PAPER • OPEN ACCESS

Enhanced Numerov Method for the Numerical Solution of Second Order Initial Value Problems

View the article online for updates and enhancements.

Enhanced Numerov Method for the Numerical Solution of Second Order Initial Value Problems

G.O. Akinlabi[1*], S. A. Bishop[1], S. O. Edeki[1]

[1]Department of Mathematics, Covenant University, Canaanland, Ota, Nigeria

Corresponding Author's e-mail: grace.akinlabi@covenantuniversity.edu.ng

Abstract. Numerov method is a multistep numerical method that is used in solving second order differential equations. In this work, we apply this method as a Boundary Value Method (BVM) for the numerical approximation of both linear and nonlinear second order initial value problems. This is achieved by constructing the Numerov method via interpolation and collocation process while utilizing data at off-step points and implementing it as a BVM. On comparing the results obtained from the solved problems, it shows that the method is accurate with high level of convergence to their exact forms and performs better than results from literature.

Keywords: Hybrid BVM; Linear Multistep Method; Initial Value Problem; Boundary Value Method

1. Introduction

Due to the needs of mathematical modelization of real-life problems, the search for a better numerical method, which can handle different problems, remain active research for numerical analysts [1-6]. In this work, we derive a new scheme based on a Linear Multistep Method (LMM) called the Numerov method. This method is a numerical method used in approximating second order differential equations. Lots of studies have been done with this method [7, 8].

Our focus is to develop a new scheme called Hybrid Boundary Value Method (HyBVM). The development will be achieved by collocating and interpolating the LMM (mentioned above) at both step and off-step points. We then implement this scheme as a Boundary Value Method (BVM) to solve second order initial value problems of the form:

$$\left. \begin{array}{l} y'' = f\left(x, y, y'\right) \\ y\left(x_0\right) = \alpha_0, \quad y'\left(x_0\right) = \beta_0 \end{array} \right\} \tag{1}$$

where α_0, β_0 are constants and f is a continuous function which satisfies the conditions for existence and uniqueness of solutions, which are guaranteed by the theorem of Henrici in [9] for Initial value problems.

In deriving this method, we will be adopting the two step Numerov method, which is a LMM of the form:

$$y_{n+k} + \sum_{i=0}^{k-1} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i f_{n+i} \qquad (2)$$

where h is the step-size and α_i, β_i are constants.

Different hybrid formulas based on LMM have been derived by authors [10 – 14], since they are more flexible the way they have been used and also possess small error constants. The BVMs were introduced to overcome some of the limitations encounter by the LMMs; they are better in approximating solutions. Several BVMs have been developed and discussed fully in literature [15–28]. The remaining sections of this study are structured as follows: In section two, the derivation and the specification of the method are discussed. In the third section, the results are presented with examples on both linear and nonlinear Initial Value Problems (IVPs). Section four offers a discussion of the results. The concluding remark is given in the last section.

2. Derivation of Methods [28]

In this section, the aim is to derive a two-step LMM of the form:

$$y_{n+k} + \sum_{i=0}^{k-1} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{v_i} \beta_{v_i} f_{n+v_i} \qquad (3)$$

And also the derivative formula of the form

$$hy'_{n+k} + \sum_{i=0}^{k-1} \alpha'_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta'_i f_{n+i} + h^2 \sum_{v_i} \beta'_{v_i} f_{n+v_i} \qquad (4)$$

using the same continuous scheme where $\alpha_k = 1$, $\beta_k \neq 0$ and α_0, β_0 do not both vanish.

We start the process of derivation by seeking to approximate the analytical solution $y(x)$ by a continuous method $Y(x)$ with its second derivative of the form:

$$Y(x) = \sum_{i=0}^{a+b-1} \beta_i P_i(x) \qquad (5)$$

$$Y''(x) = \sum_{i=0}^{a+b-1} \beta_i P_i''(x) \qquad (6)$$

where a, b are the number of interpolation and collocation points, $P_i(x)$ are the polynomial basis of degree $a+b-1$. A k-step multistep collocation method is then constructed from:

$$Y(x) = V^T (M^{-1}) P(x) \qquad (7)$$

where

$$P(x_n) = \left[P_0(x), P_1(x), P_2(x), \ldots, P_{a+b-1}(x) \right] \qquad (8)$$

$$V = \left[y_n, \ldots, y_{n+r-1}, f_n, \ldots, f_{n+b-1} \right] \tag{9}$$

$$M = \begin{pmatrix} P_0(x_n) & P_1(x_n) & \cdots & P_{a+b-1}(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ P_0(x_{n+a-1}) & P_1(x_{n+a-1}) & \cdots & P_{a+b-1}(x_{n+a-1}) \\ P_0''(x_n) & P_1''(x_n) & \cdots & P_{a+b-1}''(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ P_0''(x_{n+b-1}) & P_1''(x_{n+a-1}) & \cdots & P_{a+b-1}''(x_{n+b-1}) \end{pmatrix} \tag{10}$$

Which results into a continuous LMM

$$Y(x) = \sum_{i=0}^{a-1} \alpha_i(x) y_{n+i} + h^2 \sum_{i=0\left(\frac{1}{2}\right)}^{b-1} \beta_i(x) f_{n+i} \tag{11}$$

where $\alpha_i(x), \beta_i(x)$ are continuous coefficients to be determined. This is then used to generate the discrete LMMs of the form (3) and derivative formulas of the form (4) and other additional methods. These equations are then applied simultaneously to solve (1) above.

2.1. *Specification of the Methods [28]*

In this section, we specify the derived method.

Consider the case $k = 2$ with the specification $a = 2$ and $b = 5$ using (5) and (6) we have the following polynomials of degree $a+b-1$:

$$Y(x) = \sum_{i=0}^{6} \beta_i P_i(x) \tag{12}$$

$$Y''(x) = \sum_{i=0}^{6} \beta_i P_i''(x) \tag{13}$$

which will yield the following vectors and 7×7 collocation/interpolation matrix:

$$P = \left[1, x, x^2, x^3, x^4, x^5, x^6 \right] \tag{14}$$

$$V = \left[y_0, y_1, f_0, f_{\frac{1}{2}}, f_1, f_{\frac{3}{2}}, f_2 \right] \tag{15}$$

$$M = \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 & x_0^4 & x_0^5 & x_0^6 \\ 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 & x_1^6 \\ 0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 & 30x_0^4 \\ 0 & 0 & 2 & 6x_{\frac{1}{2}} & 12x_{\frac{1}{2}}^2 & 20x_{\frac{1}{2}}^3 & 30x_{\frac{1}{2}}^4 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 & 30x_1^4 \\ 0 & 0 & 2 & 6x_{\frac{3}{2}} & 12x_{\frac{3}{2}}^2 & 20x_{\frac{3}{2}}^3 & 30x_{\frac{1}{2}}^4 \\ 0 & 0 & 2 & 6x_2 & 12x_2^2 & 20x_2^3 & 30x_2^4 \end{pmatrix} \tag{16}$$

These are then substituted into the equation below:

$$Y(x) = V^T (M^{-1}) P(x) \tag{17}$$

which results into a continuous LMM

$$Y(x) = \begin{cases} \dfrac{(h - x + x_0) y_0}{h} + \dfrac{(x - x_0) y_1}{h} - \dfrac{f_0}{360h^4}(x - x_0)(h - x + x_0) \\[2mm] \left(\begin{array}{l} 53h^4 + 123h^2 (x - x_0)^2 - 52h(x - x_0)^3 \\ + 8(x - x_0)^4 + 127h^3(-x + x_0) \end{array} \right) \\[4mm] - \dfrac{f_{\frac{1}{2}}}{45h^4} \left(\begin{array}{l} 18h^5(x - x_0) - 60h^3(x - x_0)^3 + 65h^2(x - x_0)^4 \\ -27h(x - x_0)^5 + 4(x - x_0)^6 \end{array} \right) \\[4mm] + \dfrac{f_1}{60h^4} \left(\begin{array}{l} 5h^5(x - x_0) - 60h^3(x - x_0)^3 + 95h^2(x - x_0)^4 \\ -48h(x - x_0)^5 + 8(x - x_0)^6 \end{array} \right) \\[4mm] - \dfrac{f_{\frac{3}{2}}}{45h^4} \left(\begin{array}{l} 2h^5(x - x_0) - 20h^3(x - x_0)^3 + 35h^2(x - x_0)^4 \\ -21h(x - x_0)^5 + 4(x - x_0)^6 \end{array} \right) \\[4mm] + \dfrac{f_2}{360h^4} \left(\begin{array}{l} 3h^5(x - x_0) - 30h^3(x - x_0)^3 + 55h^2(x - x_0)^4 \\ -36h(x - x_0)^5 + 8(x - x_0)^6 \end{array} \right) \end{cases} \tag{18}$$

The main method (19) is then obtained by evaluating (18) at x_{n+2} :

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{60}\left[f_n + 26f_{n+1} + f_{n+2} + 16\left(f_{n+\frac{1}{2}} + f_{n+\frac{3}{2}} \right) \right] \tag{19}$$

which is used together with the following initial methods:

$$y_{\frac{1}{2}} - \frac{1}{2}y_0 - \frac{1}{2}y_1 = \frac{h^2}{1920}\left[-19f_0 - 14f_1 + f_2 - 204f_{\frac{1}{2}} - 204f_{\frac{3}{2}} \right] \tag{20}$$

$$y_{\frac{3}{2}} + \tfrac{1}{2} y_0 - \tfrac{3}{2} y_1 = \frac{h^2}{1920}\left[17 f_0 + 402 f_1 - 3 f_2 + 252 f_{\frac{1}{2}} + 52 f_{\frac{3}{2}}\right] \tag{21}$$

$$y_{\frac{3}{2}} + \tfrac{1}{2} y_0 - \tfrac{3}{2} y_1 = \frac{h^2}{1920}\left[17 f_0 + 402 f_1 - 3 f_2 + 252 f_{\frac{1}{2}} + 52 f_{\frac{3}{2}}\right] \tag{22}$$

and with the following derivative formulas:

$$h y'_{\frac{1}{2}} + y_0 - y_1 = h^2\left[\frac{13 f_0}{480} - \frac{f_1}{10} - \frac{7 f_2}{1440} + \frac{7 f_{\frac{1}{2}}}{144} + \frac{7 f_{\frac{3}{2}}}{240}\right] \tag{23}$$

$$h y'_1 + y_0 - y_1 = h^2\left[\frac{f_0}{72} + \frac{13 f_1}{60} + \frac{f_2}{360} + \frac{13 f_{\frac{1}{2}}}{45} - \frac{f_{\frac{3}{2}}}{45}\right] \tag{24}$$

$$h y'_{\frac{3}{2}} + y_0 - y_1 = h^2\left[\frac{31 f_0}{1440} + \frac{8 f_1}{15} - \frac{f_2}{96} + \frac{19 f_{\frac{1}{2}}}{80} + \frac{157 f_{\frac{3}{2}}}{720}\right] \tag{25}$$

$$h y'_{n+2} + y_n - y_{n+1} = h^2\left[\frac{f_n}{120} + \frac{7 f_{n+1}}{20} + \frac{59 f_{n+2}}{360} + \frac{14 f_{n+\frac{1}{2}}}{45} + \frac{2 f_{n+\frac{3}{2}}}{3}\right] \tag{26}$$

3. Numerical Examples

In this section, we apply the main method and additional method derived in the previous section to two (2) second order initial value problems. The obtained results are compared with their exact solutions and also with results from [28]. These are shown in the graphs (Figure 1 and 2) and Table 1 and 2.

Problem 3.1: Consider the linear second order IVP [28]:

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 8 y = x^3 \ , \quad x \in (0,\ 1)$$

with initial conditions:

$$y(0) = 2, \quad y'(0) = 4$$

with exact solution: $\quad y(x) = e^{2x}\left[2\cos 2x - \frac{3}{64}\sin 2x\right] + \frac{3}{32} x + \frac{3}{16} x^2 + \frac{1}{8} x^3$
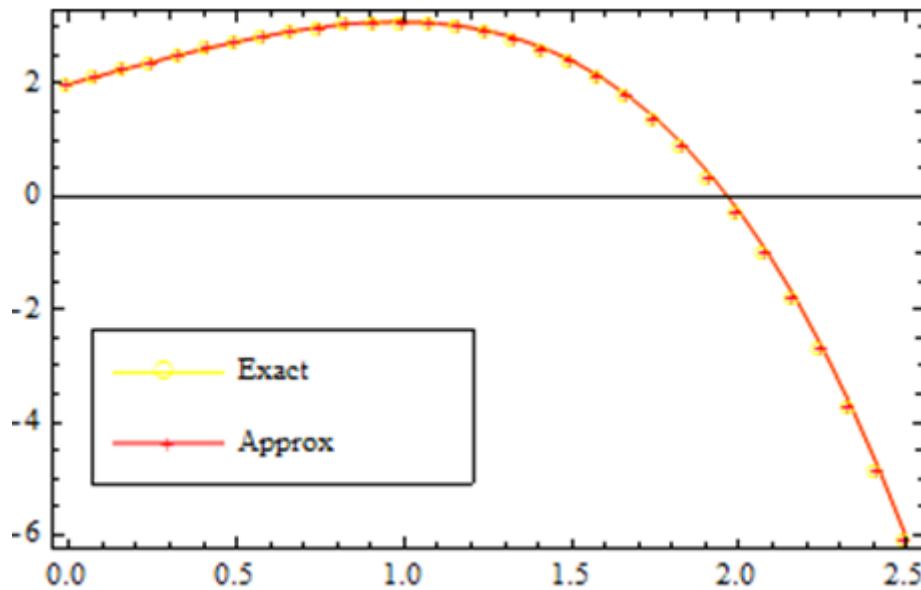
Figure 1: Solution of Problem 3.1 computed with HyBVM (k =2, h = 0.03)

Table 1: Absolute errors for Problem 3.1 ($h = 0.1$)

x	SSM(k=4)	BVM(k=4)	BVM(k=5)	HyBVM(k=2)
0.0	0.00000	0.00000	0.00000	0.00000
0.1	5.11×10^{-6}	6.13×10^{-7}	8.14×10^{-8}	7.14×10^{-8}
0.2	1.50×10^{-5}	1.85×10^{-6}	2.44×10^{-7}	1.75×10^{-7}
0.3	2.79×10^{-5}	3.42×10^{-6}	4.55×10^{-7}	2.94×10^{-7}
0.4	4.29×10^{-5}	5.55×10^{-6}	7.29×10^{-7}	4.02×10^{-7}
0.5	6.70×10^{-5}	8.39×10^{-6}	1.06×10^{-6}	4.72×10^{-7}
0.6	1.03×10^{-4}	1.23×10^{-5}	1.45×10^{-6}	4.66×10^{-7}
0.7	1.45×10^{-4}	1.74×10^{-5}	1.93×10^{-6}	3.36×10^{-7}
0.8	1.91×10^{-4}	2.35×10^{-5}	2.47×10^{-6}	2.24×10^{-8}
0.9	2.40×10^{-4}	3.09×10^{-5}	3.08×10^{-6}	5.39×10^{-7}
1.0	2.95×10^{-4}	3.86×10^{-5}	4.06×10^{-6}	1.42×10^{-6}

Problem 3.2: Consider the nonlinear second order IVP [28]:

$$\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 = 0 \quad , \quad x \in (0,\ 1)$$

with initial conditions:

$$y(0) = 1, \quad y'(0) = \frac{1}{2}$$

with exact solution:
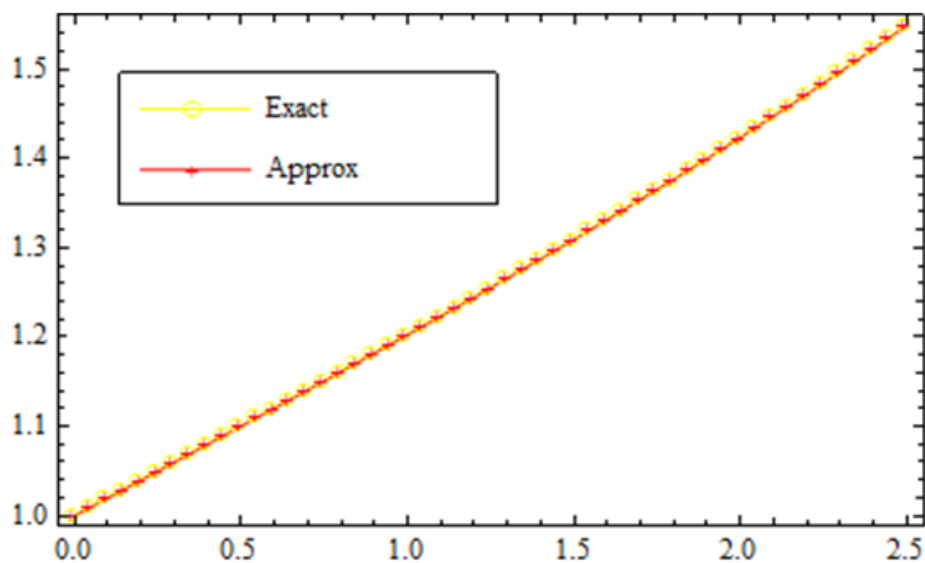$$y(x) = 1 + \frac{1}{2}\ln\left[\frac{2+x}{2-x}\right]$$

Figure 2: Solution of Problem 3.2 computed with HyBVM ($k=2$, $h=0.05$)

Table 2: Absolute errors for Problem 3.2 ($h = 0.1$)

x	SSM($k=4$)	BVM($k=4$)	BVM($k=4$)	HyBVM($k=2$)
0.1	7.51×10^{-9}	6.67×10^{-9}	1.03×10^{-9}	1.18×10^{-10}
0.2	1.80×10^{-8}	1.60×10^{-8}	2.51×10^{-9}	2.37×10^{-10}
0.3	2.88×10^{-8}	2.55×10^{-8}	3.97×10^{-9}	3.56×10^{-10}
0.4	3.65×10^{-8}	3.23×10^{-8}	5.71×10^{-9}	4.70×10^{-10}
0.5	7.05×10^{-8}	4.45×10^{-8}	7.62×10^{-9}	5.77×10^{-10}
0.6	1.20×10^{-7}	4.82×10^{-8}	9.31×10^{-9}	6.64×10^{-10}
0.7	1.73×10^{-7}	6.27×10^{-8}	1.27×10^{-8}	7.11×10^{-10}
0.8	2.14×10^{-7}	6.09×10^{-8}	1.26×10^{-8}	6.73×10^{-10}
0.9	5.82×10^{-7}	7.42×10^{-8}	2.00×10^{-8}	4.51×10^{-10}
1.0	1.15×10^{-6}	5.55×10^{-8}	3.46×10^{-8}	1.73×10^{-10}

4. Discussion of Result

In this work, a new method referred to as HyBVM has been applied to two second-order Initial Value Problems (linear and nonlinear). Figure 1 and Figure 2 show the comparison between the approximate solutions obtained from the two cases and their exact solutions.

Tables I and II also show the absolute errors from Boundary Value Method (BVM $k = 4, 5$) and Self Starting Method (SSM) obtained in [28] and also the absolute error from our proposed method for problems 3.1 and 3.2, respectively. It was observed that the HyBVM performed better than these other methods for the two cases.

5. Conclusion

In this paper, we have extended the Numerov method by applying and implementing them as Boundary Value Method (BVM). This was achieved by constructing the Numerov method via collocation and interpolation procedure while utilizing data at off-step point. We call the new method: Hybrid BVM (HyBVM). This new scheme was applied to two second order initial value problems, and the numerical tests confirmed that it of high accuracy when compared to the one in literature.

References
[1] J. G. Oghonyon, S. A. Okunuga and Bishop. S. A., A 5-step block predictor and 4-step corrector methods for solving general second order ordinary differential equations, *Global Journal of Pure and Applied Mathematics*, 11 (5), 2015, 3847-3862.
[2] T. A. Anake, Awoyemi D. O. and Adeyemi A. A., (2012) A one-step method for the solution of second order ordinary differential equations, International Journal of Science and Technology, 2(4), 159-163.
[3] S. O. Edeki and Akinlabi, G.O., Zhou Method for the Solutions of System of Proportional Delay Differential Equations, *MATEC Web of Conferences* 125, 02001, 2017.
[4] S. O. Edeki, and Akinlabi G. O., Coupled method for solving time-fractional navier-stokes equation, *Journal of Circuits, Systems and Signal Processing*, 12, 2018, 27-34.
[5] S. O. and G. O. Akinlabi, Analytic solutions of the Navier-Stokes model by He's polynomials, Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering 2016, WCE 2016, 29 June – 1 July, 2016, London, U.K., pp 16-19.
[6] S.O. Edeki, G.O. Akinlabi, A.O. Akeju, "A Handy Approximation Technique for Closed-form and Approximate Solutions of Time-Fractional Heat and Heat-Like Equations with Variable Coefficients," Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering 2016, WCE 2016, 29 June – 1 July, 2016, London, U.K., pp 88-92.
[7] E. A. Adebile, T. A. Anake, A. O. Adesanya, G. J. Oghonyon, Numerov Method for the solution of second order initial value problems of ordinary differential equations using modified block method, *The Nigerian Journal of Education*, 9(2), 45-54, 2011.
[8] L. K. Bieniasz, Improving the accuracy of the spatial discretization in finite-difference electrochemical kinetic simulations, by means of the extended Numerov method, *J. Comput Chem.* 25(8), 2004, 1075-1083.
[9] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, John Wiley and Sons, New York, 1956.

[10] G. O. Akinlabi, R. B. Adeniyi, E. A. Owoloko, Hybrid Boundary Value Methods for the solution of first order stiff systems, *International Journal of Circuits, Systems and Signal Processing*, 12(1), 2017, 332-337.

[11] C. W. Gear, Hybrid Methods for Initial Value Problems in Ordinary Differential Equations, *Math. Comp.*, 21, (1967), 146-156.

[12] J. C. Butcher, A modified multistep method for the numerical integration of ordinary differential equations, *J. Assoc. Comput. Mach.*, 12, (1965), 124-135.

[13] A. K. Ezzeddine and G. Hojjati, Hybrid extended backward differentiation formulas for stiff systems, *International Journal of Nonlinear Science*, 1292, (2011), 196-204.

[14] W. B. Gragg and H. I. Stetter, Generalized multistep predictor-corrector methods, *J. Assoc. Comput Mach.*, 11, (1964), 188-209.

[15] T. A. Biala, A Computational Study of the Boundary Value Methods and the Block Unification Methods for $y'' = f(x, y, y')$, *Abstract and Applied Analysis*. vol. 2016, Article ID 8465103, 14 pages, 2016.

[16] F. Mazzia, "Boundary Value Methods for the Numerical Solution of Boundary Value Problems in Differential-Algebraic Equations". Bolletino della Unione Matematica Italiana, (1997), 579-593.

[17] L. Brugnano and D. Trigiante, High-Order Multistep Methods for Boundary Value Problems, *Applied Numerical Mathematics*, 18, (1995), 79-94.

[18] L. Brugnano and D. Trigiante, Block Boundary Value Methods for Linear Hamiltonian Systems, *Appl. Math. Comput.*, 81, (1997), 49-68.

[19] P. Amodio and F. Iavernaro, Symmetric boundary value methods for second initial and boundary value problems, *Medit. J. Maths.*, 3, (2006), 383-398.

[20] T. A. Biala and S. N. Jator, A boundary value approach for solving three-dimensional elliptic and hyperbolic Partial Differential Equations, *SpringerPlus Journals*, vol. 4, article no 588, 2015

[21] P. Amodio and F. Mazzia, A Boundary Value Approach to the Numerical Solution of Initial Value Problems by Multistep Methods, *J Difference Eq. Appl.*, 1, (1995), 353-367.

[22] L. Aceto, P. Ghelardoni and C. Magherini, PGSCM: A family of P-Stable Boundary Value Methods for Second Order Initial Value Problems, *Journal of Computational and Applied Mathematics*, 236, (2012), 3857-3868.

[23] T. A. Biala, S. N. Jator and R. B. Adeniyi, Boundary Value Methods for Second-Order PDEs via the Lanczos-Chebyshev Reduction Technique, *Mathematical Problems in Engineering*. Vol. 2017, Article ID 5945080, 11 pages.

[24] T. A. Biala, S. N. Jator and R. B. Adeniyi, Numerical approximations of second order PDEs by boundary value methods and the method of lines, *Afrika Matematika*, 2016.

[25] L. Brugnano and D. Trigiante, *Solving Differential Problems by Multistep Initial and Boundary Value Methods*. Gordon and Breach Science, Amsterdam, 1998.

[26] G. O. Akinlabi, R. B. Adeniyi, Sixth- order and fourth- order hybrid boundary value method for systems of boundary value problems, *WSEAS Transactions on Mathematics*, 17, 2018, 258-264.

[27] G. O. Akinlabi, R. B. Adeniyi, E. A. Owoloko, The solution of boundary value problems with mixed boundary conditions via boundary value methods, *International Journal of Circuits, Systems and Signal Processing*, 12, 2018, 1-6.

[28] S. N. Jator, J. Li, Boundary Value Methods via a multistep method with variable coefficients for second order initial and boundary value problems, *International journal of pure and applied mathematics*, 50(3), 2009, 403-420.