 : 1\right]$$

and

$$\mathcal{F}_0^d(\lambda) = \left[\Omega_0^d(\lambda)\right] = \left[(y^d + xz^{d-1})\alpha + \lambda yz^{d-1}\beta\right] = \big[\underbrace{0 : 0 : \cdots : 0}_{2d} : 1 : \underbrace{0 : 0 : \cdots : 0}_{d^2+d-5} : 1 : 0 : 0 : \lambda : \underbrace{0 : 0 : \cdots : 0}_{d+3}\big].$$

In addition, the orbit of a foliation $\mathcal{F} = [\Omega] \in \mathbf{F}(d)$ is

$$O(\mathcal{F}) = \left\{ [\varphi^*\Omega] \;\middle|\; \varphi = [a_1x + a_2y + a_3z : a_4x + a_5y + a_6z : a_7x + a_8y + a_9z] \in \mathrm{Aut}(\mathbb{P}_{\mathbb{C}}^2) \right\}.$$

Let $[x_1 : x_2 : \cdots : x_{d^2+4d+3}]$ be a system of homogeneous coordinates in $\mathbb{P}_{\mathbb{C}}^{d^2+4d+2}$. For $d = 3$, let us consider the following homogeneous polynomial in $x_1, x_2, \ldots, x_{24}$ of degree 5:

$$\begin{aligned}
P_3 = {} & -90x_2\Big(x_1(294x_1 - 269x_4) + 10x_2(29x_3 + 4x_6) + 86x_4^2\Big)x_{22}x_{24} - 1125x_2^2(21x_1 - 23x_4)x_{23}x_{24}\\
& + 45x_2\Big(2x_3(294x_1 + 13x_4) - x_6(552x_1 - 271x_4) + 1125x_2x_5\Big)x_{21}x_{24} + 28125x_2x_{10}x_{21}x_{23}x_{24}\\
& + 25\Big(108(x_9 - 2x_{12})(3x_1 - 4x_4) + 9x_{10}(112x_3 - 93x_6) + 675x_2x_{11}\Big)x_{21}^2x_{24} - 6000x_2x_{10}x_{22}^2x_{24}\\
& - 5625x_5x_{11}x_{21}^3 + 20\Big((2x_1 - x_4)(41x_9 - 7x_{12}) + 30x_{10}(2x_3 - 3x_6) + 50x_2x_{11}\Big)x_{22}^3 - 50625x_2^3x_{24}^2\\
& - 5\Big(2x_9(207x_1 - 116x_4) - x_{12}(153x_1 - 314x_4) + 5x_{10}(356x_3 - 359x_6) + 1350x_2x_{11}\Big)x_{21}x_{22}x_{23}\\
& + 1875\Big(x_{11}(2x_3 - x_6) + x_5(2x_9 - x_{12})\Big)x_{21}^2x_{22} - 375x_2\Big(2x_1(3x_1 - 7x_4) - x_2(3x_3 - 2x_6) + 8x_4^2\Big)x_{23}^2\\
& + 50\Big(5x_{10}(39x_1 - 38x_4) - 3x_2(x_9 - 32x_{12})\Big)x_{21}x_{23}^2 - 50\Big(x_{10}(14x_1 - 37x_4) - 3x_2(7x_9 + x_{12})\Big)x_{22}^2x_{23}\\
& + 15\Big(5x_{11}(21x_1 + 22x_4) - 8x_3(14x_9 - 43x_{12}) + 6x_6(13x_9 - 56x_{12}) - 350x_5x_{10}\Big)x_{21}^2x_{23} + Rx_{21}^2\\
& - 5\Big(20x_{11}(24x_1 - 7x_4) + 4x_9(97x_3 - 43x_6) + x_{12}(94x_3 - 211x_6) - 600x_5x_{10}\Big)x_{21}x_{22}^2 + Sx_{21}x_{22}\\
& - 75\Big(2x_{10}(78x_1 - 29x_4) - 15x_2(2x_9 - 19x_{12})\Big)x_{21}x_{22}x_{24} + 125x_2x_{10}x_{22}x_{23}^2 + Tx_{22}^2 + Ux_{21}x_{23}\\
& + Vx_{22}x_{23},
\end{aligned}$$

where

$$\begin{aligned}
R &= 5568x_6x_5\,(3x_1-4x_4)-18x_3x_5\,(1612x_1-1941x_4)+6x_3^2\,(1952x_3-4389x_6)+3x_6^2\,(7057x_3-2136x_6)-11250x_2x_5^2\\
&\quad+2700x_7\,(3x_1-4x_4)^2+54x_8\,(3x_1-4x_4)\,(106x_3-89x_6)\,,\\
S &= 27000x_2x_7\,(3x_1-4x_4)-24x_3^2\,(658x_1-249x_4)+1512x_4x_8\,(11x_1-4x_4)+252x_1^2\,(83x_5-36x_8)-90x_2x_3\,(329x_5-318x_8)\\
&\quad-2x_4x_5\,(17073x_1-6047x_4)+3x_1x_6\,(8712x_3-3599x_6)-x_4x_6\,(11658x_3-6041x_6)+90x_2x_6\,(226x_5-267x_8)\,,\\
T &= 20x_1x_3\,(294x_1-253x_4)-40x_1x_6\,(159x_1-152x_4)+1900x_2x_3\,(x_3-x_6)+20x_4^2\,(68x_3-95x_6)-25x_2x_6\,(40x_3-33x_6)\\
&\quad+60x_1x_2\,(361x_5-252x_8)-10x_2x_4\,(983x_5-756x_8)+67500x_2^2x_7\,,\\
U &= 90x_1x_3\,(98x_1-117x_4)-30x_1x_6\,(171x_1-284x_4)-150x_2x_6\,(68x_3-35x_6)-30x_2x_4\,(167x_5+396x_8)+7050x_2x_3^2\\
&\quad+20x_4^2\,(73x_3-157x_6)+270x_1x_2\,(41x_5+33x_8)\,,\\
V &= 5x_2x_4\,(1604x_3-611x_6)-30x_1^2\,(294x_1-563x_4)-30x_4^2\,(355x_1-86x_4)-30x_1x_2\,(463x_3-242x_6)-75x_2^2\,(109x_5-198x_8)\,.
\end{aligned}$$

A computation carried out with `Maple` shows that evaluating P_3 at an arbitrary element $[\xi_1:\xi_2:\cdots:\xi_{24}]$ of $\mathcal{O}\big(\mathcal{F}_0^3(\lambda)\big)$, we find $P_3\big([\xi_1:\xi_2:\cdots:\xi_{24}]\big)=0$, *i.e.* $\mathcal{O}\big(\mathcal{F}_0^3(\lambda)\big)$ is contained in the zero locus of P_3

$$\mathrm{Zeros}(P_3):=\Big\{[x_1:x_2:\cdots:x_{24}]\in\mathbb{P}_{\mathbb{C}}^{23}\mid P_3\big([x_1:x_2:\cdots:x_{24}]\big)=0\Big\}\,,$$

which is a ZARISKI closed subset of $\mathbb{P}_{\mathbb{C}}^{23}$. Therefore we have $\overline{\mathcal{O}\big(\mathcal{F}_0^3(\lambda)\big)}\subset\mathrm{Zeros}(P_3)$ for any $\lambda\in\mathbb{C}^*$. Moreover, we have

$$P_3\,(0,1,0,0,\cdots,0,0,1)=-50625\neq 0,$$

hence $\mathcal{F}_2^3\not\in\mathrm{Zeros}(P_3)$. It follows that, for any $\lambda\in\mathbb{C}^*$, we have $\mathcal{F}_2^3\not\in\overline{\mathcal{O}\big(\mathcal{F}_0^3(\lambda)\big)}$, so that $\mathcal{F}_0^3(\lambda)$ does not degenerate onto $\mathcal{F}_2^3$. Consequently, according to Lemma 4.3, the orbit $\mathcal{O}\big(\mathcal{F}_0^3(\lambda)\big)$ is closed in $\mathbf{F}(3)$.
To show that the orbit $\mathcal{O}\big(\mathcal{F}_0^4(\lambda)\big)$, resp. $\mathcal{O}\big(\mathcal{F}_0^5(\lambda)\big)$, is closed in $\mathbf{F}(4)$, resp. $\mathbf{F}(5)$, it suffices to argue as in degree $d=3$, replacing the polynomial P_3 by the following polynomial P_4, resp. P_5:

$$\begin{aligned}
P_4 &= \Big(3x_3\,(129x_3-212x_6)+3x_4\,(178x_5+15x_8)+12x_1\,(22x_5-3x_8)+5184x_2x_7-20x_6^2\Big)x_{31}+1728x_{15}x_{31}^2\\
&\quad-432\,(2x_{13}-x_{16})\,x_{31}x_{32}+48\,(42x_{11}-31x_{14})\,x_{31}x_{33}-18\,(24x_{11}-19x_{14})\,x_{32}^2-162x_2\,(4x_1-15x_4)\,x_{34}\\
&\quad-18\Big(2x_1\,(27x_3-20x_6)-x_4\,(15x_3-x_6)+x_2\,(170x_5-69x_8)\Big)x_{32}+4212x_{12}x_{31}x_{34}-486x_{12}x_{32}x_{33}\\
&\quad+36\Big(3\,(x_1-x_4)\,(12x_1-x_4)+22x_2\,(3x_3-2x_6)\Big)x_{33}-10368x_2^2x_{35}\,,
\end{aligned}$$

resp.
$$\begin{aligned}
P_5 &= \Big(50x_7\,(4906x_1-4749x_4)-27040x_{10}\,(5x_1-6x_4)-5x_5\,(10596x_3-13469x_6)+20x_8\,(1019x_3-2028x_6)\\
&\quad+569100x_2x_9\Big)x_{43}+142275x_{19}x_{43}^2-11690x_{17}x_{43}x_{44}+98140x_{14}x_{43}x_{47}-140x_2\,(2180x_1-1691x_4)\,x_{47}\\
&\quad+35\,(1564x_{13}-1645x_{16})\,x_{43}x_{46}+\Big(8620x_8\,(2x_1-x_4)-50x_5\,(141x_1-11x_4)+10x_3\,(513x_3-1580x_6)\\
&\quad+70x_2\,(2779x_7-2704x_{10})+9875x_6^2\Big)x_{44}-35\Big(\,(x_1-x_4)\,(295x_1+683x_4)-x_2\,(3776x_3-4427x_6)\Big)x_{46}\\
&\quad+70\,(323x_{18}-253x_{15})\,x_{43}x_{45}+7\,(686x_{13}-293x_{16})\,x_{44}x_{45}-2975x_{15}x_{44}^2-15946x_{14}x_{45}^2-1422750x_2^2x_{48}\\
&\quad+\Big(14x_3\,(15x_1+1124x_4)-14x_6\,(10x_1+1129x_4)-595x_2\,(221x_5-250x_8)\Big)x_{45}+49210x_{14}x_{44}x_{46}.
\end{aligned}$$

$\square$

For $d\geq 6$, we propose:

Conjecture 1. — *Let d be an integer greater than or equal to 6 and λ a nonzero complex number. A homogeneous coordinate system $[x_1:x_2:\cdots:x_{d^2+4d+3}]$ being fixed in $\mathbb{P}_{\mathbb{C}}^{d^2+4d+2}$, there exists a homogeneous polynomial $Q_d\in\mathbb{C}[x_1,x_2,\cdots,x_{d^2+4d+3}]$ of degree 3, not depending on λ, which vanishes on the orbit $O\big(\mathcal{F}_0^d(\lambda)\big)$ and does not vanish at the point $\mathcal{F}_2^d=\big[0:1:0:0:\cdots:0:0:1\big]$.*

Computations made with `Maple` by the first author show the validity of this conjecture for d small ($d\leq 30$) by taking the polynomial Q_d in the following form:

$$Q_d=x_{d^2+3d+3}\left(\sum_{i=1}^{d-1}\alpha_i x_{2d+2i+1}x_{d^2+4d+2-i}+\sum_{i=0}^{4}\beta_i x_{2d+2i+4}x_{d^2+4d+2-i}\right)+(x_1\quad x_2\quad\cdots\quad x_{d+1})M\begin{pmatrix}x_{d^2+4d+3}\\ x_{d^2+4d+2}\\ \vdots\\ x_{d^2+3d+3}\end{pmatrix}$$
$$+x_{d^2+3d+4}\left(\delta_0 x_{2d+4}x_{d^2+4d+1}+\delta_1 x_{2d+6}x_{d^2+4d}+\sum_{i=1}^{d-3}\gamma_i x_{2d+2i+1}x_{d^2+4d+1-i}\right),$$

where $M=\begin{pmatrix}L_1\\ L_2\\ \vdots\\ L_{d+1}\end{pmatrix}$ is a square matrix of order $d+1$ whose lines are of the form:

$$L_1=\begin{bmatrix}0 & 0 & a_{1,3}x_1+b_{1,3}x_4 & a_{1,4}x_3+b_{1,4}x_6 & a_{1,5}x_5+b_{1,5}x_8 & \cdots & a_{1,d+1}x_{2d-3}+b_{1,d+1}x_{2d}\end{bmatrix}$$
$$L_2=\begin{bmatrix}b_{2,1}x_2 & a_{2,2}x_1+b_{2,2}x_4 & a_{2,3}x_3+b_{2,3}x_6 & a_{2,4}x_5+b_{2,4}x_8 & a_{2,5}x_7+b_{2,5}x_{10} & \cdots & a_{2,d+1}x_{2d-1}+b_{2,d+1}x_{2d+2}\end{bmatrix}$$
$$\vdots$$
$$L_{2k-1}=\begin{bmatrix}\underbrace{0\ 0\ \cdots\ 0}_{\min(2k,d+1)} & a_{2k-1,2k+1}x_{2k-1}+b_{2k-1,2k+1}x_{2k+2} & a_{2k-1,2k+2}x_{2k+1}+b_{2k-1,2k+2}x_{2k+4} & \cdots & a_{2k-1,d+1}x_{2d-2k-1}+b_{2k-1,d+1}x_{2d-2k+2}\end{bmatrix}$$
$$L_{2k}=\begin{bmatrix}\underbrace{0\ 0\ \cdots\ 0}_{2k-2} & b_{2k,2k-1}x_{2k} & a_{2k,2k}x_{2k-1}+b_{2k,2k}x_{2k+2} & a_{2k,2k+1}x_{2k+1}+b_{2k,2k+1}x_{2k+4} & \cdots & a_{2k,d+1}x_{2d-2k+1}+b_{2k,d+1}x_{2d-2k+4}\end{bmatrix},$$

where $\alpha_i,\beta_i,\gamma_i,\delta_i,a_{i,j},b_{i,j}\in\mathbb{C}$ with $b_{2,1}\neq 0$.

It is clear that Conjecture 1 and Lemma 4.3 imply the following conjecture.

Conjecture 2. — *For any integer $d\geq 6$ and any $\lambda\in\mathbb{C}^*$, the orbit $O\big(\mathcal{F}_0^d(\lambda)\big)$ is closed in $\mathbf{F}(d)$.*

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