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Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*
Cotutelle internationale avec *Universität Potsdam*

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Jens Walter FISCHER

Random Dynamics in Collective Behavior - Consensus, Clustering & Extinction of Populations

JURY

Vincent BANSAYE	École Polytechnique	Rapporteur
Patrick CATTIAUX	Université Paul Sabatier	Directeur de Thèse
Aldéric JOULIN	Université Paul Sabatier	Examinateur
Matthias KELLER	Universität Potsdam	Examinateur
Han Cheng LIE	Universität Potsdam	Examinateur
Jan LORENZ	Jacobs Universität	Rapporteur
Sylvie ROELLY	Universität Potsdam	Directrice de Thèse

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Directeur(s) de Thèse :
Patrick CATTIAUX et *Sylvie ROELLY*
Rapporteurs :
Jan LORENZ et *Vincent BANSAYE*

From this hour I ordain myself loos'd of limits and imaginary lines,
Going where I list, my own master total and absolute,
Listening to others, considering well what they say,
Pausing, searching, receiving, contemplating,
Gently,but with undeniable will, divesting myself of the holds that would hold me.
I inhale great draughts of space,
The east and the west are mine, and the north and the south are mine.

I am larger, better than I thought,
I did not know I held so much goodness.

All seems beautiful to me,
I can repeat over to men and women
You have done such good to me
I would do the same to you,
I will recruit for myself and you as I go,
I will scatter myself among men and women as I go,
I will toss a new gladness and roughness among them,
Whoever denies me it shall not trouble me,
Whoever accepts me he or she shall be blessed and shall bless me.

Song of the Open Road, Stanza Five
By Walt Whitman, 1856

Remerciements

Embarking on a PhD Thesis in Mathematics will be in years to come most likely be the best example for my dear research subject, complex systems. Initial conditions well fixed and not too complicated, I could have never envisioned the trajectory that I would take over the ensuing three years of my PhD. External factors as well as frustrating professional hurdles, blockages and successes but also the people I met, my personal development and the moments I got to live made it a ever so humbling experience. Being submerged in the everyday work, the fiddly nuances and the exhausting attempt to produce something that is worth called research, one can easily get the impression of being on one's own. But now that the dust starts to settle I want to take the time to express my gratitude towards those who accompanied, supported and sustained me.

Making a joke about the availability of not only german and english but also french literature on Markov chains during my first course on Markov chains, Prof. Sylvie Roelly was not aware yet that there was a student who was in dire need of such literature in said class... me. She has supported me during the ensuing ERASMUS semester, my master thesis and then my PhD unconditionally in my mathematical endeavors in scientific as well as administrative ways and, especially, gave personal support in times where I did not see the end of the PhD tunnel. For her efforts as my PhD advisor but also for her personal investment during all this time, which were far beyond from what is to be expected, I want to express my heartfelt gratitude. My life would probably look very differently today without her.

Secondly, I would like to thank Prof. Patrick Cattiaux for the energy he put into making this PhD thesis happen as well as putting up with all the little and bigger issues I encountered when I came to this foreign land called ”La France” and its system which was beyond my understanding. The impact of the opportunity to do my PhD in Toulouse as well as the freedom of making major changes during this time will have an ever lasting impact on my life. I will be forever grateful that I had this opportunity.

Partir à l'étranger pour faire une thèse, cela implique toujours deux changements principaux: apprendre une nouvelle langue et faire des rencontres. La langue se met en place naturellement mais les rencontres peuvent boulverser la vie. Je dédie mon premier remerciement ”personel” à la rencontre la plus importante de ma vie, à ma conjointe, ma chérie, mon âme soeur. Depuis mon arrivée en France elle fait partie de ma vie, nous sommes entrelacés dans une aventure que je n'aurais jamais pu imaginer. Je la remercie infiniment pour son soutien, de m'avoir supporté pendant les périodes où j'étais complètement immergé dans mon travail, de m'avoir amené dans la montagne pour que mes résultats soient au sommet et pour tous les moments qui m'ont permis de retrouver l'énergie pour finir ce travail. Une nouvelle aventure à trois nous attend bientôt...

Des Weiteren möchte ich meinen Eltern und Großeltern danken, die mich in mehr Arten, als ich hier aufzählen kann, unterstützt haben, einen langen und bezeiten beschwerlichen Weg bis zum Ende zu gehen, und die mehr als ihnen vielleicht bewusst ist zum erfolgreichen Abschluss dieser Arbeit beigetragen haben. Ich könnte mir keine besseren wünschen.

Mein abschliessender persönlicher Dank ist für meinen Bruder bestimmt. Seine Art sich zu faszinieren, wenn mich etwas fasziniert, meine Erfolge zu feieren, seien sie noch so trivial, offen zu sein, wenn ich etwas teilen muss, und mir immer das Gefühl zu geben, present zu sein unabhängig von der Distanz, haben mich an vielen Punkten in dieser Arbeit enorm unterstützt. Für all das und mehr möchte ich ihm hiermit danken.

STOCHASTISCHE DYNAMIKEN IN KOLLEKTIVEM VERHALTEN: KONSENS, GRUPPENBILDUNG, AUSSTERBEN VON POPULATIONEN

Beziehungen und damit Interaktion sowie Diskussion, aber auch Konflikt und Opposition bilden die Grundbausteine einer jeden Gesellschaft. Häufig wird Kommunikation als der übergreigende Begriff zur Beschreibung interner Strukturen einer Gesellschaft identifiziert. Dabei muss es sich aber nicht um eine Gesellschaft im Sinne von Nationen handeln, sondern kann auch schlicht eine Gruppe von Menschen umfassen, die miteinander strukturiert interagieren, beispielsweise, eine Gruppe von Angestellten, die an einem gemeinsamen Projekt arbeiten, oder die Mitglieder eines sozialen Netzwerks. In dieser Arbeit befassen wir uns mit der mathematischen Beschreibung solcher Prozesse innerhalb von Gruppen und Gesellschaften und legen dabei unseren Fokus auf die Bildung eines Konsens durch Interaktion aber auch die Konsequenzen von Konflikt und das potentielle Aussterben einer Population. Dabei werden zwei Modelle im Fokus des Interesses stehen: Das Echokammer Model sowie eine Erweiterung des Geburts-Todes Prozesses, die die Möglichkeit eines radikalen Abfalls der Populationsgröße miteinschließt. Wir beginnen mit einer Einführung in Part I und teilen die verbleibende Arbeit in drei Teile auf, wobei sich die ersten beiden technischen Abschnitte, Part II und III, mit einer ausführlichen Analyse der Bausteine des Echokammer Models befassen und im dritten Abschnitt, in Part IV, der erweiterte Geburts-Todes Prozess untersucht wird. Dieser wird im Folgenden als Geburts-Todes Prozess mit teilweiser Katastrophe bezeichnet werden.

Das Echokammer Model beschreibt die Entwicklung von Gruppen in zunächst heterogenen sozialen Netzwerken. Unter einem heterogenen sozialen Netzwerk verstehen wir dabei eine Menge von Individuen, von denen jedes exakt eine Meinungen vertritt. Meinungen werden vereinfacht durch Werte in $[0, 1]$ modelliert. Bestehende Beziehungen unter den Individuen können dann durch einen Graphen dargestellt werden. Es handelt sich bei dem Echokammer Modell um ein zeit-diskretes Modell, das entsprechend, ähnlich einem Brettspiel, in Zügen abläuft. In jedem Zug wird zufällig gleichverteilt eine bestehende Beziehung aus dem Netzwerk ausgewählt und die beiden verbundenen Individuen interagieren. Dabei kann es zu zwei verschiedenen Interaktionen kommen. Sind die Meinungen der betroffenen Individuen hinreichend ähnlich, so nähern sie sich weiter in ihren Meinungen an, während sie im Fall von Meinungen, die zu weit von einander liegen, ihre Beziehung auflösen und sich eines der Individuen eine neue Beziehung sucht.

In dieser Arbeit untersuchen wir theoretisch die Bausteine dieses Modells. Dabei legen wir die Beobachtung zu Grunde, dass die Veränderungen der Beziehungsstruktur im Netzwerk durch einen System von interagierenden Partikeln auf einem abstrakteren Raum beschrieben werden kann. Dies erlaubt es insbesondere graphentheoretische Überlegungen in die Analyse einfließen zu lassen. Diese Überlegungen werden ausührlich in Part II diskutiert und führen zur Definition eines neuen, abstrahierten Graphens, der alle möglichen Beziehungskonfigurationen des sozialen Netzwerks umfasst. Dies erlaubt es uns einen Ähnlichkeitsbegriff für Beziehungskonfigurationen auf Basis der benachbarten Knoten in besagtem Graphen zu definieren. Dies liefert uns das notwendige geometrische Verständnis um in Part III die dynamischen Komponenten des Echokammer models zu analysieren. Insbesondere fokusieren wir uns dabei auf die Dynamik der Kanten, für die bisher in der Literatur noch keine Ergebnisse existieren.

Wir lassen zunächst in Abschnitt 7 die Meinungen der Individuen beiseite und nehmen an, dass die Position der Kanten sich in jedem Zug wie zuvor beschrieben ändert, um eine grundlegendes Verständnis der unterliegenden Dynamik zu erhalten. Unter der Verwendung der Theorie von Markovketten finden wir obere Schranken an die Konvergenzgeschwindigkeit einer assoziierten Markovkette gegen ihre eindeutige stationäre Verteilung und zeigen, dass es Netzwerke gibt, die miteinander identifizierbar und unter der analysierten Dynamik daheingehend ununterscheinbar sind, dass die stationäre Verteilung der assozierten Markovkette diesen Netzwerken dasselbe Gewicht zuordnet. Anschließend beweisen wir eine Reihe von quantitativen Resultaten, die sich insbesondere in Fällen, in denen die assozierte Markovkette reversibel ist, als berechenbar herausstellen. Insbesondere die explizite Form der stationären Verteilung sowie untere Schranken an die Cheeger Konstante zur Beschreibung der Konvergenzgeschwindigkeit stehen dabei im Fokus und werden ausführlich diskutiert.

Nach dieser vertieften Analyse des reduzierten Modells, fügen wir die Meinungen unserer Betrachtung wieder hinzu. Das abschließende Result in Abschnitt 8, basierend auf absorbierenden Markovketten, liefert dann, dass in einer reduzierte Version des Echokammer Modells, in dem sich Individuen ähnlicher Meinung nicht annähern, eine hierarchische Struktur der Anzahl der konfliktreichen Beziehung identifiziert werden kann. Dies können wir ausnutzen, um eine obere Schranke an die erwartete Absorptionszeit, unter Zuhilfenahme einer quasi-stationären Verteilung, zu bestimmen. Diese hierarchische Struktur bildet außerdem eine Brücke zu klassischen Theorien von Geburts-Todes und, insbesondere, reinen Todes-Prozessen, für die eine reiche Literatur existiert. Wir zeigen abschließend auf, wie künftige Forschung diese Verbindung ausnutzen kann und diskutieren die Wichtigkeit der Ergbenisse als Bausteine eines vollständigen theoretischen Verständnisses des Echokammer Modells.

Part IV stellt abschließend einen veröffentlichten Artikel vor, der sich dem Geburts-Todes Prozess mit teilweiser Katastrophe widmet. Besagter Artikel steht dabei auf zwei Säulen. Zum Einen der expliziten Berechnung des ersten Zeitpunkts einer Katastrophe, wenn die Population zu Beginn der Beobachtung von instabiler Größe ist.

Dieser erste Teil basiert vollständig auf einer analytischen Herangehensweise an rekursiv definierte Folgen sowie der Bestimmung von Lösungen von Rekursionsgleichungen zweiten Grades mit linearen Koeffizienten. Konvergenz gegen 0 der resultierenden Folge sowie die Konvergenzgeschwindigkeit werden charakterisiert und bewiesen.
Zum Anderen der Bestimmung oberer Schranken des Erwartungswerts der Populationsgröße sowie der Varianz selbiger und der Differenz zwischen der bestimmten oberen Schranke und dem tatsächlichen Wert des Erwartungswerts. Dies erlaubt es uns Konfidenzintervalle für die Populationsgröße zu jedem Zeitpunkt anzugeben, wenn wir sie bei Beobachtungsbeginn exakt bestimmen. Wir verwenden für diese Resultate nahezu ausschließlich die Theorie gewöhnlicher nichtlinearer Differentialgleichungen und bestimmen, insbesondere, Nullclinen für die Charakterisierung des Langzeitverhaltens der betrachteten Größen. Wir schließen Part IV mit einem Ausblick, der ein verallgemeinertes Modell vorstellt, das zur Modellierung von Katastrophen in einer Population herangezogen werden kann. Es basiert auf der Idee endlicher selbstverstärkender Prozesse und liefert eine größere Flexibilität als sie mit Geburts-Todes Prozessen mit teilweiser Katastrophe erreichbar ist. Diese Flexibilität kommt mit analytischen Herausforderungen, die wir uns für spätere Forschung vorbehalten.

DYNAMIQUES ALÉATOIRES AU SEIN DE COMPORTEMENTS COLLECTIFS : CONSENSUS, CLUSTERING, EXTINCTION DE POPULATION

Les relations, et donc les interactions et les discussions, mais aussi les conflits et les oppositions, constituent les éléments de base de toute société. En sociologie, la communication est souvent identifiée comme le terme générique décrivant les structures internes d'une société. Il ne s'agit pas nécessairement d'une société au sens de nation, mais d'un simple groupe de personnes qui interagissent entre elles de manière structurée, par exemple un groupe d'employés travaillant sur un projet commun ou les membres d'un réseau social. Dans ce travail, nous nous intéressons à la description mathématique de tels processus au sein de groupes et de sociétés, en mettant l'accent sur la formation d'un consensus par l'interaction, mais aussi sur les conséquences du conflit et l'extinction potentielle d'une population. Deux modèles seront étudiés: le modèle des chambres d'écho et une extension du processus de naissance-mort qui inclut la possibilité d'une chute radicale de la taille de la population. Nous commencerons par une introduction dans la partie I et diviserons le reste du travail en trois parties, les deux premières parties techniques, les parties II et III, étant consacrées à une analyse détaillée des éléments constitutifs du modèle de la chambre d'écho et la troisième partie, la partie IV, à l'étude de l'extension du processus de naissance-mort. Ce processus sera appelé processus de naissance-mort avec catastrophe partielle.

Le modèle de la chambre d'écho décrit le développement de groupes dans des réseaux sociaux hétérogènes. Par réseau social hétérogène, nous entendons un ensemble d'individus dont chacun représente exactement une opinion. Les opinions sont modélisées de manière simplifiée par des valeurs comprises dans l'intervalle $[0, 1]$. Les relations existantes entre les individus peuvent alors être représentées par un graphique. Le modèle de la chambre d'écho est un modèle discret dans le temps qui, à l'instar d'un jeu de société, se déroule par coups. A chaque tour, une relation existante est sélectionnée de manière aléatoire et équidistante dans le réseau et les deux individus reliés interagissent. Deux interactions différentes peuvent se produire. Si les opinions des individus concernés sont suffisamment similaires, ils continuent à se rapprocher dans leurs opinions, alors que dans le cas d'opinions trop éloignées, ils rompent leur relation et un des individus cherche une nouvelle relation. Dans ce travail, nous examinons théoriquement les éléments constitutifs de ce modèle. Nous partons de l'observation que les changements de structure des relations dans le réseau peuvent être décrits par un système de particules en interaction dans un espace plus abstrait. Cela permet notamment d'intégrer certaines techniques de la théorie des graphes dans l'analyse. Ces réflexions sont discutées en détail dans la partie II et conduisent à la définition d'un nouveau graphe abstrait qui englobe toutes les configurations relationnelles possibles du réseau social. Cela nous permet de définir un concept de similarité pour les configurations de relations sur la base des nœuds voisins dans ce graphe. Cela nous fournit la compréhension géométrique nécessaire pour analyser les composantes dynamiques du modèle de chambre d'écho dans la partie III. Nous nous concentrons en particulier sur la dynamique des arêtes, pour laquelle il n'existe pas encore de résultats dans la littérature. Dans un premier temps, dans la partie 7, nous laissons de côté les opinions des inidividus et supposons que la position des arêtes change à chaque coup comme décrit précédemment, afin d'obtenir une compréhension de base de la dynamique sous-jacente. En utilisant la théorie des chaînes de Markov, nous trouvons des limites supérieures à la vitesse de convergence d'une chaîne de Markov associée vers sa distribution stationnaire unique et montrons qu'il existe des réseaux identifiables entre eux et non apparents dans la dynamique analysée, en ce sens que la distribution stationnaire de la chaîne de Markov associée attribue le même poids à ces réseaux. Nous prouvons ensuite une série de résultats quantitatifs qui s'avèrent calculables, en particulier dans les cas où la chaîne de Markov associée est réversible. Nous nous concentrons en particulier sur la forme explicite de la distribution stationnaire ainsi que sur les limites inférieures de la constante de Cheeger pour décrire la vitesse de convergence et nous en discutons en détail.

Après cette analyse approfondie du modèle réduit, nous incluons de nouveau les opinions dans nos considérations. Le résultat final de la section 8, basé sur les chaînes de Markov absorbantes, montre que dans une version réduite du modèle de la chambre d'écho, dans laquelle les individus d'opinions similaires ne se rapprochent pas, une structure hiérarchique du nombre de relations conflictuelles peut être identifiée. Nous pouvons utiliser cette structure pour déterminer une limite supérieure au temps d'absorption attendu, à l'aide d'une distribution quasi-stationnaire. Cette hiérarchie de la structure constitue également un pont vers les théories classiques des processus de naissance-mort et, en particulier, vers les processus de mort purs, pour lesquels il existe une littérature abondante. Nous concluons en montrant comment les recherches futures peuvent exploiter ce lien et en discutant de l'importance des résultats comme éléments constitutifs d'une compréhension théorique complète du modèle de la chambre d'écho.

Enfin, la partie IV présente un article publié consacré au processus de naissance-mort avec catastrophe partielle. L'article en question repose sur deux piliers. D'une part, le calcul explicite du premier moment d'une catastrophe lorsque la population est de taille instable au début de l'observation. Cette première partie est entièrement basée sur une approche analytique des suites définies par récurrence et sur la détermination des solutions des équations de récurrence du second degré à coefficients linéaires. La convergence vers 0 de la suite résultante ainsi que la vitesse de convergence sont caractérisées et prouvées. D'autre part, la détermination des limites supérieures de la valeur attendue de la taille de la population ainsi que de la variance de celle-ci et de la différence entre la limite supérieure déterminée et la valeur réelle de la valeur attendue. Cela nous permet d'indiquer des intervalles de confiance pour la taille de la population à tout moment, si nous la déterminons avec précision au début de l'observation. Pour ces résultats, nous utilisons presque exclusivement la théorie des équations différentielles non linéaires ordinaires et déterminons, en particulier, des null-clines pour la caractérisation du comportement à long terme des grandeurs considérées. Nous terminons la partie IV par une perspective qui présente un modèle généralisé pouvant être utilisé pour modéliser les catastrophes dans une population. Il est basé sur l'idée de processus finis auto-amplificateurs et offre une plus grande flexibilité que les processus de naissance-mort avec catastrophe partielle. Cette flexibilité s'accompagne de défis analytiques que nous réservons pour des recherches ultérieures.

Random Dynamics in Collective Behaviour
-
Consensus, Clustering
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PART I

INTRODUCTION

1 The Echo Chamber model

Models of complex interacting particle systems have been relevant in the scientific literature for several decades, in particular, for mathematicians and physicists in the light of interacting particle systems. Recently, they have become also of central interest for the social sciences and their applications. This is mostly due to the possibility to simulate and analyze ever more complex models and interactions, thanks to increased computing power. Examples for this research current, in particular linked to communication and conflict in population, are [BhBaRi08], [GrHiBi17], [Squa12] and [Eps12], to name a few. One model which, while quite simple in its form, has intrigued researchers, is the Echo Chamber Model. In [HolNew06] the authors propose the model which is a expansion of the voter model of opinion forming, see [HoLi75] and [Ligg99] for more on the subject. It consists of a set of individuals or agents, their personal opinions and their relationships and relies on two main features. Either the agents influence the opinions of their "friends", defined by the relationships, or change their relationships by breaking some up and recreating others. In Figure 1, one can see multiple steps until convergence of a simulation of the Echo Chamber Model with a discrete set of opinions. The group forming becomes evident rather rapidly but complete separation of the groups and, therefore, convergence can take some time, depending on the willingness to change their opinion. The authors call it ”open-mindedness” and it will be denoted by θ in this work. The model has seen extensive interest from a statistical physics

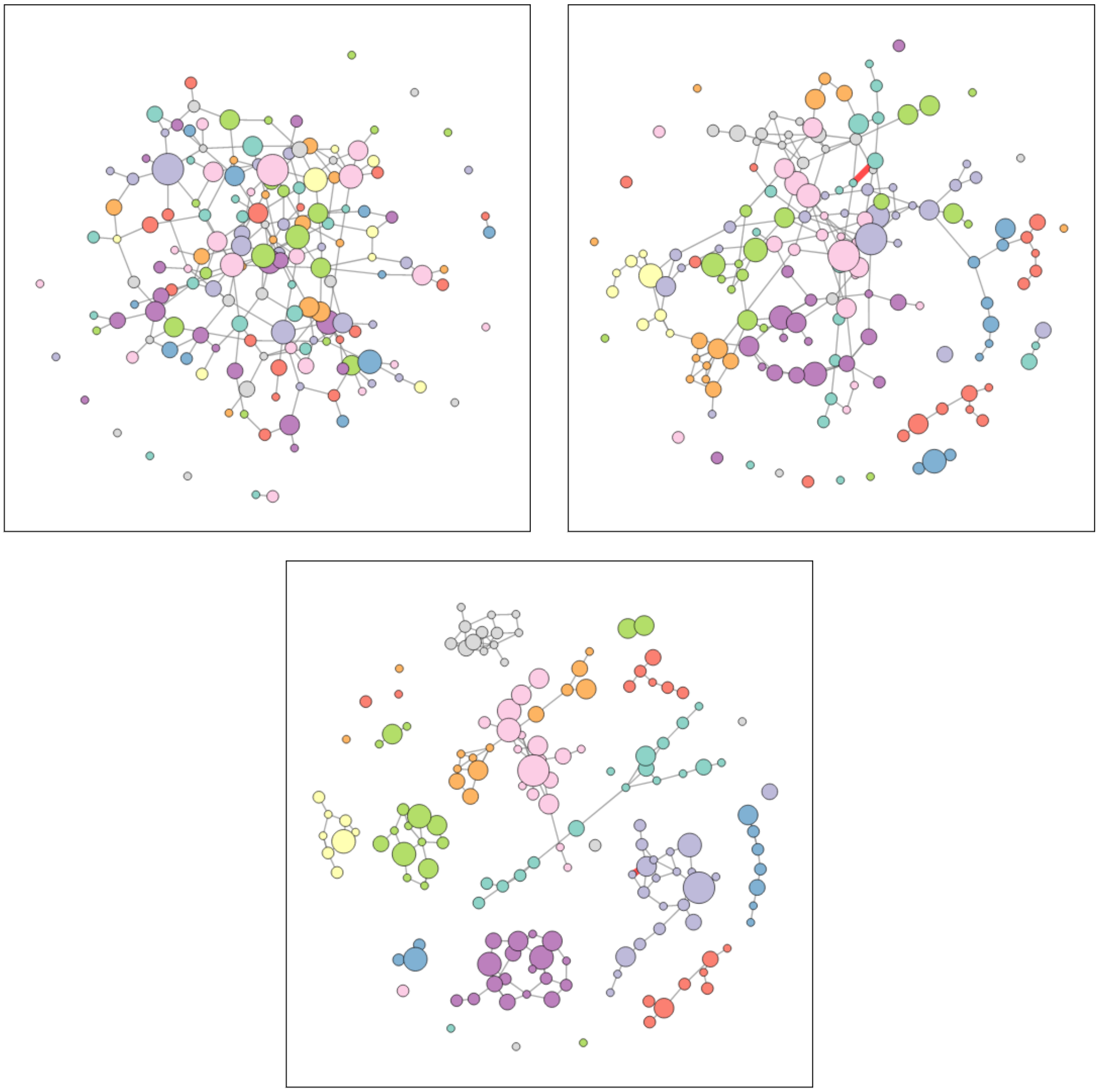

Figure 1: Evolution of the Echo Chamber Model with discrete opinions. Even though the initial network looks heterogenous, one can already discern groups in the second image and finally the separated network into several groups of uniform opinion. Screenshots taken from https://www.complexity-explorables.org/explorables/echo-chambers/ .

perspective under various assumptions on the behavior of the agents, as for example in [HolNew06], [Vesp12] or [PaTraNo06], and reviewed for example in [CaFoLo09] and [ThiBer07] to name some. At the time of composition of this work the cited articles had several thousand citations, but almost all results and observations are based on simulations and a complete theoretical analysis is lacking to this day. In this work we present a qualitative link between the Echo Chamber Model and generalized exclusion processes and attempt a quantitative approach under canonical assumptions on the probability distributions involved in the model.

Throughout this work, when considering the Echo Chamber Model, we understand it as the following discrete time, rule based process of graphs where in every time step the following rules are applied to the current graph.

Denote by $G_t = (\mathcal{V}, \mathcal{E}_t)$ the graph representing the network of individuals and relationships at some time $t \in \mathbb{N}$ and by $\mathcal{O} \in \{[0,1], \{\frac{i}{N} | i \in \{1, \dots, N\}\}\}$ for some $N \in \mathbb{N}$. Assume that the individual $a \in \mathcal{V}$ carries an opinion $X_a^t \in \mathcal{O}$ at time $t \in \mathbb{N}$. Furthermore, there is a uniform tolerance threshold $\theta \in [0,1]$ for all individuals as well as a uniform willingness to change its opinion $\mu \in \left[0, \frac{1}{2}\right)$. Then, the **Echo Chamber Model** evolves as follows.

- Draw uniformly an edge $\langle A, B \rangle \in \mathcal{E}_t$.
- If $|X_A^t - X_B^t| < \theta$, then
  - if $\mathcal{O} = [0,1]$ set
    - firstly, $X_A^{t+1} = X_A^{t+1} + \mu(X_B^{t+1} - X_A^{t+1})$ and,
    - secondly, $X_B^{t+1} = X_B^{t+1} + \mu(X_A^{t+1} - X_B^{t+1})$.
  - if $\mathcal{O} = \{\frac{i}{N} | i \in \{1, \dots, N\}\}$ draw $\mathcal{Z} \sim \text{Ber}(0.5)$ set
    - set, firstly, $X_A^{t+1} = X_A^{t+1} + \mathcal{Z}(X_B^{t+1} - X_A^{t+1})$ and,
    - secondly, $X_B^{t+1} = X_B^{t+1} + (1 - \mathcal{Z})(X_A^{t+1} - X_B^{t+1})$.
- If $|X_A^t - X_B^t| >= \theta$, then
  - Define $N_{\langle A,B \rangle} := \{e = \langle c, d \rangle \notin \mathcal{E}_t \backslash \{\langle A, B \rangle\} | k \in \{A, B\} \text{ or } l \in \{A, B\}\}$.
  - draw uniformly E from $N_{\langle A,B \rangle}$,
  - set $\mathcal{E}_{t+1} = (\mathcal{E}_t \backslash \{\langle A, B \rangle\}) \cup \{E\}$.

We make the distinction between the discrete opinion Echo Chamber Model if $\mathcal{O} = \{\frac{i}{N} | i \in \{1, \dots, N\}\}$ and the continuous opinion Echo Chamber Model if $\mathcal{O} = [0,1]$. The dynamics implied by rules and the discrete time renders the totality of the time-dependent opinions and the graph structure a Markov chain in discrete time. Note that the first half of the rules can be considered as an independent process of the relationships if we keep the opinions constant, i.e., $\theta = 0$, and on the other hand the second half may be used to imply only a change of the opinions when keeping the relationships fixed. To render the theoretical analysis more accessible, it seems judicious to first reduce the complexity of the dynamics by splitting it up into its essential parts before attacking the whole system.

Considering only the dynamics of the opinions under the assumptions that the edges of the graph are constant, the continuous opinion Echo Chamber Model reduces to the Deffuant model. We will review the Deffuant model in Subsection 5.2 and discuss some alternative approaches in the context of non-complete interaction graphs.

The dynamics of the moving edges, under the assumption of constant opinions, isolated, is analyzed in [HePraZha11] using population limits and a certain clustering of the opinions, leaving the inherent finiteness of the system aside. Our approach is based to a large part on combinatorics and graph theoretical techniques. Furthermore, we make a different choice when it comes to the distribution of the edge E drawn from $N_{\langle A,B\rangle}$, due to the following reasons.

Some sources, for example [HePraZha11], propose a two step process, drawing first uniformly a vertex among $\{A, B\}$ and then connecting the drawn vertex with one of its neighbors to which it is not connected. The corresponding neighbor is drawn uniformly among all available ones. This leads to a bias of recreating the old edge again. To illustrate, denote by N_A^c all vertices in G which are not neighbors of A. Then, the probability to create any edge which is not $\langle A, B\rangle$ is either $(2N_A^c)^{-1}$ or $(2N_B^c)^{-1}$ while the probability to recreate the old edge is $(2N_A^c)^{-1} + (2N_B^c)^{-1}$ which satisfies

$$\max\{(2N_A^c)^{-1}, (2N_B^c)^{-1}\} \leq (2N_A^c)^{-1} + (2N_A^c)^{-1}.$$

To remove this bias, we chose the uniform distribution among all possible edges for the dynamics of the edges. This leads, in particular, to results which are coherent with established theories of social systems, as discussed in [Luh84] and [Luh98]. We dedicate the whole Section 8 to an analysis of the finite system and obtain upper bounds on the time to convergence of the model under suitable assumptions. Section 8 can at the same time be seen as a synthesis of all the results in the preceding sections.

2 Summary of this work

The central objective of this work is the characterization of the dynamics presented hereinabove via a generalized version of a particle system which has been object to research for a long time, the exclusion process in discrete time. For a complete analysis we need first some preliminary results on a state space of said generalized exclusion process, which will be a graph whose vertices consist of sets, before going on to a

Markov chain theory based in-depth analysis of its behavior. Therefore, we split the part of this work, dedicated to the Echo Chamber Model as defined above, into to sub-parts, firstly, "Graphs and Combinatorial Structures" and, secondly, "Generalized Exclusion Processes & Absorbing Environments and Social Network Dynamics". The remainder of this introduction is dedicated to an outline of these two sub-parts as well as a short overview of the results obtained and their relevance in the greater context of this work. In particular, we show the necessity of Part II for the rest of the work and, therefore, its integration into the topic, even though in itself it can stand alone for an audience interested in Graph Theory.

Part II: Graphs and Combinatorial Structures

The first part of this work forms the graph and set theoretical foundation. Geometrically, most challenges in the approach, which we chose for the analysis of the Echo Chamber Model, arise from an intricate structure implied by graphs defined on the sub-graphs of regular connected graphs. This includes the necessity to use multi-sets in some proofs. To avoid confusion or lacking notions, we introduce the basic definitions from Graph and Set Theory in Section 3. We start with the fundamental definition of a graph and obtain an important result on the existence of specific sub-graphs, relevant for this work, in Theorem 3.14 supported by Lemma 3.8 and Lemma 3.12 which cover additional cases, out of the scope of Theorem 3.14.

From this specific search of sub-graph geometries, we go on to the more general theory of Matchings in Definition 3.15 and discuss some of the well established results on the topic. Matchings will allows us to characterize dispersed communities via a more general structure which is the central focus of Section 4. Building on the notion of density of sub-graphs, we suggest a characterization of the boundary between the graphs containing sub-graphs of large average density and all other graphs in Conjecture 4.25 where the definition for sub-graphs of large average density is given by Definition 4.24. These will play an essential role later on in Section 6 in the context of interacting particle systems.

After having done this work on graphs and sub-graphs we move on to sets and graphs of sets in Subsection 3.2. We recall basic operations from set theory with an emphasis on the symmetric difference. It will serve as the main tool for identifying possible transitions between two particle configurations on a graph. Already, it is possible to define a family of graphs, which have fixed size subsets of some set as vertices and two subsets are adjacent if their symmetric difference satisfy a size condition. Examples, known in the literature, are the Kneser and the Johnson graph which we recall for the reader and show Proposition 3.33 that a generalized version of the Johnson graph gives indeed rise to this family of graphs. We prove this result even though the author is certain that it has been done elsewhere but wasn't successful in finding a source to cite. Indeed, in Section 4 we come to the conclusion that generalized Johnson graphs are a particular case of an even richer graph family which can serve as a state space of particle processes of exclusion type, i.e., at most one particle can occupy a given vertex.

Section 4 comprehends the central results of Part II obtained by the author. Everything is based on the notion of the k-particle graph $\mathfrak{L}_k$, associated to some underlying graph L which we define in Definition 4.1. Subsequently, we prove equivalencies for connectedness and bipartiteness of L and $\mathfrak{L}_k$ in Proposition 4.2 and Proposition 4.7, respectively. Additionally, we obtain in Lemma 4.3 and Proposition 4.4 a link to the Johnson graph as a special case of the pair $(L, \mathfrak{L}_k)$. This gives us the general framework in which we work but, nonetheless, we enter unconquered territory. It turns out that the vertices of $\mathfrak{L}_k$ have a natural link to sub-graphs of L, which we exploit in Subsection 4.3 to characterize its degree sequence if the underlying graph is a regular graph. In Proposition 4.9 we obtain the size of the edge set and in Proposition 4.11 the average degree of each vertex, which will allow us to make estimates on the flow speed of currents in said graph. A finer analysis reveals symmetries in $\mathfrak{L}_k$ and an characterization if its automorphism group is partially obtained in Proposition 4.8. Throughout Section 4 we make links to sub-graphs of L and their properties as well as importance for the structure of $\mathfrak{L}_k$. Finally, we add to the section a conjecture on the vertex connectivity given by Conjecture 4.22 which the author managed to prove partially but a complete proof was not achieved to this day in the general form. A discussion of two special examples for L, the cycle and the almost complete graph, are given and discussed in further detail, needed for Section 6, Section 7 and Section 8. This discussion completes Part II of this work.

Part III: Absorbing Environments & Social Network Dynamics

Part III represents the main part of this work. It focuses on the building blocks of the Echo Chamber Model and builds a link with a generalized version of a well known process in statistical physics and the theory of interacting particle systems. This allows us to introduce techniques from Markov chain theory as well as propose Markov chain Monte Carlo methods for applications related to this topic. We assume that all interacting particle systems are considered on some underlying finite graph L. Due to the nested structure of Part III, the reader finds in Figure 2 the dependency structure of the sections presented in this part. Note the central role of Part II, Section 4, The k-particle Graph, as it lays the topological foundation of the state space.

Section 5 serves as a collection of multiple topics which are necessary for the analysis of the Echo Chamber Model. Subsection 5.1 is dedicated to a review of classical

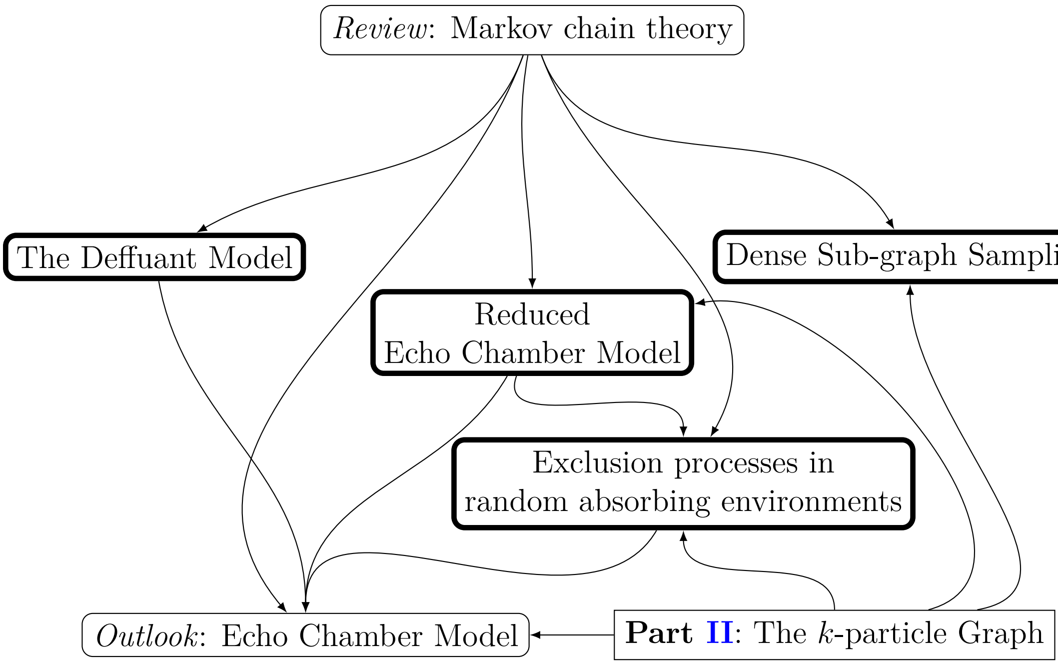

Figure 2: Dependencies of the Subsection of Part III.

Markov chain theory with its definitions and the results on convergence as well as the convergence speed of a Markov chain. Evidently, this review cannot be exhaustive and covers the topics which we need later on for the results obtained in this work. We, then, go on to a discussion of the Deffuant model in Subsection 5.2, where the emphasis is, at first, put on the evolution of the recursively defined opinion distributions over time as well as its convergence which yields a link to differential geometry in Conjecture 5.24. We then discuss a notion of cluster forming called maximum confidence clusters in the context of non-complete interaction graphs, see which gives rise to an invariant structure under the Deffuant model in Definition 5.26. Finally, we obtain a result on the convergence speed of the expected opinion profile after stabilization of the clusters in Theorem 5.29. A short informal discussion on the convergence completes Subsection 5.2.

The proceeding Subsections 5.4 and 5.5 then treat a well known interacting particle system, the exclusion process, as well as a generalization, which is necessary for this work. In Subsection 5.4 we recall the exclusion process, which is classically considered as a continuous-time process and build a bridge to a discrete time interpretation which has also been considered in the literature but to a minor extend than its continuous-time counterpart. This allows for a transition into the same framework as the Echo Chamber Model, which is a discrete time process. It turns out that it is possible to identify parts of the Echo Chamber Model with exclusion type processes but in a generalized form. Therefore, we discuss in Subsection 5.5 the necessary generalization as well as geometric interpretations in terms of sub-graphs. This forms another building block for the analysis done in Section 7 but is, likewise, indispensable for Section 6.

The first example of a generalized exclusion process is discussed in Section 6. It is constructed in such a way that its stationary distribution gives distinct weights to configurations which span sub-graphs with different densities in the underlying graph. We discuss this property in Theorem 6.2, and it follows, in particular, from Proposition

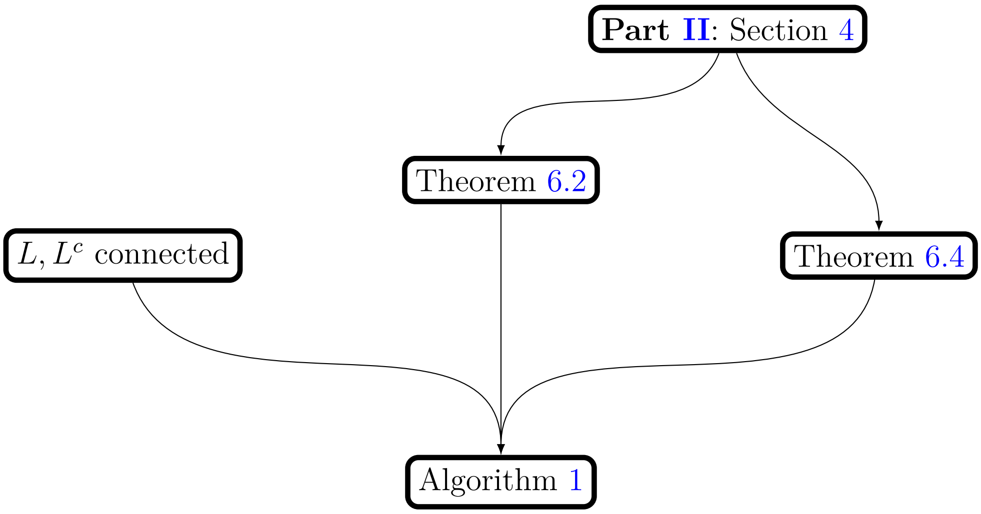

Figure 3: Dependency structure of the main parts of Section 6 whith the final goal of obtaining an alogrithm for dense sub-graph sampling via MCMC.

4.14 and Proposition 6.1. Under conditions on the connectivity of the underlying graph L and its complement we obtain Algorithm 1 for sampling of dense subgraphs of L. The links between the main results of this section are show in Figure 3. Techniques from Markov chain theory combined with results from Section 4 allow us to find bounds on the convergence speed in Theorem 6.4, which turns out to be polynomial in all model parameters and, therefore, sufficiently fast to yield a useful Markov chain Monte Carlo approach for the problem of finding densest subgraphs, discussed in Subsection 6.1. This section serves, aside from its mathematical contribution, as a first taste of the power of possible applications of generalized exclusion processes, before going on to the central one discussed in this work, which arises from the Echo Chamber Model.

Section 7 focuses on a generalized exclusion process associated to the Echo Chamber Model. After having established a canonical Markov chain on the k-particle graph

$\mathfrak{L}_k$, in Subsection 7.2 Theorem 7.1, which represents said exclusion process, we focus completely on the behavior of this Markov chain. We obtain its transition matrix in Lemma 7.2 and Theorem 7.3 and follow a classical analysis pattern for Markov chains, where we establish basic properties as well as its convergence behavior. This will be done throughout Subsection 7.3.2, starting with Theorem 7.4, which discusses irreducibilty and aperiodicity of said Markov chain, and Theorem 7.5 yields the existence of a unique limit. We can conclude this Subsection by characterizing the level sets of said limit in Theorem 7.8 as well as the expected hitting times in Proposition 7.10. The result is based on various geometrical observations as well as a natural reduction of the state space. We lay out the dependencies in Figure 4 because it is one of the more involved proof structures, as were, before, the dependencies of the Algorithm 1. Both results are based on preliminary observations on the lumpability of the Markov

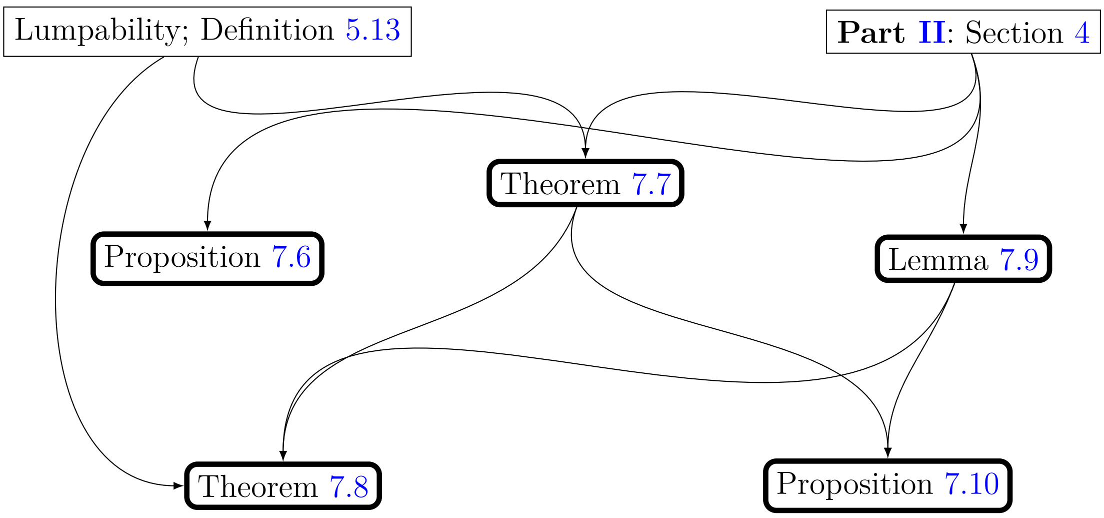

Figure 4: Dependency structure of results leading to a natural reduction of the state space $\mathfrak{L}_k$.

chain as well as the importance of isomorphic subgraphs and cycles in $\mathfrak{L}_k$.

From these qualitative observations, we move on to quantitative results in Subsection 7.3.4. Therein, we focus on the explicit form of the stationary distribution in the cases where the Markov chain is reversible. We establish in Propositions 7.11 to 7.14 for some graphs and parameter choices this property but demonstrate in Theorems 7.18 to 7.20 that this is, depending on the geometry of the underlying graph, for almost no parameter choices the case. In Subsection 7.3.6 reversibility will give us, in the few cases where the Markov chain exhibits this property, a possibility to characterize the convergence speed via the Cheeger constant or bottleneck ratio, as reviewed in Subsection 5.1.3. This is mostly due to the implied geometry of the graph $\mathfrak{L}_k$ in these cases, which renders the necessary calculations accessible. Methods from discrete optimization complete the necessary tool box. In the non-reversible cases, we focus on characterizations of the convergence speed of Markov chains, which do only depend on the structure of the path space of $\mathfrak{L}_k$. We give in Theorem 7.30 a result on this topic which follows from Theorem 5.23 and only depends on the number of paths between vertices in $\mathfrak{L}_k$, i.e., a geometrical property of $\mathfrak{L}_k$.

We complete this section by dedicating Subsection 7.5 to the comparison of the here present Markov chain and the classical discrete time exclusion process as discussed in [DiaSal93]. This reveals the inherent difference in the dynamics due to the transition structure arising from the Echo Chamber Model as well as the qualitative difference between the two associated Markov chains. This brings us, finally, back to the special case of the Echo Chamber Model in Subsection 7.5 where we apply the in Section 7 obtained results to the special underlying structure of the state space under the model assumptions and attempt an interpretation for dynamic social networks.

Part III is completed by a section which reintroduces the opinions, present in the Echo Chamber Model, into the picture. Section 8 is the synthesis of the understanding of the process underlying the movement of the edges in the Echo Chamber Model, obtained in Sections 4 and 7. We work under the assumption that only the edges move and the opinions are constant and, in particular, discrete. Noting that this implies that edges between individuals who have similar opinions persists for all time, one rapidly realizes that this leads to absorbing sites for the associated interacting particle system. We find in Subsection 8.2 and, in particular, in Lemma 8.2 and Lemma 8.3 that absorbing sites do not necessarily lead to the absorption of all particles over long time, even if the number of absorbing sites exceeds largely the number of particles. We, therefore, focus on natural geometric conditions of the random environment for absorption of the whole process and search for upper bounds on the expected time to absorption. One such natural condition is, effectively, defined by the implied dynamics of the Echo Chamber Model, which we call cell-free environments. It represents that non-absorbed particles cannot get stuck among absorbed ones.

The state space $\mathfrak{L}_k$ has to be adjusted to this new situation and we find a canonical extension in Definition 8.9. It consists of the same vertex set as $\mathfrak{L}_k$ but its edge set is a real subset of the edge set of $\mathfrak{L}_k$ and is inherently defined by the position of the absorbing sites. Considering the number of absorbing sites in each configuration, we find a hierarchical structure in the transition graph of an associated Markov chain and calculate the exact number of transitions between layers in Proposition 8.10. This allows us, under assumption of the opinion distribution and using the notion of quasi-stationary distribution, to find a general upper bound on the expected time to

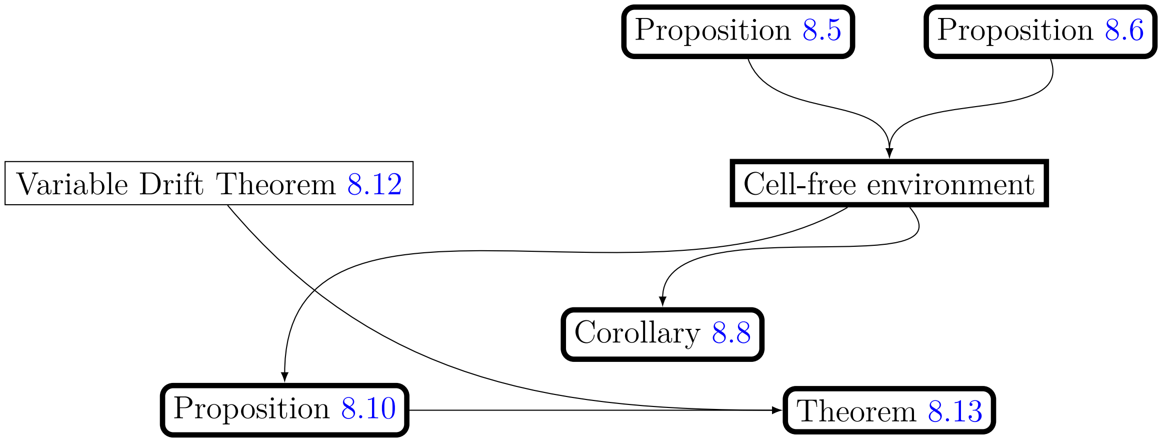

Figure 5: Dependency structure of the main results of Section 8.

absorption in Theorem 8.13. In Figure 5 we lay out the dependency of the results in Section 8 which lead to the final result Theorem 8.13.
In Subsection 8.4 we propose a comparison with a pure death process and obtain a conjecture on an upper bound of the expected time to absorption of all particles via this death process in Conjecture 8.15. This shows, again, the necessity of being able to compare Markov chains quantitatively and we leave this open for further research.

We close this final section of the second part with considerations on the opinion distributions for the Echo Chamber Model and show, that, without the change in opinions, the model does not converge with positive probability in Proposition 8.16. This leads us to the conclusion, that both parts of the Echo Chamber Model can be considered separately but their interaction gives rise to a new behavior. Finally, we propose a possibility to integrate the analysis done in this work into a complete theoretical perspective on the Echo Chamber Model but leave the development of this path to further research.

Part IV: Random Population Dynamics under Catastrophic Events

The third and last part of this work presents the article "Random Population Dynamics Under Catastrophic Events" published by the author of this work in collaboration with Patrick Cattiaux, Sylvie Roelly and Samuel Sindayigaya. The article takes a macroscopic perspective on population dynamics on the level of the population size in contrast to the microscopic scale, which we used in Part III by considering interactions between individuals. In the article we introduce a new kind of continuous time Markov chain called $BD + C_{\mathfrak{n}}$, which is an extension of a birth-death process and includes the possibility of an instantaneous reduction of the population size to a fixed size $\mathfrak{n}$. We

pursue two goals in the article. Firstly, the characterization to the expected first time the population size falls to the fixed size $\mathfrak{n}$ if the population has a size larger than $\mathfrak{n}$ at the beginning of the observation. We obtain an identity for this time as a function of the number of individuals by which the population exceeds initially the fixed size $\mathfrak{n}$. Writing the initial population size as $\mathfrak{n} + i$ for some $i \in \mathbb{N}$, we achieve, furthermore, a convergence result of said time to 0 as $i \to \infty$ with a characterization of the convergence speed as well. Using these results, we find positive recurrence of the Markov chain and, therefore, existence and uniqueness of a stationary distribution using Lyapunov functions.
The second goal consists in characterizing the expected population size as well as its variance at any fixed time t. Using techniques for ordinary differential equations combined with the master equation for moments of Markov chains, we obtain meaningful upper bounds on both the expected population size and variance. We quantify the difference between the actual solution for the expected value and the upper bound, show convergence for $t \to \infty$ to some finite limit and obtain an explicit upper bound on said limit. The whole analysis is based around nullclines and the uniqueness of positive zeros of an associated polynomial function. We finish Part IV with an outlook on a process which models the same behavior but where the instantaneous reset is replaced by a reduction in the population based on individuals.

PART II

GRAPHS AND COMBINATORIAL STRUCTURES

Contents

3 Graphs and sets: Definitions and Notations

Outline of this section: This section contains two major parts. In the first part, we review definitions and results from graph theory as well as add and prove results on sub-graphs of regular and strongly regular graphs. We start with the most basic properties to set the scene before focusing on vertex induced sub-graphs and their properties with a particular focus on regular and strongly regular graphs. A short review of matchings and the implications of the existence of a matching in a graph on vertex induced sub-graphs will be discussed shortly. We identify certain types of sub-graphs in regular graphs and define a family of graphs which is particularly densely packed. It can be shown that the set of $\bar{d}$-regular graphs is separated by a particular curve which converges nicely as the size of the vertex set converges to infinity. We finish the section with a review of paths and their defining role in graphs.

In the second part of this section, we turn to sets and multi-sets as well as graphs defined on subsets, where the neighbors are defined by some specified set operations. In particular, intersections and symmetric differences will play a central role in this context. We review and outline some of the properties of graphs defined on sub-sets of some set, which are using the intersection to define neighborhoods of vertices. The family of Johnson graphs and Kneser graphs will be discussed with a focus on their link. We extend this link graphs on subsets where the symmetric difference defines the neighborhood relationship. We show that this family of graphs is exactly given by the family of Johnson graphs but both perspectives have their utility depending on the context. The section will be finished by a short comment on generalizations of these graphs.

3.1 Graphs and graph properties

Graphs are a discrete structure, which are, aside from the deep theoretical research on their properties, often used to model social networks and other relationship based interactions of particles or individuals. Furthermore, they may serve as discrete state spaces for certain stochastic processes, most prominently random walks, dynamic particle systems like the Glauber dynamics of the Ising model, exclusion processes and many more. In this section we are going to recall various definitions from graph theory, lead the way towards the tools and structures we need in later sections and present some new results and insights. All definitions and results for which no proof is provided are taken from the classical literature on graph theory, see for classical examples [Wil96] or [Bol98], and for more recent ones [New10], [Bol12] or [PemSki09].

Definition 3.1. *Let V be any set. The simple undirected graph $L = (V, E)$ is the tuple* *of sets V, E where V are the vertices and $E \subset \{\langle v, w\rangle | v, w \in V\}$ are the edges. The unordered tuple of elements $\langle v, w\rangle$ represents an undirected edge, i.e., $\langle v, w\rangle = \langle w, v\rangle$. The Matrix $A = (a_{v,w})_{v,w\in V}$ with $a_{v,w} = \mathbb{1}_{\langle v,w\rangle \in E}$ is called the adjacency matrix of L.*

On graphs we may define paths, a distance as well as various forms of sub-graphs. A prominent sub-graph class are vertex induced sub-graphs which are spanned by a certain subset of the original vertex set and all edges included therein.

Definition 3.2. *Let $L = (V, E)$ be any simple graph and $\mathfrak{v} \subseteq V$. Then the graph $L_{\mathfrak{v}} = (\mathfrak{v}, E_{\mathfrak{v}})$ with $\langle v, w\rangle \in E_{\mathfrak{v}}$ if and only if $v, w \in \mathfrak{v}$ and $\langle v, w\rangle \in E$ is called the vertex induced sub-graph of L on $\mathfrak{v}$.*

While graphs may be of any form in general, finding certain properties or substructures may shed light on the behavior of stochastic processes, in particular, Markov chains as for example random walks. While for random walks the degree sequence of a graph, i.e., the ordered set of degrees of each vertex, yield interesting insights, we might look for more subtle structures to analyze more complex processes. Two, for this work important ones, shall be defined in Definitions 3.3, 3.11 and 3.13.

Definition 3.3. *Consider a triangle $\tau = \{\tau_1, \tau_2, \tau_3\}$ and an supplementary vertex set $\beta = \{\beta_1, \beta_2, \beta_3\}$. We define a tri-star as a graph given by $\mathcal{T} = (V_{\mathcal{T}} = \tau \cup \beta, E_{\mathcal{T}})$ where for $v, w \in V_{\mathcal{T}}$ we define an edge as $\langle v, w\rangle \in E_{\mathcal{T}}$ if and only if either $v, w \in \tau$ or $v = \beta_i$ and $w = \tau_i$.*
Consider any graph $G = (V, E)$ and assume there is a sub-graph $G_s = (V_s, E_s)$ with $V_s = \{v_1, v_2, v_3, w_1, w_2, w_3\} \subset V$ of G which is isomorphic to $\mathcal{T}$ by an isomorphism Φ which preserves the neighborhood relationships of $\beta_i = \Phi(v_i)$ and $\tau_j = \Phi(w_j)$. Then we say that G contains a tri-star $\mathcal{T}$.

A tri-star is basically a triangle with beams at every vertex such that the end points of the beams are only connected with the respective corner of the triangle. This can be made clearer by the visualization seen in Figure 6. For certain classes of graphs a

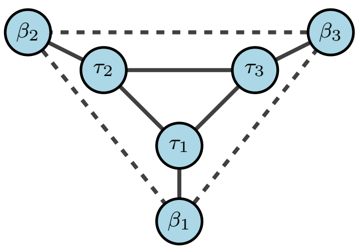

Figure 6: The tri-star $\mathcal{T}$. Dashed lines represent possible edges in G, if G contains a tri-star.

tri-star may never be found as a sub-graph while for others it is inevitable. To discuss these properties later on we recall the definitions of bipartite, regular as well as strongly regular graphs. We start with regular graphs.

Definition 3.4. *A graph $L = (V, E)$ is called a $\bar{d}$-regular graph on $\bar{n}$ vertices if*

- $|V| = \bar{n}$,
- $\deg(v) = \bar{d}$ *for all* $v \in V$.

Evidently, due to the fact that the sum over all degrees in a graph is equal to twice the number of edges, in a regular graph we are faced with the constraint that either $\bar{d}$ or $\bar{n}$ has to be even. When it is evident that a specific one of the two has to be even due to additional conditions we do not necessarily recall this fact, in particular, when we consider graphs with $\bar{d} = \bar{n} - 2$. Even though this seems like a strong restriction, one can consider graphs which are even more constrained but which yield, nonetheless, a rich structure.
The stronger form of regularity we will use in this work is strong regularity which implies particularly nice results for the Markov chain we are going to discuss later on.

Definition 3.5. *A graph $L = (V, E)$ is called a $(\bar{n}, \bar{d}, \alpha, \beta)$ strongly regular graph on $\bar{n}$ vertices if*

- $|V| = \bar{n}$,
- $\deg(v) = \bar{d}$ *for all* $v \in V$,
- α *- number of common neighbors of two adjacent vertices,*
- β *- number of common neighbors of two nonadjacent vertices.*

A common example of a strongly regular graph arises from the line graph of a complete graph. We recall the definition of a line graph.

Definition 3.6. *Let $G = (\mathcal{V}, \mathcal{E})$ be a simple graph. Its associated line graph is the defined as the graph L with vertex set $\mathcal{E}$ and edge set E where for $v, w \in \mathcal{E}$ we have $\langle v, w \rangle \in E$ if and only if v and w are incident edges in G.*

While it is a quite abstract definition it becomes rather intuitive when considering an example. In Figure 7 we illustrate the construction of a line graph from the complete graph on 4 vertices. It turns out that a line graph L of a complete graph G_c is a strongly regular graph with parameters $\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right)$, where n is the number of vertices of G_c. So, while being a class of graphs with a very demanding structure, strongly regular graphs arise naturally from other graphs. Indeed, their structure allows the characterization of certain vertex induced sub-graphs. For vertex induced sub-graphs of size 3 a complete characterization is possible.

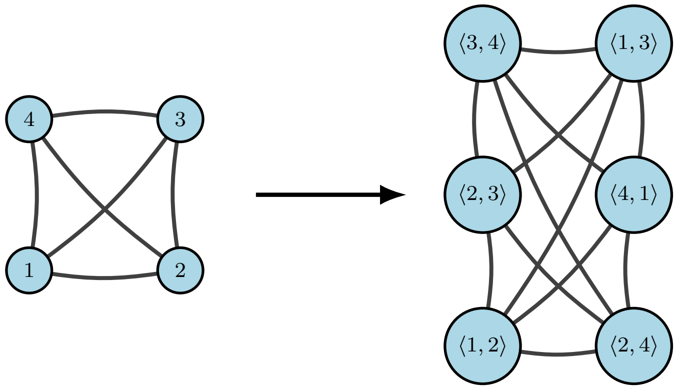

Figure 7: From complete graph on 4 vertices to its line graph.

Lemma 3.7. *Let L be a strongly regular graph with parameters $(\bar{n}, \bar{d}, \alpha, \beta)$. Then depending on the parameters on L, there are at most 4 distinct vertex induced sub-graphs of size 3, a triangle $\mathfrak{t}$, a path $\mathfrak{p}$ of length 2, the disjoint union $\mathfrak{b}$ of an edge and a vertex as well as a completely disconnected graph $\mathfrak{d}$. Additionally, L contains*

- $\frac{\bar{n}\bar{d}\alpha}{6}$ *copies of* $\mathfrak{t}$,
- $\frac{\bar{n}(\bar{n}-\bar{d}-1)\beta}{2}$ *copies of* $\mathfrak{p}$,
- $\frac{\bar{n}\bar{d}(\bar{n}-(2\bar{d}-\alpha))}{2}$ *copies of* $\mathfrak{b}$ *as well as*
- $\frac{\bar{n}(\bar{n}-\bar{d}-1)(n-2\bar{d}+\beta-2)}{6}$ *copies of* $\mathfrak{d}$.

Proof. The existence of the 4 different sub-graphs follows directly by construction, since these are the only 4 possible distinct vertex induced sub-graphs on 3 vertices for any connected simple graph. It remains, consequently, to prove the number of such sub-graphs. We start with the number of triangles.

Each edge induces α triangles, since the end points of each edge have α common neighbors. Since a triangle contains exactly 3 edges, Summing over all edges counts every triangle 3 times. Therefore, there are $\frac{1}{3}\frac{\bar{n}\bar{d}}{6}\alpha$ triangles, which proves the first claim.

Arguing along the same lines, any two non adjacent vertices have β common neighbors and there are exactly $(\bar{n} - \bar{d} - 1)$ non-adjacent vertices to any fixed vertex. Since being non-adjacent is a symmetric property, we obtain by summing over all pairs of non-adjacent vertices that there are $\frac{1}{2}\bar{n}(\bar{n} - \bar{d} - 1)\beta$ copies of $\mathfrak{p}$ in L.

We note that the complement L^c is a strongly regular graph with parameters $(\bar{n}, (\bar{n}-$

$\bar{d}-1), \bar{n}-2\bar{d}+\beta-2, \bar{n}-2\bar{d}+\alpha)$. Additionally, any $\mathfrak{p}'$ in L^c induces a $\mathfrak{b}$ in L. Consequently, there are $\frac{\bar{n}\bar{d}(\bar{n}-(2\bar{d}-\alpha))}{2}$ copies of $\mathfrak{b}$ in L by applying the previous point to L^c.

Again, by the structure of L^c and the fact that any triangle $\mathfrak{t}'$ in L^c induces a $\mathfrak{d}$ in L, we obtain that there are $\frac{\bar{n}(\bar{n}-\bar{d}-1)(n-2\bar{d}+\beta-2)}{6}$ copies of $\mathfrak{d}$ in L. □

Having seen this example one can wonder about the general structure of a strongly regular graph and it turns out that it is, in fact, inherently linked to the tri-star.

Lemma 3.8. *Let L be a strongly regular graph with parameters $(\bar{n}, \bar{d}, \alpha, \beta)$ satisfying $\bar{d} \in \{3, \dots, \bar{n}-3\}$ and $\alpha \geq 1$. Then, L contains a tri-star as defined in Definition 3.3.*

Proof. Let L as in the lemma. Then, there is a triangle $\tau = \{\tau_1, \tau_2, \tau_3\}$ due to $\alpha \geq 1$. It remains to show that each τ_i has a neighbor β_i which is not adjacent to the remaining two corners of the triangle. Assuming that one pair τ_i, τ_j has only common neighbors. Then $\alpha = \bar{d} - 1$ since τ_i and τ_j are neighbors. By strong regularity each τ_i and τ_j have, hence, $\alpha = \bar{d} - 1$ neighbors with the remaining corner τ_k. This implies that $N_{\tau_1} \backslash \tau = N_{\tau_2} \backslash \tau = N_{\tau_3} \backslash \tau$. Pick $v \in N_{\tau_1} \backslash \tau$. Then v is adjacent to τ_1, τ_2, τ_3. Hence, $\tau_v = \{v, \tau_1, \tau_2\}$ is a triangle and τ_1, τ_2 have only common neighbors by the previous observation. Therefore, by the same arguments as before we obtain $N_{\tau_1} \backslash \tau_v = N_{\tau_2} \backslash \tau_v = N_v \backslash \tau_v$. Since v was arbitrary, we obtain that there is a connected component of L of size $\bar{d}$ and as L is connected $\bar{n} = \bar{d} + 1 \leq \bar{n} - 2$ which yields a contradiction to the claim that there is one pair τ_i, τ_j which has only common neighbors. □

Note that $\alpha \geq 1$ is always satisfied for the line graph of a complete graph with more than two vertices, giving rise to the parameters $\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right)$.

On the other hand, bipartite graphs never contain a tri-star.

Definition 3.9. *Let $L = (V, E)$ be a simple graph. L is called bipartite if there are sets $V_1, V_2 \subset V$ such that $V = V_1 \sqcup V_2$ and $\langle v, w \rangle \in E$ if and only if $v \in V_1$ and $w \in V_2$.*

A direct consequence of this definition is the following property.

Lemma 3.10. *A graph L is bipartite if and only if all closed paths have even length.*

This immediately rules out the existence of a tri-star since it contains a closed path of length 3, the triangle $\{\tau_1, \tau_2, \tau_3\}$.

The tri-star can, obviously, be generalized to any circle of arbitrary length instead of the triangle τ. It turns out that for our purposes the case of a cube is sufficient since all other cases can then be discussed thanks to a way simpler structure. We start with the definition of the generalization of the tri-star graph to a cube-star graph.

Definition 3.11. *Consider a cube $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_3\}$ and a set of vertices $\beta = \{\beta_1, \beta_2, \beta_3, \beta_4\}$. We define a cube-star as a graph given by $\mathcal{C} = (V_\mathcal{C} = \alpha \cup \beta, E_\mathcal{C})$ where for $v, w \in V_\mathcal{C}$ we define an edge as $\langle v, w \rangle \in E_\mathcal{C}$ if and only if either $v = \alpha_i, w = \alpha_{(i \bmod 4)+1}$ or $v = \beta_i$ and $w = \alpha_i$.*
Consider any graph $G = (V, E)$ and assume there is a sub-graph $G_s = (V_s, E_s)$ with $V_s = \{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\} \subset V$ of G which is isomorphic to $\mathcal{C}$ by an isomorphism Φ which preserves the neighborhood relationships of $\beta_i = \Phi(v_i)$ and $\alpha_j = \Phi(w_j)$. Then we say that G contains a cube-star $\mathcal{C}$.

Definition 3.11 supplements Definition 3.3 and to visualize we only exchange the triangle in Figure 6 with a cycle of length 4. But it is indeed an addition which yields an extension to of the tristar to the discrete torus which is considered in the context of exclusion processes, see for example [Morris04] or [LaOlVa02]. Indeed, we find that a cube-star graph is contained in any discrete torus $(\mathbb{Z}/a\mathbb{Z})^m$ for $a \geq 5$.

Lemma 3.12. *Let $m \in \mathbb{N}$, $m \geq 2$ and $a \in \mathbb{N}$, $a \geq 5$. Then $(\mathbb{Z}/a\mathbb{Z})^m$ contains a cube-star graph in the sense of Definition 3.11.*

Proof. We proof the claim inductively over m. Let for now $m = 2$. Figure 8 illustrates that $(\mathbb{Z}/a\mathbb{Z})^2$ contains a cube-star graph for $a = 5$ and, therefore, also for all $a \geq 5$. For

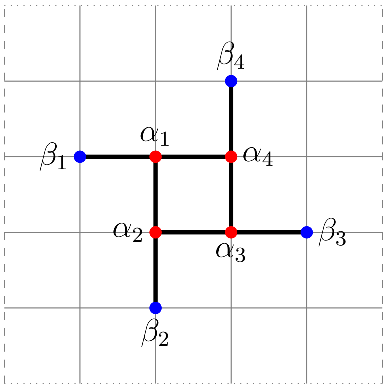

Figure 8: The cube-star $\mathcal{C}$ in $(\mathbb{Z}/a\mathbb{Z})^2$. The dotted lines are identified with each other due to the quotient, as are the dashed lines.

any $m > 2$ consider the m dimensional discrete hyper-cube of length a. Choose one two dimensional slice. By the previous observations this slice contains a cube-star. The identification of vertices due to the quotient $\mathbb{Z}/a\mathbb{Z}$ does not change the neighborhood relationship of any of the vertices present in $\mathcal{C}$ in the chosen slice. Consequently, the cube-star $\mathcal{C}$ is also contained in $(\mathbb{Z}/a\mathbb{Z})^m$. □

Obviously, generalizing $\mathcal{T}$ and $\mathcal{C}$ further using larger and larger cycles becomes more and more restrictive on the class of $\bar{d}$-regular graphs containing such structures.

Indeed, for our purposes, it suffices to characterize a small sub-graph of 6 vertices, which allows to separate neighborhoods of two vertices.

Definition 3.13. *Consider two path graphs* $\gamma = \{\gamma_1, \xi, \gamma_2\}$ *and* $\gamma' = \{\gamma'_1, \xi', \gamma'_2\}$ *of three vertices each. We define a double-pitchfork as a graph given by* $\mathcal{D} = (V_\mathcal{D} = \gamma \cup \gamma', E_\mathcal{D})$ *where for* $v, w \in V_\mathcal{D}$ *an unordered pair satisfies* $\langle v, w\rangle \in E_\mathcal{D}$ *if and only if* $\langle v, w\rangle$ *is either an edge in* γ *or* γ' *or* $v = \xi, w = \xi'$.
Consider any graph $G = (V, E)$ *and assume there is a sub-graph* $G_s = (V_s, E_s)$ *with* $V_s = \{v_1, w, v_2, v'_1, w', v'_2\} \subset V$ *of* G *which is isomorphic to* $\mathcal{D}$ *by an isomorphism* Φ *which preserves the neighborhood relationships of* $\gamma_i = \Phi(v_i)$ *and* $\xi' = \Phi(w')$, *of* $\gamma_i = \Phi(v_i)$ *and* $\gamma'_j = \Phi(v'_j)$ *with* $i \neq j$ *as well as of* $\gamma'_i = \Phi(v'_i)$ *and* $\xi = \Phi(w)$. *Then we say that* G *contains a double-pitchfork* $\mathcal{D}$.

We illustrate the double-pitchfork in Figure 9. Note that the double-pitchfork is

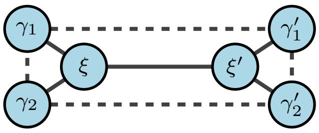

Figure 9: The double-pitchfork $\mathcal{D}$. Dashed lines represent possible edges in G, if G contains a double-pitchfork.

a quite general object which might be found in a wide variety of regular graphs, for example in Figure 8. From Figure 8 pick the vertices α_1 to α_4 as well as β_1 and β_2 and set $\xi := \alpha_1$, $\xi' := \alpha_2$, $\gamma_1 := \beta_1$, $\gamma_2 := \alpha_4$, $\gamma'_1 := \beta_2$ and $\gamma'_2 := \alpha_3$. The same observation

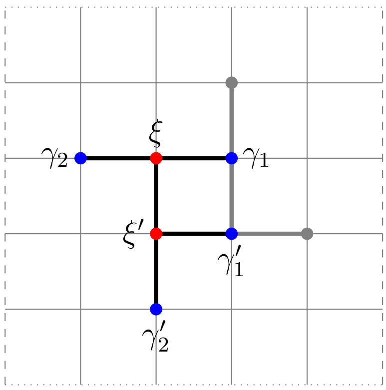

Figure 10: The double pitchfork $\mathcal{D}$ embedded into the cube-star $\mathcal{C}$ in $(\mathbb{Z}/a\mathbb{Z})^2$. The dotted and dashed lines are each identified with each other as before.

can then be made when considering objects similar to the cube-star or tri-star with larger cycles. Nonetheless, both the cube-star and the tri-star are not covered by solely considering the double pitchfork. In particular, the existence of $\mathcal{D}$ can only be assured if there is no cycle of length 4 in L which induces also as a vertex induced sub-graph a cycle of length 4 to avoid γ_i and γ'_j being connected for $i \neq j$. We show now that this is sufficient for the existence of $\mathcal{D}$.

Theorem 3.14. *Let $\bar{n} \in \mathbb{N}$ and $3 \leq \bar{d} \leq n-3$. Assume that L does not contain any vertex induced sub-graph of size 4 which is a cycle graph. Then, $\bar{d}$-regular graph $L = (V, E)$ on n vertices contains a double-pitchfork in the sense of Definition 3.13.*

Indeed, the claim of Theorem 3.14 is equivalent to the assertion that there is a connected pair of vertices v, w in L which has at most $\bar{d}-3$ common neighbors. Note that by regularity of L for each edge $\langle v, w\rangle \in E$ the neighborhood of $|N_v \backslash N_w| = |N_w \backslash N_v|$. We are going to exploit this property, which is equivalent to $|N_v \triangle N_w| = 2|N_v \backslash N_w| = 2|N_w \backslash N_v|$, in the proof of Theorem 3.14 ad noceam.

Proof. Assume that for all edges $\langle v, w\rangle \in E$ the vertices v, w have at least $\bar{d}-2$ common neighbors, i.e., there is at most one neighbor of v and one of w which are not neighbors of w and v, respectively. Considering a first case where all vertices have $\bar{d}-1$ neighbors in common, the graph L is complete and, hence, $\bar{n} = \bar{d}+1 \leq \bar{n}-2$ we arrive at a contradiction.

So, assume there is an edge $\langle v, w\rangle$ such that v and w have exactly $\bar{d}-2$ common neighbors. Therefore, there are $u_v, u_w \in V$ such that $\langle u_v, v\rangle \in E$, $\langle u_w, w\rangle \in E$, $\langle u_w, v\rangle \notin E$ and $\langle u_v, w\rangle \notin E$. Then, since u_v is a neighbor of v but not a neighbor of w, u_v has a neighbor u'_v which is not a neighbor of v. The complete structure is shown in Figure 11. We define the set $\tilde{V} := \{u'_v, u_v, v, w, u_w\}$ and arrive at the equality for

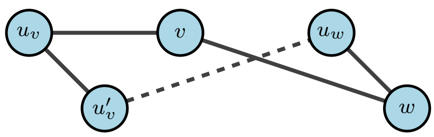

Figure 11: Configuration of u'_v, u_v, v, w and u_w where the dashed edge represents the possibility to identify u'_v and u_w in some graphs.

the neighborhoods N_v, N_w and N_{u_v}

$$N_v \backslash \{u_v, w\} = N_w \backslash \{u_w, v\}, \quad N_v \backslash \{u_v, w\} = N_{u_v} \backslash \{u'_v, v\}$$

which allows us to conclude $N_w \backslash \{u_w, v\} = N_{u_v} \backslash \{u'_v, v\}$. Therefore, there is a set $\bar{V}$ of size $\bar{d}-2$ such that

$$\bar{V} = N_v \backslash \{u_v, w\} = N_w \backslash \{u_w, v\} = N_{u_v} \backslash \{u'_v, v\}. \tag{3.1}$$

In the case where $u'_v \neq u_w$ it follows that any $m \in \bar{V}$ and v have at most $\bar{d}-3$ common neighbors, since u'_v and u_w are neighbors of m but not of v and $\langle m, v\rangle \in E$. Hence, we arrive at a contradiction to the minimal number of common neighbors being $\bar{d}-2$.

On the other hand, if $u'_v = u_w$, then for any $m \in \bar{V}$ the vertex v has a neighbor v_m which is not a neighbor of m, since $\langle m, u_w \rangle \in E$, and $v_m \in \bar{V}$. Using the assumption on the minimal number of common neighbors, we arrive at the conclusion that v_m is unique given m. Therefore, in the vertex induced sub-graph $L_{\tilde{V} \cup \bar{V}}$ any vertex $m \in \bar{V}$ satisfies

$$\deg^{L_{\tilde{V} \cup \bar{V}}}(m) = 4 + (\bar{d} - 3) - 1 = \bar{d}$$

where we us that $|\bar{V} \backslash \{m\}| = \bar{d} - 3$ and there is exactly one $v_m \in \bar{V}$ which is not neighbor of m. Additionally, for all $x \in \tilde{V}$ we know

$$\deg^{L_{\tilde{V} \cup \bar{V}}}(x) = \bar{d}$$

such that $L_{\tilde{V} \cup \bar{V}}$ forms a connected sub-graph of L where all vertices have degree $\bar{d}$. Therefore, there are no edges pointing outward of $\tilde{V} \cup \bar{V}$ such that $V = \tilde{V} \cup \bar{V}$ and, therefore,

$$\bar{n} = |V| = |\tilde{V} \cup \bar{V}| = \bar{d} - 2 + 4 = \bar{d} + 2 \leq \bar{n} - 3 + 2 = \bar{n} - 1. \tag{3.2}$$

Finally, we arrive also in this case at a contradiction such that the assumption that for all edges $\langle v, w \rangle \in E$ the vertices v, w have at least $\bar{d} - 2$ common neighbors is incorrect. Combined with the absence of a vertex induced sub-graph which is isomorphic to a cycle graph of length 4, we obtain the claim of the theorem. □

Note that this result breaks down when we deal with non-regular graphs. To illustrate this case, consider any regular graph which satisfies the conditions of the theorem. Pick a pitchfork, which you find necessarily under the assumptions of the theorem. Then, replace the edge $\langle \xi, \xi' \rangle$ by a path of length two, introducing a vertex of degree 2 between the two vertices ξ and ξ'. Further analysis is possible by using bounds on the degree sequence of a non-regular graph. While an interesting direction to pursue for a closer understanding of local structures of graphs it is out of the scope of this work and is left for later research.

Another important but, in contrast to $\mathcal{D}$ and $\mathcal{T}$, well established class of sub-graphs are matchings. A matching contains exclusively connected components of size 2, i.e., no two edges in a matching are incident to one another in L, see [LovPlu09]. We turn to the formal definition.

Definition 3.15 ([LovPlu09])**.** *Let $L = (V, E)$ be a connected loop-free simple graph and $\mathcal{M} \subset E$. The set $\mathcal{M}$ is called a matching if no two edges $e_1, e_2 \in \mathcal{M}$ are adjacent in L. A maximal matching $\mathcal{M}$ is a matching which is not subset of any other matching and a perfect matching contains for any $v \in V$ an edge $e_v \in \mathcal{M}$ such that e_v is incident to v.*

Since we focus mostly on regular and strongly regular graphs in the context on this work, we can employ the following results, in particular setting the focus on bipartite regular graphs.

Proposition 3.16 (Petersen's theorem, [LovPlu09])**.** *Let L a $\bar{d}$ connected regular graph with edge connectivity at least $\bar{d} - 1$. Then L contains a perfect matching $\mathcal{M}$.*

The following corollary on regular bipartite graphs will be more important to us since it also to vary the degree from $\bar{d} = 2$ up to $\bar{d} = \bar{n} - 1$ where $\bar{n}$ is the size of each of the two vertex sets forming L.

Corollary 3.17 ([LovPlu09])**.** *Let L a $\bar{d}$ connected regular bipartite graph. Then L contains a perfect matching $\mathcal{M}$.*

Corollary 3.17 allows us to start with some initial connected $\bar{d}$-regular bipartite graph L and then reduce, iteratively, the degree by 1 to $\bar{d} - 1$ by choosing a perfect matching $\mathcal{M}$ in L and then defining the graph $L' = (V, E \backslash \mathcal{M})$. The graph L' is then by definition of a perfect matching a $\bar{d} - 1$-regular graph. We are going to employ this property when discussing the impact of the density of a graph on movement of particle systems of exclusion in later sections.

Another approach to the density of a graph is the calculation of the number of walks from any vertex v to another vertex w. A walk in contrast to a path allows to visit edges as well as vertices multiple times and is, thus, the corresponding graph theoretic object to capture the trajectories of Markov chains.

Definition 3.18 (Walks and paths in graphs)**.** *Let $L = (V, E)$ be a simple graph. A walk from $v \in V$ to $w \in V$ of length n is a sequence $v_0, \ldots, v_n \in V$ with $v_0 = v$ and $v_n = w$. The walk is called closed, if $v = w$ and open otherwise. A path is a walk where any two vertices are distinct and a closed path is a path with $v = w$.*

In general, if A is the adjacency matrix of the graph L, then the number of walks from v to w of length n is the (v, w)-th entry of A^n. In general, it remains difficult to give explicit expressions for the number of walks and the number of paths. Due to their omnipresence in various graph problems, see for example [FePeKo01], it is an important object in research which can be approached from a spectral theoretical or combinatorial perspective to obtain at least meaningful upper and lower bounds.

Special classes of graphs allow for a description of their paths. We want to present to examples, which we need later on in the context of connected regular graphs which exhibit a connected complement graph. We start with the first in its general form and present some properties for regular graphs.

Definition 3.19. *Let $L = (V, E)$ be a simple connected graph. If L contains an Eulerian cycle, i.e., a path with same start and end point which only uses each edge exactly once, we call L an Eulerian graph.*

The central result which is much easier to verify then the property given in the definition is the following which only focuses on the degree sequence of L.

Proposition 3.20. *Let L be a simple connected graph. Then, it is an Eulerian graph if and only if it has no vertex of odd degree.*

This statement has first been claimed to be true by Euler, who gave a partial proof, a complete proof was found by Carl Hierholzer, as described in [BiLlWi86], and gives immediately the existence of an Eulerian path in L by looking at each vertex and its degree. Another graph property which is defined by paths and cycles are idem called by the name of their inventor, the class of Hamilton graphs.

Definition 3.21 ([PemSki09])**.** *Let $L = (V, E)$ be a simple connected graph. If L contains a Hamiltonian cycle, i.e., a path with same start and end point which visits each vertex exactly once, we call L a Hamiltonian graph.*

Like the Eulerian graphs, there is a characterization for Hamiltonian graphs using the degrees of the graph.

Theorem 3.22 (Dirac's Theorem)**.** *Let $L = (V, E)$ be a simple connected graph with $|V| \geq 3$ and $\deg(v) \geq \frac{|V|}{2}$ for all $v \in V$. Then, L is a Hamiltonian graph.*

Furthermore, for regular graphs both properties to be Eulerian or Hamilton have a link and a fundamental importance when considering regulars graphs and their complements, looking for connectivity in both of them.

Lemma 3.23. *Let $L = (V, E)$ be a $\bar{d}$-regular connected graph and assume that L^c is connected. Then*

- *L or L^c is Eulerian,*
- *L or L^c is Hamiltonian.*

Proof. To proof the first point, we only have to check the degrees of L and L^c. If $\bar{d}$ is even, then L is Eulerian by Proposition 3.20. On the other hand, note that L^c is a $|V| - \bar{d} - 1$ regular graph and, hence, if $\bar{d}$ is odd, then $|V|$ is even by the properties of regular graphs and $|V| - \bar{d} - 1$ is even. Consequently, by Proposition 3.20 the complement graph L^c is Eulerian which concludes the first claim.
Now, assume that both L and L^c are not Hamiltonian. Consequently, using Dirac's Theorem both L and L^c satisfy $\bar{d} < \frac{|V|}{2}$ and $|V| - \bar{d} - 1 < \frac{|V|}{2}$. Consequently, we obtain

$$|V| - 1 = \bar{d} + |V| - \bar{d} - 1 \leq \frac{|V|}{2} - 1 + \frac{|V|}{2} - 1 = |V| - 2$$

which leads to a contradiction. Consequently, L or L^c is a Hamiltonian graph. □

The property presented in Lemma 3.23 will, in fact, be limiting to results we obtain in Section **??** and we will discuss it further in that context. A last definition we want to add to this section will be later linked the identifiability of subsets in a graph and, therefore, a possible reduction of the state space of some Markov chain. A natural way is to use quotient graphs which are defined as follows.

Definition 3.24 ([San12]). *Let $G = (\mathcal{V}, \mathcal{E})$ be a simple graph and $\sim$ be a equivalence relation on $\mathcal{V}$. Then, the graph ${}^{G}/\sim$ with vertex set ${}^{V}/\sim$ and edge set*

$$\tilde{\mathcal{E}} := \{\langle [v]_\sim, [u]_\sim \rangle | \exists \tilde{v} \in [v]_\sim, \tilde{u} \in [u]_\sim : \langle \tilde{v}, \tilde{u} \rangle \in \mathcal{E}\} \tag{3.3}$$

is called the quotient graph of G with respect to $\sim$.

Given a suitable equivalence relation $\sim$ one can reduce the graph to its "essential" components or combine sets of vertices which are identifiable in view of some application. The question on how to find this identifiability will occupy us for some time in Subsection 7.3.3 in light of a Markov chain arising from the Echo Chamber Model.

In the next steps we are going to consider more specific class of graphs where the vertices consist of subsets of some other mathematical object which may bring its own topology with it. This will influence the edge structure, in particular, in Section 4 when the subsets, defining the vertices, will be sub-graphs of some other graph. This will lead us to the definition of a neighborhood relationship of vertex induced sub-graphs which has a direct interpretation using interacting particle systems.

3.2 Sets, multi-sets and graphs of sets

Central object to the analysis of exclusion processes presented in this work are sets, especially subsets of the vertex set of some graph on which the particles move. In the first part of this section we aim at recalling a few definitions from set theory based on [Hein03], [Fer08] and [LeuChe92]. In particular, we consider the notion of multi-sets and functions on such sets as well as elementary operations on multi-sets like the intersection, union and difference. It turns out that these are the canonical generalizations of the well known operations on sets. Furthermore, we are going to recall one classic operation which might not be as omnipresent as others, namely the symmetric difference.

Definition 3.25. *Let A, B two sets. The symmetric difference of A and B is defined by*

$$A \triangle B = (A \backslash B) \cup (B \backslash A). \tag{3.4}$$

The symmetric difference gives for two given sets the elements which belong exclusively to one or the other. Hence, it serves well as representation of the way a process of sets behaves when in each time step at most one element of the set may change.

The symmetric difference captures the difference between the state of such a process at one time step and the next and, moreover, clearly describes the element which has changed. In particular, for dynamic particle system with a fixed number of particles the symmetric difference will always be formed between sets with identical cardinality. A simple result which we want to recall is on the cardinality of the symmetric difference of two sets with identical cardinality.

Lemma 3.26. *Let X be any set and $A, B \subset X$ such that $|A| = |B|$. Then $|A \triangle B|$ is even or equals zero.*

Proof. By direct calculations the claim follows.

$$\begin{aligned} |A \triangle B| &= |(A \backslash B) \cup (B \backslash A)| = |(A \backslash B)| + |(B \backslash A)| \\ &= |A| - |A \cap B| + |B| - |A \cap B| = 2(|A| - |A \cap B|). \end{aligned}$$

□

This operation will be the center point of various constructions and the perspective of an exclusion process as a Markov chain on a graph of sets. Hence, it is worth it to focus on a more involved and quite aesthetic graph theoretic problem namely several possibilities to define graphs via sets. This leads to a blow up in the state space but eventually will allow us access to Markov chain theory and the calculation of explicit results.

Multi-sets are the direct generalization of sets which allows to contain multiple times the same element. In contrast to a vector a multi-set remains without order. This implies that an element of which appear multiple copies in the same multi-set cannot be distinguished from the remaining copies.

Definition 3.27 (Multi-sets)**.** *Let A be a set and $m_A : A \to \mathbb{Z}^+$ a map. Then the multi-set M with elements is defined by*

$$M := \{(a, m_A(a)) | a \in A\}. \tag{3.5}$$

Evidently, a multi-set M constructed from $A = \{a, b\}$ and $m_A : A \to \mathbb{Z}^+$ with $m(a) = 1$ and $m(b) = 2$ can be written as $\{(a, 1), (b, 2)\} = \{a, b, b\}$ without ambiguity. The set-operations on such multi-sets are then defined in a canonical way, applying the operations the underlying sets and adapting the corresponding maps. For us the only operations of interest are the union, intersection and difference, for which we recall the definition in what follows.

Definition 3.28. *Let A, B sets and $m_A : A \to \mathbb{Z}^+$, $m_B : B \to \mathbb{Z}^+$ two maps. Extend the maps m_A and m_B to zero for elements in $B \backslash A$ and in $A \backslash B$, respectively. Then the set operations union, intersection, difference and symmetric difference of the multi-sets M_A, M_B are defined as*

- $M_A \cup M_B := \{(x, \max\{m_A(x), m_B(x)\})|x \in A \cup B\}$,
- $M_A \cap M_B := \{(x, \min\{m_A(x), m_B(x)\})|x \in A \cup B\}$,
- $M_A \backslash M_B := \{(x, \max(m_A(x) - m_B(x), 0))|x \in A \cup B\}$.

where elements of the form $(x, 0)$ *are dropped for convenience.*

Multi-sets will come in handy to show that exclusion processes in their Markov chain representation are, in fact, irreducible under weak assumptions on the transition probabilities linked to the symmetric difference. We are going to analyze this Markov chain representation on a graph of sets which arises canonically from the underlying graph L on which we consider the particle positions. Graphs defined via sets and set-operations have a long history and one of the more prominent examples is the Kneser graph, named after Martin Kneser. See [ReTsch21] for more information on Martin Kneser and his work including the following definition.

Definition 3.29. *Let* A *be any set with* $\bar{n} = |A|$ *and consider some* $k \in \mathbb{N}^*$ *with* $k \leq \bar{n}$. *Set* $V_{A,k}$ *the set of subsets of* A *of size* k. *The Kneser graph* $\mathcal{K}(\bar{n}, k) = (V_{\mathcal{K}}, E_{\mathcal{K}})$ *is defined as* $V_{\mathcal{K}} := V_{A,k}$ *and* $\langle \mathfrak{v}, \mathfrak{w} \rangle \in E_{\mathcal{K}}$ *if and only if* $\mathfrak{v} \cap \mathfrak{w} = \emptyset$.

Note that the particular structure of A and the nature of its elements do not play a substantial role in the definition but rather the parameters $\bar{n}$ and k. Two sets with identical cardinality give for fixed k rise to the same $K(\bar{n}, k)$. Simple examples of a Kneser graph are $K(\bar{n}, 1)$ which corresponds to the complete graph or $K(5, 2)$ which is isomorphic to the Peterson graph, which is defined in [HolShe93] and the previously stated result being stated in the same source. A representation of the Peterson graph can be seen in Figure 12. Note the regular structure, every vertex having degree 3. A

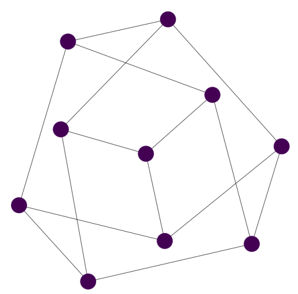

Figure 12: The Peterson graph which corresponds to $K(5, 2)$.

direct extension of the Kneser graph is the Johnson graph, see [HolShe93] for further reading, which allows for more variability when it comes to the intersections of two sets defining their neighborhood relationship. We present in definition 3.30 its most general form to show afterwards its connection to another graph defined on sets by the symmetric difference as set operation.

Definition 3.30. *Let A be any set with $\bar{n} = |A|$ and let $l, k \in \mathbb{N}^*$ with $l \leq k \leq \bar{n}$. Set $V_{A,k}$ the set of subsets of A of size k. The generalized Johnson graph $\mathcal{J}(\bar{n}, k, l) = (V_{\mathcal{J}}, E_{\mathcal{J}})$ is defined as $V_{\mathcal{J}} := V_{A,k}$ and $\langle \mathfrak{v}, \mathfrak{w} \rangle \in E_{\mathcal{J}}$ if and only if $|\mathfrak{v} \cap \mathfrak{w}| = l$.*

The simple Johnson graph can be obtained by setting $t = k - 1$ in the previous definition 3.30. We illustrate the Johnson graph $\mathcal{J}(5, 2, 1)$ in Figure 13. Having based

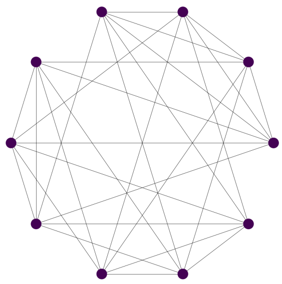

Figure 13: The generalized Johnson graph $\mathcal{J}(5, 2, 1)$.

the definition of the edges both in the Kneser and in the generalized Johnson graph on the intersection of sets on can wonder which other sensible possibilities there are to define graphs on subsets of some set V. We propose the symmetric difference which generates a neighborhood relationship that can be interpreted as follows. Considering two subsets of fixed size of some set V they are considered as adjacent if their symmetric difference has a fixed size t. This implies that the number of elements in which the two sets differ is fixed and, hence, forms the counter part to the generalized Johnson graph. We proceed with the formal definition, a visualization and a couple of results.

Definition 3.31. *Let $\bar{n} \in \mathbb{N}$, $k \in \{1, \ldots, \bar{n} - 1\}$, $t \in \{1, \ldots, \bar{n} - 1\}$ and $\mathcal{V} = \{\mathfrak{v} \subset \{1, \ldots, n\} |\; |\mathfrak{v}| = k\}$. Define $\mathcal{O}(\bar{n}, k, t) := (\mathcal{V}_{\bar{n};k;t}, \mathcal{E}_{\bar{n};k;t})$ and $(\mathfrak{v}, \mathfrak{w}) \in \mathcal{E}_{\bar{n};k;t}$ iff $|\mathfrak{v} \triangle \mathfrak{w}| = t$.*

In contrast to the Johnson graph, the graph $\mathcal{O}(\bar{n}, k, t)$ measures similarity when t is small since the symmetric difference is then small. On the other hand, if t is big, then the symmetric difference of two neighbors is equally big and, consequently, neighbors do not have many elements in common. Consequently, one might find a link between the two. We first establish a link in a particular case between $\mathcal{O}(\bar{n}, k, t)$ and the Kneser graph. The link is due to the fact that one can rewrite the symmetric difference in terms of intersections of the involved sets. We review this result in Proposition 3.32 and the result involving Johnson graphs in Proposition 3.33 with proofs but they are not original in this work and the ideas can be find elsewhere.

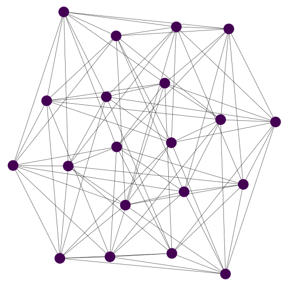

Figure 14: The graph $\mathcal{O}(6,3,2)$, a representative of the class $\mathcal{O}(\bar{n},k,t)$.

Proposition 3.32. *Let $\bar{n} \in \mathbb{N}$, $k \in \{1,\dots,\bar{n}-1\}$ and denote by $\mathcal{K}(\bar{n},k) = (V_{\mathcal{K}}, E_{\mathcal{K}})$ a Kneser graph. Then $\mathcal{K}(\bar{n},k) \cong \mathcal{O}(\bar{n},k,2k)$. Therefore, the Kneser graphs are a subclass of the previously defined graphs $\mathcal{O}$.*

Proof. Let X be any set of size $\bar{n}$. First of all $V_{\mathcal{K}} = \mathcal{V}_{\bar{n};k;t}$ independently of t. Let $A, B \subset X$ be subsets of size k such that $A \cap B = \emptyset$. Hence, $(A,B) \in E_{\mathcal{K}}$ and using the calculations in the proof of Lemma 3.26 we obtain

$$|A \triangle B| = 2(|A| - |A \cap B|) = 2(k-0) = 2k. \tag{3.6}$$

Therefore, $(A,B) \in \mathcal{E}_{\bar{n};k;2k}$. Evidently, the statement $|A \triangle B| = 2k$ is equivalent to $A \cap B = \emptyset$ for subsets $A, B \subset X$ of size k. Hence $(A,B) \in \mathcal{E}_{\bar{n};k;2k}$ if and only if $(A,B) \in E_{\mathcal{K}}$. Thus, the identity map of subsets of X provides an isomorphism. □

The Kneser graph being related to the Johnson grah $\mathcal{J}(\bar{n},k,0)$ by construction, one can wonder about the implications of the result in Proposition 3.32 for the relationship between $\mathcal{J}(\bar{n},k,l)$ and $\mathcal{O}(\bar{n},k,t)$. We establish it in what follows.

Proposition 3.33. *Let $\bar{n} \in \mathbb{N}$, $k \in \{1,\dots,\bar{n}-1\}$ and denote by $\mathcal{J}(\bar{n},k,l) = (V_{\mathcal{J}}, E_{\mathcal{J}})$ a generalized Johnson graph. Then $\mathcal{J}(\bar{n},k,l) \cong \mathcal{O}(\bar{n},k,2(k-l))$.*

Proof. Let X be any set of size $\bar{n}$. As stated previously, $V_{\mathcal{J}} = \mathcal{V}_{\bar{n};k;t}$ independently of t. Furthermore, let $A, B \subset X$ be subsets of size k such that $|A \cap B| = l$. Hence, $(A,B) \in E_{\mathcal{J}}$ and using the calculations in the proof of Lemma 3.26 we obtain

$$|A \triangle B| = 2(|A| - |A \cap B|) = 2(k-l). \tag{3.7}$$

Therefore, $(A,B) \in \mathcal{E}_{\bar{n};k;2(k-l)}$. Evidently, the statement $|A \triangle B| = 2(k-l)$ is equivalent to $|A \cap B| = l$ for subsets $A, B \subset X$ of size k. Hence $(A,B) \in \mathcal{E}_{\bar{n};n-k;2k}$ if and only if $(A,B) \in E_{\mathcal{K}}$. Thus, the identity map of subsets of X provides an isomorphism. □

Finally, this allows us to make a well known link between the Kneser graph $\mathcal{K}(\bar{n}, k)$ and the Johnson graph $\mathcal{J}(\bar{n}, k, 0)$.

Corollary 3.34. *For any parameters* $\bar{n}, k \in \mathbb{N}$ *with* $k \leq \bar{n}$ *one finds* $\mathcal{J}(\bar{n}, k, 0) \cong \mathcal{K}(\bar{n}, k)$.

Proof. Setting $l = 0$ in Proposition 3.33 and using Proposition 3.32 we obtain

$$\mathcal{J}(\bar{n}, k, 0) \cong \mathcal{O}(\bar{n}, k, 2(k-0)) \cong \mathcal{K}(\bar{n}, k) \tag{3.8}$$

□

Indeed, the class of $\mathcal{O}(\bar{n}, k, t)$ graphs are exactly the Johnson graphs $\mathcal{J}(\bar{n}, k, l)$ since for any parameters $\bar{n}, k, t, l$ the equation $2(k-l) = t$ always has a unique solution when fixing two of the three parameters involved.

While in itself interesting objects to analyze, the families $\mathcal{J}(\bar{n}, k, l)$ and $\mathcal{O}(\bar{n}, k, l)$ are restrictive in the sense that they do not necessarily include the structure of the space, their vertices are constructed from. In the following section we introduce the central graph theoretical object of this work. It is constructed from some underlying graph using the ideas underlying the construction of $\mathcal{J}(\bar{n}, k, l)$ and it turns out that we retrieve $\mathcal{J}(\bar{n}, k, l)$ in a special case.

4 k-Particle Graph (kPG) and its Properties

Outline of this section: This section is dedicated to the topology of graphs underlying exclusion processes with indistinguishable particles on a simple connected graph L. We define a new graph $\mathfrak{L}_k$ which represents the possible configurations of particles on the underlying simple graph. It turns out that the neighborhood relationship in $\mathfrak{L}_k$ defined by the symmetric difference of two configurations corresponds to the natural transition behavior of exclusion processes with indistinguishable particles in discrete or continuous time because at each time step or jump time at most one particle may change its position. Consequently, in the case of a transition, the start and end configuration differ by exactly one element. Hence, the symmetric difference of said configurations has size two. The family $\mathcal{O}(\bar{n}, k, 2)$ of graphs, presented in Subsection 3.2 Definition 3.31 is, therefore, a good candidate to look into. We are going to make a short remark on the case of distinguishable particles and the arising complications at the end of this section.
In this work we only consider next neighbor transitions for the particles on said graph L additionally constrained by the structure of this underlying graph L which is not taken into account by the construction of $\mathcal{O}(\bar{n}, k, t)$. Hence, we have to add this constraint which solely amounts to restricting the existing edges. We present in what follows the complete Definition 4.1. It will serve as the state space of Markov chains associated to generalized exclusion processes and is part of our central contribution to the understanding of these processes.

4.1 Definiton of kPG

We ant the k-particle graph induced by some graph L and a number k to capture the properties we have mentioned in the introduction of this section, i.e., the displacement of a particle creates an edge, every vertex is a configuration of particles on L and no two particles may occupy the same vertex in L. We capture those properties by defining the following graph $\mathfrak{L}_k$.

Definition 4.1. *Consider a simple graph* $L = (V, E)$*. Define* $\mathfrak{V}_k = \{\mathfrak{v} \subset v |\ |\mathfrak{v}| = k\}$ *and* $\mathfrak{E}_k = \{\langle \mathfrak{v}, \mathfrak{w} \rangle | \mathfrak{v} \triangle \mathfrak{w} = \{v, w\}; \langle v, w \rangle \in E\}$*. We call* $\mathfrak{L}_k = (\mathfrak{V}_k, \mathfrak{E}_k)$ *the* k*-particle graph and denote by* $\deg_k(\mathfrak{v})$ *the degree of* $\mathfrak{v} \in \mathfrak{V}_k$ *in* $\mathfrak{L}_k$*.*

To the best of our knowledge, this graph, based on some underlying graph L, has not yet been considered in the literature. We discuss its rich structure and possible implications of a deeper understanding of it on the analysis of vertex induced subgraphs of L throughout the remainder of this section. In Figure 15 we show how to construct $\mathfrak{L}_k$ from an underlying graph L to illustrate its structure. Evidently,

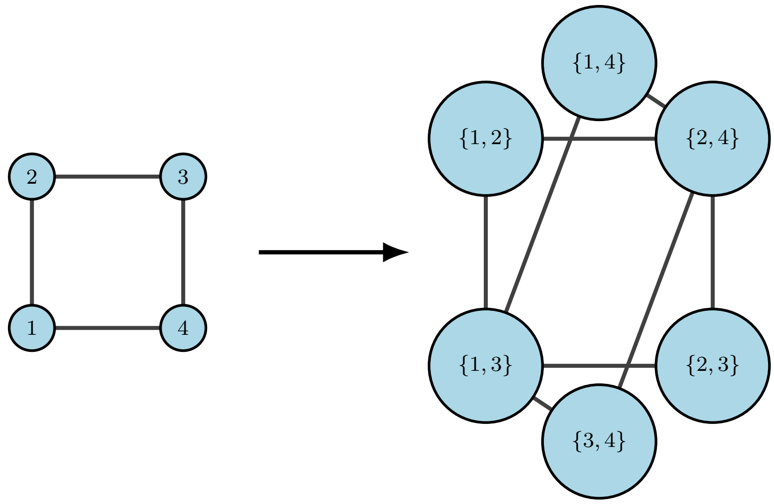

Figure 15: Construction of $\mathfrak{L}_k$ from a 2-regular graph on 4 vertices for $k = 2$. Every vertex on the right hand side corresponds to a possible configuration of 2 particles on L and the edges represent possible transitions. Note that $\mathfrak{L}_k$ is not regular.

the graph $\mathfrak{L}_k$ captures all properties of L, i.e, we can reconstruct L from $\mathfrak{L}_k$ up to isomorphisms. To this end, recall that each vertex can be interpreted as a vertex induced sub-graph of L and all edges incident to a specific vertex in $\mathfrak{L}_k$ corresponds to an edge which points outward of the induced sub-graph. Iterating through all vertices in $\mathfrak{L}_k$ we can reconstruct all neighborhood relationships in L and, therefore, L up to isomorphisms. While being the foundation to our understanding of families of exclusion processes in later sections, it is far from its sole purpose and it is, indeed, in itself an interesting object.

The goal for the remainder of this section is to establish properties of $\mathfrak{L}_k$ needed later on for the analysis of a generalized exclusion process. In particular, we are going to analyze the connectivity of $\mathfrak{L}_k$ depending on L and the role of L being bipartite. This will play a role in the proofs of irreducibility of Markov chains on $\mathfrak{L}_k$ as well as apperiodicity of said chains. The methods, which we employ, rely heavily on the notion of multi-sets which we reviewed in Subsection 3.2.

4.2 General Properties of kPG

While we consider exclusively connected graphs L throughout this work, certain properties are difficult to derive while staying in this unconstrained setting. Nonetheless, some important ones may be proven nonetheless, in particular those, which show the distinction to the case of marked particles. We start with the connectivity of $\mathfrak{L}_k$.

Proposition 4.2. *Let $L = (V, E)$ be any connected simple graph, $\bar{n} := |V|$ and $k \in \{1, \ldots, \bar{n} - 1\}$. Then, the associated graph $\mathfrak{L}_k = (\mathfrak{V}_k, \mathfrak{E}_k)$ is connected.*

Proof. For this proof we use the concept of multi-sets, i.e., sets which allow the appearance of the same element multiple times. E.g. $\{1, 1, 2\} \neq \{1, 2\}$. If a function acts on a multi-set in such a way that it changes one specific element which appears multiple times only one of them is altered. Assume for the rest of this proof that any set is a multi-set and any function mapping from sets to sets maps from multi-sets to multi-sets instead. In particular, any $\mathfrak{v} \in \mathfrak{V}_k$ will be considered as a multi-set.

Since L is connected, there is for any pair $v, w \in V$ a self-avoiding path $\phi_{v,w}$ between v and w. Let $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$. We are going to prove that there is path between $\mathfrak{v}$ and $\mathfrak{w}$. Define $\bar{\mathfrak{v}} := \mathfrak{v} \backslash \mathfrak{w}$ and $\bar{\mathfrak{w}}$, analogously. Since $|\mathfrak{v}| = k = |\mathfrak{w}|$ also $|\bar{\mathfrak{v}}| = |\bar{\mathfrak{w}}|$.
Fix $v_1 \in \bar{\mathfrak{v}}$ and $w_1 \in \bar{\mathfrak{w}}$. Then there exists a path $\phi^1_{v_1,w_1}$ in L. We want to construct iteratively a sequence of maps $(\Phi^1_i)_{i=1}^{|\phi_{v_1,w_1}|}$ by

$$\Phi^1_1(\mathfrak{u}^1_0) = \left(\mathfrak{v} \backslash \{\phi^1_{v_1,w_1}(0)\}\right) \cup \{\phi^1_{v_1,w_1}(1)\} =: \mathfrak{u}^1_1 \tag{4.1}$$

with $u^1_0 = \mathfrak{v}$. Furthermore, $\mathfrak{u}^1_i = \Phi^1_{i-1}(u^1_{i-1})$. With the same procedure for $j = 2, \ldots, |\bar{w}|$ and $\mathfrak{u}^j_0 = \mathfrak{u}^{j-1}_{|\phi_{v_{j-1},w_{j-1}}|}$ we obtain a sequence of maps

$$\Psi = \left(\Phi^1_1, \ldots, \Phi^1_{|\phi_{v_1,w_1}|}, \Phi^2_1, \ldots, \Phi^{|\bar{\mathfrak{w}}|}_{|\phi_{v_{|\bar{\mathfrak{w}}|},w_{|\bar{\mathfrak{w}}|}}|}\right)$$

which maps the $\mathfrak{v}$ to $\mathfrak{w}$ by $\left(\Phi^{|\bar{\mathfrak{w}}|}_{|\phi_{v_{|\bar{\mathfrak{w}}|},w_{|\bar{\mathfrak{w}}|}}|} \circ \ldots \circ \Phi^2_1 \circ \Phi^1_{|\phi_{v_1,w_1}|} \circ \ldots \circ \Phi^1_1\right)(\mathfrak{v}) = \mathfrak{w}$. Elements might appear twice in the same $\mathfrak{u}^j_i$. Denote by τ the first entry in Ψ such that $(\Psi_\tau \circ \ldots \circ \Psi_1)(\mathfrak{v})$ contains the same entry twice and by κ the largest number such that $(\Psi_{\tau+\iota} \circ \ldots \circ \Psi_1)(\mathfrak{v})$ contains one entry twice for $\iota = 0, \ldots, \kappa$. Transform the vector $(\Psi_{\iota'})_{\iota'=1}^{\tau+\kappa}$ into

$$(\Psi'_{\iota'})_{\iota'=1}^{\tau+\kappa} := (\Psi_1, \ldots, \Psi_{\tau+\kappa}, \Psi_{\tau+\kappa-1}, \ldots, \Psi_\tau). \tag{4.2}$$

Then $\left(\Psi'_{\tau+\iota} \circ \ldots \circ \Psi'_1\right)(\mathfrak{v})$ contains all elements only once for $\iota = 0, \ldots, \kappa$. Iterate this procedure until $\Psi' = (\Psi'_{\iota'})_{\iota'=1}^{|\Psi|}$ such that for all $\iota' \in \{1, \ldots, |\Psi|\}$ we have $(\Psi'_{\iota'} \circ \ldots \circ \Psi'_1)(\mathfrak{v}) \in \mathfrak{V}_k$ and for $\iota' \in \{1, \ldots, |\Psi| - 1\}$

$$(\Psi'_{\iota'} \circ \ldots \circ \Psi'_1)(\mathfrak{v}) \triangle (\Psi'_{\iota'+1} \circ \ldots \circ \Psi'_1)(\mathfrak{v}) = \{u^j_i, u^j_{i+1}\} \tag{4.3}$$

with $\langle u^j_i, u^j_{i+1}\rangle \in E$ and $(\Psi'_{|\Psi|} \circ \ldots \circ \Psi'_1)(\mathfrak{v}) = \mathfrak{w}$. Hence $((\Psi'_{\iota'} \circ \ldots \circ \Psi'_1)(\mathfrak{v}))_{\iota'=0}^{|\Psi|}$ defines a path from $\mathfrak{v}$ to $\mathfrak{w}$ in $\mathfrak{L}_k$. □

Having shown that $\mathfrak{L}_k$ is connected if and only if L is connected, one can wonder which other fundamental properties of L are inherited by $\mathfrak{L}_k$. To continue the discussion on the global structure of $\mathfrak{L}_k$ for simple connected graphs L, we consider the complement of L denoted by L^c and the relationship between the induced graph $\mathfrak{L}_k^c$ and $(\mathfrak{L}_k)^c$ where the complement is taken with respect to the Johnson graph $J(\bar{n}, k)$.

Lemma 4.3. *Let $L = (V, E)$ be a simple connected graph on $\bar{n} := |V|$ vertices and $k \in \{1, \ldots, \bar{n}-1\}$. Denote by $L_{\mathfrak{v}} = (\mathfrak{v}, E_{\mathfrak{v}})$ the vertex induced sub-graph of $\mathfrak{v}$ in L and by $L_{\mathfrak{v}}^c = (\mathfrak{v}, E_{\mathfrak{v}}^c)$ the vertex induced sub-graph of $\mathfrak{v}$ in $L^c = (V, E_c)$. Denote by $\mathfrak{L}_k$ and $\mathfrak{L}_k^c$ the k-particle graphs of L and L^c, respectively. Moreover, denote by $\deg_k(\mathfrak{v})$ the degree of $\mathfrak{v}$ in $\mathfrak{L}_k$ and by $\deg_k^c(\mathfrak{v})$ the degree of $\mathfrak{v}$ in $\mathfrak{L}_k^c$. Then,*

$$\deg_k(\mathfrak{v}) + \deg_k^c(\mathfrak{v}) = k(\bar{n} - k).$$

Proof. For any subset $\mathfrak{v} \subset V$ with $|\mathfrak{v}| = k$ the size of the induced sub-graphs of L and L^c satisfy

$$|E_{\mathfrak{v}}| + |E_{\mathfrak{v}}^c| = \frac{k(k-1)}{2}.$$

Additionally, we derive

$$\begin{aligned}\deg_k(\mathfrak{v}) + \deg_k^c(\mathfrak{v}) &= \sum_{v \in \mathfrak{v}} \deg(v) - \deg^{L_{\mathfrak{v}}}(v) + (\bar{n} - 1 - \deg(v)) - \deg^{L_{\mathfrak{v}}^c}(v) \\ &= k(\bar{n}-1) - 2(|E_{\mathfrak{v}}| + |E_{\mathfrak{v}}^c|) = k(\bar{n}-1) - k(k-1) = k(\bar{n}-k).\end{aligned}$$

□

Hence, adding up the degrees of $\mathfrak{v} \in \mathfrak{V}_k$ in $\mathfrak{L}_k$ and $\mathfrak{L}_k^c$ gives a constant which corresponds to the degrees in $J(\bar{n}, k)$. We prove that this link between $\mathfrak{L}_k$ and $\mathfrak{L}_k^c$ is in fact natural and the Johnson graph the natural overarching structure for the analysis of the graph $\mathfrak{L}_k$. In fact, the symmetric difference, which we used to define edges in $\mathfrak{L}_k$, allows for an even deeper identification, now for the graphs $\mathfrak{L}_k$ and $\mathfrak{L}_{\bar{n}-k}$ being practically identical as shall be shown in Proposition 4.4.

Proposition 4.4. *Let L be a simple connected graph on $\bar{n}$ vertices and $k \in \{1, \ldots, \bar{n}-1\}$. Using the notations from Lemma 4.3 the graphs $\mathfrak{L}_k^c$ and $(\mathfrak{L}_k)^c$ satisfy*

$$\mathfrak{L}_k^c \cong (\mathfrak{L}_k)^c \tag{4.4}$$

where the complement is taken with respect to the Johnson graph $J(\bar{n}, k)$. Furthermore, the graph $\mathfrak{L}_k$ satisfies $\mathfrak{L}_k \cong \mathfrak{L}_{\bar{n}-k}$.

Proof. Consider $(\mathfrak{L}_k)^c$. Then an edge $\langle \mathfrak{v}, \mathfrak{w} \rangle$ in $J(\bar{n}, k)$ is an edge in $(\mathfrak{L}_k)^c$ if and only if $\mathfrak{v} \triangle \mathfrak{w} = \{v, w\}$ and $\langle v, w \rangle \notin E$. Consequently, $\langle \mathfrak{v}, \mathfrak{w} \rangle$ is an edge in $(\mathfrak{L}_k)^c$ if and only if $\mathfrak{v} \triangle \mathfrak{w} = \{v, w\}$ and $\langle v, w \rangle \in E^c$. Therefore, $\langle \mathfrak{v}, \mathfrak{w} \rangle$ is an edge in $\mathfrak{L}_k^c$. Since the vertex sets are identical and by Lemma 4.3 each edge in $J(\bar{n}, k)$ is either an edge in $\mathfrak{L}_k$ or $\mathfrak{L}_k^c$, we conclude $\mathfrak{L}_k^c \cong (\mathfrak{L}_k)^c$.

We turn now to the second claim. By basic combinatorics of drawing without repetition $|\mathfrak{V}_k| = \binom{\bar{n}}{k} = |\mathfrak{V}_{\bar{n}-k}|$ holds true. Let $\mathfrak{v} \in \mathfrak{V}_k$ and define $\mathfrak{v}^c := V \backslash \mathfrak{v} \in \mathfrak{V}_{\bar{n}-k}$.

Define the map $\Phi_k^{\bar{n}-k}(\mathfrak{v}) = \mathfrak{v}^c$ which is bijective due to the preceding observations. Let $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ such that $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$. Hence, $\mathfrak{v} \triangle \mathfrak{w} = \{v, w\}$ and $\langle v, w \rangle \in E$ and $\mathfrak{v}^c \triangle \mathfrak{w}^c = \{v, w\}$ and $\langle v, w \rangle \in E$. Therefore, also $\Phi_k^{\bar{n}-k}(\mathfrak{v}) = \mathfrak{v}^c \sim \mathfrak{w}^c = \Phi_k^{\bar{n}-k}(\mathfrak{w})$. The map $\Phi_k^{\bar{n}-k}$, therefore, defines an isomorphism between $\mathfrak{L}_k$ and $\mathfrak{L}_{\bar{n}-k}$ and, consequently, $\mathfrak{L}_k \cong \mathfrak{L}_{\bar{n}-k}$. □

More properties can be inherited from L but some question are of special interest for later part of this work. In particular, does a bipartite graph L render $\mathfrak{L}_k$ also bipartite? We are going to find the response to this question in Propositions 4.5 and 4.7.

Proposition 4.5. *If L is bipartite so is $\mathfrak{L}_k$.*

Proof. We again use the notion of multi-sets. Assume $\mathfrak{L}_k$ is not bipartite and let $\phi = (\mathfrak{v}_1, \mathfrak{v}_2, \dots, \mathfrak{v}_l, \mathfrak{v}_1)$ be a cycle of odd length, i.e, l is an odd positive integer. Let $\mathfrak{v}_{l+1} = \mathfrak{v}_1$ and define the multi-set E_ϕ of edges in L by

$$E_\phi = \{\langle v, w \rangle \in E | \exists i \in \{1, \dots, l\}, \mathfrak{v}_i \triangle \mathfrak{v}_{i+1} = \{v, w\}\}.$$

Since ϕ is a cycle we can construct, using the edges in E_ϕ, for any $v \in \mathfrak{v}_1$ a cycle ϕ_v in L such that the edges in all cycles ϕ_v combined correspond to E_ϕ. Since L is bipartite $|\phi_w|$ is even for every w and $|E_\phi| = |\phi|$ is odd but

$$|\phi| = |E_\phi| = \sum_{v \in \mathfrak{v}_1} |\phi_v|$$

where the left hand side of the equation is odd and the right hand side is even which leads to a contradiction. □

Before we can continue with the proof of an equivalence for bipartite graphs we need a preliminary lemma, which adds additional information about a lower bound on the length of odd cycles in L if $\mathfrak{L}_k$ is bipartite. This, in turn, yields a construction of a odd cycle in $\mathfrak{L}_k$ which will lead in Proposition 4.7 to the conclusion that $\mathfrak{L}_k$ bipartite implies also L bipartite.

Lemma 4.6. *If $\mathfrak{L}_k$ is bipartite then the shortest odd cycle in L is longer than $\bar{n} - k + 1$.*

Proof. Let $k \in \{1, \dots, \bar{n} - 1\}$ and assume there is an odd cycle $\phi = (v_1, v_2, \dots, v_l, v_1)$ in L with $|\phi| \leq \bar{n} - k + 1$. Then choose a set $\mathfrak{v} \subset V \backslash (\{\phi_2, \dots, \phi_l\})$ and $v_1 \in \mathfrak{v}$. Note that $|V \backslash (\{\phi_2, \dots, \phi_l\})| \geq k$ under the condition of the length of ϕ. Define the cycle

$\Phi = (\mathfrak{v}_1 = \mathfrak{v}, \mathfrak{v}_2, \ldots, \mathfrak{v}_l, \mathfrak{v}_1)$ by $\mathfrak{v}_{i+1} = \mathfrak{v}_i \backslash \{v_i\} \cup \{v_{i+1}\}$. Then, Φ is an odd cycle which is a contradiction to the fact that $\mathfrak{L}_k$ is bipartite. □

The goal is to show that the lower bound shown in Lemma 4.6 is rather conservative, the length of a non-existing walk being defined as infinity. Nonetheless, the intermediate step, deriving a lower bound on the length of odd cycles in L if $\mathfrak{L}_k$ is bipartite, allows us the construction of a cycle in $\mathfrak{L}_k$ based on any odd cycle in L with identical length. This yields the following result.

Proposition 4.7. *If $\mathfrak{L}_k$ is bipartite for some $k \in \{1, \ldots, \bar{n} - 1\}$ then so is L.*

Proof. Also for this proof we use the concept of multi-sets and consider all $\mathfrak{v}$ as multi-sets. Consider $\mathfrak{L}_k$ as bipartite and assume $k \leq \frac{\bar{n}}{2}$ as well as L not bipartite. Then there exists an odd cycle ϕ in L and by assumption as well as Lemma 4.6 we have $|\phi| \geq \bar{n} - k + 1 \geq \frac{\bar{n}}{2} + 1 \geq |\mathfrak{v}|$ for all $\mathfrak{v} \in \mathfrak{V}_k$. Consider $\mathfrak{v} \in \mathfrak{V}_k$ with $\phi_1 \in \mathfrak{v}$. We define a sequence of maps $(\Psi_i)_{i=1}^{|\phi|}$ on multi-sets by

$$\Psi_i(\mathfrak{w}) = \begin{cases} (\mathfrak{w} \backslash \{\phi_i\}) \cup \{\phi_{i+1}\}, & \text{if } \phi_i \in \mathfrak{w} \\ \mathfrak{w}, & \text{otherwise.} \end{cases}$$

and define $\Psi = (\Psi_1, \ldots, \Psi_{|\phi|} \circ \ldots \circ \Psi_1)$. This way $(\Psi_{|\phi|} \circ \ldots \circ \Psi_1)(\mathfrak{v}) = \mathfrak{v}$ and $|(\Psi_i \circ \ldots \circ \Psi_1)(\mathfrak{v})| = k$ for all $i = 1, \ldots, |\phi|$. Along the lines of the proof of Proposition 4.2 denote by τ the first entry in Ψ such that $(\Psi_\tau \circ \ldots \circ \Psi_1)(\mathfrak{v})$ contains the same entry twice and by κ the largest number such that $(\Psi_{\tau+\iota} \circ \ldots \circ \Psi_1)(\mathfrak{v})$ contains one entry twice for $\iota = 0, \ldots, \kappa$. Remark that both τ and κ are well defined and finite due to $|\phi| \geq k$. Transform the vector $(\Psi_{\iota'})_{\iota'=1}^{\tau+\kappa}$ into

$$(\Psi'_{\iota'})_{\iota'=1}^{\tau+\kappa} := (\Psi_1, \ldots, \Psi_{\tau+\kappa}, \Psi_{\tau+\kappa-1}, \ldots, \Psi_\tau). \tag{4.5}$$

Then $\left(\Psi'_{\tau+\iota} \circ \ldots \circ \Psi'_1\right)(\mathfrak{v})$ contains all elements only once for $\iota = 0, \ldots, \kappa$. Iterate this procedure until $\Psi' = (\Psi'_{\iota'} \circ \ldots \circ \Psi'_1)_{\iota'=1}^{|\phi|}$. Indeed, this leads to $(\Psi'_{|\phi|} \circ \ldots \circ \Psi'_1)(\mathfrak{v}) = \mathfrak{v}$ and $(\Psi'_{\iota'} \circ \ldots \circ \Psi'_1)(\mathfrak{v}) \in \mathfrak{V}_k$ for all $\iota' = 1, \ldots, |\phi|$. Hence, the vector $((\Psi'_{\iota'} \circ \ldots \circ \Psi'_1)(\mathfrak{v}))_{\iota'=1}^{|\phi|}$ defines a cycle of length $|\phi|$ in $\mathfrak{L}_k$. But ϕ is an odd cycle which leads to a contradiction. The claim follows since $\mathfrak{L}_k \cong \mathfrak{L}_{\bar{n}-k}$. □

Actually, the kPG $\mathfrak{L}_k$ carries all symmetries which are also present in L. This gives us an insight into the automorphism group of $\mathfrak{L}_k$.

Proposition 4.8. *Let L be a simple connected graph $k \in \{1, \ldots, \bar{n} - 1\}$. Then, the automorphism group* $\mathrm{Aut}(L)$ *of L induces a sub-group of the automorphism group* $\mathrm{Aut}(\mathfrak{L}_k)$ *of $\mathfrak{L}_k$.*

Proof. We construct in what follows the explicit corresponding automorphism to any automorphism $\phi \in \mathrm{Aut}(L)$. Let $\phi \in \mathrm{Aut}(L)$ and define the map Φ on $\mathfrak{V}_k$ by $\Phi(\mathfrak{v}) = \{\phi(v) | v \in \mathfrak{v}\}$. Then, if $\Phi(\mathfrak{v}) \triangle \Phi(\mathfrak{w}) = \{u, \bar{u}\}$ we have $\langle \Phi(\mathfrak{v}), \Phi(\mathfrak{w}) \rangle \in \mathfrak{E}_k$ if and only if $\langle u, \bar{u} \rangle \in E$. Now assume that $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$ and $\mathfrak{v} \triangle \mathfrak{w} = \{v, w\}$. Then, we obtain that $\Phi(\mathfrak{v}) \triangle \Phi(\mathfrak{w}) = \{u, \bar{u}\} = \{\phi(v), \phi(w)\}$ and since ϕ is an isomorphism on L we arrive, moreover, at $\langle u, \bar{u} \rangle \in E$. Therefore, we can conclude $\langle \Phi(\mathfrak{v}), \Phi(\mathfrak{w}) \rangle \in \mathfrak{E}_k$ which implies that any automorphism on L defines an automorphism on $\mathfrak{L}_k$. □

Indeed, the automorphism group of $\mathfrak{L}_k$ is larger then the sub-group induced by $\mathrm{Aut}(L)$. To this end, we consider identifiable sub-graphs $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$. Define for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ the equivalence relation $\mathfrak{v} \sim \mathfrak{w}$ if and only if $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$. For fixed $\mathfrak{v}_i$ define $[\mathfrak{v}_i] := \{\mathfrak{u} \in \mathfrak{V}_k | \mathfrak{u} \sim \mathfrak{v}_i\}$ the equivalence class of $\mathfrak{v}_i$. We consider a fixed equivalence class $[\mathfrak{v}_i]$ and define for $\mathfrak{v}, \mathfrak{w} \in [\mathfrak{v}_i]$ by $\Phi_{\mathfrak{v}}^{\mathfrak{w}}$ the isomorphism $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$. We can extend $\Phi_{\mathfrak{v}}^{\mathfrak{w}}$ to an automorphism of $\mathfrak{L}_k$ which is not induced by an automorphism on L. First, consider the neighborhood of $\mathfrak{v}$. Since all vertices in the neighborhood of $\mathfrak{v}$ consist of subsets of V of size k which differ from $\mathfrak{v}$ by exactly one vertex in L and $L_{\mathfrak{v},\mathfrak{v}^c}$ represents all possible transitions to neighbors of $\mathfrak{v}$, we exploit $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$ to obtain a mapping from the neighborhood of $\mathfrak{v}$ to the neighborhood of $\mathfrak{w}$ under which the equivalence classes with respect to $\sim$ are invariant. Iteratively, by this construction we obtain a map $\hat{\Phi}_{\mathfrak{v}}^{\mathfrak{w}} : \mathfrak{L}_k \to \mathfrak{L}_k$ which preserves the neighborhood property. Consequently, $\hat{\Phi}_{\mathfrak{v}}^{\mathfrak{w}}$ defines an automorphism of $\mathfrak{L}_k$. By changing $\mathfrak{v}$ and $\mathfrak{w}$ we obtain another such map since $\hat{\Phi}_{\mathfrak{v}}^{\mathfrak{w}}(\mathfrak{v}) = \mathfrak{w}$ and, hence, $\hat{\Phi}_{\mathfrak{v}}^{\mathfrak{w}'}(\mathfrak{v}) = \mathfrak{w}'$ for another $\mathfrak{w}' \in [\mathfrak{v}_i]$ which implies that $\mathrm{Aut}(\mathfrak{L}_k)$ is larger then the induced automorphism from L.

4.3 kPGs induced by regular graphs

In what follows we focus on regular graphs L. Hence, we assume for the reminder of this section that L is a $\bar{d}$-regular graph. While this may seem restrictive at first, regular graphs having at first glance a rather forgiving structure, their role and the arising problems are already important. Consider to this end the graph in Figure 16. One can wonder about dense communities in such graphs, i.e., communities which have a maximal amount of in-group relationships. This is already a non-trivial problem as can be seen for example in [KhuSah09], [Chara15] and [FePeKo01]. We come back to the applications and an approach how to find dense communities in Section 6, since they play a defining role for various properties of the exclusion process, which we are going to consider. A preliminary result will be shown in Proposition 4.17. In later sections it will turn out that the relationship dynamics, discussed in this work, even give rise to particle dynamics on strongly regular graphs such that the following results are even in a more general form than what we actually need. This will allow us to analyze the particle dynamics on more general graphs and, hence, implies a generalization beyond the model motivated dynamics.

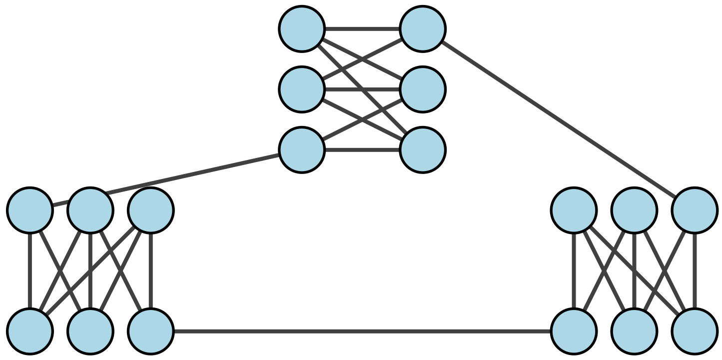

Figure 16: An exemple of a highly clustered, 3-regular, non-bipartite graph. Interpreting the edges as relationships or the possibility of communication, one can imagine that such a network could represent a rather segregated community even though the underlying graph is regular.

4.3.1 Combinatorial properties of kPGs

We start by calculating the size of the vertex and the edge set its size which follow from purely combinatoric but, in the case of the edge set, technically involved considerations.

Proposition 4.9. *Consider the graph* $\mathfrak{L}_k = (\mathfrak{V}_k, \mathfrak{E}_k)$ *for* $k \in \{1, \dots, \bar{n}-1\}$*. Then* $|\mathfrak{V}_k| = \binom{\bar{n}}{k}$ *and*

$$|\mathfrak{E}_k| = \frac{1}{2}\left(\bar{d}k\binom{\bar{n}}{k} - \bar{n}\bar{d}\binom{\bar{n}-2}{k-2}\right) = k(\bar{n}-k)\binom{\bar{n}}{k}\frac{\bar{d}}{2(\bar{n}-1)}. \tag{4.6}$$

Proof. The first claim follows from drawing without replacement. For the edge set, by the formula for the degree of a $\mathfrak{v} \in \mathfrak{V}_k$, we obtain

$$\begin{aligned} 2|\mathfrak{E}_k| &= \sum_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v}) = \sum_{\mathfrak{v}\in\mathfrak{V}_k} k\bar{d} - \sum_{\mathfrak{v}\in\mathfrak{V}_k}\sum_{v\in\mathfrak{v}} \deg^{L_\mathfrak{v}}(v) \\ &= k\bar{d}\binom{\bar{n}}{k} - \sum_{v\in V}\sum_{\mathfrak{v}\in\mathfrak{V}_k} \deg^{L_\mathfrak{v}}(v)\mathbb{1}_{v\in\mathfrak{v}}. \end{aligned}$$

Fixing a $v \in V$ we obtain for $s_v = \sum_{\mathfrak{v}\in\mathfrak{V}_k} \deg^{L_\mathfrak{v}}(v)\mathbb{1}_{v\in\mathfrak{v}}$ that we have to redistribute the $k-1$ remaining particles among the $\bar{d}$ neighbors and the $\bar{n}-1-\bar{d}$ non-adjacent vertices. Assuming that l particles are in the neighborhood of $\mathfrak{v}$ and one in v, there are $\binom{\bar{d}}{l}\binom{\bar{n}-1-\bar{d}}{k-1-l}$ ways to redistribute the particles in the aforementioned way and each contributes l to to the sum s_v. Furthermore, l ranges from 0 to $\min\{\bar{d}, k-1\}$ such that

$$s_v = \sum_{l=0}^{\min\{\bar{d},k-1\}} \binom{\bar{d}}{l}\binom{\bar{n}-1-\bar{d}}{k-1-l} l$$

independently of v. Consequently, we obtain the identity

$$|\mathfrak{E}_k| = \frac{1}{2}\left(\bar{d}k\binom{\bar{n}}{k} - \bar{n}\sum_{l=1}^{\min\{k-1,\bar{d}\}} \binom{\bar{d}}{l}\binom{\bar{n}-1-\bar{d}}{k-1-l} l\right). \tag{4.7}$$

For the second term, we remark that $\binom{\bar{d}}{l} = 0$ for $l > \bar{d}$ and $\binom{\bar{n}-1-\bar{d}}{k-1-l} = 0$ for $l > k-1$. Consequently, in the sums all summands with $l > \min\{k-1, \bar{d}\}$ are zero and, therefore,

$$\sum_{l=1}^{k-1} \binom{\bar{d}}{l}\binom{\bar{n}-1-\bar{d}}{k-1-l} l = \sum_{l=1}^{\min\{k-1,\bar{d}\}} \binom{\bar{d}}{l}\binom{\bar{n}-1-\bar{d}}{k-1-l} l = \sum_{l=1}^{\bar{d}} \binom{\bar{d}}{l}\binom{\bar{n}-1-\bar{d}}{k-1-l} l.$$

From this we can continue with a preliminary observation that

$$\sum_{l=1}^{k-1} \binom{\bar{d}}{l}\binom{\bar{n}-1-\bar{d}}{k-1-l} l = \bar{d}\sum_{l=1}^{k-1} \binom{\bar{d}-1}{l-1}\binom{\bar{n}-1-\bar{d}}{k-1-l}.$$

We omit $\bar{d}$ in what follows and prove only equality of the remaining terms. Note first that

$$\begin{aligned}\sum_{l=1}^{k-1} \binom{\bar{d}-1}{l-1}\binom{\bar{n}-1-\bar{d}}{k-1-l} &= \sum_{l=1}^{k-1} \binom{\bar{d}-1}{l-1}\binom{\bar{n}-2-(\bar{d}-1)}{k-2-(l-1)} \\ &= \sum_{l=0}^{k-2} \binom{\bar{d}-1}{l}\binom{\bar{n}-2-(\bar{d}-1)}{k-2-l}\end{aligned}$$

such that we can apply the Zhu–Vandermonde identity to obtain

$$\sum_{l=1}^{k-1} \binom{\bar{d}-1}{l-1}\binom{\bar{n}-1-\bar{d}}{k-1-l} = \binom{\bar{n}-2}{k-2}$$

which finishes the proof. □

Hence, we obtain an equality for the size of the vertex sets for k and $\bar{n}-k$ as well as for the size of the edge sets in the two cases. Indeed, it is not obvious that the value given in Equation (4.6) is an integer. When checking this property one arrives at the following conclusion.

Corollary 4.10. *Let $L = (V, E)$ be a $\bar{d}$-regular graph. Consider the graph $\mathfrak{L}_k$ for $k \in \{1, \ldots, \bar{n}-1\}$. Then,*

$$|\mathfrak{E}_k| = \binom{\bar{n}-2}{k-1}|E|. \tag{4.8}$$

Proof. We employ Equation (4.6) to derive the claim.

$$\begin{aligned}|\mathfrak{E}_k| &= k(\bar{n}-k)\binom{\bar{n}}{k}\frac{\bar{d}}{2(\bar{n}-1)} = \bar{n}\frac{(\bar{n}-2)!}{(k-1)!(\bar{n}-2-(k-1))!}\frac{\bar{d}}{2} \\ &= \binom{\bar{n}-2}{k-1}\frac{\bar{n}\bar{d}}{2} = \binom{\bar{n}-2}{k-1}|E|.\end{aligned}$$

□

On the other hand, comparing the result presented in Equation (4.6) with the size of the edge set $E_{\mathcal{J}}$ of a $\mathcal{J}(\bar{n}, k)$ Johnson graph, we obtain that

$$\frac{|\mathfrak{E}_k|}{|E_{\mathcal{J}}|} = \frac{\bar{d}}{\bar{n}-1}$$

which is independent of k and only dependent on the underlying graph L through the prescribed degree $\bar{d}$. Based on this remark we can also make the following core observation which will be central later on for estimating convergence speeds of some Markov chains.

Proposition 4.11. *Let L be a $\bar{d}$-regular graph on $\bar{n}$ vertices and $k \in \{1, \ldots, \bar{n}-1\}$. Denote by* avg deg$(\mathfrak{L}_k)$ *the average degree of $\mathfrak{L}_k$, i.e.,*

$$\text{avg deg}(\mathfrak{L}_k) := \frac{1}{|\mathfrak{V}_k|}\sum_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v}).$$

The average degree satisfies

$$\frac{\text{avg deg}(\mathfrak{L}_k)}{k(\bar{n}-k)} = \frac{\bar{d}}{\bar{n}-1} \leq 1. \tag{4.9}$$

Proof. We start by recalling that the average degree of any graph is given by the quotient of the size of its edge set times two and the size of its vertex set. Therefore,

$$\begin{aligned}\frac{\text{avg deg}(\mathfrak{L}_k)}{k(\bar{n}-k)} &= \frac{2|\mathfrak{E}_k|}{|\mathfrak{V}_k|\, k(\bar{n}-k)} = \frac{\bar{d}}{\bar{n}-k} - \frac{\bar{d}}{\binom{\bar{n}-1}{k-1}}\frac{\binom{\bar{n}-2}{k-2}}{\bar{n}-k} \\ &= \frac{\bar{d}}{\bar{n}-k}\left(1-\frac{k-1}{\bar{n}-1}\right) = \frac{\bar{d}}{\bar{n}-1}.\end{aligned}$$

The second claim follows from $\bar{d} \leq \bar{n}-1$ and equality if and only if L is a complete graph.
□

The average degree is not the only object, defined by the degree sequence, which we are interested in. The isomorphism we found in Proposition 4.4 provides a tool to analyze the difference of exclusion processes defined by different transition probabilities of each particle. In particular, we can define a certain type of homogeneity based on invariance of the transition probabilities of an associated Markov chain under the isomorphism.

4.3.2 The degree sequence of kPGs

Indeed, for regular graphs we obtain a range of structural results on $\mathfrak{L}_k$ which go beyond connectedness and focus on the local properties of the vertices. In particular, the degree of a vertex plays a central role defining in later sections the transition probabilities and stationary distributions of Markov chains on $\mathfrak{L}_k$ induced by a variety of exclusion processes.

Proposition 4.12. *Let* $k \in \{1, \dots, \bar{n}-1\}$ *and* $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$ *and write* $\mathfrak{v} \triangle \mathfrak{w} = \{v, w\}$*. Then* $\deg_k(\mathfrak{v}) = \deg_k(\mathfrak{w})$ *if and only if* $\deg^{L_\mathfrak{v}}(v) = \deg^{L_\mathfrak{w}}(w)$*.*
Moreover, any $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ *with* $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$ *and* $\mathfrak{v} \triangle \mathfrak{w} = \{v, w\}$ *satisfy*

$$\deg^{L_\mathfrak{v}}(v) + \deg^{L_{V \setminus \mathfrak{w}}}(v) = \bar{d} - 1. \tag{4.10}$$

Proof. First of all, note that $\mathfrak{v} \backslash \{v\} = \mathfrak{w} \backslash \{w\}$ and, hence, $\deg_{k-1}(\mathfrak{v} \backslash \{v\}) = \deg_{k-1}(\mathfrak{w} \backslash \{w\})$. Furthermore, removing v from $\mathfrak{v}$ removes $\deg^{L_\mathfrak{v}}(v)$ edges from the induced subgraph $L_\mathfrak{v}$. Consequently, we obtain $\deg_k(\mathfrak{v}) = \deg_{k-1}(\mathfrak{v} \backslash \{v\}) + \bar{d} - 2\deg^{L_\mathfrak{v}}(v)$. Equivalent claims are satisfied by $\mathfrak{w}$ and w. Hence, we can conclude

$$\begin{aligned} \deg_k(\mathfrak{v}) - \deg_k(\mathfrak{w}) &= \deg_{k-1}(\mathfrak{v} \backslash \{v\}) - 2\deg^{L_\mathfrak{v}}(v) - (\deg_{k-1}(\mathfrak{w} \backslash \{w\}) - 2\deg^{L_\mathfrak{w}}(w)) \\ &= 2\left(\deg^{L_\mathfrak{w}}(w) - \deg^{L_\mathfrak{v}}(v)\right) \end{aligned}$$

which is equivalent to the first claim.
Secondly, note that $\mathfrak{v} \backslash \{v\} = \mathfrak{w} \backslash \{w\}$ and, hence, $\deg_{k-1}(\mathfrak{v} \backslash \{v\}) = \deg_{k-1}(\mathfrak{w} \backslash \{w\})$. Again, removing v from $\mathfrak{v}$ removes $\deg^{L_\mathfrak{v}}(v)$ edges from the induced subgraph $L_\mathfrak{v}$. Consequently, we obtain $\deg_k(\mathfrak{v}) = \deg_{k-1}(\mathfrak{v} \backslash \{v\}) + \bar{d} - 2\deg^{L_\mathfrak{v}}(v)$. Equivalent claims are satisfied by $\mathfrak{w}$ and w. Hence, we can conclude

$$\begin{aligned} \deg_k(\mathfrak{v}) - \deg_k(\mathfrak{w}) &= \deg_{k-1}(\mathfrak{v} \backslash \{v\}) - 2\deg^{L_\mathfrak{v}}(v) - (\deg_{k-1}(\mathfrak{w} \backslash \{w\}) - 2\deg^{L_\mathfrak{w}}(w)) \\ &= 2\left(\deg^{L_\mathfrak{w}}(w) - \deg^{L_\mathfrak{v}}(v)\right) \end{aligned}$$

which is equivalent to the claim. □

Proposition 4.12 gives a perspective on the graph $\mathfrak{L}_k$ that in fact the k vertex induced sub-graphs of L are the defining objects. While their analysis is a classically difficult subject, see again for example [KhuSah09], [Chara15] and [FePeKo01], we obtain nonetheless properties based on construction of $\mathfrak{L}_k$. Later on, we can make even more conclusions due to the Markov chains on $\mathfrak{L}_k$.

Corollary 4.13. *Let $k \in \{1, \dots, \bar{n}-1\}$ and $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$. Then, $\deg_k(\mathfrak{v}) - \deg_k(\mathfrak{w})$ is an even number.*

Proof. Consider first $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$. Then, by the proof of Proposition 4.12 we have $\deg_k(\mathfrak{v}) - \deg_k(\mathfrak{w}) = 2\,(\deg^{L_\mathfrak{w}}(w) - \deg^{L_\mathfrak{v}}(v))$. For arbitrary $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ we can construct a path from $\mathfrak{v}$ to $\mathfrak{w}$ by Proposition 4.2 and for any segment the difference of degrees is even. Hence, by a bootstrap argument also $\deg_k(\mathfrak{v}) - \deg_k(\mathfrak{w})$ is even. □

We investigate in what follows the link between the edges in vertex induced sub-graphs and the degree of a vertex in $\mathfrak{L}_k$.

Proposition 4.14. *Let L be a $\bar{d}$-regular graph, $\mathfrak{v} \in \mathfrak{L}_k$ and denote by $L_\mathfrak{v} = (\mathfrak{v}, E_\mathfrak{v})$ the vertex induced sub-graph of L. Then*

$$\deg_k(\mathfrak{v}) = k \cdot \bar{d} - 2|E_\mathfrak{v}|. \tag{4.11}$$

Additionally, denote by $L_{\min;k} = (V_{\min;k}, E_{\min;k})$ a least dense vertex induced sub-graph on k vertices of L and by $L_{\max;k} = (V_{\max;k}, E_{\max;k})$ a densest vertex induced sub-graph on k vertices. Then

$$\min_{\mathfrak{v} \in \mathfrak{V}_k} \deg_k(\mathfrak{v}) = k \cdot \bar{d} - 2|E_{\max;k}|,$$
$$\max_{\mathfrak{v} \in \mathfrak{V}_k} \deg_k(\mathfrak{v}) = k \cdot \bar{d} - 2|E_{\min;k}|.$$

Proof. For $\mathfrak{v} \in \mathfrak{V}_k$ consider the vertex induced sub-graph $L_\mathfrak{v} = (V_\mathfrak{v}, E_\mathfrak{v})$ of L. Any $v \in V$ has $\bar{d}$ neighbors. Hence, the degree of $\mathfrak{v}$ in $\mathfrak{L}_k$ has the form $\deg_k^r(\mathfrak{v}) = k \cdot \bar{d} - m(\mathfrak{v})$ where $m(\mathfrak{v})$ is defined by the constrains given through the definition of $\mathfrak{E}_k$ and remains to be determined. Since $\mathfrak{v} \sim \mathfrak{w}$ if and only if $\mathfrak{v} \triangle \mathfrak{w} = \{v, w\}$ and $\langle v, w \rangle \in E$ any edge in L between $v, v' \in \mathfrak{v}$ reduce the degree by two due to symmetry of the edge $\{v, v'\}$. Consequently,

$$\deg_k(\mathfrak{v}) = k \cdot \bar{d} - 2|E_\mathfrak{v}|.$$

By the first result we can use the identity

$$\deg_k(\mathfrak{v}) = k \cdot \bar{d} - 2|E_\mathfrak{v}| \tag{4.12}$$

for any $\mathfrak{v} \in \mathfrak{V}_k$. Therefore,

$$\tilde{\delta}_{k;*} := \min_{\mathfrak{v} \in \mathfrak{V}_k} \deg_k(\mathfrak{v}) = k \cdot \bar{d} - 2 \max_{\mathfrak{v} \in \mathfrak{V}_k} |E_\mathfrak{v}| = k \cdot \bar{d} - 2|E_{\max;k}|. \tag{4.13}$$

The same argumentation is valid for $\tilde{\delta}^{k;*}$. □

Having established the relation between the k vertex induced sub-graphs and in particular the size of their edge sets with the degrees in $\mathfrak{L}_k$ we can then come back to the question about the degrees for varying k and the fact that the degrees seem to be either all even or all odd. We show this property first before going on to the former.

Lemma 4.15. *Let $k \in \{1, \dots, \bar{n}-1\}$ and let $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$. Then, $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$ implies $\deg_k(\mathfrak{v}) = \deg_k(\mathfrak{w})$. Additionally, for any $\mathfrak{v} \in \mathfrak{V}_k$ its degree $\deg_k(\mathfrak{v})$ is even if and only if $k\bar{d}$ is even. Hence, if there is $\mathfrak{v} \in \mathfrak{V}_k$ such that $\deg_k(\mathfrak{v})$ is even it is true that for all $\mathfrak{w} \in \mathfrak{V}_k$ the number $\deg_k(\mathfrak{w})$ is even.*

Proof. The first claim follows immediately since $\deg_k(\mathfrak{v}) = |E_{\mathfrak{v},\mathfrak{v}^c}|$ and $|E_{\mathfrak{v},\mathfrak{v}^c}| = |E_{\mathfrak{w},\mathfrak{w}^c}|$.

Drawing from Proposition 4.14 we obtain that

$$\deg_k(\mathfrak{v}) + 2|E_{\mathfrak{v}}| = k \cdot \bar{d}$$

which yields the first claim. The second claim follows by Corollary 4.13. □

Indeed, the importance of the graph $L_{\mathfrak{v},\mathfrak{v}^c} = ((\mathfrak{v}, \mathfrak{v}^c), E_{\mathfrak{v},\mathfrak{v}^c})$ does not only reduce to the fact that it may be a minimizer as discussed in Lemma 4.15 but it does in fact characterize both $\mathfrak{v}$ and $\mathfrak{v}^c$ at the same time. Unfortunately, obtaining further identities would amount to solving questions about sizes of sub-graphs of a $\bar{d}$-regular graph L. This is a well known problem in various fields but aside from approximations via algorithmic approaches it remains out of reach of being solved. Nonetheless, further qualitative claims can be made about $\mathfrak{L}_k$ which are in particular symmetry based observations.

4.3.3 Symmetries k-$(\bar{n}-k)$ and under complements

The first symmetry which is an immediate consequence of the previously established isomorphism between $\mathfrak{L}_k$ and $\mathfrak{L}_{\bar{n}-k}$ concerns the degree set which is an invariant under isomorphisms.

Proposition 4.16. *Let $k \in \{1, \dots, \bar{n}-1\}$ and $D_k := \{\deg_k^r(v) | v \in \mathfrak{V}_k\}$. Then $D_k = D_{\bar{n}-k}$.*
Now, let $k, k' \in \left\{1, \dots, \left\lfloor \frac{\bar{n}}{2} \right\rfloor\right\}$, $k' \leq k$. Then $|D_{k'}| \leq |D_k|$. If in turn $k, k' \in \left\{\left\lceil \frac{\bar{n}}{2} \right\rceil, \dots, \bar{n}-1\right\}$, $k \leq k'$ then $|D_k| \geq |D_{k'}|$.

Proof. The first result is a corollary of Proposition 4.4 because isomorphisms preserve the set of degrees.

Now, let $k \leq \left\lfloor \frac{\bar{n}}{2} \right\rfloor$. Then, for any $\hat{\mathfrak{v}} \in \mathfrak{V}_{k-1}$ we can define the set $\hat{\mathfrak{V}}_k := \{\mathfrak{v} \in \mathfrak{V}_k | \mathfrak{v} \cap \hat{\mathfrak{v}} = \hat{\mathfrak{v}}\}$ and $|\hat{\mathfrak{V}}_k| = \bar{n} - (k-1) \geq 1$. For any $\mathfrak{v} \in \hat{\mathfrak{V}}$ the degree of $\mathfrak{v}$ is given by the proof of Proposition 4.12 by $\deg_k(\mathfrak{v}) = \deg_{k-1}(\hat{\mathfrak{v}}) + \bar{d} - 2\deg^{L_{\mathfrak{v}}}(v)$ where $\{v\} := \mathfrak{v} \backslash \hat{\mathfrak{v}}$. Hence, any degree $d \in D_{k-1}$ defines at least one degree $d' \in D_k$. Consequently, we obtain $|D_{k-1}| \leq |D_k|$ and by a bootstrap argument also $|D_{k'}| \leq |D_k|$ for any $k' \leq k$.

The second claim of the Lemma follows from Proposition 4.4. □

Note that the inequalities in Proposition 4.16 do not have any implications on the inclusion of the degree sets. Consider to this end the graphs depicted in Figure 17 and Figure 18. In Figure 17 the underlying graph is the cycle graph on 8 vertices. Then,

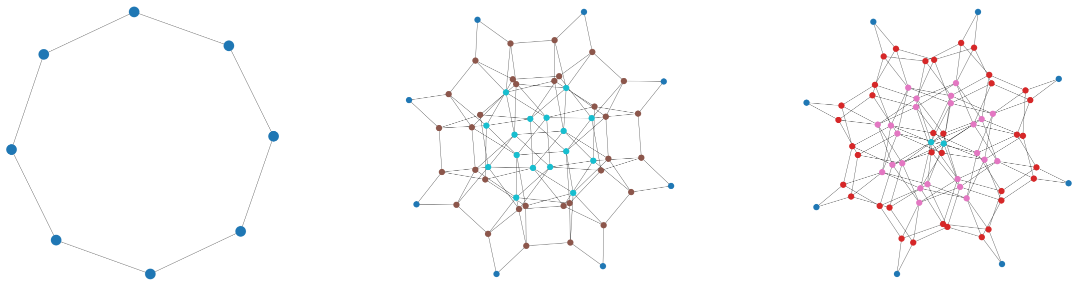

Figure 17: From left to right the underlying cycle graph, the associated graphs $\mathfrak{L}_3$ and $\mathfrak{L}_4$. Indeed, due to the configurations of the vertex induced subgraphs, inclusion $D_3 \subset D_4$ is given.

the degree sets for $k = 3$ and $k = 4$ satisfy

$$D_3 = \{2, 4, 6\} \subset \{2, 4, 6, 8\} = D_4.$$

But, already by considering a 3-regular graph, this property is no longer satisfied. We consider one example in Figure 18. Inclusion of the vertex sets would imply that by adding an additional vertex to the sub-graph we would not only increase the number of possible configuration but can also construct from more vertices always configurations in such a way that they have the same density as some configuration on less vertices. It is intuitively understandable that this cannot be possible if the degree of the underlying graph L is greater than 2 and sufficiently many vertices are used to span the sub-graph. Indeed, in the case of Figure 18 we obtain for the degree sets

$$D_3 = \{3, 5, 7, 9\} \not\subset \{4, 6, 8, 10\} = D_4 \tag{4.14}$$

and even $D_3 \cap D_4 = \emptyset$. Nonetheless, we can find implications of degrees in $\mathfrak{L}_k$ for any k exploiting the symmetry of the binomial coefficient and the construction via the

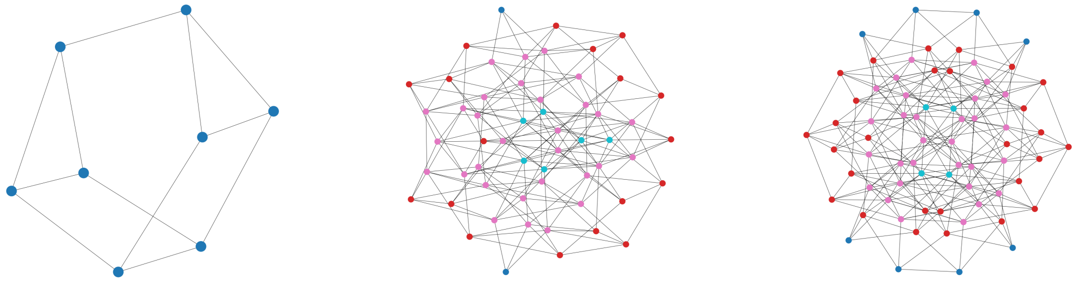

Figure 18: From left to right the underlying 3-regular graph on 8 vertices, the associated graphs $\mathfrak{L}_3$ and $\mathfrak{L}_4$. In this case we encounter $D_3 \cap D_4 = \emptyset$.

symmetric difference which yields, in particular, a symmetry for k and $\bar{n} - k$ densest sub-graphs.

Proposition 4.17. *Let $k \in \{1, \ldots, \bar{n} - 1\}$ and let $\mathfrak{v} \in \mathfrak{V}_k$ such that $L_\mathfrak{v}$ defines a densest vertex induced sub-graph on k vertices in L. Then $\mathfrak{v}^c := V \setminus \mathfrak{v}$ defines a densest sub-graph in L on $\bar{n} - k$ vertices.*
Moreover, let $\mathfrak{v} \in \mathfrak{V}_k$ such that $L_\mathfrak{v}$ defines a densest vertex induced sub-graph on k vertices in L. Then, the bipartite sub-graph $L_{\mathfrak{v},\mathfrak{v}^c} = ((\mathfrak{v}, \mathfrak{v}^c), E_{\mathfrak{v},\mathfrak{v}^c})$ satisfies

$$|E_{\mathfrak{v},\mathfrak{v}^c}| = \min_{\mathfrak{w} \in \mathfrak{V}_k} |E_{\mathfrak{w},\mathfrak{w}^c}|$$

Proof. The proof of the first claim follows by the previously observed structure of the degrees of $\mathfrak{v}$ in $\mathfrak{L}_k$ and the property $D_k = D_{\bar{n}-k}$.

$$\begin{aligned}(\bar{n} - k)\bar{d} - 2|E_{\mathfrak{v}^c}| = \deg_{\bar{n}-k}(\mathfrak{v}^c) = \deg_k(\mathfrak{v}) = \min_{\mathfrak{w} \in \mathfrak{V}_k} \bar{d}k - 2|E_\mathfrak{w}| \\ = \min_{\bar{\mathfrak{w}} \in \mathfrak{V}_{\bar{n}-k}} (\bar{n} - k)\bar{d} - 2|E_{\bar{\mathfrak{w}}}| = (\bar{n} - k)\bar{d} - 2 \max_{\bar{\mathfrak{w}} \in \mathfrak{V}_{\bar{n}-k}} |E_{\bar{\mathfrak{w}}}|.\end{aligned}$$

Turning to the second claim, we observe by symmetry of the densest sub-graphs shown in the first claim, for $\mathfrak{v}$ a densest sub-graph, that $|E_\mathfrak{v}| + |E_{\mathfrak{v}^c}| = \max_{\mathfrak{w} \in \mathfrak{V}_k}(|E_\mathfrak{w}| + |E_{\mathfrak{w}^c}|)$ and, consequently,

$$|E_{\mathfrak{v},\mathfrak{v}^c}| = |E| - (|E_\mathfrak{v}| + |E_{\mathfrak{v}^c}|) = \min_{\mathfrak{w} \in \mathfrak{V}_k} (|E| - (|E_\mathfrak{w}| + |E_{\mathfrak{w}^c}|)) = \min_{\mathfrak{w} \in \mathfrak{V}_k} |E_{\mathfrak{w},\mathfrak{w}^c}|.$$

□

More symmetries of $\mathfrak{L}_k$ and links to k induced sub-graphs of some underlying graph can be deduced from the previous properties. In particular, a link between the number of sub-graphs on k and $\bar{n} - k$ vertices containing a fixed number of edges can be established.

Proposition 4.18. *Let $l \in D_k$, $\mathfrak{V}_{D_k;l} := \{\mathfrak{v} \in \mathfrak{V}_k | \deg_k(\mathfrak{v}) = l\}$ and $\mathfrak{L}_{k;l'}$ for $l' \in \mathbb{N}$ the set of vertex induced sub-graphs of L on k vertices containing exactly l' edges. Then*

$$|\mathfrak{V}_{D_k;l}| = \left|\mathfrak{L}_{k;\frac{k\bar{d}-l}{2}}\right|. \tag{4.15}$$

Furthermore,

$$\left|\mathfrak{L}_{k;\frac{k\bar{d}-l}{2}}\right| = \left|\mathfrak{L}_{\bar{n}-k;\frac{(\bar{n}-k)\bar{d}-l}{2}}\right|. \tag{4.16}$$

Proof. Consider the identity given in Proposition 4.14 for $\mathfrak{v} \in \mathfrak{V}_k$ and let $l \in D_k$. Then

$$|\mathfrak{V}_{D_k;l}| = |\{\mathfrak{v} \in \mathfrak{V}_k | \deg_k(\mathfrak{v}) = l\}| = \left|\left\{\mathfrak{v} \in \mathfrak{V}_k \middle| |E_\mathfrak{v}| = \frac{k\bar{d}-l}{2}\right\}\right|.$$

Moreover, the set $\mathfrak{V}_k$ contains all subsets $\mathfrak{v} \subset V$ of size k. Hence,

$$\left|\left\{\mathfrak{v} \in \mathfrak{V}_k \middle| |E_\mathfrak{v}| = \frac{k\bar{d}-l}{2}\right\}\right| = |\mathfrak{L}_{k;\frac{k\bar{d}-l}{2}}|$$

and the first claim follows. The second claim follows by $\mathfrak{L}_k \cong \mathfrak{L}_{\bar{n}-k}$ and the identity given in Proposition 4.18. □

While the preceding results give insights in the links between $\mathfrak{L}_k$ and the extremal properties of k-sub-graphs of some underlying $\bar{d}$-regular graph, we are also interested in the local combinatorial properties of vertices in a k-sub-graph. This will help us quantify more properties of $\mathfrak{L}_k$ as well as establish the bases for results on Markov chains on $\mathfrak{L}_k$ induced by certain types of exclusion processes. The following result gives an identify for the average degree and the density of vertices in some sub-graph $L_\mathfrak{v}$ induced by a vertex $\mathfrak{v} \in \mathfrak{V}_k$ as well as the term $\left||E_\mathfrak{v}| - |E_\mathfrak{v}^c|\right|$ as a function of $\bar{n}, \bar{d}$ and k. It turns out that this is in fact a constant for $\bar{d}$-regular graphs.

Lemma 4.19. *Let L be a $\bar{d}$-regular graph on $\bar{n}$ vertices and $k \in \{1, \ldots, \bar{n}-1\}$. Then, for any $\mathfrak{v} \in \mathfrak{V}_k$, we have*

$$\text{avg deg}_{\bar{n}-k}(L_{\mathfrak{v}^c}) = \frac{\bar{d}(\bar{n}-2k) + k \,\text{avg deg}_k(L_\mathfrak{v})}{\bar{n}-k} \tag{4.17}$$

and

$$\left||E_\mathfrak{v}| - |E_{\mathfrak{v}^c}|\right| = \frac{\bar{d}(\bar{n}-2k)}{2}. \tag{4.18}$$

Proof. Using the isomorphism between $\mathfrak{L}_k$ and $\mathfrak{L}_{\bar{n}-k}$ we obtain that for $\mathfrak{v} \in \mathfrak{V}_k$ due to the equality of the degrees of $\mathfrak{v}$ and $\mathfrak{v}^c$

$$k(\bar{d} - \text{avg deg}_k(L_{\mathfrak{v}})) = \deg_k(\mathfrak{v}) = \deg_{\bar{n}-k}(\mathfrak{v}^{\mathfrak{c}}) = (\bar{n} - k)(\bar{d} - \text{avg deg}_{\bar{n}-k}(L_{\mathfrak{v}^c}))$$

and, equivalently,

$$\text{avg deg}_{\bar{n}-k}(L_{\mathfrak{v}^c}) = \frac{\bar{d}(\bar{n} - 2k) + k \,\text{avg deg}_k(L_{\mathfrak{v}})}{\bar{n} - k}.$$

For the second claim, we use that $k\text{avg deg}_k(L_{\mathfrak{v}}) = 2|E_{\mathfrak{v}}|$ such that by the first claim

$$2(|E_{\mathfrak{v}^c}| - |E_{\mathfrak{v}}|) = (\bar{n} - k)\,\text{avg deg}_{\bar{n}-k}(L_{\mathfrak{v}^c}) - k\,\text{avg deg}_k(L_{\mathfrak{v}}) = \bar{d}(\bar{n} - 2k)$$

which yields the second claim. □

Indeed, some well-known results may be derived directly from the degree formula obtained in Proposition 4.14. In particular, relationships between sub-graphs of L and their complements with respect to L become easily accessible.

Lemma 4.20. *Let $L = (V, E)$ be a simple connected graph on n vertices. Assume that $\mathfrak{v} \subset V$ is a least dense k-sub-graph of L. Then $\mathfrak{v}$ is a densest k-sub-graph of L^c.*

Proof. Let $\mathfrak{v} \subset V$, $|\mathfrak{v}| = k$ and assume that $(\mathfrak{v}, E_{\mathfrak{v}})$ is a least dense sub-graph in L. Then,

$$|E_{\mathfrak{v}}| = \min_{\mathfrak{w} \subset V, |\mathfrak{w}|=k} |E_{\mathfrak{w}}| = \frac{k(k-1)}{2} - \max_{\mathfrak{w} \subset V, |\mathfrak{w}|=k} |E^c_{\mathfrak{w}}|. \tag{4.19}$$

The property $|E_{\mathfrak{v}}| + |E^c_{\mathfrak{v}}| = \frac{k(k-1)}{2}$ yields the claim. □

While Lemma 4.20 gives a result on densest sub-graphs and their interpretation in the complement graph one has to wonder about the structure of said complement. This becomes particularly important when one tries to exploit the structure of L^c when investigating sub-graphs of L using Monte Carlo simulation. To assure convergence of such methods using L^c one should be at least in the presence of L connected and L^c connected. This brings some constraints with it. As discussed in Lemma 3.23 this brings a set of constraints with it which reduce the number of possible applications.

We turn now to geometric properties of $\mathfrak{L}_k$ which are useful in various contexts later on and are in themselves aesthetically nice results.

4.3.4 Geometric Properties of $\mathfrak{L}_k$

Having established various arithmetic properties of $\mathfrak{L}_k$ we turn to certain properties which are, in particular, linked to moving particles which block each other from occupying states. The vertex connectivity of L, i.e., the minimal number of vertices which have to be removed to render the graph disconnected, plays an important role in that regard. It may be interpreted as the minimal number of particles needed to block completely the transition of a particle from one "half" of the graph to the other "half" if the number of particles exceeds the connectivity of L. Additionally, still if the number particles exceeds the connectivity, we can use it to obtain bounds on cover times of L. This is based on a construction of easier sub-problems by conditioning on configurations $\mathfrak{v}$ which disconnect the graph $L\backslash\mathfrak{v}$ optimally in an almost fractal way. Unfortunately, it turns out that the connectivity of L and $\mathfrak{L}_k$ are not connected by a function defined on the natural numbers but their link is more intricate. We are going to explore this in what follows as preparation for further results in later sections.

To illustrate the difficulties, we first consider two 5 regular graphs on 8 vertices both with connectivity 4. We call them $L^{(1)}$ and $L^{(2)}$, respectively. They are illustrated with their corresponding k particle graphs $\mathfrak{L}_k^{(i)}$ with $k = 4$ in Figure 19. The two graphs $L^{(1)}$ and $L^{(2)}$ depicted in Figure 19 are not isomorphic since then, also all the associated graphs $\mathfrak{L}_k$ were isomorphic for all k.

The connectivity of $\mathfrak{L}_k$ and L is indeed linked by multiple properties. In particular, constructing the graph $\mathfrak{L}_k(L\backslash V_\sigma)$ based on the disconnected graph $L\backslash V_\sigma$, if V_σ is a vertex cut of size σ of L is not the same as removing a vertex cut from $\mathfrak{L}_k$ constructed from L. For graphs with a simple structure we can make direct deductions on the connectivity. To this end we consider first the example of a star graph to illustrate how disconnecting L translates to disconnecting $\mathfrak{L}_k$.

Lemma 4.21. *Let L be the star graph on $\bar{n}+1$ vertices and $\bar{n}$ beams. Let $k \in \{1,\dots,\bar{n}\}$. The star graph L has vertex connectivity $\kappa(L) = 1$ and $\mathfrak{L}_k$ satisfies*

$$\kappa(\mathfrak{L}_k) = \begin{cases} k, & k \le \dfrac{\bar{n}+1}{2}, \\ \bar{n}+1-k, & k \ge \dfrac{\bar{n}+1}{2}. \end{cases} \tag{4.20}$$

Proof. We only consider the case $k \le \frac{\bar{n}+1}{2}$. The second case follows by Proposition 4.4. We denote by α the center of the star and by $\beta_1,\dots,\beta_n$ the vertices at the ends of the beams. Let $\mathfrak{v} \in \mathfrak{V}_k$. We have to consider two cases, namely $\alpha \in \mathfrak{v}$ and $\alpha \notin \mathfrak{v}$. If $\alpha \notin \mathfrak{v}$, i.e, $\mathfrak{v} \subseteq \{\beta_1,\dots,\beta_n\}$, then any path $(\mathfrak{v},\mathfrak{u},\mathfrak{w})$ from $\mathfrak{v}$ to $\mathfrak{w} \subseteq \{\beta_1,\dots,\beta_n\}$ of length 2 has to include α in $\mathfrak{u}$. Consequently, any $\mathfrak{v} \subseteq \{\beta_1,\dots,\beta_n\}$ has exactly k neighbors and by the same argument any $\mathfrak{v}$ containing α has exactly $n+1-k$

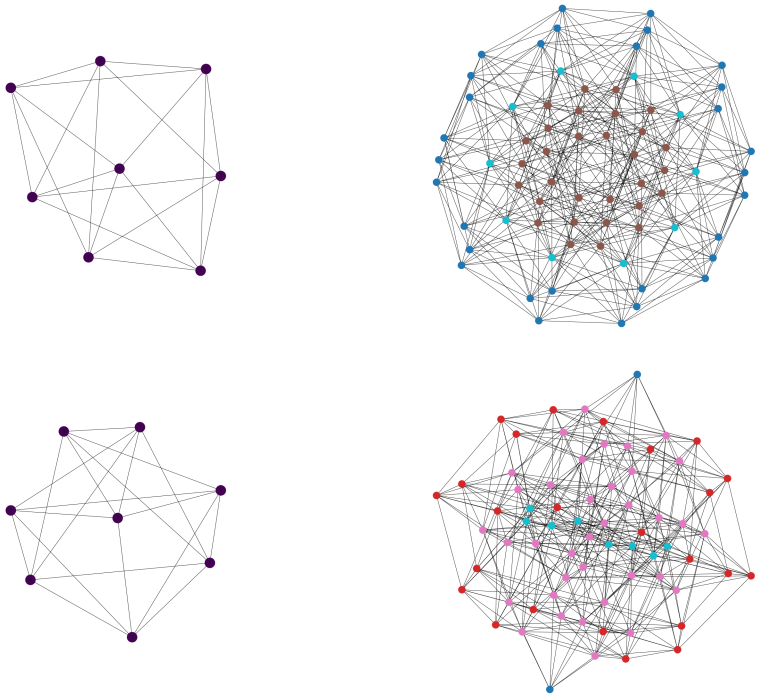

Figure 19: First row: 5-regular graph on 8 vertices with connectivity 5. Corresponding $\mathfrak{L}_4$ has connectivity 10.
Second row: 5-regular graph on 8 vertices with connectivity 5. Corresponding $\mathfrak{L}_4$ has connectivity 8.

neighbors. Furthermore, for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ with $\mathfrak{v} \neq \mathfrak{w}$ and $\alpha \in \mathfrak{v}$ as well as $\alpha \in \mathfrak{w}$ we have $\langle \mathfrak{v}, \mathfrak{w} \rangle \notin \mathfrak{E}_k$. Consequently, an edge $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$ if and only if $\alpha \in \mathfrak{v}$ and $\alpha \notin \mathfrak{w}$ or vice versa. Therefore, as long as $\mathfrak{v}$ is connected to at least one other vertex $\mathfrak{w}$ there is a path to any other $\mathfrak{v}' \in \mathfrak{V}_k$. We conclude that the vertex connectivity is given by the minimal degree which is $\min\{k, \bar{n} + 1 - k\}$. This yields the claim. □

We find that this property carries over to any $\mathfrak{L}_k$ associated to an arbitrary $\bar{d}$-regular graph for any $k \in \{1, \ldots, \bar{n} - 1\}$. Unfortunately, the proof remains incomplete, even though the intuition from the proof for the star-graph remains valid. In combination with the techniques using multisets from Proposition 4.2 and the impossibility to block paths for configurations if not all neighboring configurations in $\mathfrak{L}_k$ are blocked, should yield the proof of the following conjecture. Special cases and a lack of time force us to leave this problem open.

Conjecture 4.22. *Let L be a $\bar{d}$-regular graph on $\bar{n}$ vertices and let $k \in \{1, \dots, \bar{n}-1\}$. Then,* $\kappa(\mathfrak{L}_k) = \min_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v})$ *and the corresponding vertex cut is the neighborhood of* $\mathfrak{v}_* \in \mathfrak{V}_k$ *with* $\deg_k(\mathfrak{v}_*) = \min_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v})$.

Conjecture 4.22 captures the high connectivity of $\mathfrak{L}_k$ which is only implicitly linked to the connectivity L. Already the star graph shows that the connectivity of $\mathfrak{L}_k$ can be much larger than the one of L. On the other hand, finding the vertex connectivity of a graph is a well known problem, for which fast algorithms have been developed over the years. Conjecture 4.22 links this topic to the topic of finding the density of the densest k sub-graph of L. In [Henz00] the authors present an algorithm for finding the connectivity of a graph which is asymptotical as $\mathcal{O}(nm)$ to the number of vertices n and the number of edges m. This implies that there is an algorithm for finding the densest k vertex induced subgraph which is asymptotical as $\mathcal{O}\left(\binom{\bar{n}}{k}^2 \cdot \frac{\bar{d}k(\bar{n}-k)}{2(\bar{n}-1)}\right)$. Further research into this link might yield additional insights on upper bounds of the complexity of finding dense sub-graphs of regular graphs.

4.3.5 Special cases for $\bar{d}$-regular graphs and fixed k

To begin with, we consider the cases $k = 2$ and $k = \bar{n} - 2$. In these cases due to the isomorphism $\mathfrak{L}_k \cong \mathfrak{L}_{\bar{n}-k}$ we only have to make claims about one or the other. It turns out that the degree set D_k is given by $D_k = \{2\bar{d}, 2(\bar{d}-1)\}$ since for $k = 2$ the two particles can be neighbors and block, therefore, an adjacent vertex mutually, or not. We define the level sets with respect to the degree as $\mathfrak{V}_{D_k;2l} := \{\mathfrak{v} \in \mathfrak{V}_k | \deg_k(\mathfrak{v}) = 2l\}$ for $l \in \{\bar{d}, \bar{d}-1\}$. Remark that $|\mathfrak{V}_{D_k;2(\bar{d}-1)}| = |E| = \frac{\bar{n}\bar{d}}{2}$. An important question to ask is how large subsets of $\mathfrak{V}_k$ can be chosen when taking into account weights given by the degree sequence on said subset. We consider to this end for $\mathfrak{U} \subset \mathfrak{V}_k$ the map

$$\mathfrak{f}(\mathfrak{U}) := \frac{\bar{d}|\mathfrak{V}_{D_k;2(\bar{d}-1)} \cap \mathfrak{U}| + (\bar{d}+1)|\mathfrak{V}_{D_k;2\bar{d}} \cap \mathfrak{U}|}{\frac{\bar{n}\bar{d}^2}{2} + \frac{\bar{n}(\bar{n}-1-\bar{d})(\bar{d}+1)}{2}}$$

which represents the weighted sum of the vertices in $\mathfrak{U}$ where each weight corresponds to the quotient of the number of smaller and larger degrees then $\deg_k(\mathfrak{v})$. As there are only two sets $\mathfrak{V}_{D_k;2(\bar{d}-1)}$ and $\mathfrak{V}_{D_k;2\bar{d}}$ which constitutes $\mathfrak{V}_k$ and vertices in $\mathfrak{V}_{D_k;2(\bar{d}-1)}$ have a smaller weight than vertices in $\mathfrak{V}_{D_k;2\bar{d}}$, the size of $|\mathfrak{U}|$ is maximal subject to the constraint $\mathfrak{f}(\mathfrak{U}) \leq 2^{-1}$ if $\mathfrak{U}$ contains as many states with small degree as possible. Indeed, we find the size of the maximal $\mathfrak{U}$ with respect to the condition $\mathfrak{f}(\mathfrak{U}) \leq 2^{-1}$ as

$$\max_{\mathfrak{U}\subseteq\mathfrak{V}_k, \mathfrak{f}(\mathfrak{U})\leq 2^{-1}} |\mathfrak{U}| = \begin{cases} \frac{\bar{d}\bar{n}}{2} + y_*, & 0 \leq \bar{d} \leq \frac{1}{2}\left(\bar{n} - 1 + \sqrt{\bar{n}-1}\sqrt{\bar{n}-1+4}\right) \\ y^*, & \textit{otherwise} \end{cases} \tag{4.21}$$

where we define

$$y_* = \left\lfloor \frac{\bar{n}(\bar{n}-1-\bar{d})}{4} - \frac{\bar{d}^2\bar{n}}{4(\bar{d}+1)} \right\rfloor; \quad y^* = \left\lfloor \frac{\bar{d}+1}{\bar{d}} \frac{\bar{n}(\bar{n}-1-\bar{d})}{4} + \frac{\bar{d}\bar{n}}{4} \right\rfloor.$$

We outline a proof in what follows. First assume that

$$\bar{d} \leq \frac{1}{2}\left(\bar{n}-1+\sqrt{\bar{n}-1}\sqrt{\bar{n}-1+4}\right).$$

Recall that $D_k = \{d \in \mathbb{N} | \exists \mathfrak{v} \in \mathfrak{V}_k : \deg_k(\mathfrak{v}) = d\}$ and for $l \in D_k$ that $\mathfrak{V}_{D_k;l} := \{\mathfrak{v} \in \mathfrak{V}_k | \deg_k(\mathfrak{v}) = l\}$. In the case $k = 2$, there are exactly two distinct degrees in $\mathfrak{V}_k$ and, hence, only two sets $\mathfrak{V}_{D_k;l}$, namely, $\mathfrak{V}_{D_k;2(\bar{d}-1)}$ and $\mathfrak{V}_{D_k;2\bar{d}}$. Indeed, in the discussed case, we can derive $|\mathfrak{U}^*| := \max_{\mathfrak{U}\subseteq\mathfrak{V}_k, \mathfrak{f}(\mathfrak{U})\leq 2^{-1}} |\mathfrak{U}|$ explicitly, which becomes important since $|\mathfrak{S}| \leq |\mathfrak{U}^*|$. Considering the representation of sub-sets of size 2 of a vertex set as an edge, we obtain that $|\mathfrak{V}_{D_k;2(\bar{d}-1)}| = \frac{\bar{n}\bar{d}}{2}$ and, consequently, $|\mathfrak{V}_{D_k;2(\bar{d}-1)}| = \frac{\bar{n}(\bar{n}-1-\bar{d})}{2}$. We can now consider the two cases for $\max_{\mathfrak{U}\subseteq\mathfrak{V}_k, \mathfrak{f}(\mathfrak{U})\leq 2^{-1}} |\mathfrak{U}|$ which correspond to the cases $\frac{\bar{n}\bar{d}^2}{2} \leq \frac{\bar{n}(\bar{n}-1-\bar{d})(\bar{d}+1)}{2}$ and $\frac{\bar{n}\bar{d}^2}{2} \geq \frac{\bar{n}(\bar{n}-1-\bar{d})(\bar{d}+1)}{2}$, respectively.

Consequently, in the first case, the maximal set $\mathfrak{U}$ satisfies $|\mathfrak{V}_{D_k;2(\bar{d}-1)}| \leq |\mathfrak{U}|$ and writing $|\mathfrak{U}| = |\mathfrak{V}_{D_k;2(\bar{d}-1)}| + y = \frac{\bar{d}\bar{n}}{2} + y$ for some non-negative integer y we obtain that for all $y \leq y_*$, for y_* as in the claim, the inequality

$$\mathfrak{f}(\mathfrak{U}) = \frac{\frac{\bar{n}\bar{d}^2}{2} + (\bar{d}+1)y}{\frac{\bar{n}\bar{d}^2}{2} + \frac{\bar{n}(\bar{n}-1-\bar{d})(\bar{d}+1)}{2}} \leq \frac{\frac{\bar{n}\bar{d}^2}{2} + (\bar{d}+1)\left(\frac{\bar{n}(\bar{n}-1-\bar{d})}{4} - \frac{\bar{d}^2\bar{n}}{4(\bar{d}+1)}\right)}{\frac{\bar{n}\bar{d}^2}{2} + \frac{\bar{n}(\bar{n}-1-\bar{d})(\bar{d}+1)}{2}}$$

is satisfied where the last term is smaller or equal one half. Let $\mathfrak{A}_{y_*} \subset \mathfrak{V}_{D_k;2\bar{d}}$ be any subset of size y_*. Then, choosing $\mathfrak{U} = \mathfrak{V}_{D_k;2(\bar{d}-1)} \cup \mathfrak{A}_{y_*}$ we obtain a subset $\mathfrak{U} \subset \mathfrak{V}_k$ of size $\frac{\bar{d}\bar{n}}{2}$ with $\mathfrak{f}(\mathfrak{U}) \leq 2^{-1}$ and any larger subset $\mathfrak{W}$ gives $\mathfrak{f}(\mathfrak{W}) > 2^{-1}$. We have therefore found the size maximum of any subset $\mathfrak{U}$ satisfying $\mathfrak{f}(\mathfrak{U}) \leq 2^{-1}$. In the second case, we can choose $\mathfrak{U} \subset \mathfrak{V}_{D_k;2(\bar{d}-1)}$ and

$$\mathfrak{f}(\mathfrak{U}) = \frac{\frac{\bar{n}\bar{d}y}{2}}{\frac{\bar{n}\bar{d}^2}{2} + \frac{\bar{n}(\bar{n}-1-\bar{d})(\bar{d}+1)}{2}}.$$

The remaining arguments can be made analogously to the first case.

We discuss now special cases $\bar{d} \in \{2, \bar{n}-2\}$. In these cases explicit quantitative statements can be made about the structure of $\mathfrak{L}_k$ for any $k \in \{1, \ldots, \bar{n}-1\}$. We lead into this section with the case $\bar{d} = 2$ and analyze the structure of the graph $\mathfrak{L}_k$. This means we are working with a kPG defined on a cycle graph as illustrated in Figure 20. We assume that there are $k \leq \frac{\bar{n}}{2}$ particles since by symmetry $\mathfrak{L}_k \cong \mathfrak{L}_{\bar{n}-k}$ we cover the remaining cases as well. To find the degrees in $\mathfrak{L}_k$ we can write any configuration $\mathfrak{v}$

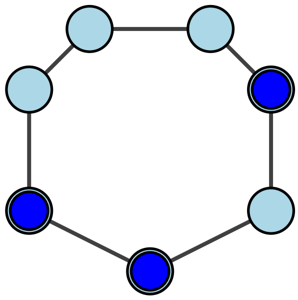

Figure 20: A particle configuration of 3 particles on a cycle graph of length 7.

of particles as a vector which contains as entries the size of a connected component of $L_{\mathfrak{v}}$. Assume that there are $c'_{\mathfrak{v}}$ connected components of $L_{\mathfrak{v}}$ and denote their respective sizes by k_i for $i = 1, \ldots, c'_{\mathfrak{v}}$. Each connected component contains exactly $k_i - 1$ edges due to the structure of the underlying cycle graph L. Therefore, we obtain that

$$\deg_k(\mathfrak{v}) = 2k - 2\sum_{i=1}^{c'_{\mathfrak{v}}}(k_i - 1) = 2c'_{\mathfrak{v}}.$$

Furthermore, since $k \leq \frac{\bar{n}}{2}$ we can construct for any $l \in \{1, \ldots, k\}$ a configuration $\mathfrak{v}$ satisfying $c'_{\mathfrak{v}} = l$. This implies that $D_k = \{2l | l \in \{1, \ldots, k\}\}$. The remaining task consist in finding $|\mathfrak{V}_{D_k;d}|$ for any $d \in D_k$ which is equivalent to finding the number of vertex induced k sub-graphs of L containing exactly $\frac{d}{2}$ edges. Figure 20 shows that this problem is related to finding all unique configurations of two colored balls on a necklace with a fixed number of each color. To explain this further, we recall that any configuration $\mathfrak{v}$ can be translated to a vector of length $c'_{\mathfrak{v}}$ where each entry corresponds to the length of one connected component of $L_{\mathfrak{v}}$. We can, hence, identify the problem with finding all distinct colored necklaces of length $c'_{\mathfrak{v}}$ with colorings $\{(x_i, y_i)_{i=1}^{c'_{\mathfrak{v}}} | \sum_{i=1}^{c'_{\mathfrak{v}}}(x_i, y_i) = (k, \bar{n} - k)\}$, a problem solved by Pòlya's enumeration theorem. The number of configurations induced by each distinct necklace is then given by the size of its orbit under the symmetric group $S_{c'_{\mathfrak{v}}}$. From this we can derive $|\mathfrak{V}_{D_k;2c'_{\mathfrak{v}}}|$ which gives the first part of the puzzle.

Lemma 4.23. *Let L be a cycle on $\bar{n}$ vertices and $k \in \left\{1, \ldots, \left\lfloor \frac{\bar{n}}{2} \right\rfloor\right\}$. Then, for $l \in \{0, \ldots, k\}$ the identity*

$$|\mathfrak{V}_{D_k;2l}| = \binom{k-1}{l-1}\binom{\bar{n}-k-1}{l-1}\frac{\bar{n}}{l} \tag{4.22}$$

is satisfied.

Proof. First, note that

$$\begin{aligned}\binom{k-1}{l-1}\binom{\bar{n}-k-1}{l-1}\frac{\bar{n}}{l} &= \binom{k-1}{l-1}\binom{\bar{n}-k-1}{l-1}\frac{\bar{n}-k+k}{l}\\ &= \binom{k-1}{l-1}\binom{\bar{n}-k-1}{l-1}\frac{k}{l}+\binom{k-1}{l-1}\binom{\bar{n}-k-1}{l-1}\frac{\bar{n}-k}{l}\\ &= \binom{k}{l}\binom{\bar{n}-k-1}{l-1}+\binom{k-1}{l-1}\binom{\bar{n}-k}{l}.\end{aligned}$$

This underlines the symmetry of the problem in k and $\bar{n}-k$. Based on Figure 20 we identify the problem with counting necklaces of two colors, which are fully defined by the sets $\mathfrak{v} \in \mathfrak{V}_k$. The colors are assumed to be blue and red, where blue represents an occupied site and red an unoccupied site. We want to calculate the number of distinct blue-red colored necklaces respecting rotational symmetry, i.e., two necklaces $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ are considered identical if and only if $\mathfrak{v} = \mathfrak{w}$. By fixing one vertex of the cycle, without loss of generality $v_1 \in V$, we can consider necklaces starting with a blue beat, continuing then counterclockwise.
Assume that v_1 is occupied, i.e, carries a blue beat. Place k blue beats on a line of length k and select l of them without replacement. There are $\binom{k}{l}$. Now, separate $\bar{n}-k$ in l positive parts, i.e, create a vector $(n_1, \ldots, n_l)$ with $n_i \in \mathbb{N}^*$ and $\sum_{i=1}^{l} n_i = \bar{n}-k$, which is possible in $\binom{\bar{n}-k-1}{l-1}$ ways. Place n_i red beats after the i-th drawn blue beat. This construction gives, consequently, all necklaces starting with a blue beat and, thus, there are $\binom{k}{l}\binom{\bar{n}-k-1}{l-1}$ such necklaces.
By symmetry there are $\binom{k-1}{l-1}\binom{\bar{n}-k}{l}$ necklaces starting with a red beat. Summing both terms yields the claim. □

We can exploit these results to make claims about densities of sub-graphs. In particular, we discuss the question when vertex induced sub-graphs become "dense". In what follows we are going to consider only regular graphs since they arise naturally, later on in this work, in the context of particle systems and graph theoretic considerations are beyond the scope of this work. To this end, we define for $\bar{d} \in \mathbb{N}$ and $k \in \{1, \ldots, \bar{n}-1\}$ the set $\mathfrak{V}_k := \{\mathfrak{v} \subset V | \ |\mathfrak{v}| = k\}$ which will play an essential role later on and serves now as the set of all possible vertex induced sub-graph configurations. We continue now with the analysis of graphs with large average density as defined in Definition 4.24 which represent graphs where particles are "forced" to interact a lot due to high edge density.

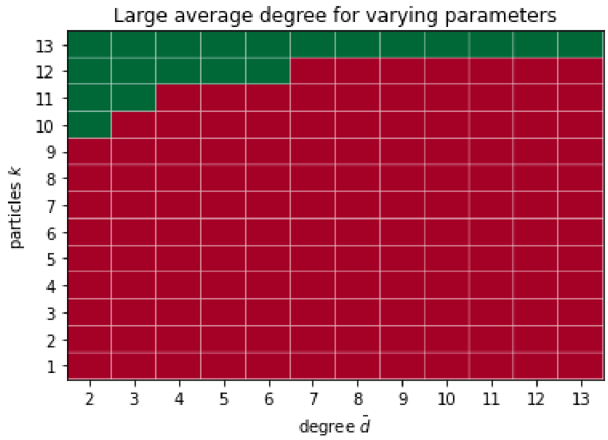

Figure 21: Grid over k and $\bar{d}$ varying between their minimal and maximal values for a given $\bar{n}$. Green blocks correspond to pairs $(\bar{d}, k)$ for which the condition $\min_{\mathfrak{v}\in\mathfrak{V}_k} \text{avg deg}_k(L_{\mathfrak{v}}) \geq \bar{d} - 1$ is satisfied for all $\bar{d}$-regular graphs L.

Definition 4.24. *Denote by $\mathcal{G}(\bar{n}, \bar{d})$ the set of all $\bar{d}$-regular graphs on $\bar{n}$ vertices and and consider for $L \in \mathcal{G}(\bar{n}, \bar{d})$ and $k \in \mathbb{N}$ the k-particle graph $\mathfrak{L}_k$. We define*

$$\Gamma_{\mathfrak{l}}^{(k)} := \bigcup_{\bar{n},\bar{d}\in\mathbb{N}} \left\{ L \in \mathcal{G}(\bar{n}, \bar{d}) \middle| \min_{\mathfrak{v}\in\mathfrak{V}_k} \text{avg deg}_k(L_{\mathfrak{v}}) \geq \bar{d} - 1. \right\} \tag{4.23}$$

and call $L \in \Gamma_{\mathfrak{l}}^{(k)}$ a graph of large average k-density.

Since the characterization of $\Gamma_{\mathfrak{l}}^{(k)}$ remains in the realm of k vertex induced sub-graphs and their properties, it is hard to get precise results on avg $\text{deg}_k(L_{\mathfrak{v}})$ for arbitrary $\mathfrak{v} \in \mathfrak{V}_k$ but in what follows we are going to illustrate the implications for $\bar{d}$ and k in the special case when $\bar{n} = 14$. We choose $\bar{n} = 14$ since it gives a qualitatively representative insight into the dependence between k and $\bar{d}$ while being still computationally approachable. In Figure 21 we show the functional dependence of

$$L \in \Gamma_{\mathfrak{l}}^{(14,k)} := \Gamma_{\mathfrak{l}}^{(k)} \cap \bigcup_{\bar{d}\in\mathbb{N}} \mathcal{G}(14, \bar{d})$$

when varying $\bar{d}$ and k. Green squares represent pairs $(\bar{d}, k)$ for which there is a graph L on $\bar{n} = 14$ vertices which satisfies $L \in \Gamma_{\mathfrak{l}}^{(14,k)}$ while red squares imply that $\Gamma_{\mathfrak{l}}^{(14,k)} = \emptyset$. One property which is striking in Figure 21 is the monotonicity of k, i.e., the number of particles necessary to have sub-graphs with high density, as the degree $\bar{d}$ grows to ensure that there is a graph $L \in \Gamma_{\mathfrak{l}}^{(14,k)}$. Evidently, the complete graph is a element of $\Gamma_{\mathfrak{l}}^{(14,k)}$ if and only if $k = \bar{n} - 1$ since every k-sub-graph of the complete graph has density $k-1$. But, Figure 21 shows also that for $k = \bar{n}-1$ for any degree $\bar{d} \in \{2,\dots,k\}$ there is a $\bar{d}$-regular graph $L \in \Gamma_{\mathfrak{l}}^{(14,k)}$. Indeed, the boundary between the red and green squares resembles a bounded growth behavior, starting from the k-sub-graphs with minimal average density on a cycle and being bounded by $\bar{n}-1$.

Conjecture 4.25. *For fixed $(\bar{n}, \bar{d}, k) \in \mathbb{N}^3$ we define a slice in $\Gamma_{\mathfrak{l}}^{(k)}$ as $\Gamma_{\mathfrak{l}}^{(\bar{n},\bar{d},k)} := \Gamma_{\mathfrak{l}}^{(k)} \cap \mathcal{G}(\bar{n}, \bar{d})$ and as well as*

$$\Gamma_{\mathfrak{l}}^{(\bar{n},\bar{d},\cdot)} := \bigcup_{k\in\mathbb{N}} \Gamma_{\mathfrak{l}}^{(\bar{n},\bar{d},k)} \tag{4.24}$$

and the functions $f_{\bar{n}} : \left[\frac{2}{\bar{n}}, 1\right] \to [0,1]$ for $x \in \left[\frac{i}{\bar{n}}, \frac{i+1}{\bar{n}}\right)$ with $i \in \{2,\dots,\bar{n}-1\}$ as

$$f_{\bar{n}}(x) := \frac{1}{\bar{n}} \min\left\{ k \in \{1,\dots,\bar{n}-1\} \,\middle|\, \min_{\substack{L\in\Gamma_{\mathfrak{l}}^{(\bar{n},i,\cdot)}\\ \mathfrak{v}\in\mathfrak{V}_k}} \operatorname{avg}\deg_k(L_{\mathfrak{v}}) \geq i-1 \right\}. \tag{4.25}$$

Let $\bar{n} = 2\bar{n}'$. Then, $f_{2\bar{n}'}$ converges pointwise for $x \in [0,1]$ and there is a monotonous function $g \in C(0,1)$ with $g(0) = 0$, $g(1) = 1$ such that the limit is given by

$$\lim_{\bar{n}'\to\infty} f_{2\bar{n}'}(x) = \frac{g(x)}{3} + \frac{2}{3}. \tag{4.26}$$

This remains a conjecture, since pointwise convergence and the continuity of the limit are out of reach for the time being but since they are central to the statement, reducing the claim would not be of interest. We manage, nonetheless, to prove the initial and finial value as well as the monotony using the following arguments.

Initial & final values: Let $x \in \left[\frac{2}{\bar{n}}, \frac{3}{\bar{n}}\right)$. We, hence, focus on the case $\bar{d} = 2$ and we find the minimal average degree in any $L_{\mathfrak{v}}$ by considering $\max_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v})$. Since for $k \leq \frac{\bar{n}}{2}$ the degree set of $\mathfrak{L}_k$ is given by $D_k = \{2l | l \in \{1,\dots,k\}\}$ if $\bar{d} = 2$ we obtain that

$$\max_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v}) = 2k = k\bar{d} - 0 \tag{4.27}$$

such that we can deduce that

$$\min_{\substack{L\in\Gamma_{\mathfrak{l}}^{(\bar{n},i,\cdot)}\\ \mathfrak{v}\in\mathfrak{V}_k}} \operatorname{avg}\deg(L_{\mathfrak{v}}) = 0 < \bar{d} - 1. \tag{4.28}$$

Consider, therefore, $k \geq \frac{\bar{n}}{2}$ from which we obtain by $\mathfrak{L}_k \cong \mathfrak{L}_{\bar{n}-k}$ and the uniqueness of the 2-regular connected graph, which is the cycle, that

$$2 - \min_{\substack{L\in\Gamma_{\mathfrak{l}}^{(\bar{n},i,\cdot)}\\ \mathfrak{v}\in\mathfrak{V}_k}} \operatorname{avg}\deg(L_{\mathfrak{v}}) = k^{-1} \max_{v\in\mathfrak{V}_k} \deg_k(\mathfrak{v}) = k^{-1} \max_{v\in\mathfrak{V}_{\bar{n}-k}} \deg_{\bar{n}-k}(\mathfrak{v}) = 2(\bar{n}-k)k^{-1}$$

which is equivalent to

$$\min_{\substack{L\in\Gamma_{\mathfrak{l}}^{(\bar{n},i,\cdot)}\\ \mathfrak{v}\in\mathfrak{V}_k}} \operatorname{avg}\deg(L_{\mathfrak{v}}) = 2 - \frac{2(\bar{n}-k)}{k}$$

and $2 - \frac{2(\bar{n}-k)}{k} \geq 1$ if and only if $k \geq \frac{2\bar{n}}{3}$. Therefore, the minimal k is given by $\left\lceil \frac{2\bar{n}}{3} \right\rceil$ and, consequently, we obtain for $x \in \left[\frac{2}{\bar{n}}, \frac{3}{\bar{n}}\right)$ that

$$f_{\bar{n}}(x) = \frac{1}{\bar{n}}\left\lceil \frac{2\bar{n}}{3} \right\rceil \to \frac{2}{3}, \qquad \bar{n}\to\infty. \tag{4.29}$$

On the other hand, if $\bar{d} = \bar{n}-1$ every sub-graph on k vertices is a complete graph on k vertices. Therefore, the average degree in all subgraphs is $k-1$ such that

$$k-1 \min_{\substack{L\in\Gamma_{\mathfrak{l}}^{(\bar{n},\bar{d},\cdot)}\\ \mathfrak{v}\in\mathfrak{V}_k}} \operatorname{avg}\deg(L_{\mathfrak{v}}) \geq \bar{d}-1$$

is equivalent to $k \geq \bar{d}$ and, therefore, $k = \bar{d} = \bar{n}-1$. Hence, we conclude for $x \in \left(\frac{\bar{n}-1}{\bar{n}}, 1\right]$ that $f_{\bar{n}}(x) = 1 - \frac{1}{\bar{n}} \to 1$ as $\bar{n}\to\infty$. Consequently, we obtain that $g(1) = 1$.

Monotony of g: We are proving now the monotony of g. To that end, fix $\bar{n}\in\mathbb{N}$ even and consider $x, y \in [0,1)$, $x \leq y$ such that $\lceil y\bar{n}\rceil = \lceil x\bar{n}\rceil + 1$. Let $f_{\bar{n}}(y) = \bar{k}\bar{n}^{-1}$. Then, all $L \in \Gamma_{\mathfrak{l}}^{(\bar{n},\lceil y\bar{n}\rceil,\cdot)}$ and $\mathfrak{v}\in\mathfrak{V}_{\bar{k}}$ satisfy

$$\frac{1}{\bar{k}}\sum_{v\in\mathfrak{v}} \deg_{\lceil y\bar{n}\rceil}^{L_{\mathfrak{v}}}(v) \geq \lceil y\bar{n}\rceil - 1.$$

Now consider $L \in \Gamma_{\mathfrak{l}}^{(\bar{n},\lceil x\bar{n}\rceil,\cdot)}$ and $\mathfrak{v}\in\mathfrak{V}_{\bar{k}}$. Then, we have to consider two cases. First, assume that there is a $\lceil y\bar{n}\rceil$-regular graph $\bar{L} = (V,\bar{E})$ which contains a perfect matching $\mathcal{M}$ such that $L \overset{\sim}{=} (V, \bar{E}\backslash\mathcal{M})$. Then, we find

$$\lceil y\bar{n}\rceil - 1 \leq \frac{1}{\bar{k}}\sum_{v\in\mathfrak{v}} \deg_{\lceil y\bar{n}\rceil}^{\bar{L}_{\mathfrak{v}}}(v) = \frac{1}{\bar{k}}\sum_{v\in\mathfrak{v}} \deg_{\lceil x\bar{n}\rceil}^{L_{\mathfrak{v}}}(v) + 1$$

and using $\lceil y\bar{n}\rceil = \lceil x\bar{n}\rceil + 1$ we obtain

$$\lceil x\bar{n}\rceil - 1 \leq \frac{1}{\bar{k}}\sum_{v\in\mathfrak{v}} \deg_{\lceil x\bar{n}\rceil}^{L_{\mathfrak{v}}}(v).$$

Secondly, if there is no such $\bar{L}$ we can construct by the Handshake Lemma and the fact that $\bar{n}$ is even a graph $\tilde{L} = (V,\tilde{E})$ as described in the following construction. Consider again $\mathfrak{v}\in\mathfrak{V}_{\bar{k}}$ and note that if $y \in \left[1-\frac{1}{\bar{n}}, 1\right)$ then $f_{\bar{n}}(x) \leq f_{\bar{n}}(y)$ since $\bar{n}f_{\bar{n}}(x) \in \{1,\ldots,\bar{n}-1\}$. Hence, we may assume $\bar{n}f_{\bar{n}}(y) \leq \bar{n}-2$. Now, we add edges to L such that for all $u \in V$ we have $\deg^L(u) \leq \deg^{\tilde{L}}(u)$ and there are exactly two vertices $v, w \in V\backslash\mathfrak{v}$ such that all $u \in V\backslash\{v,w\}$ satisfy $\deg^{\tilde{L}}(u) = \lceil x\bar{n}\rceil + 1 = \lceil y\bar{n}\rceil$ and $\deg^{\tilde{L}}(v) = \lceil x\bar{n}\rceil$ as well as $\deg^{\tilde{L}}(w) = \lceil x\bar{n}\rceil + 2$. Now, pick a neighbor u of w which is not a neighbor of v and remove the edge between u and w and add the edge between u and v. Call the resulting graph $\bar{L}$ which is a $\lceil y\bar{n}\rceil$ regular graph and observe

$$\lceil y\bar{n}\rceil - 1 \leq \frac{1}{\bar{k}} \sum_{v\in\mathfrak{v}} \deg^{L_{\mathfrak{v}}}_{\lceil y\bar{n}\rceil}(v) = \frac{1}{\bar{k}} \sum_{v\in\mathfrak{v}} \deg^{\tilde{L}_{\mathfrak{v}}}(v) \leq \frac{1}{\bar{k}} \sum_{v\in\mathfrak{v}} \deg^{L_{\mathfrak{v}}}_{\lceil x\bar{n}\rceil}(v) + 1.$$

Consequently, using $\bar{k}$ the condition $\lceil x\bar{n}\rceil - 1 \leq \bar{k}^{-1}\sum_{v\in\mathfrak{v}} \deg^{L_{\mathfrak{v}}}_{\lceil x\bar{n}\rceil}(v)$ is satisfied for all $L \in \Gamma_{\mathfrak{l}}^{(\bar{n},\lceil x\bar{n}\rceil,\cdot)}$ and $\mathfrak{v} \in \mathfrak{V}_{\bar{k}}$. Consequently, $f_{\bar{n}}(x) \leq \bar{k} = f_{\bar{n}}(y)$. Therefore, the function $f_{\bar{n}}$ is monotonous for all $\bar{n}$ and, therefore, the limit function g is also monotonous.

One may try to extend the result from Conjecture 4.25 by replacing the $\bar{d}-1$ with any $l \in \{1,\ldots,\bar{d}\}$. To illustrate, consider Figure 22. Note, in particular, the symmetric

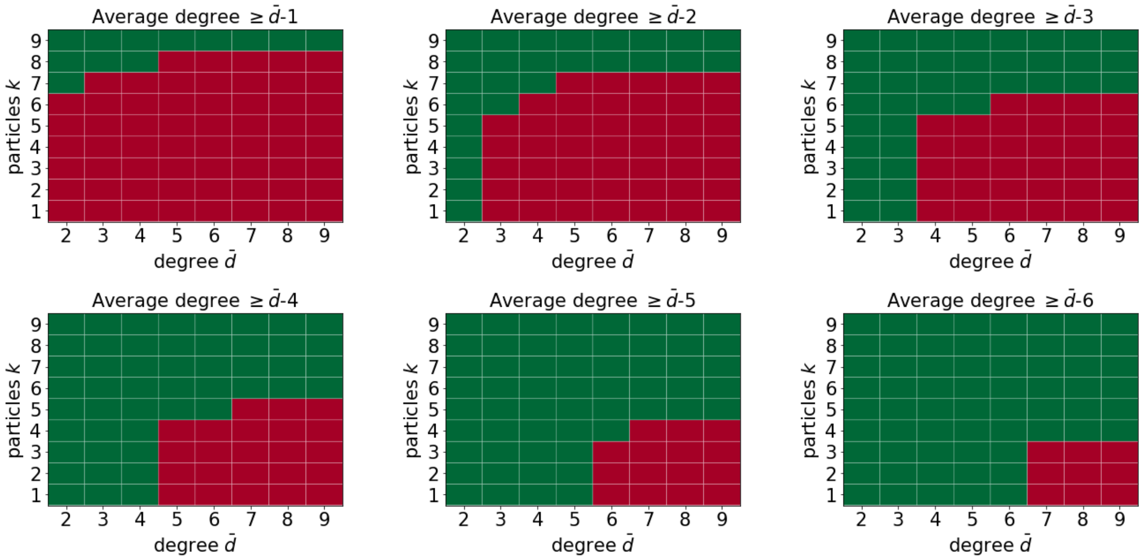

Figure 22: Change in the curve illustrated in Figure 21 using for illustrative reasons $\bar{n} = 10$. One can see that the area where the vertices of k vertex induced subgraphs of $\bar{d}$-regular graphs have average degree greater or equal $\bar{d} - i$ is, naturally, decreasing as i increases.

behavior in which this happens both on the k and $\bar{d}$ axis, which might be an indicator for regularities which might be exploited to obtain a deeper understanding of k vertex sub-graphs of $\bar{d}$-regular graphs. We leave this question open to move on to relevant tools for the analysis of interacting particle systems, presented in this work.

It turns out that the case $\bar{d} = \bar{n} - 2$ is much more forgiving in terms of the difficulty of deriving $|\mathfrak{V}_{D_k;d}|$ for some fixed $d \in D_k$ which in the previous case demanded the utilization of results from combinatorics. In what follows we can reduce the problem to considerations on the graph complement of a $\bar{d}$-regular graph with $\bar{d} = \bar{n} - 2$, following the ideas in the proof of Proposition 7.14. For example in Figure 23 one can see the translation from the graph L to its complement L^c in the sense that L^c contains exactly the edges which are not contained in L while preserving the vertex set. This renders L^c a disjoint union of paths of length 1, exhibiting $\frac{\bar{n}}{2}$ connected components.

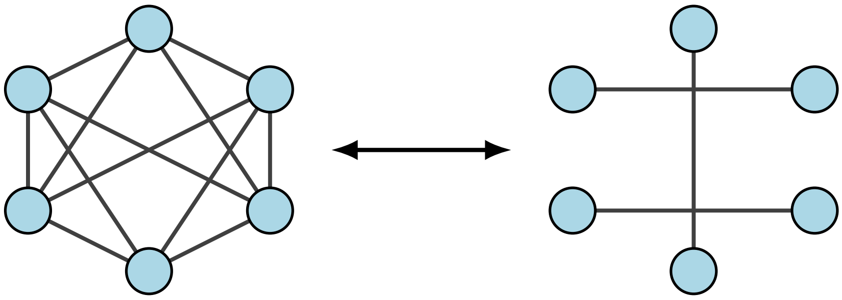

Figure 23: Translation from the 4-regular graph L on six vertices to the disjoint union of three paths of length 1, which corresponds to the complement graph of L, denoted by L^c.

Lemma 4.26. *Let L be a $\bar{n} - 2$ regular graph and $k \in \left\{1, \dots, \frac{\bar{n}}{2}\right\}$. Then,*

$$D_k = \left\{ k(\bar{n} - 2) - k(k-1) + 2l \;\middle|\; l \in \left\{0, \dots, \left\lfloor \frac{k}{2} \right\rfloor \right\} \right\} \tag{4.30}$$

and, consequently, any degree is even.

Proof. We employ the idea based on Figure 23 that it is sufficient to consider the graph complement L^c. Let $d \in D_k$. Then there exists a $\mathfrak{v} \in \mathfrak{V}_k$ such that $\deg_k(\mathfrak{v}) = d$. Since $\mathfrak{v} \subset V$ it also spans a vertex induced sub-graph in L^c. Denote by l the number of paths in L^c which are fully contained in $\mathfrak{v}$, i.e., both vertices of the path are elements of $\mathfrak{v}$. Then, $l \in \left\{0, \dots, \left\lfloor \frac{k}{2} \right\rfloor \right\}$ and

$$d = \deg_k(\mathfrak{v}) = k(\bar{n} - 2) - k(k-1) + 2l. \tag{4.31}$$

Since $\bar{n} - 2$ is even, $k(k-1)$ is even and so is $2l$ for any positive integer l we obtain that all degrees are even. □

Based on this correspondence between L and L^c we can also derive the size of each $\mathfrak{V}_{D_k;d}$ for $d \in D_k$.

Lemma 4.27. *Let L be a $\bar{n}-2$ regular graph and $k \in \left\{1, \dots, \left\lfloor \frac{\bar{n}}{2} \right\rfloor\right\}$. Then, for $j_d \in \left\{0, \dots, \left\lfloor \frac{k}{2} \right\rfloor\right\}$ the set, corresponding to the degree $d = k(\bar{n}-2)-k(k-1)+2j_d \in D_k$, satisfies the identity*

$$|\mathfrak{V}_{D_k;d}| = \sum_{l=0}^{k} \binom{\left\lfloor \frac{\bar{n}}{2} \right\rfloor}{l} \binom{l}{j_d} \binom{\left\lfloor \frac{\bar{n}}{2} \right\rfloor - l}{k-l-j_d}. \tag{4.32}$$

Proof. Fix $j_d \in \left\{0, \dots, \left\lfloor \frac{k}{2} \right\rfloor\right\}$. We write the vertices $v \in V$ in $\left\lfloor \frac{\bar{n}}{2} \right\rfloor$ columns and two rows where two vertices connected by an edge in L^c belong to the same column. First, place l particles in the first row. We have $\binom{\lfloor \frac{\bar{n}}{2} \rfloor}{l}$ possibilities to do so. To obtain a configuration $\mathfrak{v}$ with $\deg_k(\mathfrak{v}) = k(\bar{n}-2) - k(k-1) + 2j_d$ we have to place precisely j_d of the remaining $k-l$ particles in the second row among the neighbors of the already occupied vertices. We have $\binom{l}{j_d}$ to obtain this. Finally, we distribute the remaining $k-l-j_d$ particles on the remaining $\frac{\bar{n}}{2} - l$ vertices in the second row which is possible in $\binom{\lfloor \frac{\bar{n}}{2} \rfloor - l}{k-l-j_d}$ ways. To obtain the total, we sum over all possibilities for $l \in \{0, \dots, k\}$ which leads to the formula

$$|\mathfrak{V}_{D_k;d}| = \sum_{l=0}^{k} \binom{\left\lfloor \frac{\bar{n}}{2} \right\rfloor}{l} \binom{l}{j_d} \binom{\left\lfloor \frac{\bar{n}}{2} \right\rfloor - l}{k-l-j_d}.$$

□

We could demonstrate that for certain special cases of k and $\bar{d}$ it is possible to obtain explicit results on the structure of $\mathfrak{L}_k$ associated to the underlying graph L. In particular, we exploited in the cases where we fixed $\bar{d}$, that there is exactly one regular graph with this degree. Deviating from these special cases immediately yields non-uniqueness of the underlying graph L. For example, consider the case $\bar{n} = 6$ and $\bar{d} = 3$. Then, there are already two graphs with vastly different properties, which satisfy this condition. One of them even is bipartite, while the other is not, which has by Proposition 4.5 important influences on the structure of the associated kPG. We come back to this example in Subsection 7.3.4.

To end this section, we briefly review another possibility to consider particles on graphs. In this case, the particles will be distinguishable such that the size of the state space increases and even connectivity is no longer certain. The discussion will be based on an example inspired by a puzzle which is discussed in [Wil74].

4.4 kPGs for marked particles

So far, we have analyzed the structure of $\mathfrak{L}_k$ if we consider simply subsets of V of size k. This can be seen as indistinguishable particles and we record the configurations of the particles as unordered sets in the graph $\mathfrak{L}_k$. The problem becomes more involved as soon as we make the particles distinguishable. We can, then, differentiate the particles by names or give them numbers from 1 to k. This leads to a natural vertex set, which is no longer $\mathfrak{V}_k$ but V^k. The discussion in [Wil74] is the basis for what follows. For $\bar{v} \in V^k$ with $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_k)$ we can interpret the entry $\bar{v}_i$ as the position of particle i in L. Note that now configurations which are permutations of one another are distinguishable. A relevant example, which shows a possible application of marked particles on some finite graph, is the 15-puzzle. This puzzle consist of a quadratic box which can fit 16 quadratic tiles of adequate size and 15 small enumerated quadratic tiles which correspond to this size carrying numbers 1 to 15. The game then consists in ordering the tiles row by row only by sliding one tile at the time using the remaining free slot, the first row containing the numbers 1 to 4, the second one the numbers 5 to 8 and so on. One can wonder about the existence of a solution to this task depending on the initial configuration of the tiles. Indeed, this example was already of interest for the authors of [GreLov74] and it arises now naturally as a state space in the context of interacting particles.

We consider a simple connected graph $L = (V, E)$ as well as k occupied sites $\{v_1, \ldots, v_k\} \subset V$. For $t \geq 0$ one of the occupied sides is chosen and the occupant moves to an adjacent side uniformly chosen among the free sides. If no free sides are available the configuration remains the same.

Consequently, we can associate the dynamics with exclusion process $\eta = (\eta_t)_{t\in\mathbb{N}}$ on L. We reformulate the exclusion process on L as a Markov chain on the a sub-graph of the Hamming graph.

Consider the following sub-graph of the Hamming graph $\mathcal{H}(\bar{n}, k) = (V_{\mathcal{H}}, E_{\mathcal{H}})$ and its vertex set

$$V_{\mathcal{H}} := \{\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_k) \in \mathrm{V}^k | \forall i, j \in \{1, \ldots, k\}, i \neq j : \mathrm{v}_i \neq \mathrm{v}_j\}. \tag{4.33}$$

Due to fact that no two entries of any vector may have the same value and, therefore, any vertex in $V_{\mathcal{H}}$ represents a subset of size k of V, we can conclude by the same reasoning as for $\mathfrak{V}_k$ that $V_{\mathcal{H}}$ is the correct choice to capture the configurations of the exclusion process. On the other hand, the edge set of $\mathcal{H}(\bar{n}, k)$ is too large, since it does not take into consideration the structure of L, or, it corresponds to the case where L is the complete graph. Thus, we have to make the following restriction to the edge set and arrive at

$$\mathfrak{E}_k^{(\mathfrak{m})} := \{\langle v, w\rangle \in E_{\mathcal{H}} | \exists! i,\ 1 \leq i \leq k : v_i \neq w_i,\ v_j = w_j \forall j \neq i,\ \langle v_i, w_i\rangle \in E\}.$$

We write $\mathfrak{V}_k^{(\mathfrak{m})} := V_{\mathcal{H}}$ and denote by $\mathfrak{L}_k^{(\mathfrak{m})} := (\mathfrak{V}_k^{(\mathfrak{m})}, \mathfrak{E}_k^{(\mathfrak{m})})$ the marked kPG associated to L. The graph $\mathfrak{L}_k^{(\mathfrak{m})}$ can, hence, be understood as the state space of a particle system where a transition to a different configuration corresponds to the displacement of exactly one of the k particles, each particle having a distinguishable mark.

To construct the graph representing the state space of a k-particle exclusion process with marked particles we remove all nodes from $\mathcal{H}(\bar{n}, k)$ for which two coordinates coincide. Naturally all edges to which such a point is incident are also being removed. For $v \in \mathfrak{V}_k^{(\mathfrak{m})}$ we denote by $\deg_k^{(\mathfrak{m})}(v)$ its degree in $\mathfrak{L}_k^{(\mathfrak{m})}$.

Some properties of $\mathfrak{L}_k^{(\mathfrak{m})}$ may be deduced from combinatorial considerations.

Lemma 4.28. *The graph $\mathfrak{L}_k^{(\mathfrak{m})}$ has the following properties.*

- *The vertex set $\mathfrak{V}_k^{(\mathfrak{m})}$ has size $|\mathfrak{V}_k^{(\mathfrak{m})}| = \dfrac{\bar{n}!}{(\bar{n}-k)!}$*
- *For $\mathrm{v} \in \mathfrak{V}_k^{(\mathfrak{m})}$ denote by L_v the vertex induced subgraph of v in L. Then, the degree of v in $\mathfrak{L}_k^{(\mathfrak{m})}$ is given by $\deg_k^{(\mathfrak{m})}(\mathrm{v}) = \sum_{i=1}^k \deg(v_i) - \deg^{L_\mathrm{v}}(v_i)$.*

Proof. Since every vertex $\mathfrak{v} \in \mathfrak{V}_k^{(\mathfrak{m})}$ may be seen as a word of length k constructed from an alphabet of length $\bar{n} > k$ where each letter may be used at most once, the property

$$|\mathfrak{V}_k^{(\mathfrak{m})}| = \frac{\bar{n}!}{(\bar{n}-k)!}$$

follows from basic combinatorial considerations on drawing without replacement.
To respect the geometry of L, two configurations can only be neighbors if the entries which differ are connected by an edge in L. Hence, any $v \in \mathrm{v}$ contributes $\deg(v)$ to the overall degree of $\deg_k^{(\mathfrak{m})}(\mathrm{v})$ when we do not take into account that at most one particle may occupy any vertex in L. Therefore, all k-tuples $\mathfrak{v} \in V^k$ with multiple identical entries are removed from V^k to construct $\mathfrak{V}_k^{(\mathfrak{m})}$ and each vertex $\mathfrak{w} \in \mathfrak{V}_k^{(\mathfrak{m})}$ loses all such neighbors $\mathfrak{v}$. Therefore, for any $v \in \mathrm{v}$ we obtain its final contribution to $\deg_k^{(\mathfrak{m})}(\mathrm{v})$ given by $\deg(v) - \deg^{L_\mathrm{v}}(v)$. By summing over all $v \in \mathrm{v}$ we obtain $\deg^{L_\mathfrak{v}} = \sum_{i=1}^k \deg(v_i) - \deg^{L_\mathrm{v}}(v_i)$. □

Since marked particles are not the focus of this work, we only want to give a brief insight into the differences to unmarked particles. In particular, we no longer have connectedness of $\mathfrak{L}_k^{(\mathfrak{m})}$ if L is connected. In fact, this has been shown in [Wil74] where the author gives conditions on the graphs for which connectedness follows and we can use this result due to an isomorphism which we establish in Proposition 4.29.

Proposition 4.29. *Let $k = \bar{n} - 1$. Then $\mathfrak{L}_k^{(\mathfrak{m})} \simeq \mathrm{puz}(L)$, as defined in [Wil74], and is therefore connected if and only if $\mathrm{puz}(L)$ is connected.*

Proof. The existence of an isomorphism follows directly by the construction of the two objects $\mathfrak{L}_k^{(\mathfrak{m})}$ and $\mathrm{puz}(L)$. □

We can, therefore, use the result found in [Wil74] to make claims about the connectivity of not only $\mathfrak{L}_k^{(\mathfrak{m})}$ but also $\mathfrak{L}_{k'}^{(\mathfrak{m})}$ for all $k' < \bar{n} - 1$.

Proposition 4.30. *Let* $\mathrm{puz}(L)$ *be connected. For* $k \leq \bar{n} - 1$ *the graph* $\mathfrak{L}_k^{(\mathfrak{m})}$ *is connected. Otherwise,* $\mathfrak{L}_k^{(\mathfrak{m})}$ *is the empty graph.*

Proof. We start with the second case $k \geq \bar{n}$. Again the particle interpretation helps to make the point. Assume $k = \bar{n}$. Then, all sides of L are occupied by particles. Hence any vertex in $\mathfrak{L}_k^{(\mathfrak{m})}$ is only a permutation of the set of all vertices in L. Moreover no particle can move which implies that there are no edges between any vertices in $\mathfrak{L}_k^{(\mathfrak{m})}$. The graph $\mathfrak{L}_k^{(\mathfrak{m})}$ is therefore the empty graph.

If $k > \bar{n}$ then we find by a direct combinatorial argument that any vertex in V_k contains at least one entry twice. Hence by the deletion process in the construction of $\mathfrak{V}_k^{(\mathfrak{m})}$ we obtain $\mathfrak{V}_k^{(\mathfrak{m})} = \emptyset$.

Let $k' < \bar{n}$ be the number of particles on L. Denote by $d' := \bar{n} - 1 - k'$. Then for $k := k' + d'$ the graph $\mathfrak{L}_k^{(\mathfrak{m})}$ is connected by Lemma 4.29. Let $\nu = (\nu_1, \ldots, \nu_{k'})$ and $\tilde{\nu} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_{k'})$ be some configurations. Draw d' unoccupied vertices

$$\mu_1, \ldots, \mu_{d'} \in V \backslash \{\nu_1, \ldots, \nu_{k'}\}. \tag{4.34}$$

and d' vertices

$$\tilde{\mu}_1, \ldots, \tilde{\mu}_{d'} \in V \backslash \{\tilde{\nu}_1, \ldots, \tilde{\nu}_{k'}\}. \tag{4.35}$$

Consider the configurations $\nu^{d'} = (\nu_1, \ldots, \nu_{k'}, \mu_1, \ldots, \mu_{d'})$ and $\tilde{\nu}^{d'} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_{k'}, \tilde{\mu}_1, \ldots, \tilde{\mu}_{d'})$. Then there is a path ϕ in $\mathfrak{L}_k^{(\mathfrak{m})}$ connecting $\nu^{d'}$ and $\tilde{\nu}^{d'}$, from which can be constructed a path in $H_{k'}^r$ by projection on the first k' components. Hence $H_{k'}^r$ is connected. □

Hence, for marked particles one could use $\mathfrak{L}_k^{(\mathfrak{m})}$ and its structure to represent configurations and their connections or characterize this family of graphs in itself combinatorically. We now turn to a probabilistic application of $\mathfrak{L}_k$ for which $\mathfrak{L}_k^{(\mathfrak{m})}$ is not suited since it enforces more constraints then necessary and desired. We consider the family of processes generalized exclusion processes which describe dynamical changes of configurations of unmarked particles which are mutually exclusive, i.e., a particle cannot move to an already occupied vertex.

4.5 Outlook: The k-Particle Graph

In this first part, we reviewed a set definitions from graph and set theory as well as established in the beginning some results, in particular with a focus on sub-graphs and their role (in regular graphs). This allowed us to define the central graph theoretical object of this work, the kPG $\mathfrak{L}_k$ associated to an underlying graph L, as defined in

Definition 4.1. We managed to formulate and proof several properties of $\mathfrak{L}_k$ on connectivtiy, the combinatorial properties, the degree sequence as well as the automorphism group, always motivated and with the application in mind, which we will discuss at length in Part III. Hence, instead of an extensive Outlook which cannot be complete in any sens, we keep it short and let the graph theory proficient reader decide, which of the immense number of open problems they want to tackle.

4.5.1 Future work

Evidently, the analysis we performed in this section is not complete in any regard. Many graph theoretical questions remain open for later research. This ranges from the diameter of $\mathfrak{L}_k$, which would, indeed, push the results in the later sections in terms of their quantitative dimension, to the characterization of the Eigenvalues of the graph Laplacian associated to $\mathfrak{L}_k$ as well as its link to L. Even conjectures remain at this

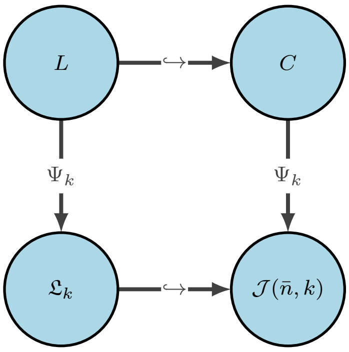

Figure 24: Link between types of graphs presented in this work. We denote by Ψ_k the construction of the kPG associated to L. Research questions on one may be considered in the setting of another along the depicted arrows.

point out of reach. Considering the diameter, from the isometry $\mathfrak{L}_k \overset{\sim}{=} \mathfrak{L}_{\bar{n}-k}$ we can, nonetheless, look for a formula which is symmetric in k and $\bar{n}-k$. On the other hand, it will be constrained by the diameter of the underlying graph L. Employing the particle view, any particle in a configuration $\mathfrak{v} \in \mathfrak{V}_k$ has to move at most the diameter of L to its new location in another configuration, if looking for the diameter of $\mathfrak{L}_k$. Therefore, a natural upper bound for $\operatorname{diam}(\mathfrak{L}_k)$ is $k \operatorname{diam}(L)$. It turns out that this bound is a lot bigger than the actual diameter of $\mathfrak{L}_k$ in cases testable via simulations.
Further topics include, for example, the link between L and $\mathfrak{L}_k$ as well as $\mathfrak{L}_k$ and $\mathcal{J}(\bar{n}, k)$. Indeed, this gives the diagram represented in Figure 24. Sub-graph relationships are preserved via the horizontal arrows, while the spanning of a kPG based on some underlying graph L is given in the vertical direction via the map Ψ_k. The potential of jumping between these images has been shown throughout this section and will,

so does the author hope, nourish the research on graph theoretic problems related to sub-graphs.

PART III

EXCLUSION PROCESSES IN ABSORBING ENVIRONMENTS - SOCIAL NETWORK DYNAMICS

Contents

5 Markov chains: Definitions and Applications

Outline of this section: Markov chains will be omnipresent in this work. The particle systems and opinion processes, which we consider, can be identified with Markov chains on some associated state space, which is, in general, larger then the problem at hand. The difficulty lies in finding a state space which is neither to large to be indescribable nor too small to capture all necessary information. Markov chain theory can than yield results on the long time behavior, convergence towards a limiting object, time reversibility of the process and links to the geometric properties of the state space. Throughout this work, we work with a finite but large state space.

The following section serves as a review of classical Markov chain theory as well as modern results on the quantitative understanding of these random processes. An example relevant to this work is given. We first recall the definition of a Markov chain, then go on to the convergence to a limiting distribution as the time tends to infinity and finish with quantitative results on the convergence speed towards the limiting distribution when it is unique and the Markov chain converges towards this limit for any initial distribution. Objects like the Cheeger constant, or bottleneck ratio, its link to the geometry of the underlying state space as well as the convergence speed of Markov chains will be discussed. A discussion of a path based result on the convergence speed, known as Doeblin's condition sheds a geometric light on this topic.

We close the section with a discussion of the Deffuant model, which represents the dynamic change of the opinions in the Echo Chamber Model in the absence of moving edges. Markov chain theory will serve as a useful tool for the analysis of the limiting opinions depending on the graph structure. We obtain a conjecture for the convergence of the model on non-complete graphs as well as bounds on the convergence speed towards the final opinion structure.

5.1 Markov chains

We begin this section with a review of Markov chain theory on finite state spaces. Evidently, we cannot review all directions of research which have been accomplished on this subject, but concentrate on the central definitions and results. The selection of the quantitative results is also highly influenced by the necessity for the results of the model analysis in later sections because this remains the main task of this work with less focus on pushing the general theoretical understanding on Markov chains.

5.1.1 Finite Markov chains

Throughout this work we consider an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we define all random objects, in particular the random processes we want to consider. Essentially, these are Markov chains, a sequence of random variables with a particular dependence structure. We review the topic using a small part of the vast literature which exists on the subject. In particular, the cited definitions and results can be found in [Norris97], [Bre13] and [Gra14].

Definition 5.1 ([Bre13])**.** *Consider a finite set S, a probability distribution ν on S and a sequence $X = (X_t)_{t\in\mathbb{N}}$ of random variables with values in S. We call X a Markov chain on S with initial distribution ν if for all $s \in S$ the equality $\mathbb{P}[X_0 = s] = \nu(s)$ holds true and for all $t \in \mathbb{N}$ and vectors $(s_0, s_1, \dots, s_{t+1}) \in S^{t+2}$ the sequence X satisfies*

$$\mathbb{P}[X_{t+1} = s_{t+1} | X_t = s_t, \dots, X_0 = s_0] = \mathbb{P}[X_{t+1} = s_{t+1} | X_t = s_t]. \tag{5.1}$$

If $\mathbb{P}[X_{t+1} = r | X_t = s]$ only depends on $s, r \in S$ and not on t for all $t \in \mathbb{N}$, then we call X a homogeneous Markov chain. Otherwise, it is called inhomogeneous.

This is the infamous property of Markov chains that the future only depends on the current situation.

Definition 5.2 ([Bre13])**.** *Consider a finite set S and a matrix $P = (p_{s,r})_{s,r\in S}$. We call P stochastic if for all $s \in S$ the equation*

$$\sum_{r\in S} p_{s,r} = 1 \tag{5.2}$$

is satisfied and doubly stochastic if and only if, additionally, for all $r \in S$

$$\sum_{s\in S} p_{s,r} = 1, \tag{5.3}$$

i.e. the transposed matrix P^T is stochastic as well.

Indeed, transition matrices are the building blocks of Markov chains, under the condition that the transition probabilities only depend on the current state and not on the time t. This makes the difference between homogeneous and inhomogeneous Markov chains.

Theorem 5.3 ([Bre13])**.** *Let S be a finite set and P a stochastic matrix on S. Then, P defines a homogeneous Markov chain X on S by*

$$\mathbb{P}[X_{t+1} = r | X_t = s] := p_{s,r} \tag{5.4}$$

for any $t \in \mathbb{N}$. In turn, if X is a homogeneous Markov chain on S, then X defines a stochastic matrix P on S by

$$p_{s,r} := \mathbb{P}[X_{t+1} = r | X_t = s] \tag{5.5}$$

In view of the preceding theorem it becomes clear that we can speak of a homogeneous Markov chain X on S with initial distribution ν and transition matrix P. Indeed, whenever we consider a Markov chain X with transition matrix P, we automatically imply that X is homogeneous due to the independence of t. The transitions of X may then fully described in terms of P and the initial distribution ν.

Theorem 5.4 ([Bre13])**.** *Let S be a finite set and X a sequence of random variables with values in S. Then, X is a homogeneous Markov chain with transition matrix P and initial distribution ν if and only if for any $t \in \mathbb{N}$ and $(s_0, \dots, s_t) \in S^{t+1}$ the sequence X satisfies*

$$\mathbb{P}[X_t = s_t, \dots, X_0 = s_0] = \nu(s_0) \cdot p_{s_0,s_1} \cdot \dots \cdot p_{s_{t-1},s_t}.$$

Additionally, in case of a homogeneous Markov chain we have access to a graphical representation of X given by a directed graph.

Definition 5.5 ([Norris97])**.** *Consider a finite set S and a Markov chain X on S with transition matrix P. Then, the matrix $A = (\mathbb{1}_{p_{s,r}>0})_{s,r\in S}$ defines a directed graph, which is called the transition graph of X.*

Transition graphs give for small examples a visual tool to examine the behavior of X even for its long term behavior, i.e, for $t \to \infty$. To emphasize this property, we consider the following example. Let $S = \{1, 2, 3\}$ and

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}. \tag{5.6}$$

Clearly, P is a stochastic matrix and, therefore, defines a Markov chain on S. While it is a fairly small state space, to build an intuition it is useful to visualize the process and possible transitions. We use the matrix A defined in Definition 5.5 to obtain this intuition. In Figure 25 one can see a directed graph with vertex set S and edges according to the possible transitions of X. The labels represent the corresponding transition probabilities. From this Figure one can deduce that X can attain any state $s \in S$ in finite time almost surely. One can also see that a full cycle in mathematically positive direction happens with greater probability, with probability $p = \frac{8}{27}$, than a cycle in the inverse direction, $q = \frac{1}{27}$. We come back to this property in the context of convergence of Markov chains. But first, we focus on the structure of the transitions given by the well known Chapman-Kolmogorov equation.

Theorem 5.6 (Chapman-Kolmogorov Equation [Bre13])**.** *Let S be a finite set and X a Markov chain on S with transition matrix P. Denote for $t \in \mathbb{N}$ the t-th power of P by $P^t = (p^{(t)}_{r,s})_{r,s\in S}$. The transition matrix satisfies for $s, r \in S$ the equation*

$$p^{(t_1+t_2)}_{s,r} = \sum_{z\in S} p^{(t_1)}_{s,z} p^{(t_2)}_{z,r} \tag{5.7}$$

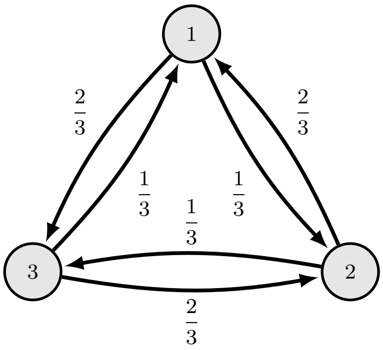

Figure 25: Transition graph of X on $S = \{1, 2, 3\}$ with transition matrix P defined in equation 5.6.

for any $t_1, t_2 \in \mathbb{N}$.

The theorem implies that going along a path of length $t_1 + t_2$ from some state s to some state r can be separated by introducing an intermediate state z and then summing up over all these intermediate steps. This allows for lower bounds on the transition probability $p_{s,r}^{(t_1+t_2)}$ when one is only able to characterize paths which visit certain states. In what follows, we will drop the statement that S is finite but we only consider this case, because it is the only relevant one for this work.

5.1.2 The Stationary distribution and reversibility

We have previously recalled the structure of a Markov chain X and its transition behavior, which depends only on the current state of the chain. Given an initial distribution ν on the state space S, one can then derived the distribution of X at any time $t \in \mathbb{N}$ by powers P^t of the associated transition matrix. This property is widely known and we omit, hence, a complete discussion, which can, by the interested reader, be found in one of the already mentioned sources [Bre13], [Norris97] or [Gra14], to name a few. An exhaustive list would be almost endless.

Furthermore, one can ask the question whether the chain X "forgets" at some point its initial distribution, i.e., the transitions of the chain have mixed the initial distribution sufficiently, that its initial form is unrecognizable. This leads to considerations about the long time-behavior of the chain, which are inherently connected with its underlying state space S, the possible sub-classes of S which might be defining for the transitions of X and, finally, symmetries of the chain in time. Various intricacies may arise when discussing the long time behavior of a Markov chain, depending also on the answers one is looking for. A review is, therefore, in order and we dedicated the following sub-sections to it.

We start with a couple of definitions characterizing the behavior on a Markov chain on its state space.

Definition 5.7 ([Bre13])**.** *Let X be a Markov chain on some state space S with transition matrix $P = (p_{x,y})_{x,y \in S}$. Write for $t \in \mathbb{N}$ the t-th power of P as $P^t = (p^{(t)}_{x,y})_{x,y \in S}$. A state $y \in S$ is said to be accessible from $x \in S$ if there is a $t \in \mathbb{N}$ such that $p^{(t)}_{x,y} > 0$. Two states $x, y \in S$ communicate with one another if x is accessible from y and vice versa.*

Indeed, communication gives a structure on the state space by separating it into classes.

Definition 5.8 ([Bre13])**.** *Let X be a Markov chain on some state space S with transition matrix $P = (p_{x,y})_{x,y \in}$. A set $C \subset S$ is called a communication class if for any pair $x, y \in C$ they, x and y, communicate. The Markov chain X is called irreducible if S is a communication class.*

Communication classes are important when discussing the long time behavior of Markov chains since they tell us about the "direction" of the chain. They work as traps since the chain can never leave such a class after having it entered once. In the applications, which we consider later on, the Markov chains will be irreducible.

Another important property of Markov chains is its periodicity which represents the existence of cycles which bring the chain back routinely back to a already attained probability distribution. For example, there are chains which jump between two distributions ν_1 and ν_2 over S, for example, in the sense $X_t \sim \nu_1$ for any even t and $X_t \sim \nu_2$ for any odd t when starting with $X_0 \sim \nu_1$. To characterize this periodicity, we use the following definition and come back to a related example afterwards.

Definition 5.9 ([Bre13])**.** *Let X be a Markov chain on some state space S with transition matrix $P = (p_{x,y})_{x,y \in}$. A state $x \in S$ has period $T_x := \gcd\{t \in \mathbb{N} | p^{(t)}_{x,x} > 0\}$. If $T_x = 1$, then x called aperiodic. If $T_x = 1$ for all $x \in S$ then X is called aperiodic.*

The previous properties showed that the transition matrix, which is a stochastic matrix, is the central algebraic object to analyze Markov chains. Making use of its Eigenvectors one can wonder about invariant objects of Markov chains. In particular, due to the fact that for $t \in \mathbb{N}$ the random variable X_t is distributed as νP^t when $X_0 \sim \nu$ and that, by stochasticity of P, there is at least one left eigenvector π to the eigenvalue 1 which is invariant in the sense $\pi = \pi P$. hence, starting with π as an initial distribution of X its distribution never changes. This object can, therefore, be seen as an equilibrium and the question about convergence to such an equilibrium arises naturally. Such π are called stationary distributions.

Theorem 5.10 ([Bre13])**.** *Let X be an irreducible and aperiodic Markov chain on some state space S. Then there is a unique stationary distribution π of X and for any* *initial distribution ν the Markov chain X converges in law to π as $t \to \infty$. In this case, X is called ergodic.*

In, both, the periodic and the reducible case one loses the independence of the initial distribution and the uniqueness of the stationary distribution. In the reducible chain, this is due to the fact "weight" given by an initial distribution to a closed communication class can never be "moved" to another communication class. On the other hand, a periodic Markov chain may show a flipping behavior moving all probability mass from one state to another. Consider as an example the Markov chain X induced by a transition matrix P which is given by

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.8}$$

Then the distribution $\pi = \left(\frac{1}{2}, \frac{1}{2}\right)$ is invariant under the action of P but if $X_0 = (0, 1)$ or $X_0 = (1, 0)$, then the chain becomes deterministic and never converges in law to π. While X is irreducible, it has period $T_x = 2$ and, therefore, falls not under the condition of Theorem 5.10.

Having established a sense in which Markov chains can converge, one can wonder about a way of identifying the limiting distribution. Since, in the most general setting we have to solve an eigenvalue problem to determine the stationary distribution, this can become complicated quickly depending on the structure of the transition matrix.

Definition 5.11 ([Norris97]). *Let X be a Markov chain on some state space S with transition matrix P. A probability distribution π on S is called reversible with respect to X if for all $x, y \in S$ the equality*

$$\pi_x p_{x,y} = \pi_y p_{y,x} \tag{5.9}$$

is satisfied. If X is ergodic with stationary distribution μ, define the reversed chain X^* with transition matrix P^* as

$$p^*_{x,y} = \frac{\mu(y)}{\mu(x)} p_{y,x}. \tag{5.10}$$

If X is ergodic and there is a reversible distribution π with respect to X, then X is called reversible.

Definition 5.11 captures the idea that a Markov chain can be considered under time reversal for the transport of the mass of a probability distribution. That any reversible distribution is also a stationary one follows directly from Equation 5.9, which is known as *detailed balance equation.* This explains that the reversibility of X is inherited immediately from the existence of a reversible probability distribution if X is ergodic. This gives a direct way to calculate the stationary distribution of an ergodic chain X

thanks to Equation 5.9. The time reversed chain as defined by equation 5.10 is in this case identical in distribution to X.

An important result, which exploits the structure of the underlying state space S and the possible transitions made by the chain, establishes reversibility of an ergodic chain X solely based on its transition matrix without the need of first finding a reversible probability distribution.

Theorem 5.12 (Kolmogorov's criterion, [Kelly11]). *Let X be an ergodic Markov chain with transition matrix P on a state space S. Then, the chain X is reversible if for all finite sequences of states $x_1, \dots, x_n \in S$ the equality*

$$p_{x_1,x_2} \cdot p_{x_2,x_3} \cdot \ldots \cdot p_{x_{n-1},x_n} = p_{x_n,x_{n-1}} \cdot p_{x_{n-1},x_{n-2}} \cdot \ldots \cdot p_{x_2,x_1} \tag{5.11}$$

is satisfied.

To illustrate Kolmogorov's criterion, we consider the example shown in Figure 25 as well as an adjusted version. Both are defined by the transition graphs shown in Figure 26 and we call the corresponding chains X and Y having the transition graph on the left and on the right, respectively. One can see by Kolmogorov's criterion, that any

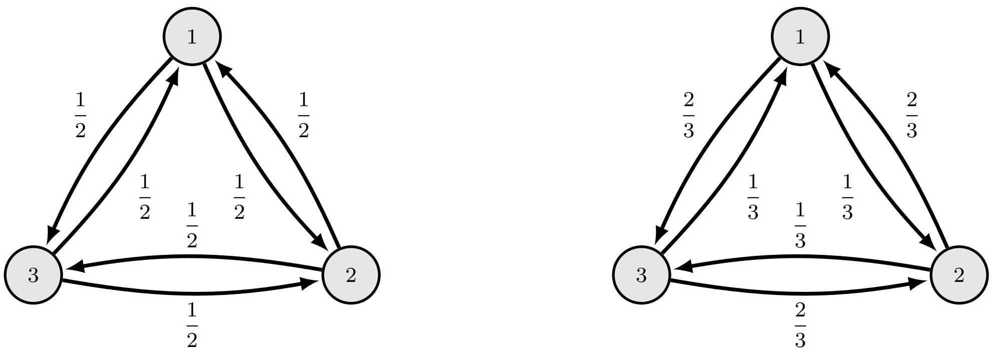

Figure 26: Two transition graphs on $S = \{1, 2, 3\}$ with defining transition matrices of Markov chains X and Y. It is easy to check that X is reversible while Y is not reversible by Kolmogorov's criterion, using the sequence $1, 2, 3$ to show the lack of equality in some cases.

path that X takes, occurs with a power of one half where the exponent corresponds to the length of the path. Qualitatively, it is indistinguishable whether the process moves in clockwise or counterclockwise direction in the triangle. It can, hence in a qualitative understanding, be reversed in time. On the other hand, the process Y has a tendency to turn counterclockwise. Choosing the sequence $1, 2, 3$ as a possible finite sequence for Kolmogorov's criterion, one finds that the chain is in fact not reversible.[1]

[1]The example was inspired by example 1.9.4 in [Norris97]

This makes reversibility to a desirable but not common property in Markov chains, if for nothing else, at least for calculation purposes of the stationary distribution.

Reversibility answered the question whether the chain looks the same when reversing it in time. This defines naturally a symmetry for the chain in the sense of time. On the other hand, one might wonder, if there are any spatial symmetries, i.e., subsets of the state space, on which the Markov chain behaves "identically". An established notion of such a symmetry is lumpability of the state space, which gives a condition for combining the states of a Markov chain while preserving its overall behavior.

Definition 5.13 ([KeSne65])**.** *Let X be a Markov chain with transition matrix P on a state space S and let $\{S_i\}_{i=1}^N$ a partition of S. Then, the chain X is called lumpable with respect to the partition $\{S_i\}_{i=1}^N$ if for all $i, j \in \{1, \ldots, N\}$ and for all $x, y \in S_i$ the equation*

$$\sum_{z \in S_j} p_{x,z} = \sum_{z \in S_j} p_{y,z}$$

and the lumped chain $\hat{X}$ has the transition matrix $\hat{P} = (\hat{p}_{i,j})_{i,j=1}^N$ with

$$\hat{p}_{i,j} = \sum_{z \in S_j} p_{x,z}$$

for an arbitrary $x \in S_i$.

Indeed, it can be proven, that the lumped chain is a Markov chain. Lumpability gives a criterion under which we can reduce the state space and possibly also facilitate the calculation of the stationary distribution. We will exploit this property in Subsection 7.3.4.

Having found ways to reduce the size of the state space as well as to calculate the stationary distribution explicitly under the assumption of reversibility and, hence, a way to calculate the limit in distribution of the chain X, the question about the speed at which the chain converges under suitable assumptions to its limit is a natural one to ask. We review the existing literature briefly in what follows.

5.1.3 Convergence speed to equilibrium

To analyze the convergence speed of ergodic Markov chains, we want to present two well known approaches. The first one is based on the eigenvalues of the associated transition matrix, the so called mixing time as well as the bottleneck ratio, also known as Cheeger constant. The second one uses the fact that powers of the transition matrix can be interpreted as weighted walks on the associated transition graph. Under a so called Doeblin condition one can then obtain upper bounds on the distance between the marginal distribution of the Markov chain at any time t and the stationary distribution. We start by reviewing a result which is deeply rooted in matrix theory.

Theorem 5.14 (Perron-Frobenius for ergodic Markov chains, [Bre13])**.** *Let P the transition matrix of an ergodic Markov chain with stationary distribution π on a state space S. Denote by Π the $|S| \times |S|$ matrix which contains in every row the entries π. Order the eigenvalues of P by $1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_{|S|}$ and consider the second largest eigenvalue $\lambda_2^* = \max\left\{|\lambda_2|, |\lambda_{|S|}|\right\}$ as well as its algebraic multiplicity m_2^*. Then, for $t \in \mathbb{N}$ the equation*

$$P^t = \Pi + O\left(t^{m_2^*-1}\,(\lambda_2^*)^t\right). \tag{5.12}$$

The task of calculating the in absolute value second largest Eigenvalue λ_2^* is not an easy one since there are in general two candidates, λ_2 and λ_n. Furthermore, to make claims about convergence speed in the case where $|\lambda_2^*|$ cannot be calculated explicitly one needs to find a meaningful upper bound for $|\lambda_2^*|$. Optimally, on obtains both meaningful lower and upper bounds. For special cases, classical Eigenvalue theory may help, nonetheless, in finding solutions.

Proposition 5.15 (Eigenvalues of symmetric matrices, Rayleigh-Ritz, [HorJoh90])**.** *Let M be a real symmetric matrix with Eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and associated Eigenvectors $x_1, \ldots, x_n$. Then*

$$\lambda_k = \max_{\substack{x \perp x_1, \ldots, x_{k-1} \\ ||x||_2 = 1}} x^T\, M\, x.$$

Under additional assumptions like positive definiteness of P as well as being symmetric the problem reduces to

$$\lambda_2^* = \lambda_2 = \max_{\substack{x \perp \mathbb{1}_n \\ ||x||_2 = 1}} x^T\, P\, x.$$

by Proposition 5.15. This can also be used to obtain bounds for said Eigenvalue. We will discuss the hereinabove laid out approach in an example in Subsection 5.2. Further established techniques to access the convergence speed of a Markov chain in a meaningful way are based on the Cheeger constant or bottleneck ratio. We are going to review briefly this approach which is discussed in detail in [LePeWi09].

Definition 5.16 (Bottleneck ratio, [LePeWi09])**.** *Let P a transition matrix inducing an ergodic Markov chain X with stationary distribution π on a state space S. Then, define for $x, y \in S$ and $A, B \subset S$*

$$Q_{x,y} = \pi(x) p_{x,y}, \qquad Q_{A,B} = \sum_{x \in A, y \in B} Q_{x,y}.$$

The bottleneck ratio Φ of a set $A \subset S$ is defined as

$$\Phi(A) := \frac{Q_{A,A^c}}{\pi(A)}$$

and the bottleneck ratio Φ^* *of the whole chain* X *is*

$$\Phi^* := \min_{A\subset S,\ 0<\pi(A)\leq\frac{1}{2}} \Phi(A). \tag{5.13}$$

For Markov chains with a uniform distribution as stationary distribution the bottleneck ratio is closely connected to the geometry of the underlying graph since

$$\begin{aligned}\Phi^* &= \min_{A\subset S,\ 0<\pi(A)\leq\frac{1}{2}} \Phi(A) = \min_{A\subset S,\ 0<0\pi(A)\leq\frac{|A|}{2}} \frac{\sum_{x\in A, y\in A^c} p_{x,y}}{|A|} \\ &= \min_{A\subset S,\ 0<0\pi(A)\leq\frac{|A|}{2}} \frac{|\partial A|}{|A|} \operatorname{avg}_{\partial A}(P)\end{aligned}$$

where for the connected simple graph $G = (S, E)$ as

$$E = \{\langle x, y\rangle \in S^2 | p_{x,y} > 0 \text{ or } p_{y,x} > 0\}$$

the boundary of A is defined as

$$\partial A := \{\langle x, y\rangle \in E | x \in A, y \in A^c\}.$$

See for example [Mohar89] for the definition of the boundary of a subset of a graph's vertex set and further comments thereon. Hence, if $\operatorname{avg}_{\partial A}(P)$ is a constant or has tight meaningful bounds independent of A, as for example in the case of a simple random walk on a r-regular graph where $p_{x,y} = r^{-1}$ is a constant independent of $x, y \in S$, we find that the defining term for Φ^* is $\min_{A\subset S,\ 0<0\pi(A)\leq\frac{|A|}{2}} \frac{|\partial A|}{|A|}$.

Definition 5.17 (Isoperimetric constant on graphs, [Mohar89])**.** *Let* $G = (\mathcal{V}, \mathcal{E})$ *be any simple connected graph. Then, the isoperimetric constant of* G *is the minimal quotient*

$$\iota(G) := \min_{A\subset S,\ 0<|A|\leq\frac{|S|}{2}} \frac{|\partial A|}{|A|}. \tag{5.14}$$

The isoperimetric constant can be seen as the permeability of a graph with respect to a flow along its edges. A large isoperimetric constant means, therefore, that all vertices in the graph are "easily" accessible and there are no large separated groups in the graph. A lower bound for the isoperimetric constant, which is itself usually hard to calculate, may, therefore, give valuable insights into what we can expect of the minimal speed of a flow on the graph.

Lemma 5.18 (Lower bound of the isoperimetric number, [Mohar89])**.** *Let* $G = (\mathcal{V}, \mathcal{E})$ *be any simple connected graph with vertex connectivity* $\kappa(G)$*. Then*

$$\iota(G) \geq \frac{2\kappa(G)}{|\mathcal{V}|}. \tag{5.15}$$

A corresponding result for the bottleneck ratio of ergodic Markov chains yields an approach to the convergence speed towards the stationary distribution. The central object for this approach are mixing times, which measure the time, from which on the stationary distribution and any initial distribution, evolving under the action of the transition matrix, are close in the total variation distance as defined in what follows.

Definition 5.19 (Total variation distance and mixing times, [LePeWi09])**.** *For any two probability distributions μ, ν on S, their variational distance is defined as*

$$||\mu - \nu||_{TV} = \sup_{A \subset S} |\mu(A) - \nu(A)|.$$

For an ergodic Markov chain with transition matrix P and stationary distribution π set

$$d(t) := \sup_{x \in S} ||P^t(x, \cdot) - \pi||_{TV}$$

and define for $\varepsilon > 0$ the mixing time $\tau_{mix}(\varepsilon)$ as

$$\tau_{mix}(\varepsilon) := \min\{t \in \mathbb{N} |\ d(t) < \varepsilon\}. \tag{5.16}$$

Mixing times yield, therefore, a measure for decay of the distance of the chain at time t from its stationary distribution. Lower bounds are, therefore, important for the understanding of the pessimistic view how long we have to wait at least before approaching the stationary distribution while good upper bounds give the optimistic perspective on how long one has to wait at most. They can be especially well characterized in the case of a lazy Markov chain, i.e., where the probability to stay in place is always larger or equal one half.

Theorem 5.20 (Cheeger Bound, [LePeWi09])**.** *Consider the transition matrix P of an lazy ergodic Markov chain, i.e, for all $x \in S$ it holds true $p_{x,x} \geq \frac{1}{2}$, with stationary distribution π. Define*

$$\pi_{min} := \min_{x \in S} \pi(x) > 0.$$

Suppose that $\lambda \neq 1$ is an eigenvalue of P. Then

$$\left(\frac{1}{1 - |\lambda|} - 1\right) \log\left(\frac{1}{2\varepsilon}\right) \leq \tau_{mix}(\varepsilon) \leq \frac{2}{(\Phi^*)^2} \log\left(\frac{1}{\pi_{min}\varepsilon}\right).$$

Investigations into the behavior of $\tau_{mix}(\varepsilon)$, for variable ε and increasing size of S, are an ongoing field of research which can yield further insights into the Markov chains' behavior.

For Markov chains, which are not lazy, i.e., $\inf_x p(x,x) \leq 2^{-1}$, other tools have to be applied. We use in this work the theory of evolving sets associated to some Markov chain as developed in [MorPer05] and used for example in [PePeSte20] in the context of random environments. For a Markov chain X with stationary distribution π on a finite set V the evolving set process $(\mathfrak{U}_t)_{t\in\mathbb{N}}$ on the subsets of V is defined in [MorPer05] as follows. If for some $t \in \mathbb{N}$ the process $\mathfrak{U}_t$ satisfies $\mathfrak{U}_t = S$ where $S \subset V$, then for $U \sim \text{Unif}([0,1])$ the next step $\mathfrak{U}_{t+1}$ is defined as

$$\mathfrak{U}_{t+1} := \{v \in V \,|\, Q(U,v) \geq U\pi(v)\} \tag{5.17}$$

where Q is defined as in Definition 5.16. As mentioned in [MorPer05] and underlined by Definition 5.16, the quotient $Q(U,v)\pi(v)^{-1}$ is the probability that the time reversed chain of X with transition matrix P is in v when starting in U. Therefore, $\mathfrak{U}_{t+1}$ represents the random set of vertices which are accessible for the time reversed process of X from $\mathfrak{U}_t$ with probability at least U. Note that $\emptyset$ and V are absorbing for $(\mathfrak{U}_t)_{t\in\mathbb{N}}$ and, additionally,

$$\mathbb{P}[v \in \mathfrak{U}_{t+1} \,|\, \mathfrak{U}_t = U] = \frac{Q(U,v)}{\pi(v)}, \text{ and } p^{(t)}(v,w) = \frac{\pi(w)}{\pi(v)}\mathbb{P}_v[w \in \mathfrak{U}_t]. \tag{5.18}$$

See [MorPer05] for more details. They obtain the following result on the convergence speed of X.

Theorem 5.21 ([MorPer05])**.** *Consider a Markov chain X with transition matrix P and stationary distribution π on some state space S. Define $\Phi(u) := \inf\{\Phi(A) \,|\, \pi(A) \leq u\}$. Suppose that for some $0 < \gamma \leq 2^{-1}$ and $p(x,x) \geq \gamma$ for all $x \in S$. If for $\varepsilon > 0$*

$$t \geq 1 + \frac{(1-\gamma)^2}{\gamma^2} \int_{4(\pi(x)\wedge\pi(y))}^{4\varepsilon^{-1}} \frac{4}{u\Phi(u)^2} \mathrm{d}u,$$

then,

$$\left| \frac{p^{(t)}(x,y) - \pi(y)}{\pi(x)} \right| \leq \varepsilon. \tag{5.19}$$

For many cases deriving the previously discussed bounds for the convergence speed is still a difficult task to resolve and a lot of research goes into finding bounds on the spectral gap for specific models. Only for very restrictive cases meaningful upper and lower bounds for Φ^* may be obtained easily. We present on example in what follows.

Lemma 5.22. *Let X be a irreducible, aperiodic, reversible Markov chain with stationary distribution π. We write $L = (S, \mathcal{E})$ for the undirected version of the transition graph of X. Moreover, denote by $p_* = \min_{\substack{x,y\in S, x\neq y \\ p(x,y)>0}} p(x,y)$ and $p^* = \max_{x,y\in S, x\neq y} p(x,y)$. Then, the Cheeger constant Φ^* satisfies*

$$\Phi^* \in \left[\frac{\min_{x\in S}\pi(x)}{\max_{x\in S}\pi(x)} \frac{2\kappa(L)p_*}{|S|}, \left(|S| - \max_{U\subset S, \pi(U)\leq 2^{-1}} |U|\right) p^* \right]. \tag{5.20}$$

Proof. We show first the upper bound. By definition of the Cheeger constant, we have

$$\Phi^* = \min_{U\subset S, \pi(U)\le 2^{-1}} \frac{Q(U\times U^c)}{\pi(U)} = \min_{U\subset S, \pi(U)\le 2^{-1}} \sum_{x\in U, y\in U^c} \frac{\pi(x)p(x,y)}{\pi(U)}$$
$$\le p^* \min_{U\subset S, \pi(U)\le 2^{-1}} \sum_{x\in U} \frac{\pi(x)}{\pi(U)} |U^c| = p^* \left(|S| - \max_{U\subset S, \pi(U)\le 2^{-1}} |U| \right).$$

The lower bound is somewhat more involved, but mostly relies on the fact that the set $S_l = \left\{ U \subset S \middle| \pi(U) \le 2^{-1}, |U| \le 2^{-1}|S| \right\}$ is not empty. Using this, we pick a $U \in S_l$ and find by using the most naive positive lower bounds of each term in the fraction

$$\frac{Q(U,U^c)}{|U|} \ge p_* \frac{\min_{x\in S}\pi(x)}{\max_{x\in S}\pi(x)} \frac{|\partial U|}{|U|} \ge p_* \frac{\min_{x\in S}\pi(x)}{\max_{x\in S}\pi(x)} \iota(L).$$

Applying the minimum over all $U \subset S$ with $\pi(U) \le 2^{-1}$ and using Lemma 5.18 gives the claim. □

Evidently, the lower bound is only useful when the estimates are not too crude. In particular, the quotient of the minimum and the maximum of the stationary distribution may become very small as a function of the parameters of the Markov chain. We consider examples in Subsection 7.3.6 for which this bound yields in some cases a good estimate while in others it does not, depending on the parameter choice. In all these cases reversibility of the underlying Markov chain is assured.

While reversibility is not always given for the Markov chains investigated in this work, the convergence speed towards the stationary distribution is of central interest in any case. It turns out that an analysis based on combinatorial arguments as well as arguments using the graph geometry allown for more precise estimates then the bound given by Lemma 5.22. A tool, which comes in handy, is Doeblin's criterion as given in Theorem 5.23, which can be found in [Gra14]. Its direct link with the graph geometry via the powers of the transition matrix, which may be translated into ensembles of weighted paths in the graph, allows a complete description of the convergence speed based on paths of $\mathfrak{L}_k$.

Theorem 5.23 (Doeblin's criterion [Gra14])**.** *Let P be a transition matrix V satisfying the Doeblin condition: there exists $k \ge 1$ and $\varepsilon > 0$ and a law $\hat{\pi}$ on V such that*

$$P^k(x,y) \ge \varepsilon\hat{\pi}(y), \qquad \forall x,y \in V$$

Then there exists a unique invariant law π of P which satisfies $\pi(x) \ge \varepsilon\hat{\pi}(x)$ for all $x \in V$ and

$$\sup_{x\in V} \sum_{y\in V} |P^n(x,y) - \pi(y)| \le 2(1-\varepsilon)^{\lfloor \frac{n}{k} \rfloor}, \quad n \ge 1. \tag{5.21}$$

The restriction of P to $\{y \in V | \pi(y) > 0\}$ is irreducible and strongly reducible if $\{y \in V | \pi(y) > 0\}$ is finite.

Indeed, this way we obtain results depending on the structure of the path space of the underlying graph, which, therefore, might be extended and refined via the theory of path spaces of graphs and C^*-algebras of graphs as introduced in [CunKri80] and discussed further, for example, in [Web13] and [BroCaWhi17]. This goes beyond the scope of this work but might yield the basis for future research, in particular, with applications to generalized exclusion processes on finite graphs. We turn now to a review of the second part of the Echo Chamber Model which is better known as the Deffuant model before considering the underlying topology of the particle system of interest of this work.

5.2 The Deffuant model

The Deffuant model is a well known model in opinion dynamics introduced in [Weis03] and discussed in various settings, for example by [LanLi20] [Lor05] or [CheSu20] without giving a complete list. We are going to recall the model structure as a complete review of the quantitative results are out of the scope of this work. Afterwards, we propose a new perspective on the convergence of the opinion distributions under assumption of the existence of a density at time 0. We close this section with a comparison of the Deffuant model on a complete graph with the non-complete graph case and show how results, obtained in the first case, may be recovered in the second case.

Consider a graph $G = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = n$ and $|\mathcal{E}| = m$ as well as two parameters $\theta \in (0,1)$ and $\mu \in \left[0, \frac{1}{2}\right)$. Each vertex $x \in \mathcal{V}$ exhibits a label $X_v \sim \mathcal{L}([0,1])$ i.i.d. with $\mathcal{L}$ absolutely continuous with respect to the Lebesgue measure. The Deffuant model model evolves as follows in time $t \in \mathbb{N}$.

- If $|X_A^t - X_B^t| < \theta$, then
  - set, firstly, $X_A^{t+1} = X_A^{t+1} + \mu(X_B^{t+1} - X_A^{t+1})$ and,
  - secondly, $X_B^{t+1} = X_B^{t+1} + \mu(X_A^{t+1} - X_B^{t+1})$.

It constitutes, therefore, the second part of the dynamics of the Echo Chamber Model. Denote the joint initial distribution of the labels X by $\mathcal{D}^0$ and assume that exhibits a density π^0. At any time $t \in \mathbb{N}$ an edge $\langle I, J \rangle$ is drawn uniformly from all edges $\mathcal{E}$ and we can rewrite the algorithmic step described hereinabove as a recurrence relation. To this end, let $A^\theta = \{x \in \mathbb{R}^2 | |x_1 - x_2| > \theta\}$, $A_{ij}^\theta = \{x \in \mathbb{R}^n | (x_i, x_j) \in A^\theta\}$ and define for $i, j \in \{1, \ldots, n\}$ the function

$$\Phi_{ij}^\mu(x_1, \ldots, x_d) = (x_1, \ldots, x_i + \mu(x_j - x_i), \ldots, x_j + \mu(x_i - x_j), \ldots, x_n).$$

Then, the labels at time $t+1$ are constructed as

$$X^{t+1} = X^t \, \mathbb{1}_{A_{IJ}^\theta}(X^t) + \Phi_{IJ}^\mu(X^t) \, \mathbb{1}_{(A_{IJ}^\theta)^c}(X^t) \tag{5.22}$$

Due to the dynamics, the density π^0 evolves over time as follows. Denote by E^t the edge drawn during the t-th step and by π^t the density of the joint distribution of the labels at time t. Then, for $t \in \mathbb{N}$, at time $t+1$ the process of labels $X^{t+1} = (X_1^{t+1}, \ldots, X_d^{t+1})$ follows the following law. Let $B \in \mathcal{B}([0,1]^n)$,

$$\begin{aligned}
\mathbb{P}\left[X^{t+1} \in B\right] &= \frac{1}{m} \sum_{<i,j>\in E} \mathbb{P}\left[X^{t+1} \in B, X^t \in A_{ij}^\theta | E^t =< i,j >\right] \\
&\qquad + \mathbb{P}\left[X^{t+1} \in B, X^t \in \left(A_{ij}^\theta\right)^c | E^t =< i,j >\right] \\
&= \frac{1}{m} \sum_{<i,j>\in E} \mathbb{P}\left[X^t \in B \cap A_{ij}^\theta\right] + \mathbb{P}\left[X^t \in \left(\Phi_{ij}^\mu\right)^{-1}(B) \cap \left(A_{ij}^\theta\right)^c\right] \\
&= \frac{1}{m} \sum_{<i,j>\in E} \int_B \pi^t(x) \mathbb{1}_{A_{ij}^\theta}(x) + \frac{\pi^t\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)}{1-2\mu} \mathbb{1}_{\left(A_{ij}^\theta\right)^c}\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right) dx \\
&= \int_B \frac{1}{m} \sum_{<i,j>\in E} \left(\pi^t(x) \mathbb{1}_{A_{ij}^\theta}(x) + \frac{\pi^t\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)}{1-2\mu} \mathbb{1}_{\left(A_{ij}^\theta\right)^c}\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)\right) dx.
\end{aligned}$$

Thus, the following equation holds

$$\pi^{t+1}(x) = \frac{1}{m} \sum_{<i,j>\in E} \left(\pi^t(x) \mathbb{1}_{A_{ij}^\theta}(x) + \frac{\pi^t\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)}{1-2\mu} \mathbb{1}_{\left(A_{ij}^\theta\right)^c}\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)\right). \quad (5.23)$$

One approach, to showing convergence of the model, would be to use the obtained equation to show convergence of the sequence $(\pi^t)_{t\in\mathbb{N}}$ as a sequence in $L^1([0,1]^n)$. We define the operator $\mathcal{T}_\theta$ for $f \in L^1([0,1]^n)$ as

$$\left(\mathcal{T}_\theta f\right)(x) := \frac{1}{m} \sum_{<i,j>\in E} \left(f(x) \mathbb{1}_{A_{ij}^\theta}(x) + \frac{f\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)}{1-2\mu} \mathbb{1}_{\left(A_{ij}^\theta\right)^c}\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)\right) \quad (5.24)$$

skipping the dependence on the underlying graph G in the notation of $\mathcal{T}_\theta$. Then, for any initial density $\pi^0 \in L^1([0,1]^n)$ the density π^t is given by $(\mathcal{T}_\theta)^t \pi^0$ and

$$\begin{aligned}
||\mathcal{T}_\theta f||_1 &= \frac{1}{m} \sum_{<i,j>\in E} \int_{[0,1]^n} \left| f(x) \mathbb{1}_{A_{ij}^\theta}(x) dx + \frac{f\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)}{1-2\mu} \mathbb{1}_{\left(A_{ij}^\theta\right)^c}\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)\right| dx \\
&\leq \frac{1}{m} \sum_{<i,j>\in E} \int_{[0,1]^n} |f(x)| \mathbb{1}_{A_{ij}^\theta}(x) + \int_{[0,1]^n} \frac{\left|f\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)\right|}{1-2\mu} \mathbb{1}_{\left(A_{ij}^\theta\right)^c}\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right) dx \\
&= \frac{1}{m} \sum_{<i,j>\in E} \int_{[0,1]^n} |f(x)| \mathbb{1}_{A_{ij}^\theta}(x) + |f(x)| \mathbb{1}_{\left(A_{ij}^\theta\right)^c}(x)\, dx \\
&= \int_{[0,1]^n} |f(x)| dx = ||f||_1.
\end{aligned}$$

Considering the constant function $\mathbb{1}(x) = 1$ we obtain that $||\mathcal{T}_\theta|| = 1$. Consequently, the operator $\mathcal{T}_\theta$ is linear and continuous but not a contraction in the sense of the Banach fixpoint theorem since the distance $||\mathcal{T}_\theta \mathbb{1} - \mathcal{T}_\theta 0||_1 = ||1||_1 = ||1 - 0||_1$ is not decreasing. Additionally, this would imply that, independently of the initially chosen density, $(\mathcal{T}_\theta)^n \pi^0$ would converge to the same limit which is the constant 0 and, therefore, not a probability density. As depicted in Figure 27 for the case $n = 2$ and a path graph of length 1, this does not seem to be the case. Figure 27 shows that regions, in

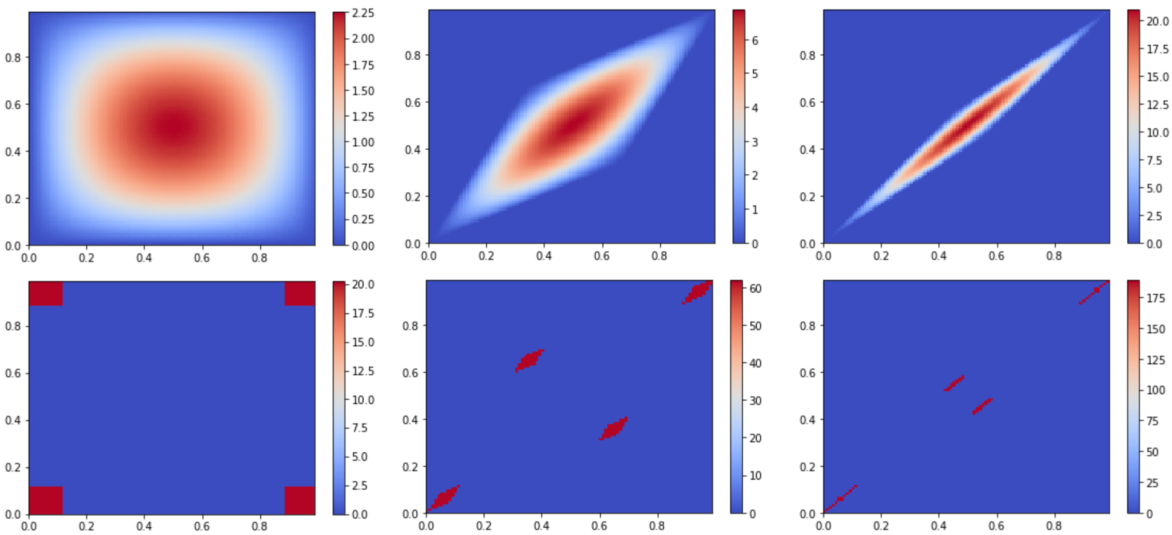

Figure 27: Evolution of densities under $\mathcal{T}_\theta$ with different initial densities on a path graph of length 1 and population size $n = 2$. In the first row, we show a pair of random variables X, Y which are i.i.d. distributed according to a triangular distribution on $[0, 1]$ evolving over time $t = 0, 5, 10$. In the second row, we show a pair of random variables X, Y which are i.i.d. distributed according to a distribution which is only supported on $[0, 0.1]$ and $[0.9, 1]$, again evolving over 10 time steps. .

which the joint initial distribution is 0, will have probability 0 for all times, such that the two cases cannot converge to the same limiting distribution. The plots suggest that in the case of convergence the limiting distribution will in any case be supported on a subset of the line $s \mapsto s \begin{pmatrix} 1, & 1 \end{pmatrix}^T$ for $s \in [0, 1]$.
We, consequently, focus subsequently on the more detailed structure of the densities of probability distributions and not only their role as a subset of $L^1([0, 1]^n)$.

5.2.1 The manifold of probability densities and opinion paths

We start this subsection by the observation that by equation (5.23) the set of densities is invariant under $\mathcal{T}_\theta$, i.e., usin

$$\partial B_1^+(0) := \{f \in L^1([0, 1]^n) | \, ||f||_1 = 1, \ f \geq 0\}$$

we have $\mathcal{T}_\theta(\partial B_1^+(0)) \subseteq \partial B_1^+(0)$. Consequently, we can consider only the restriction of $\mathcal{T}_\theta$ to $\partial B_1^+(0)$. This opens the doors to Information Geometry for our analysis, in particular the fact that $\partial B_1^+(0)$ forms a Riemannian manifold $\mathcal{M} = (\partial B_1^+(0), g)$ equipped with some metric g, which is also called statistical manifold. See [Amari12] for a more in depth discussion of the topic. Evidently, the dependence on the underlying graph G has to be integrated. We, then, can define for any $\pi \in \mathcal{M}$ a sequence of paths $\left(\gamma_\pi^{(i)}\right)_{i=1}^\infty$ where $\gamma_\pi^{(i)} : [0,1] \to \mathcal{M}$ as

$$\gamma_\pi^{(1)}(t) = \mathcal{T}_{t\theta}\pi, \qquad \gamma_\pi^{(i)}(t) = \mathcal{T}_{t\theta}\left(\mathcal{T}_\theta^{i-1}\pi\right), \ i \geq 2. \tag{5.25}$$

We call the associated paths *opinion paths* due to their representation of the process of changing opinions within the Deffuant model. Define the length of a finite path $\gamma : [0,1] \to \mathcal{M}$ in $(\mathcal{M}, g)$ as it is done on any Riemannian manifold with metric g, see for example [Jost17] by

$$l_{\mathcal{M}}(\gamma) := \int_0^1 ||\gamma'(t)||\mathrm{d}t := \int_0^1 \sqrt{g_{\gamma(t)}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}\mathrm{d}t. \tag{5.26}$$

Intuitively, if the length $l_{\mathcal{M}}\left(\gamma_\pi^{(i)}\right)$ of each $\gamma_\pi^{(i)}$ is decreasing sufficiently rapidly in i, such that $\sum_{i=1}^\infty l_{\mathcal{M}}\left(\gamma_\pi^{(i)}\right) < \infty$, then the sequence $(\pi^i)_{i\in\mathbb{N}}$ defined in equation (5.23) converges to some limit, which is attained after a final distance in the manifold $\mathcal{M}$. Using simulations we find the following connection, which remains unproven but forms the basis for future research.

Conjecture 5.24. *In the setting of the Deffuant model with n individuals and m relationships, an initial probability density π of the individual opinions as well as parameters θ and μ there are constants $C(n,m,\pi)$, $C'(\mu) > 0$ such that any sequence of opinion paths $\left(\gamma_\pi^{(i)}\right)_{i=1}^\infty$ as defined in equation 5.25 satisfies*

$$l_{\mathcal{M}}(\gamma_\pi^{(i)}) \leq C(n,m,\pi)\theta^{C'(\mu)\,i}. \tag{5.27}$$

Furthermore, the Deffuant model converges for any initial π, for any structure of G.

See Subsection 8.6 for the Outlook and the implications of this conjecture on possible approaches for the analysis of the complete Echo Chamber Model. Within the framework of the Deffuant model, a second question arises. Not only the existence of consensus and group formation but also the final pattern of the opinions, depending on the groups. We focus on this topic in what follows.

5.3 Maximum confidence clusters (MCC)

Since bubble formation is a motivation for this model one needs to define clusters which may form over time under the given dynamics. We start with the definition of Maximum Confidence Clusters as defined in [CheSu20].

Definition 5.25 ([CheSu20])**.** *Consider a set of i.i.d. uniformly distributed random variables* $X = \{X_i\}_{i=1}^n$ *and let* $\theta \in \left[0, \frac{1}{2}\right]$*. Pick any* $i_1 \in \{1, \dots, n\}$ *and define* $C_{i_1}(X)$ *as the set of* $j \in \{1, \dots, n\}$ *such that* $|X_i - X_j| < \theta$ *or there are indices* $j_1, \dots, j_k$ *such that* $|X_i - X_{j_1}| < \theta, \dots, |X_{j_k} - X_j| < \theta$*. Then, pick* $i_2 \in \{1, \dots, n\} \backslash C_i(X)$ *and construct* $C_{i_2}(X)$ *the same way. Continue until* $\{1, \dots, n\} \backslash \bigcup_{l=1}^K C_{i_l}(X) = \emptyset$*. The sets* $C_{i_l}(X)$ *are called maximum confidence clusters (MCCs) under* X *and the number of clusters is denoted by* R*.*

An implicit condition which enters into the definition is the possibility that every individual may interact with every other individual. This implies that the underlying graph is complete which is usually not the case and in particular not satisfied, if the underlying graph is dynamic. Therefore, we adapt the definition of the maximum confidence clusters as follows MCCs in the context of [CheSu20] have the advantage that

- cluster $C_i(X)$ and $C_j(X)$ define disjoint intervals for any labels X,
- for each complete cluster C_i there is a y_i such that $C_i \subset B_{\frac{\theta}{2}}(y_i)$ and $C_j \cap B_{\frac{\theta}{2}}(y_i) = \emptyset$ for all $i \neq j$.

Adjusting the definition of clusters for graphs, which are not complete, as follows, yields difficulties which we are going to discuss in this subsection.

Definition 5.26. *Consider a connected graph* $G = (V, E)$ *with i.i.d.* $\mathcal{U}[0,1]$ *distributed labels* $X = \{X_v\}_{v \in V}$ *and let* $\theta \in \left[0, \frac{1}{2}\right]$*. Pick any* $v_1 \in V$ *and define* $C_{v_1}(X)$ *as the set of* $w \in \mathcal{N}_{v_1}$ *such that* $|X_{v_1} - X_w| < \theta$ *or there is a path* $(v_1, w_1, \dots, w_k, w)$ *in* G *such that* $|X_{v_1} - X_{w_1}| < \theta, \dots, |X_{w_k} - X_w| < \theta$*. Then, pick* $v_2 \in V \backslash C_{v_1}(X)$ *and construct* $C_{v_2}(X)$ *the same way. Continue until* $V \backslash \bigcup_{l=1}^K C_{v_l}(X) = \emptyset$*. The sets* $C_{i_l}(X)$ *are called simple maximum confidence clusters (SMCCs) under* X *and the number of clusters is denoted by* K*.*

If G is complete the clusters define disjoint closed intervals in $[0, 1]$. In contrast to this consider the path graph $P = (\{1, 2, 3, 4, 5\}, E_p)$ depicted in Figure 28 where the labels of each vertex are chosen according to their position on the scale between 0 and 1. This becomes even more problematic for the case of complete clusters which are just SMCCs where the pairwise distance is smaller than $\frac{\theta}{2}$. In Figure 29 we show an example that the clusters do not capture the idea of disjoint limiting opinions considering the

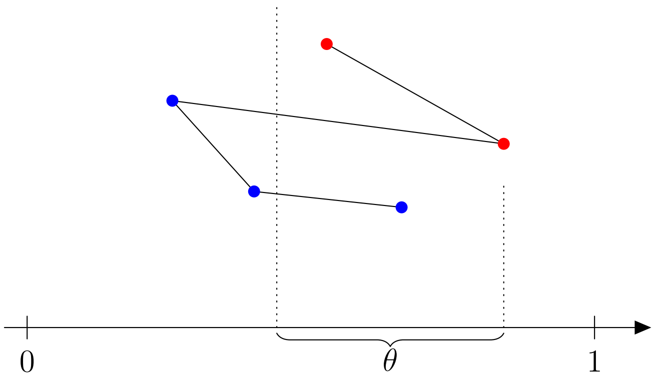

Figure 28: A path opinion graph with intersecting intervals defined by the underlying SMCCs.

idea that they converge towards a common mean but different from all the others. We show in Proposition 5.29 that each complete cluster converges to the mean of the opinions of its constituents at the time where all clusters become complete. In fact

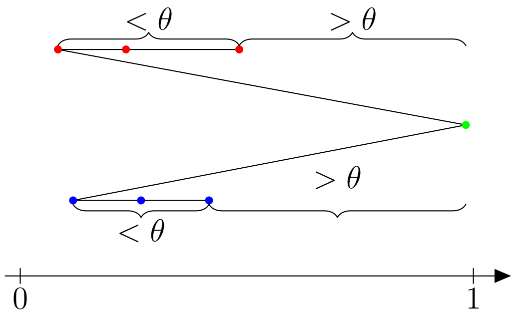

Figure 29: A path opinion graph which yields intersecting SMCCs.

assume that $G = (V, E)$ is the graph with seven vertices depicted in Figure 29 and the labels x_1, x_2, x_3 (in blue) form a complete cluster. Then there is a sufficiently small ε such that $x_4 = x_1 + \varepsilon, x_5 = x_2 - \frac{\varepsilon}{2}, x_6 = x_3 - \frac{\varepsilon}{2}$ (in red) form also a complete cluster. In particular, by the result given by Proposition 5.29 both clusters converge to the same mean opinion. In general the probability of the event

$$\{(X_1, X_2, X_3) \text{ and } (X_4, X_5, X_6) \text{ form complete clusters}\} \cap$$
$$\{X_4 + X_5 + X_6 = X_1 + X_2 + X_3 \text{ and } \min_{i \in \{1,\dots,6\}} |X_i - X_7| > \theta\}$$

is always positive. Hence there are initial conditions for which even convergence towards the same opinion in different clusters are possible. We can, nonetheless, make use of some observations from the restrictive case discussed in [CheSu20]. In fact, due to the changes of the labels, the structure within each cluster changes but no new individuals may enter a cluster. Hence, a cluster may become stationary in time in the sense that it does no longer split into sub-clusters. We use the following definition to capture this case, which is inspired by [CheSu20].

Definition 5.27. *We call a SMCC $C_{i_l}(X)$ complete if for any pair of indices $(k, h) \in C_{i_l}(X)^2$ the condition $|X_k - X_h| < \theta$ holds.*

We are mostly interested in the convergence of the process $(X^t)_{t\in\mathbb{N}}$ and hence the first time all clusters are complete gives a lot of information on that subject.

Definition 5.28. *Let $t \in \mathbb{N}$ and consider the set of maximum SMCCSs $\mathcal{C}(X^t)$. Define the first time where all clusters are complete by*

$$T^C := \inf\{t \in \mathbb{N} | \forall C_i(X^t) \in \mathcal{C}(X^t) : C_i(X^t) \text{ complete}\}. \tag{5.28}$$

Using the notion of complete SMCCs, we can make claims about the final structure of the opinions. The same result is obtained in [Lor05] using algebraic methods of matrix products, where we employ methods from Markov chain theory. We rely, nonetheless, on the result in [Lor05] that T^C, which is called t_0 in [Lor05], is almost surely finite independently of the initial distribution of the opinions. We obtain the following result on the convergence speed of the opinions within the complete clusters.

Theorem 5.29. *Consider the Deffuant model with parameters $\theta \in (0, 1)$, $\mu \in \left[0, \frac{1}{2}\right)$ on a simple graph $G = (\mathcal{V}, \mathcal{E})$ and denote by $\{\mathcal{C}_l\}_{l=1}^K$ be the limiting set of complete SMCCs of size $(\kappa_l)_{l=1}^K := (|\mathcal{C}_l|)_{l=1}^K$. Then, for any $h \in \{1, \dots, K\}$ the opinion $X_k^{t\vee T^C}$ for $k \in \mathcal{C}_h$ associated to cluster $\mathcal{C}_h$ satisfies for $t \to \infty$ almost surely*

$$\lim_{t\to\infty} \mathbb{E}\left[X_k^{t\vee T^C} \middle| \{\mathcal{C}_l\}_{l=1}^K\right] = \frac{1}{\kappa_h} \sum_{j\in\mathcal{C}_h} \mathbb{E}\left[X_j^{T^C} \middle| \{\mathcal{C}_l\}_{l=1}^K\right]. \tag{5.29}$$

Proof. Throughout the proof we consider conditional expectations of the form $\mathbb{E}[.| \{\mathcal{C}_l\}_{l=1}^K]$ to improve readability we omit the condition on $\{\mathcal{C}_l\}_{l=1}^K$ when not essential.

Denote by $E^t = \langle I_t, J_t\rangle$ the edge drawn at time t and by Φ_{ij}^μ the matrix

$$(\Phi_{ij}^\mu)_{k,l} = \begin{cases} 1 & k = l, k, l \notin \{i, j\} \\ 1-\mu & k = l, k, l \in \{i, j\} \\ \mu & (k, l) = (i, j) \text{ or } (k, l) = (j, i) \\ 0 & \text{else.} \end{cases}$$

First note that the labels X^{t+1} at time $t+1$ satisfy

$$X^{t+1} = \left(\prod_{s=1}^{t} \left(\Phi^{\mu}_{I_s J_s} \right)^{\mathbb{1}_{|X^s_{I_s} - X^s_{J_s}| < \theta}} \right) X^0 \tag{5.30}$$

and recall that T^C is almost surely finite by the result in [Lor05]. Moreover, for $t > T^C$ it holds

$$\mathbb{E}[X^{t+1}] = \mathbb{E}\left[\mathbb{E}\left[\prod_{s=T^C}^{t} \left(\Phi^{\mu}_{I_s J_s} \right)^{\mathbb{1}_{|X^s_{I_s} - X^s_{J_s}| < \theta}} \Big| X^{T^C} \right] X^{T^C} \right]$$

and by independence of the edges drawn the random matrices $\left\{ \left(\Phi^{\mu}_{I_s J_s} \right)^{\mathbb{1}_{|X^s_{I_s} - X^s_{J_s}| < \theta}} \right\}_{s=T^C}^{t}$ are independent given X^{T^C} for any $t \geq T^C$ such that

$$\mathbb{E}[X^{t+1}] = \mathbb{E}\left[\prod_{s=T^C}^{t} \mathbb{E}\left[\left(\Phi^{\mu}_{I_s J_s} \right)^{\mathbb{1}_{|X^s_{I_s} - X^s_{J_s}| < \theta}} \Big| X^{T^C} \right] X^{T^C} \right]. \tag{5.31}$$

Since we are interested in the long time behavior we have to analyze $\lim_{t\to\infty} \mathbb{E}[X^t]$. Since all $\{X^t_i\}_{i \in V} \subset [0,1]^n$ and all matrices are stochastic, we can employ dominated convergence to examine the limit. Moreover the dependence in t is uniquely determined by the sequence of random matrices within the product. There are only finitely many different matrices to consider. Fix any pair $\langle i,j \rangle \in \mathcal{E}$. Let A_{kt} denote the event that Φ^{μ}_{ij} occurs at most k times in the first t draws. Then

$$\mathbb{P}\left(\bigcap_{t \in \mathbb{N}} A_{kt} \right) = \lim_{t \to \infty} \mathbb{P}(A_{kt}) = 0$$

and hence

$$\mathbb{P}\left(\bigcup_{k=0}^{\infty} \bigcap_{t \in \mathbb{N}} A_{kt} \right) \leq \sum_{k=0}^{\infty} \mathbb{P}\left(\bigcap_{t \in \mathbb{N}} A_{kt} \right) = 0.$$

Hence, the probability that Φ^{μ}_{ij} occurs only finitely many times is zero. It follows from the finiteness of the set of all Φ^{μ}_{ij} that almost surely, all matrices in Φ^{μ}_{ij} occur infinitely many times.

For $t, t' > T^C$, $l \neq l'$ and $\langle i,j \rangle \in \mathcal{C}^2_l$, $\langle i',j' \rangle \in \mathcal{C}^2_{l'}$ the matrices $\left(\Phi^{\mu}_{ij} \right)^{\mathbb{1}_{|X^t_i - X^t_j| < \theta}}$ and $\left(\Phi^{\mu}_{i'j'} \right)^{\mathbb{1}_{|X^{t'}_{i'} - X^{t'}_{j'}| < \theta}}$ commute. This allows us to concentrate on one cluster at a time by considering the subsequences $\left(\left(\Phi^{\mu}_{ij} \right)^{\mathbb{1}_{|X^t_i - X^t_j| < \theta}} \right)_{i,j \in \mathcal{C}_l \text{ and } t > T^C}$. Moreover for $t > T^C$, given X^{T^C} it holds $\mathbb{1}_{|X^t_i - X^t_j| < \theta} = \mathbb{1}_{i,j \in \mathcal{C}_l}$ such that we can omit it. Additionally, since the multiplication of elements of the sequence $(\Phi^{\mu}_{ij})_{i,j \in \mathcal{C}_l}$ only changes rows and columns with indices in $\mathcal{C}_l$, we consider its restriction of size $\kappa_l \times \kappa_l$ defined for $p, q \in \mathcal{C}_l$ by

$$(\tilde{\Phi}^{\mu}_{ij})_{p,q} := (\Phi^{\mu}_{ij})_{p,q}.$$

The matrix $\tilde{\Phi}^{\mu}_{ij}$ is doubly stochastic for any $\langle i,j\rangle \in \mathcal{C}_l^2$. Denote by $m_l := |C_l^2 \cap \mathcal{E}|$. Since edges are drawn uniformly i.i.d. the probability to draw a fixed matrix $\tilde{\Phi}^{\mu}_{ij}$ is m^{-1}. Define

$$P_{\mathcal{C}_l} := \frac{1}{m} \sum_{\langle i,j\rangle \in \mathcal{C}_l^2 \cap \mathcal{E}} \tilde{\Phi}^{\mu}_{ij} + \frac{m - m_l}{m} \mathrm{Id}_{\kappa_l \times \kappa_l},$$

which is a doubly stochastic, aperiodic and irreducible transition matrix for some Markov chain. Hence it exhibits a unique stationary distribution $\pi_{\mathcal{C}_l} = \frac{1}{\kappa_l} \mathbb{1}_{\kappa_l}$ and $P^k_{\mathcal{C}_l} \to \frac{1}{\kappa_l} \mathbb{1}_{\kappa_l \times \kappa_l}$ as $k \to \infty$. Recall that $E_t = \langle I_t, J_t\rangle$ such that in particular for $t > T^C$ the identity

$$\mathbb{E}\left[\tilde{\Phi}^{\mu}_{I_t J_t} \middle| X^{T^C}\right] = P_{\mathcal{C}_l}.$$

holds. Using the same argument for each cluster and denoting by K the random final number of clusters, we obtain that $\prod_{s=T^C}^{t} \mathbb{E}\left[\left(\Phi^{\mu}_{I_s J_s}\right)^{\mathbb{1}_{|X^t_{I_s} - X^t_{J_s}| < \theta}} \middle| X^{T^C}\right]$ converges to a matrix Φ_∞. In fact there is a random permutation of rows and columns $\mathfrak{p}$ such that

$$\Phi_\infty = \mathfrak{p}\left(\begin{pmatrix} \frac{1}{\kappa_1}\mathbb{1}_{\kappa_1 \times \kappa_1} & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & 0 & \frac{1}{\kappa_l}\mathbb{1}_{\kappa_l \times \kappa_l} & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{\kappa_K}\mathbb{1}_{\kappa_K \times \kappa_K} \end{pmatrix}\right).$$

Hence, by dominated convergence and since Φ_∞ is measurable with respect to $\sigma\left(\{\mathcal{C}_l\}_{l=1}^K\right)$

$$\lim_{t\to\infty} \mathbb{E}\left[X^{t+1} \middle| \{\mathcal{C}_l\}_{l=1}^K\right] = \mathbb{E}\left[\Phi_\infty X^{T^C} \middle| \{\mathcal{C}_l\}_{l=1}^K\right] = \Phi_\infty \mathbb{E}\left[X^{T^C} \middle| \{\mathcal{C}_l\}_{l=1}^K\right].$$

Consequently, we obtain for $k \in \mathcal{C}_h$ the limit

$$\lim_{t\to\infty} \mathbb{E}\left[X^t_k \middle| \{\mathcal{C}_l\}_{l=1}^K\right] = \frac{1}{\kappa_h} \sum_{j \in \mathcal{C}_h} \mathbb{E}\left[X^{T^C}_j \middle| \{\mathcal{C}_l\}_{l=1}^K\right].$$

□

We can consider the evolution of each complete SMCC separately from the remaining once since they will never interact again but in expectation at any point in t all of them change. The diagonal elements of $P_{\mathcal{C}_l}$ are for $q \in \mathcal{C}_l$ given by

$$\begin{aligned}
(P_{\mathcal{C}_l})_{q,q} &= \frac{1}{m} \sum_{\langle i,j\rangle \in \mathcal{C}_l^2 \cap \mathcal{E}} \left(\tilde{\Phi}_{ij}^{\mu}\right)_{q,q} + \frac{m-m_l}{m} \\
&= \frac{\deg^{\mathcal{C}_l}(q)}{m}(1-\mu) + \frac{m_l - \deg^{\mathcal{C}_l}(q)}{m} + \frac{m-m_l}{m} \\
&= 1 - \frac{\deg^{\mathcal{C}_l}(q)}{m}(1-(1-\mu)) = 1 - \mu\frac{\deg^{\mathcal{C}_l}(q)}{m} > 1 - \frac{\deg^{\mathcal{C}_l}(q)}{2m}.
\end{aligned}$$

Consequently, we obtain that the induced Markov chain is always lazy when $\mu < \frac{1}{2}$. Moreover, by definition $\tilde{\Phi}_{ij}^{\mu}$ is symmetric and positive definite with spectrum $\sigma\left(\tilde{\Phi}_{ij}^{\mu}\right) = \{1, 1-2\mu\}$, recalling $\mu \in \left[0, \frac{1}{2}\right)$. Therefore, also $P_{\mathcal{C}_l}$ is positive definite, such that it has a positive spectrum and by the Perron-Frobenius Theorem 5.14 its convergence speed is governed by the second largest eigenvalue $\lambda_2^{\mathcal{C}_l}$ satisfying $1 = \lambda_1^{\mathcal{C}_l} > \lambda_2^{\mathcal{C}_l} \geq \lambda_3^{\mathcal{C}_l} ... > 0$. Using the formula for the diagonal elements of $P_{\mathcal{C}_l}$, we can derive a lower bound of $\lambda_2^{\mathcal{C}_l}$ via the trace $\operatorname{tr}(P_{\mathcal{C}_l})$ as follows. Obviously, we have by the eigenvalue decomposition of $P_{\mathcal{C}_l}$ the estimate $\operatorname{tr}(P_{\mathcal{C}_l}) \leq 1 + \lambda_2^{\mathcal{C}_l}(|\mathcal{C}_l| - 1)$ and, secondly, we obtain

$$\operatorname{tr}(P_{\mathcal{C}_l}) = \sum_{q \in \mathcal{C}_l} \left(1 - \mu\frac{\deg^{\mathcal{C}_l}(q)}{m}\right) = |\mathcal{C}_l| - \mu\frac{2\,m_l}{m}$$

which gives in combination with the first estimate

$$\lambda_2^{\mathcal{C}_l} \geq 1 - \mu\frac{2\,m_l}{m(|\mathcal{C}_l| - 1)}. \tag{5.32}$$

Note that this gives a lower bound for the convergence speed which satisfies

$$1 - \mu\frac{2\,m_l}{m(|\mathcal{C}_l| - 1)} > 1 - \frac{m_l}{m(|\mathcal{C}_l| - 1)}$$

and, therefore, convergence is possibly fastest on small sets $|\mathcal{C}_l|$ containing relatively many edges m_l. In general, the term $1 - \mu\frac{2\,m_l}{m(|\mathcal{C}_l|-1)}$ can be close to 1 and convergence of the group opinions to the shared opinion in expectation is, therefore, slow. This becomes particularly prevalent when there are many small groups with small internal connectivity and many edges between groups. This depends on the parameter θ which enters into the convergence speed as an implicit influence.

Using Theorem 5.20 we can make claims about the upper bound for the convergence speed of the Markov chain induced by $P_{\mathcal{C}_l}$. Note to this end that the bottleneck ratio becomes because of the uniform stationary distribution

$$\begin{aligned}
\Phi_{\mathcal{C}_l}^* &= \min_{S:\, \pi^{\mathcal{C}_l}(S) \leq \frac{1}{2}} \Phi(S) = \min_{|S| \leq \frac{|\mathcal{C}_l|}{2}} \sum_{x \in S, y \in S^c} \frac{p_{\mathcal{C}_l}(x,y)}{|S|} \\
&= \min_{|S| \leq \frac{|\mathcal{C}_l|}{2}} \frac{\mu}{m}\frac{|\partial S|}{|S|} = \frac{\mu}{m}\iota\left(G_{\mathcal{C}_l}\right)
\end{aligned}$$

Where $\iota\left(G_{\mathcal{C}_l}\right)$ is the isoperimetric constant of the vertex induced sub-graph of G on $\mathcal{C}_l$. This depends both on the graph G as well as on the final cluster $\mathcal{C}_l$ and might be as small, if for example the induced sub-graph resembles a dumbbell graph.

Being able to find an upper bound for the mixing time of the chains on each $\mathcal{C}_l$ one can obtain an upper bound for the complete mixing time $\tau_{mix}(\varepsilon)$ as

$$\tau_{mix}(\varepsilon) = \max_l \tau_{mix}^{\mathcal{C}_l}(\varepsilon) \leq \max_l \frac{m^2}{\mu^2 \iota\left(G_{\mathcal{C}_l}\right)^2} \log\left(\frac{|\mathcal{C}_l|}{\varepsilon}\right)$$

by Theorem 5.20 since all chains on each $\mathcal{C}_l$ move in each time step simultaneously. Due to the randomness of the clusters involved it is not possible to obtain further estimates, which are better then the most pessimistic ones.

Having found an upper bound on the mixing time which can be calculated algorithmically once the clusters are established, we leave the Deffuant model behind us and turn now to the particle system of interest of this work.

5.4 Exclusion processes

The exclusion process is one of the classical examples for interacting particles systems alongside spin systems like the voter model, [Ligg12]. Spitzer, [Spi70], is among the first authors to consider interacting particle systems with exclusion, calling them exclusion processes. Two possibilities are considered in [Spi70], in which exclusion may be introduced in a system of interacting particles on a countable state space. First, whenever a particle moved to an already occupied site, its movement is suppressed and it stays in place. In the second case, the particle is forced to continue its movement according to some underlying Markov chain on the state space until it arrives at an empty site, possibly its original position. The latter is further discussed in [Ligg80]. While the second possibility yields interesting results and properties, like convergence to the same stationary distribution as in the first case, many authors, as for example in [Quas92] and [Law80], considered the first case for which the definition and some results are laid out in [Ligg12], Chapter VIII.

We adhere the descriptions given in [Ligg12], Chapter VIII, as well as [DiaSal93] for the discrete time case, allowing us to show the difference between the classical exclusion process and the process discussed in this work, which will motivate the intricate combinatorial analysis of sub-graphs and their relationships from Section 4. Consequently, we consider a finite state space S and a discrete time Markov chain defined by a transition Matrix P on S. The exclusion process behaves as follows, using the description in [Ligg12], Chapter VIII. The name comes from the restriction that at any time t at most one particle may occupy a site $x \in S$, constraining the possible positions of any particle and the allowed transitions. The transitions are given by a continuous time Markov chain. Each particle waits an exponential time with parameter one and then makes a transition according to $p(x,y)$ from its current position x to y if and only if y does not already contain a particle. If y is occupied then the particle in x remains in place.

More formally, a configuration at any time t may be described as a vector $\eta \in \{0,1\}^S$ and the transition via a function η_{xy}, using the notation from [Ligg12], Chapter VIII, which switches the entries of $\eta(x)$ and $\eta(y)$. It is given for $u \in S$ by

$$\eta_{xy}(u) = \begin{cases} \eta(y), & \text{if } u = x, \\ \eta(x), & \text{if } u = y, \\ \eta(u), & \text{otherwise.} \end{cases}$$

Since throughout this work we consider finite state spaces, we assume from now on that S is finite. Additionally, this renders the discussion of particularities arising from the domain $\mathcal{D}$ of the associated generator $\mathcal{Q}$ vain. The exclusion process is then given by its generator $\mathcal{Q}$ which has for $f \in \mathcal{D}$ the form

$$(\mathcal{Q}f)(\eta) = \sum_{\substack{\eta(x)=1\\ \eta(y)=0}} p(x,y)[f(\eta_{xy}) - f(\eta)]. \tag{5.33}$$

Note that this definition implies explicitly the independence of the transition probabilities of the current configuration η, conditioned on the event that a transition occurs. The matrix P defines completely the occurring probabilities in this case.

Under assumptions on P or S various results have been obtained over the years from the existence and form of a stationary distribution, bounds on the convergence speed to equilibrium and mixing times, mostly in cases where the exclusion process turns out to be reversible, as well as the Cut-Off phenomenon. The results have been obtained both the discrete time and continuous time variants of the exclusion process by the use of functional analytic methods and combinatorial approaches. In the spirit of the latter, we are going to conduct in Section 7 an analysis of the central object of this work which is a version of the exclusion process, arising from the Echo Chamber Model. In spite of the progress made for exclusion processes, admitting a generator as in equation (5.33), where P is independent of η, most results and techniques do not apply to our case due to the fact that P is replaced by a family of transition matrices $(P^\eta)_{\eta\in\{0,1\}^S}$ and a discrete time equivalent of the generator $\mathcal{Q}'$ defined for $f \in \mathcal{D}$ by

$$(\mathcal{Q}'f)(\eta) = \sum_{\substack{\eta(x)=1\\ \eta(y)=0}} p^\eta(x,y)[f(\eta_{xy}) - f(\eta)] \tag{5.34}$$

will govern the dynamics. Hence, we are faced with an example of a generalized version of the exclusion process defined in [Ligg12], Chapter VIII. Whenever possible, we are going make the link between the two models and show their differences as well as their similarities. Evidently, due to the generality of Equation 5.34 we can only focus on examples of the family $(P^{\eta})_{\eta\in\{0,1\}^S}$, which will be motivated by two specific applications.

5.5 Generalized exclusion processes (GEP) and Markov chains

We aim at establishing a link between a system of moving particles η_k on a graph and a Markov chain in a higher dimensional space to have access to classical results on the long time behavior of the process. Due to the constraints we are going to impose on the movement of the particles the most natural choice is to associated them with exclusion processes. In contrast to the classical view on exclusion processes as discussed in [DiaMeh87] the process to be analyzed depends highly on the structure of the underlying graph as well as the current particle configuration. This renders it heterogeneous, a property arising immediately from the asymmetric structure of interactions in a social network. Consequently, we are interested in a general form of exclusion processes defined as follows.

Definition 5.30. *Let $L = (V, E)$ be a simple connected graph with $|V| = \bar{n} \in \mathbb{N}$ and $k \in \{1, \dots, \bar{n} - 1\}$. Denote by $(P^{(\eta)})_{\eta\in\{0,1\}^V,\, |\eta|=k}$ a family of stochastic matrices. A generalized exclusion process in discrete time $\eta_k := (\eta_{k;t})_{t\in\mathbb{N}}$ of k particles on L is a Markov chain on the set of configurations*

$$\mathcal{S}_k = \{\eta \in \{0,1\}^V \mid |\eta| = k\}$$

defined by the transition matrix $Q = (q_{\eta,\mu})_{\eta,\mu\in\{0,1\}^V}$ given for $\eta, \mu \in \{0,1\}^V$ by

$$q_{\eta,\mu} = \begin{cases} P^{(\eta)}(v,w)\mathbb{1}_{\substack{\eta(v)=1=\mu(w),\eta(w)=0=\mu(v),\\ \eta(u)=\mu(v)\forall u\notin\{v,w\}}}, & \eta \neq \mu \\ 1 - \sum_{\mu\neq\eta} q_{\eta,\mu}, & \eta = \mu. \end{cases} \tag{5.35}$$

Indeed, there are various ways to approach generalized exclusion processes depending on the distributions which govern the transitions of individual particles. One way is by defining a Markov chain based on a series of dynamic graphs, i.e., by interpreting η_k as a Markov chain in a random environment given by the family $(P^{\eta})_{\eta\in\{0,1\}^V,\, |\eta|=k}$. To this end define $\mathfrak{N}_{k;t} = \{v \in V | \eta_{k;t}(v) = 1\}$ and the time dependent graph $B = (B_t)$ with $B_t = ((\mathfrak{N}_{k;t}, V\backslash\mathfrak{N}_{k;t}, \Sigma_{k,t})$ with potential loops on vertices in $\mathfrak{N}_{k;t}$. The random graph B_t might be disconnected but serves as a graph theoretical representation of transitions of the generalized exclusion process η_k, i.e., the positive transition probabilities of P^{η} conditioned on η.

The central part is the change of $\Sigma_{k,t}$ when going from time t to $t + 1$. It captures the possible particle movements induced by the distribution used in Definition 5.30. To illustrate the construction, we consider a 3-regular graph on 6 vertices depicted in

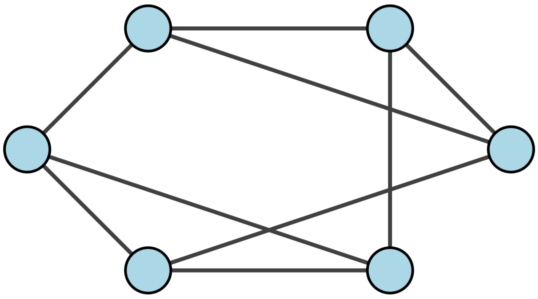

Figure 30: The underlying 3-regular graph on 6 vertices which we are going to use as example for all constructions of exclusion processes.

Figure 30. It will be the reference for missing edges which will illustrate the distinctive parts of each exclusion process. We consider two examples to illustrate the benefits of this perspective. First, consider a discrete time example of an exclusion process as discussed in [DiaSal93]. To any $\eta \in \{0,1\}^V$ we can associate a $\mathfrak{v}_\eta \subset V$ with $\mathfrak{v}_\eta = \{v \in V | \eta(v) = 1\}$ and in this sense we can write with a little abuse of notation $\eta = \mathfrak{v}_\eta$.

Assume that for some $\mathfrak{v} \subset V$, $|\mathfrak{v}| = k$ at time $t \in \mathbb{N}$ the classical exclusion process satisfies $\eta_{k;t} = \mathfrak{v}$. Set $\mathfrak{v}^c := V \backslash \mathfrak{v}$. Consider the possibly disconnected bipartite sub-graph $L_{\mathfrak{v},\mathfrak{v}^c} = ((\mathfrak{v}, \mathfrak{v}^c), E_{\mathfrak{v},\mathfrak{v}^c})$ of L. Then, add all edges to $L_{\mathfrak{v},\mathfrak{v}^c}$ which are contained in the vertex induced sub-graph $L_\mathfrak{v} = (\mathfrak{v}, E_\mathfrak{v})$ of L. We call the resulting graph $B_t = (V, \Sigma_{k,t})$ with $\Sigma_{k,t} = E_{\mathfrak{v},\mathfrak{v}^c} \sqcup E_\mathfrak{v}$. Indeed, for a $\bar{d}$-regular graph L we obtain $|\Sigma_{k,t}| = \bar{d}k$. Figure 31 illustrates one possible situation based on a 3-regular graph on six vertices. Remark that all edges incident to vertices in $\mathfrak{v}$ are also present in B_t which preserves their degree and makes it accessible to calculate the number of edges in B_t. The exclusion process now consists of drawing one edge uniformly from $\Sigma_{k,t}$ and exchanging the state of the endpoints. Note that this might lead to exchanging two occupied sites which renders P^η independent of η. As an interpretation of the exclusion process in this case one can see the exchange of states of two particles as their collision, both jumping back to the state they came from.

One can imagine changing certain steps of the construction. This leads to our second example. We keep the first step identical but do not consider, in what follows, the vertex induced sub-graph $L_\mathfrak{v}$ of L, but instead simply add loops to each site in $\mathfrak{v}$. Assume that at time t the exclusion process satisfies $\eta_{k;t} = \mathfrak{v} \subset V$. Using the previously introduced notation we consider the possibly disconnected bipartite sub-graph $L_{\mathfrak{v},\mathfrak{v}^c} = ((\mathfrak{v}, \mathfrak{v}^c), E_{\mathfrak{v},\mathfrak{v}^c})$ of L. Then, add a loop $e_{v,v}$ to any $v \in \mathfrak{v}$. We call the resulting graph $B'_t = (V, \Sigma'_{k,t})$ with $\Sigma'_{k,t} = E_{\mathfrak{v},\mathfrak{v}^c} \sqcup \{e_{v,v} | v \in \mathfrak{v}\}$. Indeed, for a $\bar{d}$-regular graph L we obtain $|\Sigma_{k,t}| = \deg_k(\mathfrak{v}) + k$. In Figure 32 we illustrate a possible example derived from the same underlying graph L as in Figure 31 and identical configurations $\mathfrak{v}$. The presence of loops and the absence of connections between vertices in $L_\mathfrak{v}$ change the behavior of the

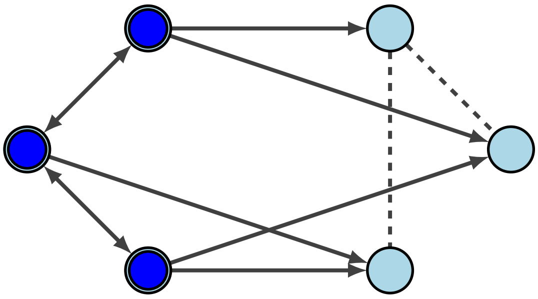

Figure 31: For an underlying 3-regular graph we apply the construction based on the classical exclusion process. The graph B_t constructed from occupied sites $\mathfrak{v}$ containing blue particles and its complement in V colored in pale blue. A particle displacement happens by drawing uniformly one of the (possibly in two directions) directed edges and exchanging the state of the vertices connected by said edge. Dashed edges cannot be used by particles.

process, simply due to changes in the allowed transitions among particles. Furthermore, the number of edges, taking also into account the loops, has changed from B_t to B'_t. The

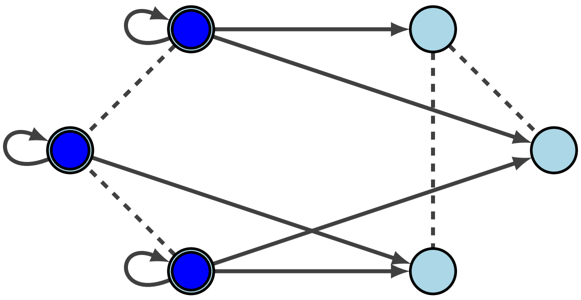

Figure 32: The graph B'_t constructed from occupied sites $\mathfrak{v}$ colored in blue and its complement in V colored in pale blue. As in Figure 31 a particle displacement happens by drawing uniformly one of the directed edges and exchanging the state of the vertices connected by said edge. When drawing a loop, everything remains the same. Again, as in Figure 31 dashed edges cannot be used by particles.

exclusion process still consists of drawing one edge uniformly from $\Sigma'_{k,t}$ and exchanging the state of the endpoints. Note, that in changing the number of present edges also the transition probabilities change due to the uniform distribution over $\Sigma'_{k,t}$. This renders the process, in particular, in-homogeneous because the transition probabilities now depend on the current configuration $\mathfrak{v}$ while for B_t only $\bar{d}$ and k played a relevant role, which are globally fixed parameters.

We reformulate the process η_k from Definition 5.30 on the set of configurations η with $|\eta| = k$ for a specific choice of $(P^\eta)_{\eta\in\{0,1\}^V}$ as a Markov chain on the kPG $\mathfrak{L}_k$.

In contrast to the context in [DiaMeh87] we put our focus on the explicit structure of the underlying graph which we have introduced previously. The group theoretic representation of the quotient of suitable symmetric groups will again play a role when we come back to convergence rates to equilibrium in the cases where said Markov chain is reversible. We come back to the two given examples in Section 6 and Subsection 7.4.

Starting in Section 7, the focus of our analysis shifts on another type of exclusion process arising from the Echo Chamber Model which incorporates a structure which represents the ability of each particle to choose among the free sites adjacent to its current site. This Markov chain will prove applicable in several contexts.

6 A Monte-Carlo method to identify densest sub-graphs

Outline of this section: Particle systems on finite graphs represent in a canonical form vertex-induced sub-graphs. This section is dedicated to the discussion of a generalized exclusion process which converges to a stationary distribution which assigns to any particle configuration a weight which is inversely proportional to its density, defined by its edge density as an induced sub-graph. We complete the chapter by proposing an algorithmic approach to sampling densest sub-graphs from regular graphs using MCMC techniques with an underlying Markov chain induced by a generalized exclusion process. We quantify its convergence speed in terms of the geometrical properties of the associated kPG and give an explicit form of the stationary distribution.

6.1 Problem of identifying densest sub-graphs

The first example of generalized exclusion processes we present in detail in this work is motivated by the task of finding k-densest sub-graphs in a graph or network of arbitrary size. This is in particular linked to the community detection problem which has received wide attention in modern research, for example in [GiNew02] and [Fort10]. The density of any vertex induced sub-graph $L_{\mathfrak{v}} = (\mathfrak{v}, E_{\mathfrak{v}})$ of some graph $L = (V, E)$ on the subset $\mathfrak{v}$ of the vertex set V is defined as $\rho(\mathfrak{v}) = \frac{|E_{\mathfrak{v}}|}{|\mathfrak{v}|}$, see for example [Mos00]. The problem of finding the densest sub-graph becomes, then, the maximization problem which aims at finding the subset $\mathfrak{v}^* \subset V$ such that $\rho(\mathfrak{v}^*) = \max_{\mathfrak{v} \in \mathcal{P}(V),\, |\mathfrak{v}| \geq 1} \rho(\mathfrak{v})$ where $\mathcal{P}(V)$ is the power set of V. In [Mos00] the author shows that this is an easy task in the sense that one can find such a sub-graph "quickly" in the sense of a notion which exceeds the scope of this work.

Fixing the size of the sub-graph renders the problem vastly more complicated as being discussed in, among many others, [CorPer84], [FePeKo01] and [KhuSah09]. In particular, the work in [KhuSah09] and [FePeKo01] discuss the difficulty, and almost impossibility, to find k-densest sub-graphs, i.e., sub-graphs on exactly k vertices, reliably "fast" in any graph. Hence, they focus on approximations of such objects using combinatorial selection methods or related families of problems which, reliably, give results which are close in density to the desired sub-graph.

We contribute to this discussion a stochastic approach using a generalized exclusion process on k particles and Monte Carlo simulation. The process is constructed in such a way that it converges most likely to a densest sub-graph and, even, gives monotonously sub-graphs of lesser density, i.e., a second densest sub-graph is obtained with second largest probability, a third densest with third larges probability and so on. We discuss the conditions on the underlying graph L and the results in what follows.

6.2 An appropriate GEP

To identify densest sub-graphs we want to construct a generalized exclusion process η_k^{MC} using on definition 5.30 by choosing an adequate family of transition matrices $(P^\theta)_{\theta\in\{0,1\}^{|V|}}$. Considering the generalized exclusion process on some graph L illustrated in Figure 32 we can propose and quantify a Markov chain Monte Carlo approach to sampling densest sub-graphs. In what follows, in order to obtain meaningful results we work under the assumption that both L and L^c are connected graphs. This assumption may be removed, imposing the difficulty of initial condition dependencies on disconnected graphs for particle systems. We leave this open to further research and remain with the assumption on L and L^c.

Consider the generalized exclusion process η_k^{MC}. We call the associated Markov chain $\mathfrak{S}_k^{MC}$ on $\mathfrak{L}_k$.

Proposition 6.1. *Let $k \in \{1, \ldots, \bar{n}-1\}$ and consider the Markov chain $\mathfrak{S}_k^{MC}$ on $\mathfrak{L}_k = (\mathfrak{V}_k, \mathfrak{E}_k)$. We call its transition matrix P_k^{MC}. Then, it has the form*

$$p^{MC}_{k;\mathfrak{v},\mathfrak{w}} = \begin{cases} \dfrac{1}{\deg_k(\mathfrak{v}) + k}, & \exists\langle x, y\rangle \in E \text{ s.t. } \mathfrak{v}\triangle\mathfrak{w} = \{x, y\}, \\ \dfrac{k}{\deg_k(\mathfrak{v}) + k}, & \mathfrak{v} = \mathfrak{w}, \\ 0, & \text{otherwise.} \end{cases}$$

The form of the transition matrix follows directly from the construction of B_t' in Figure 32 and the uniform distribution over all edges in B_t' as discussed in Subsection 5.5. The stationary distribution π_k^{MC} of $\mathfrak{S}_k^{MC}$ can consequently be derived directly and reversibility follows as well.

Theorem 6.2. *Let $k \in \{1, \ldots, \bar{n}-1\}$ and consider the Markov chain $\mathcal{S}_k^{MC}$ on $\mathfrak{L}_k = (\mathfrak{V}_k, \mathfrak{E}_k)$. Then it is aperiodic, irreducible and, hence, ergodic. Furthermore, it is a reversible chain and the stationary distribution π_k^{MC} is given in terms of $\mathfrak{v} \in \mathfrak{V}_k$ by*

$$\pi_k^{MC}(\mathfrak{v}) = \frac{\deg_k(\mathfrak{v}) + k}{2|\mathfrak{E}_k| + k\binom{\bar{n}}{k}} \tag{6.1}$$

Based on the transition matrix given by Proposition 6.1 and the stationary distribution given by Theorem 6.2 one can identify $\mathfrak{S}_k^{MC}$ with a random walk on $\mathfrak{L}_k$ where any vertex in $\mathfrak{L}_k$ has additionally to its incident edges k loops. Indeed, for almost all cases of k, this is not sufficient to render $\mathfrak{S}_k^{MC}$ a lazy random walk. Indeed, we can quantify this transition.

Lemma 6.3. *The random walk $\mathfrak{S}_k^{MC}$ is lazy if and only if*

$$\min_{\mathfrak{v} \in \mathfrak{V}_k} \text{avg deg}_k(L_{\mathfrak{v}}) \geq \bar{d} - 1.$$

Proof. Recall that the Markov chain is called lazy if and only if

$$\min_{\mathfrak{v},\mathfrak{v}} p_{k;\mathfrak{v},\mathfrak{v}}^{MC} \geq \frac{1}{2}. \tag{6.2}$$

Therefore, using the expression derived in Proposition 6.1 we obtain that $\mathfrak{S}_k^{MC}$ is lazy if and only if for all $\mathfrak{v} \in \mathfrak{V}_k$ we have

$$\frac{\deg_k(\mathfrak{v})}{k} + 1 \leq 2 \Leftrightarrow \bar{d} - \text{avg deg}_k(L_{\mathfrak{v}}) \leq 1$$

which proves the claim. □

An obvious property is $\min_{\mathfrak{v} \in \mathfrak{V}_k} \text{avg deg}_k(L_{\mathfrak{v}}) \leq \bar{d}$ since $L_{\mathfrak{v}}$ is a vertex induced sub-graph and, therefore, the degree of any vertex in $L_{\mathfrak{v}}$ is bounded by $\bar{d}$ which implies the same for the average. Geometrically, the condition given by Lemma 6.3 leaves, consequently, only little room for the parameter triple $(\bar{n}, \bar{d}, k)$. Due to the intermediate position $\mathfrak{S}_k^{MC}$ takes between the simple random walk on $\mathfrak{L}_k$ and the lazy random walk on this state space, we can in general only use that

$$\min_{\mathfrak{v} \in \mathfrak{V}_k} p_{k;\mathfrak{v},\mathfrak{v}}^{MC} \geq \gamma > 0$$

for some $\gamma \in \left(0, \frac{1}{2}\right)$ with a phase transition, if for a parameter triple $(\bar{n}, \bar{d}, k)$ we have $\min_{\mathfrak{v} \in \mathfrak{V}_k} \text{avg deg}_k(L_{\mathfrak{v}}) \geq \bar{d} - 1$ which is a geometric condition on L. We use the theory of evolving sets as well as the results which can be deduced by this perspective on the evolution of Markov chains over the state space $\mathfrak{L}_k$ which we presented in Section 4. In particular, we make use of Theorem 5.21. Note that a tiny part of the calculations is only true under the assumption, that Conjecture 4.22 is true, as well.

Theorem 6.4. *Let L be a $\bar{d}$-regular graph on $\bar{n}$ vertices and $k \in \{1, \ldots, \bar{n}-1\}$. Consider the Markov chain on $\mathfrak{S}_k^{MC}$ the k particle graph $\mathfrak{L}_k$ and $\varepsilon > 0$. Additionally, define*

$$\rho(\bar{n}, \bar{d}, k) := \frac{4(\bar{n}-1)^2(\bar{n}-k)^2}{\bar{d}^4}$$
$$\xi(\bar{n}, \bar{d}, k) := \log\left(\binom{\bar{n}}{k}\right) + \log\left(\frac{k(\bar{n}-k)}{\bar{n}-1} + \frac{k}{\bar{d}}\right).$$

If $L \in \left(\Gamma_{\mathfrak{l}}^{(k)}\right)^c$ then for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ and

$$t \geq 1 + \rho(\bar{n}, \bar{d}, k)\left(\log\left(\frac{4}{\varepsilon}\right) + \xi(\bar{n}, \bar{d}, k)\right) \tag{6.3}$$

the Markov chain $\mathfrak{S}_k^{MC}$ satisfies

$$\left|\frac{\left(p_{k;\mathfrak{v},\mathfrak{w}}^{MC}\right)^{(t)} - \pi_k^{MC}(\mathfrak{w})}{\pi_k^{MC}(\mathfrak{w})}\right| \leq \varepsilon$$

and if $L \in \Gamma_{\mathfrak{l}}^{(k)}$ then its mixing time w.r.t. the total variation distance is bounded by

$$\tau_{mix}(\varepsilon) \leq 2000 \log_2(\varepsilon^{-1})\left(\frac{\log\left(\frac{3}{4}\right)(\bar{n}-1)^2}{\min\limits_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v})^2 \bar{d}^2} + \xi(\bar{n}, \bar{d}, k)\right). \tag{6.4}$$

Proof. Let $\gamma := \left(\bar{d} + 1 - \min_{\mathfrak{v}\in\mathfrak{V}_k} \operatorname{avg} \deg_k(L_{\mathfrak{v}})\right)^{-1}$ and consider the following two terms which we call the additive constant and the multiplicative, respectively, given as, firstly,

$$\frac{4(1-\gamma)^2}{\gamma^2} \frac{\log\left(\frac{2|\mathfrak{E}_k| + k|\mathfrak{V}_k|}{4(\min\{\deg_k(\mathfrak{v}), \deg_k(\mathfrak{w})\} + k)}\right)}{\left(\iota(\mathfrak{L}_k)\frac{2|\mathfrak{E}_k|+k|\mathfrak{V}_k|}{\max_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v})+k}\right)^2} \tag{6.5}$$

and, secondly,

$$\frac{4(1-\gamma)^2}{\gamma^2}\left(\iota(\mathfrak{L}_k)\frac{2|\mathfrak{E}_k| + k|\mathfrak{V}_k|}{\max_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v}) + k}\right)^{-2}. \tag{6.6}$$

Recall that by [Mohar89], which we cited in Lemma 5.18, we can estimate the isoperimetric constant from below by a constant times the connectivity of $\mathfrak{L}_k$ in the form

$$\iota(\mathfrak{L}_k) \geq \frac{2}{|\mathfrak{V}_k|}\kappa(\mathfrak{L}_k) \tag{6.7}$$

and we have conjured in Conjecture 4.22 the value of $\kappa(\mathfrak{L}_k)$ as $\kappa(\mathfrak{L}_k) = \min_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v})$. This gives us, consequently, a lower bound for

$$\begin{aligned}\iota(\mathfrak{L}_k)\frac{2|\mathfrak{E}_k| + k|\mathfrak{V}_k|}{\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v}) + k} &\geq 2\min_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})\frac{\text{avg deg}(\mathfrak{L}_k) + k}{\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v}) + k}\\ &\geq \min_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})\frac{\text{avg deg}(\mathfrak{L}_k)}{\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})}\end{aligned}$$

using for the last inequality that $L \in \left(\Gamma_{\mathfrak{l}}^{(k)}\right)^c$. This in addition to $\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v}) \leq k(\bar{n} - k)$ and Conjecture 4.22 gives the following upper bound for the multiplicative constant by

$$\begin{aligned}\frac{4(1-\gamma)^2}{\gamma^2}\left(\frac{\iota(\mathfrak{L}_k)(2|\mathfrak{E}_k| + k|\mathfrak{V}_k|)}{\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v}) + k}\right)^{-2} &\leq \frac{4}{k^2}\left(\frac{\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})}{\text{avg deg}(\mathfrak{L}_k)}\right)^4\left(\frac{\text{avg deg}(\mathfrak{L}_k)}{\min_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})}\right)^2\\ &\leq \frac{4}{k^2}\left(\frac{\bar{n}-1}{\bar{d}}\right)^4\left(\frac{\text{avg deg}(\mathfrak{L}_k)}{\min_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})}\right)^2.\end{aligned}$$

Employing the lower bound $\min_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v}) \geq \bar{d}$, which can be derived easily by using $k \leq \bar{n} - 1$ and the fact that there is at least one empty vertex in L with degree $\bar{d}$, one can obtain the following upper bound independent of the geometry of L given by

$$\frac{4}{k^2}\left(\frac{\bar{n}-1}{\bar{d}}\right)^4\left(\frac{\text{avg deg}(\mathfrak{L}_k)}{\min_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})}\right)^2 \leq \frac{4(\bar{n}-1)^2(\bar{n}-k)^2}{\bar{d}^4}.$$

Since the additive constant equals the multiplicative constant times a logarithmic term, we are only going to focus on the logarithmic term in what follows. The average degree of $\mathfrak{L}_k$ as well as its relation to the minimal and maximal degree of $\mathfrak{L}_k$ will play an essential role and we obtain

$$\begin{aligned}\log\left(\frac{2|\mathfrak{E}_k| + k|\mathfrak{V}_k|}{4(\min\{\deg_k(\mathfrak{v}), \deg_k(\mathfrak{w})\} + k)}\right) &\leq \log\left(\frac{2|\mathfrak{E}_k| + k|\mathfrak{V}_k|}{\min_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v}) + k}\right)\\ &= \log\left(\binom{\bar{n}}{k}\right) + \log\left(\frac{\text{avg deg}(\mathfrak{L}_k) + k}{\min_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v}) + k}\right).\end{aligned}$$

For the second summand we can use the bound we used already above to obtain

$$\log\left(\frac{\text{avg deg}(\mathfrak{L}_k) + k}{\min_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v}) + k}\right) \leq \log\left(\frac{k(\bar{n}-k)}{\bar{n}-1} + \frac{k}{\bar{d}}\right).$$

Therefore, we obtain the upper bound on the additive constant

$$\frac{4(\bar{n}-1)^2(\bar{n}-k)^2}{\bar{d}^4}\left(\log\left(\binom{\bar{n}}{k}\right)+\log\left(\frac{k(\bar{n}-k)}{\bar{n}-1}+\frac{k}{\bar{d}}\right)\right)$$

We now employ some results based on evolving sets presented in [MorPer05]. To this end, we will focus on lower bounds for $\Phi_k(u) := \inf\left\{\frac{\partial(S,S^c)}{\pi_k^{MC}(S)}\middle|\,\pi_k^{MC}(S)\leq u\right\}$ for $u\in\left[\min_{\mathfrak{v}\in\mathfrak{V}_k}\pi_k^{MC}(\mathfrak{v}),\frac{1}{2}\right]$ as defined in [MorPer05]. In particular, we use that $\Phi(u)\geq\Phi\left(\frac{1}{2}\right)$. The first claim then follows by Theorem 5.21, the second one by [LovKan99]. To conclude the proof of the first claim using the previously derived bounds, note that

$$\mathcal{S} := \left\{S\subset\mathfrak{V}_k\middle|\,|S|\leq\frac{|\mathfrak{V}_k|}{2}\right\}\cap\left\{S\subset\mathfrak{V}_k\middle|\,\pi_k^{MC}(S)\leq\frac{1}{2}\right\}\neq\emptyset$$

and consider $\bar{S}\in\mathcal{S}$. Then,

$$\frac{\partial(\bar{S},\bar{S}^c)}{\pi_k^{MC}(\bar{S})}\geq\frac{\partial(\bar{S},\bar{S}^c)}{|\bar{S}|}\frac{2|\mathfrak{E}_k|+k|\mathfrak{V}_k|}{\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})+k}\geq\iota(\mathfrak{L}_k)\frac{2|\mathfrak{E}_k|+k|\mathfrak{V}_k|}{\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})+k}$$

where we used $\bar{S}\in\left\{S\subset\mathfrak{V}_k\middle|\,|S|\leq\frac{|\mathfrak{V}_k|}{2}\right\}$ for the last estimate and the definition of the isoperimetric constant of a graph given by Definition 5.17. Since, additionally, $\bar{S}\in\left\{S\subset\mathfrak{V}_k\middle|\,\pi_k^{MC}(S)\leq\frac{1}{2}\right\}$, taking the infimum over all $S\in\left\{S\subset\mathfrak{V}_k\middle|\,\pi_k^{MC}(S)\leq\frac{1}{2}\right\}$ we arrive at

$$\Phi_k\left(\frac{1}{2}\right)\geq\iota(\mathfrak{L}_k)\frac{2|\mathfrak{E}_k|+k|\mathfrak{V}_k|}{\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})+k}$$

and, therefore, for all $u\in\left[\min_{\mathfrak{v}\in\mathfrak{V}_k}\pi_k^{MC}(\mathfrak{v}),\frac{1}{2}\right]$ we obtain

$$\Phi_k(u)\geq\iota(\mathfrak{L}_k)\frac{2|\mathfrak{E}_k|+k|\mathfrak{V}_k|}{\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})+k}. \tag{6.8}$$

The second step consists in finding a lower bound for $\min_{\mathfrak{v}\in\mathfrak{V}_k}p_{k;\mathfrak{v},\mathfrak{v}}^{MC}$. We have already found in the proof of Lemma 6.3 that

$$\min_{\mathfrak{v}\in\mathfrak{V}_k}p_{k;\mathfrak{v},\mathfrak{v}}^{MC}\geq\frac{1}{\bar{d}+1-\min_{\mathfrak{v}\in\mathfrak{V}_k}\operatorname{avg}\deg_k(L_{\mathfrak{v}})}=\gamma. \tag{6.9}$$

Having found the necessary bounds we can apply Theorem 5.21, which is Theorem 5 of [MorPer05], we obtain after integration

$$\int_{4(\pi_k^{MC}(\mathfrak{v})\wedge\pi_k^{MC}(\mathfrak{w}))}^{4\varepsilon^{-1}}\frac{4\,du}{u\Phi(u)^2}\leq\frac{\left(\log\left(\frac{4}{\varepsilon}\right)-\log\left(\frac{4(\min\{\deg_k(\mathfrak{v}),\deg_k(\mathfrak{w})\}+k)}{2|\mathfrak{E}_k|+k|\mathfrak{V}_k|}\right)\right)}{\left(\iota(\mathfrak{L}_k)\frac{2|\mathfrak{E}_k|+k|\mathfrak{V}_k|}{\max_{\mathfrak{v}\in\mathfrak{V}_k}\deg_k(\mathfrak{v})+k}\right)^2}. \tag{6.10}$$

After this intermediate step, we can identify the terms in the right hand side of equation 6.10 with the additive constant and the multiplicative constant for which we have found meaningful upper bounds at the beginning of this proof, such that we obtain the first claim. The second claim follows by the same estimate for $\Phi_k(u)$ and Theorem 2.2 of [LovKan99] as well as equation (5) of [MorPer05] using the bound of the additive term by $\xi(\bar{n}, \bar{d}, k)$ as defined in the theorem. □

Note that the constants we used in Theorem 6.4, namely $\rho(\bar{n}, \bar{d}, k)$ and $\xi(\bar{n}, \bar{d}, k)$, grow at most like a polynomial as $\bar{n} \to \infty$ since $\log\left(\binom{\bar{n}}{k}\right)$ grows at most like $\bar{n}$ by Stirling's approximation and the remaining terms are polynomial in $\bar{n}$. Consequently, $\mathfrak{S}_k^{MC}$ mixes rapidly relative to the size of its actual state space, employing this term as used in [Sinc92]. Its mixing is, therefore, polynomial fast in terms of the size of the underlying graph L, which makes it a suitable process for sampling sub-graphs of L. Surprisingly, the presence of the term depending on γ in the case $L \in \left(\Gamma_{\mathfrak{l}}^{(k)}\right)^c$ allows for clearer cut constants in terms of the underlying graph L. In the second more precise bound in terms of the convergence speed, we are stuck with the term $\min_{\mathfrak{v} \in \mathfrak{V}_k} \deg_k(\mathfrak{v})$. Finding a upper bound on τ_{mix} which only depends on the parameters on L without using trivial estimates corresponds to a meaningful lower bound on $\min_{\mathfrak{v} \in \mathfrak{V}_k} \deg_k(\mathfrak{v})$ which in turn implies a meaningful upper bound on the density of densest k sub-graphs of L which is out of the scope of this work and is to the best of our knowledge not an available result which can be cited.

In the following subsection, we propose an algorithm and the idea behind sampling densest sub-graphs of L with high probability using the Markov chain $\mathfrak{S}_k^{MC}$ introduced in this section and exploiting its properties.

6.2.1 Sampling of densest k-sub-graphs

By the structure of the stationary distribution we obtain an ordering based on the degree of the vertices, where the probability of drawing a densest sub-graph as $t \to \infty$ is smallest since $\deg_k(\mathfrak{v}) = k\bar{d} - 2|E_{\mathfrak{v}}|$. On the other hand, this also implies that it is most probable under π_k^{MC} to draw a least dense sub-graph of L. Lemma 4.20 provides a link between the densest k sub-graph in L and the least dense k sub-graph in L^c. Using this, we propose a Markov chain Monte Carlo approach to finding densest k sub-graphs in a $\bar{d}$-regular graph with high probability. To this end consider a $\bar{d}$-regular simple connected graph L on $\bar{n}$ vertices and assume that its graph complement L^c is also connected. Then L^c is by Lemma 4.20 a $\bar{n} - 1 - \bar{d}$ regular simple connected graph. Let $k \in \{1, \ldots, \bar{n} - 1\}$ and define $\mathfrak{S}_k^{MC}$ as the Markov chain associated to the exclusion process on L^c. Then, by Theorem 6.2 the Markov chain $\mathfrak{S}_k^{MC}$ converges in distribution to π_k^{MC} and with maximal probability we obtain a $\mathfrak{v}_* \in \mathfrak{V}_k^c$ which induces a least dense k-sub-graph $L_{\mathfrak{v}_*}$ of L^c. Using Lemma 4.20 we obtain that $\mathfrak{v}_*$ induces a densest k-sub-graph in L.

Unfortunately, note that by Lemma 3.23 certain restrictions apply to L due to the assumption that both L and L^c are connected. After this informal discussion we now give the concrete approach to this simulation.

6.2.2 A possible algorithm

We propose the following algorithm which only uses the local information of L and the current configuration $\mathfrak{v}$ of $\mathfrak{S}_k^{MC}$. This for efficient simulations since it is not necessary to construct the whole state space $\mathfrak{L}_k$ with $\binom{\bar{n}}{k}$ vertices and the size of the edge set given by Proposition 4.9. It only depends on the neighborhood of all $v \in \mathfrak{v}$. Exploiting the fact that L is assumed to be $\bar{d}$-regular, we can bound the number of states we have to access in each turn by $\bar{d} \cdot k$ which is also a very crude upper bound for $\deg_k(\mathfrak{v})$. We, again, use multi-sets to describe the algorithm. This can be replaced in a implementation by any mutable list-type object which allows multiple times the same entry. When we define $\mathfrak{X} = \emptyset$ as a multi-set, then it is implied that all properties with respect to set operations discussed in Subsection 3.2 are satisfied, e.g., $(\mathfrak{X} \cup \{\mathfrak{v}\}) \cup \{\mathfrak{v}\} = \mathfrak{X} \cup \{\mathfrak{v}, \mathfrak{v}\}$. We can conclude this section, therefore, with an algorithm which finds rapidly densest

Algorithm 1 Particle based algorithm to simulate $\mathfrak{S}_k^{MC}$.

Require: Terminal time for simulation, e.g. t_{mix}, number of trials m, graph L
  initialize $t = 0$
  Define $\mathfrak{X} = \emptyset$ as multi-set
  while $i \leq m$ **do**
    draw initial state $\mathfrak{v} \in \mathfrak{V}_k$
    Set $\mathfrak{S}_{k,0}^{MC} = \mathfrak{v}$
    while $t \leq t_{mix}$ **do**
      Construct bipartite graph $B'(\mathfrak{v})$ as in Figure 32
      Draw uniformly an edge $\langle v, w \rangle$ from $B'(\mathfrak{v})$
      update $\mathfrak{S}_{k,t+1}^{MC} = (\mathfrak{v} \backslash \{v\}) \cup \{w\}$; $t = t + 1$
    end while
    $\mathfrak{X} = \mathfrak{X} \cup \mathfrak{S}_{k,t_{mix}}$
    $i = i + 1$; $t = 0$
  end while
  return $\mathfrak{X}$ statistic of m final states after t_{mix} simulation steps.

sub-graphs with high probability, contributing to the research on densest sub-graphs from a probabilistic point of view. Furthermore, we had the opportunity to show the importance of a detailed understanding of the underlying state space $\mathfrak{L}_k$ as a graph on subsets of the vertex set of some underlying graph. In particular, the geometric results as well as knowledge about the structure of the vertex set allowed for the bounds in

Theorem 6.4. Finally, the proposed algorithm benefits from the particle perspective in that it only needs to consider at most $k\bar{d}$ vertices in each step which is a huge reduction from considering the whole vertex set $\mathfrak{V}_k$. We turn now to a particle system which arises from a reduced version of the echo chamber model and demonstrate that with increasing complexity of the Markov chain on $\mathfrak{L}_k$ further knowledge of the structure of $\mathfrak{L}_k$ gives insights in possibly intuitive behavior of the Markov chain.

7 The Echo Chamber Model: a related exclusion process

Outline of this section: In this section we show a possibility to identify a reduced version of the Echo Chamber Model, defined in Section 1, with a generalized exclusion process on an appropriate state space. To this end, we consider first the hereinbelow described process, in which we remove the labels from the setting. This allows us to get an in depth understanding of local structures of the edges and the implications on long time behavior of the process. The properties of the kPG $\mathfrak{L}_k$ derived in Section 4 will be indispensable for our analysis, where the edges will take the role of the particles on some more abstract state space, namely the line-graph of the underlying graph.

7.1 Modeling relationship dynamics

In this section we are going to analyze the underlying relationship dynamics of the Echo Chamber Model interpreted as a particle system on a strongly regular graph. We assume that edges move always as described by the dynamics in the introduction of this section as well as hereinafter. The opinions will be reintroduced in Section 8 as a random environment which renders certain states absorbing.

We, consequently, consider, henceforth, the following sequence of random graphs $(G_t)_{t\in\mathbb{N}} = ((\mathcal{V}, \mathcal{E}_t))_{t\in\mathbb{N}}$ where the edge set undergo a transformation between two subsequent graphs G_t and G_{t+1} according to the hereinafter described procedure.

- Draw uniformly an edge $\langle A, B\rangle \in \mathcal{E}_t$.
- Define $N_{\langle A,B\rangle} = \{e = \langle c, d\rangle \notin \mathcal{E}_t\backslash\{\langle A, B\rangle\}|k \in \{A, B\} \text{ or } l \in \{A, B\}\}$.
- draw uniformly E from $N_{\langle A,B\rangle}$,
- set $\mathcal{E}_{t+1} = (\mathcal{E}_t\backslash\{\langle A, B\rangle\}) \cup \{E\}$.

We illustrate the procedure in Figure 33. This reduced process will never stop moving due to the randomness involved and the absence of absorbing states. The central question concerning long-term behavior of this process focuses, consequently, on the convergence in a probabilistic sense to some invariant distribution and the convergence speed. In Subsection 7.2 we are going to develop a new perspective on this model, translating it to a dynamic particle system on a related graph and we can deduce various properties about its long-term behavior by identification with a Markov chain and methods from Markov chain theory.

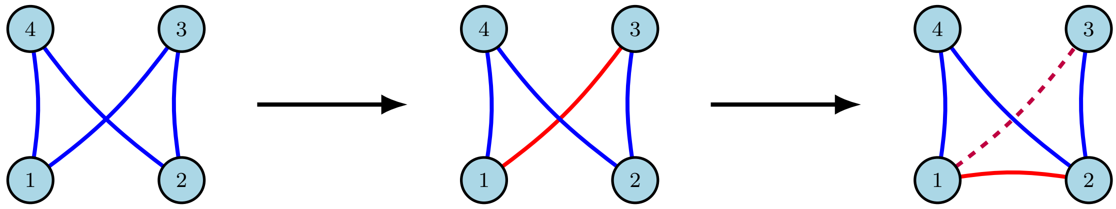

Figure 33: Change of relationships from time step t to $t+1$ on a graph with 4 vertices and 4 edges. The red edge is picked and moved according to the prescribed dynamics. The number of edges is preserved by the process.

7.2 A GEP interpretation on strongly regular graphs

We consider a graph $G = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ and the complete graph $\hat{G} = (\mathcal{V}, \hat{\mathcal{E}})$ on the vertex set $\mathcal{V}$. We can understand the edges in $\hat{G}$ as the possible positions of each edge in G under the dynamics defined in the previous section. The neighborhood relationship of edges and, hence, a direct interpretation of their transitions remains ambiguous. To shed light on this problem we consider the line graph $L = (\hat{\mathcal{E}}, E)$ of $\hat{G}$ as defined in Definition 3.6. Each $\langle v, w\rangle \in \hat{\mathcal{E}}$ carries a label in $\{0,1\}$. Since $\hat{G}$ is a complete graph its corresponding line graph $L_{\hat{G}}$ is strongly regular with parameters srg $\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right)$, see [Har71].

Consider the previously introduced process $(G_t)_{t\geq 0}$ and the line graph $L := L_{\hat{G}}$ of the complete graph $\hat{G}$ induced by any G_t for any arbitrary t. A site $e \in L$ is occupied at time t if and only if e is an edge in G_t. We find that edges in G_t can be identified with particles on the line graph of $\hat{G}$ as depicted in Figure 34. We first construct the line graph L from $\hat{G}$ and then place the edges which exist in G_t as particles on the corresponding vertices of L. Since the step from G_t to G_{t+1} corresponds to changing the place of one particular edge, we can interpret this step as a movement of one of the particles on L. This movement is constrained by two factors. First, multiple edges are not allowed in G_t for all t and, second, any edges can only be transformed to an edge which shares at least one vertex with the original one. Therefore, at most one particle can occupy a vertex in L and if a particle moves it moves to a neighboring vertex or stays in place. Consequently, we can associate the original dynamics with a particle system on a graph with state dependent transition probabilities. A possible transition is illustrated in Figure 35. Additionally, since two particles may never occupy the same site the new process corresponds to an exclusion process. Indeed, the drawing procedure of edges, defined hereinabove, leads to the following generalized exclusion process. Let $L = (V, E)$ be a simple connected graph and $k \in \{1, \ldots, |V|\}$. We consider the special case $\hat{\eta}_k$ of Definition 5.30 where the transitions at time t are defined by

1. draw uniformly $U_t \in \hat{\eta}_{k;t}$ and $W_t \in (N_{U_t} \backslash \hat{\eta}_{k;t}) \cup \{U_t\}$

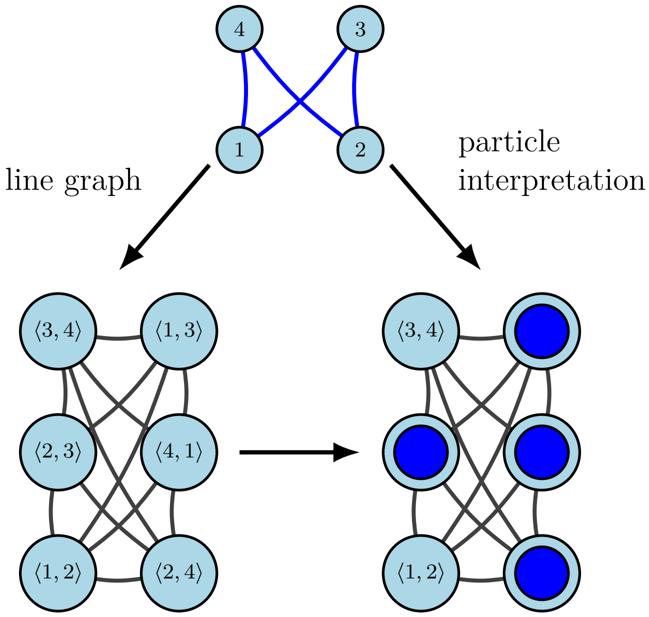

Figure 34: Transformation of existing edges (blue) in G to particles (blue) occupying vertices in the line graph L of $\hat{G}$.

2. set $\hat{\eta}_{k;t+1} = (\hat{\eta}_{k;t} \backslash \{U_t\}) \cup \{W_t\}$.

To visualize the newly defined process $\hat{\eta}_k$ on L, we come back to the example used in the previous subsection as well as in Figure 34.

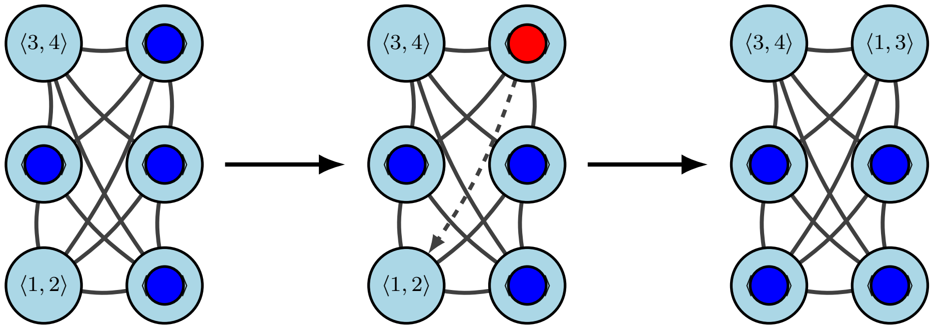

Figure 35: A particle moving from one vertex to another adjacent one, induced by the dynamics of $\hat{\eta}_k$.

7.3 Properties of the associated Markov chain

A basic but necessary result is the existence of a Markov chain on $\mathfrak{L}_k$ which represents the particle movement on L. We defined the kPG $\mathfrak{L}_k$ by considering the entire particle configuration at any time t and define its transitions by the deplacements of the particles. Indeed, the edges in $\mathfrak{L}_k$ are defined to capture the movement of the particles from a perspective of configurations.

Theorem 7.1 (Canonical representation of $\hat{\eta}_k$)**.** *Let $k \in \{1, \dots, \bar{n}-1\}$, L a connected graph, η_k the k particle exclusion process on L and $(\Omega, \mathcal{F}, \mathbb{P})$. Then there exist a probability space $(\Omega, \mathcal{F}', \mathbb{P}')$ and a Markov chain $\mathfrak{S}_k$ on it with transition Matrix $P_k^{\triangle}$ and state space $\mathfrak{L}_k$ such that for any $\mathfrak{w}, \mathfrak{v} \in \mathfrak{V}_k$ and $t \in \mathbb{N}$ the equation*

$$\begin{aligned}\mathbb{P}[\forall v \in \mathfrak{v} : \eta_{k;t}(v) = 1 | \forall w \in \mathfrak{w} : \eta_{k;0}(w) = 1] &= \mathbb{P}'[\mathfrak{S}_{k;t} = \mathfrak{v} | \mathfrak{S}_{k;0} = \mathfrak{w}] \\ &= \left(P_k^{\triangle}\right)^t_{\mathfrak{w},\mathfrak{v}}.\end{aligned}$$

is satisfied.

Proof. Let $\omega \in \Omega$ and consider for $t \geq 0$ the configurations $\hat{\eta}_{k;t}(\omega)$ and $\hat{\eta}_{k;t+1}(\omega)$. Then there is at most one $i \in \{1, \dots, k\}$ such that $\hat{\eta}^i_{k;t}(\omega) \neq \hat{\eta}^i_{k;t+1}(\omega)$ and $\hat{\eta}^i_{k;t}(\omega) \sim^L \hat{\eta}^i_{k;t+1}(\omega)$ where a site $u \in L$ always satisfies $u \sim^L u$. Since $\hat{\eta}_{k;t}(\omega) \in \mathfrak{V}_k$ for all $t \geq 0$ we can define $\mathfrak{S}_{k;0}(\omega) := \hat{\eta}_{k;0}(\omega)$. Furthermore $\hat{\eta}_{k;1}(\omega) \sim \hat{\eta}_{k;0}(\omega) = \mathfrak{S}_{k;0}(\omega)$ we can define one step of $\mathfrak{S}_k$ via $\mathfrak{S}_{k;1}(\omega) := \hat{\eta}_{k;1}(\omega)$. Define inductively $\mathfrak{S}_{k;t}(\omega)$ for all $\omega \in \Omega$ and $t \geq 0$. Then $\mathfrak{S}_k$ is a Markov chain on $\mathfrak{L}_k$. □

Having established the existence of the associated Markov chain, we can investigate the form of its transition matrix. The explicit form of its transition probabilities are given by Lemma 7.2 and Theorem 7.3.

Lemma 7.2. *Let $k \in \{1, \dots, \bar{n}-1\}$, $\mathfrak{v} \subset V$, $|\mathfrak{v}| = k$ denote by $L_{\mathfrak{v}}$ the vertex induced sub-graph of $\mathfrak{v}$ in L and for $v \in \mathfrak{v}$ write $\deg^{L_{\mathfrak{v}}}(v)$ the degree of v in $L_{\mathfrak{v}}$. Define the matrix $P_k^{\triangle}$ for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ by*

$$p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = \begin{cases} \dfrac{1}{k} \dfrac{1}{\bar{d} - \deg^{L_{\mathfrak{v}}}(v) + 1}, & \mathfrak{v} \triangle \mathfrak{w} = \{v, w\} \text{ with } v \sim^L w, \\ \displaystyle\sum_{v \in \mathfrak{v}} \frac{1}{k} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{v}}}(v) + 1}, & \mathfrak{v} = \mathfrak{w}, \\ 0, & \text{otherwise}. \end{cases} \tag{7.1}$$

Then, $P_k^{\triangle}$ is a stochastic matrix.

Proof. To verify that $P_k^{\triangle} = \left(p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}}\right)_{\mathfrak{v},\mathfrak{w} \in \mathfrak{V}_k}$ defines a stochastic matrix we calculate explicitly the row-sums.

$$\begin{aligned}\sum_{\mathfrak{w}\in\mathfrak{V}_k} p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} &= \sum_{\mathfrak{w}\sim\mathfrak{v}} p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} + \sum_{v\in\mathfrak{v}} \frac{1}{k}\frac{1}{\bar{d}-\deg^{L_{\mathfrak{v}}}(v)+1} \\ &= \sum_{v\in\mathfrak{v}} \sum_{\substack{w\sim_L v\\ w\notin\mathfrak{v}}} \frac{1}{k}\frac{1}{\bar{d}-\deg^{L_{\mathfrak{v}}}(v)+1} + \sum_{v\in\mathfrak{v}} \frac{1}{k}\frac{1}{\bar{d}-\deg^{L_{\mathfrak{v}}}(v)+1} \\ &= \sum_{v\in\mathfrak{v}} \frac{\bar{d}-\deg^{L_{\mathfrak{v}}}(v)}{k}\frac{1}{\bar{d}-\deg^{L_{\mathfrak{v}}}(v)+1} + \sum_{v\in\mathfrak{v}} \frac{1}{k}\frac{1}{\bar{d}-\deg^{L_{\mathfrak{v}}}(v)+1} = 1.\end{aligned}$$

□

We have, hence, shown that the previously defined matrix $P^{\triangle}$ is indeed a stochastic matrix from which we can define a Markov chain on $\mathfrak{L}_k$. It turns out that this Markov chain has a one to one correspondence with the generalized exclusion process defined at the beginning of this section.

Theorem 7.3. *Let $L=(V,E)$ be $\bar{d}$-regular graph with $|V|=\bar{n}$ and $k\in\{1,\dots,\bar{n}-1\}$. The transition matrix of the Markov chain $\mathfrak{S}_k$ associated to $\hat{\eta}_k$ is given by $P^{\triangle}_k$.*

Proof of Theorem 7.3. The structure of the transition matrix follows directly from considering the particle system on L and using the fact that all random variables are uniformly distributed over their respective discrete state spaces. This change of perspective is justified by Theorem 7.1.
Let $\mathfrak{v},\mathfrak{w}\in\mathfrak{V}_k$. Then the transition from $\mathfrak{v}$ to $\mathfrak{w}$ is only possible if $\mathfrak{v}\triangle\mathfrak{w}=\{v,w\}$ and $\langle v,w\rangle\in E$. Hence, $p_{k;\mathfrak{v},\mathfrak{w}}=0$ if this is not the case. Denote for $v\in\mathfrak{v}$ by $N^L_{\mathfrak{v},v}$ set of all neighbors of v in L not being elements of $\mathfrak{v}$. Then $|N^L_{\mathfrak{v},v}|=\bar{d}-\deg^{L_{\mathfrak{v}}}(v)$. Possible transitions are drawn from the set $N^L_{\mathfrak{v},v}\cup\{v\}$. Assume there is at least one edge $\langle v,w\rangle$ in L for a given v. The chain $\mathfrak{S}_k$ may only transition from $\mathfrak{v}$ to $\mathfrak{w}$ if $\mathfrak{v}\triangle\mathfrak{w}=\{v,w\}$ and $\langle v,w\rangle\in E$. Consequently, to transition from $\mathfrak{v}$ to $\mathfrak{w}$ draw first v with probability k^{-1} and then w among all neighbors of v in L, which are not included in $\mathfrak{v}$. Since drawing is done uniformly the probability to draw w is $(\bar{d}-\deg^{L_{\mathfrak{v}}}(v)+1)^{-1}$. Hence for $\mathfrak{v},\mathfrak{w}\in\mathfrak{V}_k$ with $\mathfrak{v}\triangle\mathfrak{w}=\{v,w\}$ and $\langle v,w\rangle\in E$ we obtain

$$p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = k^{-1}(\bar{d}-\deg^{L_{\mathfrak{v}}}(v)+1)^{-1}. \tag{7.2}$$

Finally if $\mathfrak{v}=\mathfrak{w}$ we are looking for the probability to stay in the same state. This happens when drawing a $v\in\mathfrak{v}$ and then again v from the set $N^L_{\mathfrak{v},v}\cup\{v\}$. This happens consequently with probability

$$p^{\triangle}_{k;\mathfrak{v},\mathfrak{v}} = \sum_{i=1}^{k}\frac{1}{k}\frac{1}{\bar{d}-\deg^{L_{\mathfrak{v}}}(v)+1}.$$

□

7.3.1 Dynamic perspective on the transition probabilities

In fact, the transition probabilities can be understood as follows. Given a certain configuration of particles $\mathfrak{v} = \{v_1, \ldots, v_k\}$, construct the following set of graphs. For any $i \in \{1, \ldots, k\}$ define $L_{\{v_i\},\mathfrak{v}^c} = (\{v_i\} \sqcup \mathfrak{v}^c, E_{\{v_i\},\mathfrak{v}^c})$ with $\langle v_i, w\rangle \in E_{\{v_i\},\mathfrak{v}^c}$ if and only if $w \in \mathfrak{v}^c$ or $w = v_i$. Then, at every time $t \in \mathbb{N}$ a transition occurs by first uniformly drawing a graph $\{L_{\{v_i\},\mathfrak{v}^c} | i = 1, \ldots, k\}$ and then uniformly an edge $e \in E_{\{v_i\},\mathfrak{v}^c}$. We visualize this procedure in Figure 36. In comparison with the constructions described

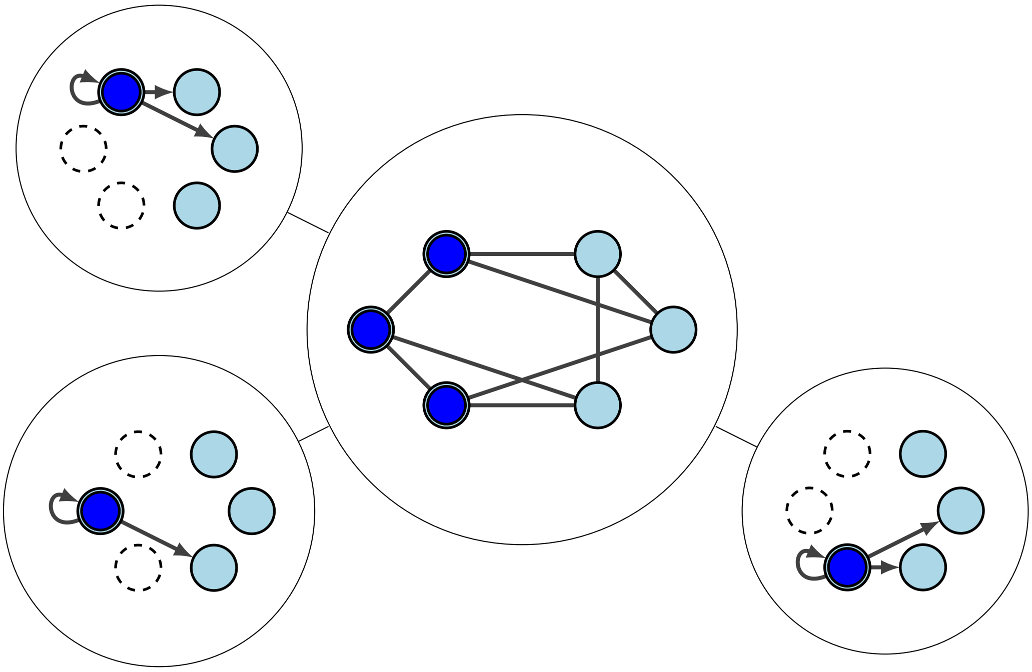

Figure 36: For an underlying 3-regular graph we apply the construction discribed in this subsection. The set of graphs $\{L_{\{v_i\},\mathfrak{v}^c} | i = 1, 2, 3\}$ constructed from occupied sites $\mathfrak{v}$ is displayed in the three outer bubbles. In each case one particular particle is allowed to move along the indicated directed edges. The corresponding bubble is chosen uniformly out of the three possible ones.

in Subsection 5.5 and underlined by Figures 31 and 32 we see that the proposed exclusion process has a stronger local dependence in its behavior than the classical ones. In particular, the transition probabilities in the classical settings do not depend directly on the situation of each particle but rather on the configuration of edges among all particles once the graphs B_t or B'_t are constructed. While still being challenging in many aspects, the combinatorial obstacles are greatly reduced by this property as we will analyze in more detail in Subsection 7.3.4. But first, we are going to develop some results for the Markov chain $\mathfrak{S}_k$ to obtain a clearer picture on its behavior and the implications of the local dependence structure.

7.3.2 Irreducibility, aperiodicity and ergodicity

In this section we are going to establish basic properties of $\mathfrak{S}_k$ independent of the graph structure as long as L is a connected simple graph. We base the results on the properties of $\mathfrak{L}_k$ proven in Section 4.

Theorem 7.4. *The Markov chain $\mathfrak{S}_k$ on $\mathfrak{L}_k$ is irreducible and aperiodic.*

Proof. Let $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ such that $p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} > 0$. Then also $p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}} > 0$ even though equality is not necessarily satisfied. Furthermore every $\langle \mathfrak{w}, \mathfrak{v} \rangle \in \mathfrak{E}_k$ implies that $p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} > 0$. Since $\mathfrak{L}_k$ is connected by Proposition 4.2 we obtain that the chain is irreducible.
The aperiodicity of $\mathfrak{S}_k$ follows directly from the fact that we can construct a state $\mathfrak{v} \in \mathfrak{V}_k$ such that $p^{\triangle}_{k;\mathfrak{v},\mathfrak{v}} > 0$. □

By Theorem 5.10 on Markov chains on finite state spaces, we can conclude from Theorem 7.4 that $\mathfrak{S}_k$ is ergodic, i.e., there is a stationary distribution and the chain converges independently of its initial distribution to said stationary distribution.

Theorem 7.5. *Let $L = (V, E)$ be a $\bar{d}$-regular graph, $\bar{n} := |V|$ and $k \in \{1, \ldots, \bar{n} - 1\}$. Let $\mathfrak{L}_k = (\mathfrak{V}_k, \mathfrak{E}_k)$ be defined as in Definition 4.1. Then there exists a distribution $\tilde{\pi}_k$ on $\mathfrak{V}_k$ such that the Markov chain $\mathfrak{S}_k$ converges in distribution to $\tilde{\pi}_k$ independently of its initial distribution ν_0.*

Having found an answer to this classical question we can wonder about the structure of $\tilde{\pi}_k$ as a function of L and other properties of $\mathfrak{S}_k$ which govern the long-time behavior, like reversibility. While the classical exclusion process is always reversible, it turns out the choice exclusion process implies a structure on the transition matrix of $\mathfrak{S}_k$ which is only reversible in rare cases. The Echo Chamber Model, therefore, gives rise to a qualitatively different kind of exclusion process as has been investigated before.

7.3.3 Lumpability and isomorph bipartite sub-graphs

In this subsection we are going to develop the idea of lumping the state space in detail, as defined in Definition 5.13. Due to the sensitivity of the Markov chain $\mathfrak{S}_k$ with respect to the number of particles as well as the geometry of the underlying graph, we have to apply sensitive tools based on as much information as possible from each sub-graph induced by a set of particles $\mathfrak{v} \in \mathfrak{V}_k$. It turns out that the bipartite sub-graphs $L_{\mathfrak{v},\mathfrak{v}^c}$ play an essential role. In fact, they provide a stronger form of lumpability for $\mathfrak{S}_k$ when lumping with respect to the degree of $\mathfrak{v}$ in $\mathfrak{L}_k$ is not possible, i.e., in almost all non-reversible cases.

Proposition 7.6. *Let $k \in \{1, \ldots, \bar{n}-1\}$ and let $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$. Consider the by decreasing size ordered vectors $(p^{\triangle}_{k;\mathfrak{v},\mathfrak{u}})_{\mathfrak{u}\in\mathfrak{V}_k}$, $(p^{\triangle}_{k;\mathfrak{w},\mathfrak{u}})_{\mathfrak{u}\in\mathfrak{V}_k}$, $(p^{\triangle}_{k;\mathfrak{u},\mathfrak{v}})_{\mathfrak{u}\in\mathfrak{V}_k}$ and $(p^{\triangle}_{k;\mathfrak{u},\mathfrak{w}})_{\mathfrak{u}\in\mathfrak{V}_k}$. Then, $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$ if and only if these vectors satisfy the identities*

$$(p^{\triangle}_{k;\mathfrak{v},\mathfrak{u}})_{\mathfrak{u}\in\mathfrak{V}_k} = (p^{\triangle}_{k;\mathfrak{w},\mathfrak{u}})_{\mathfrak{u}\in\mathfrak{V}_k}$$
$$(p^{\triangle}_{k;\mathfrak{u},\mathfrak{v}})_{\mathfrak{u}\in\mathfrak{V}_k} = (p^{\triangle}_{k;\mathfrak{u},\mathfrak{w}})_{\mathfrak{u}\in\mathfrak{V}_k}.$$

Proof. The claim follows since the bipartite graphs are isomorphic and, hence, any transition from or to $\mathfrak{v}$, defined by an edge from $\mathfrak{v}$ to $\mathfrak{v}^c$ can be translated to a transition from $\mathfrak{w}$ to $\mathfrak{w}^c$. In particular, assuming that $v \in \mathfrak{v}$ makes the transition, the number of neighbors is given by $\bar{d} - \deg^{L_\mathfrak{v}}(v)$ again using the isomorphism between $L_{\mathfrak{v},\mathfrak{v}^c}$ and $L_{\mathfrak{w},\mathfrak{w}^c}$ there is a unique $w \in \mathfrak{w}$ with $\bar{d} - \deg^{L_\mathfrak{v}}(v)$ neighbors. Consequently, for any transition probability $p^{\triangle}_{k;\mathfrak{v},\mathfrak{u}}$ we can construct a transition from $\mathfrak{w}$ to some neighbor such that $p^{\triangle}_{k;\mathfrak{v},\mathfrak{u}} = p^{\triangle}_{k;\mathfrak{v},\mathfrak{u}}$. Therefore, we obtain $(p^{\triangle}_{k;\mathfrak{v},\mathfrak{u}})_{\mathfrak{u}\in\mathfrak{V}_k} = (p^{\triangle}_{k;\mathfrak{w},\mathfrak{u}})_{\mathfrak{u}\in\mathfrak{V}_k}$.

Since $L_{\mathfrak{v},\mathfrak{v}^c}$ defines the whole neighborhood of $\mathfrak{v}$ in $\mathfrak{L}_k$ the converse for $(p^{\triangle}_{k;\mathfrak{u},\mathfrak{v}})_{\mathfrak{u}\in\mathfrak{V}_k} = (p^{\triangle}_{k;\mathfrak{u},\mathfrak{w}})_{\mathfrak{u}\in\mathfrak{V}_k}$ is also true.

The inverse direction is true since the transition probabilities $p^{\triangle}_{k;\mathfrak{v},\mathfrak{u}}$ uniquely define the number of particles in the neighborhood of any particle $v \in \mathfrak{v}$ and the inverse direction $p^{\triangle}_{k;\mathfrak{u},\mathfrak{v}}$ defines all neighbors of $\mathfrak{v}$ which can be created by displacing a single particle v. Consequently, the whole vectors $(p^{\triangle}_{k;\mathfrak{v},\mathfrak{u}})_{\mathfrak{u}\in\mathfrak{V}_k}$ and $(p^{\triangle}_{k;\mathfrak{u},\mathfrak{v}})_{\mathfrak{u}\in\mathfrak{V}_k}$ define all possible transitions and, hence, the bipartite graph $L_{\mathfrak{v},\mathfrak{v}^c}$. Consequently, if the vectors coincide for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$, then, we obtain $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$. □

Hence, two configurations $\mathfrak{v}, \mathfrak{w}$ which satisfy $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$ are seen as identical when it comes to transitions for the Markov chain $\mathfrak{S}_k$. Additionally, by defining the equivalence relation $\mathfrak{v} \sim \mathfrak{w}$ if and only if $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$ and using the statement as well as the proof of Proposition 7.6 we find that for a fixed $\mathfrak{v} \in \mathfrak{V}_k$ and $\mathfrak{w} \in \mathfrak{V}_k$ such that $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$ we have the identity $|\{\mathfrak{u} \in \mathfrak{V}_k | \mathfrak{u} \sim \mathfrak{v},\ \langle \mathfrak{u}, \mathfrak{v}\rangle \in \mathfrak{E}_k\}| = |\{\mathfrak{u}' \in \mathfrak{V}_k | \mathfrak{u}' \sim \mathfrak{w},\ \langle \mathfrak{u}', \mathfrak{w}\rangle \in \mathfrak{E}_k\}|$ and there is a bijection $\Phi : \{\mathfrak{u} \in \mathfrak{V}_k | \mathfrak{u} \sim \mathfrak{v},\ \langle \mathfrak{u}, \mathfrak{v}\rangle \in \mathfrak{E}_k\} \to \{\mathfrak{u}' \in \mathfrak{V}_k | \mathfrak{u}' \sim \mathfrak{w},\ \langle \mathfrak{u}', \mathfrak{w}\rangle \in \mathfrak{E}_k\}$ such that $p^{\triangle}_{k;\mathfrak{v},\mathfrak{u}} = p^{\triangle}_{k;\mathfrak{w},\Phi(\mathfrak{u})}$. This leads us to the realization that $\sum_{\mathfrak{u}\sim\mathfrak{v}} p^{\triangle}_{k;\mathfrak{v},\mathfrak{u}} = \sum_{\mathfrak{u}'\sim\mathfrak{w}} p^{\triangle}_{k;\mathfrak{w},\mathfrak{u}'}$ which forms the basis for the following central result.

Theorem 7.7. *Let $L = (V, E)$ be a $\bar{d}$-regular graph with $\bar{n} = |V|$ and $k \in \{1, \ldots, \bar{n}-1\}$. Denote by $\mathfrak{L}_k = (\mathfrak{V}_k, \mathfrak{E}_k)$ the k-particle graph and define for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ the equivalence relation $\mathfrak{v} \sim \mathfrak{w}$ if and only if $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$. For fixed $\mathfrak{v}$ define $[\mathfrak{v}_i] := \{\mathfrak{u} \in \mathfrak{V}_k | \mathfrak{u} \sim \mathfrak{v}_i\}$ the equivalence class of $\mathfrak{v}_i$ and denote by l the number of distinct equivalence classes. Then, the Markov chain $\mathfrak{S}_k$ is strongly lumpable with respect to the partition $\{[\mathfrak{v}_1], \ldots, [\mathfrak{v}_l]\}$.*

Proof. We have to show that for any $\mathfrak{w}_1, \mathfrak{w}_2 \in [\mathfrak{v}_i]$

$$\sum_{\mathfrak{u} \in [\mathfrak{v}_j]} p^{\triangle}_{k;\mathfrak{w}_1,\mathfrak{u}} = \sum_{\mathfrak{u} \in \mathfrak{u} \in [\mathfrak{v}_j]} p^{\triangle}_{k;\mathfrak{w}_2,\mathfrak{u}}.$$

The equality follows directly by Proposition 7.6 and the following observation. For $\mathfrak{w}_1, \mathfrak{w}_2 \in [\mathfrak{v}_i]$ and $\mathfrak{u}_1 \in [\mathfrak{v}_j]$ with $\langle \mathfrak{w}_1, \mathfrak{u}_1 \rangle \in \mathfrak{E}_k$ we have $\mathfrak{w}_1 \triangle \mathfrak{u}_1 = \{w, u\}$. Let Φ be the isomorphism between $L_{\mathfrak{w}_1, \mathfrak{w}_1^c}$ and $L_{\mathfrak{w}_2, \mathfrak{w}_2^c}$. There is a unique edge in $L_{\mathfrak{w}_2, \mathfrak{w}_2^c}$, namely $\langle \Phi(w), \Phi(u) \rangle$, $\deg^{L_{\mathfrak{w}_1}}(w) = \deg^{L_{\mathfrak{w}_2}}(\Phi(w))$ and for $\mathfrak{u}_2 := (\{\Phi(w_1) | w_1 \in \mathfrak{w}_1\} \backslash \Phi(w)) \cup \{\Phi(u)\}$ we have $\mathfrak{u}_1 \sim \mathfrak{u}_2$ by construction. Consequently, for any summand in the first sum, we find a corresponding summand in the second sum and vice versa such that they are equal. □

Lumpability gives rise to the possibility to consider the Markov chain on a smaller state space and in an aggregated form. While, usually, this makes it impossible to make local statements for example on the specific value of $\tilde{\pi}_k(\mathfrak{v})$ for some $\mathfrak{v} \in \mathfrak{V}_k$ about $\mathfrak{S}_k$ based on the lumped chain, the stronger form of lumpability implied by Proposition 7.6 allows to formulate further deductions. A central one is the structure of the stationary distribution $\tilde{\pi}_k$ of $\mathfrak{S}_k$.

Theorem 7.8. *Let $L = (V, E)$ be a $\bar{d}$-regular graph with $\bar{n} = |V|$ and $k \in \{1, \ldots, \bar{n} - 1\}$. Denote by $\tilde{\pi}_k$ the stationary distribution of $\mathfrak{S}_k$. Then, for all equivalence classes $[\mathfrak{v}]$ under the equivalence relation $\sim$ defined in Theorem 7.7 all $\mathfrak{v}, \mathfrak{w} \in [\mathfrak{v}]$ satisfy the identity $\tilde{\pi}_k(\mathfrak{v}) = \tilde{\pi}_k(\mathfrak{w})$.*

We are going to present, first, an intuitive approach, which is based on the idea that by Proposition 7.6 two equivalent vertices $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ are "identical" when it comes to their respective probability in- and out-"flow" and the approach that the equality of the average $|[\bar{\mathfrak{v}}]|^{-1} \tilde{\pi}_k([\bar{\mathfrak{v}}])$ and $\tilde{\pi}_k(\mathfrak{w})$ for an arbitrary $\mathfrak{w} \in [\bar{\mathfrak{v}}]$ gives the claim we are looking for. The following calculations are only correct under the assumption that for any $\mathfrak{v}, \mathfrak{w} \in [\bar{\mathfrak{v}}]$ we have for any equivalence class $[\bar{\mathfrak{u}}]$ the identity

$$\sum_{\mathfrak{u} \in [\bar{\mathfrak{u}}]} \tilde{\pi}_k(\mathfrak{u}) p^{\triangle}_{k;\mathfrak{u},\mathfrak{v}} = \sum_{\mathfrak{u} \in [\bar{\mathfrak{u}}]} \tilde{\pi}_k(\mathfrak{u}) p^{\triangle}_{k;\mathfrak{u},\mathfrak{w}}. \tag{7.3}$$

While there are $\mathfrak{u}, \mathfrak{u}' \in [\bar{\mathfrak{u}}]$ such that the equality $p^{\triangle}_{k;\mathfrak{u},\mathfrak{v}} = p^{\triangle}_{k;\mathfrak{u}',\mathfrak{w}}$ is satisfied we cannot make a similar claim about $\tilde{\pi}_k(\mathfrak{u}) p^{\triangle}_{k;\mathfrak{u},\mathfrak{v}}$ and $\tilde{\pi}_k(\mathfrak{u}') p^{\triangle}_{k;\mathfrak{u}',\mathfrak{w}}$. This leads to a recursive statement where the truth of the claim for one equivalence class $[\bar{\mathfrak{v}}]$ depends on the truth of the claim for all neighboring equivalence classes $[\bar{\mathfrak{u}}]$. Assuming, then, 7.3 is satisfied we obtain for a fixed $\mathfrak{w} \in [\bar{\mathfrak{v}}]$

$$\begin{aligned}\tilde{\pi}_k([\bar{\mathfrak{v}}]) &= \sum_{[\bar{\mathfrak{u}}]\in \mathfrak{V}_k/\sim} \tilde{\pi}_k([\bar{\mathfrak{u}}])\hat{p}_{[\bar{\mathfrak{u}}],[\bar{\mathfrak{v}}]} = \sum_{[\bar{\mathfrak{u}}]\in \mathfrak{V}_k/\sim} \sum_{\mathfrak{u}\in[\bar{\mathfrak{u}}]} \tilde{\pi}_k(\mathfrak{u}) \sum_{\mathfrak{v}\in[\bar{\mathfrak{v}}]} p^{\triangle}_{k,\mathfrak{u},\mathfrak{v}} \\ &= \sum_{[\bar{\mathfrak{u}}]\in \mathfrak{V}_k/\sim} \sum_{\mathfrak{u}\in[\bar{\mathfrak{u}}]} \sum_{\mathfrak{v}\in[\bar{\mathfrak{v}}]} \tilde{\pi}_k(\mathfrak{u}) p^{\triangle}_{k,\mathfrak{u},\mathfrak{v}} = \sum_{[\bar{\mathfrak{u}}]\in \mathfrak{V}_k/\sim} \sum_{\mathfrak{v}\in[\bar{\mathfrak{v}}]} \sum_{\mathfrak{u}\in[\bar{\mathfrak{u}}]} \tilde{\pi}_k(\mathfrak{u}) p^{\triangle}_{k,\mathfrak{u},\mathfrak{v}} \\ &\overset{7.3}{=} \sum_{[\bar{\mathfrak{u}}]\in \mathfrak{V}_k/\sim} \sum_{\mathfrak{v}\in[\bar{\mathfrak{v}}]} \sum_{\mathfrak{u}\in[\bar{\mathfrak{u}}]} \tilde{\pi}_k(\mathfrak{u}) p^{\triangle}_{k,\mathfrak{u},\mathfrak{w}} = |[\bar{\mathfrak{v}}]| \sum_{\mathfrak{u}\in \mathfrak{V}_k} \tilde{\pi}_k(\mathfrak{u}) p^{\triangle}_{k,\mathfrak{u},\mathfrak{w}} = |[\bar{\mathfrak{v}}]| \tilde{\pi}_k(\mathfrak{w}).\end{aligned}$$

This would lead to the claim of Theorem 7.8 but the problem now lies in proving 7.3. This poses, again, a recursive problem in terms of the neighborhoods of each $\mathfrak{v} \in [\bar{\mathfrak{v}}]$ which is due to the missing information on the local structure of $\mathfrak{L}_k$ inaccessible. Indeed, we need a global perspective on $\mathfrak{L}_k$, which is based on cycles and their identification for equivalent vertices $\mathfrak{v}, \mathfrak{w}$.

Lemma 7.9. *Let $L = (V, E)$ be a $\bar{d}$-regular graph and $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ and $\mathfrak{v}, \mathfrak{w} \in [\bar{\mathfrak{v}}]$ for some $[\bar{\mathfrak{v}}] \in \mathfrak{V}_k/\sim$. Let $l \in \mathbb{N}$. Then for any cycle of length l from $\mathfrak{v}$ to $\mathfrak{v}$ there is a cycle from $\mathfrak{w}$ to $\mathfrak{w}$ of length l. Additionally, the number of distinct cycles from $\mathfrak{v}$ to $\mathfrak{v}$ equals the number of distinct cycles from $\mathfrak{w}$ to $\mathfrak{w}$.*

Proof. The proof works constructively, based on the fact that for any neighbor $\mathfrak{u}$ of $\mathfrak{v}$ we can find a neighbor $\hat{\mathfrak{u}}$ of $\mathfrak{w}$ such that $L_{\mathfrak{u},\mathfrak{u}^c} \cong L_{\hat{\mathfrak{u}},\hat{\mathfrak{u}}^c}$. This allows us to iteratively construct a cycle from $\mathfrak{w}$ to $\mathfrak{w}$ based on a cycle from $\mathfrak{v}$ to $\mathfrak{v}$.

For $l \in \mathbb{N}$ denote by $\phi_{\mathfrak{v}}$ a cycle from $\mathfrak{v}$ to $\mathfrak{v}$. As mentioned in the introduction, we find to $\phi_{\mathfrak{v}}(2)$ a neighbor $\mathfrak{u}$ of $\mathfrak{w}$ such that $L_{\phi_{\mathfrak{v}}(2),\phi_{\mathfrak{v}}(2)^c} \cong L_{\mathfrak{u},\mathfrak{u}^c}$ by replacing $w \in \mathfrak{w}$ by $u \in \mathfrak{w}^c$ where w is the image of $v \in \mathfrak{v}$ and u the image of $v' \in \mathfrak{v}^c$ under the isomorphism given by $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$ and $\mathfrak{v}\triangle\phi_{\mathfrak{v}}(2) = \{v, v'\}$. We call $\phi_{\mathfrak{w}}(1) := \mathfrak{w}$ and $\phi_{\mathfrak{w}}(2) := \mathfrak{u}$. Restarting the same procedure for $\phi_{\mathfrak{v}}(2)$ and $\phi_{\mathfrak{v}}(3)$ in combination with $\phi_{\mathfrak{w}}(2)$ we obtain iteratively a cycle from $\mathfrak{w}$ to $\mathfrak{w}$ and this cycle has length l since, otherwise, there is an index $l' < l$ such that $\phi_{\mathfrak{w}}(l') = \mathfrak{w}$ and applying the previous construction to $\phi_{\mathfrak{w}}(l'-1)$ leads to $\phi_{\mathfrak{v}}(l') = \mathfrak{v}$ which is a contradiction.

By the previous construction, there are at least as many cycles from $\mathfrak{w}$ to $\mathfrak{w}$ as are from $\mathfrak{v}$ to $\mathfrak{v}$. But by exchanging the roles of $\mathfrak{v}$ and $\mathfrak{w}$, which is possible due to the symmetry of the equivalence relation, we obtain also the inverse sense of the inequality such that there are as many cycles from $\mathfrak{w}$ to $\mathfrak{w}$ as are from $\mathfrak{v}$ to $\mathfrak{v}$. □

Based on this geometric property of $\mathfrak{L}_k$ we can analyze the stationary distribution of $\mathfrak{S}_k$ as discussed in Theorem 7.8 using first return times.

Proof of Theorem 7.8. Let $\mathfrak{v}, \mathfrak{w} \in [\mathfrak{v}]$ and consider the expected first return time $\mathbb{E}_{\mathfrak{v}}[T_{\mathfrak{v}}]$.

Then, by definition

$$\mathbb{E}_{\mathfrak{v}}[T_{\mathfrak{v}}] = \sum_{l=1}^{\infty} l\mathbb{P}[T_{\mathfrak{v}} = l | \mathfrak{S}_{k;0} = \mathfrak{v}]$$

and

$$\mathbb{P}[T_{\mathfrak{v}} = l | \mathfrak{S}_{k;0} = \mathfrak{v}] = \sum_{\substack{\phi_{\mathfrak{v}} \text{ cycle}, |\phi_{\mathfrak{v}}|=l \\ \phi_{\mathfrak{v}}(1)=\mathfrak{v}=\phi_{\mathfrak{v}}(l)}} p^{\triangle}_{k;\phi_{\mathfrak{v}}(1),\phi_{\mathfrak{v}}(2)} \cdot \ldots \cdot p^{\triangle}_{k;\phi_{\mathfrak{v}}(l-1),\phi_{\mathfrak{v}}(l)}.$$

Using the construction from the proof of Lemma 7.9 and the property stated in Proposition 7.6 iteratively along the constructed path $\phi_{\mathfrak{v}}$ we obtain that for any cycle $\phi_{\mathfrak{v}}$ the associated cycle $\phi_{\mathfrak{w}}$ gives rise to the equality

$$p^{\triangle}_{k;\phi_{\mathfrak{v}}(1),\phi_{\mathfrak{v}}(2)} \cdot \ldots \cdot p^{\triangle}_{k;\phi_{\mathfrak{v}}(l-1),\phi_{\mathfrak{v}}(l)} = p^{\triangle}_{k;\phi_{\mathfrak{w}}(1),\phi_{\mathfrak{w}}(2)} \cdot \ldots \cdot p^{\triangle}_{k;\phi_{\mathfrak{w}}(l-1),\phi_{\mathfrak{w}}(l)}$$

and, moreover,

$$\begin{aligned}
\mathbb{P}[T_{\mathfrak{v}} = l | \mathfrak{S}_{k;0} = \mathfrak{v}] &= \sum_{\substack{\phi_{\mathfrak{v}} \text{ cycle}, |\phi_{\mathfrak{v}}|=l \\ \phi_{\mathfrak{v}}(1)=\mathfrak{v}=\phi_{\mathfrak{v}}(l)}} p^{\triangle}_{k;\phi_{\mathfrak{v}}(1),\phi_{\mathfrak{v}}(2)} \cdot \ldots \cdot p^{\triangle}_{k;\phi_{\mathfrak{v}}(l-1),\phi_{\mathfrak{v}}(l)} \\
&= \sum_{\substack{\phi_{\mathfrak{w}} \text{ cycle}, |\phi_{\mathfrak{w}}|=l \\ \phi_{\mathfrak{w}}(1)=\mathfrak{w}=\phi_{\mathfrak{w}}(l)}} p^{\triangle}_{k;\phi_{\mathfrak{w}}(1),\phi_{\mathfrak{w}}(2)} \cdot \ldots \cdot p^{\triangle}_{k;\phi_{\mathfrak{w}}(l-1),\phi_{\mathfrak{w}}(l)} \\
&= \mathbb{P}[T_{\mathfrak{w}} = l | \mathfrak{S}_{k;0} = \mathfrak{w}].
\end{aligned}$$

Consequently, we can conclude

$$\mathbb{E}_{\mathfrak{v}}[T_{\mathfrak{v}}] = \sum_{l=1}^{\infty} l\mathbb{P}[T_{\mathfrak{v}} = l | \mathfrak{S}_{k;0} = \mathfrak{v}] = \sum_{l=1}^{\infty} l\mathbb{P}[T_{\mathfrak{w}} = l | \mathfrak{S}_{k;0} = \mathfrak{w}] = \mathbb{E}_{\mathfrak{w}}[T_{\mathfrak{w}}] \tag{7.4}$$

and by $\tilde{\pi}_k(\mathfrak{v}) = \mathbb{E}_{\mathfrak{v}}[T_{\mathfrak{v}}]^{-1}$ we obtain the claim. □

From Theorem 7.8 we can conclude that it is in fact not the internal structure of a configuration, which is defining for the values of the stationary distribution but their connection with the remainder of the graph. In particular, this separates the vertex set of $\mathfrak{L}_k$ more finely than can be done simply by the degree of each vertex. Since the level sets, defined with respect to the stationary distribution, represent the elements on which $\mathfrak{S}_k$ acts interchangeably, we can conclude that the Markov chain, which we defined, with its intricate transition probabilities, sees the difference between a configuration and its environment and not only the internal structure of the configuration. This is a standing theory in systems theory, a sub-field of the social sciences. For example, the work by Niklas Luhmann, see for example [Luh84] or [Luh98] for more details, speaks exactly of this property in social systems. Luhmann focuses on communication and, therefore, communication paths, i.e., relationships, as the central property which forms a social system. He proposed in his work a paradigm shift from the consideration of a constituent as part of a whole to the difference between the whole system and its environment as the defining feature of a social system. We find that our process exhibits exactly this property. Two configurations are different if their respective differences with their respective environments are not congruent. This underlines that our choice for the process' transition probabilities are not simply arbitrary but lead in fact to results which are coherent with considerations in the social sciences, by which the model is motivated.

After a short excursion into the social sciences and systems theory, we return to the mathematical properties of $\mathfrak{S}_k$. Indeed, the technique used in the proof of Theorem 7.8 can be extended to characterizing hitting times of $\mathfrak{S}_k$ in $\mathfrak{L}_k$.

Proposition 7.10. *Let* $L = (V, E)$ *be a* $\bar{d}$*-regular graph and* $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ *and* $\mathfrak{v}, \mathfrak{w} \in [\bar{\mathfrak{v}}]$ *for some* $[\bar{\mathfrak{v}}] \in \mathfrak{V}_k/\sim$*. Then, for any* $\mathfrak{u} \in \mathfrak{V}_k$ *there is a* $\mathfrak{u}' \in [\mathfrak{u}]$ *such that*

$$\mathbb{E}_{\mathfrak{v}}[T_{\mathfrak{u}}] = \mathbb{E}_{\mathfrak{w}}[T_{\mathfrak{u}'}]. \tag{7.5}$$

Proof. The proof follows since the constructions made in the previous two proofs may be extended to paths instead of cycles. To this end, take path ϕ as in the proof of Lemma 7.9 from $\mathfrak{v}$ to $\mathfrak{u}$ and construct a corresponding path ϕ iteratively from $\mathfrak{w}$ to some $\mathfrak{u}' \in \mathfrak{V}_k$. Indeed, one can observe that $\mathfrak{u} \sim \mathfrak{u}'$ since $\phi(0) = \mathfrak{v} \sim \mathfrak{w} = \phi'(0)$ and by the same argument as in the proof of Lemma 7.9 we obtain $\phi(1) \sim \phi'(1)$ and, inductively, $\phi(i) \sim \phi'(i)$ for any $i \in \{1, \ldots, |\phi|\}$, which implies $\mathfrak{u} \sim \mathfrak{u}'$. By the same argument as in the proof of Theorem 7.8, writing the expected value as sum of transition probabilities along paths an Proposition 7.6 we obtain the claim. □

Since further properties of hitting times can be found when the Markov chain is reversible, as for example discussed at length in [AldFi02], therefore, on the form of the stationary distribution, we now turn to the analysis of the reversible cases of $\mathfrak{S}_k$.

7.3.4 Stationary distribution and reversibility

Being most interested in the long time behavior of $\mathfrak{S}_k$, we focus now on its limiting behavior for $t \to \infty$. Evidently, the previous section provides the necessary properties to show that indeed the convergence towards a stationary distribution is satisfied. We proved this, nonetheless, briefly but rigorously recalling the classical results from Markov chain theory in Theorem 7.5. Additionally, we obtained a geometric characterization of the stationary distribution but without the possibility of giving an explicit expression due to the complicated structure implied by k sub-graphs. Under additional assumptions we can, nonetheless, find the desired explicit form and implications on the geometry of the graph L. The identification of the aforementioned stationary distribution in special cases will, subsequently, be our focus. To this end, we try to establish reversibility but fail to do so on a large class of graphs and even show that, in contrast to the classical exclusion process discussed in [DiaSal93], our adapted version is not always reversible. This task will keep us occupied until the end of this section where we draw multiple conclusions and point towards implications for the initially motivating social network model.

Additionally, for an explicit expression of the stationary distribution as well as characterizations of the hitting times, reversibility of $\mathfrak{S}_k$ may give an approach to the analysis. It will turn out, that the reversibility $\mathfrak{S}_k$ depends on the choice of k and $\bar{d}$. We are going to discuss in what follows a range of cases depending on $\bar{d}$ and k for which reversibility can be proven directly. This will, later in Subsection 7.3.6 be the starting point for the analysis of convergence speeds in the reversible case, in particular, in the sense of deriving bounds on the Cheeger constant of the process.

Proposition 7.11. *Let $k \in \{1,2\}$ and recall that $\mathfrak{D}_l^k := \{\mathfrak{v} \in \mathfrak{V}_k | \deg_k^r(\mathfrak{v}) = l\}$. Then $\mathfrak{S}_k$ is reversible. The corresponding stationary distributions are given by*

$$k = 1: \qquad \tilde{\pi}_k(\mathfrak{v}) = \frac{\bar{d}+1}{2|\mathfrak{E}_k| + |\mathfrak{V}_k|}, \tag{7.6}$$

$$k = 2: \qquad \tilde{\pi}_k(\mathfrak{v}) = \begin{cases} \dfrac{\bar{d}+1}{C_2^{\bar{d}}}, & \text{if } \deg_k^r(\mathfrak{v}) = 2\bar{d}, \\ \dfrac{\bar{d}}{C_2^{\bar{d}}}, & \text{if } \deg_k^r(\mathfrak{v}) = 2(\bar{d}-1) \end{cases} \tag{7.7}$$

where $C_2^{\bar{d}} = |\mathfrak{D}_{2\bar{d}}^2|(\bar{d}+1) + |\mathfrak{D}_{2(\bar{d}-1)}^2|\bar{d}$.

Proof. If $k = 1$ the process $\mathfrak{S}_k$ corresponds to a random walk on a regular graph with uniform probability to leave along any edge or stay at the current vertex. The statement follows, hence, by classical Markov chain theory.
In the case $k = 2$ we check that the probability vector $\tilde{\pi}_k$ satisfies the detailed balance equation. To this end, consider first the transition matrix $P^{\triangle}$. If $k = 2$ it takes for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ with $\mathfrak{v} \sim \mathfrak{w}$ the form

$$p^{\triangle}_{2;\mathfrak{v},\mathfrak{w}} = \begin{cases} \dfrac{1}{2(\bar{d}+1)}, & \text{if } \deg_k^r(\mathfrak{v}) = 2\bar{d} \\ \dfrac{1}{2\bar{d}}, & \text{if } \deg_k^r(\mathfrak{v}) = 2(\bar{d}-1). \end{cases}$$

It is easy to check that these cases are exhaustive. Evidently, if $\deg_k^r(\mathfrak{v}) = \deg_k^r(\mathfrak{w})$ detailed balanced equation is satisfied. Assume now $\deg_k^r(\mathfrak{v}) \neq \deg_k^r(\mathfrak{w})$ and without loss of generality $\deg_k^r(\mathfrak{v}) = 2\bar{d}$. Then

$$\tilde{\pi}_k(\mathfrak{v}) p^{\triangle}_{2;\mathfrak{v},\mathfrak{w}} = \frac{\bar{d}+1}{2C_2^{\bar{d}}} \frac{1}{\bar{d}+1} = \frac{1}{2C_2^{\bar{d}}} = \frac{\bar{d}}{2C_2^{\bar{d}}} \frac{1}{\bar{d}} = \tilde{\pi}_k(\mathfrak{w}) p^{\triangle}_{2;\mathfrak{w},\mathfrak{v}}.$$

Hence, $\tilde{\pi}_k$ defines a reversible stationary measure which is unique by ergodicity of $\mathfrak{S}_k$. The term $C_2^{\bar{d}} = |\mathfrak{D}_{2\bar{d}}|(\bar{d}+1) + |\mathfrak{D}_{2(\bar{d}-1)}|\bar{d}$ indeed normalizes $\tilde{\pi}_k$ such that it is, in fact, a stationary distribution. □

Evidently, in this case we can exploit the structure of the set of degrees of $\mathfrak{L}_k$, since there are only two distinct ones. This structure arises also when $k = \bar{n}-2$, which we discuss in what follows.

Proposition 7.12. *Let $k \in \{\bar{n}-1, \bar{n}-2\}$ and recall that $\mathfrak{D}_l^k = \{\mathfrak{v} \in \mathfrak{V}_k | \deg_k^r(\mathfrak{v}) = l\}$. Then $\mathfrak{S}_k$ is reversible. The corresponding stationary distributions are given by*

$$k = \bar{n}-1: \qquad \tilde{\pi}_k(\mathfrak{v}) = \frac{\bar{d}+1}{2|\mathfrak{E}_k| + |\mathfrak{V}_k|}, \tag{7.8}$$

$$k = \bar{n}-2: \qquad \tilde{\pi}_k(\mathfrak{v}) = \begin{cases} \dfrac{3}{C_{\bar{n}-2}^{\bar{d}}}, & \textit{if } \deg_k^r(\mathfrak{v}) = 2\bar{d}, \\ \dfrac{2}{C_{\bar{n}-2}^{\bar{d}}}, & \textit{if } \deg_k^r(\mathfrak{v}) = 2(\bar{d}-1) \end{cases} \tag{7.9}$$

where $C_{\bar{n}-2}^{\bar{d}} := |\mathfrak{D}_{2\bar{d}}^k| 3 + |\mathfrak{D}_{2(\bar{d}-1)}^k| 2$.

Proof. First, recall that $\mathfrak{L}_k \cong \mathfrak{L}_{\bar{n}-k}$ by Proposition 4.4. Hence, we observe that $\mathfrak{L}_{\bar{n}-1}$ is a regular graph with degree $\bar{d}$ and $\mathfrak{S}_k$ a random walk. The formula for the stationary distribution follows by classical theory.

In the case $k = \bar{n}-2$ we distinguish again two cases based on the degrees of the considered vertex $\mathfrak{v}$. For two vertices $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ with $\deg_k^r(\mathfrak{v}) = \deg_k^r(\mathfrak{w})$ the detailed balance equation is satisfied for the given $\tilde{\pi}_k$. Assume now that $\deg_k^r(\mathfrak{v}) \neq \deg_k^r(\mathfrak{w})$ and without loss of generality $\deg_k^r(\mathfrak{v}) = 2\bar{d}$ and $\deg_k^r(\mathfrak{w}) = 2(\bar{d}-1)$. Then

$$p^{\triangle}_{\bar{n}-2;\mathfrak{v},\mathfrak{w}} = \frac{1}{\bar{n}-2} \frac{1}{\bar{d}-(\bar{d}-2)+1} = \frac{1}{3(\bar{n}-2)}$$

and

$$p^{\triangle}_{\bar{n}-2;\mathfrak{w},\mathfrak{v}} = \frac{1}{\bar{n}-2} \frac{1}{\bar{d}-(\bar{d}-1)+1} = \frac{1}{2(\bar{n}-2)}.$$

Hence,

$$\tilde{\pi}_k(\mathfrak{v}) p^{\triangle}_{2;\mathfrak{v},\mathfrak{w}} = \frac{1}{C_{\bar{n}-2}^{\bar{d}}(\bar{n}-2)} = \tilde{\pi}_k(\mathfrak{v}) p^{\triangle}_{\bar{n}-2;\mathfrak{w},\mathfrak{v}}. \tag{7.10}$$

Consequently, $\tilde{\pi}_k$ is a reversible measure and normalized, hence, a reversible distribution and, therefore, the unique stationary distribution. □

Even though the transition probabilities for the cases $k = 2$ and $k = \bar{n} - 2$ are not identical, the structure of $\mathfrak{L}_k$ provides, nonetheless, the symmetry to obtain reversibility in both cases by similar arguments as can be seen in the corresponding proofs. The idea remains to exploit that transition probabilities between states of same degree are identical. Hence, only transitions between states of different degree are to be considered to prove detailed balance.
This idea can be extended to any case where transition probabilities between states of different degree only depend on the degree and not on the local structure of the specific states. This yields the following results.

Proposition 7.13. *Let $k \in \{1, \ldots, \bar{n} - 1\}$ and consider the cycle $\mathcal{C}_{\bar{n}}$. For $\mathfrak{v} \in \mathfrak{V}_k$ define $l_\mathfrak{v} = |\{\deg_k(\mathfrak{w})|\deg_k(\mathfrak{w}) > \deg_k(\mathfrak{v})\}|$ and $d_k = |\{\deg_k(\mathfrak{w})|\mathfrak{w} \in \mathfrak{V}_k\}|$. The stationary distribution $\tilde{\pi}_k$ of $\mathfrak{S}_k$ on $\mathfrak{L}_k$ induced by the k-particle exclusion process on $\mathcal{C}_{\bar{n}}$ is given by*

$$\tilde{\pi}_k(\mathfrak{v}) = \frac{2^{l_\mathfrak{v}} 3^{d_k - l_\mathfrak{v} - 1}}{C_k^2} \tag{7.11}$$

where C_k^2 is a normalization constant. Additionally, $\mathfrak{S}_k$ is reversible.

See Lemma 4.23, Lemma 4.26 and Lemma 4.27 for explicit expressions of d_k and $l_\mathfrak{v}$.

Proof of Proposition 7.13. First of all, note that for $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$ we have $p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} \in \{(2k)^{-1}, (3k)^{-1}\}$ and $p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} \neq p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}}$ if and only if $\deg_k(\mathfrak{v}) = \deg_k(\mathfrak{w}) \pm 2$. Without loss of generality, assume that $p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = (2k)^{-1}$ and $p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}} = (3k)^{-1}$, i.e., $\deg_k(\mathfrak{v}) = \deg_k(\mathfrak{w}) - 2$. Then, $l_\mathfrak{v} = l_\mathfrak{w} - 1$ and, hence, we obtain

$$\tilde{\pi}_k(\mathfrak{v}) p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = \frac{2^{l_\mathfrak{v} - 1} 3^{d_k - l_\mathfrak{v} - 1}}{k C_k^2} = \frac{2^{l_\mathfrak{w}} 3^{d_k - l_\mathfrak{w} - 1}}{3k C_k^2} = \tilde{\pi}_k(\mathfrak{w}) p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}}. \tag{7.12}$$

□

Taking Proposition 7.13 as a departure point one can also wonder about the case $\bar{d} = \bar{n} - 2$. Indeed, the result still remains valid while its proof becomes more involved due to combinatoric complications.

Proposition 7.14. *Let $k \in \{1, \ldots, \bar{n} - 1\}$ and $\bar{d} = \bar{n} - 2$. For $\mathfrak{v} \in \mathfrak{V}_k$ define $l_\mathfrak{v} = |\{\deg_k(\mathfrak{w})|\deg_k(\mathfrak{w}) > \deg_k(\mathfrak{v})\}|$ and $d_k = |\{\deg_k(\mathfrak{w})|\mathfrak{w} \in \mathfrak{V}_k\}|$. The stationary distribution $\tilde{\pi}_k$ of $\mathfrak{S}_k$ on $\mathfrak{L}_k$ induced by the k-particle exclusion process on L is given by*

$$\tilde{\pi}_k(\mathfrak{v}) = \frac{(\bar{d} - k + 2)^{l_\mathfrak{v}} (\bar{d} - k + 3)^{d_k - l_\mathfrak{v} - 1}}{C_k^{\bar{n}-2}} \tag{7.13}$$

where $C_k^{\bar{n}-2}$ is a normalization constant. Additionally, $\mathfrak{S}_k$ is reversible.

Proof of Proposition 7.14. First note that for $\mathfrak{v} \in \mathfrak{V}_k$ with $\deg_k(\mathfrak{v}) = k(\bar{n}-2) - k(k-1) + 2l$ we have

$$l_{\mathfrak{v}} = \left\lfloor \frac{k}{2} \right\rfloor - l. \tag{7.14}$$

Furthermore, since $\bar{d} = \bar{n} - 2$ for any $v \in V$ there is a unique $u_v \in V$ such that $\langle v, u \rangle \notin E$. For $\mathfrak{v} \in \mathfrak{V}_k$ and $v \in \mathfrak{v}$ define the function $e^c_{\mathfrak{v}}$ as

$$e^c_{\mathfrak{v}}(v) = \begin{cases} 1, & u_v \in \mathfrak{v}, \\ 0, & u_v \notin \mathfrak{v}. \end{cases}$$

Then, the degree of $\mathfrak{v} \in \mathfrak{V}_k$ is given by $\deg_k(\mathfrak{v}) = k\bar{d} - k(k-1) + \sum_{v \in \mathfrak{v}} e^c_{\mathfrak{v}}(v)$. Moreover, for $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$ with $\mathfrak{v} \triangle \mathfrak{w} = \{v, w\}$ we have

$$p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = \begin{cases} \dfrac{1}{k} \dfrac{1}{\bar{d} - k + 3}, & e^c_{\mathfrak{v}}(v) = 1 \\ \dfrac{1}{k} \dfrac{1}{\bar{d} - k + 2}, & e^c_{\mathfrak{v}}(v) = 0. \end{cases}$$

Denote the distinct degrees in $\mathfrak{L}_k$ by $d_1 < \ldots < d_l$. If $p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} > 0$, $\deg_k(\mathfrak{v}) = d_i$ and $e^c_{\mathfrak{v}}(v) = 1$ then $\deg_k(\mathfrak{w}) \in \{d_i, d_{i+1}\}$. On the other hand, if $e^c_{\mathfrak{v}}(v) = 0$ then $\deg_k(\mathfrak{w}) \in \{d_i, d_{i-1}\}$. In case of a transition from $\mathfrak{v}$ to $\mathfrak{w}$ the degrees are preserved if and only if $e^c_{\mathfrak{v}}(v) = e^c_{\mathfrak{w}}(w)$. Therefore, the properties $\deg_k(\mathfrak{v}) = \deg_k(\mathfrak{w})$ and $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$ are equivalent to $p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}}$. We now apply the obtained steps to the detailed balance equation. First of all by the preceding discussion and the form of equation 7.13 detailed balanced is satisfied for any $\mathfrak{v}, \mathfrak{w}$ with same degree. Furthermore, we only have to consider vertices with neighboring degrees since, otherwise, the transition probabilities equal zero. Hence, consider $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$ and assume without loss of generality that $\deg_k(\mathfrak{v}) = d_i$ and $\deg_k(\mathfrak{w}) = d_{i+1}$. Then,

$$\begin{aligned} \tilde{\pi}_k(\mathfrak{v}) p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} &= \frac{(\bar{d} - k + 2)^{l_{\mathfrak{v}} - 1} (\bar{d} - k + 3)^{d_k - l_{\mathfrak{v}} - 1}}{k C^{\bar{n}-2}_k} \\ &= \frac{(\bar{d} - k + 2)^{l_{\mathfrak{w}}} (\bar{d} - k + 3)^{d_k - l_{\mathfrak{w}} - 1}}{k (\bar{d} - k + 3) C^{\bar{n}-2}_k} = \tilde{\pi}_k(\mathfrak{w}) p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}}. \end{aligned}$$

□

Indeed, when comparing the results in Propositions 7.12 and 7.13 to one may realize that the stationary distributions only depend on the degree of each state $\mathfrak{v}$ and also the transition probabilities only depend on the degrees. While in Propositions 7.11 and 7.12 their forms seam to differ substantially from the results in Propositions 7.13 and 7.14, they all can in fact be rewritten to obtain a consistent closed form in all proven reversible cases. Define to this end $p_* = \min_{\mathfrak{v},\mathfrak{w}\in\mathfrak{V}_k}\{p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}}\}$, $p^* = \max_{\mathfrak{v},\mathfrak{w}\in\mathfrak{V}_k}\{p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}}\}$ and for $\mathfrak{v} \in \mathfrak{V}_k$ define $l_\mathfrak{v} = |\{\deg_k(\mathfrak{w})|\deg_k(\mathfrak{w}) > \deg_k(\mathfrak{v})\}|$ and $d_k = |\{\deg_k(\mathfrak{w})|\mathfrak{w} \in \mathfrak{V}_k\}|$. The stationary distribution $\tilde{\pi}_k$ takes in the proven reversible cases the form

$$\tilde{\pi}_k(\mathfrak{v}) = \frac{(kp^*)^{l_\mathfrak{v}}(kp_*)^{d_k-l_\mathfrak{v}-1}}{C} \tag{7.15}$$

where C is a normalization constant. While it is aesthetic to recover a closed form of the stationary distribution, which also underlines the consistency of our results in the cases where both $\bar{d} \in \{2, \bar{n}-2\}$ and $k \in \{2, \bar{n}-2\}$, its interpretation based on equation 7.15 is all the while more interesting. We need a preliminary result to proceed in this sense, which explains the behavior of the Markov chain $\mathfrak{S}_k$ while moving between states of same degree.

Lemma 7.15. *Let $k \in \{1, \dots, \bar{n}-1\}$ and consider the Markov chain $\mathfrak{S}_k$ on $\mathfrak{L}_k$. Then, for $\langle\mathfrak{v}, \mathfrak{w}\rangle \in \mathfrak{E}_k$ the assertions $p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}}$ and $\deg_k(\mathfrak{v}) = \deg_k(\mathfrak{w})$ are equivalent.*

Proof. Note that $p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}}$ is equivalent to $\deg^{L_\mathfrak{v}}(v) = \deg^{L_\mathfrak{w}}(w)$ if $\mathfrak{v}\triangle\mathfrak{w} = \{v, w\}$. The equivalency then follows by Proposition **??**. □

A stronger implication can be found based on the bipartite graph $L_{\mathfrak{v},\mathfrak{v}^c}$ which will play an important role in the analysis of the non-reversible cases.

Lemma 7.16. *Let $k \in \{1, \dots, \bar{n}-1\}$ and consider the Markov chain $\mathfrak{S}_k$ on $\mathfrak{L}_k$ as well as $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ with $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$. Then, $\langle\mathfrak{v}, \mathfrak{w}\rangle \in \mathfrak{E}_k$ implies $p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}}$.*

Proof. The proof follows by Lemma 4.15 and Lemma 7.15. □

Hence, transitions within the class of vertices of same degree happen with equal probability and, if possible, increasing the degree of the current state happens with identical probability for all $\mathfrak{v} \in \mathfrak{V}_k$ and so does decreasing the degree, if possible. Hence, the stationary distribution is fully defined by another chain on a reduced graph where states are combined based on their degree in $\mathfrak{L}_k$. Evidently, this renders the problem easier since we can write the stationary distribution as a function of one variable which is the degree of each state. Indeed, this can be extended to more complex graphs if k is arbitrary. We are going to focus on the complete bipartite graph.

Theorem 7.17. *Let $L = ((V_1, V_2), E)$ be a complete bipartite graph with $n_1 = |V_1|$ and $n_2 = |V_2|$. For $\mathfrak{v} \in \mathfrak{V}_k$ we define $\mathfrak{v}_* = \{\mathfrak{u} \in \{\mathfrak{v} \cap V_1, \mathfrak{v} \cap V_2\} \big| |\mathfrak{u}| = \min\{|\mathfrak{v} \cap V_1|, |\mathfrak{v} \cap V_2|\}\}$ and $\mathfrak{v}^*$ analogously replacing* min *with* max. *Furthermore, we define $V_*^{\mathfrak{v}} = \{V' \in \{V_1, V_2\} | \mathfrak{v}_* \subset V'\}$ and analogously $V_{\mathfrak{v}}^*$. Then, the Markov chain $\mathfrak{S}_k$ is reversible and its stationary distribution is given by*

$$\tilde{\pi}_k(\mathfrak{v}) = \prod_{i=1}^{|\mathfrak{v}_*|} (|V_{\mathfrak{v}}^*| - (k-i) + 1) \prod_{j=1}^{|\mathfrak{v}^*|} (|V_*^{\mathfrak{v}}| - (k-j) + 1). \tag{7.16}$$

Proof. Let $\mathfrak{v} \in \mathfrak{V}_k$. First, we discuss the case that $|\mathfrak{v}^*| \geq \left\lceil \frac{k}{2} \right\rceil + 1$. Consider $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k$ such that $|\mathfrak{w}^*| \in \{|\mathfrak{v}^*| \pm 1\}$. Assume without loss of generality that $|\mathfrak{w}^*| = |\mathfrak{v}^*| - 1$. Then

$$p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = \frac{1}{k} \frac{1}{|V_*^{\mathfrak{v}}| - (k - |\mathfrak{v}^*|) + 1}, \; p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}} = \frac{1}{k} \frac{1}{|V_{\mathfrak{w}}^*| - (k - |\mathfrak{w}_*|) + 1}.$$

Therefore, we obtain

$$\begin{aligned}
k\tilde{\pi}_k(\mathfrak{v}) p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} &= \prod_{i=1}^{|\mathfrak{v}_*|} (|V_{\mathfrak{v}}^*| - (k-i) + 1) \prod_{j=1}^{|\mathfrak{v}^*|-1} (|V_*^{\mathfrak{v}}| - (k-j) + 1) \\
&= \frac{1}{|V_{\mathfrak{v}}^*| - (k - (|\mathfrak{v}_*| + 1)) + 1} \prod_{i=1}^{|\mathfrak{v}_*|+1} (|V_{\mathfrak{v}}^*| - (k-i) + 1) \prod_{j=1}^{|\mathfrak{v}^*|-1} (|V_*^{\mathfrak{v}}| - (k-j) + 1) \\
&= \frac{1}{|V_{\mathfrak{w}}^*| - (k - |\mathfrak{w}_*|) + 1} \prod_{i=1}^{|\mathfrak{w}_*|} (|V_{\mathfrak{w}}^*| - (k-i) + 1) \prod_{j=1}^{|\mathfrak{w}^*|} (|V_*^{\mathfrak{w}}| - (k-j) + 1) \\
&= k\tilde{\pi}_k(\mathfrak{w}) p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}}.
\end{aligned}$$

In the case of k even this shows the claim even if $|\mathfrak{v}^*| = \frac{k}{2}$, since then $|\mathfrak{w}^*| \geq \frac{k}{2} + 1$ and we can choose $|V_{\mathfrak{v}}^*| = |V_{\mathfrak{w}}^*|$. Hence, by symmetry of $\mathfrak{v}_*$ and $\mathfrak{v}^*$ we obtain the claim.
We turn to the case of k being odd. Assume $|\mathfrak{v}^*| = \left\lceil \frac{k}{2} \right\rceil$ and $|\mathfrak{v}_*| = \left\lfloor \frac{k}{2} \right\rfloor$ as well as $|\mathfrak{w}^*| = \left\lceil \frac{k}{2} \right\rceil$ and $|\mathfrak{w}_*| = \left\lfloor \frac{k}{2} \right\rfloor$ with $V_{\mathfrak{v}}^* \neq V_{\mathfrak{w}}^*$. Then $|\mathfrak{v}_*| + 1 = |\mathfrak{w}^*|$, $|\mathfrak{v}^*| - 1 = |\mathfrak{w}_*|$, $V_{\mathfrak{v}}^* = V_*^{\mathfrak{w}}$ and $V_*^{\mathfrak{v}} = V_{\mathfrak{w}}^*$. Consequently, we arrive at the conclusion

$$\begin{aligned}
k\tilde{\pi}_k(\mathfrak{v}) p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} &= \prod_{i=1}^{|\mathfrak{v}_*|} (|V_{\mathfrak{v}}^*| - (k-i) + 1) \prod_{j=1}^{|\mathfrak{v}^*|-1} (|V_*^{\mathfrak{v}}| - (k-j) + 1) \\
&= \frac{1}{|V_{\mathfrak{v}}^*| - (k - (|\mathfrak{v}_*| + 1)) + 1} \prod_{i=1}^{|\mathfrak{v}_*|+1} (|V_{\mathfrak{v}}^*| - (k-i) + 1) \prod_{j=1}^{|\mathfrak{v}^*|-1} (|V_*^{\mathfrak{v}}| - (k-j) + 1) \\
&= \frac{1}{|V_*^{\mathfrak{w}}| - (k - |\mathfrak{w}^*|) + 1} \prod_{i=1}^{|\mathfrak{w}^*|} (|V_*^{\mathfrak{w}}| - (k-i) + 1) \prod_{j=1}^{|\mathfrak{w}_*|} (|V_{\mathfrak{w}}^*| - (k-j) + 1) \\
&= k\tilde{\pi}_k(\mathfrak{w}) p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}}
\end{aligned}$$

which completes the proof. □

Regular bipartite graphs might, hence, be a subclass which could be considered separately, but we will see later on in this section, that examples can easily be constructed, on which $\mathfrak{S}_k$ is not reversible for the cases $k \in \{3, \ldots, \bar{n} - 3\}$.

Figure 37 compares for $k = 4$ and the cycle graph $L = C_8$ the two graphs $\mathfrak{L}_k$ and its quotient graph ${}^{\mathfrak{L}_k}\!/\!_{\sim}$ as defined in Definition 3.24 with respect to the equivalence relation $\mathfrak{v} \sim \mathfrak{w}$ if and only if $\deg_k(\mathfrak{v}) = \deg_k(\mathfrak{w})$. We illustrate in Figure 38 the

Figure 37: On the left $\mathfrak{L}_k$ constructed for $k = 4$ based on the cycle graph on 8 vertices. On the right the quotient graph ${}^{\mathfrak{L}_k}\!/\!_{\sim}$.

transition structure of a projected version of $\mathfrak{S}_k$ from $\mathfrak{L}_k$ to ${}^{\mathfrak{L}_k}\!/\!_{\sim}$. Indeed, it turns out

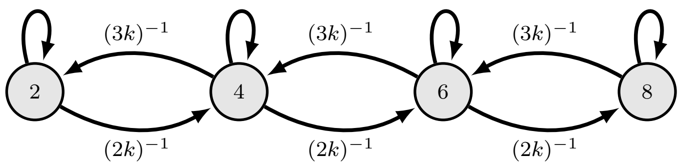

Figure 38: Transitions on the quotient graph of $\mathfrak{L}_4$ based on the cycle graph with 8 vertices. The number on every vertex represents the corresponding degree.

that $\mathfrak{S}_k$ is lumpable with respect to the map $\deg_k : \mathfrak{V}_k \to \mathbb{N}$ in the proven cases of reversibility. In fact, the chain $\mathfrak{S}_k$ is almost always non-reversible when considering as underlying graph a regular graph L.

The preceding results rely heavily on the structure of the graph $\mathfrak{L}_k$ and we will employ Definitions 3.3, 3.11 and 3.13 to prove our results on lack of reversibility in many other cases. While all proofs rely on Kolmogorov's criterion, the details differ and show the dependence on the underlying graph topology.

Theorem 7.18. *Let L be any simple connected $\bar{d}$-regular graph on $\bar{n}$ vertices with $\bar{d} \in \{3, \dots, \bar{n} - 3\}$. Assume that L contains a tri-star $\mathcal{T}$ as defined in Definition 3.3. Then, the Markov chain $\mathfrak{S}_k$ is reversible on $\mathfrak{L}_k$ if and only if $k \in \{1, 2, \bar{n} - 2, \bar{n} - 1\}$.*

Based on the tri-star we want to explain briefly and informally our way of proceeding. It yields a minimal counterexample for reversibility based on Kolmogorovs theorem. Nonetheless, it is not artificial in a sense that the set of graphs which contain such a structure in the sense of Definition 3.3 are rare or have to be constructed. Indeed, drawing upon the social network motivation, a tri-graph can be seen as a cluster of six individuals, three of whom know each other and any of these three also has a friend who is not friends with the remaining two. The particles can then be seen as presents that are given randomly to a friend if that friend does not have a present already. This usually occurs in contexts like Secret Santa or thelike.

Before beginning with the proof we discuss the idea behind the counterexample, represented in Figure 39. Under the transition probabilities shown in Theorem 7.3 the

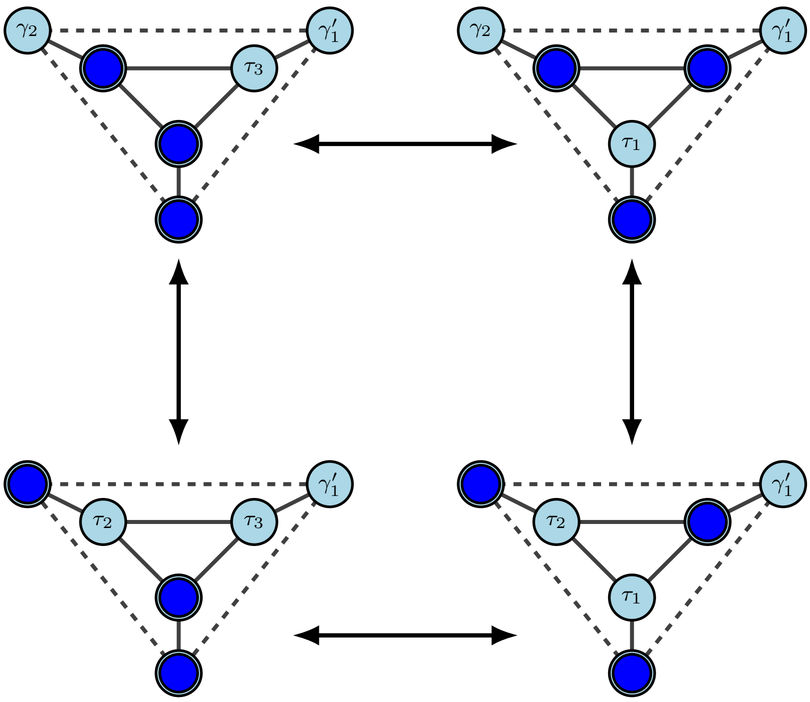

Figure 39: A closed path of particle configurations in $\mathfrak{L}_k$, which shows the lack of reversibility of $\mathfrak{S}_k$ if a tri-star is present, shown in L.

probability to move a particle highly depends on his neighborhood in the current configuration. Hence, moving first a particle from a dense cluster within the configuration at time t, in this case the particle on vertex τ_1, and secondly a particle from a less densely populated region of the cluster, in this case from τ_2, does not commute in the sense of the transition probabilities if there are more than two particles in the cluster. We use this idea to prove Theorem 7.18.

Proof of Theorem 7.18. We have already shown in Propositions 7.11 and 7.12 that for $k \in \{1, 2, \bar{n}-2, \bar{n}-1\}$ the Markov chain $\mathfrak{S}_k$ is reversible on $\mathfrak{L}_k$. Hence, we assume $k \in \{3, \ldots, \bar{n}-3\}$ and show that $\mathfrak{S}_k$ is not reversible.

Recall from Definition 3.3 that the vertex set of $\mathcal{T}$ is given by $V_\mathcal{T} = \{\tau_1, \tau_2, \tau_3, \gamma_1, \gamma_2, \gamma_1'\}$. We define the following cycle $\mathfrak{p}$ in $\mathfrak{L}_k$. First denote by $\mathfrak{u}_{k-3} \subset V \backslash V_\mathcal{T}$ a subset of size $k-3$ and define the cycle ϕ as well as $\mathfrak{p}$ as

$$\begin{aligned}&\phi := (\{\tau_1, \tau_2, \gamma_1\}, \{\tau_3, \tau_2, \gamma_1\}, \{\tau_3, \gamma_2, \gamma_1\}, \{\tau_1, \gamma_2, \gamma_1\}, \{\tau_1, \tau_2, \gamma_1\}),\\ &\mathfrak{p} := (\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4); \qquad \mathfrak{p}_i := \mathfrak{u}_{k-3} \cup \phi_i,\ i = 1, \ldots, 4.\end{aligned}$$

The transition probability for $\mathfrak{S}_k$ to go along $\mathfrak{p}$ is given by

$$\begin{aligned}&\frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\tau_1) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_2}}(\tau_2) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\tau_3) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_4}}(\gamma_2) + 1}\\ =&\frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\tau_1) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\tau_2) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\tau_3) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_4}}(\gamma_2) + 1}\end{aligned}$$

and for the reverse path

$$\begin{aligned}&\frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\tau_2) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_4}}(\tau_1) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_2) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_2}}(\tau_3) + 1}\\ =&\frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\tau_2) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\tau_1) + 2} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_4}}(\gamma_2) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\tau_3)}.\end{aligned}$$

By comparison we see that both terms are equal if and only if

$$(\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\tau_1) + 1)(\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\tau_3) + 1) = (\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\tau_1) + 2)(\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\tau_3))$$

which is equivalent to

$$\deg^{L_{\mathfrak{p}_1}}(\tau_1) = \deg^{L_{\mathfrak{p}_3}}(\tau_3) + 1. \tag{7.17}$$

Since $\gamma_1 \in \mathfrak{p}_i$ for $i = 1, \ldots, 4$, $\deg^{L_{V_\mathcal{T} \cap \mathfrak{p}_1}}(\tau_1) = 2$ and $\deg^{L_{V_\mathcal{T} \cap \mathfrak{p}_3}}(\tau_2) = 0$ we can redistribute all particles in $\mathfrak{u}_{k-3}$ on L equally in the neighborhoods of τ_1 and τ_2 in V outside of $\mathcal{T}$ such that we obtain a contradiction to the equation 7.17. By Kolmogorov's criterion the claim follows. □

If L does not contain a tri-star there is still no certainty that reversibility would follow. In fact, for various other cases this is not the case.

Theorem 7.19. *Let L be any simple connected $\bar{d}$-regular graph on $\bar{n}$ vertices with $\bar{d} \in \{3, \dots, \bar{n}-3\}$. Assume that L contains a cube-star $\mathcal{C}$ as defined in Definition 3.11. Then, the Markov chain $\mathfrak{S}_k$ is reversible on $\mathfrak{L}_k$ if and only if $k \in \{1, 2, \bar{n}-2, \bar{n}-1\}$.*

The method of constructing a suitable counterexample based on the assumed subgraph remains the main tool for the proof of Theorem 7.19. .

Proof of Theorem 7.19. We have already shown in Propositions 7.11 and 7.12 that for $k \in \{1, 2, \bar{n}-2, \bar{n}-1\}$ the Markov chain $\mathfrak{S}_k$ is reversible on $\mathfrak{L}_k$. Hence, we assume $k \in \{3, \dots, \bar{n}-3\}$ and show that $\mathfrak{S}_k$ is not reversible.

Using Definition 3.11 define the following cycle $\mathfrak{p}$ in $\mathfrak{L}_k$. Again, we denote by $\mathfrak{u}_{k-3} \subset V \backslash \{\alpha_1, \xi', \alpha_3, \alpha_4, \gamma_1, \gamma_2'\}$ a subset of size $k-3$ and define

$$\begin{aligned} &\phi := (\{\alpha_1, \alpha_4, \gamma_1\}, \{\xi', \alpha_4, \gamma_1\}, \{\xi', \alpha_3, \gamma_1\}, \{\alpha_1, \alpha_3, \gamma_1\}, \{\alpha_1, \alpha_4, \gamma_1\}), \\ &\mathfrak{p} := (\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4); \qquad \mathfrak{p}_i := \mathfrak{u}_{k-3} \cup \phi_i, \ i = 1, \dots, 4. \end{aligned}$$

Since $k \leq \bar{n}-3$ it is possible to redistribute the remaining $k-3$ particles in $\mathfrak{u}_{k-3}$ on $V \backslash \{\alpha_1, \xi', \alpha_3, \alpha_4, \gamma_1, \gamma_2'\}$ such that $\deg^{\mathfrak{p}_1}(\alpha_1) > \deg^{\mathfrak{p}_1}(\alpha_4)$ and $\deg^{\mathfrak{p}_3}(\alpha_3) \geq \deg^{\mathfrak{p}_3}(\xi')$. The transition probability for $\mathfrak{S}_k$ to go along $\mathfrak{p}$ is proportional to

$$\frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\alpha_1) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\alpha_4) + 2} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\xi') + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\alpha_3) + 2} \tag{7.18}$$

and for the reverse path

$$\frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\alpha_4) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\alpha_1) + 2} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\alpha_3) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\xi') + 2} \tag{7.19}$$

where we leave out the factor k^{-4} in each expression. We define

$$\begin{aligned} x &:= \bar{d} - \deg^{L_{\mathfrak{p}_1}}(\alpha_1), \quad y := \bar{d} - \deg^{L_{\mathfrak{p}_1}}(\alpha_4), \\ p &:= \bar{d} - \deg^{L_{\mathfrak{p}_3}}(\alpha_3), \quad q := \bar{d} - \deg^{L_{\mathfrak{p}_3}}(\xi'). \end{aligned}$$

such that 7.18 equals 7.19 if and only if

$$pq(x-y) + xy(p-q) + 3(px - qy) + 2(x - y + p - q) = 0. \tag{7.20}$$

Since by the assumed type of the configuration we have $\deg^{\mathfrak{p}_1}(\alpha_1) > \deg^{\mathfrak{p}_1}(\alpha_4)$ and $\deg^{\mathfrak{p}_3}(\alpha_3) \geq \deg^{\mathfrak{p}_3}(\xi')$ which implies $x < y$ and $p \leq q$. Ergo, $pq(x-y) + xy(p-q) + 3(px-qy) + 2(x-y+p-q) < 0$ such that equality in 7.20 is impossible. By Kolmogorov's criterion it follows that $\mathfrak{S}_k$ is not reversible. □

A direct generalization of this method would include replacing the triangle or the square by larger cycles. It turns out that this is not necessary since an even simpler structure, the double-pitchfork, already yields the needed implication. Indeed, it covers also generalized cases of the tri-star by cycles of length larger than 4.

Theorem 7.20. *Let L be any simple connected $\bar{d}$-regular graph on $\bar{n}$ vertices with $\bar{d} \in \{3, \dots, \bar{n}-3\}$, which is not a complete bipartite graph. Assume that L contains a double-pitchfork $\mathcal{D}$ as defined in Definition 3.13. Then, the Markov chain $\mathfrak{S}_k$ is reversible on $\mathfrak{L}_k$ if and only if $k \in \{1, 2, \bar{n}-2, \bar{n}-1\}$.*

Again, the main idea of the proof remains identical. But the construction becomes in fact more simple since the particles which move now are only interacting in the initial configuration $\mathfrak{p}_1$ of the path.

Proof of Theorem 7.20. We have already shown in Propositions 7.11 and 7.12 that for $k \in \{1, 2, \bar{n}-2, \bar{n}-1\}$ the Markov chain $\mathfrak{S}_k$ is reversible on $\mathfrak{L}_k$. Furthermore, in Theorem 7.17 we have shown reversibility in the case of a complete bipartite graph, which contains, nonetheless, by Theorem 3.14 a double pitchfork. Hence, we assume $k \in \{3, \dots, \bar{n}-3\}$ as well as L not a complete bipartite graph and show that $\mathfrak{S}_k$ is not reversible.

Now employing Definition 3.13 we define the following cycle $\mathfrak{p}$ in $\mathfrak{L}_k$. As before, we denote by $\mathfrak{u}_{k-3} \subset V \backslash \{\gamma_1, \xi, \gamma_2, \gamma_1', \xi', \gamma_2'\}$ a subset of size $k-3$ and define

$$\begin{aligned} \phi &:= (\{\xi, \xi', \gamma_1\}, \{\gamma_2, \xi', \gamma_1\}, \{\gamma_2, \gamma_1', \gamma_1\}, \{\xi, \gamma_1', \gamma_1\}, \{\xi, \xi', \gamma_1\}), \\ \mathfrak{p} &:= (\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4); \qquad \mathfrak{p}_i := \mathfrak{u}_{k-3} \cup \phi_i, \; i = 1, \dots, 4. \end{aligned}$$

Since $\langle \gamma_2, \gamma_1' \rangle \notin E$, the transition probability for $\mathfrak{S}_k$ to go along $\mathfrak{p}$ is proportional to

$$\frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi') + 2} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_2) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_1') + 1} \tag{7.21}$$

and for the reverse path

$$\frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi') + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi) + 2} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_1') + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_2) + 1} \tag{7.22}$$

where we, again, leave out the factor k^{-4} in each expression. We consider in this case only the reduced set of parameters

$$a := \bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi), \quad b := \bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi').$$

such that 7.21 equals 7.22 if and only if

$$(a+1)(b+2) = (a+2)(b+1). \tag{7.23}$$

Therefore, only $a = b$ implies reversibility. Again, using the same arguments as in the previous proofs, we find always a configuration of particles such that $a \leq b - 1$ and by Kolmogorov's criterion, it follows again that $\mathfrak{S}_k$ is not reversible. □

Indeed, if we do not exclude the edge $\langle \gamma_2, \gamma_1' \rangle$ from $\mathcal{D}$, reversibility cannot be disproved via the given structure. Assume, that $\langle \gamma_2, \gamma_1' \rangle \in E$. Then, using the same notation as in the proof of Theorem 7.20, the transition probability for $\mathfrak{S}_k$ to go along $\mathfrak{p}$ is proportional to

$$\frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi') + 2} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_2) + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_1') + 2} \tag{7.24}$$

and for the reverse path

$$\frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi') + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi) + 2} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_1') + 1} \frac{1}{\bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_2) + 2} \tag{7.25}$$

where we leave out the factor k^{-4} in each expression. We define

$$a := \bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi) + 1, \quad b := \bar{d} - \deg^{L_{\mathfrak{p}_1}}(\xi') + 1,$$
$$c := \bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_2) + 1, \quad d := \bar{d} - \deg^{L_{\mathfrak{p}_3}}(\gamma_1') + 1,$$

such that 7.24 equals 7.25 if and only if

$$a(b+1)c(d+1) = (a+1)b(c+1)d. \tag{7.26}$$

We discuss two cases of integer solutions to this equation and the corresponding implications in a graph theoretical sense. Firstly, if $a = d$ and $b = c$ the equation is satisfied. Since ξ has two empty neighbors in L with respect to $\mathfrak{p}_1$ and γ_1' has at least one empty neighbor in L with respect to $\mathfrak{p}_3$, we can put in all adjacent vertices of γ_1' one of the remaining $k-3$ particles except for ξ' and possibly γ_2'. Then, $a = d$ if and only if ξ and γ_1' share all neighbors. Arguing along the lines of the proof of Theorem 3.14, we find that this is only possible, if L is a complete bipartite graph. In turn, if L is a complete bipartite graph, then ξ and γ_1' belong to the same independent set and share all neighbors. This implies $a = d$ and analogous remarks yield $b = c$. Therefore, we have $a = d$ and $b = c$, as was to be expected by Theorem 7.17.
Secondly, with $a = b$ and $c = d$ we find a solution to equation (7.26). Due to the fact that γ_1 always contains a particle, but γ_1' as well as γ_2' do not in $\mathfrak{p}_1$, we obtain $a \leq b-1$. Using that L is a $\bar{d}$-regular graph, we can redistribute the remaining particles such that equality in (7.26) is impossible. By Kolmogorov's criterion, it follows again that $\mathfrak{S}_k$ is not reversible. Other solutions to equation (7.26) might give additional insights into reversibility. We find that equation (7.26) is equivalent to

$$-abc + abd - acd - ac + bcd + bd = 0.$$

From this we can conclude that with $q(a,b,c,d) := \frac{a(b+d+1)}{d(a+c+1)}$ a quadruple (a,b,c,d) is a solution to equation (7.26) if and only if

$$\frac{-q(a,b,c,d)\cdot 1+\frac{b}{c}}{q(a,b,c,d)\cdot 1+\frac{b}{c}}=0 \iff \frac{\frac{2b}{c}}{q(a,b,c,d)\cdot 1+\frac{b}{c}}=1.$$

Therefore, we find the equivalence with a fixed point problem of a Möbius transform $\mathcal{A}$, see [Bear12] for the more details, for $z\in\mathbb{C}$ induced by the matrix following A with

$$A=\begin{pmatrix} 0 & 2bd(a+c+1) \\ ac(b+d+1) & b. \end{pmatrix}, \qquad \mathcal{A}: z\mapsto \frac{0\cdot z+2bd(a+c+1)}{ac(b+d+1)z+b} \tag{7.27}$$

and $\mathcal{A}(1)=1$. Consequently, the problem of finding integer solutions to equation (7.26) is equivalent to finding Möbius transforms with integer entries under certain constraints which have 1 as fixed point. From this we can, in particular, define a new notion of independence of two solutions (a,b,c,d) and (a',b',c',d') if their induced Möbius transform $\mathcal{A}$ and $\mathcal{B}$, respectively, do not define a new solution via $\mathcal{A}\circ\mathcal{B}$. Unfortunately, the scope of this work as well as the the author's knowledge are not sufficient to cover the whole analysis, the latter not being extendable on this subject in a timely manner. Therefore, further solutions to (7.26) remain due to the complexity out of reach for the moment and we leave this problem open.

In fact, the double pitchfork also exists in a wide variety of trees which do not necessarily satisfy the regularity condition. Hence, we can also make claims about reversibility for these trees along the lines of the proof of Theorem 7.20

Theorem 7.21. *Let L be a tree on $\bar{n}$ vertices where non-leave vertices have degree at least 2. Assume that L contains a double-pitchfork $\mathcal{D}$ as defined in Definition 3.13. Then, the Markov chain $\mathfrak{S}_k$ is reversible on $\mathfrak{L}_k$ if and only if $k\in\{1,\bar{n}-1\}$ or $k\in\{2,\bar{n}-2\}$ and L is a $\bar{d}$-ary tree, i.e., non-leave vertices have identical degree $\bar{d}$.*

Proof. The proof works along the lines of the proof of Theorem 7.20. Since L is not necessary regular in the case of a tree, we assume without loss of generality that $\deg(\xi)\le\deg(\xi')$ to construct the counterexample. Symmetry of the double-pitchfork assures that this is, in fact, not a restriction.
We obtain the identical equations as in the proof of Theorem 7.20 such that for

$$x=\deg(\xi)-\deg^{L_{\mathfrak{p}_1}}(\xi), \quad y=\deg(\xi')-\deg^{L_{\mathfrak{p}_1}}(\xi').$$

we obtain the condition for reversibility

$$0=(x+1)(y+2)-(x+2)(y+1)=x-y. \tag{7.28}$$

This is equivalent to

$$\deg(\xi) - \deg^{L_{\mathfrak{p}_1}}(\xi) = \deg(\xi') - \deg^{L_{\mathfrak{p}_1}}(\xi'). \tag{7.29}$$

Note that the left hand side corresponds to the number of neighbors of ξ in L which are not occupied by a particle. An analogous observation can be made for the right hand side. The neighborhood of ξ contains one vertex γ_1 which is occupied by a particle and one γ_2 which is not and the neighborhood of ξ' contains two vertices which are not occupied by particles, namely γ_1' and γ_2'. Furthermore, $\deg(\xi) \leq \deg(\xi')$ such that by putting the remaining $k-3$ particles in $N_\xi \backslash \{\gamma_1, \gamma_2\}$ before filling $N_{\xi'} \backslash \{\gamma_1', \gamma_2'\}$ we obtain $\deg(\xi') - \deg^{L_{\mathfrak{p}_1}}(\xi') < \deg(\xi') - 1$ if and only if $\deg(\xi) - \deg^{L_{\mathfrak{p}_1}}(\xi) = 1$. In this case,

$$\deg(\xi) - \deg^{L_{\mathfrak{p}_1}}(\xi) = 1 < 2 \leq \deg(\xi') - \deg^{L_{\mathfrak{p}_1}}(\xi')$$

and, otherwise,

$$\deg(\xi) - \deg^{L_{\mathfrak{p}_1}}(\xi) \leq \deg(\xi) - 2 < \deg(\xi') - 1 = \deg(\xi') - \deg^{L_{\mathfrak{p}_1}}(\xi').$$

which contradicts in both cases equality in 7.29 and, hence, reversibility of $\mathfrak{S}_k$ if $k \in \{3, \ldots, \bar{n} - 3\}$.

The cases $k = 1$ and $k = \bar{n} - 1$ are covered by classical theory on random walks on graphs and reversibility is, therefore, given.

Let $k = 2$ and assume $\mathfrak{v} = \{v, w\}$ for any non-leave vertices in V with $\langle v, w \rangle \in E$. Then, using the same construction as for cases $k \in \{3, \ldots, \bar{n} - 3\}$ by equation 7.29 we obtain that reversibility is given if and only if $\deg(v) = \deg(w)$ implying the same degree of all non-leave vertices due to the minimal degree of 2. Indeed, this implies that the degree of non-leave vertices is at least 3 by the form of the double-pitchfork. The case $k = \bar{n} - 2$ can be proven analogously by symmetry of $\mathfrak{L}_k \overset{\sim}{=} \mathfrak{L}_{\bar{n}-k}$. □

The preceding results yield a characterization for reversibility of the process $\mathfrak{S}_k$ and show the roles of both the degree in L and the number of particles. The region for these parameters is depicted in Figure 40. Moreover, any strongly regular graph L with parameters $(\bar{n}, \bar{d}, \alpha, \beta)$ satisfying $\alpha \geq 1$ and $\bar{d} \leq \bar{n} - 3$ contains a tri-star, such that Theorem 7.18 yields an equivalence for reversibility for any strongly regular graph satisfying $\alpha \geq 1$.

7.3.5 Considerations for $\bar{d} = 3$ and $k = 3$

Indeed, the smallest case where we can see a divergence of the behavior of $\mathfrak{S}_k$ based on the geometry of the underlying graph L is for $\bar{n} = 6$, $\bar{d} = 3$ and $k = 3$. Note that for $\bar{n} = 6$ and $\bar{d} = 3$ there are two non-isomorphic graphs. We present them in Figure 41. One of them is in fact a complete bipartite graph while the other is not. We call them $L_{||}$ and $L_\circ$, respectively. This is a geometric difference between the two graphs which

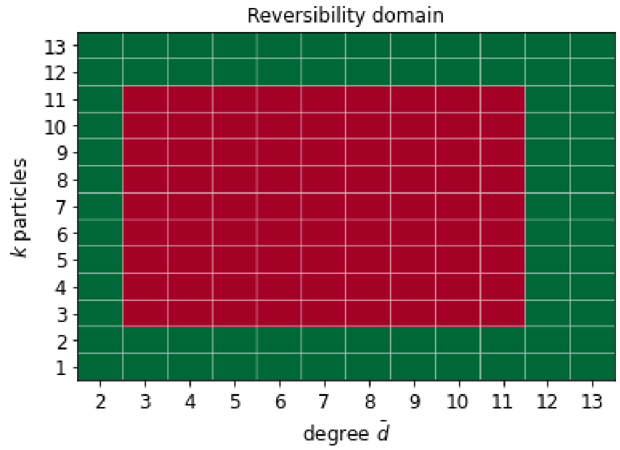

Figure 40: Parameter pairs $(\bar{d}, k)$ depicted in the cube of side length $\bar{n} = 14$. Green pairs imply reversibility while red pairs imply the lack of reversibility.

we are going to translate in quantitative properties in terms of their diameter and eigenvalues. Already, the Markov chain $\mathfrak{S}_3$ has very different properties depending on the underlying graph. On $L_{||}$ it is reversible by Theorem 7.17 while it is not on $L_\circ$ by Theorem 7.18. Additionally, $L_\circ$ is the smallest graph containing a tri-star. Comparing

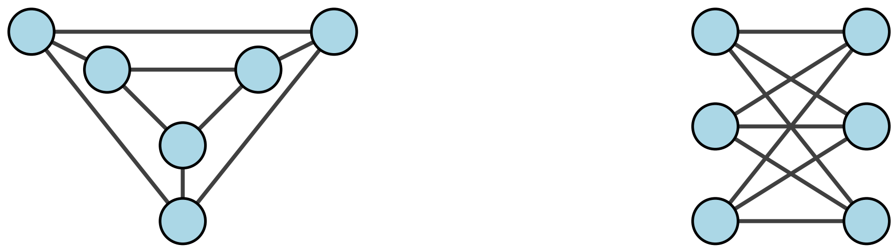

Figure 41: Both $\bar{3}$-regular graphs on 6 vertices. The graph $L_\circ$ can be seen on the left and the graph $L_{||}$ on the right.

the degree sets, we find that $D_3^{||} = \{5, 9\}$ and $D_3^\circ = \{3, 5, 7\}$. Furthermore, for $L_\circ$ we observe that for any $\mathfrak{v}, \mathfrak{w} \in \mathfrak{L}_k^\circ$ with $p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle;\circ} > 0$ if either $\deg_k^\circ(\mathfrak{v}) = 7$ or $\deg_k^\circ(\mathfrak{w}) = 7$. Since the stationary distribution and reversibility of $\mathfrak{S}_k^{||}$ are shown in Theorem 7.17, we are going to focus on $\mathfrak{S}_k^\circ$. In what follows we are going to approach the stationary distribution constructively.

Simulations show that for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ the stationary distribution $\tilde{\pi}_k^\circ$ satisfies

$$\tilde{\pi}_k^\circ(\mathfrak{v}) = \tilde{\pi}_k^\circ(\mathfrak{w}) \iff \deg_k^\circ(\mathfrak{v}) = \deg_k^\circ(\mathfrak{w}). \tag{7.30}$$

Figure 42 underlines the fact that $\mathfrak{S}_k$ is, indeed, lumpable with respect to the degree. Therefore, we obtain by Theorem 7.8 that the equivalence given by (7.30) is indeed satisfied. We can consider a similar reduction of the state space as shown in Figure 38.

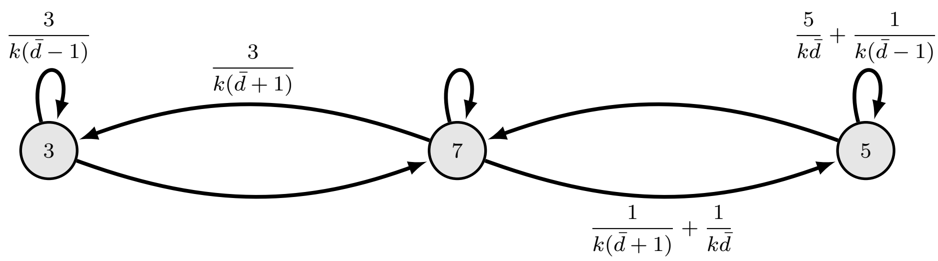

Figure 42: Transitions on the quotient graph of $\mathfrak{L}_3^\circ$ based on the non-bipartite 3 regular graph on 6 vertices. The number on every vertex represents the corresponding degree.

Unfortunately, transitions do no longer only happen between vertices of next higher or lower degree. This also leads to involved holding conditions implied by the local structure of the sub-graphs induced by any $\mathfrak{v} \in \mathfrak{V}_k$.

To derive the transition probabilities depicted in Figure 42 we consider the graph $\mathfrak{L}_k^\circ$ with the goal of deriving an explicit form of the equations $\tilde{\pi}_k^\circ(\mathfrak{v}) = \sum_{\mathfrak{w}\in\mathfrak{V}_k} \tilde{\pi}_k^\circ(\mathfrak{w}) p_{k;\mathfrak{w},\mathfrak{v}}^{\triangle;\circ}$. Analyzing the neighborhood of each vertex in Figure 43, we obtain that the neighborhoods of vertices of identical degree are identical. For example any vertex of degree 3 has exactly 3 neighbors with degree 7, while any vertex with degree 5 has 2 neighbors with degree 7 and 3 neighbors with degree 5. This observation underlines also the structure visualized in Figure 42. Hence, for any vertices $\mathfrak{v}^{(3)}$, $\mathfrak{v}^{(5)}$ and $\mathfrak{v}^{(7)}$ with degree

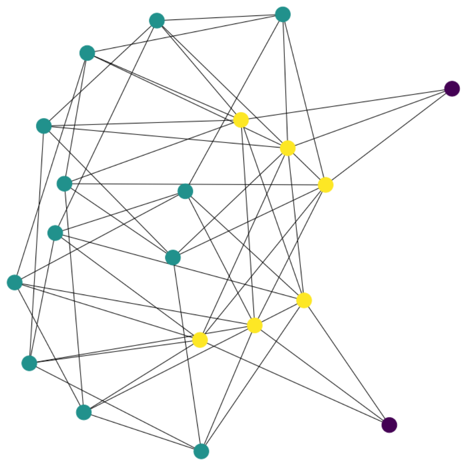

Figure 43: The graph $\mathfrak{L}_3^\circ$ for the underlying non-binary 3-regular graph on 6 vertices. The nodes with degree 3 are colored purple, those with degree 5 are colored pale blue and those with degree 7 are colored yellow.

3, degree 5 and degree 7, respectively, we obtain

$$\tilde{\pi}_k^\circ(\mathfrak{v}^{(3)}) = \frac{3}{k(\bar{d}+1)}\tilde{\pi}_k^\circ(\mathfrak{v}^{(7)}) + \frac{3}{k(\bar{d}-1)}\tilde{\pi}_k^\circ(\mathfrak{v}^{(3)}),$$
$$\tilde{\pi}_k^\circ(\mathfrak{v}^{(5)}) = \left(\frac{1}{k(\bar{d}+1)} + \frac{1}{k\bar{d}}\right)\tilde{\pi}_k^\circ(\mathfrak{v}^{(7)}) + \left(\frac{5}{k\bar{d}} + \frac{1}{k\bar{d}}\right)\tilde{\pi}_k^\circ(\mathfrak{v}^{(5)}),$$
$$1 = |\mathfrak{V}_{D_k;3}|\tilde{\pi}_k^\circ(\mathfrak{v}^{(3)}) + |\mathfrak{V}_{D_k;5}|\tilde{\pi}_k^\circ(\mathfrak{v}^{(5)}) + |\mathfrak{V}_{D_k;7}|\tilde{\pi}_k^\circ(\mathfrak{v}^{(7)})$$

using the transition graph 42. We do not yet replace the parameters $\bar{n}, \bar{d}$ and k with the actual values to facilitate the interpretation later on. It remains to derive $|\mathfrak{V}_{D_k;l}|$ for $l \in \{3,5,7\}$. First, $|\mathfrak{V}_{D_k;3}|$ corresponds to the number of induced sub-graphs $\mathfrak{v}^{(3)}$ with $\deg_k^\circ = k\bar{d} - 2\cdot 3$ and, hence, to the number of triangles in L. There are 2 triangles in $\mathfrak{L}_k^{circ}$ such that $|\mathfrak{V}_{D_k;3}| = 2$. Secondly, $|\mathfrak{V}_{D_k;5}|$ is defined by the number of paths of length 2 in L, which do not induce a triangle as vertex induced sub-graph. Indeed, there are $\bar{n}$ paths of length 2 which induce a triangle, one for each vertex in V since each vertex may be place in the middle of such a path. Therefore, $|\mathfrak{V}_{D_k;5}| = \bar{n}\left(\binom{\bar{n}}{2} - 1\right)$. Finally, this gives $|\mathfrak{V}_{D_k;7}| = \binom{\bar{n}}{k} - \bar{n}\left(\binom{\bar{n}}{2} - 1\right) - 2$. This results in the following values for the stationary distribution

$$\tilde{\pi}_k^\circ(\mathfrak{v}^{(3)}) = \frac{\frac{2\bar{d}+1}{k(\bar{d}+1)\bar{d}}}{1 - \frac{5}{k\bar{d}} - \frac{1}{k(\bar{d}-1)}}\tilde{\pi}_k^\circ(\mathfrak{v}^{(7)}), \qquad \tilde{\pi}_k^\circ(\mathfrak{v}^{(5)}) = \frac{\frac{3}{k(\bar{d}+1)}}{1 - \frac{3}{k(\bar{d}-1)}}\tilde{\pi}_k^\circ(\mathfrak{v}^{(7)}), \tag{7.31}$$

$$\tilde{\pi}_k^\circ(\mathfrak{v}^{(7)}) = \left(\binom{\bar{n}}{k} - \bar{n}\binom{\bar{n}}{2} + \bar{n} - 2 + \frac{\frac{6}{k(\bar{d}+1)}}{1 - \frac{3}{k(\bar{d}-1)}} + \frac{\bar{n}\left(\binom{\bar{d}}{2} - 1\right)\frac{2\bar{d}+1}{k(\bar{d}+1)\bar{d}}}{1 - \frac{5}{k\bar{d}} - \frac{1}{k(\bar{d}-1)}}\right)^{-1}. \tag{7.32}$$

We can now interpret the form of the stationary distribution in terms of the geometry of $L_\circ$ where we are going to focus on $\tilde{\pi}_k^\circ(\mathfrak{v}^{(7)})$. We skip the first summands, which are independent of k, and jump directly to the fraction $\frac{2}{k(\bar{d}+1)}\left(1 - \frac{3}{k(\bar{d}-1)}\right)^{-1}$. This is the quotient of the probabilities to create a triangle from a disconnected sub-graph of $L_\circ$ with 2 connected components divided by the probability not to remain a triangle when the current configuration induces a triangle in $L_\circ$. A disconnected sub-graph is represented in Figure 44. Consequently, this quotient describes the balance of the

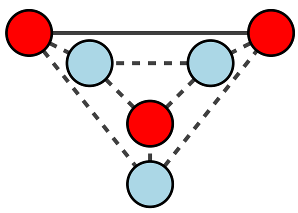

Figure 44: The graph $L_\circ$ and a sub-graph which consists of two connected components.

probabilities of going back and forth between sub-graph geometries. The factor 3 represents the number of such transitions. In the last summand, we find that paths of length 2, which do not induce a triangle in L and are visualized in Figure 45, play the central role. The quotient still describes the analogous balance times the number of existing paths of length 2. The first summands $\binom{\bar{n}}{k} - \bar{n}\binom{\bar{n}}{2} + \bar{n}$ is the balance of the

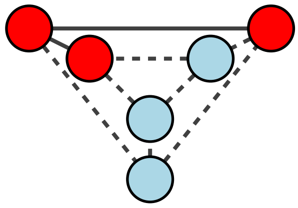

Figure 45: The graph $L_\circ$ with an induced sub-graph which is isomorphic to a path of length 2.

probabilities to create a different sub-graph from a disconnected sub-graph with 2 connected components in $L_\circ$ and the probability not to remain a disconnected sub-graph with 2 connected components in $L_\circ$, which is simply 1.
Indeed, this analysis gives an approach to deriving stationary distributions in non-reversible cases based on induced sub-graph geometries. These are much richer then a simple reduction to the degree set D_k since multiple non-isomorphic induced sub-graphs may give rise to the same degree. Indeed, for $k = 3$ we can extend the hereinabove discussed analysis to any arbitrary graphs since there are at most 4 different non-isomorphic vertex induced sub-graphs of size 3, the triangle, a path of length 2, a disconnected graph with 2 connected components and a disconnected graph with 3 components. The latter does not exist in the previously discussed case and the transitions between these sub-graphs become more intricate than those depicted in Figure 42 for the special case $\bar{n} = 6$, $\bar{d} = 3$ and $k = 3$.

7.3.6 Convergence speed to equilibrium for reversible case

In what follows, we are going to discuss the cases where $\mathfrak{S}_k$ is reversible as well as the implications of this property both on the behavior of the process as well as the geometry of the underlying graph. We have already seen that reversibility is in this case a boolean function of k and $\bar{d}$ and is, consequently, a structural property given by not only the transition probabilities.

We are going to use reversibility and laziness to make conclusions about the convergence speed of $\mathfrak{S}_k$ in the rare cases where these are given. We denote the Cheeger constant for $\mathfrak{S}_k$ by $\Phi_k^{\bar{d}}$ to highlight the dependence on the degree and the number of particles. We recall the Cheeger bounds from 5.20 for the spectral gap of a reversible

Markov chain given by

$$1 - 2\Phi_k^{\bar{d}} \leq \lambda_2 \leq 1 - \frac{(\Phi_k^{\bar{d}})^2}{2}. \tag{7.33}$$

Our discussion will, hence, focus on conclusions based on the Propositions 7.11 to 7.14. We ignore the cases $k = 1$ and $k = \bar{n} - 1$ since they amount to classical random walks on regular graphs and are, hence, covered by classical results, which can be found for example in [LePeWi09]. Again, we need to employ Conjecture 4.22 and have to assume that it is true.

Corollary 7.22. *Let $L = (V, E)$ be a $\bar{d}$-regular connected simple graph and $k = 2$. Then, the Cheeger constant of $\mathfrak{S}_2$ satisfies*

$$\Phi_2^{\bar{d}} \geq \frac{2(\bar{d} - 1)}{\bar{d} + 1} \frac{1}{\bar{n}(\bar{n} - 1)}. \tag{7.34}$$

Proof. Let $\mathfrak{U} \in \{\bar{\mathfrak{U}} \subset \mathfrak{V}_k \,|\, |\bar{\mathfrak{U}}| \leq 2^{-1}|\mathfrak{V}_k|,\ \tilde{\pi}_k(\bar{\mathfrak{U}}) \leq 2^{-1}\}$. Note that $\tilde{\pi}_k(\mathfrak{v})p_{k;\mathfrak{v},\mathfrak{w}} = \frac{1}{2C}$ for all neighbors $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$. Therefore, we obtain

$$\Phi(\mathfrak{U}) = \frac{1}{2} \frac{|\partial \mathfrak{S}|}{\bar{d}\,|\mathfrak{V}_{D_k;2(\bar{d}-1)} \cap \mathfrak{U}| + (\bar{d}+1)|\mathfrak{V}_{D_k;2\bar{d}} \cap \mathfrak{U}|} \in \left[\frac{1}{2(\bar{d}+1)} \frac{|\partial \mathfrak{U}|}{|\mathfrak{U}|}, \frac{1}{2\bar{d}} \frac{|\partial \mathfrak{U}|}{|\mathfrak{U}|}\right].$$

Applying Lemma 5.18 and Conjecture 4.22 we obtain that

$$\Phi_2^{\bar{d}} \geq \frac{1}{2(\bar{d}+1)} \iota(\mathfrak{L}_2) \geq \frac{2(\bar{d}-1)}{\bar{d}+1} \frac{1}{\bar{n}(\bar{n}-1)}$$

which completes the proof. □

In the sense discussed in [Sinc92] the chain $\mathfrak{S}_2$ mixes rapidly as a Markovian particle system on L since the Cheeger constant is bounded by a term which is an inverse polynomial in the size of the state space.
We continue by considering the properties of $\mathfrak{S}_{\bar{n}-2}$ and discuss afterwards the role of the isomorphism $\mathfrak{L}_k \cong \mathfrak{L}_{\bar{n}-k}$ in the quantitative behavior of $\mathfrak{S}_k$.

Corollary 7.23. *Let L be a $\bar{d}$-regular connected simple graph and $k = \bar{n} - 2$. Then, the Cheeger constant of $\mathfrak{S}_{\bar{n}-2}$ satisfies*

$$\Phi_{\bar{n}-2}^{\bar{d}} \geq \frac{2(\bar{d}-1)}{3(\bar{n}-2)} \frac{1}{\bar{n}(\bar{n}-1)}. \tag{7.35}$$

Proof. The proof follows analogously to the proof of Corollary 7.22 exploiting that for all neighbors $\mathfrak{v}, \mathfrak{w}$ we have

$$\tilde{\pi}_k(\mathfrak{v}) p_{k;\mathfrak{v},\mathfrak{w}} = \frac{1}{C(\bar{n}-2)} = \tilde{\pi}_k(\mathfrak{w}) p_{k;\mathfrak{w},\mathfrak{v}}$$

□

By considering $p_* := \min_{\mathfrak{v} \in \mathfrak{V}_k} p^{\triangle}_{k;\mathfrak{v},\mathfrak{v}}$ in both cases, we find that for $k = 2$ it amounts to $\bar{d} \leq 1$ and for $k = \bar{n} - 2$ we arrive at $\bar{n} < 0$ as equivalent condition for $p_* \geq 2^{-1}$. Therefore, in both cases $\mathfrak{S}_k$ is under no choice of parameters $\bar{n}, \bar{d}$ lazy. We want to emphasize here the difference in the quantitative behavior of $\mathfrak{S}_k$ and $\mathfrak{S}_{\bar{n}-k}$ which are not identifiable in spite of the isomorphism between their respective state spaces and the defining underlying structure of the dynamics.

Now we turn to the remaining reversible cases, i.e., we fix first $\bar{d} = 2$ and then $\bar{d} = \bar{n} - 2$. We have already seen that we can derive the stationary distributions of $\mathfrak{S}_k$ in these two cases explicitly. We obtain the following result which is based on Lemma 5.22.

Corollary 7.24. *Let L a $\bar{d}$-regular graph on $\bar{n}$ vertices and $k \in \{1, \dots, \bar{n}-1\}$. Then, the Cheeger constant associated to $\mathfrak{S}_k$ satisfies*

$$\Phi_k^{\bar{d}} \geq \begin{cases} \frac{2}{k}\left(\frac{2}{3}\right)^{k+1}\binom{\bar{n}}{k}^{-1}, & \bar{d} = 2 \\ \frac{4}{k}\frac{(\bar{n}-k)^k}{(\bar{n}-k+1)^{k+1}}\binom{\bar{n}}{k}^{-1}, & \bar{d} = \bar{n}-2. \end{cases} \tag{7.36}$$

Proof. The result follows from Propositions 7.13 and 7.14 using the form of the stationary distributions in each case as well as the minimum of the transition probabilities, as well as Lemma 5.22. The latter combines the quantitative value obtained in Propositions 7.13 and 7.14 and yields the claim. □

We see that the lower bounds we find in these two cases do not imply the rapid mixing in the sense used in [Sinc92] in contrast to the previous cases $k = 2$ and $k = \bar{n} - 2$. This is due to the crude nature of the estimate made to obtain Lemma 5.22 and exponential nature of the quotient of the minimum and the maximum of the stationary distributions. There is, intuitively, no reason why we shouldn't obtain a bound on $\Phi_k^{\bar{d}}$ which is polynomial as a function of $\bar{n}$ and k. Methods from discrete optimization, applied to results from Subsection 4.3 on the special forms of $\mathfrak{L}_k$ in these to cases, might yield further insights and improved bounds, exploiting further the explicit form of the stationary distributions. Indeed, depending on the number of particles we can even deduce further properties which support the usefulness of the Cheeger constant.

Theorem 7.25. *Let L be a cycle on $\bar{n}$ vertices and $k \in \{1, \dots, \bar{n}-1\}$. Define $\bar{k} := \left\lceil \frac{2\bar{n}}{3} \right\rceil$. Then, for $k < \bar{k}$ the Markov chain $\mathfrak{S}_k$ on $\mathfrak{L}_k$ is not a lazy Markov chain and for $k \geq \bar{k}$ the chain $\mathfrak{S}_k$ is a lazy Markov chain.*

Proof. First, let $k \in \left\{1, \dots, \left\lfloor \frac{\bar{n}}{2} \right\rfloor\right\}$. Then, there exist a $\mathfrak{v} \in \mathfrak{V}_k$ such that $|E_{\mathfrak{v}}| = 0$. Consequently,

$$p^{\triangle}_{k;\mathfrak{v},\mathfrak{v}} = \sum_{v \in \mathfrak{v}} \frac{1}{k} \frac{1}{2 - 0 + 1} = \frac{1}{3} < \frac{1}{2}$$

and, thus, $\mathfrak{S}_k$ is not lazy.

Now, let $k \in \left\{\left\lceil \frac{\bar{n}}{2} \right\rceil, \dots, \bar{n}-1\right\}$. Consider $\mathfrak{v}_* \in \mathfrak{V}_k$ such that the vertex induced sub-graph $L_{\mathfrak{v}}$ has $\bar{n}-k$ connected components, where $x_1 = (\bar{n}-k)\left(\left\lfloor \frac{k}{\bar{n}-k} \right\rfloor + 1\right) - k$ components have size $\kappa_1 := \left\lfloor \frac{k}{\bar{n}-k} \right\rfloor$ and $x_2 = \bar{n} - k - x_1$ components have size $\kappa_2 := \left\lfloor \frac{k}{\bar{n}-k} \right\rfloor + 1$. We are going to prove that

$$p^{\triangle}_{k;\mathfrak{v}_*,\mathfrak{v}_*} = \min_{\mathfrak{w} \in \mathfrak{V}_k} p^{\triangle}_{k;\mathfrak{w},\mathfrak{w}} \tag{7.37}$$

such that $p^{\triangle}_{k;\mathfrak{v}_*,\mathfrak{v}_*}$ can be used to derive $\bar{k}$. Note that

$$\min_{\mathfrak{w} \in \mathfrak{V}_k} p^{\triangle}_{k;\mathfrak{w},\mathfrak{w}} = \min_{\{a,b,c \in \mathbb{N}, a+b+c=k\}} \frac{a}{k} + \frac{b}{2k} + \frac{c}{3k} =: \min_{\{a,b,c \in \mathbb{N}, a+b+c=k\}} \Gamma(a,b,c)$$

where the first summand represents all vertices with degree two inside a connected component of $L_{\mathfrak{v}}$, the second one all vertices with degree one inside a connected component of $L_{\mathfrak{v}}$ and the third one the vertices who form their a connected component of size one. Since we consider the case $k \geq \left\lceil \frac{\bar{n}}{2} \right\rceil$, we obtain that by symmetry for any configuration $\mathfrak{v}$ with $c \geq 1$ and $a \geq 1$ we find another configuration $\mathfrak{v}'$ with $c' = c - 1$, $a' = a - 1$ and $b' = b + 2$. In particular, $\Gamma(a,b,c) > \Gamma(a',b',c')$. Hence, the minimizing triple satisfies $(a,b,c) \in \{(x,y,0),(0,y',z') | x + y = k = y' + z'\}$.

In the first case with $(a,b,c) = (x,y,0)$ it follows that $y = 2(\bar{n}-k)$ and we only look for a configuration which minimizes x, i.e., the number of vertices with degree two inside a connected component of said configuration. This can be obtained if the sizes of all connected components are as balanced as possible, i.e., each connected component has either size $\left\lfloor \frac{k}{\bar{n}-k} \right\rfloor$ or $\left\lfloor \frac{k}{\bar{n}-k} \right\rfloor + 1$. Furthermore, summing up the sizes of the connected components one has to obtain k, i.e.,

$$\begin{aligned} k &= x_1 \left\lfloor \frac{k}{\bar{n}-k} \right\rfloor + (\bar{n}-k-x_1)\left(\left\lfloor \frac{k}{\bar{n}-k} \right\rfloor + 1 \right) \\ &= (\bar{n}-k)\left(\left\lfloor \frac{k}{\bar{n}-k} \right\rfloor + 1 \right) - x_1 \end{aligned}$$

which yields $x_1 = (\bar{n}-k)\left(\left\lfloor \frac{k}{\bar{n}-k}\right\rfloor + 1\right) - k$ such that $\mathfrak{v}_*$ is the minimizer of $\min_{\mathfrak{w}\in\mathfrak{V}_k} p^{\triangle}_{k;\mathfrak{w},\mathfrak{w}}$. The minimum follows analogously in the case $(a,b,c) = (0, y', z')$.

It only remains to calculate $p^{\triangle}_{k;\mathfrak{v}_*,\mathfrak{v}_*}$ by

$$\begin{aligned} p^{\triangle}_{k;\mathfrak{v}_*,\mathfrak{v}_*} &= \frac{2(\bar{n}-k)}{2k} + \frac{x_1(\kappa_1 - 2)}{k} + \frac{x_2(\kappa_1-1)}{k} \\ &= \frac{\bar{n}-k}{k} + \frac{3k-2\bar{n})}{k} = \frac{2k-\bar{n}}{k} \end{aligned}$$

such that $p^{\triangle}_{k;\mathfrak{v}_*,\mathfrak{v}_*} \geq 2^{-1}$ if and only if $k \geq \frac{2\bar{n}}{3}$, which proves the claim. □

So, while the state spaces are identifiable for parameters k and $\bar{n}-k$ the behaviors of the corresponding Markov chains $\mathfrak{S}_k$ and $\mathfrak{S}_{\bar{n}-k}$ differ in certain aspects.

We turn now also in this case to the lazyness of the Markov chain which makes the bounds on the Cheeger constant all the while more useful in light of Theorem 5.20.

Theorem 7.26. *Let L be a $\bar{d}$-regular graph on $\bar{n} \geq 2$ vertices with $\bar{d} = \bar{n}-2$ and $k \in \{1, \ldots, \bar{n}-1\}$. Then, the Markov chain $\mathfrak{S}_k$ on $\mathfrak{L}_k$ is a lazy Markov chain if and only if $k = \bar{n}-1$.*

Proof. We are going to proceed as in the proof of Theorem 7.25 calculating $\min_{\mathfrak{w}\in\mathfrak{V}_k} p^{\triangle}_{k;\mathfrak{w},\mathfrak{w}}$ but using the idea based on Figure 23. Furthermore, we exclude the case $k=1$ and since it is analogous to a random walk on a L such that it is covered by classical results. Assume that $\mathfrak{v}_* \in \mathfrak{V}_k$ is the minimizer of $p^{\triangle}_{k;\mathfrak{w},\mathfrak{w}}$. Then,

$$p^{\triangle}_{k;\mathfrak{v}_*,\mathfrak{v}_*} = \frac{1}{k}\left(1 \cdot a + \frac{b}{\bar{n}-k} + \frac{c}{\bar{n}-k+1}\right) \tag{7.38}$$

for some $a, b, c \in \mathbb{N}$ with $a+b+c = k$ to be determined. The number a represents the number of $v \in \mathfrak{v}_*$ which stay in place with probability 1 if they are drawn among all vertices in $\mathfrak{v}_*$. The quantity b represents the number of particles containing $\bar{d}-1$ particles in their neighborhood and c the number of particles containing $\bar{d}-2$ particles in their neighborhood. In what follows, we derive a, b and c which minimize equation 7.38. First of all, the case $a > 0$ implies that $\deg^{\mathfrak{v}}(v) = \bar{n}-2$ since $\bar{d} = \bar{n}-2$ and all neighbors of v have to be occupied. Consequently, $k = \bar{n}-1$. We are going to analyze

this case first.
Assuming for now $k = \bar{n} - 1$, there can be at most one free site in the neighborhood of any occupied vertex. Consequently, $c = 0$. Furthermore, since there is in any configuration exactly one vertex which is not occupied we obtain that $b = \bar{n} - 2$ and $a = 1$ where the vertex which implies $a = 1$ is given by the neighbor of the unoccupied vertex in L^c. Consequently, we obtain

$$p^{\triangle}_{\bar{n}-1;\mathfrak{v}_*,\mathfrak{v}_*} = \frac{1}{\bar{n}-1}\left(1 + \frac{\bar{n}-2}{\bar{n}-2-(\bar{n}-3)+1}\right) = \frac{1}{2} + \frac{1}{2(\bar{n}-1)} > \frac{1}{2}.$$

Therefore, $\mathfrak{S}_{\bar{n}-1}$ is lazy.
In the second case, if $k < \bar{n} - 1$ whenever we consider a configuration $\mathfrak{v}$ with $a > 0$ we obtain due to the condition $\bar{d} = \bar{n} - 2$ that $k = \bar{n} - 1$ which is a contradiction to the underlying assumption. Hence, in what follows, we can set $a = 0$ such that

$$p^{\triangle}_{k;\mathfrak{v}_*,\mathfrak{v}_*} = \frac{1}{k}\left(\frac{b}{\bar{n}-k} + \frac{c}{\bar{n}-k+1}\right).$$

Obviously, by maximizing c and, in turn, minimizing b, we obtain the desired result. Using Figure 23 and the fact that transitions of particles of type c are only possible if there is another particle such that the pair of particles occupies both ends of an edge in L^c. Consequently, c is even and the maximal number of pairs we can create from k particles is $\left\lfloor \frac{k}{2} \right\rfloor$ and, hence, $c = 2\left\lfloor \frac{k}{2} \right\rfloor$ which implies $b = k \bmod 2$. To conclude these calculations we obtain the minimal value for the probabilities to stay in place in the form.

$$p^{\triangle}_{k;\mathfrak{v}_*,\mathfrak{v}_*} = \begin{cases} \frac{1}{k}\left(\frac{1}{\bar{n}-k} + \frac{k-1}{\bar{n}-k+1}\right), & \text{if } k \text{ is odd,} \\ \frac{1}{\bar{n}-k+1}, & \text{otherwise.} \end{cases} \tag{7.39}$$

In the first case, we rewrite the transition probabilities using $k = 2l + 1$ for some adequate $l \in \mathbb{N}$ to obtain

$$\frac{1}{k}\left(\frac{1}{\bar{n}-k} + \frac{k-1}{\bar{n}-k+1}\right) = \frac{1}{2l+1}\left(\frac{1}{\bar{n}-2l-1} + \frac{2l}{\bar{n}-2l}\right) =: q_l$$

Varying l between 1 and $\frac{\bar{n}-4}{2}$ is then equivalent to considering all possible odd $k > 1$. Indeed, by considering the quotient

$$\frac{q_l}{q_{l-1}} = 1 + \frac{8(l-1)l}{(2l+1)(4l-1)((2l-1)\bar{n}-4l^2+4l)} + \frac{6(2l-1)l}{(2l+1)(\bar{n}-(2l+1))(4l-1)} + \frac{2(2l-1)l}{(2l+1)(\bar{n}-2l)} > 1$$

we obtain that q_l is growing in l such that $p^{\triangle}_{k;\mathfrak{v}_*,\mathfrak{v}_*} > p^{\triangle}_{k';\mathfrak{v}_*,\mathfrak{v}_*}$ for k, k' odd with $k > k'$. Additionally,

$$p^{\triangle}_{\bar{n}-3;\mathfrak{v}_*,\mathfrak{v}_*} = \frac{1}{\bar{n}-3}\left(\frac{1}{4} + \frac{1}{12(\bar{n}-3)}\right) < \frac{1}{2}$$

which implies that $\mathfrak{S}_k$ is not lazy for $k = \bar{n} - 3$ and by the monotony of q_l we obtain that $\mathfrak{S}_k$ is for k odd only lazy if $k = \bar{n} - 1$. In the second case, for k even, we arrive at $\bar{n} - k + 1 \leq 2$ and, consequently, $k = \bar{n} - 1$ which is never satisfied for k even since $\bar{n}$ is even. □

To complete this discussion for particular cases, we are going to consider the complete graph on $\bar{n}$ vertices before going on to discussing the evolution of the properties of $\mathfrak{S}_k$ when varying $\bar{d}$ in a way where we add edges to L while preserving the regular graph property. We are going to employ perfect matchings to approach the arising questions.

Proposition 7.27. *Let L be the complete graph on $\bar{n}$ vertices and $k \in \{1, \dots, \bar{n} - 1\}$. The Markov chain $\mathfrak{S}_k$ is a lazy Markov chain if and only if $k = \bar{n} - 1$.*

Proof. For the complete graph, we obtain

$$p^{\triangle}_{k,\mathfrak{v},\mathfrak{v}} = \frac{1}{k}\frac{k}{\bar{n}-k+1} = \frac{1}{\bar{n}-k+1}$$

which is greater or equal one half if and only if $k \geq \bar{n} - 1$ and, therefore, only if $k = \bar{n} - 1$. □

Hereinabove, we have established results on the convergence speed of $\mathfrak{S}_k$ for specific values of $\bar{d}$ for which it was also possible to establish reversibility. Based on the results in Theorem 7.25 and Theorem 7.26 one can intuitively guess that by increasing $\bar{d}$ also the threshold $\bar{k}$ for the number of particles, above which $\mathfrak{S}_k$ becomes lazy, might increase until attaining its maximal possible value $\bar{k} = \bar{n} - 1$. Unfortunately, the idea of increasing the degree $\bar{d}$ is not necessarily well defined for a regular graph L if we want to preserve the neighborhood relationships of the vertices in L which exist for degree $\bar{d}$ when going to $\bar{d} + 1$. In the inverse sense, we want to ensure that going from a $\bar{d} + 1$-regular graph on $\bar{n}$ vertices, we can generate a $\bar{d}$-regular graph on $\bar{n}$ vertices by removing edges in a suitable way.

Proposition 7.28. *Let $L = (V, E)$ be a $\bar{d}$-regular graph on $\bar{n}$ vertices. Consider $(L_i)_{i=0}^{\bar{n}-1-\bar{d}} = ((V, E_i))_{i=0}^{\bar{n}-1-\bar{d}}$ a vector of regular graphs where $L_0 = L$, L_i is $\bar{d}_i := \bar{d}+i$-regular and for any $i \in \{1, \dots, \bar{n}-1-\bar{d}\}$ there is a perfect matching $\mathcal{M}_i$ in L_i such that $L_{i-1} = (V, E_i \backslash \mathcal{M}_i)$. Denote by $\mathfrak{L}_k^{\bar{d}_i} = (\mathfrak{V}_k, \mathfrak{E}_k^{\bar{d}_i})$ the to L_i corresponding particle graph and by $P_k^{\triangle,\bar{d}_i}$ the transition matrix of $\mathfrak{S}_k^{\bar{d}_i}$ on the graph $\mathfrak{L}_k^{\bar{d}_i}$.*
Then, for fixed $k \in \{1, \dots, \bar{n}-1\}$ any configuration $\mathfrak{v} \in \mathfrak{V}_k$ satisfies for $i' \geq i$

$$p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle,\bar{d}_{i'}} \leq p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle,\bar{d}_i} \tag{7.40}$$

and for fixed $\mathfrak{w} \in \mathfrak{V}_k$ such that $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k^{\bar{d}_i}$

$$p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle,\bar{d}_{i'}} \leq p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle,\bar{d}_i}. \tag{7.41}$$

Proof. We start the proof with the second claim. Since $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k^{d_i}$ they satisfy $\mathfrak{v} \triangle \mathfrak{w} = \{v, w\}$ for some $v, w \in V$, $v \neq w$ which implies

$$\begin{aligned} p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle,\bar{d}_{i+1}} - p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle,\bar{d}_i} &= \frac{1}{k} \frac{1}{(\bar{d}_i + 1) - \deg_{d_{i+1}}^{\mathfrak{v}}(v) + 1} - \frac{1}{k} \frac{1}{\bar{d}_i - \deg_{d_i}^{\mathfrak{v}}(v) + 1} \\ &= \frac{1}{k} \frac{\deg_{d_{i+1}}^{\mathfrak{v}}(v) - \deg_{d_i}^{\mathfrak{v}}(v) - 1}{((\bar{d}_i + 1) - \deg_{d_{i+1}}^{\mathfrak{v}}(v) + 1)(\bar{d}_i - \deg_{d_i}^{\mathfrak{v}}(v) + 1)} \end{aligned} \tag{7.42}$$

and the numerator takes either the value 0 or -1 since the increase of the degree by 1 might connect v with another vertex in $\mathfrak{v}$ but never disconnects from vertices in $\mathfrak{v}$ due to the edge difference being the matching $\mathcal{M}_{i+1}$. Consequently, $p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle,\bar{d}_{i+1}} \leq p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle,\bar{d}_i}$ and by a bootstrap argument for $i' \geq i$ we obtain $p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle,\bar{d}_{i'}} \leq p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle,\bar{d}_i}$ which proves the second claim.
For the first claim, we emphasize that $p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle,\bar{d}_i}$ is the sum of all possible transition probabilities for $v \in \mathfrak{v}$. Consequently, the conclusion made from equation 7.42 is valid for each summand of $p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle,\bar{d}_i}$ and each summand is positive. Therefore, we conclude

$$p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle,\bar{d}_{i'}} = \frac{1}{k} \sum_{v \in \mathfrak{v}} \frac{1}{\bar{d}_i' - \deg_{d_{i'}}^{\mathfrak{v}}(v) + 1} \leq \frac{1}{k} \sum_{v \in \mathfrak{v}} \frac{1}{\bar{d}_i - \deg_{d_i}^{\mathfrak{v}}(v) + 1} = p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle,\bar{d}_i} \tag{7.43}$$

and equality holds if and only if $\{v \in \mathfrak{v} | \deg_{d_i}^{\mathfrak{v}}(v) < \deg_{d_{i'}}^{\mathfrak{v}}(v)\} = \emptyset$. We have proven, hence, the first claim. □

Indeed, Proposition 7.28 applies by Corollary 3.17 applies to bipartite graphs, from which vectors of $\bar{d}_i$-regular graphs can be constructed. This gives rise to the question about the importance of bipartite graphs in the context of Markov chain $\mathfrak{S}_k$. We have already seen, that $\mathfrak{S}_k$ is reversible if the underlying graph L is a complete bipartite graph. This remains an open direction for further research on the behavior of $\mathfrak{S}_k$ under varying degrees $\bar{d}$ of the underlying $\bar{d}$-regular graph L.

7.3.7 Convergence Speed in Non-Reversible Cases

We have already seen that, up to some simple cases ,we can never exploit results on convergence speeds of reversible Markov chains. Additionally, due to the inaccessible form of the transition matrix $P_k^{\triangle}$ we encounter huge difficulties when trying to approach this problem from a spectral theoretical perspective. In particular, the dependence on induced sub-graphs and adjacency of induced sub-graphs based on the symmetric difference renders a meaningful spectral theoretical result out of reach. On the other hand, we have already shown in the previous sections that combinatorial approaches are quite fruitful and there is, in particular, a back and forth when it comes to claims about sub-graphs and $\mathfrak{S}_k$.
One result, which we can exploit from a combinatorial perspective is Doeblin's criterion on the convergence speed of Markov chains, which we presented in Theorem 5.23. It transforms the problem into a uniform lower bound on the transition probabilities for a fixed length paths between two states and some probability distribution on the state space $\mathfrak{L}_k$. We need the following direct but necessary preliminary result.

Lemma 7.29. *Let L be a $\bar{d}$-regular graph on $\bar{n}$ vertices and $k \in \{1, \dots, \bar{n}-1\}$ with $\bar{d}+k+1 \leq \bar{n}$. Then, for all $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ we have $p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle} \geq \frac{1}{2k(\bar{n}-k)}$ and $p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle} \geq \frac{1}{2(\bar{n}-k)}$.*

Proof. The proof follows since under the condition $\bar{d} + k + 1 \leq \bar{n}$ we obtain for some suitable $v \in \mathfrak{v}$ that $p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle} = \frac{1}{k(\bar{d}-\deg^{L_{\mathfrak{v}}}(v)+1)} \geq \frac{1}{k(\bar{n}-k+1)}$ if and only if for all $v \in \mathfrak{v}$ the inequality $\deg^{L_{\mathfrak{v}}}(v) < \bar{d} + k + 1 - \bar{n} \leq 0$ is satisfied, which is never the case, and $\frac{1}{k(\bar{n}-k+1)} \geq \frac{1}{2k(\bar{n}-k)}$ since $k \leq \bar{n} - 1$. The second claim follows since $p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle} = \sum_{v\in\mathfrak{v}} \frac{1}{k(\bar{d}-\deg^{L_{\mathfrak{v}}}(v)+1)}$ and by using the first part of the proof. □

Using this bound on the transition probabilities, we obtain the following general result which has only a simple condition on the parameter choices, which is almost always satisfied in the applications.

Theorem 7.30. *Let L be a connected $\bar{d}$-regular graph on $\bar{n}$ vertices and $k \in \{1, \dots, \bar{n}-1\}$ with $\bar{d}+k+1 \leq \bar{n}$. Consider the k-particle graph $\mathfrak{L}_k$ of L, the Markov chain $\mathfrak{S}_k$ on $\mathfrak{L}_k$, its stationary distribution π as well as its transition matrix $P_k^{\triangle}$. Denote for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ and $l \in \mathbb{N}$ by $\omega_l^{\mathfrak{L}_k}(\mathfrak{v}, \mathfrak{w})$ the number of walks of length l from $\mathfrak{v}$ to $\mathfrak{w}$ along the edges in $\mathfrak{L}_k$. Let $\delta := \operatorname{diam}(\mathfrak{L}_k)$,*

$$\operatorname{conv}(W) := \operatorname{conv}\left(\left\{\left(\omega_\iota^{\mathfrak{L}_k}(\mathfrak{v}, \mathfrak{w})^2\right)_{\iota=0}^{\delta} | \mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k\right\}\right)$$

the convex hull in $\mathbb{R}^{\delta+1}$ of all possible numbers of walks with maximal length δ from any $\mathfrak{v}$ to any $\mathfrak{w}$ and for $y \in \mathbb{R}_+^{\delta+1}$ the function f be defined as

$$f(y) := \sum_{\iota=0}^{\delta} \frac{\sqrt{y_\iota}}{(2k(\bar{n}-k))^\iota} \left(\frac{\iota}{2(\bar{n}-k)}\right)^{\delta-\iota}.$$

Define

$$C := \frac{\min_{y \in \operatorname{conv}(W)} f(y)}{\max_{y \in \operatorname{conv}(W)} f(y)}, \qquad \varepsilon := \frac{C}{2^\delta} \sum_{\iota=0}^{\delta} \left(\frac{\bar{d}}{\bar{n}-1}\right)^\iota \left(\frac{\iota}{\bar{n}-k}\right)^{\delta-\iota}.$$

Then, $\varepsilon > 0$ and the transition matrix $P_k^{\triangle}$ satisfies

$$\sup_{\mathfrak{v} \in \mathfrak{V}_k} \sum_{\mathfrak{w} \in \mathfrak{V}_k} |p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle;(n)} - \tilde{\pi}_k(\mathfrak{w})| \leq 2(1-\varepsilon)^{\lfloor \frac{n}{\delta} \rfloor}, \quad n \geq 1. \tag{7.44}$$

In what follows, we present the idea of the central result in this subsection. We use rarely satisfied conditions which render the steps more understandable before discussing the details of the general result in the proof of the Theorem 7.30. To this end, recall the following notations. For $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ and $l \in \mathbb{N}$ we denote by $\omega_l^{\mathfrak{L}_k}(\mathfrak{v}, \mathfrak{w})$ the number of walks of length l from $\mathfrak{v}$ to $\mathfrak{w}$ along the edges in $\mathfrak{L}_k$. Note that $\mathfrak{L}_k$ is an indirected graph and, consequently, we have $\omega_l^{\mathfrak{L}_k}(\mathfrak{v}, \mathfrak{w}) = \omega_l^{\mathfrak{L}_k}(\mathfrak{w}, \mathfrak{v})$. Additionally, we define $\omega_l^{\mathfrak{L}_k,\circ}(\mathfrak{v}, \mathfrak{w})$ as the number of walks from $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ in the extended graph $\mathfrak{L}_k^\circ$ where a loop is added to every vertex. The graph $\mathfrak{L}_k^\circ$ is the graph which includes all pairs of vertices in $\mathfrak{V}_k^2$ as edges which show a positive transition probability under $P_k^{\triangle}$. A important result for us, which can be found in [FePeKo01], Lemma 4.2, states that for a fixed l there is a pair $\bar{\mathfrak{v}}, \bar{\mathfrak{w}} \in \mathfrak{V}_k$ such that

$$\omega_l^{\mathfrak{L}_k}(\bar{\mathfrak{v}}, \bar{\mathfrak{w}}) \geq \frac{(\operatorname{avg}\deg(\mathfrak{L}_k))^l}{|\mathfrak{V}_k|}. \tag{7.45}$$

Furthermore, drawing upon Proposition 7.28, we find that using $k \leq \bar{n}-1$ and Lemma

7.29 that

$$p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} \geq \frac{1}{2k(\bar{n}-k)}, \qquad p^{\triangle}_{k;\mathfrak{v},\mathfrak{v}} \geq \frac{1}{2(\bar{n}-k)}.$$

Moreover, since at any point a transition from a vertex to any of its neighbors in $\mathfrak{L}_k$ is possible with positive probability the transition matrix satisfies for $l \geq \operatorname{diam}(\mathfrak{L}_k)$ the property $(P^{\triangle}_k)^l > 0$. Therefore, we can estimate $(P^{\triangle}_k)^l$ for fixed $l \geq \operatorname{diam}(\mathfrak{L}_k)$ and $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ with $\mathfrak{v} \neq \mathfrak{w}$ by

$$p^{\triangle;(l)}_{k;\mathfrak{v},\mathfrak{w}} \geq \sum_{\iota=0}^{l} \frac{w^{\mathfrak{L}_k}_{\iota}(\mathfrak{v},\mathfrak{w})}{(2k(\bar{n}-k))^{\iota}} \sum_{l_1+\ldots+l_\iota=l-\iota} \binom{l-\iota}{l_1,\ldots,l_\iota} \frac{1}{(2(\bar{n}-k))^{l-\iota}}$$

where every summand represents a path with ι real steps and $l-\iota$ loops taken by $\mathfrak{S}_k$. Under the illustrative assumption that for any $\iota \in \mathbb{N}$ we have $w^{\mathfrak{L}_k}_{\iota}(\mathfrak{v},\mathfrak{w}) > 0$, which is almost never true, we find that there is by continuity of the map $x \mapsto x^{\iota}$ on $\mathbb{R}$ a constant $a > 0$ for all $\iota \in \mathbb{N}$ such that $w^{\mathfrak{L}_k}_{\iota}(\mathfrak{v},\mathfrak{w}) \geq \frac{a^{\iota}}{|\mathfrak{V}_k|}$ with $a \leq \operatorname{avg}\deg(\mathfrak{L}_k)$ by equation 7.45. This gives

$$\begin{aligned} p^{\triangle;(l)}_{k;\mathfrak{v},\mathfrak{w}} &\geq \sum_{\iota=0}^{l} \frac{w^{\mathfrak{L}_k}_{\iota}(\mathfrak{v},\mathfrak{w})}{(2k(\bar{n}-k))^{\iota}} \sum_{l_1+\ldots+l_\iota=l-\iota} \binom{l-\iota}{l_1,\ldots,l_\iota} \frac{1}{(2(\bar{n}-k))^{l-\iota}} \\ &\geq \frac{1}{|\mathfrak{V}_k|} \sum_{\iota=1}^{l} \frac{a^{\iota}}{(2k(\bar{n}-k))^{\iota}} \left(\frac{\iota}{2(\bar{n}-k)}\right)^{l-\iota} \geq \frac{1}{|\mathfrak{V}_k|} \sum_{\iota=1}^{l} \frac{a^{\iota}}{(2k(\bar{n}-k))^{\iota}} \left(\frac{1}{2(\bar{n}-k)}\right)^{l-\iota}. \end{aligned}$$

By Proposition 4.11 we obtain that $\frac{a}{2k(\bar{n}-k)} \leq \frac{1}{2}$ and, consequently, for all $l \in \mathbb{N}$

$$\sum_{\iota=1}^{l} \frac{a^{\iota}}{(2k(\bar{n}-k))^{\iota}} \left(\frac{1}{2(\bar{n}-k)}\right)^{l-\iota} \leq \sum_{\iota=1}^{l} \frac{a^{\iota}}{(2k(\bar{n}-k))^{\iota}} \leq \sum_{\iota=1}^{l} 2^{-\iota} = 1 - 2^{-l} < 1.$$

Hence, using Doeblin's criterion from Theorem 5.23 with

$$\varepsilon = \sum_{\iota=1}^{\operatorname{diam}(\mathfrak{L}_k)} \frac{a^{\iota}}{(2k(\bar{n}-k))^{\iota}} \left(\frac{1}{2(\bar{n}-k)}\right)^{l-\iota}$$

and $\delta = \operatorname{diam}(\mathfrak{L}_k)$ as well as $\hat{\pi}(\mathfrak{v}) = |\mathfrak{V}_k|^{-1}$, the uniform distribution on $\mathfrak{V}_k$, we obtain that

$$\sup_{\mathfrak{v} \in \mathfrak{V}_k} \sum_{\mathfrak{w} \in \mathfrak{V}_k} |p^{\triangle;(n)}_{k;\mathfrak{v},\mathfrak{w}} - \tilde{\pi}_k(\mathfrak{w})| \leq 2(1-\varepsilon)^{\lfloor \frac{n}{\delta} \rfloor}, \quad n \geq 1$$

and, therefore, a quantitative description of the exponential convergence speed of $\mathfrak{S}_k$ to its stationary distribution. While these formal calculations convey the idea correctly, already the assumption $w^{\mathfrak{L}_k}_{\iota}(\mathfrak{v},\mathfrak{w}) > 0$ is far from applicable. On the other hand, the lower bound $(2k(\bar{n}-k))^{-1}$ for the transition probabilities in Proposition 7.28 only applies to cases which satisfy the conditions in Proposition 7.28 if we remove the condition on $\bar{n}, k$ and $\bar{d}$. To overcome this hurdle the proof of Theorem 7.30 demands additional assumptions on $\bar{n}, k$ and $\bar{d}$. We now present the complete proof.

Proof of Theorem 7.30. We start the proof by finding a uniform lower bound for

$$\sum_{\iota=0}^{l} \frac{\omega_\iota^{\mathfrak{L}_k}(\mathfrak{v},\mathfrak{w})}{(2k(\bar{n}-k))^\iota} \left(\frac{\iota}{(2(\bar{n}-k)}\right)^{l-\iota} \tag{7.46}$$

before coming back to the heuristic arguments made hereinbefore which then become mathematically rigorous thanks to this lower bound. To this end, we consider the function f and $\text{conv}(W)$ as defined in Theorem 7.30. Note that $f(y) > 0$ for all $y \in \text{conv}(W)$ and $\inf_{y\in\text{conv}(W)} f(y) = \min_{y\in\text{conv}(W)} f(y) > 0$ since all $\omega \in \text{conv}(W)$ have at least one positive entry which is bounded away from 0. In what follows we denote for $\iota \in \{1,\dots,\delta\}$ by α_ι the term

$$\alpha_\iota = \frac{1}{(2k(\bar{n}-k))^\iota} \left(\frac{\iota}{(2(\bar{n}-k)}\right)^{\delta-\iota}. \tag{7.47}$$

Leaning on the proof of Lemma 4.2. in [FePeKo01] we obtain that

$$\sum_{\iota=0}^{\delta} \frac{\alpha_\iota}{|\mathfrak{V}_k|}(\text{avg deg}(\mathfrak{L}_k))^\iota \le \sum_{\iota=0}^{\delta} \alpha_\iota \sqrt{\frac{\sum_{\mathfrak{v},\mathfrak{w}} \omega_\iota^{\mathfrak{L}_k}(\mathfrak{v},\mathfrak{w})^2}{|\mathfrak{V}_k|^2}}.$$

Since $\sum_{\mathfrak{v},\mathfrak{w}} \frac{1}{|\mathfrak{V}_k|^2} = 1$ we deduce

$$(\bar{\omega}_\iota)_{\iota=0}^{\delta} := \left(\frac{1}{|\mathfrak{V}_k|^2}\sum_{\mathfrak{v},\mathfrak{w}} \omega_\iota^{\mathfrak{L}_k}(\mathfrak{v},\mathfrak{w})^2\right)_{\iota=0}^{\delta} \in \text{conv}(W) \tag{7.48}$$

which allows us to conclude

$$\begin{aligned}\sum_{\iota=0}^{\delta} \alpha_\iota \sqrt{\frac{\sum_{\mathfrak{v},\mathfrak{w}} \omega_\iota^{\mathfrak{L}_k}(\mathfrak{v},\mathfrak{w})^2}{|\mathfrak{V}_k|^2}} &= f\left((\bar{\omega}_\iota)_{\iota=0}^{\delta}\right) \le \frac{\max_{y\in\text{conv}(W)} f(y)}{\min_{y\in\text{conv}(W)} f(y)} \min_{y\in\text{conv}(W)} f(y) \\ &\le C^{-1} \min_{\mathfrak{v},\mathfrak{w}\in\mathfrak{V}_k} f\left((\omega_\iota^{\mathfrak{L}_k}(\mathfrak{v},\mathfrak{w})^2)_{\iota=0}^{\delta}\right).\end{aligned}$$

Furthermore, under the condition $\bar{d}+k+1 \le \bar{n}$ we obtain by $k \le \bar{n}-1$ the already mentioned estimates

$$p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle} \ge \frac{1}{2k(\bar{n}-k)}, \qquad p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle} \ge \frac{1}{2(\bar{n}-k)}.$$

The estimates which we have given in the heuristic discussion hereinbefore and which are in fact rigorous imply that

$$\begin{aligned} p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle;(l)} &\ge \sum_{\iota=0}^{l} \frac{\omega_\iota^{\mathfrak{L}_k}(\mathfrak{v},\mathfrak{w})}{(2k(\bar{n}-k))^\iota} \sum_{l_1+\ldots+l_\iota=l-\iota} \binom{l-\iota}{l_1,\dots,l_\iota} \frac{1}{(2(\bar{n}-k))^{l-\iota}} \\ &= \sum_{\iota=0}^{l} \frac{\omega_\iota^{\mathfrak{L}_k}(\mathfrak{v},\mathfrak{w})}{(2k(\bar{n}-k))^\iota} \left(\frac{\iota}{2(\bar{n}-k)}\right)^{l-\iota} \\ &\ge \min_{\mathfrak{v},\mathfrak{w}\in\mathfrak{V}_k} \sum_{\iota=0}^{l} \frac{\omega_\iota^{\mathfrak{L}_k}(\mathfrak{v},\mathfrak{w})}{(2k(\bar{n}-k))^\iota} \left(\frac{\iota}{2(\bar{n}-k)}\right)^{l-\iota} = \min_{\mathfrak{v},\mathfrak{w}\in\mathfrak{V}_k} f\left((\omega_\iota^{\mathfrak{L}_k}(\mathfrak{v},\mathfrak{w})^2)_{\iota=0}^{l}\right).\end{aligned}$$

Using this in combination with the previously obtained lower bound we arrive at

$$p_{k;\mathfrak{v},\mathfrak{w}}^{\triangle;(\delta)} \geq \frac{C}{|\mathfrak{V}_k|} \sum_{\iota=0}^{\delta} \alpha_\iota (\text{avg deg}(\mathfrak{L}_k))^\iota. \tag{7.49}$$

Applying Doeblin's criterion, see Theorem 5.23, we obtain the claim using $\hat{\pi}(\mathfrak{v}) = |\mathfrak{V}_k|^{-1}$ and ε as given in Theorem 7.30 by applying Proposition 4.11 to the right hand side of equation 7.49. □

Indeed, considering the constant C one can wonder about its quality depending on the underlying graph L, the implied geometry of $\mathfrak{L}_k$ and whether it can be improved, which would automatically increase the value for the convergence speed of $\mathfrak{S}_k$. The discussion in the proof of Theorem 7.30 focuses on one pillar when it comes to the definition of C, the convex hull of the squares of the number of paths in combination with f. It turns out that C is the best possible constant when using the general arguments we employed, which do not take into account the difficult geometry of $\mathfrak{L}_k$, but work in general for arbitrary underlying regular connected graphs L. Indeed, assuming for illustrative purposes that $\mathfrak{L}_k$ is the complete graph we find that $\text{conv}(W)$ becomes just one single point such that f is a constant function on $\text{conv}(W)$. This leads us to the conclusion that $C = 1$ and in any other case $C \leq 1$. Consequently, we find a graph for which we have equality in the estimates concerning f, and C can, thus, not be improved without more information on the geometry of $\mathfrak{L}_k$. An expression for the diameter of $\mathfrak{L}_k$ remains subject of active research on the authors site, but one for the moment without a satisfying answer.

Within this section we have obtained structural and quantitative properties of $\mathfrak{S}_k$ as a function of the underlying graph L and the number of particles k. We have seen that the methods we needed to apply vary greatly depending on k and L due to the difficulties arising from k sub-graph problems and the multi-layer dependency of the transition probabilities of $\mathfrak{S}_k$ on both vertices from $\mathfrak{L}_k$ and L. Combinatorial approaches allowed us to derive, nonetheless, bounds which are explicit up to the state of the art in the research of sub-graph problems, the number of walks in graphs and discrete optimization. Advancements in these fields can, therefore, also push the quality of our results and fill in gaps which need new results outside of the scope of this work.

In the next subsection we compare the behavior of the chain $\mathfrak{S}_k$ with an example of a discrete time exlcusion processes which has been subject to research for some time. It is one of the examples given in [DiaSal93] to underline the difference which may arise depending on the local and global interaction of particles.

7.4 Comparison to classical discrete time exclusion process

In [DiaSal93] the authors use a certain form of an exclusion process as an example to show the possibility to apply their main result of the paper on the convergence speed of reversible Markov chains. It is a discrete time model which captures the characteristics of the exclusion process using the transition matrix for an associated Markov chain. We recall some results on this model, which we described in Subsection 5.5 geometrically, before going on to a comparison with $\mathfrak{S}_k$. Firstly, recall that we called the associated Markov chain $\mathfrak{S}_k^{\mathfrak{c}}$ on $\mathfrak{L}_k$ and its transition matrix $P^{\mathfrak{c}}$ which has the form

$$p_{k;\mathfrak{v},\mathfrak{w}}^{\mathfrak{c}} = \begin{cases} \dfrac{1}{k \cdot \bar{d}}, & \mathfrak{v} \triangle \mathfrak{v} = \{x, y\},\ \langle x, y \rangle \in E, \\ \dfrac{\sum_{v \in \mathfrak{v}} \deg^{L_{\mathfrak{v}}}(v)}{k \cdot \bar{d}}, & \mathfrak{v} = \mathfrak{w}, \\ 0, & \text{otherwise.} \end{cases}$$

Conditioned on a transition, the form of $P^{\mathfrak{c}}$ is given by the uniform distribution. The stationary distribution $\pi_k^{\mathfrak{c}}$ of $\mathfrak{S}_k^{\mathfrak{c}}$ can consequently be derived directly and reversibility follows as well.

Corollary 7.31. *Let L be a $\bar{d}$-regular graph as well as $k \in \{1, \ldots, \bar{n}-1\}$ and consider the Markov chain $\mathfrak{S}_k^{\mathfrak{c}}$ on $\mathfrak{L}_k$. Then it is aperiodic, irreducible and, hence, ergodic. Furthermore, it is a reversible chain and the stationary distribution $\pi_k^{\mathfrak{c}}$ is given in terms of subsets $\mathfrak{v} \in \mathfrak{V}_k$ by*

$$\pi_k^{\mathfrak{c}}(\mathfrak{v}) = \frac{1}{\binom{\bar{n}}{k}}. \tag{7.50}$$

While further analysis is an interesting topic it has been subject to a variety of approaches and the results are quite complete, see [DiaSal93] and [Fill91]. We are going to recall some of them briefly here before going on to further properties we can derive based on $\mathfrak{L}_k$.

Since $\mathfrak{S}_k^{\mathfrak{c}}$ is reversible independently of the underlying graph, we are interested in the cases where $\mathfrak{S}_k$ is lazy as a function of k and the underlying graph L. To this end we focus exclusively on $\min_{\mathfrak{v} \in \mathfrak{V}_k} p_{k;\mathfrak{v},\mathfrak{v}}^{\mathfrak{c}}$ and discuss the implications based on sub-graphs of L.

Lemma 7.32. *Let L be a $\bar{d}$-regular graph on $\bar{n}$ vertices and $k \in \{1, \ldots, \bar{n}-1\}$. If*

$$\frac{\bar{d}}{2} \geq \max_{\mathfrak{v} \in \mathfrak{V}_k} \frac{\deg_k(\mathfrak{v})}{k} \tag{7.51}$$

then, $\mathfrak{S}_k^{\mathfrak{c}}$ is lazy.

Proof. The result follows directly from Proposition 6.1 and the definition of $\deg_k(\mathfrak{v})$. □

To compare with the generalized exclusion process we consider in what follows the case of the cycle graph L on $\bar{n}$ vertices, i.e., the regular graph with $\bar{d} = 2$. By the discussion in the previous sections, we have for $k \leq \left\lfloor \frac{\bar{n}}{2} \right\rfloor$ the characterization of the degree set of $\mathfrak{L}_k$

$$D_k = \{2\,l|\; l \in \{1, \ldots, k\}\}$$

and by symmetry $\mathfrak{L}_k \overset{\sim}{=} \mathfrak{L}_{\bar{n}-k}$ we obtain for $k \geq \left\lceil \frac{\bar{n}}{2} \right\rceil$

$$D_k = \{2\,l|\; l \in \{1, \ldots, \bar{n} - k\}\}.$$

Note that for $k \leq \left\lfloor \frac{\bar{n}}{2} \right\rfloor$ we have $\max_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v}) = 2k$ and, therefore, $\mathfrak{S}_k^{\mathfrak{c}}$ is not lazy. On the other hand for $k \geq \left\lceil \frac{\bar{n}}{2} \right\rceil$ we obtain the condition $1 \geq \frac{2(\bar{n}-k)}{k}$ which is equivalent to $3 \geq \frac{2\bar{n}}{k}$ and, therefore, $k \geq \frac{2\bar{n}}{3}$. Therefore, we obtain on the cycle graph that $\mathfrak{S}_k^{\mathfrak{c}}$ is a lazy Markov chain if and only if $\mathfrak{S}_k$ is also a lazy Markov chain. Nonetheless, the stationary distributions have nothing in common. When comparing to the second explicit case of $\bar{d} = \bar{n} - 2$ we see already that the previously observed property is more of a coincidence for a particular graph.

By Lemma 4.26 we have $D_k = \left\{ k(\bar{n} - 2) - k(k-1) + 2l \;\middle|\; l \in \left\{0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \right\} \right\}$ and, therefore, for $k \leq \frac{\bar{n}}{2}$ the equivalence

$$\frac{\bar{n}-2}{2} \geq \frac{\max_{\mathfrak{v}\in\mathfrak{V}_k} \deg_k(\mathfrak{v})}{k} \;\Leftrightarrow\; \frac{\bar{n}-2}{2} \geq (\bar{n}-2) - (k-1) + \frac{2}{k}\left\lfloor \frac{k}{2} \right\rfloor. \tag{7.52}$$

For k even this implies that

$$0 \geq \frac{(\bar{n}-2)}{2} - (k-1) + 1 = \frac{(\bar{n}-2)}{2} - k + 2$$

which is the case if and only if $k \geq \frac{\bar{n}}{2} + 1$. On the other hand for k odd we have

$$0 \geq \frac{(\bar{n}-2)}{2} - (k-1) + 1 - \frac{1}{k} = \frac{\bar{n}}{2} + 1 - k - \frac{1}{k}$$

which amounts to

$$k \leq \frac{1}{4}(-\sqrt{(\bar{n}+2)^2 - 16} + \bar{n} + 2) < 1$$

for $\bar{n} > 2$ or

$$k \geq \frac{1}{4}(\sqrt{(\bar{n}+2)^2 - 16} + \bar{n} + 2) > \frac{\bar{n}}{2}$$

for $\bar{n} > 2$. We conclude that the chain is never lazy if $k \leq \frac{\bar{n}}{2}$. We, therefore, turn to the case $k > \frac{\bar{n}}{2}$ which leads to the equality

$$D_k = \left\{ (\bar{n}-k)(\bar{n}-2) - (\bar{n}-k)((\bar{n}-k)-1) + 2l \;\middle|\; l \in \left\{0, \ldots, \left\lfloor \frac{\bar{n}-k}{2} \right\rfloor \right\} \right\}.$$

Again, for k even this implies that

$$0 \geq \left((\bar{n}-k) - \frac{k}{2}\right)(\bar{n}-2) - (\bar{n}-k)((\bar{n}-k)-1) + (\bar{n}-k)$$
$$= \left(\bar{n} - \frac{3\,k}{2}\right)(\bar{n}-2) - (\bar{n}-k)((\bar{n}-k)-2) = k\left(\frac{\bar{n}}{2} - k + 1\right)$$

which is satisfied for any $k > \frac{\bar{n}}{2}$. Hence, for $\bar{d} = \bar{n} - 2$ we see a strong difference in the characteristic properties of $\mathfrak{S}_k$ and $\mathfrak{S}_k^{\mathfrak{c}}$. While $\mathfrak{S}_k$ is according to Theorem 7.26 only for $k = \bar{n} - 1$ lazy, $\mathfrak{S}_k^{\mathfrak{c}}$ is lazy for any $k > \frac{\bar{n}}{2}$.
We are going to explore the divergence in behavior of the two Markov chains further. To this end, we turn to the behavior of the transition probabilities when varying the degree $\bar{d}$. In Figure 46 we illustrate the changes for growing $\bar{d}$ for the minimal transition probabilities. One can observe monotony for the resulting curves but while they are

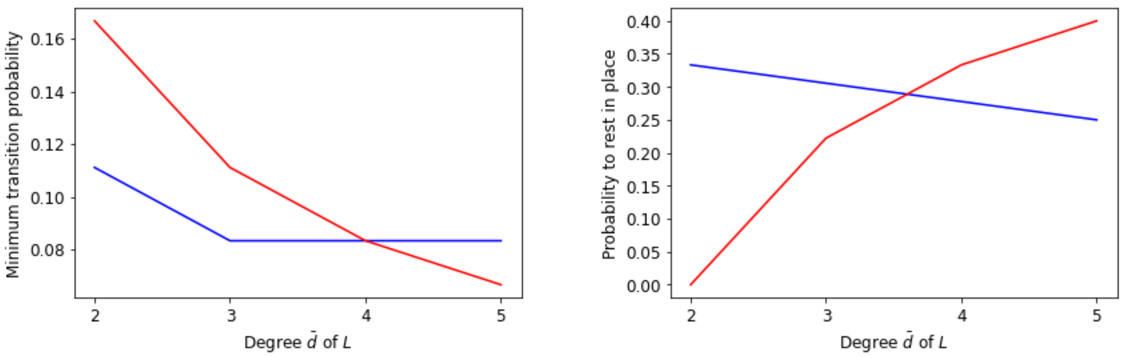

Figure 46: For an underlying $\bar{d}$ regular graph on 6 vertices we show the behavior of the minimal transition probabilities when varying $\bar{d}$ between 2 and 5. On the righthandside, we consider the case when the start and end configuration are different. On the lefthandside we consider identical ones, i.e., the probability to stay put. The blue curve represents $\mathfrak{S}_k$ and the red one $\mathfrak{S}_k^{\mathfrak{c}}$.

decreasing in both cases for $\mathfrak{S}_k$ they show different monotony types for $\mathfrak{S}_k^{\mathfrak{c}}$. Evidently, under the assumptions of Proposition 7.28 we can, again, make consistent quantitative claims when varying $\bar{d}$. Adapting the notation from Proposition 7.28 the transition probabilities $P_k^{\mathfrak{c},\bar{d}}$ satisfy for any $\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k^{\bar{d}}$ independently of $\mathfrak{v}$ and $\mathfrak{w}$

$$p_{k;\mathfrak{v},\mathfrak{w}}^{\mathfrak{c},\bar{d}} = \frac{1}{k\bar{d}} > \frac{1}{k(\bar{d}+1)} = p_{k;\mathfrak{v},\mathfrak{w}}^{\mathfrak{c},\bar{d}+1}.$$

This is still in accordance with the property derived for $\mathfrak{S}_k^{\bar{d}}$ derived in Proposition 7.28. We consider now the probability of $\mathfrak{S}_k^{\mathfrak{c},\bar{d}}$ to stay put at any time step. We have seen that for $\mathfrak{S}_k^{\bar{d}}$ under variation of $\bar{d}$ it satisfies $p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle,\bar{d}+1} \leq p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle,\bar{d}}$ which is not true for $\mathfrak{S}_k^{\mathfrak{c},\bar{d}}$. To this end consider the following derivations where $\mathfrak{v}^+ = \{v \in \mathfrak{v} | \deg_{\bar{d}}^{\mathfrak{v}}(v) < \deg_{\bar{d}+1}^{\mathfrak{v}}(v)\}$.

$$\begin{aligned} p_{k;\mathfrak{v},\mathfrak{v}}^{\mathfrak{c},\bar{d}} - p_{k;\mathfrak{v},\mathfrak{v}}^{\mathfrak{c},\bar{d}+1} &= \sum_{v \in \mathfrak{v}} \frac{\deg_{\bar{d}}^{\mathfrak{v}}(v)}{k\bar{d}} - \frac{\deg_{\bar{d}+1}^{\mathfrak{v}}(v)}{k(\bar{d}+1)} = \sum_{v \in \mathfrak{v}^+} \frac{\deg_{\bar{d}}^{\mathfrak{v}}(v)}{k\bar{d}} - \frac{\deg_{\bar{d}+1}^{\mathfrak{v}}(v)}{k(\bar{d}+1)} \\ &= \sum_{v \in \mathfrak{v}^+} \frac{\deg_{\bar{d}}^{\mathfrak{v}}(v)(\bar{d}+1) - (\deg_{\bar{d}}^{\mathfrak{v}}(v)+1)\bar{d}}{k(\bar{d}+1)\bar{d}} = \sum_{v \in \mathfrak{v}^+} \frac{\deg_{\bar{d}}^{\mathfrak{v}}(v) - \bar{d}}{k(\bar{d}+1)\bar{d}} \leq 0. \end{aligned}$$

Moreover, equality holds if for all $v \in \mathfrak{v}$ the identity $\deg_{\bar{d}}^{\mathfrak{v}}(v) = \bar{d}$ is satisfied, which implies immediately $k = \bar{n}$ and is, therefore, never possible. Consequently, $p_{k;\mathfrak{v},\mathfrak{v}}^{\mathfrak{c},\bar{d}}$ is strictly increasing in $\bar{d}$, which is the opposite of the behavior established in Proposition 7.28 for $p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle,\bar{d}}$. Hence, we can conclude that while for $\bar{d} = 2$ both chains have identical thresholds for k beyond which they become lazy Markov chains afterwards they diverge from one another as $\bar{d}$ increases, the threshold for $\mathfrak{S}_k^{\bar{d}}$ increasing due to the decreasing behavior $p_{k;\mathfrak{v},\mathfrak{v}}^{\triangle,\bar{d}}$ while the threshold for $\mathfrak{S}_k^{\mathfrak{c},\bar{d}}$ decreases due to the increasing behavior $p_{k;\mathfrak{v},\mathfrak{v}}^{\mathfrak{c},\bar{d}}$.

To conclude, we can say that the choices we made for the transition probabilities lead to a Markov chain which diverges hugely in its behavior from this classic example. Hopes to compare them beyond the reversible cases, for which the tools developed in would come in handy, if they were not analytically approachable, are, consequently, in vain. This leaves us with the question of comparability of Markov chains to obtain results on generalized exclusion processes as was done for multiple random walks by comparing with single random walks, see [Abd12] and classical exclusion processes by comparing also single particles, see for example [Oliv13].

In the next subsection we come back to the motivating example of this whole section, the reduced Echo Chamber Model and the behavior of its edges over long time. We discuss the results of this subsection in light of the Johnson graph as underlying graph L and make further observations based on the strongly regular structure of this graph.

7.5 Interpretation for relationship dynamics

In this subsection we come back to the initially posed problem to analyze the exclusion process which arises form the reduced version of the Echo Chamber Model. The underlying graph L takes, therefore, a particular form, namely for some $n \in \mathbb{N}$ it satisfies $L = \operatorname{srg}\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right)$. In particular, it is a strongly regular graph with $\alpha \geq 1$ if $n \geq 3$. With the idea in mind, that we want consider large but finite populations in the context of the Echo Chamber Model, we can ignore the case $n = 2$, which amounts L being a graph consisting of a single vertex. Theorem 7.18 gives the following corollary.

Corollary 7.33. *Consider the particles process* $\mathfrak{S}_k$ *and the Markov chain* $S = (S_t)_t$ *induced by the reduced Echo Chamber Model*

1. *Let* L *be a strongly regular graph with parameters* $(\bar{n}, \bar{d}, \alpha, \beta)$*. Then process* $\mathfrak{S}_k$ *is reversible if and only if* $\bar{d} \in \{2, \bar{n}-2, \bar{n}-1\}$ *or* $k \in \{1, 2, \bar{n}-2, \bar{n}-1\}$*.*
2. *Consider the process* $S = (S_t)_t$ *on a population of* n *individuals with* k *relationships. Then the associated Markov chain* $\mathfrak{S}_k$ *is reversible if and only if* $n = 3$ *or* $k \in \{1, 2, \frac{n(n-1)}{2} - 2, \frac{n(n-1)}{2} - 1\}$*.*

Proof. Both claims follow from Theorem 7.18, Theorem 7.20 and the fact that if $\alpha = 0$, L contains a double-pitchfork since no two vertices have common neighbors. □

One would be tempted to make additional claims given that the problem is so much more structured than the general problem on regular graphs which we considered beforehand. Let us begin with the structure of the stationary distribution of $\mathfrak{S}_k$. By Theorem 7.8 two configuration $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$ carry the same weight with respect to $\tilde{\pi}_k$ if $L_{\mathfrak{v},\mathfrak{v}^c} \cong L_{\mathfrak{w},\mathfrak{w}^c}$. Finding the number of distinct values of $\tilde{\pi}_k$ amounts, consequently, to a more involved task than searching the number of non-isomorphic graphs on n vertices and k edges. This task has been resolved and can be found in [Har71]. The whole expression can also be found in [Weis22]. Therefore, we find the number of level sets of $\tilde{\pi}_k$ using this identification of the problems in the form of a highly complicated formula. Further results on the stationary distribution, in particular the explicit values are out of reach for now.

Turning now to the convergence speed in the sense of Theorem 7.30, we want to recall that the underlying graph is $L = \operatorname{srg}\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right)$. In this setting, the condition $\bar{d} + k + 1 \le \bar{n}$ is equivalent to

$$k \le \frac{n(n-1)}{2} - 2n - 3 = \frac{1}{2}(n^2 - 5n - 6) = \frac{1}{2}\left(n - \frac{5}{2}\right)^2 - \frac{49}{8} \tag{7.53}$$

such that k can even be of the order of n^2. Publications on the density of social networks and the internet at large show that the degree of each vertex is distributed like a power law distribution, see for example [MuPei13] and [Hub99]. This implies that k is in these cases of order n^α for some $\alpha \in (1, 2)$. Therefore, this restriction is not really one, and we can assume that it is met, ignoring some theoretical corner cases where the graph contains almost all its edges. Therefore, we can apply Theorem 7.30. Firstly, we find the trivial bound that $\delta = \operatorname{diam}(\mathfrak{L}_k) \le k \operatorname{diag}(L) = 2k$ since the diameter of all strongly regular graphs with $\beta > 0$ is equal to 2 and k particles have to be moved along the edges of L. Therefore, finding the convergence speed is linked to the problem of characterizing the set of vectors which describe the number of walks in $\mathfrak{L}_k$ between any two configurations. In spite of the very clear structure of srg $\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right)$, this turns out to be a computationally demanding combinatorial problem and the author did not manage to find a satisfying description of the convex hull of these vectors for a complete analysis of the constants involved in Theorem 7.30. Indeed, this leads into the field of analyzing sub-graphs of the Johnson graph $\mathcal{J}(n, 2)$ and the path space of the associate graph $\mathfrak{L}_k$ which can be of interest in itself and would also contribute graph theory aside from its application to this example of the Markov chain $\mathfrak{S}_k$. This might be another fruitful connection between $\mathfrak{S}_k$ and the analysis of sub-graphs, which is, unfortunately, out of scope of this work.

8 Exclusion processes in random absorbing environments

Outline of this section: In the previous section, we have analyzed in detail the behavior of $\mathfrak{S}_k$ on $\mathfrak{L}_k$ as a finite Markov chain. In this section, we constrain the state space by introducing a random environment on L which is absorbing for the particles of $\mathfrak{S}_k$. We discuss the link between the geometric properties of the absorbing environment and the finiteness of the time to absorption for $\mathfrak{S}_k$ as a whole particle system. Additionally, we describe explicitly the resulting state space of the implied Markov chain $\mathfrak{S}_k^{\text{abs}}$ and find an upper bound on the expected time to absorption as a function of its quasi-stationary distribution. Finally, we discuss the implications for the Echo Chamber Model with continuous opinions with constant opinions and find negative results for the time to convergence independent of the number k of particles.

8.1 From echo chambers to random absorbing environments

We consider a graph which changes randomly over time due to conflict between the individuals. To this end, consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of simple but not necessarily connected graphs $(G_t)_{t\in\mathbb{N}} = ((\mathcal{V}, \mathcal{E}_t))_{t\in\mathbb{N}}$ with for $t \in \mathbb{N}$ a vertex set $\mathcal{V}$ with $|\mathcal{V}| = n$ and an edge set $\mathcal{E}_t$ with $|\mathcal{E}_t| = k$. Every $a \in \mathcal{V}$ carries a label given by a random variable $X_a \sim \mathcal{L}([0,1])$ on $(\Omega, \mathcal{F}, \mathbb{P})$ representing an individual's own opinion. The set $\{X_a | a \in \mathcal{V}\}$ is assumed to a set of i.i.d. random variables. Furthermore, let $\theta \in [0,1]$. The value θ is the individual tolerance threshold, which we assume to be identical for every vertex, i.e., individual in the population. The graph then evolves in discrete time following the hereinafter described rule at each time step form t to $t+1$

- Draw uniformly an edge $\langle A, B\rangle \in \mathcal{E}_t$.
- Define $N_{\langle A,B\rangle} = \{e = \langle c, d\rangle \notin \mathcal{E}_t \backslash \{\langle A, B\rangle\} | k \in \{A, B\} \text{ or } l \in \{A, B\}\}$.
- If $|X_A - X_B| \geq \theta$
  - remove the edge with probability $p \in [0,1]$,
  - draw uniformly E from $N_{\langle A,B\rangle}$,
  - set $\mathcal{E}_{t+1} = (\mathcal{E}_t \backslash \{\langle A, B\rangle\}) \cup \{E\}$.

Choosing instead for some $N \in \mathbb{N}$ and $X_a \sim \mathcal{L}\left(\left\{\frac{i}{N} | i \in \{1, \ldots, N\}\right\}\right)$ as well as $\theta < N^{-1}$ one can transform the model into a discrete opinion one. In both cases, this leads to a dynamic graph process which is governed by the vertex labels, which may be interpreted as the opinions of the corresponding individual within the network. Due to the condition on the difference of opinions, some connections will always be preserved once establish. This leads to the question if a network may separate into homogeneous groups of identical opinions under this dynamic. In what follows we are going to consider the dynamics again in the context of particles on the line graph of the complete graph and make in particular distinctions between certain types of label distributions.

Consider, therefore, the line graph $L = (V, E)$ of the complete graph $\hat{G}$ on n vertices with vertex labels $Y = (Y_v)_{v \in V}$ with $Y_v = ||X_a - X_b||_2$ if $v = \langle a, b \rangle$ such that $Y_v \sim \mathcal{L}([0,1])$ with $||.||_2$ being the usual Euclidean norm. Define a new Markov chain based on an appropriate transition matrix $P_\theta^\triangle$ where $P_0^\triangle := P^\triangle$. Consider $\mathfrak{S}_{k,\theta}$ as defining the dynamics of the whole configuration on $\mathfrak{L}_k$. Given a fixed set of vertex labels it becomes a Markov chain on $\mathfrak{L}_k$. Let $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k$. Then the transition between confiurations, capturing the more involved dynamics induced by the labels is defined by the transition probability

$$p^\theta_{k;\mathfrak{v},\mathfrak{w}} := p^\triangle_{k;\mathfrak{v},\mathfrak{w}} \mathbb{1}_{Y_v \geq \theta} \mathbb{1}_{\langle \mathfrak{v},\mathfrak{w} \rangle \in \mathfrak{E}_k} \tag{8.1}$$

and $p^\theta_{k;\mathfrak{v},\mathfrak{v}} := 1 - \sum_{\mathfrak{w} \neq \mathfrak{v}} p^\theta_{\mathfrak{v},\mathfrak{w}}$.

Due to the introduced factor $\mathbb{1}_{Y_v \geq \theta}$ in the transition probabilities, cases may arise where $p^\theta_{k;\mathfrak{v},\mathfrak{v}} = 1$. This renders the state $\mathfrak{v}$ absorbing. This may happen for the following two reasons.

- If for all $v \in \mathfrak{v}$ the condition $Y_v < \theta$ is satisfied.
- If for some subset $\mathfrak{v}' \subset \mathfrak{v}$ we have $\forall v \in \mathfrak{v}' : Y_v < \theta$ and for all $w \in \mathfrak{v} \setminus \mathfrak{v}'$ the identity $\deg^{L_\mathfrak{v}}(w) = \deg(w)$ is satisfied.

To illustrate this further, we consider again the case we touched upon in Figure 34. Assume that there are 4 absorbing states as visualized in Figure 47. The red vertices symbolize said absorbing sites and bright blue vertices represent empty sites. Assume

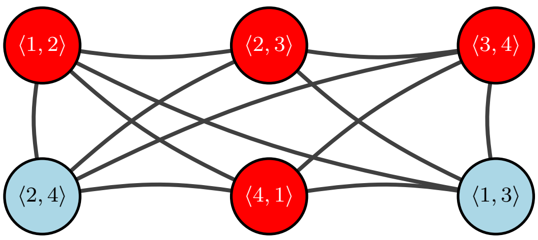

Figure 47: Four absorbing sites in the line graph of K_4. These are represented as red vertices. Particles behave as usual on blue vertices.

that $k = 5$, i.e., we consider a generalized exclusion process on k particles in the random environment induced by the labels and assume, furthermore, that for some $t \in \mathbb{N}$ we have $\mathfrak{S}_{k,\theta;t} = \mathfrak{v}$ with $\mathfrak{v} = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 2,4\rangle, \langle 1,4\rangle\}$ as shown in Figure 48. Then $\{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 1,4\rangle\}$ are absorbing states occupied by particles. Those particles will, therefore, remain forever in their state. But also $\langle 2,4\rangle$ becomes an absorbing state since all possibilities to leave are blocked for the particle occupying $\langle 2,4\rangle$ since all its neighbors are absorbing states occupied by particles. Therefore, we can deduce, that,

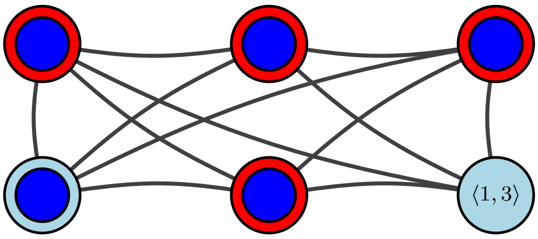

Figure 48: Particle configuration of $k = 5$ particles on the line graph of K_4. Four particles are in absorbing states while the final one, which is not in an absorbing state, is blocked by the others.

since some states are absorbing the process may stop to move at some point but this is not necessarily enforced by the absorbing states in themselves but also their position in the network. Figure 48 illustrates that the Markov chain $\mathfrak{S}_{k,\theta}$ may even converge when k is larger than the actual number of absorbing vertices in L. This is in fact a general property of multi-particle dynamics in random absorbing environments and is equally fascinating in the context of the Echo Chamber Model and the underlying assumptions on the opinions. We are going to discuss this in what follows.

8.2 Implied topology and finite time to absorption

We are going to consider the problem on the graph $L = (V, E)$ in a rather general form. To this end, consider random labels $(Z_v)_{v\in V}$ with $Z_v \in \{0, 1\}$ almost surely. A vertex $v \in V$ is considered absorbing if $Z_v = 0$. This corresponds to $Z_v = \mathbb{1}_{Y_v \geq \theta}$ in the previously discussed setting of the Echo Chamber Model.
To include the case illustrated in Figure 48 in the setting of absorbing Markov chains we need the following definitions and observations.

Definition 8.1. *Consider a $\bar{d}$-regular graph $L = (V, E)$ with $|V| = \bar{n}$ and let $k \in \{1, \ldots, \bar{n} - 1\}$. We define the time to absorption of the Markov chain $\mathfrak{S}_{k,\theta}$ on $\mathfrak{L}_k$ by*

$$T^0_{k,\theta} := \inf\{t \geq 0 | \exists \mathfrak{v} \in \mathfrak{V}_k : \mathfrak{S}_{k,\theta;t} = \mathfrak{v},\ p^\theta_{k;\mathfrak{v},\mathfrak{v}} = 1\}. \tag{8.2}$$

We want to emphasize with Lemma 8.2 that this does neither mean that all particles are in absorbing sites, i.e., for all $v \in \mathfrak{v}$ we have $Z_v = 0$, nor that all absorbing states are occupied even if the number of particles is larger then the number of absorbing sites.

Lemma 8.2. *Consider a $\bar{d}$-regular graph $L = (V, E)$ with $\bar{n} := |V|$ and let $k \in \{1, \ldots, \bar{n} - 1\}$. Assume the vertices of L carry labels $(Z_v)_{v \in V}$ with $Z_v \in \{0, 1\}$ such that v is absorbing if and only if $Z_v = 0$. Denote by $\mathcal{Z} := \{v \in V | Z_v = 0\}$. If $k \leq \bar{d}$ then either $|\mathcal{Z}| < k$ and $T^0_{k,\theta} = \infty$ or $T^0_{k,\theta} < \infty$ almost surely.*

Proof. Assume first that $|\mathcal{Z}| < k$. Then, for all $\mathfrak{v} \in \mathfrak{V}_k$ we have $|\mathcal{Z} \cap \mathfrak{v}| < k$. Since $k \leq \bar{d}$ the number of absorbing states is given by $|\mathcal{Z}|$ since no vertex $v \in V$ may have only absorbing neighbors all of them being occupied by particles. By assumption $|\mathcal{Z}| < k$ such that there remains for all $\mathfrak{v} \in \mathfrak{V}_k$ always at least one particle $v \in \mathfrak{v}$ which is not in an absorbing state such that for all $\mathfrak{v} \in \mathfrak{V}_k$ we have $p^{\theta}_{k;\mathfrak{v},\mathfrak{v}} < 1$ which implies $T^0_{k,\theta} = \infty$.
On the other hand if $|\mathcal{Z}| < k$ then there exists a set $\mathfrak{v} \in \mathfrak{V}_k$ such that for all $v \in \mathfrak{v}$ we have $Z_v = 0$. Furthermore, since $k \leq \bar{d}$ the minimal number of accessible states $v \in \mathfrak{L}$, i.e, states which satisfy that there is a $w \in N_v$ such that $w \notin \mathcal{Z}$, is greater or equal k. Consequently, there is a state in $\mathfrak{V}_k$ with positive probability to be entered and by finiteness of the state space of $\mathfrak{S}_{k,\theta}$ we obtain $T^0_{k,\theta} < \infty$ almost surely. □

Evidently, setting $k = 1$ we come back to the classical setting of a random walk on a graph with random absorbing vertices. Hence, for $k \leq \bar{d}$ we can make a distinction of the finiteness of $T^0_{k,\theta}$ only based on the size of the set $\mathcal{Z}$. For an i.i.d. environment we can make immediately the following claim.

Lemma 8.3. *Consider a $\bar{d}$-regular graph $L = (V, E)$ with $|V| = \bar{n}$ and let $k \in \{1, \ldots, \bar{n} - 1\}$, $k < \bar{d}$. Assume the vertices of L carry i.i.d. labels $(Z_v)_{v \in V}$ with $Z_v \sim \mathcal{B}(p)$ with $p \in (0, 1)$ such that v is absorbing if and only if $Z_v = 0$. Then*

$$\mathbb{P}[T^0_{k,\theta} < \infty] = 1 - \sum_{j=0}^{k-1} \binom{\bar{n} - 1}{j} (1 - p)^j p^{\bar{n} - j} \tag{8.3}$$

and, hence, it follows $\mathbb{E}[T^0_{k,\theta}] = \infty$.

Proof. The result follows since $\sum_{v \in V} Z_v \sim \mathrm{Bin}(\bar{n}, p)$ and under the condition $k < \bar{d}$ the inequality $T^0_{k,\theta} < \infty$ is equivalent to $\sum_{v \in V} Z_v = |\mathcal{Z}| \geq k$. □

It turns out that $k < \bar{d}$ is not necessarily an interesting case. Consider for example the Echo Chamber Model and, hence, L as the line graph of a complete graph on n vertices. Then $\bar{n} = \dfrac{n(n-1)}{2}$ and $\bar{d} = 2(n - 2)$ such that $\bar{d}$ grows as $\sqrt{\bar{n}}$. In turn, recalling that k is the number of existing edges in the network at each time t, applications, as already mentioned see for example [MuPei13] and [Hub99], motivate rather a growth behavior of k in the order of $\bar{n}^\alpha$ with $\alpha \in (1,2)$. Hence, we rather have to focus on the case $k > \bar{d}$. On the other hand, if $k > \bar{d}$ the process may run into various complications which we are going to discuss in what follows. In particular, we need the following definition underlining the fact that $\mathcal{Z}$ may have profound implications on the chain $\mathfrak{S}_{k,\theta}$ beyond the fact that it renders states absorbing.

Definition 8.4. *Consider a $\bar{d}$-regular graph $L = (V, E)$ with $|V| = \bar{n}$, labels $Z = (Z_v)_{v\in V}$ on the vertex set and let $k \in \{1, \dots, \bar{n} - 1\}$. Recall $\mathcal{Z} = \{v \in V | Z_v = 0\}$. We call a vertex $v \in V$ inaccessible if $N_v \subseteq \mathcal{Z}$, denote the set of all inaccessible vertices of L with respect to Z by $\mathcal{I}$ and define $\mathcal{Z}_\iota := \mathcal{Z} \cap \mathcal{I}$ as the set of blocked absorbing states.*

We illustrate the notion of inaccessible vertices in Figure 49, showing the possibility that both non-absorbing and absorbing states may be inaccessible. Drawing from

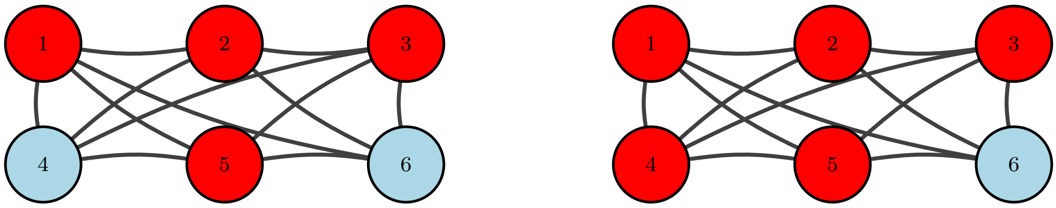

Figure 49: Inaccessible states may both be absorbing or not. All neighbors are absorbing such that a particle cannot enter an inaccessible state. The vertex 4 shows the two possibilities in the two graphs.

Figure 49 we see that, the non-absorbing accessible states may yield counter-intuitive interpretations because they are in some sense absorbed while not being in an absorbing state based on their label. The bigger problem arises nonetheless from the vertices in $\mathcal{Z}_\iota$ since they render the set of actual absorbing states more difficult to reach. To illustrate this point assume that $\bar{n} - |\mathcal{Z}| > k$, and for some $t \in \mathbb{N}$ we have $\mathfrak{S}_{k,\theta;t} = \mathfrak{v} \subset V \backslash \mathcal{Z}$ and $|\mathcal{Z} \backslash \mathcal{Z}_\iota| < k$. Then, $\mathfrak{S}_{k,\theta}$ can never be completely absorbed, since only the state $\mathcal{Z} \backslash \mathcal{Z}_\iota$ can be reached from $V \backslash \mathcal{Z}$. Hence, there will always be $k - |\mathcal{Z} \backslash \mathcal{Z}_\iota|$ free particles such that $T^0_{k,\theta} = \infty$ almost surely, conditioned on these assumptions.

8.3 Expected time to absorption in cell free environments

In the previous subsection we have discussed, based on Figure 49, that both non-absorbing accessible states and the vertices in $\mathcal{Z}_\iota$ may influence greatly the behavior of the process. While these structures may theoretically arise with positive probability based on the labels $(Z_v)_{v\in V}$ they may not be "natural" in the sense inspired by an underlying model. To this end, we consider the Echo Chamber Model on a network with distinct colors presented in [HePraZha11]. Edges are only preserved if they connect two individuals of identical opinion. Otherwise, they are terminated and recreated according to the dynamics described in the beginning of this whole section. The opinions are drawn independently for each individual and according to the same distribution. Assuming that there are N colors on the graph $G = (\mathcal{V}, \mathcal{E})$, we can translate the dynamics in [HePraZha11] to our setting, by considering any probability measure $\mathcal{L}(\{\frac{i}{N} | i = 1, \dots, N\})$ for the labels $(X_v)_{v \in \mathcal{V}}$ and $\theta < \frac{1}{N}$. Consequently, Assuming that there is a path $(v_1, \dots, v_l)$ from $v \in \mathcal{V}$ to $w \in \mathcal{V}$ of short edges, then $X_v = X_w$. This idea forms the basis for the following preliminary result.

Proposition 8.5. *Let $\hat{G} = (\mathcal{V}, \mathcal{E})$ be the complete graph on n vertices with labels $(X_a)_{a \in \mathcal{V}}$ distributed i.i.d. as $\mathcal{L}(\{\frac{i}{N} | i = 1, \dots, N\})$ and let $\theta < \frac{1}{N}$. Denote by $L = (V, E)$ the line-graph of the complete graph on n vertices and define by $(Y_{\langle a,b \rangle})_{\langle a,b \rangle \in V} = (|X_a - X_b|)_{(a,b) \in \mathcal{V}^2,\, a \neq b}$ the labels on the vertices of L and by $Z = (Z_{\langle a,b \rangle})_{\langle a,b \rangle \in V} = (\mathbb{1}_{Y_{\langle a,b \rangle} \geq \theta})_{\langle a,b \rangle \in V}$. Then, the set $\mathcal{I}$ of inaccessible states of L with respect to Z is almost surely empty or V.*

Proof. Let $v \in \mathcal{I} \subseteq V$. Then, by definition $N_v \subseteq \mathcal{I}$ and there is an edge $\langle a, b \rangle$ in G such that $v = \langle a, b \rangle$. Since $v \in \mathcal{I}$ we obtain that all $a', b' \in \mathcal{V} \backslash \{a, b\}$ satisfy $|X_a - X_{a'}| = 0$ and $|X_b - X_{b'}| = 0$ such that for all $a, b \in \mathcal{V}$ we conclude $X_a = X_b$, i.e., all vertices in G have the same color. Consequently, $\mathbb{1}_{Y_{\langle a,b \rangle} \geq \theta} = 0$ for any $\langle a, b \rangle \in V$ such that $\mathcal{I} = V$. On the other hand, if not all vertices in G have the same color, we arrive at a contradiction such that $\mathcal{I} = \emptyset$. □

Hence, for the case of distinct colors, which represent the opinions, we can get rid of such structures and, indeed, it turns out that under these conditions removing all $v \in \mathcal{Z}$ from L we still have a connected graph L'. This gives us the following percolation-like result.

Proposition 8.6. *Let $\hat{G} = (\mathcal{V}, \mathcal{E})$ be the complete graph on n vertices with labels $(X_a)_{a \in \mathcal{V}}$ distributed as $\mathcal{L}(\{\frac{i}{N} | i = 1, \dots, N\})$ and let $\theta < \frac{1}{N}$. Denote by $L = (V, E)$ the line-graph of the complete graph on n vertices and define by $(Y_{\langle a,b \rangle})_{\langle a,b \rangle \in V} = (|X_a - X_b|)_{(a,b) \in \mathcal{V}^2,\, a \neq b}$ the labels on the vertices of L and by $Z = (Z_{\langle a,b \rangle})_{\langle a,b \rangle \in V} = (\mathbb{1}_{Y_{\langle a,b \rangle} \geq \theta})_{\langle a,b \rangle \in V}$. Then, the vertex induced sub-graph $L_{V \backslash \mathcal{Z}}$ is almost surely connected.*

Proof. By Proposition 8.5 we have to consider the two cases $\mathcal{I} = V$ and $\mathcal{I} = \emptyset$. In the first case, $L_{V \backslash \mathcal{Z}}$ is the empty graph with no vertices and no edges. Discussing connectedness of this graph is not center of this work and we assume that it is without loss of generality connected.

In the second case, $\mathcal{I} = \emptyset$, we assume that $L_{V\setminus\mathcal{Z}}$ is disconnected, i.e., there are $V_1, V_2 \subseteq V\setminus\mathcal{Z}$ such that $V\setminus\mathcal{Z} = V_1 \sqcup V_2$. Let $v_1 \in V_1$ and $v_2 \in V_2$. Then there is a path ϕ of length l between v_1 and v_2 in L such that for some a $i \in \{1, \ldots, l\}$ the segment $\phi(i) \in \mathcal{Z}$ and $\phi(i-1) \in V_1$ as well as $\phi(i+1) \in V_2$ or $\phi(i+1) \in \mathcal{Z}$ and $\phi(i+2) \in V_2$ because $\mathcal{I} = \emptyset$. We focus on the first case where $\phi(i+1) \in V_2$ since the second case

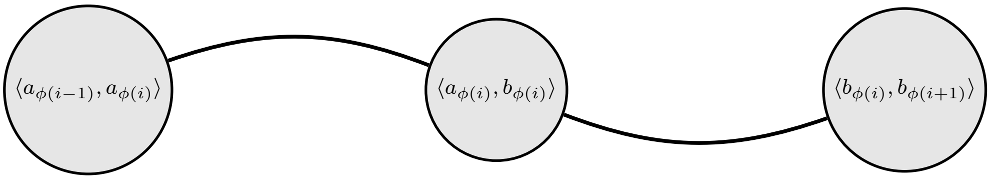

Figure 50: Neighborhood relationship of $\phi(i) = \langle a_{\phi(i)}, b_{\phi(i)}\rangle$, $\phi(i-1) = \langle a_{\phi(i-1)}, a_{\phi(i)}\rangle$ and $\phi(i+1) = \langle b_{\phi(i)}, b_{\phi(i+1)}\rangle$.

follows using the same argument on $\phi(i+2) \in V_2$.

Rewriting $\phi(i \pm 1)$ and $\phi(i)$ as edges in G, we obtain links as depicted in Figure 50, when writing $\phi(i) = \langle a_{\phi(i)}, b_{\phi(i)}\rangle$, $\phi(i-1) = \langle a_{\phi(i-1)}, a_{\phi(i)}\rangle$ and $\phi(i+1) = \langle b_{\phi(i)}, b_{\phi(i+1)}\rangle$. Then, since $\hat{G}$ is the complete $v = \langle a_{\phi(i-1)}, b_{\phi(i)}\rangle \in V$ and $(\phi(i-1), v, \phi(i+1))$ is a path connecting $\phi(i-1)$ and $\phi(i+1)$ in L. Consequently, $v \in \mathcal{Z}$. Considering the labels X in G, we obtain that, based on the preceding discussion, $X_{a_{\phi(i)}} = X_{b_{\phi(i)}}$ and $X_{a_{\phi(i-1)}} = X_{b_{\phi(i)}} = X_{a_{\phi(i)}}$. Therefore, $\phi(i-1) \in \mathcal{Z}$ which is a contradiction and we obtain that $L_{V\setminus\mathcal{Z}}$ is almost surely connected. □

Indeed, the result presented in Proposition 8.6 ensures that any absorbing vertex in L can be reached at any time no matter the particle configuration. Also, comparing to the discussion in the previous subsection, we see that the implied topology of the absorbing states due to the underlying graph G gives the intuitive properties we are looking for while labeling only the graph L may lead to blockages in various ways. In the interpretation of the Echo Chamber Model, this implies that the relationships always stabilize over time and at some point only individuals with identical opinions maintain relationships, as long as the total number of relationships is not too large. We find a sufficient condition on the total number of edges in G for them to be not "too many".

Proposition 8.7. *Let $\hat{G} = (\mathcal{V}, \mathcal{E})$ be the complete graph with $n = |\mathcal{V}|$, $N \in \mathbb{N}$, $N \geq 1$ and labels $X_a \sim \mathcal{L}(\{\frac{i}{N}|i = 1, \ldots, N\})$ for $a \in \mathcal{V}$. Then, under the assumption $\theta < N^{-1}$ the inequality*

$$\left|\left\{\langle a,b\rangle \in \mathcal{E} \middle| \, |X_a - X_b| < \theta\right\}\right| \geq \frac{1}{2}\left(N\left\lfloor\frac{n}{N}\right\rfloor^2 - n\right) \tag{8.4}$$

is almost surely satisfied.

Proof. Fix the labels of the vertices $\mathcal{V}$ and call them $(X_a)_{a\in\mathcal{V}}$. Let $n_i := \{a \in \mathcal{V}|X_a = \frac{i}{N}\}$ for $i = 1, \ldots, N$. Hence, almost surely $\sum_{i=1}^N n_i = n$. Therefore, the number of short edges in G is given by

$$\sum_{i=1}^{N} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\left(\sum_{i=1}^{N} n_i^2 - n\right).$$

The sum $\sum_{i=1}^N n_i^2$ is minimized by the set $\{n_i^*|i \in \{1, \ldots, N\}\}$ if the n_i^* are as equal as possible, i.e., $|n_i^* - n_j^*| \in \{0, 1\}$ for all i, j, and in particular $n_i^* \geq \left\lfloor \frac{n}{N} \right\rfloor$. We conclude that

$$\left|\left\{\langle a,b\rangle \in \mathcal{E} \middle| \, |X_u - X_v| < \theta\right\}\right| = \sum_{i=1}^{N} \frac{n_i(n_i-1)}{2} \geq \frac{1}{2}\left(N\left\lfloor\frac{n}{N}\right\rfloor^2 - n\right).$$

□

Translating this condition to the line graph L and the k particle system, we obtain that under the condition $k \leq \frac{1}{2}\left(N\left\lfloor\frac{n}{N}\right\rfloor^2 - n\right)$ the number of absorbing sites $\mathcal{Z}$ satisfies $|\mathcal{Z}| \geq k$. Combining Proposition 8.7 and Proposition 8.6 we obtain that $T_{k,\theta}^0$ corresponds to the first time where all edges are short and, therefore, only individuals of identical opinions are connected.

Corollary 8.8. *Let $G_0 = (\mathcal{V}, \mathcal{E}_0)$ be some initial graph with labels $(X_a)_{a\in\mathcal{V}}$. Fix $N \in \mathbb{N}$, $N \geq 1$, let $\theta < N^{-1}$ and assume that the labels are distributed as $X_a \sim \mathcal{L}(\{\frac{i}{N}|i = 1, \ldots, N\})$ for $a \in \mathcal{V}$. Then, if $k < \frac{1}{2}\left(N\left\lfloor\frac{n}{N}\right\rfloor^2 - n\right)$*

$$T_{k,\theta}^0 = \inf\{t \geq 0|\, \exists \mathfrak{v} \subset V : \mathfrak{S}_{k,\theta;t} = \mathfrak{v}, \mathfrak{v} \subset \mathcal{Z}\} \tag{8.5}$$

and $T_{k,\theta}^0 < \infty$ almost surely.

Proof. The proof follows from Proposition 8.6, which implies that a the probability that $\mathfrak{S}_{k,\theta;t}$ is not absorbed is dominated by a geometric distribution with t loosing draws, and Proposition 8.7. □

Indeed, the structure defined by Proposition 8.6, i.e., $L_{V\setminus\mathcal{Z}}$ is connected, allows for a natural, billiard like interpretation of the absorbing sites and the particles on the graph. We use this notion to define for general exclusion processes cell-free-environments which imply the natural interpretation of the time to absorption $T^0_{k,\theta}$ given by equation 8.5. For the rest of the subsection we assume that $\theta < N^{-1}$ and write $\mathfrak{S}^{\text{abs}}_k := \mathfrak{S}_{k,\theta}$ as well as $T^{\text{abs}}_k := T^0_{k,\theta}$.

Having established the preceding qualitative result in Corollary 8.8, the next questions concerns a quantitative description of the stopping time. In particular, the expected value is of interest. We are going to give an upper bound in Theorem 8.13 for which we need some preliminary ideas. Firstly, we use that the line graph has diameter 2 such that any particle is at most at 2 steps from a free absorbing site. Secondly, we note that any path in $\mathfrak{L}_k$ can be dissected into multiple paths of multiple independent Markov chain for each single particle in L.

Geometrically, assuming that the current configuration is given by some $\mathfrak{v} \in \mathfrak{V}_k$, we consider for any $v \in \mathfrak{v}$ the graph $\tilde{L}^{(\mathfrak{v})}_{\mathcal{Z},\mathcal{Z}^c} = (\mathcal{Z}\setminus\mathfrak{v} \cup V\setminus\mathcal{Z}, \tilde{E}_{\mathcal{Z},\mathcal{Z}^c})$ where $\langle v, w\rangle \in E_{\mathcal{Z},\mathcal{Z}^c}$ if and only if $\langle v, w\rangle \in E$, $v \in \mathcal{Z}$ or $w \in \mathcal{Z}$ and $\{v, w\} \not\subset \mathcal{Z}$, or $\{v, w\} \subset V\setminus\mathcal{Z}$, i.e., $\tilde{L}^{(\mathfrak{v})}_{\mathcal{Z},\mathcal{Z}^c}$ represents all possible transition of each particle before being absorbed. Note that $\tilde{L}^{(\mathfrak{v})}_{\mathcal{Z},\mathcal{Z}^c} \simeq \tilde{L}^{(\mathfrak{w})}_{\mathcal{Z},\mathcal{Z}^c}$ if and only if $\mathcal{Z}\setminus\mathfrak{v} = \mathcal{Z}\setminus\mathfrak{w}$. Figure 51 shows one possible case of this construction for the line graph of the complete graph on 4 vertices.

Due to separating the particles into ”independent” graphs, we lose the information

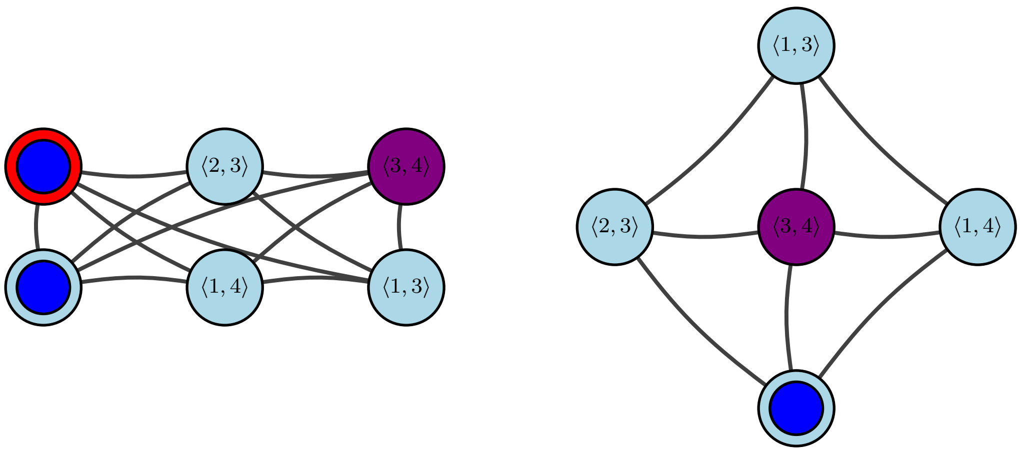

Figure 51: Two absorbing sites in the line graph of K_4 represented as red and violet vertices. We consider the case of $k = 3$ particles. The current particle positions are depicted as blue dots on the vertices.

on the local structures and, hence, on the transition probabilities which are highly dependent on the local structure of $\mathfrak{v}$, see 7.1. Nonetheless, using the same estimates as in Theorem 7.30 for $\mathfrak{S}_k$ we can bound the transition probabilities uniformly from below. A final remark concerns the fact that after the absorption of any particle, the structure of choosing a particle changes, since the number of particles, which can still move freely, is reduced by one. The construction described previously combined with the preceding remark lead to the following definition which projects them both on the transition graph of $\mathfrak{S}_k^{\mathrm{abs}}$ defined on $\mathfrak{L}_k$ combined with the absorbing sites $\mathcal{Z}$.

Definition 8.9. *Let $L = (V, E)$ a simple connected graph and $\mathcal{Z} \subset V$ the set of designated vertices. We define the hierarchical particle graph $\vec{\mathfrak{L}}_k = (\mathfrak{V}_k, \vec{\mathfrak{E}}_k)$ as a directed graph with $(\mathfrak{v}, \mathfrak{w}) \in \vec{\mathfrak{E}}_k$ if and only if*

$$\langle \mathfrak{v}, \mathfrak{w} \rangle \in \mathfrak{E}_k, \ \mathfrak{v} \triangle \mathfrak{w} = \{v, w\} \text{ with } v \notin \mathcal{Z}.$$

Additionally, for $i \in \{1, \ldots, k-1\}$, with

$$\mathfrak{V}_k^{(i)} := \left\{ \mathfrak{v} \in \mathfrak{V}_k \middle| |\mathfrak{v} \cap \mathcal{Z}| = i \right\}$$

we define

$$\vec{\mathfrak{E}}_k^{(i,i+1)} = \left\{ (\mathfrak{v}, \mathfrak{w}) \in \vec{\mathfrak{E}}_k \middle| \mathfrak{v} \in \mathfrak{V}_k^{(i)}, \mathfrak{w} \in \mathfrak{V}_k^{(i+1)} \right\}.$$

Evidently, we are have in particular the case in mind, where L and $\mathcal{Z}$ arise as follows, inspired by the Echo Chamber Model. Let $G = (\mathcal{V}, \mathcal{E})$ be the complete graph on n vertices with i.i.d. labels $(X_a)_{a \in \mathcal{V}}$. Fix $N \in \mathbb{N}$, $N \geq 1$ and assume that the labels are distributed as $X_a \sim \mathcal{L}(\{\frac{i}{N} | i = 1, \ldots, N\}) = (p_1, \ldots, p_N)$ for $a \in \mathcal{V}$. Denote by $L = (V, E)$ the line graph associated to G with labels on vertex $v = \langle a, b \rangle$ given $Y_v = \mathbb{1}_{X_a = X_b}$. Denote by $\mathcal{Z} = \{v \in E | Y_v = 0\}$. Note that the set $\mathcal{Z}$ represents the set of absorbing sites for edges under the previously described absorbing dynamics represented in L. Indeed, a mix between a local, particle based perspective and the mesoscopic properties of $\mathfrak{E}_k^{(i,i+1)}$ allows for precise calculations when it comes to the probability of $\mathfrak{S}_k^{\mathrm{abs}}$ transitioning from $\mathfrak{V}_k^i$ to $\mathfrak{V}_k^{i+1}$. In the previous section we have seen the various symmetries of $\mathfrak{L}_k$ and we can recover some of them in this context. We illustrate in Figure 52 first using a hierarchical layout the layered structure of $\vec{\mathfrak{L}}_k$ and then its symmetries based on the spectral plot, using the two maximal eigenvalues of the adjacency matrix. The second plot shows the graph in a layout defined by the eigenvectors of its adjacency matrix which are hard to obtain explicitly in this setting. But aside form its esthetic value it may reveal something about the behavior of $\mathfrak{S}_k^{\mathrm{abs}}$, which leaves a possibility for future research.

Apart from the necessary estimates on the transition probabilities of $\mathfrak{S}_k^{\mathrm{abs}}$ due to intricate local structures of vertex induced sub-graphs, we can avoid estimates on the expected time to absorption thanks to the properties of $\vec{\mathfrak{L}}_k$, in particular, the properties of $\vec{\mathfrak{E}}_k^{(i,i+1)}$.

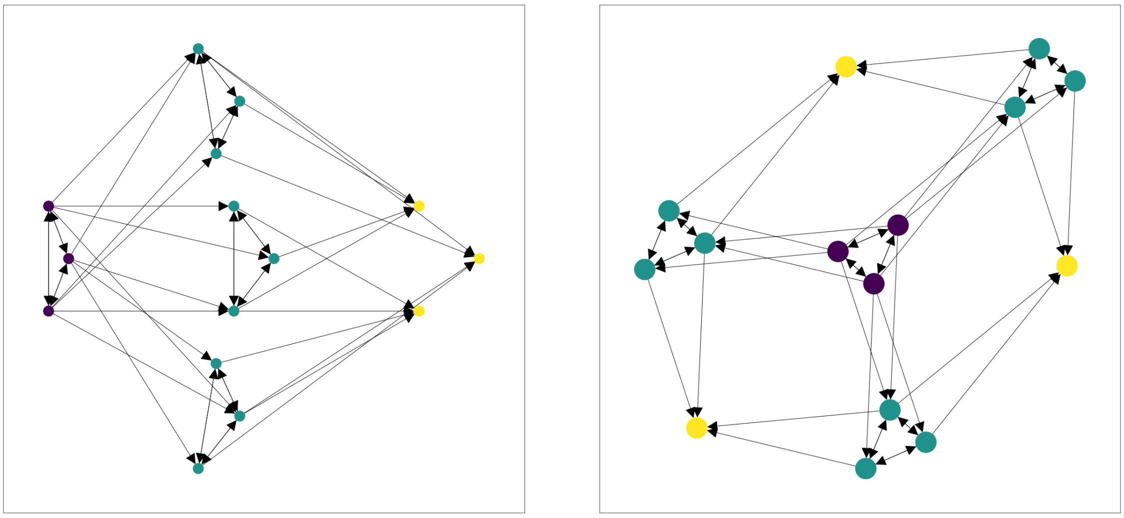

Figure 52: For an underlying complete graph on 4 vertices and 2 edges with 2 distinct opinions. We compare the graph $\vec{\mathfrak{L}}_k$ in two layouts. Note in the first one, that there are only connections within the same level set $\mathfrak{V}_k^{(i)}$ or pointing from $\mathfrak{V}_k^{(i)}$ to $\mathfrak{V}_k^{(i+1)}$. Second one is layed out according to the eigenvectors associated to its to largest eigenvalues. Note the symmetry which arises naturally from the eigenvector based layout.

Proposition 8.10. *Let $L = (V, E)$ a simple connected graph and $\mathcal{Z} \subset V$ the set of designated vertices. Then, the sets $\mathfrak{V}_k^{(i)}$ satisfy $|\mathfrak{V}_k^{(i)}| = \binom{|\mathcal{Z}|}{i}\binom{\bar{n}-|\mathcal{Z}|}{k-i}$ and the sets $\vec{\mathfrak{E}}_k^{(i,i+1)}$ have size*

$$\left|\vec{\mathfrak{E}}_k^{(i,i+1)}\right| = 2\left|\mathfrak{V}_k^{(i+1)}\right| \frac{(i+1)}{|\mathcal{Z}|}\left(1 - \frac{k-(i+1)}{\bar{n}-|\mathcal{Z}|}\right)|E_{\mathcal{Z},\mathcal{Z}^c}|\,.$$

In the case, where $G = (\mathcal{V}, \mathcal{E})$ is the complete graph on n vertices with i.i.d. labels $(X_a)_{a\in\mathcal{V}}$ we find the following relation. With $\bar{n} = \frac{n(n-1)}{2}$ and $\bar{d} = 2(n-2)$, fix $N \in \mathbb{N}$, $N \geq 1$ and assume that the labels are distributed as $X_a \sim \mathcal{L}(\{\frac{i}{N}|i = 1, \ldots, N\}) = (p_1, \ldots, p_N)$ for $a \in \mathcal{V}$. Then, denoting by $n_\alpha = |\{a \in \mathcal{V}|X_a = \alpha\}|$ we obtain

$$|E_{\mathcal{Z},\mathcal{Z}^c}| = n|\mathcal{Z}| - \sum_{\alpha=1}^{N} \frac{n_\alpha^2(n_\alpha - 1)}{2}$$

giving us a explicit form for the state space underlying the Echo Chamber Model. We turn now to the proof.

Proof. First, we start by considering the number of edges which point from $\mathfrak{V}_k^{(i+1)}$ into $\mathfrak{V}_k^{(i+1)}$ to calculate $\left|\vec{\mathfrak{E}}_k^{(i,i+1)}\right|$. To this end, we consider any absorbed $w \in \mathfrak{w}$ for

$\mathfrak{w} \in \mathfrak{V}_k^{(i+1)}$ and count the number of $\mathfrak{v} \in$ such that $\mathfrak{w} \cap \mathfrak{v} = \mathfrak{w}\backslash\{w\}$.

For any configuration $\mathfrak{w} \in \mathfrak{V}_k^{(i+1)}$ there are $i+1$ particles in $\mathfrak{w}$ which are absorbed, i.e., we consider sites $w \in \mathfrak{w} \cap \mathcal{Z}$. Any one of them has $\deg(w)$ neighbors in L. To any such w we can associate a color n_{Y_w} induced by the connected components of $L_{\mathcal{Z}}$. Therefore, the particle w has $\deg^{L_{\mathcal{Z}}}(w)$ neighbors which have the same color. None of those sites is accessible from w since once being in w or one of those sites a particle is blocked by definition. Hence, there are $\deg(w) - \deg^{L_{\mathcal{Z}}}(w)$ neighbors of w from which a blocked particle in w can come from. Additionally, any non-blocked neighbors of w which is occupied by a particle in $\mathfrak{w}$ cannot be the origin of the particle in w. Therefore, we obtain the formula

$$\left|\overrightarrow{\mathfrak{E}}_k^{(i,i+1)}\right| = \sum_{\mathfrak{w}\in\mathfrak{V}_k^{(i+1)}} \sum_{w\in\mathfrak{w}\cap\mathcal{Z}} (\deg(w) - \deg^{L_{\mathcal{Z}}}(w) - \deg^{L_{\mathfrak{w}\backslash\mathcal{Z}}}(w)).$$

We can derive explicit expressions for each of the terms in the difference as functions of the parameters. This leads to

$$\begin{aligned}
\sum_{\mathfrak{w}\in\mathfrak{V}_k^{(i+1)}} \sum_{w\in\mathfrak{w}\cap\mathcal{Z}} (\deg(w) - \deg^{L_{\mathcal{Z}}}(w)) &= \sum_{w\in\mathcal{Z}} \sum_{\substack{\mathfrak{w}\in\mathfrak{V}_k^{(i+1)} \\ w\in\mathfrak{w}}} (\deg(w) - \deg^{L_{\mathcal{Z}}}(w)) \\
&= \sum_{w\in\mathcal{Z}} (\deg(w) - \deg^{L_{\mathcal{Z}}}(w)) \sum_{\mathfrak{w}\in\mathfrak{V}_k^{(i+1)}} \mathbb{1}_{w\in\mathfrak{w}} = \frac{i+1}{|\mathcal{Z}|}\left|\mathfrak{V}_k^{(i+1)}\right| \sum_{w\in\mathcal{Z}} (\deg(w) - \deg^{L_{\mathcal{Z}}}(w)) \\
&= 2\frac{i+1}{|\mathcal{Z}|}\left|\mathfrak{V}_k^{(i+1)}\right| |E_{\mathcal{Z},\mathcal{Z}^c}|
\end{aligned}$$

as well as

$$\begin{aligned}
\sum_{\mathfrak{w}\in\mathfrak{V}_k^{(i+1)}} \sum_{w\in\mathfrak{w}\cap\mathcal{Z}} \deg^{L_{\mathfrak{w}\backslash\mathcal{Z}}}(w) &= \sum_{w\in\mathcal{Z}} \sum_{\substack{\mathfrak{w}\in\mathfrak{V}_k^{(i+1)} \\ w\in\mathfrak{w}}} \deg^{L_{\mathfrak{w}\backslash\mathcal{Z}}}(w) \\
&= \sum_{w\in\mathcal{Z}} \binom{|\mathcal{Z}|-1}{i} \sum_{j=0}^{\min\{k-i, \deg(w)-\deg^{L_{\mathcal{Z}}}(w)\}\}} j \binom{\deg(w)--\deg^{L_{\mathcal{Z}}}(w)}{j} \binom{\bar{n}-|\mathcal{Z}|-(\deg(w)-\deg^{L_{\mathcal{Z}}}(w)\})}{k-(i+1)-j} \\
&= \binom{|\mathcal{Z}|-1}{i} \sum_{w\in\mathcal{Z}} \sum_{j=0}^{\min\{k-i, \deg(w)-\deg^{L_{\mathcal{Z}}}(w)\}} j \binom{\deg(w)-\deg^{L_{\mathcal{Z}}}(w)}{j} \binom{\bar{n}-|\mathcal{Z}|-(\deg(w)-\deg^{L_{\mathcal{Z}}}(w))}{k-(i+1)-j}.
\end{aligned}$$

Additionally, noting that for $j > k-i$, since $k-(i+1)-j \leq 0$, and for $j > \deg(w) - \deg^{L_{\mathcal{Z}}}(w)$ we have

$$\binom{\bar{n}-|\mathcal{Z}|-(\deg(w)-\deg^{L_{\mathcal{Z}}}(w))}{k-(i+1)-j} = 0 \quad \text{and} \quad \binom{\deg(w)-\deg^{L_{\mathcal{Z}}}(w)}{j} = 0,$$

respectively. Consequently, using the notation

$$\iota_{k,i} := \min\{k-(i+1), \deg(w) - \deg^{L_{\mathcal{Z}}}(w)\}$$
$$\kappa_{\bar{n},\mathcal{Z}} := \bar{n} - |\mathcal{Z}| - (\deg(w) - \deg^{L_{\mathcal{Z}}}(w))$$

we can conclude

$$\sum_{j=0}^{\iota_{k,i}} j \binom{\deg(w) - \deg^{L_{\mathcal{Z}}}(w)}{j} \binom{\kappa_{\bar{n},\mathcal{Z}}}{k-(i+1)-j}$$
$$= \sum_{j=0}^{\deg(w)-\deg^{L_{\mathcal{Z}}}(w)} j \binom{\deg(w) - \deg^{L_{\mathcal{Z}}}(w)}{j} \binom{\kappa_{\bar{n},\mathcal{Z}}}{k-(i+1)-j}.$$

Now, realizing that the term on the right-hand-side equals $\binom{\bar{n}-|\mathcal{Z}|}{k-(i+1)}\mathbb{E}[\chi_w]$ where χ_w is hypergeometrically distributed with parameter $\text{Hyp}(k-(i+1), \deg(w)-\deg^{L_{\mathcal{Z}}}(w), \bar{n}-|\mathcal{Z}|)$ we find

$$\sum_{j=0}^{\deg(w)-\deg^{L_{\mathcal{Z}}}(w)} j \binom{\deg(w) - \deg^{L_{\mathcal{Z}}}(w)}{j} \binom{\kappa_{\bar{n},\mathcal{Z}}}{k-(i+1)-j}$$
$$= \binom{\bar{n}-|\mathcal{Z}|}{k-(i+1)} \frac{(\deg(w) - \deg^{L_{\mathcal{Z}}}(w))(k-(i+1))}{\bar{n}-|\mathcal{Z}|}.$$

Summing over $w \in \mathcal{Z}$ we find

$$\sum_{w\in\mathcal{Z}} \binom{\bar{n}-|\mathcal{Z}|}{k-(i+1)} \frac{(\deg(w) - \deg^{L_{\mathcal{Z}}}(w))(k-(i+1))}{\bar{n}-|\mathcal{Z}|}$$
$$= 2|E_{\mathcal{Z},\mathcal{Z}^c}| \binom{\bar{n}-|\mathcal{Z}|}{k-(i+1)} \frac{k-(i+1)}{\bar{n}-|\mathcal{Z}|}$$

and combining all calculated terms, we obtain the claim. □

Aside from the edge structure defining transitions between the level set, we can make deeper geometrical observations about the connectivity of $\vec{\mathfrak{L}}_k$. Since each level set is disconnected due to the fact that only non-absorbed particles may move around and so any configuration of a fixed number $j \in \{0, \dots, k\}$ of absorbed particles defines an equivalence class which is not connected to any other equivalence class defined by the same j. On the other hand, we find that variations in j may give paths which define common neighbors of two equivalence classes at a deeper level. We formalize this observation in the following Proposition.

Proposition 8.11. *Under the assumptions of Proposition 8.10 consider* $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k^{(i)}$ *for some* $i, j \in \{0, \ldots, k\}$ *and* $j \geq i$. *Then, there is a* $\mathfrak{u} \in \mathfrak{V}_k^{(j)}$ *and paths* $\phi_{\mathfrak{v},\mathfrak{u}}, \phi_{\mathfrak{w},\mathfrak{u}}$ *in* $\vec{\mathfrak{L}}_k$ *from, respectively,* $\mathfrak{v}$ *and* $\mathfrak{w}$ *to* $\mathfrak{u}$ *if and only if*

$$\frac{1}{2}|(\mathfrak{v} \cap \mathcal{Z}) \triangle (\mathfrak{w} \cap \mathcal{Z})| \leq j - i. \tag{8.6}$$

Additionally, assume that $k \leq \bar{n} - |\mathcal{Z}|$. *Then, the transition matrix* $P^{\triangle,\mathrm{abs}}$ *of* $\mathfrak{S}_k^{\mathrm{abs}}$ *takes, after ordering the states according to the number of absorbed vertices, the form*

$$P_k^{\triangle,\mathrm{abs}} = \begin{pmatrix} Q_0 & R_{0,1} & 0 & 0 & \ldots \\ 0 & Q_1 & R_{1,2} & 0 & \ldots \\ \vdots & 0 & \vdots & \ddots & \\ & & 0 & Q_{k-1} & R_{k-1,k} \\ & & & 0 & I_k \end{pmatrix}$$

Proof. The claim follows since the labels define a cell-free environment on L and, consequently, when the absorbed particles in two configurations $\mathfrak{v}, \mathfrak{w} \in \mathfrak{V}_k^{(i)}$ differ by less then $j-i$ particles, putting first up to $j-i$ non-absorbed particles in $\mathfrak{v}$ in absorbing sites, which are occupied in $\mathfrak{w}$, we obtain a path to some $\mathfrak{u} \in \mathfrak{V}_k^{(j)}$. Doing the same for $\mathfrak{w}$ and moving the remaining non-absorbed particles in the correct spots, we obtain the claim.
Inversely, if there is such a $\mathfrak{u}$ and the corresponding paths, we are able to reconstruct the absorbed configuration $\mathfrak{v} \cap \mathcal{Z}$ and $\mathfrak{w} \cap \mathcal{Z}$ of $\mathfrak{v}$ and $\mathfrak{w}$, respectively, by removing absorbed particles, since we cannot change the configuration of already existing absorbed particles in another way. Consequently, the sets $(\mathfrak{v} \cap \mathcal{Z})$ and $(\mathfrak{w} \cap \mathcal{Z})$ may only differ by at most $j-i$ elements which implies

$$\frac{1}{2}|(\mathfrak{v} \cap \mathcal{Z}) \triangle (\mathfrak{w} \cap \mathcal{Z})| \leq j - i. \tag{8.7}$$

For the second claim, note that $k \leq \bar{n} - |\mathcal{Z}|$ implies that all particles may be absorbed. Additionally, since particles are absorbed one after another, either the chain remains in a level set $\mathfrak{V}_k^{(j)}$ according to the transitions of a matrix Q_j or transitions to the next one $\mathfrak{V}_k^{(j+1)}$ according to some matrix $R_{j,j+1}$. We may, therefore, write $P^{\triangle,\mathrm{abs}}$ as an upper triangular matrix (or, equivalently, a lower triangular matrix.) □

Note that by the form of $P^{\triangle,\mathrm{abs}}$ shown in Proposition 8.11 we can identify $\mathfrak{S}_k^{\mathrm{abs}}$ with a quasi-death process as discussed in [DoPol07]. Satisfying the condition of a possibly reducible state space in [DoPol07] and further discussed [DoPol08] the Markov chain $\mathfrak{S}_k^{\mathrm{abs}}$ exhibits at least one quasi-stationary distribution $\pi_{\mathfrak{q}}$ satisfying

$$\mathbb{P}_{\pi_{\mathfrak{q}}}\left[\mathfrak{S}_{k;t}^{\mathrm{abs}} = \mathfrak{v} | T_k^{\mathrm{abs}} > t\right] = \pi_{\mathfrak{q}}(\mathfrak{v}).$$

This allows us to make claims about the upper bound of the expected time to absorption thanks to our detailed analysis of the state space $\vec{\mathfrak{L}}_k$. Combined with the following Theorem 8.12, which is known as Variable Drift Theorem and discussed at length in [RoSud14], [RoMiCa09] and [BaSte96] we can make quantitative statements about the time to absorption. We employ the following formulation which can be found in [Leng20].

Theorem 8.12 (Variable Drift Theorem [Leng20])**.** *Let $(X_t)_{t\in\mathbb{N}}$ a sequence of non-negative random variables with a finite state space $S \subset \mathbb{R}_0^+$ with $0 \in S$. Let $s_{min} := \min(S\backslash\{0\})$ and $T := \inf\{t \in \mathbb{N} | X_t = 0\}$ and for $t \in \mathbb{N}$, $s \in S$ let $\Delta_t(s) := \mathbb{E}[X_t - X_{t+1} | X_t = s]$. If there is a monotonously increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $s \in S\backslash\{0\}$ and all $t \in \mathbb{N}$ the lower bound $\Delta_t(s) \geq h(s)$ is satisfied, then*

$$\mathbb{E}[T] \leq \frac{s_{min}}{h(s_{min})} + \mathbb{E}\left[\int_{s_{min}}^{X_0} \frac{1}{h(z)} \mathrm{d}z\right]. \tag{8.8}$$

We obtain the following upper bound as a function of $\pi_\mathfrak{q}$, which remains out of reach of further analysis due to the intricate transition structure between vertex induced subgraphs with partially fixated vertices.

Theorem 8.13. *Let $G_0 = (\mathcal{V}, \mathcal{E}_0)$ be some initial graph with i.i.d. labels $(X_a)_{a\in\mathcal{V}}$. Fix $N \in \mathbb{N}$, $N \geq 1$ and assume that the labels are distributed as $X_a \sim \mathcal{L}(\{\frac{i}{N} | i = 1, \ldots, N\}) = (p_1, \ldots, p_N)$ for $a \in \mathcal{V}$. Let $\pi_\mathfrak{q}$ be a quasi-stationary distribution of $\mathfrak{S}_k^{\text{abs}}$, i.e., $\mathbb{P}_{\pi_\mathfrak{q}}\left[\mathfrak{S}_{k;t}^{\text{abs}} = \mathfrak{v} | T_k^{\text{abs}} > t\right] = \pi_\mathfrak{q}(\mathfrak{v})$. Define $\operatorname{rank}(\mathfrak{S}_{k;t}^{\text{abs}}) := |\mathfrak{S}_{k;t}^{\text{abs}} \cap \mathcal{Z}|$. Then, under the condition $k < |\mathcal{Z}|$, the expected time to absorption satisfies*

$$\begin{aligned}\mathbb{E}_{\pi_\mathfrak{q}}\left[T_k^{\text{abs}}\right] \leq & 1 + \frac{k(\bar{d}+1)(\bar{n} - |\mathcal{Z}|)}{2\,|E_{\mathcal{Z},\mathcal{Z}^c}|} \sup_i \frac{\operatorname{avg}_{\mathfrak{V}_k^i} \pi_\mathfrak{q}}{\inf_{\mathfrak{v}\in\mathfrak{V}_k^i} \pi_\mathfrak{q}(\mathfrak{v})} \\ & \cdot \mathbb{E}_{\pi_\mathfrak{q}}\left[\int_1^{\operatorname{rank}(\mathfrak{S}_{k;0}^{\text{abs}})} \frac{1}{s(|\mathcal{Z}| - (s-k))} \mathrm{d}s\right].\end{aligned}$$

Proof. Fix some random environment $\mathcal{Z}$ and condition on $k < |\mathcal{Z}|$ We drop the condition on the random environment in the conditional expectations and probabilities to increase readability. First, note that for any $t \in \mathbb{N}$ and a quasi-stationary distribution $\pi_\mathfrak{q}$ of $\mathfrak{S}_k^{\text{abs}}$ the equation

$$\begin{aligned}\pi_\mathfrak{q}(\mathfrak{v}) &= \mathbb{P}_{\pi_\mathfrak{q}}\left[\mathfrak{S}_{k;t}^{\text{abs}} = \mathfrak{v} | T_k^{\text{abs}} > t\right] \\ &= \mathbb{P}_{\pi_\mathfrak{q}}\left[\mathfrak{S}_{k;t}^{\text{abs}} = \mathfrak{v} | \mathfrak{S}_{k;t}^{\text{abs}} \in \mathfrak{V}_k^{(i)},\, T_k^{\text{abs}} > t\right] \mathbb{P}_{\pi_\mathfrak{q}}\left[\mathfrak{S}_{k;t}^{\text{abs}} \in \mathfrak{V}_k^{(i)} | T_k^{\text{abs}} > t\right] \\ &= \mathbb{P}_{\pi_\mathfrak{q}}\left[\mathfrak{S}_{k;t}^{\text{abs}} = \mathfrak{v} | \mathfrak{S}_{k;t}^{\text{abs}} \in \mathfrak{V}_k^{(i)}\right] \mathbb{P}_{\pi_\mathfrak{q}}\left[\mathfrak{S}_{k;t}^{\text{abs}} \in \mathfrak{V}_k^{(i)} | T_k^{\text{abs}} > t\right]\end{aligned}$$

is satisfied and, therefore, we obtain

$$\mathbb{P}_{\pi_{\mathfrak{q}}}\left[\mathfrak{S}^{\text{abs}}_{k;t}=\mathfrak{v}|\mathfrak{S}^{\text{abs}}_{k;t}\in\mathfrak{V}^{(i)}_k\right]=\frac{\pi_{\mathfrak{q}}(\mathfrak{v})}{\sum_{\mathfrak{u}\in\mathfrak{V}^{(i)}_k}\pi_{\mathfrak{q}}(\mathfrak{u})}.$$

Furthermore, we find that for any $t\in\mathbb{N}$ and $i\in\{0,\dots,k-1\}$ that

$$\begin{aligned}
&\mathbb{E}_{\pi_{\mathfrak{q}}}\left[\operatorname{rank}(\mathfrak{S}^{\text{abs}}_{k;t+1})-\operatorname{rank}(\mathfrak{S}^{\text{abs}}_{k;t})|\operatorname{rank}(\mathfrak{S}^{\text{abs}}_{k;t})=i\right]\\
&\quad=\mathbb{P}_{\pi_{\mathfrak{q}}}\left[\operatorname{rank}(\mathfrak{S}^{\text{abs}}_{k;t+1})-\operatorname{rank}(\mathfrak{S}^{\text{abs}}_{k;t})=1|\operatorname{rank}(\mathfrak{S}^{\text{abs}}_{k;t})=i\right]\\
&\quad=\sum_{(\mathfrak{v},\mathfrak{w})\in\vec{\mathfrak{E}}^{(i,i+1)}_k}\mathbb{P}_{\pi_{\mathfrak{q}}}\left[\mathfrak{S}^{\text{abs}}_{k;t+1}=\mathfrak{w}|\mathfrak{S}^{\text{abs}}_{k;t}=\mathfrak{v}\right]\mathbb{P}_{\pi_{\mathfrak{q}}}\left[\mathfrak{S}^{\text{abs}}_{k;t}=\mathfrak{v}|\mathfrak{S}^{\text{abs}}_{k;t}\in\mathfrak{V}^{(i)}_k\right]\\
&\quad\geq\frac{1}{k(\bar{d}+1)}\sum_{\mathfrak{v}\in\mathfrak{V}^{(i)}_k}\sum_{\mathfrak{w}\in\mathfrak{V}^{(i+1)}_k}\frac{\pi_{\mathfrak{q}}(\mathfrak{v})}{\sum_{\mathfrak{u}\in\mathfrak{V}^{(i)}_k}\pi_{\mathfrak{q}}(\mathfrak{u})}\mathbb{1}_{(\mathfrak{v},\mathfrak{w})\in\vec{\mathfrak{E}}^{(i,i+1)}_k}\\
&\quad\geq\frac{1}{k(\bar{d}+1)}\left(\sup_i\frac{\operatorname{avg}_{\mathfrak{V}^i_k}\pi_{\mathfrak{q}}}{\inf_{\mathfrak{v}\in\mathfrak{V}^i_k}\pi_{\mathfrak{q}}(\mathfrak{v})}\right)^{-1}\frac{\left|\vec{\mathfrak{E}}^{(i,i+1)}_k\right|}{|\mathfrak{V}^{(i)}_k|}\\
&\quad=2\left(\sup_i\frac{\operatorname{avg}_{\mathfrak{V}^i_k}\pi_{\mathfrak{q}}}{\inf_{\mathfrak{v}\in\mathfrak{V}^i_k}\pi_{\mathfrak{q}}(\mathfrak{v})}\right)^{-1}\frac{(i+1)}{|\mathcal{Z}|}\left(1-\frac{k-(i+1)}{\bar{n}-|\mathcal{Z}|}\right)|E_{\mathcal{Z},\mathcal{Z}^c}|
\end{aligned}$$

using the equality derived in Proposition 8.10 for the calculations in the last line. Applying the Variable Drift Theorem 8.12 to the previous estimate we obtain for the process $(k-\operatorname{rank}(\mathfrak{S}^{\text{abs}}_{k;t}))_{t\in\mathbb{N}}$ that

$$\mathbb{E}_{\pi_{\mathfrak{q}}}\left[T^{\text{abs}}_k\right]\leq1+\frac{k(\bar{d}+1)(\bar{n}-|\mathcal{Z}|)}{2\,|E_{\mathcal{Z},\mathcal{Z}^c}|}\sup_i\frac{\operatorname{avg}_{\mathfrak{V}^i_k}\pi_{\mathfrak{q}}}{\inf_{\mathfrak{v}\in\mathfrak{V}^i_k}\pi_{\mathfrak{q}}(\mathfrak{v})}\mathbb{E}_{\pi_{\mathfrak{q}}}\left[\int_1^{\operatorname{rank}(\mathfrak{S}^{\text{abs}}_{k;0})}\frac{1}{s(|\mathcal{Z}|-(s-k))}\mathrm{d}s\right]$$

which yields, consequently, the estimate under the condition on the fixed random environment. □

The hereinabove calculated upper bound is, while theoretically useful and aesthetically pleasing, out of reach for improvement or precise analysis since the quasi-stationary distribution $\pi_{\mathfrak{q}}$ remains difficult to approach. While the authors of [DoPol07] give somewhat specific conditions for various properties of $\pi_{\mathfrak{q}}$, they are all based on the spectrum of the Q_j which form the transition matrix $P^{\triangle,\text{abs}}$. In particular, they depend in the context, which we consider, on the explicit random environment $\mathcal{Z}$ which has to be taken into account for the analysis. Due to these complications, we propose another approach in what follows, which is based on the comparison of two Markov chains based on their transition probabilities. Indeed, we propose to compare a quasi-death process with a pure death process.

8.4 Comparison to pure death process

In this subsection we are going to compare $\mathfrak{S}_{k;t}^{\text{abs}}$ with a pure death process. To this end, we analyze the macroscopic features of $\mathfrak{S}_{k;t}^{\text{abs}}$ on the sets $\mathfrak{V}_k^{(i)}$ and the transitions from $\mathfrak{V}_k^{(i)}$ to $\mathfrak{V}_k^{(i+1)}$. Recall that by Proposition 8.11 for $k \le \bar{n} - |\mathcal{Z}|$ the transition matrix $P_k^{\triangle,\text{abs}}$ of $\mathfrak{S}_k^{\text{abs}}$ takes the form

$$P_k^{\triangle,\text{abs}} = \begin{pmatrix} Q_0 & R_{0,1} & 0 & 0 & \dots \\ 0 & Q_1 & R_{1,2} & 0 & \dots \\ \vdots & 0 & \vdots & \ddots & \\ & & 0 & Q_{k-1} & R_{k-1,k} \\ & & & 0 & I_k \end{pmatrix}.$$

Using this form of $P_k^{\triangle,\text{abs}}$ we find that for $\mathfrak{v} \in \mathfrak{V}_k^{(i)}$ and $\mathfrak{w} \in \mathfrak{V}_k^{(j)}$ with $j \ge i$, $t \ge 0$ the transition probability $\mathbb{P}[\mathfrak{S}_{k;t}^{\text{abs}} = \mathfrak{w} | \mathfrak{S}_{k;0}^{\text{abs}} = \mathfrak{v}]$ satisfies

$$\mathbb{P}[\mathfrak{S}_{k;t}^{\text{abs}} = \mathfrak{w} | \mathfrak{S}_{k;0}^{\text{abs}} = \mathfrak{v}] = \mathrm{e}_{\mathfrak{v}} \left(\sum_{\substack{(t_1,\dots,t_{j-i}) \in \mathbb{N}^{j-i} \\ \sum_l t_l = t-(j-i)}} \prod_{l=1}^{j-i} \left(Q_{i+l-1}^{t_l} R_{i+l-1,i+l} \right) \right) \mathrm{e}_{\mathfrak{w}}.$$

While this seems straight forward, it turns out that the state space of $\mathfrak{S}_{k;t}^{\text{abs}}$ is highly disconnected and each of its level sets consists of an explicitly derivable amount of connected components.

Lemma 8.14. *The induced sub-graph* $\vec{\mathfrak{L}}_{\mathfrak{V}_k^{(i)}}$ *has* $\binom{|\mathcal{Z}|}{i}$ *connected components and each component contains* $\binom{\bar{n}-|\mathcal{Z}|}{k-i}$ *vertices.*

Proof. The claim follows since there are $\binom{|\mathcal{Z}|}{i}$ possibilities to draw i absorbed particles from the absorbing vertices and each connected component is defined by a fixed set of i absorbed particles. Furthermore, any connected component contains exactly $\binom{\bar{n}-|\mathcal{Z}|}{k-i}$ vertices since each vertex in a connected component is defined by a fixed set of non-absorbed particles of size $k - i$. □

Therefore, any Q_i has the form

$$Q_i = \operatorname{diag}\left(Q_i^{(1)}, \dots, Q_i^{\left(\binom{|\mathcal{Z}|}{i}\right)} \right)$$

where each $Q_i^{(j)}$ is a matrix of size $\binom{\bar{n}-|\mathcal{Z}|}{k-i} \times \binom{\bar{n}-|\mathcal{Z}|}{k-i}$. Note for any pair i, j the matrix $Q_i^{(j)}$ is not a stochastic matrix under the condition $k < \frac{1}{2}\left(N \left\lfloor \frac{n}{N} \right\rfloor^2 - n \right)$.

Combining the states in $\mathfrak{V}_k^{(i)}$ to a single state i while preserving the Markov structure of $\mathfrak{S}_k^{\text{abs}}$ is not possible due to the dependence of the transition probabilities on the current state. Assuming, it would be possible, we could reduce $\mathfrak{S}_{k;t}^{\text{abs}}$ to a pure birth-process with maximal population k, which is absorbing, since from any state i one can only stay in i or transition to $i+1$. We use the following birth process as a comparison.

Let $G = (\mathcal{V}, \mathcal{V}_0)$ be some graph with i.i.d. labels $(X_a)_{a\in\mathcal{V}}$. Fix $N \in \mathbb{N}$, $N \geq 1$ and assume that the labels are distributed as $X_a \sim \mathcal{L}(\{\frac{i}{N}|i = 1, \ldots, N\}) = (p_1, \ldots, p_N)$ for $a \in \mathcal{V}$. Let $n := |\mathcal{V}|$ and define $\bar{n} := \frac{n(n-1)}{2}$ as well as $\bar{d} := 2(n-2)$ and C as the discussion after Proposition 8.10. Furthermore, consider the set of absorbing states $\mathcal{Z}$ of $\mathfrak{S}_k^{\text{abs}}$. Finally, we define a Markov chain R^B on $\{0, \ldots, k\}$ with transition probabilities given by

$$q_{i,i+1} := \frac{C}{k(\bar{d}-1)} \frac{(k-i)(|\mathcal{Z}|-i)}{\bar{n}-|\mathcal{Z}|},$$
$$q_{i,i} := 1 - q_{i,i+1}.$$

Then, the expected time to absorption T_R^{abs} of R^B is given by

$$\mathbb{E}_i[T_R^{\text{abs}}] = \sum_{j=i}^{k-1} \frac{k(\bar{d}-1)(\bar{n}-|\mathcal{Z}|)}{k(\bar{d}-1)(\bar{n}-|\mathcal{Z}|)-(k-j)(|\mathcal{Z}|-j)}. \tag{8.9}$$

To make the link with R^B more obvious, we define $R^D := k - R^B$, which is then a pure death process, absorbing in 0 and can, therefore, be compared with the process given at time t by $|\{v \in \mathfrak{S}_{k;t}^{\text{abs}} | Y_v = 0\}|$. In Figure 53 one can see that R^D respects the symmetry given by the distribution of the opinions in the graph in the case when there are 2 opinions, with a maximum in the time to absorption when both opinions are as equally distributed as possible. We conjecture the following link between the expected time to absorption which was out of scope to prove in the framework of this work.

Conjecture 8.15. *Under the previously discussed conditions assume that $\pi_{\mathfrak{q}}$ is a quasi-stationary distribution of $\mathfrak{S}_k^{\text{abs}}$. Then, the expected absorption times of R^D and $\mathfrak{S}_k^{\text{abs}}$ satisfy*

$$\mathbb{E}_{\pi_{\mathfrak{q}}}[T_k^{abs}] \leq \mathbb{E}_{\pi_{\mathfrak{q}}\circ\text{rank}}[T_R^{\text{abs}}]. \tag{8.10}$$

In fact, one would assume that if there is an overwhelming group of individuals with the same opinion, that for the Markov chain $\mathfrak{S}_k^{\text{abs}}$ there are many traps from which particles may not move on to a configuration with more absorbed particles. The chain R^D does not see these traps since it is constructed from averaging over all transition probabilities. As visualized in Figure 54, this is indeed false, the Markov chain $\mathfrak{S}_k^{\text{abs}}$ being always at least as fast absorbed in expectation as the pure death process R^D. A more in depth analysis of this estimate based on a quantitative comparison of absorbing Markov chains seems to be in order. Aggregation approaches, for example discussed in [LeRubSe94] could yield compatible Markov chains in some sense.

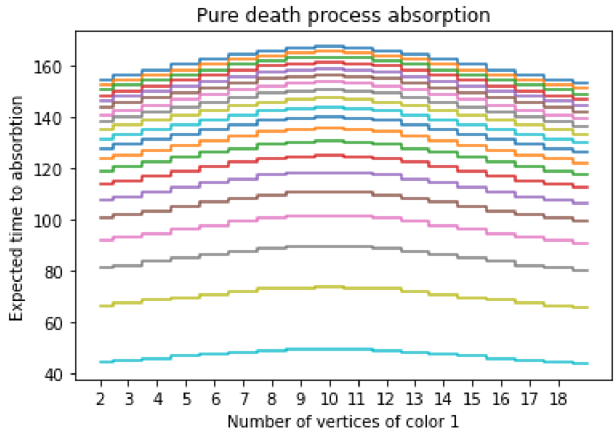

Figure 53: For an underlying graph on 20 vertices, 20 edges, and 2 opinions, we illustrate the dependence of $\mathbb{E}_i[T_R^{\text{abs}}]$ on the structure of the opinions. Note the symmetry around $N = \frac{n}{2}$ which maixmizes the number of non-absorbing sites. Each curve represents a specific initial value $i \in \{0, \ldots, k-1\}$ of the chain.

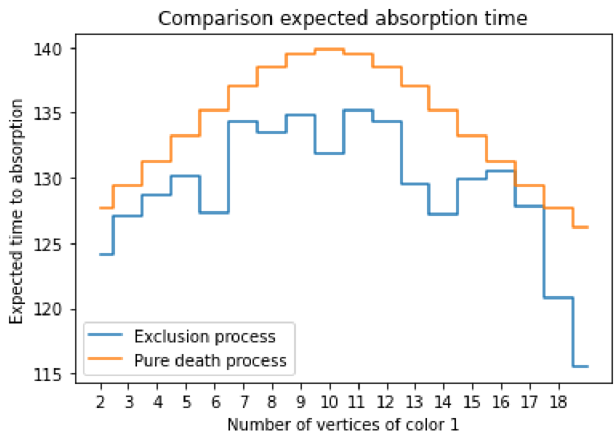

Figure 54: For an underlying graph on 20 vertices, 20 edges, and 2 opinions, we illustrate the dependence of $\mathbb{E}_i[T_R^{\text{abs}}]$ on the structure of the opinions. Note the symmetry around $N = \frac{n}{2}$ for both curves. For the plots the initial condition was fixed to be $i = 1$. The estimate in the conjecture seems to be unaltered by this lack of precision.

Comparison methods for ergodic Markov chains are not useful due to the absorbing setting. Lower bounds may be achieved by using techniques as discussed, for example, in [ErmGom14].

8.5 Considerations on the case of continuous opinions

We turn now to the setting of [HePraZha11] where the labels are not discrete but continuous, taking values in $[-1,1]$. Since the explicit choice of the interval is not essential to the behavior of the model but only the fact that it remains a compact interval, we use $[0,1]$ instead. This will allow us a short discussion of the limit of the discrete N color case as $N \to \infty$. In what follows we discuss properties of the continuous color model, which arise in the finite population setting and which are qualitatively different to the results obtained in [HePraZha11] for an idealized infinite population.

Firstly, we can extend the result from Proposition 8.7 to the continuous color setting for any arbitrary distribution $\mathcal{L}([0,1]^n)$ of the vertices of G. This gives rise to hopes to prove $T^0_{k,\theta} < \infty$ almost surely along the same lines as it was done in the previous section.

Proposition 8.16. *Let $G = (\mathcal{V}, \mathcal{E})$ be a graph with labels $(X_a)_{a\in\mathcal{V}} \sim \mathcal{L}([0,1])$ and let $\theta \in (0,1]$. Then, with $N := \lceil \theta^{-1} \rceil$ the inequality*

$$\left| \left\{ \langle a,b \rangle \in \mathcal{E} \middle| |X_a - X_b| < \theta \right\} \right| \geq \frac{1}{2} \left(N \left\lfloor \frac{n}{N} \right\rfloor^2 - n \right) \tag{8.11}$$

is almost surely satisfied.

Proof. Let $N := \lceil \theta^{-1} \rceil$. There is a finite set $\mathcal{Z} := \{z_i | i = 1, \dots, N\} \subset [0,1]$ and a finite set $\mathcal{R} \subset (0,1)$ such that

$$[0,1] \backslash \mathcal{R} = \bigsqcup_{i=1}^{N} B_{\frac{\theta}{2}}(z_i). \tag{8.12}$$

Note that $\mathbb{P}[\{X_a | a \in \mathcal{V}\} \subset [0,1] \backslash \mathcal{R}] = 1$. Denote by

$$n_i^{\mathcal{K}} := \operatorname{card}\left(\left\{ X_a | a \in \mathcal{V},\ X_a \in B_{\frac{\theta}{2}}(z_i) \right\} \right)$$

and hence almost surely

$$\sum_{i=1}^{N} n_i^{\mathcal{K}} = n.$$

For an arbitrary vector $\left(\tilde{X}_a\right)_{a\in\mathcal{V}} \in [0,1]^n$ with

$$n_i := \operatorname{card}(\{\tilde{X}_a | a \in \mathcal{V},\ \tilde{X}_a \in B_{\frac{\theta}{2}}(z_i)\}) \tag{8.13}$$

the combined amount of edges within the complete graphs within each ball

$$\sum_{i=1}^{N} \frac{n_i(n_i - 1)}{2} = \frac{1}{2} \left(\sum_{i=1}^{N} n_i^2 - n \right)$$

is minimized by the set $\{n_i^*|i \in \{1,\dots,N\}\}$ if the n_i^* are as equal as possible, i.e., $|n_i^* - n_j^*| \in \{0,1\}$, and in particular $n_i^* \geq \left\lfloor \frac{n}{N} \right\rfloor$. We conclude

$$\sum_{i=1}^{N} \frac{n_i^{\mathcal{K}}(n_i^{\mathcal{K}}-1)}{2} \geq \frac{1}{2}\left(N\left\lfloor \frac{n}{N}\right\rfloor^2 - n\right). \tag{8.14}$$

□

Since we could conclude from Theorem 8.13 based on Proposition 8.7 that we have an almost surely finite time to absorption, one might imagine that it is possible to deduce a similar result from Proposition 8.16 for the case of continuous colors. Unfortunately, even if the number of edges is linear in the number of vertices, such a claim does not hold if $\theta \in (0,1)$. To this end we consider the following example. Let $\theta \in (0,1)$ and fix some number of vertices n. Assume that two vertices v, w have labels X_v, X_w with $|X_v - X_w| \in \left(\theta, \frac{3}{2}\theta\right)$, i.e., the intersection $B_\theta(X_v) \cap B_\theta(X_w)$ is not empty. Assume that all remaining vertices $u_1, \dots, u_{n-2} \in B_\theta(X_v) \cap B_\theta(X_w)$. Then, all edges $\langle u_i, u_j \rangle$ in the complete graph on n vertices are short but it is sufficient to assume that some initial graph G has $2(n-2)+1$ edges to enforce absorbed configurations with at least one long edge. This is in particular the case, where the initial graph contains the edges $\{\langle X_v, X_{u_i}\rangle | i = 1,\dots,n-2\rangle\} \cup \{\langle X_w, X_{u_i}\rangle | i = 1,\dots,n-2\rangle\} \cup \{\langle X_v, X_w\rangle\}$ which is for clarification depicted in Figure 55. Since all edges $\langle X_v, X_{u_i}\rangle$ and $\langle X_w, X_{u_i}\rangle$

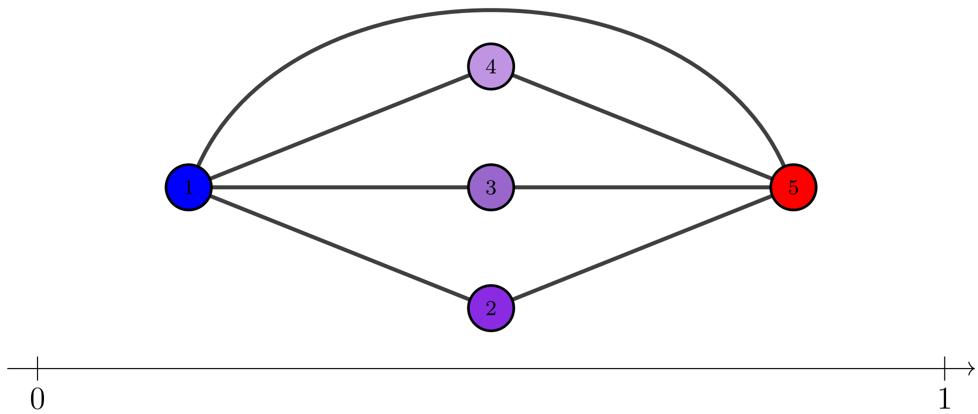

Figure 55: Illustration of the problem of persistent long edges even if the number of edges is very small, e.g., of the order of the number of vertices. Assume that $|X_1 - X_5| \in \left(\theta, \frac{3}{2}\theta\right)$ and $X_2, X_3, X_4 \in B_\theta(X_1) \cap B_\theta(X_5)$.

for $i = 1,\dots,n-2$ are short, they will never change their position under the echo chamber dynamics. On the other hand, $\langle X_v, X_w\rangle$ is a long edge but since all other possible edges incident to X_v or X_w already exist, deleting $\langle X_v, X_w \rangle$ only leads to recreating it. Hence, the graph never changes and the edge $\langle X_v, X_w \rangle$ is preserved for all times. Consequently, for any continuous opinion distribution supported on the whole interval $[0, 1]$, we encounter with positive probability blocked edges, such that the corresponding random environment on L is not cell-free. This seems counter-intuitive under the model motivation since it implies that an individual can not break up a relationship with someone of opposing opinion if said individual has to many like-minded friends.

Hence, the choice of continuous distributions which are supported by the whole interval $[0, 1]$ is a questionable one for a version of the model in which the opinions do not change over time. On the other hand, it allows for a straightforward way of including opinions which change over time due to interaction and exchange of individuals in a population represented by the vertices of the graph G. One way of including this opinion change is based on the Deffuant model, which we discussed and reviewed briefly in Subsection 5.2.

The combination of both models yields one possible complete Echo Chamber Model which we will discuss further in the outlook section, demonstrating at the same time possible while challenging but fruitful applications of the structures we introduced in this work.

8.6 Outlook: The Echo Chamber Model

In Part III of this work we have extensively discussed parts of the Echo Chamber Model, each on its on, leaving their interaction aside for the moment. This allowed us to get deep insights into the behavior and the relevant structures for these processes without being too much influenced by additional phenomena, which may occur. We have shown the complexity which arises already from this reduced perspective, for example particle configurations being only equivalent with respect to the process $\mathfrak{S}_k$ if their situation relative to the environment is identical, as seen in Theorem 7.8. Furthermore, we have seen that the qualitative properties of $\mathfrak{S}_k$ are dependent on the parameter choice as well as the geometry of the underlying graph, inducing in the widest sense phase transitions, as discussed at length in Theorems 7.18 to 7.20. Open problems in this context were already mentioned before.

In the final section of Part III, we managed to obtain an upper bound for the time to absorption in Theorem 8.13 of a reduced version of the Echo Chamber Model in terms of a quasi-stationary distribution of the process $\mathfrak{S}_k^{\text{abs}}$. A fully quantitative characterization is lacking in this work and will be a starting point for the outlook. Again, we touched upon some subjects only superficially, giving possible directions for future research, for example the comparison with a pure death process in Conjecture 8.15. This remains in the realm of the comparison of Markov chains which will also play in Part IV a big role but for which there is still no general theory. This subject is, therefore, promising for further research both from a theoretical point of view on Markov chains but also for the complexity reduction of applied problems when finding an easier Markov chain which behaves almost as the one of interest.

8.6.1 Future work

Let us recall the definition as well as some of the results we have already obtained on parts of the Echo Chamber Model. The Echo Chamber Model with continuous opinions was defined as a process of opinions and relationships within a network. The evolution of this network can be seen as the evolution of a labeled graph process $(G_t)_{t\in\mathbb{N}}$ with $G_t = (\mathcal{V}, \mathcal{E}_t)$ combined with an evolution of the labels $(X_t)_{t\in\mathbb{N}}$ where the changes per timestep are given by the following process

- Draw uniformly an edge $\langle A, B\rangle \in \mathcal{E}_t$.
- If $|X_A^t - X_B^t| >= \theta$, then
  - Define $N_{\langle A,B\rangle} := \{e = \langle c, d\rangle \notin \mathcal{E}_t \backslash \{\langle A, B\rangle\} | k \in \{A, B\} \text{ or } l \in \{A, B\}\}$.
  - draw uniformly E from $N_{\langle A,B\rangle}$,
  - set $\mathcal{E}_{t+1} = (\mathcal{E}_t \backslash \{\langle A, B\rangle\}) \cup \{E\}$.
- If $|X_A^t - X_B^t| < \theta$, then
  - set, firstly, $X_A^{t+1} = X_A^{t+1} + \mu(X_B^{t+1} - X_A^{t+1})$ and,
  - secondly, $X_B^{t+1} = X_B^{t+1} + \mu(X_A^{t+1} - X_B^{t+1})$.

In section, we have shown that the first half of the dynamics can be interpreted as the evolution of the probability distributions of the opinions, which under the assumption of the existence of a density π^0 at time 0 leads to the recursion

$$\pi^{t+1}(x) = \frac{1}{m} \sum_{<i,j>\in E} \left(\pi^t(x) \mathbb{1}_{A_{ij}^\theta}(x) + \frac{\pi^t\left(\left(\Phi_{ij}^\mu\right)^{-1}(x)\right)}{1-2\mu} \mathbb{1}_{\left(A_{ij}^\theta\right)^c} \left(\left(\Phi_{ij}^\mu\right)^{-1}(x) \right) \right).$$

which we have already derived in (5.23). The second half of the dynamics in itself is then given by the process $\mathfrak{S}_k^{\text{abs}}$, which is a conditioned version of $\mathfrak{S}_k$. Combining the two leads to the Echo Chamber Model but it turns out that the single vertex perspective used in equation (5.23) combined with the particle perspective used in the analysis of $\mathfrak{S}_k^{\text{abs}}$ is not sufficient. Indeed, we have to consider every particle in its role as an edge in G to capture the evolution of the whole process.

From this we obtain the following evolution equation of the joint distribution of the two parts of the Echo Chamber Model. Fix $\mathfrak{w} \in \mathfrak{V}_k$ and B a measurable subset of

$[0,1]^n$. Recall that k is the number of edges and n the number of vertices in the model as well as $A^\theta_{i,j} = \{x \in [0,1]^n \,|\, |x_i - x_j| > \theta\}$. Denoting by $\mathbb{P}_{v,w}[\cdot] = \mathbb{P}[\cdot|E^t = \langle v,w\rangle]$ we obtain for $t \in \mathbb{N}$

$$\begin{aligned}\mathbb{P}[\mathfrak{S}^{\text{abs}}_{k;t+1} = \mathfrak{w}, X^{t+1} \in B] &= \sum_{\mathfrak{v}\in\mathfrak{V}_k} \mathbb{P}[\mathfrak{S}^{\text{abs}}_{k;t+1} = \mathfrak{w}, \mathfrak{S}^{\text{abs}}_{k;t} = \mathfrak{v},\ X^{t+1} \in B] \\ &= \sum_{\mathfrak{v}\in\mathfrak{V}_k} \left(\frac{1}{k} \sum_{\langle v,w\rangle\in\mathfrak{v}} \mathbb{P}_{v,w}[\mathfrak{S}^{\text{abs}}_{k;t+1} = \mathfrak{w}, \mathfrak{S}^{\text{abs}}_{k;t} = \mathfrak{v}, X^t \in A^\theta_{v,w} \cap B] \right. \\ &\left. \quad + \mathbb{P}_{v,w}\left[\mathfrak{S}^{\text{abs}}_{k;t} = \mathfrak{v}, X^t \in \left(A^\theta_{v,w}\right)^c \cap \left(\Phi^\mu_{ij}\right)^{-1}(B)\right]\right). \end{aligned} \tag{8.15}$$

The two resulting summands can be considered as a single step in the respective submodels, i.e., the Deffuant model and the Markov chain $\mathfrak{S}^{\text{abs}}_k$, conditioned on the other one. This transfers both models in the setting of random dynamic environments where the environment of one is always defined by the other part of the dynamics. Equation 8.15 illustrates this entanglement of the two processes.

Results on the convergence of the model as a function of the parameters θ, k and μ might be approachable via similar ideas as proposed in Conjecture 5.24 by finding a sufficiently rapid decay of a recursion formula towards a limit. Future work on quantitative properties of the Echo Chamber Model could, therefore, be obtained by adapting the results from Theorem 5.29 and Theorem 8.13 to the new situation of random environments. Difficulties will arise in finding meaningful bounds and estimates for the objects involved to avoid worst case estimates.

PART IV

RANDOM POPULATION DYNAMICS UNDER CATASTROPHES

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In Part IV, we present a publication on a new model which extends the birth death process and takes, therefore, a macroscopic perspective, only counting the number of individuals in a population. This is in contrast to the micro- to mesoscopic perspective of population dynamics, meaning the consideration of individuals or structured populations. Our model includes the occurrence of catastrophic events as soon as the population exceeds a certain size. On the occurrence of such an event, the population size resets to the predefined size. This can be seen as a level beyond which the population becomes unstable or susceptible events which reduce the population size instantaneously and drastically to a ”stable” level.

In the Outlook of this part we propose a model which combines a birth-death process with a fast timescale process which is triggered by some initial event, replacing the instantaneous reset. The idea is based on the class of self-exciting processes, also known as Hawkes process, which have been most prominently used in earthquake modeling. Recent to the publication of this work, a variation, adapted to modeling infectious disease outbreaks in finite population, was presented by different authors, including the possibility of endogenous shocks into the model. We expand on this in the setting where only parts of the population are susceptible.

RANDOM POPULATION DYNAMICS UNDER CATASTROPHIC EVENTS.

PATRICK CATTIAUX ♠ , **JENS FISCHER**♠,♣ , **SYLVIE RŒLLY** ♣ , AND **SAMUEL SINDAYIGAYA** ♢

♠ Université de Toulouse

♣ Universität Potsdam

♢ Institut d'Enseignement Supérieur de Ruhengeri

Abstract. In this paper we introduce new Birth-and-Death processes with partial catastrophe and study some of their properties. In particular we obtain some estimates for the mean catastrophe time, and the first and second moments of the distribution of the process at a fixed time t. This is completed by some asymptotic results.

Key words : Birth-and-Death process; Population Dynamics; Extinction Time; Birth, Death and Catastrophe Process

MSC 2010 : 60J28, 60J80, 65Q30, 34D45, 35B40.

1. Introduction

The aim of this work is to propose a model for the evolution of the size of a population submitted to exceptional conditions, like a genocide, see [Sind16]. To this end, we introduce a new Birth-and-Death type process with *partial catastrophe*. Indeed, Birth-and-Death processes (BD-processes for short) are the more standard stochastic models for the description of the evolution of a population's size.

A BD-process assigns arbitrary non-negative Birth-and-Death rate pairs to a birth or a death of an individual in the population. Hence, whenever the population size changes, it grows or decreases exactly by one individual. The BD-process is under suitable assumptions a continuous time Markov chain (CTMC) on the discrete state space $\mathbb{N}$ and jump size ± 1. For a more in depth discussion on continuous time Markov chains and BD-processes see the Textbooks [And91] or [Nor97].

BD-processes have a long history and were first discussed, amongst others, for arbitrary Birth-and-Death rate by Feller [Fel39] and Kendall [Ken48]. Note that BD-processes can also modelize immigration and emigration of individuals. Nonetheless, changes in the population size which concern not only individuals but groups cannot be considered. Brockwell and his coauthors pioneered in [BGR82], [Bro85] and [Bro86] an extension of the BD-process including the possibility of a catastrophe captured by a sudden and exceptionally large decrease in the population size. They were in particular concerned with the probability of extinction as well as the time to extinction in such population models.

Mathematically one can capture catastrophes by allowing the process to make larger jumps downwards rather than the jump size of 1 in both directions in the classical framework of a BD-process. The classic Birth-and-Death rate pair is denoted by (λ_i, μ_i) for a population of size $i \in \mathbb{N}$. These are complemented by catastrophe rates $(\gamma_i)_{i\in\mathbb{N}}$ as well as corresponding law of the catastrophe sizes $(d_i(j))_{j\leq i}$ where $d_i(j)$ is the probability that a catastrophe in a population of size i leads to j deaths. The infinitesimal generator of the process is described by Brockwell in [Bro85] under classical assumptions on the coefficients as follows.

Definition 1.1 (BD-process with catastrophes, [Bro85]). *A BD-process $X = (X_t)_{t\geq 0}$ with general catastrophes is a Continuous Time Markov Chain with values in $\mathbb{N}$ associated to an infinitesimal generator $\tilde{Q} = (\tilde{q}_{ij})_{i,j\in\mathbb{N}}$, of the form*

$$\begin{cases} \tilde{q}_{ij} = \gamma_i\, d_i(i-j)\mathbb{1}_{[0,i)}(j) + \mu_i \mathbb{1}_{i-1}(j) + \lambda_i \mathbb{1}_{i+1}(j), & j \neq i, \\ \tilde{q}_{ii} = -(\lambda_i + \mu_i + \gamma_i) + \gamma_i d_i(0), \end{cases}$$

with for any $i \in \mathbb{N}$, $\lambda_i, \mu_i, \gamma_i, d_i(k) \in \mathbb{R}_+$ and $\sum_{k=0}^{i} d_i(k) = 1$. Moreover $\lambda_0 = \mu_0 = \gamma_0 = 0$ and $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = +\infty$.

An important class of the BD-processes with general catastrophes considers exclusively *total catastrophes* by setting $d_i(j) = \mathbb{1}_{\{j=i\}}$, i.e., in case of a catastrophe, the process jumps from its current state i to the state 0. Note that in Definition 1.1, since$\tilde{q}_{00} = 0$, the state 0 is absorbing. Therefore a total catastrophe may happen at most once before the population dies out. Moreover, without immigration ($\lambda_0 = 0$), a catastrophe leads to the extinction of the population. Note that the infinitesimal generator Q retains a tridiagonal form, if one only considers the states $i \geq 1$. Van Doorn and Zeifmann use this fact in [vDZ04] and [vDZ05] to investigate the transition probabilities at any time t and to extend the classical representation result of the transition probabilities of a BD-process in terms of associated orthogonal polynomials by Karlin and McGregor in [KM58]. Assuming constant catastrophe rates $\gamma_i \equiv \gamma$, Swift obtains in [Swi01] explicit expressions for the transition probabilities in terms of their generating function.

In [BGR82] the authors introduce BD-processes with different types of catastrophes: Geometric catastrophes, Uniform catastrophes and Binomial catastrophes; see also [Dicetal08]. The binomial model, later studied in [Kapetal16], considers a binomial redistribution of the population on the set of integers up to the current one. This induces a new expected population size concentrated around a fixed proportion $p \in$

$[0,1]$ of the previous one, which does not seem to correspond to the data we want to fit. The practical and statistical aspects of the problem will however not be discussed here.

In the present paper, we extend the study of populations under total catastrophes by considering populations which are subject to *partial catastrophes*. This means that, with positive rate γ_i, the population size i can be drastically reduced to a distinguished state $\mathfrak{n} \geq 1$ - as soon as i exceeds $\mathfrak{n}$. Remark that, in contrast to a total catastrophe, the population can not die out as a consequence of a partial catastrophe. Of course this model is the simplest one in this spirit, and instead of only one catastrophic new state $\mathfrak{n}$, we could consider a new distribution concentrated around $\mathfrak{n}$. Nevertheless, the study of this simple mathematical model is the first necessary step.

We define in Section 2 the exact class of processes we will study and present our main results. We are interested in the first hitting time $T_X^{(\mathfrak{n})}$ of the (catastrophic) population size $\mathfrak{n}$. Under the assumption of linear birth, death and catastrophe rates, we will use the tools introduced by Brockwell in [Bro86], in order to obtain explicit expressions for the expected catastrophe time $\mathbb{E}[T_X^{(\mathfrak{n})}|X_0 = n_0]$, $n_0 \geq \mathfrak{n}$. We also identify its limiting behavior for large initial population i.e. when $n_0 \to \infty$. Proofs are presented in Section 3. We then study the first two moments of the population size at a fixed time t. After establishing positive recurrence of the BD-process with partial catastrophe in Section 4 we compute and discuss explicit upper bounds for the process' first and second moment in Section 5.

2. Birth-and-Death Process with Partial Catastrophe: our main results

We fix a *catastrophic state* $\mathfrak{n} \in \mathbb{N}^*$ and set $d_i(j) = \mathbb{1}_{\{j=i-\mathfrak{n}\}}, i \geq \mathfrak{n}$. We introduce a positive rate ν to model a (minimal) immigration phenomenon when the population vanishes, ensuring the irreducibility of the process. $(\gamma_i)_{i\in\mathbb{N}}$, $(\lambda_i)_{i\in\mathbb{N}}$, $(\mu_i)_{i\in\mathbb{N}}$ are respectively the catastrophe, the birth and the death rates, where the index i represents the size of the population. All rates are assumed to be linear in the population size with proportionality coefficients respectively $\gamma, \lambda, \mu > 0$, see (2.1). Finally, throughout all this paper we assume that $\lambda > \mu$, that is the individual birth rate exceeds the individual death rate. Hence, in the absence of catastrophic event, the basic Birth-and-Death process with immigration would model a growing population.

We consider the CTMC on the state space $\mathbb{N}$ denoted by $X = (X_t)_{t\geq 0}$ whose infinitesimal generator $Q = (q_{ij})_{i,j\in\mathbb{N}}$ is given by

$$q_{ij} = \begin{cases} \lambda_0 = \nu, & \\ \lambda_i = \lambda \cdot i, & j = i+1, \\ \mu_i = \mu \cdot i, & j = i-1,\ i \geq 1, i \neq \mathfrak{n}+1, \\ \gamma_i = \gamma \cdot i, & j = \mathfrak{n},\ i > \mathfrak{n}+1, \\ \mu_i + \gamma_i, & j = \mathfrak{n},\ i = \mathfrak{n}+1, \\ -\sum_{j\neq i} q_{ij}, & j = i. \end{cases} \tag{2.1}$$

The choice of linear dependence with respect to the size of the population for the birth and death rates is very natural. On the other hand, the linear rate of the partial catastrophes means that the risk grows with the number of individuals. This situation appears in a population where any individual carries a risk of γ which might lead to a catastrophe, e.g., transmitting a deadly disease which once it happens kills on a very fast time scale a huge part of the population. The cardinality $\mathfrak{n}$ may be seen as the one of the population of possible transmitters, so they contribute to the rate γ_i, but at the same time individuals immune to the effects of the catastrophe, e.g., by vaccination or natural resistance, such that they do not vanish when the catastrophe sets in.

Mathematically, these assumptions lead to explicitly derivable and particularly nice results which were our main goal for this first analysis of this kind of population models to emphasize their qualitative behavior.

Its transition graph is depicted in Figure 1.

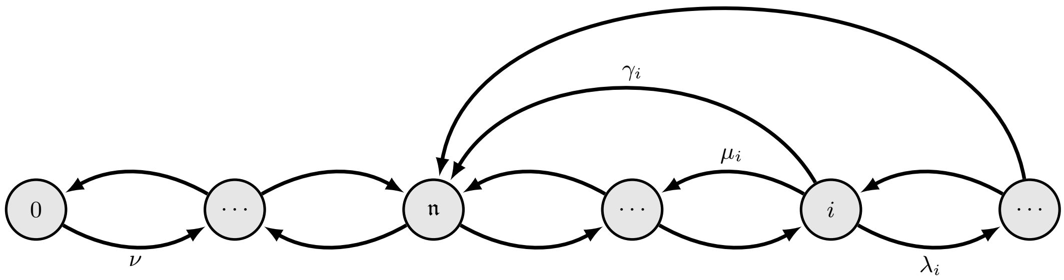

FIGURE 1. Transition graph of X.

Definition 2.1. *A Birth-and-Death process with partial catastrophe (BD+$C_{\mathfrak{n}}$ process) X is a CTMC with infinitesimal generator Q defined by the equations (2.1) and $X_0 = n_0 > \mathfrak{n}$.*

Its so-called catastrophe time $T_X^{(\mathfrak{n})}$ *is defined as its hitting time of the catastrophic state $\mathfrak{n}$:*

$$T_X^{(\mathfrak{n})} := \inf\{t \geq 0 | X_t = \mathfrak{n}\}. \tag{2.2}$$

Our results on the catastrophe time $T_X^{(\mathfrak{n})}$ are presented in the next three theorems. We state in Theorem 2.2 its almost sure finiteness and go on to study in Theorems 2.3 and 2.4 its expectation and its asymptotic behavior as the initial size of the population tends to ∞.

Theorem 2.2 (Finiteness of catastrophe time)**.** *Let X be a BD+$C_{\mathfrak{n}}$ process whose infinitesimal generator Q is given by (2.1). Then its catastrophe time $T_X^{(\mathfrak{n})}$ is almost surely finite.*

Better, we can compute the first moment of $T_X^{(\mathfrak{n})}$. Using the following notation

$$\mathbb{E}_i[T_X^{(\mathfrak{n})}] := \mathbb{E}[T_X^{(\mathfrak{n})} | X_0 = i + \mathfrak{n}], \tag{2.3}$$

we find an explicit expression for $\mathbb{E}_i[T_X^{(\mathfrak{n})}]$ and infer its asymptotic behavior for $i \to \infty$.

Theorem 2.3 (Explicit computation of the mean catastrophe time)**.** *Let X be a BD+$C_\mathfrak{n}$ process whose infinitesimal generator Q is given by (2.1). Denote by $\underline{a} < 1 < \bar{a}$ the distinct real zeros of the polynomial $\mathbf{x} \mapsto \mu\mathbf{x}^2 - (\lambda + \mu + \gamma)\mathbf{x} + \lambda$. The mean catastrophe time - defined in (2.3) - is given by*

$$\mathbb{E}_i[T_X^{(\mathfrak{n})}] = c\left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right)\sum_{k=1}^{\infty}\frac{\underline{a}^k}{k+\mathfrak{n}} + c\sum_{k=1}^{i-1}\frac{1}{k+\mathfrak{n}}\left(\frac{1}{\bar{a}^{i-k}} - \frac{1}{\underline{a}^{i-k}}\right), \quad i \geq 1 \tag{2.4}$$

with $c = (\sqrt{(\lambda + \mu + \gamma)^2 - 4\lambda\mu})^{-1}$.

Moreover, we obtain an explicit decreasing rate of the mean catastrophe time for large initial populations, which is in some sense counterintuitive.

Corollary 2.4. *Let X be the above BD+$C_\mathfrak{n}$ process. The asymptotic behavior of its mean catastrophe time for large initial populations is:*

$$\mathbb{E}_i[T_X^{(\mathfrak{n})}] = \mathcal{O}(i^{-1}).$$

Proofs of Theorems 2.2 - 2.3 and Corollary 2.4 are postponed to Section 3.

After establishing positive recurrence of the BD+$C_\mathfrak{n}$ process in Section 4 we will present in Section 5 the proofs of the following properties of - upper bounds for - the process' first and second moment.

Theorem 2.5 (Upper bound for the mean)**.** *Consider $(X_t)_{t\geq 0}$ the BD+$C_\mathfrak{n}$ process whose infinitesimal generator Q is given by (2.1). Then, the following upper bound holds:*

$$\mathbb{E}[X_t] \leq \bar{m}(t), \quad t \geq 0, \tag{2.5}$$

where the function $\bar{m}$ is the solution to the differential equation

$$\begin{cases} \bar{m}(0) = n_0, \\ \bar{m}'(t) = -\gamma\,\bar{m}(t)^2 + (\lambda - \mu + \gamma\,\mathfrak{n})\,\bar{m}(t) + \nu, \quad t > 0. \end{cases} \tag{2.6}$$

We also obtain a similar result for the second moment.

Theorem 2.6 (Upper bound for the second moment)**.** *Let $(X_t)_{t\geq 0}$ be the BD+$C_\mathfrak{n}$ process whose infinitesimal generator Q is given by (2.1). Consider the function $\bar{m}$ solution to (2.6) and assume that the initial size of the population is larger than a constant $\bar{m}_e$ computed in (5.4). Then, the second moment admits the following upper bound:*

$$\mathbb{E}[X_t^2] \leq \bar{v}(t), \quad t \geq 0, \tag{2.7}$$

the function $\bar{v}(t)$ being solution to the differential equation

$$\begin{cases} \bar{v}(0) = n_0^2, \\ \bar{v}'(t) = 2(\lambda - \mu)\bar{v}(t) + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\bar{m}(t) - \gamma\bar{v}(t)^{\frac{3}{2}} + \nu, \quad t > 0. \end{cases}$$

Such a $\bar{v}$ is bounded uniformly in time and so is, thus, $\mathbb{E}[X_t^2]$.

We close Section 5 with a discussion of the quality of the bounds $\bar{m}$ and $\bar{v}$ as defined in the previous theorems with a focus on their long time behavior.

3. Expected Catastrophe Time

This section consists of three subsections which lead through the proofs of Theorem 2.2, Theorem 2.3 and Corollary 2.4.
The proofs are based on various lemmas and propositions, which shed light on recurrence relations of order 2. In particular, we examine the limit behavior of their solutions in Lemma 3.3, the speed of divergence in Lemma 3.4, the dependence on the initial values in Lemma 3.5, Lemma 3.6 as well as possible explicit expressions for the minimal solution in Lemma 3.7 and Lemma 3.9. A road-map for the whole proof structure can be seen in Figure 2.

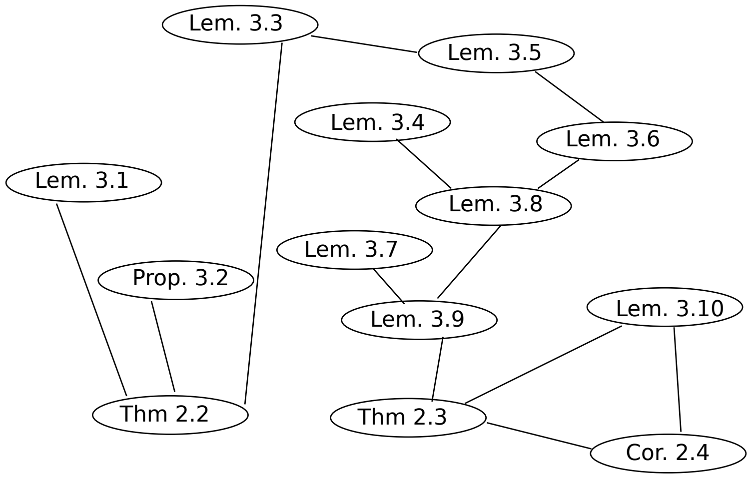

Figure 2. Dependencies and implications of results in Section 3.

3.1. **Finiteness of Time of Catastrophe.**

We analyze the catastrophe time $T_X^{(\mathfrak{n})}$ using the auxiliary process $Y = (Y_t)_{t\geq 0}$ defined by shifting the process X by $\mathfrak{n}$ and stopping it at catastrophe time $T_X^{(\mathfrak{n})}$:

$$Y_t := X_{t\wedge T_X^{(\mathfrak{n})}} - \mathfrak{n}, \qquad t \geq 0. \tag{3.1}$$

We discuss its properties and their implications for $T_X^{(\mathfrak{n})}$ in the following straightforward lemma.

Lemma 3.1. *Let X be the BD+$C_{\mathfrak{n}}$ process whose generator Q satisfies (2.1) with initial condition $X_0 = n_0 > \mathfrak{n}$. Then Y is a BD-process with total catastrophe whose birth, death and catastrophe rates are affine and given respectively by*

$$\tilde{\lambda}_i := \lambda\,(i+\mathfrak{n}), \quad \tilde{\mu}_i := \mu\,(i+\mathfrak{n}) \text{ and } \tilde{\gamma}_i := \gamma\,(i+\mathfrak{n}), \quad i \in \mathbb{N}. \tag{3.2}$$

The catastrophe time of X corresponds to the extinction time for the process Y.

While the proof of this lemma is evident, it yields a helpful tool for the further analysis. That way we switch from characterizing the first hitting time of the state $\mathfrak{n}$ for the process X to the more classical study of the extinction time of the process Y. In particular, the state 0 is *absorbing* for Y and is a *boundary state*. This gives us an advantage compared to analyzing $T_X^{(\mathfrak{n})}$ directly, since X may leave the state $\mathfrak{n}$ again both to $\mathfrak{n}+1$ or $\mathfrak{n}-1$. We may thus directly use the tools developed by Brockwell to analyze extinction times for Birth-and-Death processes with general catastrophes, see [Bro86] Lemma 3.1. We recall this result in the following proposition.

Proposition 3.2 ([Bro86] Lemma 3.1)**.** *For fixed $u \geq 0$ consider the sequence $(\alpha_i(u))_{i\in\mathbb{N}}$ defined per iteration by*

$$\alpha_0(u) = 0, \alpha_1(u) = 1, \quad \sum_{j=1}^{i+1} \tilde{q}_{ij}\,\alpha_j(u) = u\,\alpha_i(u), \quad i \in \mathbb{N}^*.$$

Let $\alpha_\infty(0) := \lim_{i\to\infty} \alpha_i(0)$. Let $(Z_t)_{t\geq 0}$ be a BD-process with general catastrophes as defined in Definition 1.1. Its time to extinction $T_Z^0 := \inf\{t \geq 0 | Z_t = 0\}$ verifies

(1) $\mathbb{P}[T_Z^0 < \infty | Z_0 = i] = 1 - \frac{\alpha_i(0)}{\alpha_\infty(0)}$, $i \in \mathbb{N}$,
(2) $\mathbb{P}[T_Z^0 < \infty | Z_0 = i] = 1\ \forall i \in \mathbb{N}^* \Leftrightarrow \mathbb{P}[T_Z^0 < \infty | Z_0 = 1] = 1 \Leftrightarrow \alpha_\infty(0) = \infty$.

It is worth noticing that, while Brockwell studied in details the linear case $\tilde{\lambda}_i = \lambda\, i$ in [Bro85], we have to consider here the shifted (affine) case $\tilde{\lambda}_i = \lambda i + \lambda\mathfrak{n}$, so that we have to perform all calculations when applying Proposition 3.2 to the process Y.
Using Proposition 3.2 to study T_Y^0 we only have to analyze the behavior of the associated sequence $(a_i(0))_{i\in\mathbb{N}}$. It is the aim of the following lemma.

Lemma 3.3. *Using the notations (3.2) consider for $u \geq 0$ and $\mathrm{a} > 0$ the sequence $(a_i(u))_{i\in\mathbb{N}}$ defined by the recurrence relation*

$$\begin{cases} (a_0(u),\, a_1(u)) = (0, \mathrm{a}) \\ \tilde{\lambda}_i a_{i+1}(u) = (u + \tilde{\lambda}_i + \tilde{\mu}_i + \tilde{\gamma}_i)\, a_i(u) - \tilde{\mu}_i\, a_{i-1}(u), \quad i \geq 1. \end{cases} \tag{3.3}$$

Then $(a_i(u))_{i\in\mathbb{N}}$ is non-decreasing and $a_\infty(0) = \infty$.

Proof. Fix $u \geq 0$ and set $a_i := a_i(u)$ to improve readability. Note that (3.3) is equivalent to

$$\tilde{\lambda}_i(a_{i+1} - a_i) = u a_i + \tilde{\mu}_i(a_i - a_{i-1}) + \tilde{\gamma}_i(a_i - a_0).$$

First, $a_1 - a_0 = \mathrm{a} > 0$ by assumption. Fix $i \geq 1$ and suppose that for $j \leq i-1$ the inequality $a_{j+1} - a_j \geq 0$ holds. Hence, in particular, $a_i \geq 0$. Moreover

$$\tilde{\lambda}_i(a_{i+1} - a_i) = ua_i + \tilde{\mu}_i(a_i - a_{i-1}) + \tilde{\gamma}_i \sum_{j=1}^{i}(a_j - a_{j-1}) \geq ua_i \geq 0.$$

By induction $(a_i)_{i\in\mathbb{N}}$ is non-decreasing and, thus, $a_i \geq 0$ for all $i \in \mathbb{N}$. Moreover, we obtain $\tilde{\lambda}_i a_{i+1} \geq (\tilde{\lambda}_i + \tilde{\gamma}_i)a_i$. Hence, for all $i \in \mathbb{N}$,

$$a_{i+1} \geq \frac{\tilde{\lambda}_i + \tilde{\gamma}_i}{\tilde{\lambda}_i} a_i = \left(1 + \frac{\gamma}{\lambda}\right) a_i \geq \ldots \geq \left(1 + \frac{\gamma}{\lambda}\right)^i \mathrm{a}$$

and, thus, $a_\infty(0) = \lim_{i\to\infty} a_i = \infty$ with at least a geometric rate. □

Indeed, in the following lemma, we quantify the exact speed of divergence for the sequence $(a_i(0))_{i\in\mathbb{N}}$.

Lemma 3.4. *Take $u = 0$ in (3.3). Then the sequence $\left(\frac{a_i(0)}{a_{i+1}(0)}\right)_{i\in\mathbb{N}}$ converges to the smaller real zero $\underset{\sim}{a} \in (0,1)$ of the polynomial*

$$\mathbf{x} \mapsto \mu\mathbf{x}^2 - (\lambda + \mu + \gamma)\mathbf{x} + \lambda.$$

Proof. Set $a_i := a_i(0)$ to improve readability. It holds for $i \geq 2$

$$\begin{aligned} a_{i-1}a_i^{-1} &= \frac{\tilde{\lambda}_{i-1}a_{i-1}}{(\tilde{\lambda}_{i-1} + \tilde{\gamma}_{i-1} + \tilde{\mu}_{i-1})a_{i-1} - \tilde{\mu}_{i-1}a_{i-2}} \\ &= \frac{\lambda a_{i-1}}{(\lambda + \gamma + \mu)a_{i-1} - \mu\, a_{i-2}}. \end{aligned}$$

Set $z_i = a_i a_{i+1}^{-1}$ for $i \geq 1$ we therefore have

$$z_1 = \frac{\lambda}{\lambda + \gamma + \mu}, \qquad z_i = \frac{\lambda}{\lambda + \gamma + \mu - \mu z_{i-1}}.$$

Consider the map $\phi : [0,1] \to [0,1]$ defined by $\phi(x) = \dfrac{\lambda}{\lambda + \gamma + \mu - \mu x}$. Since $0 < \phi(0) < \phi(1) < 1$ and ϕ is strictly increasing, ϕ has a unique fixed point $\underset{\sim}{a} \in (0,1)$ given by

$$\underset{\sim}{a} = \frac{\lambda + \mu + \gamma - \sqrt{(\lambda + \mu + \gamma)^2 - 4\lambda\mu}}{2\mu}.$$

Moreover $\lim_{i\to\infty} z_i = \underset{\sim}{a}$. □

Applying Proposition 3.2 and Lemma 3.3 to the shifted process Y defined in Lemma 3.1 we obtain that the extinction time T_Y^0 is almost surely finite. Hence, the catastrophe time $T_X^{(\mathfrak{n})}$ associated to $(X_t)_{t\geq 0}$ is almost surely finite. This completes the proof of Theorem 2.2.

3.2. **Explicit Computation of Mean Catastrophe Time.** In this subsection we prove Theorem 2.3. To that aim we first recall well a known result about the mean of hitting times, see for example [Nor97]. If Y is an irreducible CTMC with infinitesimal generator $Q = (q_{ij})_{ij}$, the sequence $(\mathbb{E}[T_Y^0 | Y_0 = i])_i$ is the minimal positive solution of the linear system

$$\begin{cases} x_i = 0, & i = 0 \\ -\sum_{j\in\mathbb{N}} q_{ij}x_j = 1, & i \neq 0 \cdot \end{cases} \tag{3.4}$$

Hence, since $T_Y^0 = T_X^{(\mathfrak{n})}$ a.s. the sequence $(\mathbb{E}_i[T_X^{(\mathfrak{n})}])_{i\geq 0}$ satisfies the recurrence relation

$$\tilde{\lambda}_i x_{i+1} = -1 + (\tilde{\gamma}_i + \tilde{\lambda}_i + \tilde{\mu}_i)x_i - \tilde{\mu}_i x_{i-1}, \ i \geq 1, \tag{3.5}$$

where $x_0 = 0$ and the value of x_1 has to be determined.
In what follows, we use Lemma 3.3 extensively, always fixing $u = 0$ and abbreviating $a_i := a_i(0)$. We focus firstly on the dependence of the solution of (3.5) with respect to the value of $x_1 \in \mathbb{R}$.

Lemma 3.5. *Let $(x_i)_{i\in\mathbb{N}}$ be a sequence solution to the recurrence relation (3.5) where the coefficients are given by (3.2). Then there is at most one possible value for x_1 such that the sequence $(x_i)_{i\in\mathbb{N}}$ is bounded.*

Proof. Consider two solutions $(x_i)_{i\in\mathbb{N}}$ and $(x_i')_{i\in\mathbb{N}}$ of the recurrence relation (3.5) satisfying $x_1 = x$ resp. $x_1' = x' < x$. The sequence $(\Delta_i := x_i - x_i', i \in \mathbb{N})$ satisfies (3.3) with $\Delta_1 = x - x'$ and $u = 0$. By Lemma 3.3, $(\Delta_i)_{i\in\mathbb{N}}$ is a non-decreasing sequence tending to ∞ as $i \to \infty$.
If there were two bounded sequences with different values x and x' for $i = 1$ also their difference would be bounded which is a contradiction. □

The following lemmas yield step by step the value of x_1 for which the sequence solution of (3.5) is uniformly bounded. As a first step we show in Lemma 3.6 a dichotomy of the divergence behavior around a certain initial value $\hat{x}$. This is going to provide a direct argument for the convergence radius of an associate generating function in Lemma 3.8.

Lemma 3.6. *There exists a unique value $\hat{x} > 0$ such that,*

(1) if $x_1 < \hat{x}$, $\lim_{i\to\infty} x_i = -\infty$,
(2) if $x_1 > \hat{x}$, $\lim_{i\to\infty} x_i = +\infty$.

Proof. Consider unbounded solutions $(x_i)_{i\in\mathbb{N}}$ to (3.5). As in the proof of Lemma 3.5 take two solutions $(x_i)_{i\in\mathbb{N}}$ and $(x_i')_{i\in\mathbb{N}}$ of (3.5) with $x_1' = x' < x = x_1$. Since the sequence $(\Delta_i = x_i - x_i', \ i \in \mathbb{N})$ is non-decreasing by Lemma 3.3 we obtain that $x_i > x_i'$ for all $i \geq 1$. Therefore solutions preserve for any i the order of their values for $i = 1$.
Consider the case where there is a $i_0 \in \mathbb{N}$ such that $x_{i_0-1} \geq 0$ and $x_{i_0} < 0$. Note that if there is a $k \in \mathbb{N}$ such that $x_k < 0$ such a pair (x_{i_0-1}, x_{i_0}) may always be found since $x_0 = 0$. Then, because $\tilde{\gamma}_i > 0$ for all $i \in \mathbb{N}$, we have

$$\tilde{\lambda}_{i_0}(x_{i_0+1} - x_{i_0}) = -1 + \tilde{\gamma}_{i_0}x_{i_0} + \tilde{\mu}_{i_0}(x_{i_0} - x_{i_0-1}) < 0. \tag{3.6}$$

Thus, $x_{i_0+1} < x_{i_0} < 0$ and

$$0 > \tilde{\gamma}_{i_0} x_{i_0} > \tilde{\gamma}_{i_0} x_{i_0+1} > \tilde{\gamma}_{i_0+1} x_{i_0+1}.$$

Hence, inductively we see that starting from i_0 the sequence $(x_i)_{i \geq i_0}$ is decreasing and negative. In particular, multiplying the recurrence relation (3.6) with -1 and using Lemma 3.3 with $u = 1$, we obtain that $(x_i)_{i \geq i_0}$ diverges to $-\infty$ as $i \to \infty$. In particular, once the sequence $(x_i)_{i \in \mathbb{N}}$ becomes negative, it stays negative.
Secondly, consider a positive unbounded solution $(x_i')_{i \geq 0}$ of (3.5). Then, for any level $C > 0$, there is an index $I \in \mathbb{N}$ such that $x_i' < C$ for all $i < I$ and $x_I' \geq C$. Let $C > 0$ be sufficiently large such that $C\tilde{\gamma}_I > 1$ with $I = \inf\{i : x_I' \geq C\}$. Then, by the recurrence relation (3.5) we obtain

$$\tilde{\lambda}_I (x_{I+1}' - x_I') \;=\; \tilde{\mu}_I (x_I' - x_{I-1}') + \tilde{\gamma}_I x_I' - 1 > C\tilde{\gamma}_I - 1 > 0.$$

Hence, $x_{I+1}' > x_I'$ and, thus, $\tilde{\gamma}_{I+1} x_{I+1}' > \tilde{\gamma}_I x_I' > 1$. Furthermore, there is a $\varepsilon > 0$ such that $x_{I+1}' \geq C + \varepsilon$ and $x_I' < C + \varepsilon$. Applying the same arguments to x_{I+1}' with a new constant C' set to $C + \varepsilon$ yields inductively that $(x_i')_{i \geq N}$ is strictly increasing, by assumption unbounded and, therefore, divergent to $+\infty$.
Therefore, since two sequences solution of (3.5) preserve the order of their initial values, using Lemma 3.5, there exists a critical value $\hat{x} > 0$ such that for $x_1 > \hat{x}$, the solution to (3.5) tends to ∞, while for $x_1 < \hat{x}$ it tends to $-\infty$. □

In fact, we will be able to compute the critical value $\hat{x}$ employing generating functions as tool.

Lemma 3.7. *Denote by $(x_i)_{i \in \mathbb{N}}$ the sequence solution to the recurrence relation (3.5) with $x_1 := x$. Its generating function $\mathcal{E}(z) := \sum_{i=0}^{\infty} x_i z^i$ satisfies*

$$\mathcal{E}(z) = z \frac{\lambda x - \sum_{k=1}^{\infty} \frac{z^k}{k+\mathfrak{n}}}{\mu z^2 - qz + \lambda} \tag{3.7}$$

within its radius of convergence $R \in [0, \infty)$, where $q := \lambda + \mu + \gamma$.

Proof. Set $\tilde{q}_i := q(i + \mathfrak{n})$ such that (3.5) becomes

$$\tilde{q}_i x_i = \tilde{\lambda}_i x_{i+1} + \tilde{\mu}_i x_{i-1} + 1.$$

By exploiting this formulation we obtain

$$\begin{aligned}
\mathcal{E}(z) &= \sum_{i=1}^{\infty} x_i z^i = \sum_{i=1}^{\infty} \left(\tilde{\lambda}_i x_{i+1} + \tilde{\mu}_i x_{i-1} + 1\right) \frac{z^i}{\tilde{q}_i} \\
&= \frac{\lambda}{q} \sum_{i=1}^{\infty} x_{i+1} z^i + \frac{\mu}{q} \sum_{i=1}^{\infty} x_{i-1} z^i + \sum_{i=1}^{\infty} \frac{z^i}{\tilde{q}_i} \\
&= \frac{\lambda}{qz} \sum_{i=1}^{\infty} x_{i+1} z^{i+1} + \frac{\mu}{q} z \sum_{i=1}^{\infty} x_i z^i + \sum_{i=1}^{\infty} \frac{z^i}{\tilde{q}_i} \\
&= \frac{\lambda}{qz} (\mathcal{E}(z) - xz) + \frac{\mu}{q} z \mathcal{E}(z) + \sum_{i=1}^{\infty} \frac{z^i}{\tilde{q}_i}
\end{aligned}$$

which leads to the result (3.7).

□

Based on the explicit generating function we derive an expression of the coefficients by differentiating in 0. To justify this approach we have to establish that $\mathcal{E}$ converges within a positive radius of convergence.

Lemma 3.8. *The generating function $\mathcal{E}$ given by (3.7) has a positive radius of convergence.*

Proof. Let $\hat{x}$ as in Lemma 3.6. Consider solutions $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ to (3.5) with $x'_1 = x' < \hat{x} < x = x_1$. Since there is a $i_0 \in \mathbb{N}$ such that $x'_i < 0$ for all $i \geq i_0$ it follows for $\Delta_i := x_i - x'_i$ that

$$\Delta_i \in [\max\{x_i, |x'_i|\}, 2\max\{x_i, |x'_i|\}], \quad i \geq i_0.$$

Additionally $(\Delta_i)_{i\in\mathbb{N}}$ satisfies (3.3) with $u = 0$; so according to Lemma 3.4 it grows like $\underline{a}^{-i}$ as $i \to \infty$. By the upper and lower bound of Δ_i for $i \geq i_0$ we obtain that both $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ grow asymptotically at most like $\underline{a}^{-i}$. Note that the solution $(\hat{x}_i)_{i\in\mathbb{N}}$ of (3.5) with $\hat{x}_1 = \hat{x}$ cannot grow faster than $\underline{a}^{-i}$ as $i \to \infty$ by positiveness of Δ_i for all $i \geq 1$. Thus, the series

$$\mathcal{E}(z) = \sum_{i=1}^{\infty} x_i z^i$$

converges for any initial value at least within the radius of convergence $R := \underline{a} > 0$. □

We proceed with a direct calculation of the coefficients. Recall that $\underline{a} < \bar{a}$ are the distinct real zeros of the polynomial $\mathbf{x} \mapsto \mu\mathbf{x}^2 - (\lambda + \mu + \gamma)\mathbf{x} + \lambda$.

Lemma 3.9. *Any solution* $(x_i)_{i\in\mathbb{N}}$ *to equation* (3.5) *with* $x_1 = x > 0$ *has the form*

$$x_i = \lambda\, c\, x \left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - c\sum_{k=1}^{i-1}\frac{1}{k+\mathfrak{n}}\left(\frac{1}{\underline{a}^{i-k}} - \frac{1}{\bar{a}^{i-k}}\right), \tag{3.8}$$

with $c = \left(\sqrt{(\lambda+\mu+\gamma)^2 - 4\lambda\mu}\right)^{-1}$.

Proof. According to Lemma 3.7 and Lemma 3.8 the introduced generating function $\mathcal{E}$ has a positive radius of convergence and the particular form

$$\mathcal{E}(z) = z\frac{f(z)}{g(z)} \text{ with } f(z) := \lambda x - \sum_{k=1}^{\infty}\frac{z^k}{k+\mathfrak{n}} \text{ and } g(z) := \mu z^2 - qz + \lambda.$$

Note first, that by that form we have

$$\partial_z^i \mathcal{E}(z)\Big|_{z=0} = i\,\partial_z^{i-1}\left(\frac{f(z)}{g(z)}\right)\Big|_{z=0} = i\sum_{k=0}^{i-1}\binom{i-1}{k}\left(\partial_z^k f(z)\partial_z^{i-1-k}(\frac{1}{g})(z)\right)\Big|_{z=0}.$$

Due to the form of g we have

$$\frac{1}{g(z)} = \frac{c}{z-\bar{a}} - \frac{c}{z-\underline{a}},$$

and therefore any derivative of order n, namely,

$$\partial_z^i \frac{1}{g(z)}\Big|_{z=0} = i!\,c\left(\frac{1}{\underline{a}^{i+1}} - \frac{1}{\bar{a}^{i+1}}\right). \tag{3.9}$$

Considering the numerator $f(z)$, we have

$$\partial_z^i f(z)\Big|_{z=0} = -\sum_{k=i}^{\infty}\frac{z^{k-i}}{k+\mathfrak{n}}k\cdot(k-1)\cdot\ldots\cdot(k-i+1)\Big|_{z=0} = -\frac{i!}{i+\mathfrak{n}}.$$

Hence,

$$\begin{aligned}
\partial_z^i \mathcal{E}(z)\Big|_{z=0} &= i\,\partial_z^{i-1}\left(\frac{f(z)}{g(z)}\right)\Big|_{z=0} \\
&= i\left(\lambda\, x\,(i-1)!c\left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right)\right. \\
&\qquad \left. - \sum_{k=1}^{i-1}\binom{i-1}{k}\frac{k!}{k+\mathfrak{n}}c(i-1-k)!\left(-\frac{1}{\bar{a}^{i-k}} + \frac{1}{\underline{a}^{i-k}}\right)\right) \\
&= i!\left(\lambda\, x\, c\left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - \sum_{k=1}^{i-1}\frac{c}{k+\mathfrak{n}}\left(\frac{1}{\underline{a}^{i-k}} - \frac{1}{\bar{a}^{i-k}}\right)\right)
\end{aligned}$$

Since $x_i = i!^{-1}\partial_z^i \mathcal{E}(z)\Big|_{z=0}$, we find that for any $i \geq 1$

$$x_i = \lambda\, c\, x \left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - c\sum_{k=1}^{i-1} \frac{1}{k+\mathfrak{n}} \left(-\frac{1}{\bar{a}^{i-k}} + \frac{1}{\underline{a}^{i-k}}\right).$$

□

We can now proceed with the proof of Theorem 2.3 by combining as follows the previously presented results. The sequence of expected hitting times $(\mathbb{E}_i[T_X^{(\mathfrak{n})}])_{i\in\mathbb{N}}$ is a solution to (3.5). Lemma 3.9 shows that the expected value for $i \geq 0$ is in fact given by

$$\mathbb{E}_i[T_X^{(\mathfrak{n})}] = \lambda\, \mathbb{E}_1[T_X^{(\mathfrak{n})}]\, c \left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - c\sum_{k=1}^{i-1} \frac{1}{k+\mathfrak{n}} \left(\frac{1}{\underline{a}^{i-k}} - \frac{1}{\bar{a}^{i-k}}\right),$$

with $c = \left(\sqrt{(\lambda+\mu+\gamma)^2 - 4\lambda\mu}\right)^{-1}$ where we followed the usual convention that an empty sum equals 0. It remains to derive the value of $\mathbb{E}_1[T_X^{(\mathfrak{n})}]$ to complete the proof of Theorem 2.3.

To improve the readability in what continuous, we introduce the following notation:

$$\Phi_n(z) := \sum_{k=0}^{\infty} \frac{z^k}{k+n}, \quad |z| < 1,\ n \in \mathbb{N}^*. \tag{3.10}$$

Therefore

$$\mathcal{E}(z) = z\frac{\lambda x_1 - \Phi_{\mathfrak{n}}(z) + \mathfrak{n}^{-1}}{\mu z^2 - qz + \lambda}$$

In fact, it turns out that the value $x_1 = \frac{1}{\lambda}\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1}\right)$ yields the minimal positive solution of (3.5) which does not tend to $+\infty$, as proved in the following lemma.

Lemma 3.10. *The sequence* $(x_i)_{i\in\mathbb{N}}$ *satisfying* (3.5) *with* $x_1 = \frac{1}{\lambda}\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1}\right)$ *is the minimal positive solution to* (3.5)*. Moreover* $x_i = \mathcal{O}(i^{-1})$ *as* $i \to \infty$*. Therefore*

$$\hat{x} = \frac{1}{\lambda}\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1}\right).$$

Proof. By Lemma 3.9,

$$
\begin{aligned}
x_i &= \lambda\, c\, x_1 \left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - c \sum_{k=1}^{i-1} \frac{1}{k+\mathfrak{n}} \left(\frac{1}{\underline{a}^{i-k}} - \frac{1}{\bar{a}^{i-k}}\right) \\
&= c \sum_{k=1}^{\infty} \frac{\underline{a}^k}{k+\mathfrak{n}} \left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - c \sum_{k=1}^{i-1} \frac{1}{k+\mathfrak{n}} \left(\frac{1}{\underline{a}^{i-k}} - \frac{1}{\bar{a}^{i-k}}\right) \\
&= \frac{c}{\underline{a}^i} \sum_{k=i}^{\infty} \frac{\underline{a}^k}{k+\mathfrak{n}} - \frac{c}{\bar{a}^i} \sum_{k=1}^{\infty} \frac{\underline{a}^k}{k+\mathfrak{n}} + c \sum_{k=1}^{i-1} \frac{1}{k+\mathfrak{n}} \frac{1}{\bar{a}^{i-k}}. \qquad (3.11)
\end{aligned}
$$

Using the notation $b_i := \sum_{k=1}^{i-1} \frac{\bar{a}^k}{k+\mathfrak{n}}$, then $\sum_{k=1}^{i-1} \frac{1}{k+\mathfrak{n}} \frac{1}{\bar{a}^{i-k}} = \frac{b_i}{\bar{a}^i}$ and

$$
\frac{b_{i+1} - b_i}{\bar{a}^{i+1} - \bar{a}^i} = \frac{1}{(\bar{a}-1)(i+\mathfrak{n})} \to 0, \qquad i \to \infty.
$$

Using the fact that $\bar{a} > 1$ and applying Stolz-Césaro Theorem we obtain that

$$
\frac{b_i}{\bar{a}^i} = \frac{1}{\bar{a}^i} \sum_{k=1}^{i-1} \frac{\bar{a}^k}{k+\mathfrak{n}} \to 0, \text{ as } i \to \infty.
$$

Thus, the asymptotic behavior of $(x_i)_{i\geq 0}$ is the same as of

$$
\frac{c}{\underline{a}^i} \sum_{k=i}^{\infty} \frac{\underline{a}^k}{k+\mathfrak{n}} \geq \frac{c}{i+\mathfrak{n}}. \qquad (3.12)
$$

Let us check an upper bound. Using the integral comparison,

$$
\begin{aligned}
\sum_{k=i}^{\infty} \frac{\underline{a}^k}{k+\mathfrak{n}} &\leq \frac{\underline{a}^i}{i+\mathfrak{n}} + \int_i^{\infty} \frac{\underline{a}^s}{s+\mathfrak{n}} ds \\
&\leq \frac{\underline{a}^i}{i+\mathfrak{n}} + \frac{\underline{a}^{-\mathfrak{n}}}{i+\mathfrak{n}} \int_{i+\mathfrak{n}}^{\infty} \exp(\log \underline{a} s) ds \\
&= \frac{\underline{a}^i}{i+\mathfrak{n}} + \frac{\underline{a}^i}{(i+\mathfrak{n})(-\log \underline{a})}
\end{aligned}
$$

which implies that $\frac{1}{\underline{a}^i} \sum_{k=i}^{\infty} \frac{\underline{a}^k}{k+\mathfrak{n}}$ vanishes as $i \to \infty$. Hence $x_i \to 0$ as $i \to \infty$ which implies in particular that the sequence is bounded.

Furthermore, the sequence $(x_i)_{i\in\mathbb{N}}$ is positive since otherwise it would diverge to $-\infty$ by the same arguments used in the proof of Lemma 3.6. Moreover, the lower bound (3.12) and the upper bound of order $\mathcal{O}(i^{-1})$ implies the convergence rate $x_i = \mathcal{O}(i^{-1})$ as $i \to \infty$.

Finally, by Lemma 3.6, the sequence $(x_i)_{i\geq 0}$ with $x_1 = \frac{1}{\lambda}\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1}\right)$ is indeed the minimal positive solution to (3.5), which determines uniquely the value of $\hat{x}$. □

Note that the Lemma 3.10 completes the proof of Theorem 2.3 and we can proceed to the proof of Corollary 2.4. The sequence of expected hitting times $(\mathbb{E}_i[T_X^{(\mathfrak{n})}])_{i\in\mathbb{N}}$ is the solution of

$$\begin{cases} x_0 = 0, \;\; x_1 = \hat{x}, \\ \lambda_i x_{i+1} = -1 + (\gamma_i + \lambda_i + \mu_i)x_i - \mu_i x_{i-1}, \quad i \geq 1, \end{cases} \tag{3.13}$$

with $\hat{x} = \frac{1}{\lambda}\sum_{k=1}^{\infty}\frac{\underline{a}^k}{k+\mathfrak{n}}$ where from Lemma 3.10 follows the rate of convergence of $\mathbb{E}_i[T_X^{(\mathfrak{n})}]$ to 0 as $i \to \infty$.

4. Recurrence Property and Stationary Distribution

Based on the information we obtained on the expected hitting times of the catastrophe state, we now prove the following theorem.

Theorem 4.1 (Positive Recurrence)**.** *Let $X = (X_t)_{t\geq 0}$ be the BD+$C_{\mathfrak{n}}$ process whose infinitesimal generator Q is given by (2.1). Then X is positive recurrent, it exhibits a unique non degenerated stationary distribution π and, for any initial distribution and for large time, X_t converges in distribution to π.*

Note that this is radically different from the behavior of a BD-process with immigration satisfying $\lambda > \mu$. In that case the population grows in expectation over time and no stationary distribution exists, see e.g. [And91]. Hence, by introducing a partial catastrophe to the model its behavior changes drastically, ensuring for any catastrophe rate $\gamma > 0$ the existence of a unique stationary distribution.
To prove the theorem, we apply the existence of Lyapunov functions.

Proof of Theorem 4.1. Recall that a function $V : \mathbb{N} \to \mathbb{R}^+ \cup \{+\infty\}$ is a Lyapunov function associated to the CTMC X if V satisfies

$$QV(n) \leq -1 + \mathbb{1}_A(n)$$

where Q is the generator of X and A is some *petite set*, see e.g. [MLU93]. Take here $A = \{0, \ldots, \mathfrak{n}\}$. By the previous section, $\sup_{i\in\mathbb{N}}\mathbb{E}[T_X^A|X_0 = i] < \infty$ where T_X^A is the first time the process X hits the set A. Hence, $i \mapsto 1 + \mathbb{E}[T^A|X_0 = i]$ yields a Lyapunov function. It follows directly by irreducibility of X that the process X admits a stationary distribution π and that it is positive recurrent. The uniqueness of π follows from the irreducibility of X and the limiting behavior follows from classical results on non-explosive continuous time Markov chains, see e.g. [Nor97]. □

5. Expected Population Size at Fixed Times

In this last section we analyse the first two moments of the BD+C$_\mathfrak{n}$ process at a fixed time $t \geq 0$ and prove Theorem 2.5 and Theorem 2.6. Due to the asymmetric form of the generator (2.1) we focus on upper bounds of the first and second moment of $(X_t)_{t\geq 0}$.
In contraposition to the last section, the parameter ν representing the immigration rate in case of extinction of the population now plays an important role, making the state 0 non-absorbing.

5.1. **Associated Kolmogorov Equations.** Let us consider the Kolmogorov equations associated to the BD+C$_\mathfrak{n}$ process $(X_t)_{t\geq 0}$ to analyze its behavior at fixed time. With $P_n(t) := \mathbb{P}[X_t = n | X_0 = n_0], n_0 > \mathfrak{n}$, the following identities hold:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} P_0(t) &= \mu P_1(t) - \nu P_0(t) \\ \frac{\mathrm{d}}{\mathrm{d}t} P_n(t) &= \lambda_{n-1} P_{n-1}(t) - n(\mu+\lambda) P_n(t) + (n+1)\mu P_{n+1}(t), \qquad 0 < n < \mathfrak{n}, \\ \frac{\mathrm{d}}{\mathrm{d}t} P_\mathfrak{n}(t) &= (\mathfrak{n}-1)\lambda P_{\mathfrak{n}-1}(t) - \mathfrak{n}(\mu+\lambda) P_\mathfrak{n}(t) + (\mathfrak{n}+1)\mu P_{\mathfrak{n}+1}(t) + \gamma \sum_{i=\mathfrak{n}+1}^\infty i P_i(t), \\ \frac{\mathrm{d}}{\mathrm{d}t} P_n(t) &= (n-1)\lambda P_{n-1}(t) - n(\gamma+\mu+\lambda) P_n(t) + (n+1)\mu P_{n+1}(t), \quad n > \mathfrak{n}, \end{cases} \tag{5.1}$$

with initial condition $P_n(0) = \delta_{n_0,n}$. Recall that $\lambda_n = n\,\lambda$ if $n \geq 1$ but $\lambda_0 = \nu$.

5.2. **Upper bounds for First and Second Moment.** We approach the first and second moment by analyzing their corresponding ODEs.

Proof of Theorem 2.5. Using (5.1), we obtain

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[X_t] &= \lambda \sum_{n=1}^\infty n(n-1)P_{n-1}(t) - (\mu+\lambda)\sum_{n=1}^\infty n^2 P_n(t) \\ &\quad + \mu \sum_{n=1}^\infty n(n+1)P_{n+1}(t) - \gamma \sum_{n=\mathfrak{n}+1}^\infty (n-\mathfrak{n}) n P_n(t) + \nu P_0(t) \\ &= (\lambda-\mu)\mathbb{E}[X_t] + \gamma \sum_{n=\mathfrak{n}+1}^\infty (\mathfrak{n}-n) n P_n(t) + \nu P_0(t). \end{aligned}$$

If we denote the first moment by $m(t) := \mathbb{E}[X_t]$ then it solves the ODE

$$m'(t) = (\lambda-\mu)\, m(t) - \gamma \sum_{n=\mathfrak{n}+1}^\infty (n-\mathfrak{n}) n P_n(t) + \nu P_0(t), \quad m(0) = n_0 \tag{5.2}$$

In particular, by Jensen's inequality,

$$\begin{aligned} m'(t) &\leq (\lambda-\mu)\, m(t) + \gamma \sum_{n=0}^{\infty} (\mathfrak{n}-n) n P_n(t) + \nu P_0(t) \\ &\leq (\lambda-\mu)\, m(t) + \gamma(\mathfrak{n} - m(t))\, m(t) + \nu. \end{aligned}$$

The solution $\bar{m}$ to the ODE

$$\bar{m}'(t) = (\lambda-\mu)\bar{m}(t) + \gamma(\mathfrak{n} - \bar{m}(t))\, \bar{m}(t) + \nu, \quad \bar{m}(0) = n_0, \tag{5.3}$$

is, therefore, a candidate for an upper bound of m. To that aim, we will show that the difference $\bar{m} - m$ is a non-negative function on $[0, \infty)$.
We first prove that $\bar{m} - m$ has a strict local minimum at $t = 0$. Note that solutions of both ODEs (5.2) and (5.3) exist globally on $(0, \infty)$. Since both right hand sides are continuous in t for $t \geq 0$ we can extend definition of the derivative to the set $[0, \infty)$. Note that if $n_0 > \mathfrak{n}$ then

$$\lim_{t\to 0^+} m'(t) \;=\; (\lambda-\mu)n_0 + \gamma(\mathfrak{n} - n_0)n_0 = \lim_{t\to 0^+} \bar{m}'(t) - \nu < \lim_{t\to 0^+} \bar{m}'(t)$$

and $\bar{m}(0) = m(0)$. Thus, during some time, $\bar{m}$ dominates m:

$$\exists t_0 > 0: \quad m(t) \leq \bar{m}(t) \quad \forall t \in [0, t_0).$$

Assume that this domination only holds locally, or equivalently, that there is a $t_2 > t_0$ such that $\bar{m}(t_2) < m(t_2)$. By continuity there exists an interval $I_\delta := (t_2 - \delta, t_2 + \delta)$ such that $\bar{m}(t) < m(t)$ for all $t \in I_\delta$. Without loss of generality we may assume that $\bar{m}(t) \geq m(t)$ for all $t \in [0, t_2 - \delta)$. Set $t_1 := t_2 - \delta$. By continuity of $\bar{m}$ and m, $\bar{m}(t_1) = m(t_1)$. Additionally, since $\bar{m}$ and m are C^1-functions, $\bar{m}'(t_1) < m'(t_1)$. But, on the other hand

$$\bar{m}'(t_1) = g(\bar{m}(t_1)) = g(m(t_1)) \geq m'(t_1),$$

where $g(x) := (\lambda-\mu)x + \gamma(\mathfrak{n} - x)x + \nu$, which leads to a contradiction. Thus, $\bar{m}$ dominates m globally.

□

Note that the ODE (5.3) is the same than (2.6), whose solution is a logistic growth function. In particular $\bar{m}(t)$ converges for t large to its equilibrium value $\bar{m}_e$ given by the largest zero of the polynomial $\mathbf{x} \mapsto \gamma\, \mathbf{x}^2 - (\lambda - \mu + \gamma\, \mathfrak{n})\, \mathbf{x} - \nu$,

$$\bar{m}_e := \frac{(\lambda - \mu + \gamma\, \mathfrak{n}) + \sqrt{(\lambda - \mu + \gamma\, \mathfrak{n})^2 + 4\gamma\nu}}{2\gamma}. \tag{5.4}$$

Moreover $t \mapsto \bar{m}(t)$ is decreasing whenever it is larger than its equilibrium $\bar{m}_e$.

Proof of Theorem 2.6. The dynamics of the process' second moment $v(t) := \mathbb{E}[X_t^2]$ can be derived in the same way as the one for the first moment and it reads

$$\begin{aligned} v'(t) &= 2(\lambda-\mu)v(t) + (\lambda+\mu)m(t) - \gamma \sum_{n=\mathfrak{n}+1}^{\infty} (n^3 - \mathfrak{n}^2 n)P_n(t) \\ &\leq 2(\lambda-\mu)v(t) + (\lambda+\mu)m(t) - \gamma \sum_{n=0}^{\infty}(n^3 - \mathfrak{n}^2 n)P_n(t) + \nu P_0(t) \\ &\leq 2(\lambda-\mu)v(t) + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\bar{m}(t) - \gamma v(t)^{\frac{3}{2}} + \nu \end{aligned}$$

The last inequality, due to Jensen's inequality, is even strict for $t > 0$:

$$v'(t) < 2(\lambda-\mu)v(t) + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\bar{m}(t) - \gamma v(t)^{\frac{3}{2}} + \nu, \quad t > 0.$$

Consider now the function $\bar{v}$ being solution of the ODE

$$\bar{v}'(t) = 2(\lambda-\mu)\,\bar{v}(t) + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\,\bar{m}(t) - \gamma\,\bar{v}(t)^{\frac{3}{2}} + \nu, \quad \bar{v}(0) = n_0^2. \tag{5.5}$$

It is a candidate for an upper bound of v. Compare the right limits in 0 of the first derivatives:

$$\lim_{t\to 0^+} v'(t) < 2(\lambda-\mu)n_0^2 + (\lambda+\mu+\gamma\,\mathfrak{n}^2)n_0 - \gamma n_0^3 + \nu = \lim_{t\to 0^+} \bar{v}'(t). \tag{5.6}$$

By the same argumentation as for the m, we obtain the domination of v by the function $\bar{v}$.

The positive function $\bar{v}$ is indeed bounded, as we will prove now.
Assume conversely, that any large value can be taken by $\bar{v}$. In particular choose any c_0 large enough such that

$$2(\lambda-\mu)c_0 + (\lambda+\mu+\gamma\,\mathfrak{n}^2)n_0 - \gamma c_0^{\frac{3}{2}} + \nu < 0 \tag{5.7}$$

and suppose that there exists a time t_0 such that $\bar{v}(t_0) = c_0$. Since by assumption $n_0 > \bar{m}_e$, the function $\bar{m}$ decreases from $\bar{m}(0) = n_0$. It follows that

$$\begin{aligned} \bar{v}'(t_0) &= 2(\lambda-\mu)\bar{v}(t_0) + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\bar{m}(t_0) - \gamma\bar{v}(t_0)^{\frac{3}{2}} + \nu \\ &\leq 2(\lambda-\mu)c_0 + (\lambda+\mu+\gamma\,\mathfrak{n}^2)n_0 - \gamma c_0^{\frac{3}{2}} + \nu < 0. \end{aligned}$$

Hence, whenever $\bar{v}$ takes the value c_0, it has a negative derivative. Thus, for $t > t_0$, $\bar{v}(t)$ is uniformly bounded by c_0 which is contradictory. □

We also compare $\bar{v}$ and $\bar{m}^2$ and obtain informations on the long time behavior of $\bar{v}$.

Proposition 5.1. *Let $\bar{v}$ be solution of the ODE (5.5) with $n_0 > \bar{m}_e$ and $\lambda + \mu \geq 2\nu$. Then $\bar{v}(t) \geq \bar{m}^2(t)$ for all $t \geq 0$, where $\bar{m}$ is solution of the ODE (5.3). Moreover, as $t \to \infty$, $\bar{v}(t)$ tends to the solution $\bar{v}_\infty$ of the following equation:*

$$0 = 2(\lambda-\mu)\bar{v}_\infty + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\bar{m}_e - \gamma\bar{v}_\infty^{\frac{3}{2}} + \nu. \tag{5.8}$$

To prove these properties, we analyze the nullcline, i.e., the function $\mathbf{v}(t)$ solution of the limit equation obtained from (5.5) with vanishing l.h.s.:

$$0 = 2(\lambda-\mu)\mathbf{v}(t) + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\bar{m}(t) - \gamma(\mathbf{v}(t))^{\frac{3}{2}} + \nu. \tag{5.9}$$

To prove the uniqueness of the solution of the equation (5.8) we use Descarts' rule of signs, which we now recall (see [HK91]):

Lemma 5.2. *The number of positive roots counted with multiplicities of a polynomial with real coefficients is equal to the number of changes of sign in the list of coefficients, or is less than this number by a multiple of 2.*

We also obtain the C^1-regularity of the function $\mathbf{v}$ by the following lemma.

Lemma 5.3. *Let $t \geq 0$ and $\mathbf{P}^t : \mathbf{x} \mapsto a_0(t) + \sum_{k=1}^n a_k \mathbf{x}^k$ be a real polynomial with $a_0 \in C^1([0,\infty),\mathbb{R})$ and $a_0'(t) < 0$ for all $t > 0$. If for all $t \geq 0$ the polynomial $\mathbf{P}^t$ has a unique positive simple zero denoted by x_0^t then $t \mapsto x_0^t \in C^1((0,\infty),\mathbb{R})$.*

Proof of Lemma 5.3. For any $t > 0$ and sufficiently small $h > 0$,

$$\begin{aligned} 0 &= \sum_{k=1}^n a_k\Big((x_0^{t+h})^k - (x_0^t)^k\Big) + a_0(t+h) - a_0(t) \\ &= \left(x_0^{t+h} - x_0^t\right) \sum_{k=1}^n a_k \sum_{l=1}^{k-1} \left(x_0^{t+h}\right)^l \left(x_0^t\right)^{k-1-l} + a_0(t+h) - a_0(t). \end{aligned}$$

Note that $\sum_{k=1}^n a_k \sum_{l=1}^{k-1} \left(x_0^{t+h}\right)^l \left(x_0^t\right)^{k-1-l} \neq 0$ and $x_0^{t+h} - x_0^t \neq 0$, because a_0 is decreasing by assumption. Thus,

$$\frac{x_0^{t+h} - x_0^t}{h} = \frac{a_0(t) - a_0(t+h)}{h} \left(\sum_{k=1}^n a_k \sum_{l=1}^{k-1} \left(x_0^{t+h}\right)^l \left(x_0^t\right)^{k-1-l}\right)^{-1} \tag{5.10}$$

and, hence, $t \mapsto x_0^t \in C^1((0,\infty),\mathbb{R})$. □

Proof of Proposition 5.1. Define a positive function ψ by $\psi(t)^2 := \mathbf{v}(t)$ where $\mathbf{v}(t)$ solves the equation (5.9). Then $\psi(t)$ is a positive zero of the polynomial

$$\mathbf{x} \mapsto -\gamma\mathbf{x}^3 + 2(\lambda-\mu)\mathbf{x}^2 + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\bar{m}(t) + \nu.$$

By Lemma 5.2, for fixed t it exists, is simple and unique. Therefore the nullcline is defined pointwise by $\mathbf{v}(t) := \sqrt{\psi(t)}, t \geq 0$. Since $t \mapsto \bar{m}(t)$ is continuous, $\mathbf{v}$ is also continuous. The function $\mathbf{v}$ is even continuously differentiable by positivity of ψ, smoothness of $\bar{m}$ and Lemma 5.3. Moreover, since $\bar{m}$ converges to m_e as $t \to \infty$, $\mathbf{v}$ converges as $t \to \infty$ to the solution of the equation (5.8). Furthermore, differentiating (5.9), one obtains

$$0 = 2(\lambda-\mu)\mathbf{v}'(t) + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\bar{m}'(t) - \frac{3}{2}\gamma\mathbf{v}'(t)\mathbf{v}(t)^{\frac{1}{2}}. \tag{5.11}$$

Since $\bar{m}'$ tends to 0 as $t \to \infty$, therefore $\mathbf{v}'$ converges to some value denoted by $\mathbf{v}'(\infty)$, which solves the equation

$$0 = 2(\lambda - \mu)\mathbf{v}'(\infty) - \frac{3}{2}\gamma \mathbf{v}'(\infty)\bar{v}_\infty^{\frac{1}{2}}. \tag{5.12}$$

Thus, either $\mathbf{v}'(\infty) = 0$ or $\bar{v}_\infty = \left(\frac{4(\lambda-\mu)}{3\gamma}\right)^2$. This later value cannot solve (5.8). Hence, $\mathbf{v}'(\infty) = 0$. Note that if $x > x_0^t$, then

$$-\gamma x^3 + 2(\lambda - \mu)x^2 + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\bar{m}(t) + \nu < 0$$

and vice versa. Thus, $\bar{v}$ always tends towards the nullcline $\mathbf{v}$. Since $\bar{v}$ always tends towards the nullcline $\mathbf{v}$ and $\mathbf{v}'$ tends to 0, also $\bar{v}$ converges to $\bar{v}_\infty$, which proves the claim. □

Note by the way that one deduces from equation (5.8) that $\bar{v}_\infty \sim \mathcal{O}(\mathfrak{n}^2)$ as $\mathfrak{n} \to \infty$.

About the comparison between $\bar{v}(t)$ and $\bar{m}^2(t)$:
By assumption $\bar{v}(0) = n_0^2 = \bar{m}(0)^2$. Note that

$$\bar{m}(t) \geq \bar{m}_e > \mathfrak{n} \Rightarrow (\mathfrak{n} - \bar{m}(t))^2 > 0 \Rightarrow 2\bar{m}(t)(\mathfrak{n} - \bar{m}(t)) < \mathfrak{n}^2 - \bar{m}^2(t).$$

Therefore

$$\begin{aligned}
(\bar{m}^2)'(t) &= 2\bar{m}(t)\bar{m}'(t) = -2\gamma\bar{m}(t)^3 + 2(\lambda - \mu + \gamma\,\mathfrak{n})\bar{m}^2(t) + 2\nu\bar{m}(t) \qquad (5.13)\\
&< \gamma(\mathfrak{n}^2 - \bar{m}^2(t))\bar{m}(t) + 2(\lambda - \mu)\bar{m}^2(t) + 2\nu\bar{m}(t)\\
&= -\gamma\left(\bar{m}^2(t)\right)^{\frac{3}{2}} + 2(\lambda - \mu)\bar{m}^2(t) + (2\nu + \gamma\,\mathfrak{n}^2)\bar{m}(t)\\
&\leq -\gamma\left(\bar{m}^2(t)\right)^{\frac{3}{2}} + 2(\lambda - \mu)\bar{m}^2(t) + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\bar{m}(t), \qquad (5.14)
\end{aligned}$$

where the assumption $2\nu \leq \lambda + \mu$ is used for the last inequality. Now by (5.5) and (5.13),

$$\begin{aligned}
\bar{v}'(0) - (\bar{m}^2)'(0) &= 2(\lambda - \mu)\,\bar{v}(0) + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\,\bar{m}(0) - \gamma\,\bar{v}(0)^{\frac{3}{2}} + \nu\\
&\quad -\Big(-2\gamma\bar{m}(0)^3 + 2(\lambda - \mu + \gamma\,\mathfrak{n})\bar{m}^2(0) + 2\nu\bar{m}(0)\Big)\\
&= (\lambda + \mu - 2\nu)n_0 + \gamma n_0(n_0 - \mathfrak{n})^2 + \nu > 0.
\end{aligned}$$

This inequality propagates for all $t \geq 0$. This can be proved by the same argumentation as in the proof of Theorem 2.5, using (5.14).

5.3. **Accuracy of these upper bounds.** Previously, we found an upper bound $\bar{m}$ to the first moment m. We would like to quantify the sharpness of this bound by estimating the non negative difference function D defined by

$$D(t) := \bar{m}(t) - m(t) \;\geq 0, \quad D(0) = 0. \tag{5.15}$$

Moreover, $\limsup_{t\to\infty} D(t) \leq \bar{m}_e$ which implies that D is a bounded function.

Proposition 5.4 (Upper Bound for the difference function D). *Let m, $\bar{m}$, v and $\bar{v}$ defined as above under the same assumptions. Then, the difference function D defined in equation (5.15) is pointwise bounded from above by the solution $\bar{D}$ of the non linear ODE*

$$\begin{cases} \bar{D}'(t) = -\gamma\bar{D}(t)^2 + \big(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}(t))\big)\bar{D}(t) \\ \qquad\qquad\qquad +\gamma\big(\bar{v}(t) - \bar{m}(t)^2\big) + \gamma\dfrac{\mathfrak{n}^2}{4} + \nu, \quad t > 0, \\ \bar{D}(0) = 0. \end{cases} \tag{5.16}$$

Moreover, the function $\bar{D}$ tends for large time to the positive solution $\bar{D}_\infty$ of the equation

$$0 = -\gamma\bar{D}_\infty^2 + \big(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}_e)\big)\bar{D}_\infty + \gamma(\bar{v}_\infty - \bar{m}_e^2) + \gamma\frac{\mathfrak{n}^2}{4} + \nu. \tag{5.17}$$

Proof. By differentiating the function D defined in (5.15) one obtains

$$\begin{aligned} D'(t) &\leq (\lambda - \mu)D(t) + \mathfrak{n}\gamma\bar{m}(t) - \gamma\bar{m}^2(t) + \gamma\sum_{n\geq\mathfrak{n}+1}(n - \mathfrak{n})nP_n(t) + \nu \\ &\leq -\gamma D(t)^2 + (\lambda - \mu + \gamma\mathfrak{n})D(t) - 2\gamma D(t)m(t) + 2\gamma D(t)\bar{m}(t) \\ &\qquad +\gamma(\bar{v}(t) - \bar{m}^2(t)) + \gamma\sum_{n=0}^{\mathfrak{n}}(\mathfrak{n} - n)nP_n(t) + \nu \qquad (5.18) \\ &\leq -\gamma D(t)^2 + \big(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}(t))\big)D(t) + \gamma(\bar{v}(t) - \bar{m}^2(t)) + \gamma\frac{\mathfrak{n}^2}{4} + \nu. \end{aligned}$$

The function $\bar{D}$, solving the ODE (5.16), is an upper bound for D since $D(0) = \bar{D}(0)$ and

$$\lim_{t\to 0^+} D'(t) = 0, \qquad \lim_{t\to 0^+} \bar{D}'(t) = \gamma\frac{\mathfrak{n}^2}{4} + \nu > 0.$$

Thus, by similar arguments as for $\bar{m}$ and $\bar{v}$, it follows that $\bar{D}$ is a global upper bound of D.

To investigate the asymptotic behavior of $\bar{D}(t)$ we use the similar techniques as we applied in the proof of Proposition 5.1. This time, the equation (5.16) induces the positive nullcline $\mathbf{D}$ satisfying

$$0 = -\gamma\mathbf{D}(t)^2 + \big(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}(t))\big)\mathbf{D}(t) + \gamma\big(\bar{v}(t) - \bar{m}^2(t)\big) + \gamma\frac{\mathfrak{n}^2}{4} + \nu.$$

Applying again Lemma 5.2, one proves its existence and also the convergence of $\bar{D}$ towards this nullcline. As in Proposition 5.1, since $\bar{v}$ and $\bar{m}$ converges for t large, $\mathbf{D}(t)$ converges to $\bar{D}_\infty$, which is the unique positive solution to

$$0 = -\gamma\bar{D}_\infty^2 + \big(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}_e)\big)\bar{D}_\infty + \gamma(\bar{v}_\infty - \bar{m}_e^2) + \gamma\frac{\mathfrak{n}^2}{4} + \nu.$$

Therefore, analogously as precedently, the function $\bar{D}$ converges to $\bar{D}_\infty$ thanks the positivity of $\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}(t))$ and the C^1-regularity of $\mathbf{D}$.

Solving (5.17) explicitly, we obtain

$$\bar{D}_\infty = \frac{(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}_e)) + \sqrt{(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}_e))^2 + \gamma^2(4(\bar{v}_\infty - \bar{m}_e^2) + \mathfrak{n}^2) + 4\gamma\nu}}{2\gamma}$$

and consequently $\bar{D}_\infty \sim \mathcal{O}(\mathfrak{n})$ as $\mathfrak{n} \to \infty$. Hence, for small $\mathfrak{n}$, the small size of $\bar{D}$ leads to sufficiently good estimates by considering $\bar{m}$ instead of m. Instead, for large $\mathfrak{n}$, the large values of $\bar{D}$ do not allow to conclude if $\bar{m}$ and m are close. Finer estimates would be needed to paint a clearer picture of the sharpness of the upper bounds but they seem to be currently out of reach.

□

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Patrick CATTIAUX, Institut de Mathématiques de Toulouse, CNRS UMR 5219. Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse cedex 09, France.

Email address: `patrick.cattiaux@math.univ-toulouse.fr`

Jens FISCHER, Institut de Mathématiques de Toulouse, CNRS UMR 5219. Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse cedex 09, France; Institut für Mathematik der Universität Potsdam. Karl-Liebknecht-Str. 24-25, 14476 Potsdam OT Golm, Germany.

Email address: `jens.fischer@math.univ-toulouse.fr`

Sylvie RŒLLY, Institut für Mathematik der Universität Potsdam. Karl-Liebknecht-Str. 24-25, 14476 Potsdam OT Golm, Germany.

Email address: `roelly@math.uni-potsdam.de`

Samuel SINDAYIGAYA, Institut d'Enseignement Supérieur de Ruhengeri. Musanze, Street NM 155, Po Box: 155 Ruhengeri, Rwanda.

Email address: `sindasam12@gmail.com`

9.1 Outlook: Populations under Catastrophic Events

The previously introduced process $BD+C_{\mathfrak{n}}$ captures a reduction in the population size of a population which exhibits a fixed level of stability $\mathfrak{n}$. Deaths due to catastrophes are assumed to happen at the same time, capturing the extremely short time scale of catastrophes relative to the regular time scales of births and deaths. This short time scale is due to the acceleration of the endogenous effects which cause the catastrophes, in particular conflicts or pandemics. In the beginning they start slowly, have then a short phase with a high number of casualties before slowing down again in the intensity. Processes who capture this behavior for exogenous shocks, for example in the modeling of earthquakes and aftershocks, are Hawkes processes which are due to this property also known as self-exciting processes, see [Oga98]. Self-exciting processes are point processes which carry their name due to the recursive structure of the intensity function associated to counting measure, see [HaOa74] for the ground laying work and [JanOh21] for a review of the topic. With Hawkes process on the real line is a point process $\mathcal{T}$ on the positive real line with associated counting measure $N = (N_t)_{t\geq 0}$ with intensity

$$\lambda(t) = \mu + \sum_{\substack{t' \in \mathcal{T} \\ t' < t}} h(t - t') \tag{9.1}$$

where $\mu > 0$ and h is a non-negative real function. This leads to a clustering of events after an initial event as illustrated in Figure 56. A priori, the cluster sizes are

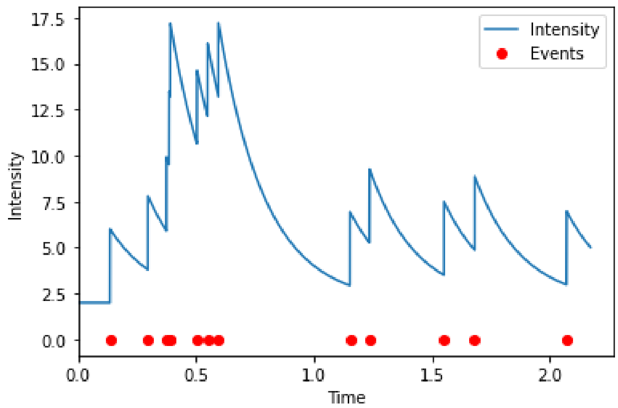

Figure 56: An illustration of a Hawkes process with $h(t) = \alpha \exp(-\beta t)$. One can see the clustering of events which is associated to the increase in the intensity function.

unbounded but finite with probability one under certain conditions, which implies the need of a reformulation to interpret the events of a Hawkes process as individuals of a population being affected by a catastrophe because we only consider finite populations. We propose in what follows such a model as an extension to the $BD + C_{\mathfrak{n}}$.

9.1.1 Future work

In the Outlook of this part, we want to consider a variation of the Hawkes process as a more flexible alternative to the $BD + C_{\mathfrak{n}}$ process.A Hawkes process on the real line is a point process $\mathcal{T}$ on the positive real line with associated counting measure $N = (N_t)_{t\geq 0}$ with intensity

$$\lambda(t) = \mu + \sum_{\substack{t' \in \mathcal{T} \\ t' < t}} h(t - t').$$

where $\mu > 0$ is a constant and h is a positive real function. Reviews and results on Hawkes processes can be found here [Oga98], [JanOh21] to name just a small selection. They are most known for the self-exciting nature, in the sense that the occurrence of one jump of the counting measure N increases the probability of a second event shortly afterwards in a cumulative fashion. This class of stochastic processes has seen a lot of interest in the modeling of earthquakes, epidemics as well as shocks in the financial markets and, therefore, systems undergoing exogenous shocks.

In what follows, we show a possibility of how the self-exciting nature of the Hawkes process can be a sensible approach to endogenous shocks in a population, for example, due to conflict. Unfortunately, Hawkes processes capture by their nature the impact of exogenous shocks in form of a Poisson process with intensity μ and subsequent induced effects on the population due to the kernels h which does not take in to account an underlying population and its individuals which form the "support" of the process. In [RizMis18] the authors propose the HawkesN process to enable the use of the self-exciting nature of Hawkes processes for population models with finite size of size $\mathfrak{n}$. The adjustment in form of the intensity function is given by

$$\tilde{\lambda}_{\mathfrak{n}}(t) = \left(1 - \frac{N_t}{\mathfrak{n}}\right)\left(\mu + \sum_{\substack{t' \in \mathcal{T} \\ t' < t}} h(t - t')\right). \tag{9.2}$$

The factor $\left(1 - \frac{N_t}{\mathfrak{n}}\right)$ ensures that, as the number of events in the HawkesN process increases, the rate of new events slows down. As soon as the process hits $\mathfrak{n}$ events it stops. The processes has been applied to real world cases in [UnRout21] allowing the forecast of disease transmission.

We propose a model using a jump process $\mathcal{Q} = (\mathcal{Q}_t)_{t\geq 0}$ based on the HawkesN process $(N_t^P)_{t\geq 0}$. It consists of a population which grows according to a birth-death process $(P_t)_{t\geq 0}$ with state dependent rates λi and μi, where λ is the individual birth rate and μ the individual death rate where i is the current value of Q_t. Additionally, there is a susceptible sub-population of size $P_t - \mathfrak{n}$. The whole population then evolves

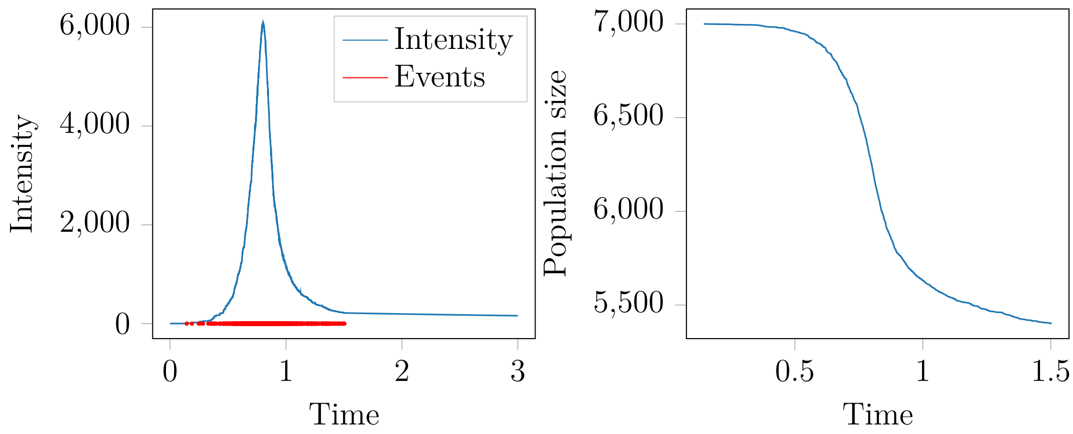

Figure 57: Left: Intensity function of N^H using an exponential kernel $h(\cdot) = \exp(-\alpha\,\cdot)$. The resistent population is of size $N = 5000$ and $P_t = 7000$. Right: The decrease of the population size from 7000 to approximately 5400. Note that the "time" here is virtual time and has no interpretative value since there was no fitting to any data involved.

like $(Q_t)_{t\geq 0} = (P_t - N_t^H)_{t\geq 0}$ where N^H is governed by the intensity function

$$\lambda_{N^H}(t) = \Phi(P_t, \mathfrak{n}, N_t^H)\left(\gamma + \sum_{\substack{t'\in\mathcal{T}\\ t'<t}} h(t-t')\right). \tag{9.3}$$

where $\gamma > 0$ is the risk factor of starting a catastrophic event, h is a positive real function and the factor Φ is a positive function and satisfies $\Phi(P_t, \mathfrak{n}, 0) = 1$ and $\Phi(P_t, \mathfrak{n}, P_t - \mathfrak{n}) = 0$. In what follows, we use the following form, which is inspired by the kernel for modeling earthquakes via Hawkes processes. We fix Φ to be

$$\Phi(P_t, \mathfrak{n}, N_t^H) = \begin{cases} \dfrac{1}{1 + \left(\frac{N_t^H (P_t-\mathfrak{n})^{-1}}{1-N_t^H (P_t-\mathfrak{n})^{-1}}\right)^k} & \text{, if } N_t^H < P_t - \mathfrak{n}, \\ 0 & \text{, otherwise} \end{cases}$$

for some $k \in \mathbb{N}$. Assuming independence of the jump times of P and N^H and choosing the function $h(\cdot) = \exp(-\alpha\,\cdot)$, one obtains a behavior as shown in Figure 57. Note the peak in the intensity function which shows the accumulation of events of N^H and the corresponding drop in the population size Q. We exploit the fact that the adjusted process is by definition always finite such that the classical constrain on Hawkes processes

$$m := \int_0^\infty h(s)ds < 1 \tag{9.4}$$

where h is the kernel of the process is no longer necessary since we enforce finiteness. Thus, we can use supercritical values $m > 1$ to model extreme events that are concentrated in time. This way, we can tackle the short coming of the $BD + C_{\mathfrak{n}}$ process, where instead of a fast time scale we introduced catastrophes as instantaneous events. Furthermore, we add to the flexibility of the process since now the parameter $\mathfrak{n}$ and, therefore, $P_t - \mathfrak{n}$ may easily be replaced in the definitions by another birth-death process, which is independent of P. This might be, in particular, be interesting for conflict models in populations, forecasting casualties and the comparison with an evolution without conflict. In this sense, the $BD + C_{\mathfrak{n}}$ is rigid, since it does not offer direct generalizations without losing the insights we obtained. Rendering the model in this way more realistic comes with the prize of a more complex process which is, in particular, no longer a Markov chain. But similar processes have been tackled in the literature, as for example in [SchoHo17] and [RizMis18] and some of the applied methods may, after some first considerations by the author of this work, be applicable to this form of the HawkesN process as well as generalizations, which, we think, add to models on populations dynamics with endogenous shocks, as for example, due to conflict within a population.

List of Symbols

Symbols	Meaning
θ	Tolerance threshold in Echo Chamber model and Deffuant model
μ	Openmindedness parameter in Echo Chamber model and Deffuant model
G	Simple undirected graph
$\mathcal{V}$	Vertex set associated to a simple graph G
$\mathcal{E}$	Edge set associated to a simple graph G
a, b	Vertices in $\mathcal{V}$
$\langle a, b \rangle$	Undirected edge in G
(a, b)	Directed edge from a to b in a directed graph
n	Size of vertex set of a graph G
$(X_a^t)_{a\in\mathcal{V};t\in\mathbb{N}}$	Opinions in the Echo Chamber or Deffuant model
G_c	Complete graph
$\kappa(G)$	Vertex connectivity of G
$\text{diam}(G)$	Diameter of G
$\omega_l^G(a, b)$	the number of walks of length l from a to b along the edges in G
L	Simple graph with interpretation generalized from line graph of G_c
V	Vertex set associated to L
E	Edge set associated to L
v, w, u	Vertices in V
$\bar{n}$	Size of vertex set of a graph L
$\bar{d}$	Vertex degree in a regular graph
α	Number of common neighbors of two neighbors in strongly graph
β	Number of common neighbors of two non-neighbors in strongly graph

$\deg(v)$ .... Degree of $v \in V$ in L

$\mathcal{M}$ .... Perfect matching in some graph L or G

$\iota(L)$ .... Isoperimetric constant of some graph L

k .... Positive integer representing number of particles on L

$\mathfrak{V}_k$ .... Subsets of V of size k representing particle configurations on L

$\mathfrak{E}_k$ .... neighborhood relationships of particle configurations in $\mathfrak{V}_k$

$\mathfrak{L}_k$ .... kPG associated to L constructed from $\mathfrak{V}_k$ and $\mathfrak{E}_k$

$\mathfrak{v}, \mathfrak{w}, \mathfrak{u}$ .... Particle configurations in $\mathfrak{V}_k$

$L_{\mathfrak{v}}$ .... Vertex induced sub-graphs of L on $\mathfrak{v} \subset V$

$\deg_k(\mathfrak{v})$ .... Degree of $\mathfrak{v}$ as vertex in $\mathfrak{L}_k$

$\deg^{L_{\mathfrak{v}}}(v)$ .... Degree of $v \in V$ in vertex induced sub-graphs $L_{\mathfrak{v}}$ in L

D_k .... Set of different degrees of $\mathfrak{L}_k$

$\mathfrak{V}_{D_k;d}$ .... Vertices in $\mathfrak{L}_k$ with degree d

$\vec{\mathfrak{L}}_k$ .... Random hierarchical kPG

$\mathfrak{V}_k^{(i)}$ .... Level set i of $\vec{\mathfrak{L}}_k$

$\vec{\mathfrak{E}}_k^{(i,i+1)}$ .... Edge set between $\mathfrak{V}_k^{(i)}$ and $\mathfrak{V}_k^{(i+1)}$ in $\vec{\mathfrak{L}}_k$

X .... Markov chain on some state space

$\mathfrak{S}_k^{MC}$ .... Markov chain associated to sampling densest sub-graphs

P_k^{MC} .... Transition matrix associated to $\mathfrak{S}_k^{MC}$

$\tilde{\pi}_k^{MC}$ .... Stationary distribution of $\mathfrak{S}_k^{MC}$

$\mathfrak{S}_k$ .... Markov chain associated to Echo Chamber model

$P_k^{\triangle}$ .... Transition matrix associated to $\mathfrak{S}_k$

$\tilde{\pi}_k$ .... Stationary distribution of $\mathfrak{S}_k$

$\Phi_k^{\bar{d}}$ .... Cheeger constant associated to $\mathfrak{S}_k$ on $\bar{d}$-regular graph

$\mathfrak{S}_k^{\mathfrak{c}}$ .... Markov chain associated to classical exclusion process

$P_k^{\mathfrak{c}}$ .... Transition matrix associated to $\mathfrak{S}_k^{\mathfrak{c}}$

$\tilde{\pi}_k^{\mathfrak{c}}$ .... Stationary distribution of $\mathfrak{S}_k^{\mathfrak{c}}$

$\mathfrak{S}_{k,\theta}$ .... Absorbing Markov chain associated to Echo Chamber model

$P_{k,\theta}^{\triangle}$ .... Transition matrix associated to $\mathfrak{S}_{k,\theta}$

$\mathcal{Z}$ .... Set of absorbing vertices in L with random absorbing environment

$T_{k,\theta}^0$ .... Time to absorption of $\mathfrak{S}_{k,\theta}$

$\mathfrak{S}_k^{\text{abs}}$ .... Discrete opinion version of $\mathfrak{S}_{k,\theta}$

T_k^{abs} .... Time to absorption of $\mathfrak{S}_k^{\mathfrak{abs}}$

$\pi_{\mathfrak{q}}$ .... Quasi-stationary distribution of $\mathfrak{S}_k^{\text{abs}}$

$BD + C_{\mathfrak{n}}$ .... Birth-death catastrophe process

Q .... Infinitesimal generator of $BD + C_{\mathfrak{n}}$

$\mathfrak{n}$ .... Maximal stable population size

ν .... Immigration rate of $BD + C_{\mathfrak{n}}$

λ .... Birth rate of $BD + C_{\mathfrak{n}}$

μ .... Death rate of $BD + C_{\mathfrak{n}}$

γ .... Catastrophe rate of $BD + C_{\mathfrak{n}}$

$T_X^{(\mathfrak{n})}$ .... Catastrophe time of $BD + C_{\mathfrak{n}}$

$\bar{m}_t$ .... Upper bound expected population size of $BD + C_{\mathfrak{n}}$ at time t

$\bar{v}_t$ .... Upper bound population size variance of $BD + C_{\mathfrak{n}}$ at time t

π .... Stationary distribution of $BD + C_{\mathfrak{n}}$

$\tilde{\lambda}_{\mathfrak{n}}$ .... Intensity function of HawkesN process

N^H .... HawkesN process counting process

$\mathcal{Q}$ .... Population size under influence of catastrophic HawkesN process

List of Figures

Code

This section contains the code used in this work to construct the graph $\mathfrak{L}_k$ and simulate the Markov chains $\mathfrak{S}_k$ and $\mathfrak{S}_k^{\text{abs}}$.

```
import networkx as nx
import itertools
import numpy as np
from networkx.algorithms import isomorphism
from collections import Counter
from scipy.special import binom
from scipy.stats import randint
from random import choice, seed, sample

seed(42)

def relabel_graph(G, size, k):
    def findsubsets(S,k):
        return itertools.combinations(S, k)

    items = set(range(size))
    int_binomial = [int(value) for value in range(int(binom(size,k)))]
    return nx.relabel_nodes(G, dict(zip(int_binomial,findsubsets(items, k))))

def edge_builder(G, size, H):
    for node1,node2 in itertools.product(G.nodes(),G.nodes()):
        sym_diff = list(set(node1).symmetric_difference(set(node2)))
        if len(sym_diff) == 2 and (sym_diff[0],sym_diff[1]) in H.edges():
            G.add_edge(node1,node2)
    return G

def quotient_with_root(v,u,w):
    v = set(v)
    u = set(u)
    w = set(w)
    if u - v == w - v:
        return True
    else:
        return False

def graph_set(size,k,deg=2,complete = False):
    D = nx.complete_graph(size)
    H = nx.line_graph(D)
    size_H = len(list(H.nodes()))
    H = nx.relabel_nodes(H, dict(zip(H.nodes(),range(size_H))))
    G = nx.empty_graph(int(binom(size_H,k)))
    G = relabel_graph(G, size_H, k)
    G = edge_builder(G, size_H, H)
    return H, G
```

```
def graph_set_echo_with_label(size,k,N,deg=2,complete = False):
        D = nx.complete_graph(size)
    H = nx.line_graph(D)
    label_D = [randint.rvs(0,N) for node in D.nodes()]
    label_H = [np.abs(label_D[e[0]]-label_D[e[1]]) == 0 for e in H.nodes()]
    size_H = len(list(H.nodes()))
    H = nx.relabel_nodes(H, dict(zip(H.nodes(),range(size_H))))
    G = nx.empty_graph(int(binom(size_H,k)))
    G = relabel_graph(G, size_H, k)
    G = edge_builder(G, size_H, H)
    return H, G, label_H, label_D

def hirarchical_absorbing_graph(G, H, label_H):
    G_hirach = G.copy()
    for edge in G.edges():
        v = set(list(edge[0]))
        w = set(list(edge[1]))
        vv = list(v-w)[0]
        ww = list(w-v)[0]
        if not ( (label_H[vv] != 0 and label_H[ww] == 0) \\
                              or (label_H[ww] != 0 and label_H[vv] == 0)):
            G_hirach.remove_edge(edge[0],edge[1])
    return G_hirach

def V_i_creator(G, label_H, k):
    def number_short_i(v, label_H):
        return sum([label_H[vv] for vv in v])

    V = [[v for v in G.nodes() if number_short_i(v, label_H) == i] \\
                    for i in range(k+1)]
    return V

def edges_i_ip1(G_hirach, V, i):
        def test(edge):
                a = ((edge[0] in V[i] and edge[1] in V[i+1])
                b = (edge[0] in V[i+1] and edge[1] in V[i]))
                return a or b

    nn = [edge for edge in G_hirach.edges() if test(edge)]
    return nn

def graph_set_from_graph(H,k):
    size_H = len(list(H.nodes()))
    H = nx.relabel_nodes(H, dict(zip(H.nodes(),range(size_H))))
    G = nx.empty_graph(int(binom(size_H,k)))
    G = relabel_graph(G, size_H, k)
    G = edge_builder(G, size_H, H)
    return H, G
```

```
def graph_set_with_complement(size,deg, k):
    # H = nx.complete_graph(size)
    # H = nx.random_regular_graph(deg, size)
    H = nx.star_graph(size)
    H_c = nx.complement(H)
    size_H = len(list(H.nodes()))
    H = nx.relabel_nodes(H, dict(zip(H.nodes(),range(size_H))))
    G = nx.empty_graph(int(binom(size_H,k)))
    G = relabel_graph(G, size_H, k)
    G = edge_builder(G, size_H, H)
    size_H_c = len(list(H_c.nodes()))
    H_c = nx.relabel_nodes(H_c, dict(zip(H_c.nodes(),range(size_H_c))))
    G_c = nx.empty_graph(int(binom(size_H_c,k)))
    G_c = relabel_graph(G_c, size_H_c, k)
    G_c = edge_builder(G_c, size_H_c, H_c)
    return H, G, H_c, G_c

def MC(G,H,v,w):
    vv = list(set(v)-set(w))
    H_sub = nx.subgraph(H,v)
    if v == w:
            x = sum([1/(H.degree(u)-int(nx.degree(H_sub, u))+1) for u in v])
        return 1/len(v)*x
    elif (v,w) in G.edges():
            y = 1/(H.degree(vv[0])-nx.degree(H_sub, vv)[vv[0]]+1)
        return 1/len(v)*y
    else:
        return 0

def MC_lazy(G,H,v,w):
    vv = list(set(v)-set(w))
    reg_deg = nx.degree(H,v[0])
    H_sub = nx.subgraph(H,v)
    x = sum([1/(reg_deg-int(nx.degree(H_sub, u))+1) for u in v])
    c_vv = 1-1/len(v)*x
    if v == w:
        return 0
    elif (v,w) in G.edges():
        return 1/len(v)*1/(reg_deg-nx.degree(H_sub, vv)[vv[0]]+1)/c_vv
    else:
        return 0

def MC_classic(G,H,v,w):
    reg_deg = nx.degree(H,v[0])
    if v == w:
        return (reg_deg*len(v)-G.degree(v))/(reg_deg*len(v))
    elif (v,w) in G.edges():
        return 1/(reg_deg*len(v))
    else:
        return 0
```

```python
def MC_holes(G,H,v,w):
    w_hole = list(set(w)-set(v))
    H_w_sub = nx.subgraph(H,w)
    if v == w:# what is the probability to stay in place?
        return MC(G,H,v,v)
    elif (v,w) in G.edges() or (w,v) in G.edges():
        return 1/len(v)*1/(nx.degree(H_w_sub, w_hole)[w_hole[0]]+2)
    else:
        return 0

def MC_abs(H,H_label,v):
    vv = choice(v)
    if H_label[vv] != 0:
        N = [uu for uu in H.neighbors(vv) if uu not in v]
        N.append(vv)
        ww = choice(N)
        w = [u for u in v if u != vv]
        w.append(ww)
        return tuple(w)
    else:
        return v

def quotient_with_similarity(v,w):
    v = set(v)
    w = set(w)
    if len(v.intersection(w)) == 0:
        return True
    else:
        return False


def pi_stationary(G,H,v):
    # reg_deg = nx.degree(H,v[0])
    H_sub = nx.subgraph(H,v)
    value = np.prod([(H.degree(vv)-H_sub.degree[vv]+1) for vv in v])
    return value


def psi_perturbation(G,H,v,w):
    if (v,w) in G.edges() or (w,v) in G.edges():
        deg = nx.degree(H,v[0])
        v_bar = list(set(v)-set(w))[0]
        w_bar = list(set(w)-set(v))[0]
        H_sub_v = nx.subgraph(H,v)
        H_sub_w = nx.subgraph(H,w)
        H_sub_v_neigh = set(H_sub_v.neighbors(v_bar))
        H_sub_w_neigh = set(H_sub_w.neighbors(w_bar))
        elements = list(H_sub_v_neigh-H_sub_w_neigh)
```

```
        def inv_degree(v):
                return deg-H_sub_v.degree[v] + 1
        factors = [(inv_degree(v)+1)/inv_degree(v) for v in elements]
        return np.prod(factors)
    else:
        return 0

def degree_sequence_subgraph(H,v):
    H_sub = nx.subgraph(H,v)
    return tuple(sorted(list(dict(nx.degree(H_sub)).values())))

def degree_sequence_subgraph_comp(H,v):
    v_comp = list(set(H.nodes())-set(v))
    H_sub = nx.subgraph(H,v_comp)
    return tuple(sorted(list(dict(nx.degree(H_sub)).values())))

def MC_cont_generator(G,H,v,w):
    # vv = list(set(v)-set(w))
    reg_deg = nx.degree(H,v[0])
    if v == w:
        return - G.degree(v)*1/reg_deg
    elif (v,w) in G.edges():
        return 1/reg_deg
    else:
        return 0

def isomorphism_subgraph_and_compl(H,v,w):
    v_c = list(set(H.nodes())-set(v))
    w_c = list(set(H.nodes())-set(w))
    H_sub_v = nx.subgraph(H,v)
    H_sub_w = nx.subgraph(H,w)
    H_sub_v_c = nx.subgraph(H,v_c)
    H_sub_w_c = nx.subgraph(H,w_c)
    deg_seq_v = sorted([H_sub_v.degree(vv) for vv in H_sub_v.nodes()])
    deg_seq_v_c = sorted([H_sub_v_c.degree(vv) for vv in H_sub_v_c.nodes()])
    deg_seq_w = sorted([H_sub_w.degree(vv) for vv in H_sub_w.nodes()])
    deg_seq_w_c = sorted([H_sub_w_c.degree(vv) for vv in H_sub_w_c.nodes()])
    con_comp_v_w = deg_seq_v  == deg_seq_w
    con_comp_v_c_w_c = deg_seq_v_c  == deg_seq_w_c
    return  con_comp_v_w and con_comp_v_c_w_c

def quot_wrt_stat_dist(nodes,stat,v,w):
    i_v = nodes.index(v)
    i_w = nodes.index(w)
    return stat[i_v] == stat[i_w]

def get_detailed_balance(tp_vw,tp_wv,nodes,stat,v,w):
    i_v = nodes.index(v)
    i_w = nodes.index(w)
    print(stat[i_w]*tp_wv)
```

```
    print(stat[i_v]*tp_vw)
    return np.round(stat[i_v]*tp_vw,8) == np.round(stat[i_w]*tp_wv,8)

def connectivity_stat_dist(H,v,w):
    if nx.node_connectivity(H) > len(v):
        H_sub_v = nx.subgraph(H,v)
        H_sub_w = nx.subgraph(H,w)
        return nx.is_isomorphic(H_sub_v, H_sub_w)
    else:
        return False

def create_bip_graph(H,v):
    v_c = list(set(H.nodes())-set(v))
    H_sub_v = nx.subgraph(H,v)
    H_sub_v_c = nx.subgraph(H,v_c)
    H_v_v_c = nx.difference(H,nx.compose(H_sub_v,H_sub_v_c))
    return H_v_v_c

def bip_stat_test(H,nodes,v,w):
    i_v = nodes.index(v)
    i_w = nodes.index(w)
    H_v_v_c = create_bip_graph(H,v)
    H_w_w_c = create_bip_graph(H,w)
    return nx.is_isomorphic(H_v_v_c, H_w_w_c)
```

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