# THÈSE

**En vue de l'obtention du**

## DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

**Délivré par l'Université Toulouse 3 - Paul Sabatier**


**Présentée et soutenue par**

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Le 19 juillet 2022

# Gamma-convergence et singularités vortex au bord dans des films ferromagnétiques minces avec interaction de Dzyaloshinskii-Moriya.

Ecole doctorale : **EDMITT - Ecole Doctorale Mathématiques, Informatique et Télécommunications de Toulouse**

Spécialité : **Mathématiques et Applications**

Unité de recherche :
**IMT : Institut de Mathématiques de Toulouse**

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## Remerciements

Je tiens d'abord à remercier mon directeur de thèse, Radu Ignat. Merci Radu pour avoir encadré cette thèse, merci pour vos conseils et vos remarques, merci pour votre aide et votre bienveillance, notamment dans la relecture méticuleuse de mes écrits, et enfin merci pour votre bonhomie et votre humour que j'ai beaucoup appréciés pendant ces quatre années. Les années que j'ai passées à Toulouse et ses alentours ont été exceptionnelles pour moi à plusieurs titres ; je suis certain que j'en garderai des souvenirs toute ma vie.

Je remercie Lia Bronsard et Benoît Merlet pour avoir accepté de rapporter ma thèse. J'en suis très honoré et je leur en suis très reconnaissant. Merci également à François Alouges, Pierre Bousquet et Mihai Mariş pour avoir accepté de faire partie de mon jury de thèse.

Je tiens à saluer le personnel de l'IMT que j'ai pu côtoyer pendant ma thèse, aussi bien le personnel administratif que les mathématiciens et les mathématiciennes déjà experts, ou en passe de le devenir grâce à leur thèse. Je remercie particulièrement les trois docteurs avec qui j'ai partagé un bureau international : Hoang Phuong, Kuntal et Joe. Merci à vous trois pour la si bonne – et si studieuse – ambiance qui a régné dans ce bureau, pour les quelques pauses quotidiennes, pour la magie de Noël de début décembre à la mi-mars, pour les cours particuliers de langues, pour les statistiques pré-covid, et j'en oublie certainement ; simplement merci pour votre constante bonne humeur. Je remercie également les deux anciens Rennais avec qui je suis arrivé à Toulouse en master. Merci à Maxime, que j'ai mieux appris à connaître notamment grâce à de nombreuses sorties cinématographiques. Merci Valentin pour la NBA prédictive, pour la NBA amatrice, pour la NBA professionnelle, mais aussi pour les très sérieuses formations doctorales, pour la plus sérieuse théorie du potentiel, et tant d'autres choses.

J'ai une chaleureuse pensée pour mes collègues du lycée Charles de Gaulle de Poissy, qui ont contribué à permettre que ma dernière année de thèse se déroule parallèlement à ma première année d'enseignement dans le secondaire.

Je termine par remercier ma famille et mes amis, à qui je peux enfin dire fièrement que ça y est, je soutiens ma thèse. Merci à ma mère et à ma soeur, qui ont toujours été d'un soutien essentiel.

**Résumé**

Cette thèse porte sur l'analyse asymptotique d'un modèle variationnel pour les films ferromagnétiques minces. On étudie l'énergie micromagnétique, définie pour des applications d'un ouvert de $\mathbb{R}^3$ à valeurs dans la sphère $\mathbb{S}^2$, appelées aimantations, en prenant en compte l'effet antisymétrique dû à l'interaction de Dzyaloshinskii-Moriya.

Dans le premier chapitre de la thèse, on étudie la Gamma-convergence de l'énergie micromagnétique dans un régime de film mince qui favorise l'émergence de vortex au bord de taille $\varepsilon > 0$, via une pénalisation au bord obtenue dans la Gamma-limite de l'énergie. Cette limite est en fait définie pour des aimantations invariantes dans l'épaisseur du film et à valeurs dans le cercle unité $\mathbb{S}^1$, ce qui signifie que le modèle général en trois dimensions se réduit à un modèle en deux dimensions. On cherche ensuite les minimiseurs locaux de l'énergie Gamma-limite dans le demi-plan supérieur. Pour cela, on étudie d'abord ses points critiques, qui satisfont un problème de Neumann non linéaire similaire au problème de Peierls-Nabarro. On en déduit, sous certaines conditions, l'unicité des minimiseurs locaux de l'énergie au sens de De Giorgi. Ceux-ci correspondent à des applications qui tendent à présenter un vortex de taille $\varepsilon > 0$ sur le bord du domaine.

Dans le second chapitre de la thèse, on considère un autre régime de film mince favorisant les singularités vortex au bord. Grosso modo, ce régime consiste dans l'analyse du modèle précédent lorsque $\varepsilon \to 0$. On montre que l'étude du modèle général en trois dimensions se ramène ici aussi à l'étude d'un modèle intermédiaire en deux dimensions, mais dans lequel figurent à la fois une pénalisation intérieure (les applications ne sont plus à valeurs dans $\mathbb{S}^1$) et une pénalisation au bord du domaine. D'abord, on étudie ce modèle intermédiaire en deux dimensions. Grâce à la notion de Jacobien global, on obtient des résultats de compacité et un développement asymptotique au second ordre par Gamma-convergence de l'énergie micromagnétique. Le terme d'ordre un montre que les singularités vortex sont localisées au bord du domaine, tandis que le terme d'ordre deux est une énergie renormalisée, semblable à celle rencontrée dans le modèle de Ginzburg-Landau, qui permet d'estimer le coût d'interaction entre les vortex au bord. On calcule explicitement cette énergie renormalisée, influencée par l'interaction de Dzyaloshinskii-Moriya, et on étudie la structure des minimiseurs. Enfin, grâce à ces résultats, nous déduisons la Gamma-convergence à l'ordre deux dans le modèle en trois dimensions.

**Mots-clés :** Micromagnétisme, Gamma-convergence, compacité, vortex au bord, Jacobien, énergie renormalisée, interaction de Dzyaloshinskii-Moriya, théorie de Ginzburg-Landau.

**Abstract**

This thesis deals with the asymptotic analysis of a variational model for thin ferromagnetic films. We study the micromagnetic energy for three-dimensional maps with values into the unit sphere $\mathbb{S}^2$, called magnetizations, by taking into account the antisymmetric effect of the Dzyaloshinskii-Moriya interaction.

In the first chapter, we study the Gamma-convergence of the micromagnetic energy in a thin-film regime that favors boundary vortices of size $\varepsilon > 0$, via a boundary penalization in the Gamma-limit energy. This limit is in fact defined for magnetizations that are invariant in the thickness of the film and take values into the unit circle $\mathbb{S}^1$. It means that the general three-dimensional model reduces to a two-dimensional model. We then focus on local minimizers in the upper-half plane of the Gamma-limit energy. To do so, we begin with studying its critical points, that satisfy a nonlinear Neumann boundary value problem, similar to the Peierls-Nabarro problem. We deduce, under certain conditions, the uniqueness of the local minimizers of the energy in the sense of De Giorgi. These minimizers correspond to maps having a vortex of size $\varepsilon > 0$ on the boundary of the domain.

In the second chapter, we consider another thin-film regime for boundary vortices. Roughly speaking, this regime corresponds to the limit $\varepsilon \to 0$ in the previous model. The study of the three-dimensional general model reduces here to an intermediate two-dimensional model for $\mathbb{R}^2$-valued maps (not $\mathbb{S}^1$-valued maps), that combines an interior penalization and a boundary penalization on the domain. First, we study this two-dimensional intermediate model. Using the notion of global Jacobian, we prove compactness results and an asymptotic expansion at the second order by Gamma-convergence for the micromagnetic energy. The first order term indicates that the singularities are located on the boundary of the domain, while the second order term is a renormalized energy, similar to the Ginzburg-Landau model, that measures the interactions between the boundary vortices. We compute explicitely this renormalized energy, that depends on the Dzyaloshinskii-Moriya interaction, and study the structure of its minimizers. Finally, thanks to these results, we deduce the Gamma-convergence at the second order for the general three-dimensional model.



**Keywords:** Micromagnetics, Gamma-convergence, compactness, boundary vortices, Jacobian, renormalized energy, Dzyaloshinskii-Moriya interaction, Ginzburg-Landau theory.

# Contents

# Introduction

La modélisation des matériaux ferromagnétiques, utilisés par exemple pour le stockage d'information numérique, est basée sur la théorie du micromagnétisme. Ce modèle suppose qu'un matériau ferromagnétique peut être caractérisé par un champ de vecteurs en trois dimensions, appelé aimantation, à valeurs dans la sphère unité $\mathbb{S}^2$, et dont les états d'équilibres stables correspondent aux minimiseurs locaux d'une énergie appelée énergie micromagnétique (voir [20]). Le problème variationnel associé est non convexe, non local et multi-échelle. En particulier, les relations entre les paramètres de taille du matériau ferromagnétique (aussi bien les paramètres intrinsèques du matériau que les paramètres géométriques) permettent de considérer plusieurs régimes asymptotiques différents. Selon le régime asymptotique, l'un ou l'autre des termes composant l'énergie micromagnétique peut être prépondérant sur les autres. Cela conduit à la formation de singularités magnétiques pour l'aimantation, comme des parois de Néel ou de Bloch entre plusieurs domaines, des vortex à l'intérieur ou au bord d'un domaine, etc. Dans cette thèse, on s'intéresse à des régimes asymptotiques conduisant à la formation de vortex de bord en prenant en compte l'effet antisymétrique de l'interaction de Dzyaloshinskii-Moriya.

## 0.1 Le modèle général en trois dimensions

On considère un matériau ferromagnétique de forme cylindrique

$$\Omega_t^\ell = \Omega^\ell \times (0,t) \subset \mathbb{R}^3,$$

où $\Omega^\ell \subset \mathbb{R}^2$ est un ouvert borné régulier de diamètre $\ell$ (par exemple, $\Omega^\ell$ peut être assimilé à un disque ouvert de diamètre $\ell$). L'aimantation $m$ est le champ de vecteurs

$$m \colon \Omega_t^\ell \to \mathbb{S}^2,$$

où $\mathbb{S}^2$ désigne la sphère unité de $\mathbb{R}^3$. En particulier, la contrainte $|m| = 1$ impose la non-convexité du problème. L'énergie micromagnétique est classiquement définie par

$$E_{\text{class}}(m) = A^2 \int_{\Omega_t^\ell} |\nabla m|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + Q \int_{\Omega_t^\ell} \Phi(m) \,\mathrm{d}x - 2 \int_{\Omega_t^\ell} H_{\text{ext}} \cdot m \,\mathrm{d}x.$$

Examinons les quatre termes apparaissant dans l'expression de $E_{\text{class}}$.

– Le premier terme est l'énergie d'échange, générée par les interactions à faible distance dans le matériau. Cette énergie favorise l'alignement des spins voisins dans le matériau. Le paramètre $A > 0$ est intrinsèque au matériau ferromagnétique, généralement de l'ordre du nanomètre. Ce paramètre est appelé longueur d'échange.

– Le deuxième terme est appelé énergie magnétostatique ou énergie démagnétisante. Cette énergie est générée par les interactions de longue distance dans le matériau. Elle est en fait due au champ magnétique induit par l'aimantation. Plus précisément, le potentiel démagnétisant $u \in H^1(\mathbb{R}^3, \mathbb{R})$ satisfait

$$\Delta u = \operatorname{div}(m \mathbb{1}_{\Omega_t^\ell}) \text{ au sens des distributions dans } \mathbb{R}^3,$$

où $\mathbb{1}_{\Omega_t^\ell}(x) = 1$ si $x \in \Omega_t^\ell$, et $\mathbb{1}_{\Omega_t^\ell}(x) = 0$ sinon.

– Le troisième terme est l'énergie d'anisotropie, qui prend en compte les effets d'anisotropie résultant de la structure cristalline du matériau. Elle fait intervenir le facteur de qualité $Q > 0$ (qui est un deuxième paramètre intrinsèque du matériau) et la fonction $\Phi \colon \mathbb{S}^2 \to \mathbb{R}_+$ qui présente des propriétés de symétrie.

– Le quatrième et dernier terme est l'énergie de Zeeman : elle est générée par un champ magnétique extérieur, le champ de vecteurs

$$H_{\text{ext}} \colon \mathbb{R}^3 \to \mathbb{R}^3$$

qui favorise l'alignement de l'aimantation dans la même direction que ce champ magnétique extérieur.

Dans cette thèse, on considère un terme supplémentaire : l'interaction de Dzyaloshinskii-Moriya. Cette interaction a été introduite dans les années 50 (voir [17]) pour décrire l'aimantation dans des matériaux présentant de faibles propriétés de symétrie. On suppose dans cette thèse que la densité d'interaction de Dzyaloshinskii-Moriya en trois dimensions est définie par

$$D : \nabla m \wedge m = \sum_{j=1}^{3} D_j \cdot \partial_j m \wedge m, \tag{1}$$

où $D = (D_1, D_2, D_3) \in \mathbb{R}^{3\times 3}$, $\cdot$ est le produit scalaire dans $\mathbb{R}^3$, et $\wedge$ est le produit vectoriel dans $\mathbb{R}^3$. On considère finalement l'énergie micromagnétique $E(m)$ définie par

$$\begin{aligned} E(m) = A^2 \int_{\Omega_t^\ell} |\nabla m|^2 \,\mathrm{d}x + \int_{\Omega_t^\ell} D : \nabla m \wedge m \,\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x \\ + Q \int_{\Omega_t^\ell} \Phi(m) \,\mathrm{d}x - 2 \int_{\Omega_t^\ell} H_{\text{ext}} \cdot m \,\mathrm{d}x. \end{aligned} \tag{2}$$

Pour des détails supplémentaires sur les termes composant l'énergie micromagnétique, notamment à propos des interprétations physiques, on renvoie à [1], [16], [20] ou [17].

Le caractère multi-échelle de l'énergie micromagnétique (2) est simple à remarquer. En effet, en plus du tenseur $D$ et du facteur de qualité $Q$, trois paramètres de longueur liés au matériau ferromagnétique interviennent : la longueur d'échange $A$, le diamètre planaire $\ell$ et l'épaisseur $t$ du matériau. À partir de ces trois paramètres, on peut se réduire à deux paramètres sans dimension :

$$h = \frac{t}{\ell} \quad \text{et} \quad \eta = \frac{A}{\ell}.$$

Lorsqu'on fait tendre $h$ vers zéro, l'épaisseur relative du matériau ferromagnétique tend vers zéro : on parle de film mince à la limite. On peut en fait parler de film mince dès que l'épaisseur $t$ du matériau est *beaucoup plus petite* que le diamètre planaire $\ell$, et c'est justement le sens donné à « $t$ beaucoup plus petite que $\ell$ » qui donne accès à une large variété de régimes asymptotiques. Plus précisément, les différentes relations possibles entre $h$ et $\eta$ dans l'asymptotique $h \to 0$ peuvent avoir des effets divers sur l'aimantation et l'énergie micromagnétique.

## 0.2 Un régime de film mince

### 0.2.1 Adimensionnement en longueur

Afin d'étudier l'énergie micromagnétique dans un film mince, il est pertinent de faire un adimensionnement en longueur. En particulier, on réduit les trois paramètres de longueurs $A$, $\ell$ et $t$, à seulement deux paramètres sans dimension $h$ et $\eta$, définis ci-dessus. On pose

$$\Omega_h = \frac{\Omega_t^\ell}{\ell} = \Omega \times (0,h) \subset \mathbb{R}^3,$$

où $\Omega = \frac{1}{\ell}\Omega^\ell \subset \mathbb{R}^2$ est un ouvert borné régulier de diamètre 1 (par exemple, $\Omega$ peut être assimilé au disque unité de $\mathbb{R}^2$). Pour tout $x = (x_1, x_2, x_3) \in \Omega_t^\ell$, on pose $\widehat{x} = \frac{1}{\ell}x \in \Omega_h$ et $\widehat{D} = \frac{1}{\ell}D$. On considère les applications $m_h \colon \Omega_h \to \mathbb{S}^2$, $u_h \colon \mathbb{R}^3 \to \mathbb{R}$ et $H_{\mathrm{ext},h} \colon \mathbb{R}^3 \to \mathbb{R}^3$ telles que, pour tout $\widehat{x} = \frac{x}{\ell} \in \Omega_h$,

$$m_h(\widehat{x}) = m(x), \quad u_h(\widehat{x}) = \frac{1}{\ell}u(x),$$

avec

$$\Delta u_h = \operatorname{div}(m_h \mathbb{1}_{\Omega_h}) \text{ au sens des distributions dans } \mathbb{R}^3, \tag{3}$$

et

$$H_{\mathrm{ext},h}(\widehat{x}) = H_{\mathrm{ext}}(x).$$

L'énergie micromagnétique (2) peut se réécrire à l'aide de $m_h$ :

$$\begin{aligned} \widehat{E}(m_h) = \ell^3 \bigg[ \eta^2 \int_{\Omega_h} |\nabla m_h|^2 \mathrm{d}\widehat{x} + \int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \; \mathrm{d}\widehat{x} + \int_{\mathbb{R}^3} |\nabla u_h|^2 \mathrm{d}\widehat{x} \\ + Q \int_{\Omega_h} \Phi(m_h) \; \mathrm{d}\widehat{x} - 2 \int_{\Omega_h} H_{\mathrm{ext},h} \cdot m_h \; \mathrm{d}\widehat{x} \bigg]. \end{aligned} \tag{4}$$

Pour alléger les notations, on écrira $x$ au lieu de $\widehat{x}$ dans la suite.

### 0.2.2 Approche heuristique d'un régime de film mince

Le modèle de film mince est caractérisé par l'hypothèse $h = t/\ell \to 0$, c'est-à-dire que les variations de l'aimantation par rapport à la composante verticale $x_3$ sont fortement pénalisées. On suppose, pour un moment, que $m_h$ ne dépend pas de $x_3$, c'est-à-dire

$$m_h(x_1, x_2, x_3) = m_h(x_1, x_2) \colon \Omega \to \mathbb{S}^2, \tag{5}$$

et que

$$m_h \text{ varie à des échelles de longueurs } \gg h. \tag{6}$$

On suppose de plus que le champ magnétique extérieur $H_{\mathrm{ext},h}$ est planaire, invariant par rapport à $x_3$ et indépendant de $h$, c'est-à-dire

$$H_{\mathrm{ext},h}(x_1, x_2, x_3) = (H'_{\mathrm{ext},h}(x_1, x_2), 0). \tag{7}$$

L'équation de Maxwell (3) implique

$$\Delta u_h = \operatorname{div}(m_h \mathbb{1}_{\Omega_h}) = \operatorname{div}(m_h)\mathbb{1}_{\Omega_h} - (m_h \cdot \nu)\mathbb{1}_{\partial\Omega_h} = \operatorname{div}'(m'_h)\mathbb{1}_{\Omega_h} - (m_h \cdot \nu)\mathbb{1}_{\partial\Omega_h}$$

au sens des distributions dans $\mathbb{R}^3$, où $m'_h = (m_{h,1}, m_{h,2})$, $\operatorname{div}'(m'_h) = \partial_1 m_{h,1} + \partial_2 m_{h,2}$ et $\nu$ désigne le vecteur normal unitaire sortant sur $\partial\Omega_h$. En d'autres termes, $u_h$ est solution du problème

$$\begin{cases} \Delta u_h = \operatorname{div}'(m'_h) & \text{dans } \Omega_h, \\ \Delta u_h = 0 & \text{dans } \mathbb{R}^3 \setminus \Omega_h, \\ \left[\frac{\partial u_h}{\partial \nu}\right] = m_h \cdot \nu & \text{sur } \partial\Omega_h, \end{cases}$$

où $[a] = a^+ - a^-$ est le saut de $a$ par rapport au vecteur normal unitaire sortant $\nu$ sur $\partial\Omega_h$. D'après [21, Section 2.1.2] (voir aussi [16]), en supposant que $u_h \in H^1(\mathbb{R}^3)$, on peut exprimer l'énergie démagnétisante en considérant la transformée de Fourier selon les variables planaires :

$$\int_{\mathbb{R}^3} |\nabla u_h|^2 \,\mathrm{d}x = h \int_{\mathbb{R}^2} \frac{|\xi' \cdot \mathcal{F}(m_h' \mathbb{1}_\Omega)(\xi')|^2}{|\xi'|^2} (1 - g_h(|\xi'|)) \,\mathrm{d}\xi' + h \int_{\mathbb{R}^2} |\mathcal{F}(m_{h,3} \mathbb{1}_\Omega)(\xi')|^2 g_h(|\xi'|) \,\mathrm{d}\xi',$$

où $\mathcal{F}$ désigne la transformée de Fourier dans $\mathbb{R}^2$, c'est-à-dire pour tous $f \colon \mathbb{R}^2 \to \mathbb{C}$ et $\xi' \in \mathbb{R}^2$,

$$\mathcal{F}(f)(\xi') = \hat{f}(\xi') = \int_{\mathbb{R}^2} f(x') e^{-2i\pi x' \cdot \xi'} \mathrm{d}x',$$

et où

$$g_h(|\xi'|) = \frac{1 - e^{-2\pi h |\xi'|}}{2\pi h |\xi'|}. \tag{8}$$

À la limite quand $h \to 0$, on a $g_h(|\xi'|) \to 1$ et $1 - g_h(|\xi'|) \to 0$. On peut ainsi faire l'approximation

$$\int_{\mathbb{R}^3} |\nabla u_h|^2 \,\mathrm{d}x \approx h^2 \int_{\mathbb{R}^2} \frac{\pi |\xi' \cdot \mathcal{F}(m_h' \mathbb{1}_\Omega)(\xi')|^2}{|\xi'|} \,\mathrm{d}\xi' + h \int_\Omega m_{h,3}^2 \,\mathrm{d}x'.$$

Une approche plus précise (voir [16], [32]) est la suivante :

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u_h|^2 \,\mathrm{d}x \approx{}& h^2 \left\| \mathrm{div}'(m_h')_{cont} \right\|^2_{\dot{H}^{-1/2}(\mathbb{R}^2)} \\ &+ \frac{1}{2\pi} h^2 |\log h| \int_{\partial\Omega} (m_h' \cdot \nu')^2 \,\mathrm{d}\mathcal{H}^1 + h \int_\Omega m_{h,3}^2 \,\mathrm{d}x', \end{aligned} \tag{9}$$

où $\mathrm{div}'(m_h')_{cont} = \mathrm{div}'(m_h')\mathbb{1}_\Omega$ et $\nu'$ est le vecteur normal unitaire sortant sur $\partial\Omega$. L'énergie démagnétisante se décompose ainsi asymptotiquement en trois termes dans l'approximation de film mince. Le premier terme pénalise les charges volumiques avec une semi-norme homogène $\dot{H}^{-1/2}$, et favorise les parois de Néel. Le deuxième terme prend en compte les charges latérales du matériau cylindrique et favorise les vortex au bord. Le troisième terme pénalise les charges surfaciques au niveau des faces inférieure et supérieure du cylindre, et favorise les vortex intérieurs. Pour plus de détails sur les différents types de singularités qui peuvent apparaître dans les régimes de films minces, on renvoie à [16] ou [21]. En combinant (9) avec les hypothèses (5), (6) et (7), on obtient l'approximation suivante par un modèle réduit bidimensionnel de l'énergie dans un régime de film mince :

$$\begin{aligned} \widehat{E}(m_h) \approx \ell^3 \Bigg[ & h\eta^2 \int_\Omega |\nabla' m_h|^2 \mathrm{d}x' + h \int_\Omega \widehat{D}' : \nabla' m_h \wedge m_h \,\mathrm{d}x' \\ &+ \frac{h^2}{2} \left\| \mathrm{div}'(m_h')_{cont} \right\|^2_{\dot{H}^{-1/2}(\mathbb{R}^2)} \\ &+ \frac{1}{2\pi} h^2 |\log h| \int_{\partial\Omega} (m_h' \cdot \nu')^2 \mathrm{d}\mathcal{H}^1 \\ &+ h \int_\Omega \left( m_{h,3}^2 + Q\Phi(m_h) - 2H'_{\mathrm{ext},h} \cdot m_h' \right) \mathrm{d}x' \Bigg], \end{aligned} \tag{10}$$

avec $\widehat{D}' = (\widehat{D}_1, \widehat{D}_2)$ et $\widehat{D}' : \nabla' m_h \wedge m_h = \sum_{j=1}^2 \widehat{D}_j \cdot \partial_j m_h \wedge m_h$.

### 0.2.3 Un panorama de régimes de films minces

L'expression (10) a pour intérêt de mettre en évidence la diversité des régimes de films minces. Par *régime de film mince*, on entend une relation asymptotique entre $h$ et $\eta$ lorsque $h \to 0$. La plupart des résultats que nous citons dans cette partie ont été obtenus en l'absence d'anisotropie, de champ magnétique extérieur et d'interaction de Dzyaloshinskii-Moriya dans l'énergie micromagnétique.

Un premier régime de film mince qu'il est naturel de considérer est celui où $\eta$ reste constant. Ce régime a été étudié (en l'absence d'anisotropie, de champ magnétique extérieur et d'interaction de Dzyaloshinskii-Moriya) par Gioia et James, qui ont prouvé que dans ce cas, l'énergie micromagnétique est minimisée par toute aimantation $m$ constante et telle que $m_3 = 0$.

On peut ensuite considérer les films minces pour lesquels $\eta \ll 1$ : ce sont des matériaux ferromagnétiques « larges » dans le sens où le diamètre planaire $\ell$ est grand devant la longueur d'échange $A$ du matériau.

Le cas où $\eta^2 \gg h\,|\log h|$ (c'est-à-dire $\frac{\eta^2}{h|\log h|} \to +\infty$) a été étudié par Kohn-Slastikov [32], à la suite de travaux antérieurs dûs à Carbou [11]. Quitte à diviser $\widehat{E}(m_h)$ par $\ell^3 h^2\,|\log h|$ dans (10), en l'absence d'anisotropie, de champ magnétique extérieur et d'interaction de Dzyaloshinskii-Moriya, on observe qu'à la limite quand $h \to 0$, seul le terme de pénalisation au bord $\frac{1}{2\pi}\int_{\partial\Omega}(m'\cdot\nu')^2 \mathrm{d}\mathcal{H}^1$ persiste. L'aimantation limite est donc une aimantation planaire et constante dont la direction est donnée par le vecteur $\nu'$. Dans [32], Kohn et Slastikov ont prouvé ce résultat par Gamma-convergence dans le régime $\eta^2 \gg h\,|\log h|$.

Ils s'intéressent également, dans le même article, au régime $\frac{\eta^2}{h|\log h|} \to \alpha > 0$. La différence avec le régime précédent, visible dans (10), est que le terme d'échange dans l'énergie micromagnétique n'est plus négligeable à la limite, et est en compétition avec le terme de pénalisation au bord dû à l'énergie démagnétisante. Autrement dit, à la limite, l'énergie micromagnétique devient $\alpha\int_\Omega |\nabla' m|^2\,\mathrm{d}x' + \frac{1}{2\pi}\int_{\partial\Omega}(m'\cdot\nu')^2\,\mathrm{d}\mathcal{H}^1$. L'aimantation limite dans ce régime n'est plus nécessairement constante, mais reste planaire. Dans le premier chapitre de cette thèse, on s'intéresse à ce régime de film mince, en tenant compte des termes d'anisotropie, de champ magnétique extérieur et d'interaction de Dzyaloshinskii-Moriya dans l'expression de l'énergie micromagnétique. On prouve un résultat de compacité et de Gamma-convergence analogue à celui de Kohn-Slastikov.

Après avoir divisé l'énergie limite ci-dessus par $\alpha$, le cas où $\alpha \to 0$ a été étudié par Kurzke [33]. Pour des raisons topologiques, il n'existe pas de champ de vecteurs $m'\colon \Omega \to \mathbb{R}^2$ vérifiant à la fois $|m'| = 1$ et $m'\cdot\nu' = 0$ sur $\partial\Omega$, où $\Omega$ est un ouvert simplement connexe de $\mathbb{R}^2$. Le terme de pénalisation au bord ne pouvant donc pas être rendu nul, l'aimantation limite développe des vortex au bord, c'est-à-dire des points où l'aimantation est nulle. Kurzke [33] a prouvé qu'au voisinage de ces vortex, un saut de phase de $\pi$ se produit pour l'aimantation limite.

Récemment, Ignat et Kurzke [24], [25] ont unifié le double passage à la limite obtenu par combinaison des régimes de Kohn-Slastikov et Kurzke, en considérant le régime $h \ll \eta^2 \ll h\,|\log h|$. Ils ont obtenu un développement asymptotique de l'énergie micromagnétique au second ordre par Gamma-convergence, dans lequel le terme d'ordre 1 est lié au nombre de vortex au bord et le terme d'ordre 2 permet de mesurer l'interaction entre les vortex. Dans le second chapitre de cette thèse, on étudie l'énergie micromagnétique dans le régime considéré par Ignat et Kurzke en tenant compte de l'interaction de Dzyaloshinskii-Moriya. On obtient également un développement asymptotique au second ordre par Gamma-convergence, la nouveauté étant l'influence de l'interaction de Dzyaloshinskii-Moriya dans le terme d'ordre 2, qui montre que cette interaction joue un rôle dans la répartition des vortex au bord du domaine.

De nombreux autres régimes de films minces ont été étudiés. Dans le régime $\eta^2 = O(h)$, Moser [38], [39] a mis en évidence l'émergence de vortex de bord. Dans les régimes pour lesquels $\eta^2 \ll h$, étudiés par Ignat-Otto [30] ou Ignat-Knüpfer [23], d'autres types de singularités topologiques apparaissent, comme des vortex intérieurs au domaine ou des parois de Néel. Enfin, dans le régime $\eta^2 \ll \frac{h}{|\log h|}$, DeSimone, Kohn, Moser et Otto [15] ont montré que l'énergie d'échange disparaît à la limite, alors que le terme $\|\mathrm{div}'(m'_h)_{cont}\|^2_{\dot{H}^{-1/2}(\mathbb{R}^2)}$ provenant de l'énergie démagnétisante persiste. Les singularités apparaissant dans ce modèle sont des parois de Néel et des lignes de Bloch.

Dans cette thèse, on considère des régimes de films minces qui favorisent les singularités vortex au bord, tout en prenant en compte l'interaction de Dzyaloshinskii-Moriya. Les hypothèses que nous faisons sont basées sur le régime étudié par Kohn-Slastikov [32] pour le premier chapitre, et sur le régime étudié par Ignat-Kurzke [24], [25] pour le second chapitre. La nouveauté dans notre étude est que l'on prend en compte l'interaction de Dzyaloshinskii-Moriya. On peut mentionner les récents travaux de Davoli, Di Fratta, Praetorius et Ruggeri [14] pour déterminer une limite de film mince de l'énergie micromagnétique prenant en compte l'interaction de Dzyaloshinskii-Moriya, qui sont similaires à ce qui est étudié dans cette thèse. Récemment, Alama, Bronsard et Golovaty [3] ont étudié un nouveau type de vortex au bord, appelés boojums, qui apparaissent dans un modèle de film mince pour des cristaux liquides nématiques. Ce type de vortex au bord coûte moins d'énergie que ceux étudiés dans cette thèse, si bien que les vortex au bord « classiques » rencontrés ici sont absents dans leur modèle. Pour les singularités de type parois de Néel, on renvoie à Ignat [22], Ignat-Moser [28], [29], Ignat-Otto [30], [31] et Melcher [36], [37].

## 0.3 Notations

On utilise la même notation $\cdot$ pour le produit scalaire à la fois dans $\mathbb{R}^3$ et dans $\mathbb{R}^2$. On utilise la même notation $\wedge$ pour le produit vectoriel dans $\mathbb{R}^3$, c'est-à-dire

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix},$$

et pour le déterminant dans $\mathbb{R}^2$, c'est-à-dire

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1b_2 - a_2b_1.$$

Lorsque cela est judicieux, on utilise l'identification $\mathbb{R}^2 \simeq \mathbb{C}$. On note $\mathfrak{Re}(\cdot)$ et $\mathfrak{Im}(\cdot)$ les parties réelle et imaginaire d'un nombre complexe.
Remarquons que, pour $a = (a_1, a_2) = a_1 + ia_2$ et $b = (b_1, b_2) = b_1 + ib_2$, on a $a \wedge b = \mathfrak{Im}(\overline{a}b)$, où $\overline{a} = (a_1, -a_2) = a_1 - ia_2$.
On note

$$\mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty), \quad \text{et} \quad \overline{\mathbb{R}^2_+} = \mathbb{R} \times [0, +\infty).$$

Pour $r > 0$, on note $B_r$ le disque

$$B_r = B(0, r) = \left\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 < r\right\},$$

et $B_r^+$ le demi-disque supérieur

$$B_r^+ = \left\{(x_1, x_2) \in \mathbb{R}^2_+ \colon x_1^2 + x_2^2 < r\right\}.$$

Étant donné un ensemble $E$, on utilise la notation $\mathbb{1}_E$ pour la fonction caractéristique de $E$, c'est-à-dire : $\mathbb{1}_E(x) = 1$ si $x \in E$, $\mathbb{1}_E(x) = 0$ sinon.
Étant donnée une quantité en trois dimensions $q = (q_1, q_2, q_3)$, on utilise la notation prime $'$ pour désigner la composante planaire de $q$, c'est-à-dire $q' = (q_1, q_2)$ et $q = (q', q_3)$. Lorsque la situation étudiée est en deux dimensions et qu'aucune confusion n'est possible, par exemple dans la Section 1.3 et dans la Section 2.2, on ôte les primes de toutes les notations.

Comme déjà défini dans (1) pour la densité d'interaction de Dzyaloshinskii-Moriya, on note

$$F : \nabla f \wedge f = \sum_{j=1}^{3} F_j \cdot \partial_j f \wedge f$$

pour $F \in \mathbb{R}^{3\times 3}$ et $f\colon \mathbb{R}^3 \to \mathbb{R}^3$. On note également, pour $G \in \mathbb{R}^{3\times 2}$ et $f\colon \mathbb{R}^3 \to \mathbb{R}^3$,

$$G : \nabla' f \wedge f = \sum_{j=1}^{2} G_j \cdot \partial_j f \wedge f.$$

La plupart du temps, les constantes (souvent notées $C$) peuvent changer d'une ligne à l'autre dans les calculs. On note $a \ll b$ ou $a = o(b)$ lorsque $\frac{a}{b} \to 0$, $a \lesssim b$ ou $a = O(b)$ s'il existe une constante $C > 0$ telle que $a \leqslant Cb$, et $a \sim b$ lorsque $a \lesssim b$ et $b \lesssim a$. Enfin, les formulations « une famille/suite $(m_h)$ ... quand $h \to 0$ » doivent être comprises comme « une famille/suite $(m_{h_n})_{n\in\mathbb{N}}$ ... avec $h_n \to 0$ quand $n \to +\infty$ ».

## 0.4 Résultats principaux et structure de la thèse

Cette thèse comporte deux chapitres, portant chacun sur l'étude de l'énergie micromagnétique (4) dans un régime de film mince. Dans le premier chapitre, on se place dans un régime pour lequel $\frac{\eta^2}{h|\log h|} \to \alpha > 0$ (déjà considéré par Kohn-Slastikov [32]), tandis que dans le second chapitre, on considère le régime $h \ll \eta^2 \ll h\,|\log h|$ et le régime plus restrictif $h \log|\log h| \ll \eta^2 \ll h\,|\log h|$ (étudiés par Ignat-Kurzke [24], [25]). La principale nouveauté apportée dans cette thèse est la prise en compte de l'interaction de Dzyaloshinskii-Moriya dans l'énergie micromagnétique, dans des régimes où cette interaction a une influence lors du passage à la limite quand $h \to 0$.

Dans les deux chapitres de cette thèse, on établit entre autres un résultat de Gamma-convergence pour l'énergie micromagnétique. Rappelons la définition de la Gamma-convergence et son principal intérêt.

**Definition 0.4.1.** Soit $(X, \mathcal{T})$ un espace topologique à base dénombrable de voisinages. On considère une famille de fonctionnelles dépendant d'un paramètre $h > 0$, notée $\mathcal{E}_h\colon X \to \mathbb{R}\cup\{+\infty\}$, et une fonctionnelle $\mathcal{E}_0\colon X \to \mathbb{R}\cup\{+\infty\}$. On dit alors que $(\mathcal{E}_h)_{h>0}$ $\Gamma$-converge vers $\mathcal{E}_0$ pour la topologie $\mathcal{T}$ lorsque $h \to 0$ si les deux propriétés suivantes sont vérifiées pour tout $u_0 \in X$ :

– *Borne inférieure.* Pour toute famille $(u_h)$ convergeant vers $u_0$ pour la topologie $\mathcal{T}$,

$$\liminf_{h\to 0} \mathcal{E}_h(u_h) \geqslant \mathcal{E}_0(u_0).$$

– *Borne supérieure.* Il existe une famille $(u_h)$ convergeant vers $u_0$ pour la topologie $\mathcal{T}$,

$$\lim_{h\to 0} \mathcal{E}_h(u_h) = \mathcal{E}_0(u_0).$$

Le principal intérêt de la Gamma-convergence est que, combinée avec un résultat de compacité, elle garantit la convergence des minimiseurs.

**Theorem 0.4.2.** *Soit $(X, \mathcal{T})$ un espace topologique à base dénombrable de voisinages. Soient $\mathcal{E}_h\colon X \to \mathbb{R}\cup\{+\infty\}$, pour tout $h > 0$, et $\mathcal{E}_0\colon X \to \mathbb{R}\cup\{+\infty\}$. Supposons que $(\mathcal{E}_h)_{h>0}$ $\Gamma$-converge vers $\mathcal{E}_0$ pour la topologie $\mathcal{T}$ et que $(u_h)$ est une famille telle que $\mathcal{E}_h(u_h) = \min_{u\in X} \mathcal{E}_h(u)$ pour tout $h > 0$. S'il existe une sous-suite de $(u_h)$ convergeant vers un certain $u_0$ pour la topologie $\mathcal{T}$, alors $u_0$ est un minimiseur de $\mathcal{E}_0$ et*

$$\lim_{h\to 0} \mathcal{E}_h(u_h) = \mathcal{E}_0(u_0).$$

Pour plus de détails sur la Gamma-convergence, on renvoie au livre de Dal Maso [13].

### 0.4.1 Gamma-convergence de l'énergie micromagnétique dans un régime de film mince pour des vortex de bord et minimiseurs locaux de l'énergie limite dans le demi-plan supérieur

Dans ce chapitre, on étudie l'énergie $\widehat{E}(m_h)$, donnée dans (4), dans le régime de film mince suivant :

$$\begin{gathered} h \ll 1, \quad \frac{\eta^2}{h\,|\log h|} \to \alpha, \quad \frac{Q}{h\,|\log h|} \to \beta, \quad \frac{\widehat{D}_{13}}{\eta^2} \to 2\delta_1, \quad \frac{\widehat{D}_{23}}{\eta^2} \to 2\delta_2, \\ \frac{1}{h\,|\log h|} \sum_{j,k=1}^{2} |\widehat{D}_{jk}| \ll 1, \quad \frac{1}{h\,|\log h|} \sum_{k=1}^{3} |\widehat{D}_{3k}| \ll 1, \end{gathered} \tag{11}$$

avec $\alpha, \beta > 0$, $\delta_1, \delta_2 \in \mathbb{R}$ et $\widehat{D} = (\widehat{D}_{jk})_{(j,k)\in\{1,2,3\}^2} \in \mathbb{R}^{3\times 3}$. Les paramètres $\eta = \eta(h)$, $Q = Q(h)$ et $\widehat{D} = \widehat{D}(h)$ sont supposés être des fonctions dépendant de $h$.

#### 0.4.1.1 Gamma-convergence de l'énergie micromagnétique avec interaction de Dzyaloshinskii-Moriya dans un régime de film mince pour des vortex de bord

On considère l'énergie

$$E_h(m_h) = \frac{\widehat{E}(m_h)}{\ell^3 h^2\,|\log h|}$$

pour des applications $m_h \colon \Omega_h = \Omega \times (0,h) \to \mathbb{S}^2$ avec

$$\Delta u_h = \operatorname{div}(m_h \mathbb{1}_{\Omega_h}) \quad \text{au sens des distributions dans } \mathbb{R}^3.$$

Pour tout $h > 0$, on définit les applications $\widetilde{m}_h \colon \Omega_1 = \Omega \times (0,1) \to \mathbb{S}^2$ et $\widetilde{H}_{\text{ext},h} \colon \mathbb{R}^3 \to \mathbb{R}^3$ telles que, pour tout $(x', x_3) \in \Omega_1$,

$$\widetilde{m}_h(x', x_3) = m_h(x', hx_3),$$

et

$$\widetilde{H}_{\text{ext},h}(x', x_3) = H_{\text{ext},h}(x', hx_3).$$

De plus, on suppose que

$$\left(\frac{\widetilde{H}_{\text{ext},h}}{h\,|\log h|}\right) \text{ converge dans } L^1(\Omega_1) \text{ vers } \gamma \widetilde{H}_{\text{ext},0}, \tag{12}$$

où $\gamma \in \mathbb{R}$, et $\widetilde{H}_{\text{ext},0} \colon \mathbb{R}^3 \to \mathbb{R}^3$ est indépendant de $x_3$. On pose alors

$$\widetilde{E}_h(\widetilde{m}_h) = E_h(m_h),$$

et on a plus précisément (voir la Section 1.2.1 pour les détails de calcul) :

$$\begin{aligned} \widetilde{E}_h(\widetilde{m}_h) = {} & \frac{\eta^2}{h\,|\log h|} \int_{\Omega_1} \left( |\nabla' \widetilde{m}_h|^2 + \frac{1}{h^2}\,|\partial_3 \widetilde{m}_h|^2 \right) \mathrm{d}x \\ & + \frac{1}{h\,|\log h|} \int_{\Omega_1} \widehat{D}' : \nabla' \widetilde{m}_h \wedge \widetilde{m}_h \,\mathrm{d}x + \frac{1}{h^2\,|\log h|} \int_{\Omega_1} \widehat{D}_3 \cdot \partial_3 \widetilde{m}_h \wedge \widetilde{m}_h \,\mathrm{d}x \\ & + \frac{1}{h^2\,|\log h|} \int_{\mathbb{R}^3} |\nabla u_h|^2 \,\mathrm{d}x \\ & + \frac{Q}{h\,|\log h|} \int_{\Omega_1} \Phi(\widetilde{m}_h) \,\mathrm{d}x - \frac{2}{h\,|\log h|} \int_{\Omega_1} \widetilde{H}_{\text{ext},h} \cdot \widetilde{m}_h \,\mathrm{d}x. \end{aligned} \tag{13}$$

Une quantité qui intervient naturellement dans le problème est l'aimantation moyennée par rapport à $x_3$, puisqu'on se situe dans un régime de film mince : l'épaisseur du matériau tend vers zéro et la composante verticale en $x_3$ est donc pénalisée. On définit ainsi l'application $\overline{m}_h \colon \Omega \to \overline{B^2} \subset \mathbb{R}^3$ (avec $B^2$ désignant la boule unité de $\mathbb{R}^3$) par

$$\overline{m}_h(x') = \frac{1}{h} \int_0^h m_h(x', x_3) \, \mathrm{d}x_3$$

pour tout $x' \in \Omega$, et on note $\overline{u}_h \colon \mathbb{R}^3 \to \mathbb{R}$ le potentiel démagnétisant associé, défini par

$$\Delta \overline{u}_h = \mathrm{div}(\overline{m}_h \mathbb{1}_{\Omega_h}) \text{ au sens des distributions dans } \mathbb{R}^3.$$

Le but de la première partie du Chapitre 1 est de prouver un résultat de compacité et de Gamma-convergence pour l'énergie $\widetilde{E}_h$ donnée par (13) dans le régime (11)+(12). Plus précisément, on montre (voir Théorème 1.1.3) que $(\widetilde{E}_h)_{h>0}$ $\Gamma$-converge dans $H^1(\Omega_1, \mathbb{S}^2)$ muni de la topologie faible vers la fonctionnelle

$$\begin{aligned} \widetilde{E}_0(\widetilde{m}) = \alpha & \left[ \int_{\Omega_1} |\nabla' \widetilde{m}|^2 \mathrm{d}x + 2 \int_{\Omega_1} \delta \cdot \nabla' \widetilde{m}' \wedge \widetilde{m}' \, \mathrm{d}x \right] \\ & + \frac{1}{2\pi} \int_{\partial\Omega} (\widetilde{m} \cdot \nu)^2 \, \mathrm{d}\mathcal{H}^1 + \beta \int_{\Omega_1} \Phi(\widetilde{m}) \, \mathrm{d}x - 2\gamma \int_{\Omega_1} \widetilde{H}_{\mathrm{ext},0} \cdot \widetilde{m} \, \mathrm{d}x, \end{aligned} \tag{14}$$

où $\nu$ désigne le vecteur normal unitaire sortant sur $\partial\Omega_1$, qui est définie pour des applications $\widetilde{m} \in H^1(\Omega_1, \mathbb{S}^2)$ indépendantes de $x_3$ et telles que $\widetilde{m}_3 \equiv 0$, et égale à $+\infty$ sinon. Cette Gamma-limite revient donc à considérer des applications $\widetilde{m}' \colon \Omega \to \mathbb{S}^1$ définies sur une section horizontale de $\Omega_1$ et à valeurs dans le cercle unité $\mathbb{S}^1$. Le modèle tridimensionnel étudié se ramène ainsi à un modèle réduit en deux dimensions. En conséquence de la convergence des minimiseurs préservée par Gamma-convergence, on déduit le Corollaire 1.1.5, qui dit que toute suite de minimiseurs de $\widetilde{E}_h$ converge faiblement dans $H^1$ vers un minimiseur de $\widetilde{E}_0$.

#### 0.4.1.2 Sur les minimiseurs locaux de la Gamma-limite de l'énergie micromagnétique dans le demi-plan supérieur

Dans la seconde partie du Chapitre 1, on étudie les minimiseurs locaux de la Gamma-limite donnée dans (14), en l'absence des termes d'anisotropie et de champ magnétique extérieur, dans le demi-plan supérieur $\mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty)$. Pour cela, quitte à diviser par $\alpha$, on considère l'énergie

$$E^\delta_\varepsilon(\varphi; \Omega) = \frac{1}{2} \int_{\Omega \cap \mathbb{R}^2_+} \Big( |\nabla\varphi|^2 - 2\delta \cdot \nabla\varphi \Big) \, \mathrm{d}x + \frac{1}{2\varepsilon} \int_{\Omega \cap (\mathbb{R} \times \{0\})} \sin^2 \varphi \, \mathrm{d}\mathcal{H}^1,$$

où $\delta = (\delta_1, \delta_2)$, $\varepsilon = 2\pi\alpha$, $\Omega$ est un ouvert borné de $\mathbb{R}^2$, et $\varphi \colon \Omega \to \mathbb{R}$ est un relèvement de $\widetilde{m}' : \Omega \to \mathbb{S}^1$. Cette fonctionnelle a été étudiée par Kurzke [33] dans le régime $\varepsilon \to 0$, pour lequel l'aimantation associée $\widetilde{m}'$ génère des vortex au bord du domaine $\Omega$.

Avant d'étudier les minimiseurs locaux de $E^\delta_\varepsilon$ au sens de De Giorgi (voir Définition 1.1.7) dans $\mathbb{R}^2_+$, on s'intéresse aux points critiques de $E^\delta_\varepsilon$. Dans la Proposition 1.3.2, on prouve que si $\varphi_\varepsilon$ est un point critique de $E^\delta_\varepsilon$, alors $\varphi_\varepsilon \in C^\infty(\overline{\mathbb{R}^2_+})$, avec $\overline{\mathbb{R}^2_+} = \mathbb{R} \times [0, +\infty)$, et la fonction

$$\phi_\varepsilon \colon (x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto 2\varphi_\varepsilon(\varepsilon x_1, \varepsilon x_2) + \pi,$$

vérifie le problème de Neumann non linéaire ($\mathrm{PN}_\lambda$) ci-dessous avec $\lambda = 2\varepsilon\delta_2$, à condition que la fonction $(x_1, x_2) \mapsto \phi_\varepsilon(x_1, x_2) - 2\varepsilon\delta_2 x_2$ soit bornée dans $\mathbb{R}^2_+$. Le problème en question est donné par

$$\begin{cases} f \in C^\infty(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+}), \\ (x_1,x_2) \mapsto f(x_1,x_2) - \lambda x_2 \text{ est bornée dans } \mathbb{R}^2_+, \\ \Delta f = 0 \text{ dans } \mathbb{R}^2_+, \\ \partial_2 f - \lambda + \sin f = 0 \text{ sur } \mathbb{R} \times \{0\}\,, \end{cases} \tag{PN$_\lambda$}$$

où $\lambda \in \mathbb{R}$ est constant. Ce problème est une version modifiée du problème de Peierls-Nabarro, étudié par Amick et Toland ([5], [43]) dans le cas où $\lambda = 0$. En se basant sur leurs travaux, on montre dans le Théorème 1.1.8 que les solutions de (PN$_\lambda$) peuvent être de trois types. La fonction $(x_1,x_2) \mapsto \phi_\varepsilon(x_1,x_2) - 2\varepsilon\delta_2 x_2$ est soit constante égale à un multiple de $\pi$, soit périodique par rapport à la première variable, soit non périodique. Dans ce dernier cas où la fonction ci-dessus est non périodique, un changement d'échelle dans les variables $x_1$ et $x_2$ permet de voir que la fonction présente une singularité au bord du domaine lorsque $\varepsilon \to 0$, ce qui correspond à un vortex au bord pour l'aimantation. Ce sont donc ces fonctions non périodiques que l'on s'attend à voir intervenir dans le calcul des minimiseurs de $E^\delta_\varepsilon$.

On prouve dans le Théorème 1.1.9 que les seuls minimiseurs locaux de $E^\delta_\varepsilon$ dans $\mathbb{R}^2_+$ au sens de De Giorgi vérifiant les conditions

$$\lim_{x_1 \to +\infty} \varphi_\varepsilon(x_1,0) = 0, \quad \lim_{x_1 \to -\infty} \varphi_\varepsilon(x_1,0) = \pi \text{ et } [(x_1,x_2) \mapsto \varphi_\varepsilon(x_1,x_2) - \delta_2 x_2] \in L^\infty(\mathbb{R}^2_+)$$

sont les fonctions

$$\varphi_\varepsilon \colon (x_1,x_2) \in \overline{\mathbb{R}^2_+} \mapsto \frac{\pi}{2} - \arctan\left(\frac{x_1 + \varepsilon a}{x_2 + \varepsilon}\right) + \delta_2 x_2$$

où $a \in \mathbb{R}$ est une constante. Ces fonctions sont obtenues à partir des solutions non périodiques de (PN$_\lambda$) ; les solutions constantes et périodiques étant éliminées par les conditions imposées ci-dessus. Le résultat d'unicité dans le Théorème 1.1.9 est prouvé en utilisant un résultat local dû à Cabré et Solà-Morales [10].

### 0.4.2 Gamma-convergence de l'énergie micromagnétique dans un régime de film mince pour des vortex de bord et énergie renormalisée entre les vortex de bord

Dans ce chapitre, on étudie l'énergie $\widehat{E}(m_h)$ donnée dans (4), en supposant que l'anisotropie $\Phi$ et le champ magnétique extérieur $H_{\text{ext},h}$ sont nuls, dans un régime de film mince différent de celui étudié dans le Chapitre 1. Plus précisément, on considère ici le régime

$$\begin{aligned} &h \ll 1, \quad \eta \ll 1, \quad \frac{1}{|\log h|} \ll \varepsilon \ll 1, \quad \frac{\widehat{D}_{13}}{\eta^2} \to 2\delta_1, \quad \frac{\widehat{D}_{23}}{\eta^2} \to 2\delta_2, \\ &\frac{1}{\eta^2}\sum_{j,k=1}^{2} |\widehat{D}_{jk}| \ll 1, \quad \frac{1}{\eta^2}\sum_{k=1}^{3} |\widehat{D}_{3k}| \ll 1, \end{aligned} \tag{15}$$

où $\delta_1, \delta_2 \in \mathbb{R}$, $\widehat{D} = (\widehat{D}_{jk})_{(j,k)\in\{1,2,3\}^2} \in \mathbb{R}^{3\times 3}$, et

$$\varepsilon = \frac{\eta^2}{h\,|\log h|}.$$

Les paramètres $\eta = \eta(h)$, $\varepsilon = \varepsilon(h)$ et $\widehat{D} = \widehat{D}(h)$ sont supposés être des fonctions dépendant de $h$. Comme $a \ll b \ll 1$ implique $a\,|\log a| \ll b\,|\log b| \ll 1$, on en déduit que, pour $a = \frac{1}{|\log h|}$ et $b = \varepsilon$, le régime (15) implique

$$\begin{aligned}&\frac{\log|\log h|}{|\log h|} \ll \varepsilon\,|\log \varepsilon| \ll 1, \quad \frac{\widehat{D}_{13}}{\eta^2} \to 2\delta_1, \quad \frac{\widehat{D}_{23}}{\eta^2} \to 2\delta_2,\\ &\frac{1}{\eta^2}\sum_{j,k=1}^{2}|\widehat{D}_{jk}| \ll 1, \quad \frac{1}{\eta^2}\sum_{k=1}^{3}|\widehat{D}_{3k}| \ll 1.\end{aligned} \tag{16}$$

Afin de prouver un développement asymptotique au second ordre par Gamma-convergence, on va de plus considérer le régime

$$\begin{aligned}&\frac{\log|\log h|}{|\log h|} \ll \varepsilon \ll 1, \quad \left|\frac{\widehat{D}_{13}}{\eta^2} - 2\delta_1\right| \ll \frac{1}{|\log\varepsilon|}, \quad \left|\frac{\widehat{D}_{23}}{\eta^2} - 2\delta_2\right| \ll \frac{1}{|\log\varepsilon|},\\ &\frac{1}{\eta^2}\sum_{j,k=1}^{2}|\widehat{D}_{jk}| \ll 1, \quad \frac{1}{\eta^2}\sum_{k=1}^{3}|\widehat{D}_{3k}| \ll \frac{1}{|\log\varepsilon|},\end{aligned} \tag{17}$$

qui est plus restrictif que le régime (16). On suppose que $\Omega \subset \mathbb{R}^2$ est un ouvert borné, simplement connexe et de classe $C^{1,1}$. On considère l'énergie

$$\mathcal{E}_h(m_h) = \frac{\widehat{E}(m_h)}{\ell^3 h\eta^2\,|\log\varepsilon|} = \frac{\widehat{E}(m_h)}{\ell^3 h^2\,|\log h|\,\varepsilon\,|\log\varepsilon|},$$

pour des applications $m_h \colon \Omega_h = \Omega \times (0,h) \to \mathbb{S}^2$ avec

$$\Delta u_h = \operatorname{div}(m_h \mathbb{1}_{\Omega_h}) \ \text{ au sens des distributions dans } \mathbb{R}^3,$$

en supposant que l'anisotropie $\Phi$ et le champ magnétique extérieur $H_{\text{ext},h}$ sont nuls. Plus précisément, l'énergie $\mathcal{E}_h(m_h)$ est donnée par

$$\mathcal{E}_h(m_h) = \frac{1}{|\log\varepsilon|}\left(\frac{1}{h}\int_{\Omega_h}|\nabla m_h|^2\,\mathrm{d}x + \frac{1}{h\eta^2}\int_{\Omega_h}\widehat{D} : \nabla m_h \wedge m_h\ \mathrm{d}x + \frac{1}{h\eta^2}\int_{\mathbb{R}^3}|\nabla u_h|^2\,\mathrm{d}x\right).$$

L'énergie $\mathcal{E}_h$ a été étudiée par Ignat et Kurzke [25] lorsque l'interaction de Dzyaloshinskii-Moriya est supposée négligeable, c'est-à-dire $\widehat{D} = 0$. Étant donné que leurs résultats vont être utiles pour notre étude, on notera $\mathcal{E}_h^0$ la fonctionnelle d'énergie $\mathcal{E}_h$ lorsque $\widehat{D} = 0$, c'est-à-dire

$$\mathcal{E}_h^0(m_h) = \frac{1}{|\log\varepsilon|}\left(\frac{1}{h}\int_{\Omega_h}|\nabla m_h|^2\,\mathrm{d}x + \frac{1}{h\eta^2}\int_{\mathbb{R}^3}|\nabla u_h|^2\,\mathrm{d}x\right). \tag{18}$$

En comparaison avec le Chapitre 1, dans lequel on suppose que $\varepsilon = \frac{\eta^2}{h|\log h|} \to \alpha > 0$, on considère ici le cas où $\varepsilon \to 0$. Les résultats obtenus dans ce chapitre sont basés sur ceux de Ignat et Kurzke [24], [25], qui ont étudié l'énergie $\mathcal{E}_h^0$ donnée par (18).

#### 0.4.2.1 Un modèle réduit en deux dimensions pour des applications $m' \colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$

Dans la première partie de ce chapitre, on considère un modèle bidimensionnel (voir [24]) pour des applications $m' \colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$. Plus précisément, étant donné un vecteur $\delta \in \mathbb{R}^2$, on considère la fonctionnelle d'énergie

$$\begin{aligned}E_{\varepsilon,\eta}^{\delta}(m') = &\int_\Omega |\nabla' m'|^2\,\mathrm{d}x + 2\int_\Omega \delta\cdot\nabla' m' \wedge m'\ \mathrm{d}x\\ &+ \frac{1}{\eta^2}\int_\Omega \left(1-|m'|^2\right)^2\mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega}(m'\cdot\nu')^2\mathrm{d}\mathcal{H}^1,\end{aligned}$$

où $\nabla' = (\partial_1, \partial_2)$ et $\nu'$ est le vecteur normal unitaire sortant sur $\partial\Omega$. On peut remarquer que l'on retrouve partiellement l'énergie étudiée dans la deuxième partie du Chapitre 1 avec une pénalisation au bord, qui favorise les vortex au bord pour l'aimantation. Ici, une pénalisation intérieure s'ajoute, provenant de $m_3^2$, similaire à celle rencontrée dans le modèle de Ginzburg-Landau. Ce modèle réduit combinant à la fois une pénalisation intérieure (comme dans le modèle classique de Ginzburg-Landau [7]) et une pénalisation au bord (comme dans le Chapitre 1, ou dans Kurzke [32]), les singularités topologiques peuvent a priori apparaître à l'intérieur ou au bord du domaine bidimensionnel. Pour détecter ces singularités topologiques, on utilise le *Jacobien global* introduit par Ignat et Kurzke, qui est un opérateur linéaire défini pour des applications de $H^1(\Omega, \mathbb{R}^2)$. Dans le régime étudié dans ce chapitre, qui favorise l'émergence de vortex de bord, le Jacobien global limite sera une mesure supportée sur le bord $\partial\Omega$ du domaine qui dépend à la fois du nombre de vortex au bord et de la courbure du bord $\partial\Omega$. Pour des détails sur la notion de Jacobien global, on renvoie à la Section 2.1.2 de cette thèse, ou à [24], [25].

On étudie l'énergie $E^\delta_{\varepsilon,\eta}$ dans le régime asymptotique

$$\eta \ll 1, \quad \varepsilon \ll 1, \quad |\log \varepsilon| \ll |\log \eta| \,. \tag{19}$$

On peut noter que le régime de film mince pour le problème en trois dimensions (15) implique le régime (19). En effet, le régime (15) est équivalent à $h \ll \eta^2 \ll h\,|\log h| \ll 1$ en utilisant la définition de $\varepsilon$, donc $|\log h| \sim |\log \eta|$. D'après (15), on a $\frac{1}{\varepsilon} \ll |\log \eta|$ et d'après (16), on a $|\log \varepsilon| \ll \frac{1}{\varepsilon}$, et on obtient donc le régime (19).

Dans les Théorèmes 2.1.1, 2.1.3 et 2.1.4, on prouve des résultats de compacité pour le Jacobien global et pour les applications $m'$, ainsi qu'un développement asymptotique d'ordre 2 par Gamma-convergence pour l'énergie $E^\delta_{\varepsilon,\eta}$, qui montre que les singularités sont localisées sur le bord du domaine $\Omega$. Le terme d'ordre 1 dans le développement asymptotique donne une estimation de l'énergie limite par rapport au nombre de vortex au bord. Le terme d'ordre 2 dans le développement asymptotique contient une énergie renormalisée qui représente les interactions entre les vortex au bord, et permet de déduire leurs positions optimales, c'est-à-dire celles qui minimisent l'énergie renormalisée.

Dans le Théorème 2.1.5, le Corollaire 2.1.6 et le Théorème 2.1.7, on calcule explicitement cette énergie renormalisée et on prouve l'existence de minimiseurs présentant deux vortex au bord de multiplicité 1 pour cette énergie renormalisée. On prouve enfin, dans le Corollaire 2.1.8, que les minimiseurs de l'énergie $E^\delta_{\varepsilon,\eta}$ sont ceux de la configuration avec deux vortex au bord de multiplicité 1.

#### 0.4.2.2 Le modèle en trois dimensions pour des applications $m_h \colon \Omega_h \subset \mathbb{R}^3 \to \mathbb{S}^2$

Dans la seconde partie de ce chapitre, on revient au modèle en trois dimensions pour des applications $m_h \colon \Omega_h \subset \mathbb{R}^3 \to \mathbb{S}^2$ et on étudie l'énergie $\mathcal{E}_h$. Tout comme dans le Chapitre 1, puisque l'on considère ici encore un régime de film mince, on est amené à utiliser l'aimantation moyennée par rapport à $x_3$, $\overline{m}_h \colon \Omega \to \overline{B^2} \subset \mathbb{R}^3$ (avec $B^2$ désignant la boule unité de $\mathbb{R}^3$), qui permet de lever le caractère non local de l'énergie démagnétisante. Rappelons que

$$\overline{m}_h(x') = \frac{1}{h}\int_0^h m_h(x', x_3)\;\mathrm{d}x_3$$

pour tout $x' \in \Omega$, et on note $\overline{u}_h \colon \mathbb{R}^3 \to \mathbb{R}$ le potentiel démagnétisant associé, défini par

$$\Delta \overline{u}_h = \operatorname{div}(\overline{m}_h \mathbb{1}_{\Omega_h}) \;\text{ au sens des distributions dans } \mathbb{R}^3.$$

On introduit l'énergie réduite en deux dimensions

$$\overline{\mathcal{E}}_h(\overline{m}_h) = \frac{1}{|\log \varepsilon|} \left( \int_\Omega |\nabla' \overline{m}_h|^2 \, \mathrm{d}x' + \frac{1}{\eta^2} \int_\Omega \widehat{D}' \ : \nabla' \overline{m}_h \wedge \overline{m}_h \, \mathrm{d}x' \right.$$
$$\left. + \frac{1}{\eta^2} \int_\Omega \left(1 - |\overline{m}'_h|^2\right) \mathrm{d}x' + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (\overline{m}'_h \cdot \nu')^2 \, \mathrm{d}\mathcal{H}^1 \right)$$

avec $\overline{m}'_h = (\overline{m}_{h,1}, \overline{m}_{h,2})$ et $\nu'$ le vecteur normal unitaire sortant sur $\partial\Omega$. Pour utiliser les résultats obtenus par Ignat et Kurzke [25] dans le cas où $\widehat{D} = 0$, on considère également l'énergie

$$\overline{\mathcal{E}}^0_h(\overline{m}_h) = \frac{1}{|\log \varepsilon|} \left( \int_\Omega |\nabla' \overline{m}_h|^2 \, \mathrm{d}x' + \frac{1}{\eta^2} \int_\Omega \left(1 - |\overline{m}'_h|^2\right) \mathrm{d}x' + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (\overline{m}'_h \cdot \nu')^2 \, \mathrm{d}\mathcal{H}^1 \right). \tag{20}$$

Un argument qui nous incite à utiliser ces énergies moyennées est que dans le régime (15), l'énergie démagnétisante dans (18) est proche des termes de pénalisations intérieure et au bord dans (20) (voir [25, Lemmes 15 et 16]). Par conséquent, l'énergie moyennée $\overline{\mathcal{E}}^0_h$ permet de s'affranchir du caractère non local de l'énergie démagnétisante en remplaçant cette énergie par des termes locaux. Les estimations obtenues par Ignat et Kurzke sont un raffinement d'estimations de Carbou [11] et Kohn-Slastikov [32] respectivement.

Dans le Théorème 2.1.10, on compare l'énergie $\mathcal{E}_h$ avec l'énergie réduite $\overline{\mathcal{E}}_h$ et on obtient des bornes inférieures pour $\mathcal{E}_h$ faisant intervenir $\overline{\mathcal{E}}_h$. Afin de pouvoir utiliser les résultats obtenus dans la Section 2.2, on compare ensuite l'énergie $\overline{\mathcal{E}}_h$ avec l'énergie $E^\delta_{\varepsilon,\eta}$ du modèle réduit pour des applications $m' \colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$. Grâce au Corollaire 2.1.11, on obtient alors des bornes inférieures pour $\mathcal{E}_h$ en termes de $E^\delta_{\varepsilon,\eta}$.

En utilisant la compacité et la Gamma-convergence de $E^\delta_{\varepsilon,\eta}$ obtenues dans la Section 2.2, on en déduit, dans les Théorèmes 2.1.12 et 2.1.13, un résultat de compacité et une borne inférieure pour l'énergie $\mathcal{E}_h$. Pour des applications $m_h$ indépendantes de $x_3$ et vérifiant $m_h = (m'_h, 0)$, grâce au Corollaire 2.1.11, on obtient des bornes supérieures qui complètent le développement asymptotique d'ordre 2 par Gamma-convergence pour $\mathcal{E}_h$ : il s'agit du Théorème 2.1.14. Enfin, on prouve le Corollaire 2.1.15 qui établit que les minimiseurs du modèle en trois dimensions obéissent à la même configuration que ceux du modèle réduit en deux dimensions.

# Chapter 1

## Gamma-convergence of the micromagnetic energy in a thin-film regime for boundary vortices and local minimizers of the thin-film limit in the upper-half plane

**Abstract**

We consider the three-dimensional micromagnetic model with Dzyaloshinskii-Moriya interaction in a thin-film regime. We prove the Gamma-convergence of the micromagnetic energy in the considered regime, for which the Gamma-limit energy is two-dimensional and relevant for boundary vortices. We then study local minimizers of the Gamma-limit energy and prove a uniqueness result in a certain setting. This chapter is based on the preprint [35].

### Contents

## 1.1 Introduction

The modeling of small ferromagnetic materials is based on the theory of micromagnetics. The model states that a ferromagnetic device can be described by a three-dimensional vector field with values in $\mathbb{S}^2$, called magnetization, whose stable states correspond to local minimizers of the micromagnetic energy (see e.g. [20]). The associated variational problem is non-convex, non-local and multiscale. In particular, the relation between length parameters – both intrinsic parameters of the magnetic material and geometric parameters – allows to consider a lot of different asymptotic regimes. This leads to the formation of magnetic patterns, such as domain walls, vortices, etc. We are interested in a special regime relevant for boundary vortices.

### 1.1.1 The general three-dimensional model

We consider a ferromagnetic sample of cylindrical shape

$$\Omega_t^\ell = \Omega^\ell \times (0,t) \subset \mathbb{R}^3,$$

where $\Omega^\ell \subset \mathbb{R}^2$ is a smooth bounded open set of typical length $\ell$ (for example, $\Omega^\ell$ can be assumed to be an open disk of diameter $\ell$). The magnetization $m$ is a unitary three-dimensional vector field

$$m\colon \Omega_t^\ell \to \mathbb{S}^2,$$

where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. In particular, the constraint $|m| = 1$ yields the non-convexity of the problem. The classical micromagnetic energy is defined as

$$E_{\text{class}}(m) = A^2 \int_{\Omega_t^\ell} |\nabla m|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + Q \int_{\Omega_t^\ell} \Phi(m) \,\mathrm{d}x - 2 \int_{\Omega_t^\ell} H_{\text{ext}} \cdot m \,\mathrm{d}x.$$

Let us explain briefly the four terms of $E_{\text{class}}$. The first term is the exchange energy. It is generated by small-distance interactions in the sample ; roughly speaking, this energy favors the alignment of neighboring spins in the sample. The positive parameter $A$ is an intrinsic parameter of the ferromagnetic material, and is typically of the order of nanometers: we call it the exchange length. The second term is called magnetostatic or demagnetizing energy. This energy is generated by the large-distance interactions in the sample. It is in fact the energy generated by the magnetic field induced by magnetization. More precisely, the demagnetizing potential $u \in H^1(\mathbb{R}^3, \mathbb{R})$ satisfies

$$\Delta u = \operatorname{div}(m \mathbb{1}_{\Omega_t^\ell}) \text{ in the distributional sense in } \mathbb{R}^3, \tag{1.1}$$

where $\mathbb{1}_{\Omega_t^\ell}(x) = 1$ if $x \in \Omega_t^\ell$, and $\mathbb{1}_{\Omega_t^\ell}(x) = 0$ elsewhere. The third term is the anisotropy energy: it takes into account the anisotropy effects resulting from the crystalline structure of the sample. It involves the quality factor $Q > 0$ (that is a second intrinsic parameter of the sample) and the function $\Phi\colon \mathbb{S}^2 \to \mathbb{R}_+$ which has some symmetry properties. The fourth and last term is the external field or Zeeman energy: it is generated by an external magnetic field. The Zeeman energy involves the vector field

$$H_{\text{ext}}\colon \mathbb{R}^3 \to \mathbb{R}^3$$

and favours the alignment of the magnetization in the direction of the external magnetic field.

In this thesis, we consider an additional term: the Dzyaloshinskii-Moriya interaction. This interaction was introduced in the 1950s [17] to describe the magnetization in some materials with few symmetry properties. We assume here that the Dzyaloshinskii-Moriya interaction density in three dimensions is defined as

$$D : \nabla m \wedge m = \sum_{j=1}^{3} D_j \cdot \partial_j m \wedge m, \tag{1.2}$$

where $D = (D_1, D_2, D_3) \in \mathbb{R}^{3\times 3}$, $\cdot$ denotes the inner product in $\mathbb{R}^3$, and $\wedge$ denotes the cross product in $\mathbb{R}^3$. Hence, we consider the micromagnetic energy $E(m)$ given as

$$\begin{aligned} E(m) = A^2 \int_{\Omega_t^\ell} |\nabla m|^2 \,\mathrm{d}x + \int_{\Omega_t^\ell} D : \nabla m \wedge m \,\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x \\ + Q \int_{\Omega_t^\ell} \Phi(m) \,\mathrm{d}x - 2 \int_{\Omega_t^\ell} H_{\mathrm{ext}} \cdot m \,\mathrm{d}x. \end{aligned} \tag{1.3}$$

For more details about the components of the micromagnetic energy, especially physical interpretations, we refer to [1], [20] or [17].

The multiscale aspect of the micromagnetic energy (1.3) is obvious. Indeed, beside the tensor $D$ and the quality factor $Q$, three length paramaters of the ferromagnetic device interact together: the exchange length $A$, the planar diameter $\ell$ and the thickness $t$ of the sample. From these parameters, we introduce the dimensionless parameters

$$h = \frac{t}{\ell} \quad \text{and} \quad \eta = \frac{A}{\ell}.$$

By letting $h$ tend to zero, the relative thickness of the ferromagnetic device tends to zero: it is a thin-film limit. The consequences concerning the magnetization and the micromagnetic energy depend on the relations between $h$ and $\eta$, i.e. on the thin-film asymptotic regime.

### 1.1.2 A thin-film regime

#### 1.1.2.1 Nondimensionalization in length

In order to study the micromagnetic energy in a thin-film regime, it is convenient to nondimensionalize it in length ; in particular, we get from the three length parameters $A$, $\ell$ and $t$ only two dimensionless parameters $h$ and $\eta$ defined above. We set

$$\Omega_h = \frac{\Omega_t^\ell}{\ell} = \Omega \times (0, h) \subset \mathbb{R}^3,$$

where $\Omega = \frac{\Omega^\ell}{\ell} \subset \mathbb{R}^2$ is a smooth bounded open set of typical length 1 (for example, $\Omega$ can be assumed to be the unit disk in $\mathbb{R}^2$). To each $x = (x_1, x_2, x_3) \in \Omega$, we associate $\widehat{x} = \frac{x}{\ell} \in \Omega_h$ and we set $\widehat{D} = \frac{1}{\ell} D$. We also consider the maps $m_h \colon \Omega_h \to \mathbb{S}^2$, $u_h \colon \mathbb{R}^3 \to \mathbb{R}$ and $H_{\mathrm{ext},h} \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that, for every $\widehat{x} = \frac{x}{\ell} \in \Omega_h$,

$$m_h(\widehat{x}) = m(x), \quad u_h(\widehat{x}) = \frac{1}{\ell} u(x),$$

that satisfy

$$\Delta u_h = \mathrm{div}(m_h \mathbb{1}_{\Omega_h}) \ \text{ in the distributional sense in } \mathbb{R}^3, \tag{1.4}$$

and

$$H_{\mathrm{ext},h}(\widehat{x}) = H_{\mathrm{ext}}(x).$$

The micromagnetic energy (1.3) can then be written in terms of $m_h$:

$$\begin{aligned} \widehat{E}(m_h) = \ell^3 \Bigg[ \eta^2 \int_{\Omega_h} |\nabla m_h|^2 \mathrm{d}\widehat{x} + \int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \,\mathrm{d}\widehat{x} \\ + \int_{\mathbb{R}^3} |\nabla u_h|^2 \mathrm{d}\widehat{x} \\ + Q \int_{\Omega_h} \Phi(m_h) \,\mathrm{d}\widehat{x} - 2 \int_{\Omega_h} H_{\mathrm{ext},h} \cdot m_h \,\mathrm{d}\widehat{x} \Bigg]. \end{aligned} \tag{1.5}$$

For simplicity of the notations, we write $x$ instead of $\widehat{x}$ in the following.

#### 1.1.2.2 Heuristic approach of a thin-film regime

The thin-film model is characterized by the assumption $h = t/\ell \to 0$, i.e. the variations of the magnetization $m$ with respect to the vertical component $x_3$ are strongly penalized. With this in mind, we assume for a while that $m_h$ does not depend on $x_3$, i.e.

$$m_h(x_1, x_2, x_3) = m_h(x_1, x_2) \colon \Omega \to \mathbb{S}^2, \tag{1.6}$$

and that

$$m_h \text{ varies on length scales } \gg h. \tag{1.7}$$

Moreover, we also assume that the external magnetic field $H_{\text{ext},h}$ is in-plane, invariant in $x_3$ and independent of $h$, i.e.

$$H_{\text{ext},h}(x_1, x_2, x_3) = (H'_{\text{ext},h}(x_1, x_2), 0). \tag{1.8}$$

The Maxwell equation (1.4) implies

$$\Delta u_h = \operatorname{div}(m_h \mathbb{1}_{\Omega_h}) = \operatorname{div}(m_h)\mathbb{1}_{\Omega_h} - (m_h \cdot \nu)\mathbb{1}_{\partial\Omega_h} = \operatorname{div}'(m'_h)\mathbb{1}_{\Omega_h} - (m_h \cdot \nu)\mathbb{1}_{\partial\Omega_h}$$

in the distributional sense in $\mathbb{R}^3$, where $\nu$ is the outer unit normal vector on $\partial\Omega_h$, $m'_h = (m_{h,1}, m_{h,2})$ and $\operatorname{div}'(m'_h) = \partial_1 m_{h,1} + \partial_2 m_{h,2}$. In other words, $u_h$ is a solution of the problem

$$\begin{cases} \Delta u_h &= \operatorname{div}'(m'_h) & \text{in } \Omega_h, \\ \Delta u_h &= 0 & \text{in } \mathbb{R}^3 \setminus \Omega_h, \\ \left[\frac{\partial u_h}{\partial \nu}\right] &= m_h \cdot \nu & \text{on } \partial\Omega_h, \end{cases}$$

where $[a] = a^+ - a^-$ stands for the jump of $a$ with respect to the outer unit normal vector $\nu$ on $\partial\Omega_h$. From [21, Section 2.1.2] (see also [16]), by assuming $u_h \in H^1(\mathbb{R}^3)$, we can express the stray-field energy by considering the Fourier transform in the horizontal variables:

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u_h|^2 \,\mathrm{d}x = h \int_{\mathbb{R}^2} \frac{|\xi' \cdot \mathcal{F}(m'_h \mathbb{1}_\Omega)(\xi')|^2}{|\xi'|^2} \left(1 - g_h\left(|\xi'|\right)\right) \mathrm{d}\xi' \\ + h \int_{\mathbb{R}^2} |\mathcal{F}(m_{h,3}\mathbb{1}_\Omega)(\xi')|^2 \, g_h\left(|\xi'|\right) \mathrm{d}\xi', \end{aligned}$$

where $\mathcal{F}$ stands for the Fourier transform in $\mathbb{R}^2$, i.e. for every $f \colon \mathbb{R}^2 \to \mathbb{C}$ and for every $\xi' \in \mathbb{R}^2$,

$$\mathcal{F}(f)(\xi') = \hat{f}(\xi') = \int_{\mathbb{R}^2} f(x') e^{-2i\pi x' \cdot \xi'} \mathrm{d}x',$$

and

$$g_h(|\xi'|) = \frac{1 - e^{-2\pi h |\xi'|}}{2\pi h |\xi'|}. \tag{1.9}$$

*Remark* 1.1.1. The function $g_h$ satisfies the following useful properties.
For every $h > 0$ and $\xi' \in \mathbb{R}^2$, $e^{-2\pi h|\xi'|} \in (0, 1]$, hence $g_h(|\xi'|) \geqslant 0$. Moreover, for every $\xi' \in \mathbb{R}^2$,

$$e^{-2\pi h|\xi'|} = 1 - 2\pi h|\xi'| + 2\pi^2 h^2 |\xi'|^2 + o(h^2) \quad \text{when } h \to 0,$$

thus $(g_h)$ converges almost everywhere to 1 in $\mathbb{R}^2$ and, for every $\xi' \in \mathbb{R}^2$, for every $h > 0$, $g_h(|\xi'|)$ is bounded independently of $h$.

In the asymptotics $h \to 0$, we have $g_h\left(|\xi'|\right) \to 1$ and $1 - g_h\left(|\xi'|\right) \to 0$, hence we can approximate

$$\int_{\mathbb{R}^3} |\nabla u_h|^2 \,\mathrm{d}x \approx h^2 \int_{\mathbb{R}^2} \frac{\pi \left|\xi' \cdot \mathcal{F}(m'_h \mathbb{1}_\Omega)(\xi')\right|^2}{|\xi'|} \,\mathrm{d}\xi' + h \int_\Omega m_{h,3}^2 \,\mathrm{d}x'.$$

A more precise approach (see [16], [32]) is given by

$$\int_{\mathbb{R}^3} |\nabla u_h|^2 \, \mathrm{d}x \approx h^2 \left\| \mathrm{div}'(m_h')_{cont} \right\|^2_{\dot{H}^{-1/2}(\mathbb{R}^2)} + \frac{1}{2\pi} h^2 |\log h| \int_{\partial\Omega} (m_h' \cdot \nu')^2 \, \mathrm{d}\mathcal{H}^1 + h \int_\Omega m_{h,3}^2 \, \mathrm{d}x', \tag{1.10}$$

where $\mathrm{div}'(m_h')_{cont} = \mathrm{div}'(m_h')\mathbb{1}_\Omega$ and $\nu'$ is the outer unit normal vector on $\partial\Omega$. Hence, the stray-field energy is asymptotically decomposed in three terms in the thin-film approximation. The first term penalizes the volume charges, as an homogeneous $\dot{H}^{-1/2}$ seminorm, and favors Néel walls. The second term takes in account the lateral charges on the cylindrical sample and favors boundary vortices. The third term penalizes the surface charges on the top and bottom of the cylinder, and leads to interior vortices. For more details on the different types of singularities that can occur in thin-film regimes, we refer to [16] or [21]. Combining (1.10) with the assumptions (1.6), (1.7) and (1.8), we get the following approximation for the reduced two-dimensional thin-film energy:

$$\begin{aligned} \widehat{E}(m_h) \approx \ell^3 \Bigg[ & h\eta^2 \int_\Omega |\nabla' m_h|^2 \mathrm{d}x' + h \int_\Omega \widehat{D}' : \nabla' m_h \wedge m_h \, \mathrm{d}x' \\ & + \frac{h^2}{2} \left\| \mathrm{div}'(m_h')_{cont} \right\|^2_{\dot{H}^{-1/2}(\mathbb{R}^2)} \\ & + \frac{1}{2\pi} h^2 |\log h| \int_{\partial\Omega} (m_h' \cdot \nu')^2 \mathrm{d}\mathcal{H}^1 \\ & + h \int_\Omega \left( m_{h,3}^2 + Q\Phi(m_h) - 2H'_{\mathrm{ext},h} \cdot m_h' \right) \mathrm{d}x' \Bigg], \end{aligned} \tag{1.11}$$

with $\widehat{D}' = (\widehat{D}_1, \widehat{D}_2)$ and $\widehat{D}' : \nabla' m_h \wedge m_h = \sum_{j=1}^2 \widehat{D}_j \cdot \partial_j m_h \wedge m_h$.

The expression (1.11) is interesting because it allows us to see the diversity of thin-film regimes. By *thin-film regime*, we mean an asymptotic relation between $h$ and $\eta$ when $h \to 0$. In the following, we consider a thin-film regime that favors boundary vortices, while taking account of the Dzyaloshinskii-Moriya interaction, the anisotropy and the external magnetic field. Hence, regarding (1.11), we renormalize by $h^2 |\log h|$ so that

$$\frac{\eta^2}{h|\log h|}, \quad \frac{\widehat{D}_{13}}{h|\log h|}, \quad \frac{\widehat{D}_{23}}{h|\log h|}, \quad \frac{Q}{h|\log h|}, \quad \text{and} \quad \frac{H'_{\mathrm{ext},h}}{h|\log h|}$$

are of the same order, and the remaining components in $\widehat{D}$ are coefficients of terms involving $m_{h,3}$, hence they must vanish at the thin-film limit.

Our assumptions are based on the regime studied by Kohn and Slastikov [32] for which the magnetization develops boundary vortices ; the new point here being that we take in account the Dzyaloshinskii-Moriya interaction. By letting $\frac{\eta^2}{h|\log h|}$ tend to zero, we obtain the regime studied by Kurzke [33] and Ignat-Kurzke [24], [25]. Boundary vortices can occur in other thin-film regimes (see Moser [38], [39]). We should also mention the recent work of Davoli, Di Fratta, Praetorius and Ruggeri [14] for finding a thin-film limit of the micromagnetic energy with Dzyaloshinskii-Moriya interaction, that is similar to what we do in this thesis. Recently, Alama, Bronsard and Golovaty [3] investigated a special type of boundary vortices, called boojums, that appear in a thin-film model of nematic liquid crystals. That type of boundary vortices costs less energy than the one we study in this thesis, so that our "classical" boundary vortices are not present in their model. For studies on the Néel walls, we refer to Ignat [22], Ignat-Moser [28], [29], Ignat-Otto [30], [31] and Melcher [36], [37].

Coming back to a general magnetization $m_h$ depending on $x_3$ and $h$, we introduce the $x_3$-average of $m_h$ in the following notation. This quantity is very appropriate in a thin-film regime for reducing the three-dimensional general model to a two-dimensional model.

*Notation* 1.1.2. For $h > 0$, we denote by $\overline{m}_h \colon \Omega \to \mathbb{R}^3$ the $x_3$-average of $m_h$, i.e.

$$\overline{m}_h(x') = \frac{1}{h} \int_0^h m_h(x', x_3)\ \mathrm{d}x_3 \tag{1.12}$$

for every $x' \in \Omega$, and we denote by $\overline{u}_h \colon \mathbb{R}^3 \to \mathbb{R}$ the associated stray field potential given by

$$\Delta \overline{u}_h = \operatorname{div}(\overline{m}_h \mathbb{1}_{\Omega_h}) \ \text{ in the distributional sense in } \mathbb{R}^3. \tag{1.13}$$

Note that the unit-length constraint is convexified by averaging, thus $|\overline{m}_h| \leqslant 1$.

### 1.1.3 Main results of Chapter 1

In this chapter, we study the energy $\widehat{E}(m_h)$, given at (1.5), in a thin-film regime governed by the main assumptions that $\frac{\eta^2}{h|\log h|}$, $\frac{\widehat{D}_{13}}{h|\log h|}$, $\frac{\widehat{D}_{23}}{h|\log h|}$, $\frac{Q}{h|\log h|}$ and $\frac{H_{\mathrm{ext},h}}{h|\log h|}$ are of order 1. More precisely, we consider the regime

$$\begin{gathered} h \ll 1, \quad \frac{\eta^2}{h\,|\log h|} \to \alpha, \quad \frac{Q}{h\,|\log h|} \to \beta, \quad \frac{\widehat{D}_{13}}{\eta^2} \to 2\delta_1, \quad \frac{\widehat{D}_{23}}{\eta^2} \to 2\delta_2, \\ \frac{1}{h\,|\log h|} \sum_{j,k=1}^{2} |\widehat{D}_{jk}| \ll 1, \quad \frac{1}{h\,|\log h|} \sum_{k=1}^{3} |\widehat{D}_{3k}| \ll 1, \end{gathered} \tag{1.14}$$

where $\alpha, \beta > 0$, $\delta_1, \delta_2 \in \mathbb{R}$ and $\widehat{D} = (\widehat{D}_{jk})_{(j,k)\in\{1,2,3\}^2} \in \mathbb{R}^{3\times 3}$. The parameters $\eta = \eta(h)$, $Q = Q(h)$ and $\widehat{D} = \widehat{D}(h)$ are assumed to be functions in $h$.

#### 1.1.3.1 Gamma-convergence of the micromagnetic energy with Dzyaloshinskii-Moriya interaction in a thin-film regime for boundary vortices

We consider the rescaled energy

$$E_h(m_h) = \frac{\widehat{E}(m_h)}{\ell^3 h^2\,|\log h|} \tag{1.15}$$

for maps $m_h \colon \Omega_h = \Omega \times (0, h) \to \mathbb{S}^2$ with

$$\Delta u_h = \operatorname{div}(m_h \mathbb{1}_{\Omega_h}) \ \text{ in the distributional sense in } \mathbb{R}^3.$$

For every $h > 0$, we define $\widetilde{m}_h \colon \Omega_1 = \Omega \times (0,1) \to \mathbb{S}^2$ and $\widetilde{H}_{\mathrm{ext},h} \colon \mathbb{R}^3 \to \mathbb{R}^3$ to be such that, for every $(x', x_3) \in \Omega_1$,

$$\widetilde{m}_h(x', x_3) = m_h(x', hx_3), \tag{1.16}$$

and

$$\widetilde{H}_{\mathrm{ext},h}(x', x_3) = H_{\mathrm{ext},h}(x', hx_3). \tag{1.17}$$

Moreover, we assume that

$$\left(\frac{\widetilde{H}_{\mathrm{ext},h}}{h\,|\log h|}\right) \text{ converges in } L^1(\Omega_1) \text{ to } \gamma \widetilde{H}_{\mathrm{ext},0}, \tag{1.18}$$

where $\gamma \in \mathbb{R}$ and $\widetilde{H}_{\mathrm{ext},0} \colon \mathbb{R}^3 \to \mathbb{R}^3$ is independent of $x_3$. Setting finally

$$\widetilde{E}_h(\widetilde{m}_h) = E_h(m_h),$$

we have (see Section 1.2.1 for details):

$$
\begin{aligned}
\widetilde{E}_h(\widetilde{m}_h) = &\frac{\eta^2}{h\,|\log h|}\int_{\Omega_1}\left(|\nabla'\widetilde{m}_h|^2 + \frac{1}{h^2}\,|\partial_3\widetilde{m}_h|^2\right)\mathrm{d}x\\
&+\frac{1}{h\,|\log h|}\int_{\Omega_1}\widehat{D}' : \nabla'\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x + \frac{1}{h^2\,|\log h|}\int_{\Omega_1}\widehat{D}_3\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x\\
&+\frac{1}{h^2\,|\log h|}\int_{\mathbb{R}^3}|\nabla u_h|^2\,\mathrm{d}x\\
&+\frac{Q}{h\,|\log h|}\int_{\Omega_1}\Phi(\widetilde{m}_h)\ \mathrm{d}x - \frac{2}{h\,|\log h|}\int_{\Omega_1}\widetilde{H}_{\mathrm{ext},h}\cdot\widetilde{m}_h\ \mathrm{d}x.
\end{aligned}
\tag{1.19}
$$

We prove the following statement of Gamma-convergence for $\widetilde{E}_h$. It justifies that in the regime (1.14)+(1.18), we obtain a reduction from a 3D model to a 2D model by Gamma-convergence.

**Theorem 1.1.3.** *Consider the regime* (1.14)+(1.18)*. Let $\nu$ be the outer unit normal vector on $\partial\Omega_1$ and $\delta = (\delta_1, \delta_2)$. Let $\widetilde{E}_h$ be given at* (1.19)*. Then the following statements hold:*
***(i) Compactness and lower bound.***
*Assume that there exists a constant $C > 0$ such that, for every $h > 0$, $\widetilde{E}_h(\widetilde{m}_h) \leqslant C$. Then, for a subsequence, $(\widetilde{m}_h)$ converges weakly in $H^1$ to $\widetilde{m}_0 \in H^1(\Omega_1, \mathbb{S}^2)$, where $\widetilde{m}_0$ is independent of $x_3$ and satisfies $\widetilde{m}_{0,3} \equiv 0$. Moreover,*

$$
\liminf_{h\to 0}\widetilde{E}_h(\widetilde{m}_h) \geqslant \widetilde{E}_0(\widetilde{m}_0),
$$

*with*

$$
\begin{aligned}
\widetilde{E}_0(\widetilde{m}) = \alpha&\left[\int_{\Omega_1}|\nabla'\widetilde{m}|^2\mathrm{d}x + 2\int_{\Omega_1}\delta\cdot\nabla'\widetilde{m}'\wedge\widetilde{m}'\ \mathrm{d}x\right]\\
&+\frac{1}{2\pi}\int_{\partial\Omega}(\widetilde{m}\cdot\nu)^2\,\mathrm{d}\mathcal{H}^1 + \beta\int_{\Omega_1}\Phi(\widetilde{m})\ \mathrm{d}x - 2\gamma\int_{\Omega_1}\widetilde{H}_{\mathrm{ext},0}\cdot\widetilde{m}\ \mathrm{d}x,
\end{aligned}
\tag{1.20}
$$

*if $\widetilde{m} \in H^1(\Omega_1, \mathbb{S}^2)$ is independent of $x_3$ and satisfies $\widetilde{m}_3 \equiv 0$, and $\widetilde{E}_0(\widetilde{m}) = +\infty$ elsewhere.*
***(ii) Upper bound.***
*Let $\widetilde{m}_0 \in H^1(\Omega_1, \mathbb{S}^1)$ be such that $\widetilde{m}_0$ is independent of $x_3$ and $\widetilde{m}_{0,3} \equiv 0$. Then there exists a sequence $(\widetilde{m}_h)$ that converges strongly to $\widetilde{m}_0$ in $H^1(\Omega_1, \mathbb{S}^1)$ and satisfies*

$$
\lim_{h\to 0}\widetilde{E}_h(\widetilde{m}_h) = \widetilde{E}_0(\widetilde{m}_0),
$$

*where $\widetilde{E}_0$ is given at* (1.20)*.*

The proof of this theorem combines the works of Kohn-Slastikov [32] and Carbou [11], to which we add here the contribution of the anisotropy energy, the external magnetic field, and the Dzyaloshinskii-Moriya interaction.

The three-to-two-dimensions reduction takes sense in the following remark.

*Remark* 1.1.4. We can be more precise concerning the Gamma-limit $\widetilde{E}_0$ of the sequence $(\widetilde{E}_h)$ in Theorem 1.1.3. Since $\Omega_1$ has height 1, $\widetilde{H}_{\mathrm{ext},0}$ is independent of $x_3$ and $\widetilde{E}_0$ is a functional of maps $\widetilde{m} \in H^1(\Omega_1, \mathbb{S}^2)$ such that $\widetilde{m}$ is independent of $x_3$ and $\widetilde{m}_3 \equiv 0$, we have

$$
\begin{aligned}
\widetilde{E}_0(\widetilde{m}) &= \widetilde{E}_0(\widetilde{m}')\\
&= \alpha\left[\int_{\Omega}|\nabla'\widetilde{m}'|^2\,\mathrm{d}x' + 2\int_{\Omega}\delta\cdot\nabla'\widetilde{m}'\wedge\widetilde{m}'\ \mathrm{d}x'\right]\\
&\quad+\frac{1}{2\pi}\int_{\partial\Omega}(\widetilde{m}'\cdot\nu')^2\mathrm{d}\mathcal{H}^1 + \beta\int_{\Omega}\Phi(\widetilde{m}')\ \mathrm{d}x' - 2\gamma\int_{\Omega}\widetilde{H}_{\mathrm{ext},0}\cdot\widetilde{m}'\ \mathrm{d}x',
\end{aligned}
\tag{1.21}
$$

for every $\widetilde{m}' \in H^1(\Omega, \mathbb{S}^1)$, where $\nu'$ denotes here the outer unit normal vector on $\partial\Omega$. As a consequence, Theorem 1.1.3 means a reduction from a three-dimensional model to a two-dimensional model.

**Corollary 1.1.5.** *In the regime* (1.14)+(1.18)*, let $\widetilde{E}_h$ be given at* (1.19)*. For every $h > 0$, $\widetilde{E}_h$ admits a minimizer $\widetilde{m}_h \in H^1(\Omega_1, \mathbb{S}^2)$. Furthermore, for a subsequence, $(\widetilde{m}_h)$ converges weakly in $H^1$ to a minimizer $\widetilde{m}_0 \in H^1(\Omega_1, \mathbb{S}^2)$ of $\widetilde{E}_0$.*

#### 1.1.3.2 On the local minimizers of the Gamma-limit of the micromagnetic energy in the upper-half plane.

We study local minimizers of the Gamma-limit given at (1.21), without anisotropy and external magnetic field. To do so, we analyse critical points of this energy in the upper-half plane $\mathbb{R}^2_+$, and we are led to consider a rescaled version of $\widetilde{E}_0$ denoted by

$$E^\delta_\varepsilon(\varphi; \Omega) = \frac{1}{2} \int_{\Omega\cap\mathbb{R}^2_+} \left( |\nabla\varphi|^2 - 2\delta \cdot \nabla\varphi \right) \mathrm{d}x + \frac{1}{2\varepsilon} \int_{\Omega\cap(\mathbb{R}\times\{0\})} \sin^2\varphi \; \mathrm{d}\mathcal{H}^1, \tag{1.22}$$

where $\delta = (\delta_1, \delta_2)$, $\varepsilon = 2\pi\alpha$, $\Omega$ is a bounded open subset of $\mathbb{R}^2$, and $\varphi\colon \Omega \to \mathbb{R}$ is a lifting of $\widetilde{m}'\colon \Omega \to \mathbb{S}^1$.

**Definition 1.1.6.** A function $\varphi \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ is a critical point of $E^\delta_\varepsilon$ if

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E^\delta_\varepsilon(\varphi + t\psi; \mathrm{Supp}(\psi)) = 0,$$

for every $\psi \in C^1(\mathbb{R}^2)$ with compact support.

**Definition 1.1.7.** A function $\varphi \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ is a local minimizer of $E^\delta_\varepsilon$ in $\mathbb{R}^2_+$ in the sense of De Giorgi if

$$E^\delta_\varepsilon(\varphi; \mathrm{Supp}(\psi)) \leqslant E^\delta_\varepsilon(\varphi + \psi; \mathrm{Supp}(\psi))$$

for every $\psi \in C^1(\mathbb{R}^2)$ with compact support.

Consider the problem

$$\begin{cases} f \in C^\infty(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+}), \\ (x_1, x_2) \mapsto f(x_1, x_2) - \lambda x_2 \text{ is bounded in } \mathbb{R}^2_+, \\ \Delta f = 0 \text{ in } \mathbb{R}^2_+, \\ \partial_2 f - \lambda + \sin f = 0 \text{ on } \mathbb{R}\times\{0\}\,, \end{cases} \tag{PN$_\lambda$}$$

where $\lambda \in \mathbb{R}$ is a constant parameter. This problem is a modified Peierls-Nabarro problem, that has been studied by Amick and Toland ([5], [43]) in the case $\lambda = 0$. Following their work, we show the following statement.

**Theorem 1.1.8.** *Let $\lambda \in \mathbb{R}$. Solutions of* (PN$_\lambda$) *in $\overline{\mathbb{R}^2_+}$ are given by:*

– *for $n \in \mathbb{Z}$, the functions*

$$(x_1, x_2) \mapsto n\pi + \lambda x_2, \tag{1.23}$$

– *for $n \in \mathbb{Z}$ and $\alpha \in (1, 2)$, the $x_1$-periodic (with period $\frac{\pi}{\sigma}$) functions*

$$(x_1, x_2) \mapsto 2n\pi \pm 2\left[\arctan\left(\frac{\tan(\sigma x_1)}{\Gamma_\alpha(x_2)}\right) - \arctan\left(\Gamma_\alpha(x_2)\tan(\sigma x_1)\right)\right] + \lambda x_2, \tag{1.24}$$

*defined for* $x_1 \in \mathbb{R} \setminus \left(\frac{\pi}{2\sigma} + \frac{\pi}{\sigma}\mathbb{Z}\right)$ *and extended by* $(x_1, x_2) \mapsto 2n\pi + \lambda x_2$ *elsewhere, and every translation in the variable* $x_1$ *of this type of functions, where* $\sigma$ *and* $\Gamma_\alpha$ *are given by*

$$\sigma = \frac{1}{2}\sqrt{\alpha(2-\alpha)}, \quad \Gamma_\alpha(x_2) = \frac{\gamma + \tanh(\sigma x_2)}{1 + \gamma \tanh(\sigma x_2)} \quad \textit{with} \quad \gamma = \frac{\alpha}{2\sigma},$$

– *for* $n \in \mathbb{Z}$*, the non-periodic functions*

$$(x_1, x_2) \mapsto 2n\pi \pm 2\arctan\left(\frac{x_1}{1+x_2}\right) + \lambda x_2 \tag{1.25}$$

*and every translation in the variable* $x_1$ *of this type of functions.*

In Proposition 1.3.2 below, we show that if $\varphi_\varepsilon$ is a critical point of $E_\varepsilon^\delta$, then the rescaled function

$$\phi_\varepsilon \colon (x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto 2\varphi_\varepsilon(\varepsilon x_1, \varepsilon x_2) + \pi,$$

with the assumption that $(x_1, x_2) \mapsto \phi_\varepsilon(x_1, x_2) - \lambda_\varepsilon x_2$ is bounded in $\mathbb{R}^2_+$, satisfies the problem $(\mathrm{PN}_{\lambda_\varepsilon})$ in the case $\lambda_\varepsilon = 2\varepsilon\delta_2$. Coming back to local minimizers of $E_\varepsilon^\delta$ in $\mathbb{R}^2_+$ in the sense of De Giorgi, we finally get the following statement.

**Theorem 1.1.9.** *For* $a \in \mathbb{R}$*, the functions*

$$\varphi_\varepsilon \colon (x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto \frac{\pi}{2} - \arctan\left(\frac{x_1 + \varepsilon a}{x_2 + \varepsilon}\right) + \delta_2 x_2 \tag{1.26}$$

*are the only local minimizers of* $E_\varepsilon^\delta$ *in* $\mathbb{R}^2_+$ *in the sense of De Giorgi such that*

$$\lim_{x_1 \to +\infty} \varphi_\varepsilon(x_1, 0) = 0, \ \lim_{x_1 \to -\infty} \varphi_\varepsilon(x_1, 0) = \pi \ \textit{and} \ [(x_1, x_2) \mapsto \varphi_\varepsilon(x_1, x_2) - \delta_2 x_2] \in L^\infty(\mathbb{R}^2_+). \tag{1.27}$$

## 1.2 Gamma-convergence of the micromagnetic energy with Dzyaloshinskii-Moriya interaction in a thin-film regime for boundary vortices

### 1.2.1 The rescaled energy $\widetilde{E}_h$

Let $\Omega_h = \Omega \times (0, h)$ with $\Omega \subset \mathbb{R}^2$ a smooth bounded open set of typical length 1. For a magnetization $m_h \colon \Omega_h \to \mathbb{S}^2$ that satisfies

$$\Delta u_h = \operatorname{div}(m_h \mathbb{1}_{\Omega_h}) \ \text{ in the distributional sense in } \mathbb{R}^3,$$

we define $E_h(m_h)$ as in (1.15), using (1.5), as

$$\begin{aligned} E_h(m_h) = {} & \frac{\eta^2}{h\,|\log h|}\int_{\Omega_h} |\nabla m_h|^2 \,\mathrm{d}x + \frac{1}{h^2\,|\log h|}\int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \,\mathrm{d}x \\ & + \frac{1}{h^2\,|\log h|}\int_{\mathbb{R}^3} |\nabla u_h|^2 \,\mathrm{d}x \\ & + \frac{Q}{h^2\,|\log h|}\int_{\Omega_h} \Phi(m_h)\,\mathrm{d}x - \frac{2}{h^2\,|\log h|}\int_{\Omega_h} H_{\mathrm{ext},h} \cdot m_h \,\mathrm{d}x, \end{aligned} \tag{1.28}$$

where $\Phi \colon \mathbb{S}^2 \to \mathbb{R}_+$, $H_{\mathrm{ext},h} \colon \mathbb{R}^3 \to \mathbb{R}^3$ and $\widehat{D} \in \mathbb{R}^{3\times 3}$.

The energy (1.28) has been studied by Kohn and Slastikov in [32] by considering only the exchange and magnetostatic terms. If it seems clear that the exchange energy is one of the leading-order terms in the asymptotic regime (1.14)+(1.18), the work of Kohn and Slastikov shows that the limit model also keeps a contribution from the magnetostatic energy.

In order to make the energy $E_h$ easier to study, we remove the dependence on $h$ in the bounds of the involved integrals by a new rescaling. More precisely, for any $h > 0$, we set $\widetilde{m}_h \colon \Omega_1 \to \mathbb{S}^2$ and $\widetilde{H}_{\mathrm{ext},h} \colon \mathbb{R}^3 \to \mathbb{R}^3$ to be such that, for every $(x', x_3) \in \Omega_1 = \Omega \times (0,1)$,

$$\widetilde{m}_h(x', x_3) = m_h(x', hx_3) \quad \text{and} \quad \widetilde{H}_{\mathrm{ext},h}(x', x_3) = H_{\mathrm{ext},h}(x', hx_3).$$

We then have

$$\begin{aligned}
\int_{\Omega_h} |\nabla m_h|^2 \mathrm{d}x &= \int_\Omega \int_0^h \left|\nabla \left(m_h(x', x_3)\right)\right|^2 \mathrm{d}x' \mathrm{d}x_3 \\
&= h \int_\Omega \int_0^1 \left|\nabla \left(m_h(x', h\widetilde{x}_3)\right)\right|^2 \mathrm{d}x' \mathrm{d}\widetilde{x}_3 \\
&= h \int_\Omega \int_0^1 \left( \left|\nabla' \widetilde{m}_h(x', \widetilde{x}_3)\right|^2 + \frac{1}{h^2} \left|\partial_3 \widetilde{m}_h(x', \widetilde{x}_3)\right|^2 \right) \mathrm{d}x' \mathrm{d}\widetilde{x}_3 \\
&= h \int_{\Omega_1} \left( |\nabla' \widetilde{m}_h|^2 + \frac{1}{h^2} \left|\partial_3 \widetilde{m}_h\right|^2 \right) \mathrm{d}x,
\end{aligned}$$

by the change of variable $x_3 = h\widetilde{x}_3$ and using that $\partial_3 m_h(x', h\widetilde{x}_3) = \frac{1}{h}\partial_3 \widetilde{m}_h(x', \widetilde{x}_3)$. By the same change of variable,

$$\int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \, \mathrm{d}x = h \int_{\Omega_1} \widehat{D}' : \nabla' \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x + \int_{\Omega_1} \widehat{D}_3 \cdot \partial_3 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x,$$

where $\widehat{D} = (\widehat{D}', \widehat{D}_3)$,

$$\int_{\Omega_h} \Phi(m_h) \, \mathrm{d}x = h \int_{\Omega_1} \Phi(\widetilde{m}_h) \, \mathrm{d}x, \quad \text{and} \quad \int_{\Omega_h} H_{\mathrm{ext},h} \cdot m_h \, \mathrm{d}x = h \int_{\Omega_1} \widetilde{H}_{\mathrm{ext},h} \cdot \widetilde{m}_h \, \mathrm{d}x.$$

Setting $\widetilde{E}_h(\widetilde{m}_h) = E_h(m_h)$, we eventually get

$$\begin{aligned}
\widetilde{E}_h(\widetilde{m}_h) = & \frac{\eta^2}{h\,|\log h|} \int_{\Omega_1} \left( |\nabla' \widetilde{m}_h|^2 + \frac{1}{h^2} \left|\partial_3 \widetilde{m}_h\right|^2 \right) \mathrm{d}x \\
&+ \frac{1}{h\,|\log h|} \int_{\Omega_1} \widehat{D}' : \nabla' \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x \\
&+ \frac{1}{h^2\,|\log h|} \int_{\Omega_1} \widehat{D}_3 \cdot \partial_3 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x \\
&+ \frac{1}{h^2\,|\log h|} \int_{\mathbb{R}^3} \left|\nabla u_h\right|^2 \mathrm{d}x \\
&+ \frac{Q}{h\,|\log h|} \int_{\Omega_1} \Phi(\widetilde{m}_h) \, \mathrm{d}x - \frac{2}{h\,|\log h|} \int_{\Omega_1} \widetilde{H}_{\mathrm{ext},h} \cdot \widetilde{m}_h \, \mathrm{d}x
\end{aligned}$$

which is (1.19). The rest of this section is devoted to prove Theorem 1.1.3.

### 1.2.2 Coercivity

In this section, we provide a statement of coercivity concerning the energy $\widetilde{E}_h$. We begin with giving basic estimates for the Dzyaloshinskii-Moriya interaction energy, that will be useful in the sequel.

**Lemma 1.2.1.** *We have*

$$\left|\int_{\Omega_1} \widehat{D}_1 \cdot \partial_1 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x\right| \leqslant |\widehat{D}_{13}| \int_{\Omega_1} |\partial_1 \widetilde{m}_h' \wedge \widetilde{m}_h'| \, \mathrm{d}x + \left(|\widehat{D}_{11}| + |\widehat{D}_{12}|\right) \int_{\Omega_1} \left(1 + |\partial_1 \widetilde{m}_h|^2\right) \mathrm{d}x, \tag{1.29}$$

*and*

$$\left|\int_{\Omega_1} \widehat{D}_2 \cdot \partial_2 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x\right| \leqslant |\widehat{D}_{23}| \int_{\Omega_1} |\partial_2 \widetilde{m}_h' \wedge \widetilde{m}_h'| \, \mathrm{d}x + \left(|\widehat{D}_{21}| + |\widehat{D}_{22}|\right) \int_{\Omega_1} \left(1 + |\partial_2 \widetilde{m}_h|^2\right) \mathrm{d}x. \tag{1.30}$$

*Proof.* We denote by $(e_1, e_2, e_3)$ the standard orthonormal basis in $\mathbb{R}^3$.
By expanding $\widehat{D}_1$ as $\widehat{D}_1 = \widehat{D}_{11} e_1 + \widehat{D}_{12} e_2 + \widehat{D}_{13} e_3$, we have

$$\begin{aligned}\int_{\Omega_1} \widehat{D}_1 \cdot \partial_1 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x = &\int_{\Omega_1} \widehat{D}_{11} e_1 \cdot \partial_1 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x \\ &+ \int_{\Omega_1} \widehat{D}_{12} e_2 \cdot \partial_1 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x \\ &+ \int_{\Omega_1} \widehat{D}_{13} e_3 \cdot \partial_1 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x,\end{aligned} \tag{1.31}$$

and it is clear that

$$\int_{\Omega_1} \widehat{D}_{13} e_3 \cdot \partial_1 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x = \widehat{D}_{13} \int_{\Omega_1} \partial_1 \widetilde{m}_h' \wedge \widetilde{m}_h' \, \mathrm{d}x. \tag{1.32}$$

Hence, it suffices to find an upper bound for the absolute value of the integrals involving $\widehat{D}_{11}$ and $\widehat{D}_{12}$. For the first integral, note that

$$\int_{\Omega_1} \widehat{D}_{11} e_1 \cdot \partial_1 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x = \widehat{D}_{11} \left( \int_{\Omega_1} \widetilde{m}_{h,3} \partial_1 \widetilde{m}_{h,2} \, \mathrm{d}x - \int_{\Omega_1} \widetilde{m}_{h,2} \partial_1 \widetilde{m}_{h,3} \, \mathrm{d}x \right).$$

By Young's inequality,

$$\widetilde{m}_{h,3} \partial_1 \widetilde{m}_{h,2} \leqslant \frac{1}{2} \left( |\widetilde{m}_{h,3}|^2 + |\partial_1 \widetilde{m}_{h,2}|^2 \right) \leqslant \frac{1}{2} \left( 1 + |\partial_1 \widetilde{m}_h|^2 \right),$$

and similarly,

$$\widetilde{m}_{h,2} \partial_1 \widetilde{m}_{h,3} \leqslant \frac{1}{2} \left( |\widetilde{m}_{h,2}|^2 + |\partial_1 \widetilde{m}_{h,3}|^2 \right) \leqslant \frac{1}{2} \left( 1 + |\partial_1 \widetilde{m}_h|^2 \right).$$

We deduce that

$$\left|\int_{\Omega_1} \widehat{D}_{11} e_1 \cdot \partial_1 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x\right| \leqslant |\widehat{D}_{11}| \int_{\Omega_1} \left(1 + |\partial_1 \widetilde{m}_h|^2\right) \mathrm{d}x.$$

By the same arguments, we have

$$\left|\int_{\Omega_1} \widehat{D}_{12} e_2 \cdot \partial_1 \widetilde{m}_h \wedge \widetilde{m}_h \, \mathrm{d}x\right| \leqslant |\widehat{D}_{12}| \int_{\Omega_1} \left(1 + |\partial_1 \widetilde{m}_h|^2\right) \mathrm{d}x.$$

Combining the above inequalities with (1.31) and (1.32), we deduce (1.29). The proof of (1.30) is similar. □

**Lemma 1.2.2.** *We have*

$$\left|\frac{1}{h}\int_{\Omega_1} \widehat{D}_3\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x\right| \leqslant \left(|\widehat{D}_{31}|+|\widehat{D}_{32}|+\frac{|\widehat{D}_{33}|}{2}\right)\int_{\Omega_1}\left(1+\frac{1}{h^2}\left|\partial_3\widetilde{m}_h\right|^2\right)\mathrm{d}x. \tag{1.33}$$

*Proof.* We denote by $(e_1, e_2, e_3)$ the standard orthonormal basis in $\mathbb{R}^3$.
By expanding $\widehat{D}_3$ as $\widehat{D}_3 = \widehat{D}_{31}e_1 + \widehat{D}_{32}e_2 + \widehat{D}_{33}e_3$, we have

$$\begin{aligned}\frac{1}{h}\int_{\Omega_1} \widehat{D}_3\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x = &\frac{1}{h}\int_{\Omega_1} \widehat{D}_{31}e_1\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x\\ &+\frac{1}{h}\int_{\Omega_1} \widehat{D}_{32}e_2\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x\\ &+\frac{1}{h}\int_{\Omega_1} \widehat{D}_{33}e_3\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x.\end{aligned}$$

For the first integral in the right-hand side above, note that

$$\frac{1}{h}\int_{\Omega_1} \widehat{D}_{31}e_1\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x = \frac{\widehat{D}_{31}}{h}\left(\int_{\Omega_1}\widetilde{m}_{h,3}\partial_3\widetilde{m}_{h,2}\ \mathrm{d}x - \int_{\Omega_1}\widetilde{m}_{h,2}\partial_3\widetilde{m}_{h,3}\ \mathrm{d}x\right).$$

By Young's inequality,

$$\frac{1}{h}\widetilde{m}_{h,3}\partial_3\widetilde{m}_{h,2} \leqslant \frac{1}{2}\left(\left|\widetilde{m}_{h,3}\right|^2+\frac{1}{h^2}\left|\partial_3\widetilde{m}_{h,2}\right|^2\right)\leqslant\frac{1}{2}\left(1+\frac{1}{h^2}\left|\partial_3\widetilde{m}_h\right|^2\right),$$

and similarly,

$$\frac{1}{h}\widetilde{m}_{h,2}\partial_3\widetilde{m}_{h,3} \leqslant \frac{1}{2}\left(\left|\widetilde{m}_{h,2}\right|^2+\frac{1}{h^2}\left|\partial_3\widetilde{m}_{h,3}\right|^2\right)\leqslant\frac{1}{2}\left(1+\frac{1}{h^2}\left|\partial_3\widetilde{m}_h\right|^2\right).$$

We deduce that

$$\left|\frac{1}{h}\int_{\Omega_1} \widehat{D}_{31}e_1\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x\right|\leqslant|\widehat{D}_{31}|\int_{\Omega_1}\left(1+\frac{1}{h^2}\left|\partial_3\widetilde{m}_h\right|^2\right)\mathrm{d}x.$$

By the same arguments, we have

$$\left|\frac{1}{h}\int_{\Omega_1} \widehat{D}_{32}e_2\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x\right|\leqslant|\widehat{D}_{32}|\int_{\Omega_1}\left(1+\frac{1}{h^2}\left|\partial_3\widetilde{m}_h\right|^2\right)\mathrm{d}x.$$

Furthermore, we have

$$\frac{1}{h}\int_{\Omega_1} \widehat{D}_{33}e_3\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x = \widehat{D}_{33}\int_{\Omega_1}\frac{1}{h}\partial_3\widetilde{m}_h'\wedge\widetilde{m}_h'\ \mathrm{d}x,$$

and by Young's inequality,

$$\begin{aligned}\frac{1}{h}\partial_3\widetilde{m}_h'\wedge\widetilde{m}_h' = \frac{1}{h}\partial_3\widetilde{m}_h'\cdot(\widetilde{m}_h')^\perp &\leqslant \frac{1}{2}\left(\frac{1}{h^2}\left|\partial_3\widetilde{m}_h'\right|^2+\left|\widetilde{m}_h'\right|^2\right)\\ &\leqslant\frac{1}{2}\left(\frac{1}{h^2}\left|\partial_3\widetilde{m}_h\right|^2+1\right).\end{aligned}$$

Consequently,

$$\left|\frac{1}{h}\int_{\Omega_1} \widehat{D}_{33}e_3\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x\right| \leqslant \frac{|\widehat{D}_{33}|}{2}\int_{\Omega_1}\left(1+\frac{1}{h^2}|\partial_3\widetilde{m}_h|^2\right)\mathrm{d}x,$$

and the upper bound (1.33) follows. □

*Notation* 1.2.3. In the following, we denote from (1.19):

$$\widetilde{E}_h(\widetilde{m}_h) = \widetilde{E}_h^{(0)}(\widetilde{m}_h) + \widetilde{E}_h^{(1)}(\widetilde{m}_h) + \widetilde{E}_h^{(2)}(\widetilde{m}_h), \tag{1.34}$$

where

$$\begin{aligned}\widetilde{E}_h^{(0)}(\widetilde{m}_h) = \frac{\eta^2}{h\,|\log h|}&\int_{\Omega_1}\left(|\nabla'\widetilde{m}_h|^2+\frac{1}{h^2}|\partial_3\widetilde{m}_h|^2\right)\mathrm{d}x\\ &+\frac{1}{h^2\,|\log h|}\int_{\mathbb{R}^3}|\nabla u_h|^2\,\mathrm{d}x+\frac{Q}{h\,|\log h|}\int_{\Omega_1}\Phi(\widetilde{m}_h)\ \mathrm{d}x,\end{aligned} \tag{1.35}$$

$$\widetilde{E}_h^{(1)}(\widetilde{m}_h) = -\frac{2}{h\,|\log h|}\int_{\Omega_1}\widetilde{H}_{\mathrm{ext},h}\cdot\widetilde{m}_h\ \mathrm{d}x, \tag{1.36}$$

and

$$\begin{aligned}\widetilde{E}_h^{(2)}(\widetilde{m}_h) = \frac{1}{h\,|\log h|}&\int_{\Omega_1}\widehat{D}' : \nabla'\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x\\ &+\frac{1}{h^2\,|\log h|}\int_{\Omega_1}\widehat{D}_3\cdot\partial_3\widetilde{m}_h\wedge\widetilde{m}_h\ \mathrm{d}x.\end{aligned} \tag{1.37}$$

**Proposition 1.2.4** (Coercivity). *In the regime* (1.14)+(1.18)*, there exist constants* $h_0 > 0$ *and* $C > 0$ *such that, for every* $h \in (0, h_0)$*,*

$$\widetilde{E}_h(\widetilde{m}_h) \geqslant \frac{1}{2}\widetilde{E}_h^{(0)}(\widetilde{m}_h) - C. \tag{1.38}$$

*Proof.* Let $h > 0$. We use the decomposition (1.34) of $\widetilde{E}_h(\widetilde{m}_h)$. We clearly have $\widetilde{E}_h^{(0)}(\widetilde{m}_h) \geqslant 0$.

The strategy for the two remaining terms is the following. On the one hand, the energy $\widetilde{E}_h^{(1)}(\widetilde{m}_h)$ being bounded, we will absorb it in the constant $C$. On the other hand, we will distribute the contribution of the energy $\widetilde{E}_h^{(2)}(\widetilde{m}_h)$ in the energy $\widetilde{E}_h^{(0)}(\widetilde{m}_h)$ and in the constant $C$, using Lemma 1.2.1 and Lemma 1.2.2. Another example of absorbing the DMI into other terms of the micromagnetic energy can be found in [12]. In that paper, Ignat and Côte absorb the DMI into the exchange and anisotropy energies, in order to prove coercivity and then a Gamma-convergence result.

Using Hölder's inequality with $|\widetilde{m}_h| = 1$ in $\Omega_1$, we have

$$|\widetilde{E}_h^{(1)}(\widetilde{m}_h)| = \left|\frac{2}{h\,|\log h|}\int_{\Omega_1}\widetilde{H}_{\mathrm{ext},h}\cdot\widetilde{m}_h\ \mathrm{d}x\right| \leqslant 2\left\|\frac{\widetilde{H}_{\mathrm{ext},h}}{h\,|\log h|}\right\|_{L^1(\Omega_1)}.$$

Furthermore, in the regime (1.14)+(1.18), $\left\|\frac{\widetilde{H}_{\mathrm{ext},h}}{h|\log h|}\right\|_{L^1(\Omega_1)} \to |\gamma|\,\|\widetilde{H}_{\mathrm{ext},0}\|_{L^1(\Omega_1)}$, hence there exists a constant $C(\gamma) > 0$ such that, for $h > 0$ sufficiently small, we have $|\widetilde{E}_h^{(1)}(\widetilde{m}_h)| \leqslant C(\gamma)$.

By Lemma 1.2.1 and Lemma 1.2.2, we have

$$
\begin{aligned}
|\widetilde{E}_h^{(2)}(\widetilde{m}_h)| \leqslant \frac{1}{h\,|\log h|} \Bigg[ & \sum_{j,k=1}^{2} |\widehat{D}_{jk}| \int_{\Omega_1} \left(1 + |\nabla' \widetilde{m}_h|^2\right) \mathrm{d}x \\
& + |\widehat{D}_{13}| \int_{\Omega_1} |\partial_1 \widetilde{m}_h' \wedge \widetilde{m}_h'| \,\mathrm{d}x \\
& + |\widehat{D}_{23}| \int_{\Omega_1} |\partial_2 \widetilde{m}_h' \wedge \widetilde{m}_h'| \,\mathrm{d}x \\
& + \sum_{k=1}^{3} |\widehat{D}_{3k}| \int_{\Omega_1} \left(1 + \frac{1}{h^2} |\partial_3 \widetilde{m}_h|^2\right) \mathrm{d}x \Bigg].
\end{aligned}
$$

Let $\varepsilon > 0$. By Young's inequality,

$$
\begin{aligned}
\frac{|\widehat{D}_{13}|}{\eta^2} \int_{\Omega_1} |\partial_1 \widetilde{m}_h' \wedge \widetilde{m}_h'| \,\mathrm{d}x &\leqslant \varepsilon \int_{\Omega_1} |\partial_1 \widetilde{m}_h'|^2 \,\mathrm{d}x + \frac{1}{4\varepsilon} \left(\frac{|\widehat{D}_{13}|}{\eta^2}\right)^2 \int_{\Omega_1} |\widetilde{m}_h'|^2 \,\mathrm{d}x \\
&\leqslant \varepsilon \int_{\Omega_1} |\partial_1 \widetilde{m}_h'|^2 \,\mathrm{d}x + \frac{1}{4\varepsilon} \left(\frac{|\widehat{D}_{13}|}{\eta^2}\right)^2 |\Omega_1| \,,
\end{aligned}
$$

since $|\widetilde{m}_h'| \leqslant |\widetilde{m}_h| = 1$. Using the same arguments,

$$
\frac{|\widehat{D}_{23}|}{\eta^2} \int_{\Omega_1} |\partial_2 \widetilde{m}_h' \wedge \widetilde{m}_h'| \,\mathrm{d}x \leqslant \varepsilon \int_{\Omega_1} |\partial_2 \widetilde{m}_h'|^2 \,\mathrm{d}x + \frac{1}{4\varepsilon} \left(\frac{|\widehat{D}_{23}|}{\eta^2}\right)^2 |\Omega_1| \,.
$$

Furthermore, we note that, in the regime (1.14)+(1.18),

$$
\frac{|\widehat{D}_{13}|}{\eta^2} \to 2\delta_1, \quad \frac{|\widehat{D}_{23}|}{\eta^2} \to 2\delta_2, \quad \frac{1}{h\,|\log h|} \sum_{j,k=1}^{2} |\widehat{D}_{jk}| \ll 1, \quad \frac{1}{h\,|\log h|} \sum_{k=1}^{3} |\widehat{D}_{3k}| \ll 1.
$$

We deduce that, for $h > 0$ sufficiently small,

$$
\begin{aligned}
|\widetilde{E}_h^{(2)}(\widetilde{m}_h)| \leqslant{}& o_h(1) \int_{\Omega_1} \left(1 + |\nabla' \widetilde{m}_h|^2\right) \mathrm{d}x + \frac{\eta^2 \varepsilon}{h\,|\log h|} \int_{\Omega_1} \left(|\partial_1 \widetilde{m}_h'|^2 + |\partial_2 \widetilde{m}_h'|^2\right) \mathrm{d}x \\
&+ \frac{\eta^2}{h\,|\log h|} \frac{|\Omega_1|}{\varepsilon} \left(\delta_1^2 + \delta_2^2 + 1\right) + o_h(1) \int_{\Omega_1} \left(1 + \frac{1}{h^2} |\partial_3 \widetilde{m}_h|^2\right) \mathrm{d}x \\
\leqslant{}& \left(\frac{\eta^2 \varepsilon}{h\,|\log h|} + o_h(1)\right) \int_{\Omega_1} \left(|\nabla' \widetilde{m}_h|^2 + \frac{1}{h^2} |\partial_3 \widetilde{m}_h|^2\right) \mathrm{d}x \\
&+ \frac{\eta^2}{h\,|\log h|} \frac{|\Omega_1|}{\varepsilon} \left(\delta_1^2 + \delta_2^2 + 1\right) + o_h(1).
\end{aligned}
$$

We eventually combine our estimates on $\widetilde{E}_h^{(1)}(\widetilde{m}_h)$ and $\widetilde{E}_h^{(2)}(\widetilde{m}_h)$: for $h > 0$ sufficiently small, we have

$$
\begin{aligned}
\widetilde{E}_h(\widetilde{m}_h) \geqslant{}& \widetilde{E}_h^{(0)}(\widetilde{m}_h) - C(\gamma) \\
&- \left(\frac{\eta^2}{h\,|\log h|} \varepsilon + o_h(1)\right) \int_{\Omega_1} \left(|\nabla' \widetilde{m}_h|^2 + \frac{1}{h^2} |\partial_3 \widetilde{m}_h|^2\right) \mathrm{d}x \\
&- \frac{\eta^2}{h\,|\log h|} \frac{|\Omega_1|}{\varepsilon} \left(\delta_1^2 + \delta_2^2 + 1\right) - o_h(1).
\end{aligned}
$$

Since $\frac{\eta^2}{h|\log h|} \to \alpha$ in the regime (1.14)+(1.18), then choosing $\varepsilon > 0$ so small that $1-\varepsilon-o_h(1) \geqslant \frac{3}{4}$ and setting $C(\alpha,\delta_1,\delta_2) = (\alpha+1)\,|\Omega_1|\,(\delta_1^2+\delta_2^2+1)+1$, we get

$$\widetilde{E}_h(\widetilde{m}_h) \geqslant \frac{1}{2}\widetilde{E}_h^{(0)}(\widetilde{m}_h) - C(\gamma) - \frac{1}{\varepsilon}C(\alpha,\delta_1,\delta_2).$$

Setting $C = C(\gamma) + \frac{1}{\varepsilon}C(\alpha,\delta_1,\delta_2)$, we get the expected estimate. □

### 1.2.3 Gamma-convergence

As mentioned in the introduction, the stray-field energy in $\widetilde{E}_h(\widetilde{m}_h)$ has been studied by Kohn and Slastikov [32] in the regime we are considering. We cite [32, Lemma 4] and the estimate (33) from [32]:

**Theorem 1.2.5.** *For $h > 0$, set*

$$I(h) = \int_0^h \int_{\partial\Omega} \int_0^h \int_{\partial\Omega} \frac{(\overline{m}_h\cdot\nu)\,(x')\,(\overline{m}_h\cdot\nu)\,(y')}{\sqrt{|x'-y'|^2+(s-t)^2}} \mathrm{d}x'\mathrm{d}s\mathrm{d}y'\mathrm{d}t, \tag{1.39}$$

*where $\nu(x')$ is the outer unit normal vector at a point $x' \in \partial\Omega$. If $(\overline{m}_h)$ converges weakly to $\overline{m}_0$ in $H^1(\Omega)$, then*

$$\lim_{h\to 0} \frac{I(h)}{h^2\,|\log h|} = 2\int_{\partial\Omega} (\overline{m}_0\cdot\nu)^2\,\mathrm{d}\mathcal{H}^1. \tag{1.40}$$

In particular, the constant 2 in the above limit is computed by using an integral operator in [11].

**Theorem 1.2.6.** *For $h > 0$, consider $g_h$ as in* (1.9)*, $m_h\colon \Omega_h \to \mathbb{S}^2$, $u_h$ as in* (1.4)*, $\overline{m}_h$ as in* (1.12) *and $\widetilde{m}_h\colon \Omega_1 \to \mathbb{S}^2$ satisfying* (1.16)*. We have*

$$\begin{aligned}
&\frac{1}{h^2\,|\log h|}\int_{\mathbb{R}^3} |\nabla u_h|^2\,\mathrm{d}x \\
&\qquad = \frac{1}{h\,|\log h|}\int_{\mathbb{R}^2} |\mathcal{F}(\overline{m}_h\cdot e_3)(\xi')|^2\, g_h(|\xi'|)\mathrm{d}\xi' \\
&\qquad\quad + \frac{I(h)}{4\pi h^2\,|\log h|} \\
&\qquad\quad + \left(1 + \left\|\mathrm{div}'(\widetilde{m}_h)\right\|^2_{L^2(\Omega_1)} + \frac{1}{h^2}\left\|\partial_3\widetilde{m}_h\right\|^2_{L^2(\Omega_1)}\right) O\left(\frac{1}{|\log h|}\right) \quad \text{as } h\to 0,
\end{aligned} \tag{1.41}$$

*where $I(h)$ is defined in* (1.39)*, $\mathcal{F}$ denotes the Fourier transformation and $e_3$ is the third unit vector of the standard orthonormal basis in $\mathbb{R}^3$.*

Both previous theorems will be useful for proving Theorem 1.1.3. We now present and prove three theorems (compactness, lower bound and upper bound) on the Gamma-convergence of $\widetilde{E}_h$ that will lead to Theorem 1.1.3.

**Theorem 1.2.7** (Compactness)**.** *Consider the regime* (1.14)+(1.18)*. Assume that there exists a constant $C > 0$ such that, for every $h > 0$, $\widetilde{E}_h(\widetilde{m}_h) \leqslant C$. Then, for a subsequence, $(\widetilde{m}_h)$ converges weakly to $\widetilde{m}_0$ in $H^1$, where $\widetilde{m}_0 \in H^1(\Omega_1,\mathbb{S}^2)$ is independent of $x_3$ and satisfies $\widetilde{m}_{0,3} \equiv 0$.*

*Proof.* By Proposition 1.2.4 and since we assumed that $\widetilde{E}_h(\widetilde{m}_h)$ is bounded, there exist $h_0 > 0$ and $C_0 > 0$ such that, for every $h \in (0,h_0)$, $\widetilde{E}_h^{(0)}(\widetilde{m}_h) \leqslant C_0$. In particular, for every $h \in (0,h_0)$, we have

$$\int_{\Omega_1} |\nabla'\widetilde{m}_h|^2\,\mathrm{d}x \leqslant \frac{h\,|\log h|}{\eta^2}C_0,$$

and

$$\int_{\Omega_1} |\partial_3 \widetilde{m}_h|^2 \, \mathrm{d}x \leqslant h^2 \cdot \frac{h\,|\log h|}{\eta^2} C_0. \tag{1.42}$$

In the regime (1.14)+(1.18), $\lim_{h\to 0} \frac{h|\log h|}{\eta^2} = \alpha$ and $\lim_{h\to 0} h^2 \cdot \frac{h|\log h|}{\eta^2} = 0$. Hence, $(\nabla' \widetilde{m}_h)$ is bounded in $L^2$. Moreover, $|\widetilde{m}_h| = 1$ for every $h > 0$, thus the sequence $(\widetilde{m}_h)$ is bounded in $H^1$. By the Banach-Alaoglu theorem ([42, Theorem 3.15]), for a subsequence, $(\widetilde{m}_h)$ converges weakly to $\widetilde{m}_0$ in $H^1$, for some $\widetilde{m}_0 \in H^1(\Omega_1, \mathbb{R}^3)$.

It remains to show the stated properties of $\widetilde{m}_0$. By the Rellich-Kondrachov compactness theorem ([18, Section 5.7]), up to taking a further subsequence, we can assume that $(\widetilde{m}_h)$ converges strongly to $\widetilde{m}_0$ in $L^2$ and almost everywhere in$\Omega_1$. In particular, $|\widetilde{m}_0| = 1$ almost everywhere in $\Omega_1$. By (1.42), $(\partial_3 \widetilde{m}_h)$ tends to zero in $L^2$, but we also know that $(\partial_3 \widetilde{m}_h)$ converges to $\partial_3 \widetilde{m}_0$ weakly in $L^2$. By uniqueness of the weak limit, we have $\partial_3 \widetilde{m}_0 \equiv 0$ in $L^2$. It follows that $\widetilde{m}_0$ is independent of $x_3$. By Proposition 1.2.4 and since we assumed that $\widetilde{E}_h(\widetilde{m}_h)$ is bounded, there exist $h_1 > 0$ and $C_1 > 0$ such that, for every $h \in (0, h_1)$,

$$\frac{1}{h^2\,|\log h|} \int_{\mathbb{R}^2} |\nabla u_h|^2 \, \mathrm{d}x \leqslant C_1.$$

Using Remark 1.1.1 and Theorem 1.2.6, it follows that, for every $h \in (0, h_1)$,

$$0 \leqslant \int_{\mathbb{R}^2} |\mathcal{F}(\overline{m}_h \cdot e_3)(\xi')|^2 g_h(|\xi'|) \, \mathrm{d}\xi' \leqslant \left( C_1 - \frac{I(h)}{4\pi h^2\,|\log h|} \right) h\,|\log h|\,.$$

By (1.16), we have $\overline{\widetilde{m}}_h = \overline{m}_h$. By weak convergence of $(\widetilde{m}_h)$ to $\widetilde{m}_0$ in $H^1$, by Fubini's theorem, and since $\widetilde{m}_0$ is independent of $x_3$, then $(\overline{m}_h)$ converges weakly to $\widetilde{m}_0$ in $H^1$. Hence, we can use Theorem 1.2.5 and we get

$$\lim_{h\to 0} \left( C_1 - \frac{I(h)}{4\pi h^2\,|\log h|} \right) = C_1 - \frac{1}{2\pi} \int_{\partial\Omega} (\widetilde{m}_0 \cdot \nu)^2 \mathrm{d}\mathcal{H}^1 < +\infty,$$

since $|\widetilde{m}_0| = 1$ almost everywhere, where $\nu$ is the outer unit normal vector on $\partial\Omega$. We deduce that

$$\lim_{h\to 0} \int_{\mathbb{R}^2} |\mathcal{F}(\overline{m}_h \cdot e_3)(\xi')|^2 g_h(|\xi'|) \, \mathrm{d}\xi' = 0.$$

Since $m_h \equiv 0$ in $\mathbb{R}^3 \setminus \Omega_1$, then for every $h > 0$,

$$\int_{\mathbb{R}^2} |\mathcal{F}(\overline{m}_h \cdot e_3)(\xi')|^2 g_h(|\xi'|) \, \mathrm{d}\xi' = \int_{\mathbb{R}^2} |\mathcal{F}(\overline{\widetilde{m}}_{h,3} \mathbb{1}_\Omega)(\xi')|^2 g_h(|\xi'|) \, \mathrm{d}\xi'.$$

Since $(g_h)$ converges almost everywhere to 1 in $\mathbb{R}^2$ and $g_h(|\xi'|)$ is bounded for almost every $\xi' \in \mathbb{R}^2$ (see Remark 1.1.1), and $(\mathcal{F}(\overline{\widetilde{m}}_{h,3}))$ is bounded and converges almost everywhere to $\mathcal{F}(\overline{\widetilde{m}}_{0,3})$ in $\Omega$, we deduce from the dominated convergence theorem and the two above relations that

$$\int_{\mathbb{R}^2} |\mathcal{F}(\overline{\widetilde{m}}_{0,3} \mathbb{1}_\Omega)(\xi')|^2 \mathrm{d}\xi' = 0.$$

By Plancherel's formula, we get

$$\int_{\Omega} |\overline{\widetilde{m}}_{0,3}(x')|^2 \mathrm{d}x' = \int_{\mathbb{R}^2} |\mathcal{F}(\overline{\widetilde{m}}_{0,3} \mathbb{1}_\Omega)(\xi')|^2 \mathrm{d}\xi' = 0.$$

We deduce that $\overline{\widetilde{m}}_{0,3} \equiv 0$ in $\Omega$, but since $\widetilde{m}_0$ is independent of $x_3$, we firstly have $\widetilde{m}_{0,3} = \overline{\widetilde{m}}_{0,3} \equiv 0$ in $\Omega$, and secondly $\widetilde{m}_{0,3} \equiv 0$ in $\Omega_1$. □

**Theorem 1.2.8** (Lower bound)**.** *Consider the regime* (1.14)+(1.18)*. Consider a sequence* $(\widetilde{m}_h)$ *in* $H^1(\Omega_1, \mathbb{S}^2)$ *and* $\widetilde{m}_0 \in H^1(\Omega_1, \mathbb{S}^2)$ *such that* $\widetilde{m}_0$ *is independent of* $x_3$*,* $\widetilde{m}_{0,3} \equiv 0$ *and* $(\widetilde{m}_h)$ *converges weakly to* $\widetilde{m}_0$ *in* $H^1$*. Then*

$$\liminf_{h\to 0} \widetilde{E}_h(\widetilde{m}_h) \geqslant \widetilde{E}_0(\widetilde{m}_0), \tag{1.43}$$

*where* $\widetilde{E}_0$ *is given at* (1.20)*.*

*Proof.* We denote by $\nu$ the outer unit normal vector on $\partial\Omega_1$.
If $\liminf\limits_{h\to 0} \widetilde{E}_h(\widetilde{m}_h) = +\infty$, then the expected inequality is obvious. Assume that there exists a constant $C > 0$ such that $\liminf\limits_{h\to 0} \widetilde{E}_h(\widetilde{m}_h) \leqslant C$. Let $h > 0$. By (1.19), we clearly have

$$\begin{aligned}
\widetilde{E}_h(\widetilde{m}_h) \geqslant & \frac{\eta^2}{h\,|\log h|} \int_{\Omega_1} |\nabla' \widetilde{m}_h|^2 \mathrm{d}x \\
& + \frac{1}{h\,|\log h|} \int_{\Omega_1} \widehat{D}' : \nabla' \widetilde{m}_h \wedge \widetilde{m}_h \; \mathrm{d}x \\
& + \frac{1}{h^2\,|\log h|} \int_{\Omega_1} \widehat{D}_3 \cdot \partial_3 \widetilde{m}_h \wedge \widetilde{m}_h \; \mathrm{d}x \\
& + \frac{1}{h^2\,|\log h|} \int_{\mathbb{R}^3} |\nabla u_h|^2 \, \mathrm{d}x \\
& + \frac{Q}{h\,|\log h|} \int_{\Omega_1} \Phi(\widetilde{m}_h) \; \mathrm{d}x - 2 \int_{\Omega_1} \frac{\widetilde{H}_{\mathrm{ext},h}}{h\,|\log h|} \cdot \widetilde{m}_h \; \mathrm{d}x.
\end{aligned} \tag{1.44}$$

Let us examine each term of this inequality in order to prove (1.43). Since $\frac{\eta^2}{h|\log h|} \to \alpha$ and $(\nabla \widetilde{m}_h)$ converges weakly to $\nabla \widetilde{m}_0$ in $L^2$, then by weak lower semicontinuity of the Dirichlet integral, we have

$$\liminf_{h\to 0} \frac{\eta^2}{h\,|\log h|} \int_{\Omega_1} |\nabla' \widetilde{m}_h|^2 \, \mathrm{d}x \geqslant \alpha \int_{\Omega_1} |\nabla' \widetilde{m}_0|^2 \, \mathrm{d}x. \tag{1.45}$$

Recall that

$$\begin{aligned}
\frac{1}{h\,|\log h|} \int_{\Omega_1} \widehat{D}' : \nabla' \widetilde{m}_h \wedge \widetilde{m}_h \; \mathrm{d}x &= \frac{1}{h\,|\log h|} \sum_{j=1}^{2} \sum_{k=1}^{3} \int_{\Omega_1} \widehat{D}_{jk} e_k \cdot \partial_j \widetilde{m}_h \wedge \widetilde{m}_h \; \mathrm{d}x \\
&= \sum_{j,k=1}^{2} \frac{\widehat{D}_{jk}}{h\,|\log h|} \; e_k \cdot \int_{\Omega_1} \partial_j \widetilde{m}_h \wedge \widetilde{m}_h \; \mathrm{d}x \\
&\quad + \frac{\eta^2}{h\,|\log h|} \frac{\widehat{D}_{13}}{\eta^2} \; e_3 \cdot \int_{\Omega_1} \partial_1 \widetilde{m}_h \wedge \widetilde{m}_h \; \mathrm{d}x \\
&\quad + \frac{\eta^2}{h\,|\log h|} \frac{\widehat{D}_{23}}{\eta^2} \; e_3 \cdot \int_{\Omega_1} \partial_2 \widetilde{m}_h \wedge \widetilde{m}_h \; \mathrm{d}x.
\end{aligned}$$

Since $(\widetilde{m}_h)$ converges weakly to $\widetilde{m}_0$ in $H^1$, then up to a subsequence (thanks to the Rellich-Kondrachov compactness theorem [18, Section 5.7]), we can assume that $(\widetilde{m}_h)$ converges strongly to $\widetilde{m}_0$ in $L^2$. As a consequence, we deduce that, for $j \in \{1, 2\}$,

$$\lim_{h\to 0} \int_{\Omega_1} \partial_j \widetilde{m}_h \wedge \widetilde{m}_h \; \mathrm{d}x = \int_{\Omega_1} \partial_j \widetilde{m}_0 \wedge \widetilde{m}_0 \; \mathrm{d}x.$$

Combining this with the assumptions (1.14)+(1.18), we deduce that

$$\lim_{h\to 0} \frac{1}{h\,|\log h|} \int_{\Omega_1} \widehat{D}' : \nabla' \widetilde{m}_h \wedge \widetilde{m}_h \ \mathrm{d}x = 2\alpha \int_{\Omega_1} \delta \cdot \nabla' \widetilde{m}_0' \wedge \widetilde{m}_0' \ \mathrm{d}x. \tag{1.46}$$

Similarly, as $\left(\frac{1}{h}\partial_3 \widetilde{m}_h\right)$ converges weakly to some $M$ in $L^2$,

$$\lim_{h\to 0} \frac{1}{h} \int_{\Omega_1} \partial_3 \widetilde{m}_h \wedge \widetilde{m}_h \ \mathrm{d}x = \int_{\Omega_1} M \wedge \widetilde{m}_0 \ \mathrm{d}x.$$

But $\frac{1}{h|\log h|} \sum_{k=1}^{3} |\widehat{D}_{3k}| \ll 1$, hence

$$\lim_{h\to 0} \frac{1}{h^2\,|\log h|} \int_{\Omega_1} \widehat{D}_3 \cdot \partial_3 \widetilde{m}_h \wedge \widetilde{m}_h \ \mathrm{d}x = 0. \tag{1.47}$$

By Theorem 1.2.6 and since $g_h \geqslant 0$ (see Remark 1.1.1),

$$\frac{1}{h^2\,|\log h|} \int_{\mathbb{R}^3} |\nabla u_h|^2 \,\mathrm{d}x \geqslant \frac{I(h)}{4\pi h^2\,|\log h|} + o_h(1).$$

By (1.16), we have $\overline{\overline{m}}_h = \overline{m}_h$. By weak convergence of $(\widetilde{m}_h)$ to $\widetilde{m}_0$ in $H^1$, by Fubini's theorem, and since $\widetilde{m}_0$ is independent of $x_3$, then $(\overline{m}_h)$ converges weakly to $\widetilde{m}_0$ in $H^1$. Hence, we can use Theorem 1.2.5, from which it follows

$$\lim_{h\to 0} \frac{1}{h^2\,|\log h|} \int_{\mathbb{R}^3} |\nabla u_h|^2 \,\mathrm{d}x \geqslant \frac{1}{2\pi} \int_{\partial\Omega} (\widetilde{m}_0 \cdot \nu)^2 \mathrm{d}\mathcal{H}^1. \tag{1.48}$$

Since $(\widetilde{m}_h)$ converges (up to a subsequence) almost everywhere to $\widetilde{m}_0$ in $\Omega_1$ and $\Phi$ is continuous in $\mathbb{S}^2$, then $(\Phi(\widetilde{m}_h))$ converges almost everywhere to $\Phi(\widetilde{m}_0)$ and is bounded (because $\mathbb{S}^2$ is compact). Thus, by the dominated convergence theorem and since $\frac{Q}{h|\log h|} \to \beta$,

$$\lim_{h\to 0} \frac{Q}{h\,|\log h|} \int_{\Omega_1} \Phi(\widetilde{m}_h) \ \mathrm{d}x = \beta \int_{\Omega_1} \Phi(\widetilde{m}_0) \ \mathrm{d}x. \tag{1.49}$$

Finally, we have

$$\left| \int_{\Omega_1} \left( \frac{\widetilde{H}_{\mathrm{ext},h}}{h\,|\log h|} \cdot \widetilde{m}_h - \gamma \widetilde{H}_{\mathrm{ext},0} \cdot \widetilde{m}_0 \right) \mathrm{d}x \right| \leqslant \left| \int_{\Omega_1} \left( \frac{\widetilde{H}_{\mathrm{ext},h}}{h\,|\log h|} - \gamma \widetilde{H}_{\mathrm{ext},0} \right) \cdot \widetilde{m}_h \ \mathrm{d}x \right| + \left| \int_{\Omega_1} \gamma \widetilde{H}_{\mathrm{ext},0} \cdot (\widetilde{m}_h - \widetilde{m}_0) \,\mathrm{d}x \right|.$$

The first term in the right-hand side above tends to zero by Hölder's inequality, since $\left(\frac{\widetilde{H}_{\mathrm{ext},h}}{h|\log h|}\right)$ converges to $\gamma \widetilde{H}_{\mathrm{ext},0}$ in $L^1$ and $\|\widetilde{m}_h\|_{L^\infty} = 1$. The second term also tends to zero, by dominated convergence theorem: indeed, up to a subsequence, $\widetilde{m}_h \to \widetilde{m}_0$ almost everywhere in $\Omega_1$, and $|\gamma \widetilde{H}_{\mathrm{ext},0} \cdot (\widetilde{m}_h - \widetilde{m}_0)\,| \leqslant C \|\widetilde{H}_{\mathrm{ext},0}\|_{L^\infty} \leqslant C$. We deduce that

$$\lim_{h\to 0} \int_{\Omega_1} \frac{\widetilde{H}_{\mathrm{ext},h}}{h\,|\log h|} \cdot \widetilde{m}_h \ \mathrm{d}x = \gamma \int_{\Omega_1} \widetilde{H}_{\mathrm{ext},0} \cdot \widetilde{m}_0 \ \mathrm{d}x. \tag{1.50}$$

Taking the lim inf in (1.44) and using (1.45), (1.46), (1.47), (1.48), (1.49) and (1.50), we get (1.43) as expected. □

**Theorem 1.2.9** (Upper bound)**.** *Consider the regime* (1.14)+(1.18). *Consider a map* $\widetilde{m}_0$ *in* $H^1(\Omega_1, \mathbb{S}^1)$ *such that* $\widetilde{m}_0$ *is independent of* $x_3$ *and* $\widetilde{m}_{0,3} \equiv 0$*. Then there exists a sequence* $(\widetilde{m}_h)$ *in* $H^1(\Omega_1, \mathbb{S}^1)$ *such that* $(\widetilde{m}_h)$ *converges strongly to* $\widetilde{m}_0$ *in* $H^1$ *and satisfies*

$$\lim_{h\to 0} \widetilde{E}_h(\widetilde{m}_h) = \widetilde{E}_0(\widetilde{m}_0), \tag{1.51}$$

*where* $\widetilde{E}_0$ *is given at* (1.20).

*Proof.* We denote by $\nu$ the outer unit normal vector on $\partial\Omega_1$.
We consider the constant sequence $(\widetilde{m}_h) = (\widetilde{m}_0)$. By Theorem 1.2.6 and Theorem 1.2.5, we have in this case

$$\lim_{h\to 0} \frac{1}{h^2\,|\log h|} \int_{\mathbb{R}^3} |\nabla u_h|^2 \,\mathrm{d}x = \frac{1}{2\pi} \int_{\partial\Omega} (\overline{\widetilde{m}}_0 \cdot \nu)^2 \,\mathrm{d}\mathcal{H}^1 = \frac{1}{2\pi} \int_{\partial\Omega} (\widetilde{m}_0 \cdot \nu)^2 \,\mathrm{d}\mathcal{H}^1,$$

since $\widetilde{m}_0$ is independent of $x_3$ and $\widetilde{m}_{0,3} \equiv 0$. Using the decomposition (1.34), we have, in the regime (1.14)+(1.18),

$$\lim_{h\to 0} \widetilde{E}_h^{(0)}(\widetilde{m}_0) = \alpha \int_{\Omega_1} |\nabla' \widetilde{m}_0|^2 \,\mathrm{d}x + \frac{1}{2\pi} \int_{\partial\Omega} (\widetilde{m}_0 \cdot \nu)^2 \,\mathrm{d}\mathcal{H}^1 + \beta \int_{\Omega_1} \Phi(\widetilde{m}_0) \,\mathrm{d}x, \tag{1.52}$$

by the above convergence result and using again that $\widetilde{m}_0$ is independent of $x_3$. We also have

$$\lim_{h\to 0} \widetilde{E}_h^{(1)}(\widetilde{m}_0) = -2\gamma \int_{\Omega_1} \widetilde{H}_{\mathrm{ext},0} \cdot \widetilde{m}_0 \,\mathrm{d}x, \tag{1.53}$$

since by Hölder's inequality,

$$\left| \int_{\Omega_1} \frac{\widetilde{H}_{\mathrm{ext},h}}{h\,|\log h|} \cdot \widetilde{m}_0 \,\mathrm{d}x - \int_{\Omega_1} \gamma \widetilde{H}_{\mathrm{ext},0} \cdot \widetilde{m}_0 \,\mathrm{d}x \right| \leqslant \left\| \frac{\widetilde{H}_{\mathrm{ext},h}}{h\,|\log h|} - \gamma \widetilde{H}_{\mathrm{ext},0} \right\|_{L^1(\Omega_1)}$$

and $\left(\frac{\widetilde{H}_{\mathrm{ext},h}}{h|\log h|}\right)$ converges to $\gamma \widetilde{H}_{\mathrm{ext},0}$ in $L^1$. Furthermore, using that $\widetilde{m}_0$ is independent of $x_3$, we get

$$\begin{aligned}
\widetilde{E}_h^{(2)}(\widetilde{m}_0) &= \frac{1}{h\,|\log h|} \int_{\Omega_1} \widehat{D}' : \nabla' \widetilde{m}_0 \wedge \widetilde{m}_0 \,\mathrm{d}x \\
&= \frac{1}{h\,|\log h|} \int_{\Omega_1} \left( \widehat{D}_1 \cdot \partial_1 \widetilde{m}_0 \wedge \widetilde{m}_0 + \widehat{D}_2 \cdot \partial_2 \widetilde{m}_0 \wedge \widetilde{m}_0 \right) \mathrm{d}x,
\end{aligned}$$

and since $\widetilde{m}_{0,3} \equiv 0$ (and thus $\partial_j \widetilde{m}_{0,3} \equiv 0$ for $j \in \{1, 2\}$),

$$\begin{aligned}
\widetilde{E}_h^{(2)}(\widetilde{m}_0) &= \frac{1}{h\,|\log h|} \int_{\Omega_1} \left( \widehat{D}_{13} e_3 \cdot \partial_1 \widetilde{m}_0 \wedge \widetilde{m}_0 + \widehat{D}_{23} e_3 \cdot \partial_2 \widetilde{m}_0 \wedge \widetilde{m}_0 \right) \mathrm{d}x \\
&= \frac{\eta^2}{h\,|\log h|} \frac{\widehat{D}_{13}}{\eta^2} \int_{\Omega_1} \partial_1 \widetilde{m}_0' \wedge \widetilde{m}_0' \,\mathrm{d}x + \frac{\eta^2}{h\,|\log h|} \frac{\widehat{D}_{23}}{\eta^2} \int_{\Omega_1} \partial_2 \widetilde{m}_0' \wedge \widetilde{m}_0' \,\mathrm{d}x.
\end{aligned}$$

Then, in the regime (1.14)+(1.18), we have

$$\lim_{h\to 0} \widetilde{E}_h^{(2)}(\widetilde{m}_0) = 2\alpha\delta_1 \int_{\Omega_1} \partial_1 \widetilde{m}_0' \wedge \widetilde{m}_0' \,\mathrm{d}x + 2\alpha\delta_2 \int_{\Omega_1} \partial_2 \widetilde{m}_0' \wedge \widetilde{m}_0' \,\mathrm{d}x. \tag{1.54}$$

Combining (1.52), (1.53) and (1.54), we get (1.51). □

Theorem 1.1.3 is now a direct consequence from Theorems 1.2.7, 1.2.8 and 1.2.9.

Corollary 1.1.5 is a consequence from Proposition 1.2.4 and the direct method in the calculus of variations on the one hand, and from Theorem 1.1.3 and properties of Gamma-limits (see [13, Proposition 7.8]) on the other hand.

## 1.3 On the local minimizers of the Gamma-limit of the micromagnetic energy in the upper-half plane

This section is devoted to look for local minimizers of the Gamma-limit $\widetilde{E}_0$ given at (1.21). We assume here that the anisotropy $\Phi$ and the external magnetic field $\widetilde{H}_{\text{ext},0}$ are equal to zero.
**Moreover, since all quantities in this section are two-dimensional quantities, we drop the primes $'$ in the notations.**

The energy that we will study in this section, resulting from (1.21), is

$$\widetilde{E}_0(\widetilde{m};\Omega\cap\mathbb{R}^2_+)=\alpha\left[\int_{\Omega\cap\mathbb{R}^2_+}|\nabla\widetilde{m}|^2\,\mathrm{d}x+2\int_{\Omega\cap\mathbb{R}^2_+}\delta\cdot\nabla\widetilde{m}\wedge\widetilde{m}\,\mathrm{d}x\right]+\frac{1}{2\pi}\int_{\Omega\cap(\mathbb{R}\times\{0\})}(\widetilde{m}\cdot\nu)^2\,\mathrm{d}\mathcal{H}^1,$$

for every $\widetilde{m}\in H^1(\Omega,\mathbb{S}^1)$, where $\Omega$ is a smooth bounded open subset of $\mathbb{R}^2$ and $\nu$ is the outer unit normal vector on $\partial\Omega$.

### 1.3.1 The energy in the upper-half plane and its critical points

By making a blow-up near the boundary $\partial\Omega$, we are led to consider localized functionals with the integrals defined on sets of the form $\Omega\cap\mathbb{R}^2_+$, where $\Omega$ is a smooth bounded open subset of $\mathbb{R}^2$. Let $\Omega\subset\mathbb{R}^2$ be such a set. For $\widetilde{m}\in H^1(\Omega,\mathbb{S}^1)$, there exists (see [9]) a lifting $\varphi\in H^1(\Omega,\mathbb{R})$ of $\widetilde{m}$, i.e. $\widetilde{m}=e^{i\varphi}$. Note that $\nu=-e_2$, where $e_2$ is the second unit vector of the standard orthonormal basis in $\mathbb{R}^2$. Since

$$\nabla\widetilde{m}=\nabla\left(e^{i\varphi}\right)=i\nabla\varphi e^{i\varphi},$$
$$\nabla\widetilde{m}\wedge\widetilde{m}=\mathfrak{Im}\left(-i\nabla\varphi e^{-i\varphi}e^{i\varphi}\right)=-\nabla\varphi,$$

and

$$\widetilde{m}\cdot\nu=-\widetilde{m}_2=-\sin\varphi,$$

we can introduce

$$\widetilde{E}_0(\widetilde{m};\Omega\cap\mathbb{R}^2_+)=\alpha\int_{\Omega\cap\mathbb{R}^2_+}\left(|\nabla\varphi|^2-2\delta\cdot\nabla\varphi\right)\mathrm{d}x+\frac{1}{2\pi}\int_{\Omega\cap(\mathbb{R}\times\{0\})}\sin^2\varphi\,\mathrm{d}\mathcal{H}^1.$$

Setting $\varepsilon=2\pi\alpha$ and $E^\delta_\varepsilon(\varphi;\Omega)=\frac{1}{2\alpha}\widetilde{E}_0(\widetilde{m};\Omega\cap\mathbb{R}^2_+)$, we get

$$E^\delta_\varepsilon(\varphi;\Omega)=\frac{1}{2}\int_{\Omega\cap\mathbb{R}^2_+}\left(|\nabla\varphi|^2-2\delta\cdot\nabla\varphi\right)\mathrm{d}x+\frac{1}{2\varepsilon}\int_{\Omega\cap(\mathbb{R}\times\{0\})}\sin^2\varphi\,\mathrm{d}\mathcal{H}^1. \tag{1.55}$$

In the case $\delta=0$, the energy $E^\delta_\varepsilon=E^0_\varepsilon$ has been deeply studied by Kurzke [33] and Ignat-Kurzke [24].

**Proposition 1.3.1.** *If $\varphi\in H^1_{\text{loc}}(\overline{\mathbb{R}^2_+})$ is a critical point of $E^\delta_\varepsilon$, then*

$$\int_{\mathbb{R}^2_+\cap\text{Supp}(\psi)}(\nabla\varphi-\delta)\cdot\nabla\psi\,\mathrm{d}x+\frac{1}{2\varepsilon}\int_{(\mathbb{R}\times\{0\})\cap\text{Supp}(\psi)}\sin(2\varphi)\psi\,\mathrm{d}\mathcal{H}^1=0, \tag{1.56}$$

*for every $\psi\in H^1(\mathbb{R}^2)$ with compact support.*

*Proof.* Let $\varphi\in H^1_{\text{loc}}(\overline{\mathbb{R}^2_+})$, $\psi\in C^1(\mathbb{R}^2)$ with compact support, and $t\in\mathbb{R}$. We have

$$E^\delta_\varepsilon(\varphi+t\psi;\text{Supp}(\psi))=\frac{1}{2}\int_{\mathbb{R}^2_+\cap\text{Supp}(\psi)}\left(|\nabla\varphi+t\nabla\psi|^2-2\delta\cdot\nabla(\varphi+t\psi)\right)\mathrm{d}x$$
$$+\frac{1}{2\varepsilon}\int_{(\mathbb{R}\times\{0\})\cap\text{Supp}(\psi)}\sin^2(\varphi+t\psi)\,\mathrm{d}\mathcal{H}^1.$$

On the one hand, in $\mathbb{R}^2_+ \cap \mathrm{Supp}(\psi)$,

$$|\nabla\varphi + t\nabla\psi|^2 - 2\delta\cdot\nabla(\varphi+t\psi) = |\nabla\varphi|^2 + 2t(\nabla\varphi-\delta)\cdot\nabla\psi + O(t^2) \quad \text{as } t\to 0.$$

On the other hand, in $\mathbb{R}\times\{0\}$,

$$\sin^2(\varphi+t\psi) = \sin^2(\varphi) + t\psi\sin(2\varphi) + O(t^2) \quad \text{as } t\to 0.$$

Hence, as $t\to 0$,

$$\begin{aligned} E^\delta_\varepsilon(\varphi+t\psi;\mathrm{Supp}(\psi)) = E^\delta_\varepsilon(\varphi;\mathrm{Supp}(\psi)) + t\int_{\mathbb{R}^2_+\cap\mathrm{Supp}(\psi)} (\nabla\varphi-\delta)\cdot\nabla\psi \ \mathrm{d}x \\ + \frac{t}{2\varepsilon}\int_{(\mathbb{R}\times\{0\})\cap\mathrm{Supp}(\psi)} \sin(2\varphi)\psi \ \mathrm{d}\mathcal{H}^1 + O(t^2), \end{aligned}$$

and we deduce that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E^\delta_\varepsilon(\varphi+t\psi;\mathrm{Supp}(\psi)) = \int_{\mathbb{R}^2_+\cap\mathrm{Supp}(\psi)} (\nabla\varphi-\delta)\cdot\nabla\psi \ \mathrm{d}x + \frac{1}{2\varepsilon}\int_{(\mathbb{R}\times\{0\})\cap\mathrm{Supp}(\psi)} \sin(2\varphi)\psi \ \mathrm{d}\mathcal{H}^1.$$

By density and by Definition 1.1.6, we deduce (1.56). □

**Proposition 1.3.2.** *Any critical point $\varphi$ of $E^\delta_\varepsilon$ belongs to $C^\infty(\overline{\mathbb{R}^2_+})$ and satisfies*

$$\begin{cases} \Delta\varphi = 0 & \text{in } \mathbb{R}^2_+, \\ \partial_2\varphi = \frac{1}{2\varepsilon}\sin 2\varphi + \delta_2 & \text{on } \mathbb{R}\times\{0\}. \end{cases} \tag{1.57}$$

*Proof.* Let $\varphi \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ be a critical point of $E^\delta_\varepsilon$.

*Step 1:* We begin with proving (1.57) and we first assume that $\varphi\in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$.

For every $\psi\in H^1(\mathbb{R}^2)$ with compact support, we have, thanks to (1.56),

$$\int_{\mathbb{R}^2_+\cap\mathrm{Supp}(\psi)} (\nabla\varphi-\delta)\cdot\nabla\psi \ \mathrm{d}x + \frac{1}{2\varepsilon}\int_{(\mathbb{R}\times\{0\})\cap\mathrm{Supp}(\psi)} \sin(2\varphi)\psi \ \mathrm{d}\mathcal{H}^1 = 0.$$

Integrating by parts, we get

$$\begin{aligned} -\int_{\mathbb{R}^2_+\cap\mathrm{Supp}(\psi)} (\Delta\varphi)\psi \ \mathrm{d}x + \int_{\partial(\mathbb{R}^2_+\cap\mathrm{Supp}(\psi))} (\partial_\nu\varphi - \delta\cdot\nu)\,\psi \ \mathrm{d}\mathcal{H}^1 \\ + \frac{1}{2\varepsilon}\int_{(\mathbb{R}\times\{0\})\cap\mathrm{Supp}(\psi)} \sin(2\varphi)\psi \ \mathrm{d}\mathcal{H}^1 = 0. \end{aligned}$$

But $\psi\equiv 0$ on $\partial(\mathrm{Supp}(\psi))$ and $\nu = -e_2$ on $(\mathbb{R}\times\{0\})\cap\mathrm{Supp}(\psi)$, thus

$$-\int_{\mathbb{R}^2_+\cap\mathrm{Supp}(\psi)} (\Delta\varphi)\psi \ \mathrm{d}x + \int_{(\mathbb{R}\times\{0\})\cap\mathrm{Supp}(\psi)} \left(-\partial_2\varphi + \delta_2 + \frac{1}{2\varepsilon}\sin(2\varphi)\right)\psi \ \mathrm{d}\mathcal{H}^1 = 0.$$

We can choose $\psi$ such that $\psi\equiv 0$ on $\mathbb{R}\times\{0\}$, so that $\Delta\varphi = 0$ in $\mathbb{R}^2_+\cap\mathrm{Supp}(\psi)$. We then deduce that $\partial_2\varphi = \frac{1}{2\varepsilon}\sin(2\varphi)+\delta_2$ on $(\mathbb{R}\times\{0\})\cap\mathrm{Supp}(\psi)$. This equalities being true for every $\psi\in H^1(\mathbb{R}^2)$ with compact support, we deduce (1.57).

*Step 2:* Let us prove that $\varphi\in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$.

For the interior regularity, by (1.57) we have $\Delta\varphi = 0$ in the distributional sense in $\mathbb{R}^2_+$. By Weyl's lemma, $\varphi\in C^\infty(\mathbb{R}^2_+)$.

For the boundary regularity, we introduce tangential difference quotients as defined in [18, Section 5.8.2] (see also [19, Section 7.11]), and we use the ideas in [18, Section 6.3]. Our strategy consists in proving that $\varphi$ is $H^2$ around $(x_1, x_2) = (0, 0)$, and then around any point $(x_1, 0)$ with $x_1 \in \mathbb{R}$, by translation.

For $|h| > 0$ and $(x_1, x_2) \in \overline{\mathbb{R}^2_+} = \mathbb{R} \times [0, +\infty)$, set

$$\Delta_h \varphi(x_1, x_2) = \frac{1}{h} \left( \varphi(x_1 + h, x_2) - \varphi(x_1, x_2) \right).$$

Let $r \in (0, 1)$ be fixed. Let $\zeta \in C^\infty(\overline{B_r})$ be a function that satisfies $0 \leqslant \zeta \leqslant 1$, $\zeta \equiv 1$ in $\overline{B_{r/2}}$, $\zeta \equiv 0$ in $\overline{B_r} \setminus \overline{B_{3r/4}}$ and $|\nabla \zeta| \leqslant \frac{C}{r}$ for some constant $C > 0$. For $|h| > 0$, set $\psi = \Delta_{-h}(\zeta^2 \Delta_h \varphi)$, which is in $H^1(\mathbb{R}^2)$ with compact support. Thus, we can input this function $\psi$ in (1.56). By use of the integration by parts property of difference quotients (see equation (16) in [18, Section 6.3]), we get

$$\int_{B_r^+} \sum_{j=1}^{2} \Delta_h(\partial_j \varphi - \delta_j) \partial_j (\zeta^2 \Delta_h \varphi) \mathrm{d}x + \frac{1}{2\varepsilon} \int_{(-r,r)\times\{0\}} \zeta^2 (\Delta_h \varphi) \Delta_h(\sin(2\varphi)) \mathrm{d}\mathcal{H}^1 = 0.$$

Since $\delta_1, \delta_2$ are constants and expanding the derivative of $\zeta^2 \Delta_h \varphi$, the above relation can be written as $I_1 + I_2 + I_3 = 0$ with

$$I_1 = \int_{B_r^+} \zeta^2 \left| \Delta_h (\nabla \varphi) \right|^2 \mathrm{d}x,$$

$$I_2 = 2 \int_{B_r^+} \sum_{j=1}^{2} \zeta (\partial_j \zeta)(\Delta_h \varphi) \partial_j (\Delta_h \varphi) \mathrm{d}x,$$

and

$$I_3 = \frac{1}{2\varepsilon} \int_{(-r,r)\times\{0\}} \zeta^2 (\Delta_h \varphi) \Delta_h(\sin(2\varphi)) \mathrm{d}\mathcal{H}^1.$$

Let $\varepsilon_1 > 0$, that will be precised later, and $j \in \{1, 2\}$. Using the properties of $\zeta$ and Young's inequality, we have

$$\begin{aligned} \left| 2 \int_{B_r^+} \zeta (\partial_j \zeta)(\Delta_h \varphi) \partial_j (\Delta_h \varphi) \mathrm{d}x \right| &= \left| 2 \int_{B_{3r/4}^+} \zeta (\partial_j \zeta)(\Delta_h \varphi) \partial_j (\Delta_h \varphi) \mathrm{d}x \right| \\ &\leqslant \varepsilon_1 \int_{B_r^+} \zeta^2 \left| \Delta_h (\partial_j \varphi) \right|^2 \mathrm{d}x \\ &\quad + \frac{1}{\varepsilon_1} \int_{B_{3r/4}^+} \left| \partial_j \zeta \right|^2 \left| \Delta_h \varphi \right|^2 \mathrm{d}x. \end{aligned}$$

In particular, using the properties of $\zeta$ and [18, Section 5.8.2, Theorem 3(i)], we have

$$\int_{B_{3r/4}^+} \left| \partial_j \zeta \right|^2 \left| \Delta_h \varphi \right|^2 \mathrm{d}x \leqslant \frac{C}{r^2} \int_{B_{3r/4}^+} \left| \Delta_h \varphi \right|^2 \mathrm{d}x \leqslant \frac{C}{r^2} \int_{B_r^+} \left| \nabla \varphi \right|^2 \mathrm{d}x.$$

We deduce that

$$|I_2| \leqslant \varepsilon_1 \int_{B_r^+} \zeta^2 \left| \Delta_h (\nabla \varphi) \right|^2 \mathrm{d}x + \frac{C}{\varepsilon_1 r^2} \int_{B_r^+} \left| \nabla \varphi \right|^2 \mathrm{d}x. \tag{1.58}$$

By the mean-value theorem, $|\Delta_h(\sin(2\varphi))| \leqslant 2 \left| \Delta_h \varphi \right|$. Combining this with Green's formula,

$$|I_3| \leqslant \frac{1}{\varepsilon} \int_{(-r,r)\times\{0\}} \zeta^2 \left| \Delta_h \varphi \right|^2 \mathrm{d}\mathcal{H}^1 = \frac{1}{\varepsilon} \int_{B_r^+} -\partial_2 \left( (\zeta \Delta_h \varphi)^2 \right) \mathrm{d}x = -\frac{2}{\varepsilon} \int_{B_r^+} \zeta \Delta_h \varphi \partial_2 \left( \zeta \Delta_h \varphi \right) \mathrm{d}x.$$

We deduce that

$$|I_3| \leqslant \frac{2}{\varepsilon} \int_{B_r^+} \zeta \, |\Delta_h \varphi| \, |\nabla(\zeta \Delta_h \varphi)| \, \mathrm{d}x.$$

Let $\varepsilon_2 \in (0, \frac{1}{2})$, that will be precised later. By Young's inequality,

$$|I_3| \leqslant \frac{2r\varepsilon_2}{\varepsilon} \int_{B_r^+} |\nabla(\zeta \Delta_h \varphi)|^2 \, \mathrm{d}x + \frac{2}{r\varepsilon\varepsilon_2} \int_{B_r^+} \zeta^2 \, |\Delta_h \varphi|^2 \, \mathrm{d}x.$$

On the one hand, using the properties of $\zeta$, Young's inequality and [18, Section 5.8.2, Theorem 3(i)],

$$\begin{aligned}
\int_{B_r^+} |\nabla(\zeta \Delta_h \varphi)|^2 \, \mathrm{d}x &= \int_{B_{3r/4}^+} |\nabla(\zeta \Delta_h \varphi)|^2 \, \mathrm{d}x \\
&= \int_{B_{3r/4}^+} |(\Delta_h \varphi) \nabla \zeta + \zeta \Delta_h (\nabla \varphi)|^2 \, \mathrm{d}x \\
&\leqslant 2 \int_{B_{3r/4}^+} |\nabla \zeta|^2 \, |\Delta_h \varphi|^2 \, \mathrm{d}x + 2 \int_{B_{3r/4}^+} \zeta^2 \, |\Delta_h (\nabla \varphi)|^2 \, \mathrm{d}x \\
&\leqslant \frac{C}{r^2} \int_{B_r^+} |\nabla \varphi|^2 \, \mathrm{d}x + 2 \int_{B_r^+} \zeta^2 \, |\Delta_h (\nabla \varphi)|^2 \, \mathrm{d}x.
\end{aligned}$$

On the other hand, using the properties of $\zeta$ and [18, Section 5.8.2, Theorem 3(i)],

$$\int_{B_r^+} \zeta^2 \, |\Delta_h \varphi|^2 \, \mathrm{d}x \leqslant \int_{B_{3r/4}^+} |\Delta_h \varphi|^2 \, \mathrm{d}x \leqslant C \int_{B_r^+} |\nabla \varphi|^2 \, \mathrm{d}x.$$

We deduce that

$$|I_3| \leqslant C(\varepsilon) r \varepsilon_2 \int_{B_r^+} \zeta^2 \, |\Delta_h (\nabla \varphi)|^2 \, \mathrm{d}x + \frac{C(\varepsilon, \varepsilon_2)}{r} \int_{B_r^+} |\nabla \varphi|^2 \, \mathrm{d}x. \tag{1.59}$$

Combining (1.58), (1.59) with the definition of $I_1$ and the relation $I_1 + I_2 + I_3 = 0$, we deduce that

$$(1 - \varepsilon_1 - C(\varepsilon) r \varepsilon_2) \int_{B_r^+} \zeta^2 \, |\Delta_h (\nabla \varphi)|^2 \, \mathrm{d}x \leqslant \frac{(1+r) C(\varepsilon, \varepsilon_1, \varepsilon_2)}{r^2} \int_{B_r^+} |\nabla \varphi|^2 \, \mathrm{d}x.$$

Using that $r \in (0,1)$, and choosing $\varepsilon_1$ and $\varepsilon_2$ so small that $1 - \varepsilon_1 - C(\varepsilon)\varepsilon_2 \geqslant \frac{1}{2}$, we finally get

$$\int_{B_{r/2}^+} |\Delta_h (\nabla \varphi)|^2 \, \mathrm{d}x \leqslant \int_{B_r^+} \zeta^2 \, |\Delta_h (\nabla \varphi)|^2 \, \mathrm{d}x \leqslant \frac{C}{r^2} \int_{B_r^+} |\nabla \varphi|^2 \, \mathrm{d}x,$$

with the first inequality coming from the definition of $\zeta$. By [18, Section 5.8.2, Theorem 3(ii)], taking the limits when $h \to 0$, we get

$$\int_{B_{r/2}^+} |\partial_1 (\nabla \varphi)|^2 \, \mathrm{d}x \leqslant \frac{C}{r^2} \int_{B_r^+} |\nabla \varphi|^2 \, \mathrm{d}x.$$

We deduce that

$$\int_{B_{r/2}^+} \left( |\partial_{11} \varphi|^2 + |\partial_{12} \varphi|^2 \right) \mathrm{d}x \leqslant \frac{C}{r^2} \int_{B_r^+} |\nabla \varphi|^2 \, \mathrm{d}x.$$

Since $\varphi \in C^\infty(\mathbb{R}^2_+)$, we get $\partial_{21} \varphi = \partial_{12} \varphi$ by Schwarz's lemma, and $\partial_{22} \varphi = -\partial_{11} \varphi$ because $\Delta \varphi = 0$ in $\mathbb{R}^2_+$. It follows that $\varphi \in H^2(B_{r/2}^+)$. Translating the support of $\zeta$ with respect to $x_1$ in the previous calculations, we deduce that $\varphi \in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$.

*Step 3:* Let us prove that $\varphi \in C^\infty(\overline{\mathbb{R}^2_+})$.

For the interior regularity, we already noticed in Step 2 that $\varphi \in C^\infty(\mathbb{R}^2_+)$ as a harmonic function.

For the boundary regularity, we repeat the arguments of Step 2 for $\nabla^2\varphi$, etc. and prove by induction that $\varphi \in H^m_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ for every $m \in \mathbb{N}^*$, as in [18, Section 6.3, Theorem 5]. Using Sobolev embeddings (see [18, Section 5.6.3, Theorem 6(ii)]), we deduce that $\varphi \in C^\infty(\overline{U})$ for every bounded open subset $U \subset \overline{\mathbb{R}^2_+}$. It follows that $\varphi \in C^\infty(\overline{\mathbb{R}^2_+})$. □

Given a critical point $\varphi_\varepsilon$ of $E^\delta_\varepsilon$ that satisfies (1.57) by Proposition 1.3.2, we will consider in the following the rescaled functions

$$\phi_\varepsilon \colon (x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto 2\varphi_\varepsilon(\varepsilon x_1, \varepsilon x_2) + \pi, \tag{1.60}$$

that are harmonic in $\mathbb{R}^2_+$ and satisfy $\partial_2\phi_\varepsilon = -\sin(\phi_\varepsilon) + 2\varepsilon\delta_2$ on $\mathbb{R} \times \{0\}$, i.e.

$$\left\{ \begin{array}{rcll} \Delta\phi_\varepsilon & = & 0 & \text{in } \mathbb{R}^2_+, \\ \partial_2\phi_\varepsilon - \lambda_\varepsilon + \sin(\phi_\varepsilon) & = & 0 & \text{on } \mathbb{R} \times \{0\}, \end{array} \right. \tag{1.61}$$

where $\lambda_\varepsilon = 2\varepsilon\delta_2$. We now look for explicit solutions of this problem under the boundedness condition $[(x_1, x_2) \mapsto \phi_\varepsilon(x_1, x_2) - \lambda_\varepsilon x_2] \in L^\infty(\mathbb{R}^2_+)$ in order to get a modified Peierls-Nabarro problem. More precisely, we will especially look for nonconstant, nonperiodic and bounded solutions, since this type of solutions is expected to minimize $E^\delta_\varepsilon$ as for $\delta = 0$ (see [33]).

### 1.3.2 A modified Peierls-Nabarro problem

For any $\lambda \in \mathbb{R}$, we consider the problem

$$\left\{ \begin{array}{l} f \in C^\infty(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+}), \\ (x_1, x_2) \mapsto f(x_1, x_2) - \lambda x_2 \text{ is bounded in } \mathbb{R}^2_+, \\ \Delta f = 0 \text{ in } \mathbb{R}^2_+, \\ \partial_2 f - \lambda + \sin f = 0 \text{ on } \mathbb{R} \times \{0\}. \end{array} \right. \tag{PN$_\lambda$}$$

The problem (PN$_\lambda$) is a generalization of the classical Peierls-Nabarro problem (which is in fact the case $\lambda = 0$). In [43], Toland shows a link between solutions of the Peierls-Nabarro problem and solutions of the Benjamin-Ono problem, that is given by

$$\left\{ \begin{array}{l} u \in C^\infty(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+}), \\ u \text{ is bounded in } \mathbb{R}^2_+, \\ \Delta u = 0 \text{ in } \mathbb{R}^2_+, \\ \partial_2 u + u^2 - u = 0 \text{ on } \mathbb{R} \times \{0\}. \end{array} \right. \tag{BO}$$

The main point in finding solutions of the Peierls-Nabarro problem is based on the fact that all solutions of the Benjamin-Ono problem were classified by Amick and Toland in [5]. The rest of this section is devoted to determine solutions of (PN$_\lambda$), using the ideas in Toland [43]. A first observation before linking solutions of (PN$_\lambda$) with solutions of (BO) is the following remark.

*Remark* 1.3.3. Note that the problem (PN$_\lambda$) is "odd" with respect to $\lambda$, in the sense that $f$ satisfies (PN$_\lambda$) if and only if $-f$ satisfies (PN$_{-\lambda}$). In particular, $f$ satisfies (PN$_0$) if and only if $-f$ satisfies (PN$_0$).

**Linking solutions of** $(\mathrm{PN}_\lambda)$ **to solutions of** (BO)**.**

Let us formally explain the strategy for establishing Theorem 1.3.4. Given $\lambda \in \mathbb{R}$ and a solution $f$ of $(\mathrm{PN}_\lambda)$, we look for a relation between $f$ and a solution of (BO). To do so, a crucial observation – due to Toland [43] – is that the function

$$(x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto 2\arctan\left(\frac{x_1}{1+x_2}\right)$$

satisfies $(\mathrm{PN}_0)$, and the function

$$(x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto \frac{2(1+x_2)}{x_1^2 + (1+x_2)^2},$$

that satisfies (BO), is the $x_1$-derivative of the first one. It is clear that the latter function is also the $x_1$-derivative of the function

$$(x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto 2\arctan\left(\frac{x_1}{1+x_2}\right) + \lambda x_2,$$

that satisfies $(\mathrm{PN}_\lambda)$. This is a motivation for setting $u = \partial_1 f$ in $\mathbb{R}^2_+$, where $u$ is a solution of (BO).

On the one hand, using (BO), $\partial_2 u = u - u^2 = u(1-u)$ on $\mathbb{R} \times \{0\}$. On the other hand, using $(\mathrm{PN}_\lambda)$,

$$\partial_2 u = \partial_{21} f = \partial_1(\partial_2 f - \lambda) = \partial_1(-\sin f) = -\partial_1 f \cos f = -u \cos f$$

on $\mathbb{R} \times \{0\}$. Hence, $u(1 + \cos f - u) = 0$ on $\mathbb{R} \times \{0\}$ and this identity motivates the relation $u = 1 + \cos f$ on $\mathbb{R} \times \{0\}$ (in particular since (BO) has nontrivial solutions, see [43] or Theorem 1.3.5 below). Moreover,

$$\partial_1 u = \partial_1(1 + \cos f) = -\partial_1 f \sin f = \partial_1 f(\partial_2 f - \lambda)$$

on $\mathbb{R} \times \{0\}$. By harmonicity and boundedness, we can extend this equality to $\mathbb{R}^2_+$ using the Phragmén-Lindelöf principle [41, Theorem 2.3.2]. We observe that $u = \partial_1 f$ and $\partial_1 u = \partial_1 f(\partial_2 f - \lambda)$ in $\mathbb{R}^2_+$, so that $\partial_1 u = \frac{1}{2} w_{f,\lambda}$ where

$$w_{f,\lambda} = \partial_{11} f + \partial_1 f(\partial_2 f - \lambda).$$

Hence, the idea of Theorem 1.3.4 consists in setting $w_{f,\lambda}$ as above, and re-construct $u$ as an integral of $\frac{1}{2} w_{f,\lambda}$ with respect to $x_1$ such that $u$ satisfies (BO). More precisely, given $w_{f,\lambda}$, we first introduce a function $W_{f,\lambda}$ that satisfies $w_{f,\lambda} = \partial_1 W_{f,\lambda}$:

$$W_{f,\lambda}(x_1, x_2) = W_{f,\lambda}(0, x_2) + \int_0^{x_1} w_{f,\lambda}(s, x_2)\, \mathrm{d}s,$$

where $W_{f,\lambda}(0, x_2)$ is chosen such that $W_{f,\lambda}$ is harmonic. Then, we set

$$u(x_1, x_2) = \frac{1}{2} W_{f,\lambda}(x_1, x_2) + G_{f,\lambda}(x_2),$$

where $G_{f,\lambda}$ has to be precised. Since $W_{f,\lambda}$ is harmonic and $u$ must be harmonic for satisfying (BO), then $G_{f,\lambda}$ must be affine. The above relation and the calculation of $W_{f,\lambda}(x_1, 0)$ (see (1.66) below or [43, Section 4]) give

$$G_{f,\lambda}(0) = u(x_1, 0) - C + \frac{1}{2}\partial_1 f(x_1, 0) - \frac{1}{2}\cos f(x_1, 0). \tag{1.62}$$

However, as $u$ is expected to satisfy $u(x_1,0) = 1 + \cos f(x_1,0)$, we see that $G_{f,\lambda}(0)$ is a priori not constant.

For solving this difficulty, we rely on the odd symmetry of the problem ($\mathrm{PN}_\lambda$) (see Remark 1.3.3). More precisely, we consider

$$v(x_1,x_2) = \frac{1}{2} W_{-f,-\lambda}(x_1,x_2) + G_{-f,-\lambda}(x_2),$$

so that $\partial_1 v = \frac{1}{2}\partial_1 W_{-f,-\lambda} = \frac{1}{2} w_{-f,-\lambda}$. Similarly than $u$ before, $v$ is expected to satisfy the relation $v(x_1,0) = 1+\cos f(x_1,0)$, so that adding (1.62) and the analogous identity for $G_{-f,-\lambda}(0)$, the terms $\partial_1 f(x_1,0)$ cancel and we get:

$$G_{f,\lambda}(0) + G_{-f,-\lambda}(0) = C + \underbrace{\cos f(x_1,0)}_{\text{from } u(x_1,0)} + \underbrace{\cos f(x_1,0)}_{\text{from } v(x_1,0)} - \cos f(x_1,0).$$

In order to cancel the cosines completely, we replace the functions $u$ and $v$ defined above by their half, so that the sum $u(x_1,0)+v(x_1,0)$ is expected to be equal to $1+\cos f(x_1,0)$. This replacements do not change harmonicity or boundedness, and it justifies the expected relations (1.63), (1.64) and (1.65) below.

**Theorem 1.3.4.** *Let $\lambda \in \mathbb{R}$ and $f$ be a solution of* ($\mathrm{PN}_\lambda$). *Then there exist two solutions $u$ and $v$ of* (BO) *such that*

$$u - v = \partial_1 f \qquad \text{in } \mathbb{R}^2_+, \tag{1.63}$$

$$\partial_1 u + \partial_1 v = \partial_1 f(\partial_2 f - \lambda) \qquad \text{in } \mathbb{R}^2_+, \tag{1.64}$$

*and*

$$u(x_1,0) + v(x_1,0) = 1 + \cos f(x_1,0) \qquad \forall x_1 \in \mathbb{R}. \tag{1.65}$$

*Proof.* Let $\lambda \in \mathbb{R}$ and $f \colon \overline{\mathbb{R}^2_+} \to \mathbb{R}$ be a solution of ($\mathrm{PN}_\lambda$).

*Step 1:* The function $w_{f,\lambda} \colon \overline{\mathbb{R}^2_+} \to \mathbb{R}$, defined as

$$w_{f,\lambda} = \partial_{11} f + \partial_1 f(\partial_2 f - \lambda),$$

is harmonic in $\mathbb{R}^2_+$ and satisfies

$$w_{f,\lambda}(x_1,0) = \left[\partial_1\left(\partial_1 f(x_1,x_2) + \cos f(x_1,x_2)\right)\right]\Big|_{x_2=0} \qquad \forall x_1 \in \mathbb{R},$$

$$\partial_2 w_{f,\lambda}(x_1,0) = \left[-\frac{1}{2}\partial_1\left(\left(\partial_1 f(x_1,x_2) + \cos f(x_1,x_2)\right)^2\right)\right]\Bigg|_{x_2=0} \qquad \forall x_1 \in \mathbb{R},$$

and

$$-\partial_1 w_{f,\lambda} = \partial_{22}\left(\partial_1 f - \lambda f\right) + \partial_{22} f \partial_2 f - \partial_{12} f \partial_1 f \qquad \text{in } \mathbb{R}^2_+.$$

For the proofs of the harmonicity and of the three identities above, that are elementary calculations, we refer to Toland [43, Section 4].

*Step 2:* The function $W_{f,\lambda} \colon \overline{\mathbb{R}^2_+} \to \mathbb{R}$, defined as

$$W_{f,\lambda}(x_1,x_2) = \partial_1 f(0,x_2) - \frac{1}{2}\int_{x_2}^{+\infty} \left((\partial_2 f(0,t) - \lambda)^2 - \partial_1 f(0,t)^2\right) \mathrm{d}t + \int_0^{x_1} w_{f,\lambda}(s,x_2)\,\mathrm{d}s,$$

for every $(x_1,x_2) \in \overline{\mathbb{R}^2_+}$, is harmonic and bounded in $\mathbb{R}^2_+$ and satisfies

$$W_{f,\lambda}(x_1,0) = A_\lambda + \partial_1 f(x_1,0) + \cos f(x_1,0) \qquad \forall x_1 \in \mathbb{R}, \tag{1.66}$$

and

$$\partial_2 W_{f,\lambda}(x_1,0) = \frac{1}{2}\left(1 + W_{f,\lambda}(x_1,0) - A_\lambda\right)\left(1 - W_{f,\lambda}(x_1,0) + A_\lambda\right) \qquad \forall x_1 \in \mathbb{R}, \tag{1.67}$$

where

$$A_\lambda = -\cos f(0,0) - \frac{1}{2}\int_0^{+\infty} \left((\partial_2 f(0,t) - \lambda)^2 - \partial_1 f(0,t)^2\right) \mathrm{d}t.$$

Once again, we refer to Toland [43, Section 4] for the proofs of this properties of $W_{f,\lambda}$, using Step 1.

*Step 3:* Let $u\colon \overline{\mathbb{R}^2_+} \to \mathbb{R}$ be defined as

$$u(x_1,x_2) = \frac{1}{2}\left(1 + W_{f,\lambda}(x_1,x_2) - A_\lambda\right),$$

for every $(x_1,x_2) \in \overline{\mathbb{R}^2_+}$. By Step 2, $u$ is harmonic and bounded in $\mathbb{R}^2_+$, and of class $C^1$ in $\overline{\mathbb{R}^2_+}$. Using (1.67), for every $x_1 \in \mathbb{R}$,

$$\begin{aligned}
\partial_2 u(x_1,0) &= \frac{1}{2}\partial_2 W_{f,\lambda}(x_1,0) \\
&= \frac{1}{4}(1 + W_{f,\lambda}(x_1,x_2) - A_\lambda)(1 - W_{f,\lambda}(x_1,x_2) + A_\lambda) \\
&= u(x_1,0)(1 - u(x_1,0)),
\end{aligned}$$

so that $u$ is a solution of (BO).

Let $v\colon \overline{\mathbb{R}^2_+} \to \mathbb{R}$ be defined as

$$v(x_1,x_2) = \frac{1}{2}\left(1 + W_{-f,-\lambda}(x_1,x_2) - A_{-\lambda}\right),$$

for every $(x_1,x_2) \in \overline{\mathbb{R}^2_+}$. Similarly than before, $v$ is harmonic and bounded in $\mathbb{R}^2_+$, of class $C^1$ in $\overline{\mathbb{R}^2_+}$ and satisfies (BO).

For every $x_1 \in \mathbb{R}$, using (1.66),

$$\begin{aligned}
u(x_1,0) + v(x_1,0) &= 1 + \frac{1}{2}(W_{f,\lambda}(x_1,0) - A_\lambda) + \frac{1}{2}(W_{-f,-\lambda}(x_1,0) - A_{-\lambda}) \\
&= 1 + \cos f(x_1,0),
\end{aligned}$$

which shows (1.65). Using (1.66) again, for every $x_1 \in \mathbb{R}$,

$$u(x_1,0) - v(x_1,0) = \frac{1}{2}(W_{f,\lambda}(x_1,0) - A_\lambda) - \frac{1}{2}(W_{-f,-\lambda}(x_1,0) - A_{-\lambda}) = \partial_1 f(x_1,0).$$

Since $(x_1,x_2) \mapsto f(x_1,x_2) - \lambda x_2$ is harmonic and bounded in $\mathbb{R}^2_+$, as a solution of (PN$_\lambda$), then $\partial_1 f$ is bounded (see [19, Theorem 2.10]). Hence, $u - v$ and $\partial_1 f$ are both bounded harmonic functions which coincide on $\mathbb{R}\times\{0\} = \partial(\mathbb{R}^2_+)\setminus\{\infty\}$. By the Phragmén-Lindelöf principle [41, Theorem 2.3.2], the functions $u - v$ and $\partial_1 f$ coincide in $\mathbb{R}^2_+$. This proves (1.63). Finally, in $\mathbb{R}^2_+$,

$$\partial_1 u + \partial_1 v = \frac{1}{2}\left(\partial_1 W_{f,\lambda} + \partial_1 W_{-f,-\lambda}\right) = \frac{1}{2}\left(w_{f,\lambda} + w_{-f,-\lambda}\right) = \partial_1 f(\partial_2 f - \lambda),$$

which gives (1.64). □

We now quote two statements from [5].

**Theorem 1.3.5.** *Solutions of* (BO) *in* $\overline{\mathbb{R}^2_+}$ *are:*

- *the constant function* $u_0 \equiv 0$,
- *for* $\alpha \in [1,2)$, *the functions*

$$u_\alpha\colon (x_1,x_2) \mapsto \frac{2\sigma\Gamma_\alpha(x_2)}{\cos^2(\sigma x_1) + \Gamma_\alpha(x_2)^2\sin^2(\sigma x_1)} \tag{1.68}$$

  *where*

$$\sigma = \frac{1}{2}\sqrt{\alpha(2-\alpha)}, \quad \Gamma_\alpha(x_2) = \frac{\gamma + \tanh(\sigma x_2)}{1+\gamma\tanh(\sigma x_2)} \quad \text{and} \quad \gamma = \frac{\alpha}{2\sigma}, \tag{1.69}$$

  *which are non-constant periodic functions of the variable* $x_1$, *and every translation of* $u_\alpha$ *in the* $x_1$*-direction,*
- *the function*

$$u_2\colon (x_1,x_2) \mapsto \frac{2(1+x_2)}{x_1^2 + (1+x_2)^2} \tag{1.70}$$

  *which is non-constant and non-periodic in* $x_1$, *and every translation of* $u_2$ *in the* $x_1$*-direction.*

*Remark* 1.3.6. The solution $u_\alpha$ given at (1.68) is not well-defined for $\alpha = 2$, because in this case $\sigma = 0$. However, $(u_\alpha)_{\alpha\in[1,2)}$ converges pointwise to $u_2$ when $\alpha \to 2$.

**Proposition 1.3.7.** *Solutions of* (BO) *have the following properties:*

*i) For every* $\alpha \in [1,2]$,

$$u_\alpha > 0. \tag{P1$_{\mathrm{BO}}$}$$

*ii) For* $\alpha \in (1,2)$, *for every* $x_2 > 0$,

$$u_\alpha(\cdot,x_2) \text{ is } \frac{\pi}{\sigma}\text{-periodic and } u_\alpha(0,0) = \alpha = \max_{\mathbb{R}} u_\alpha(\cdot,x_2). \tag{P2$_{\mathrm{BO}}$}$$

*iii) For every* $\alpha \in (1,2]$,

$$\partial_1 u_\alpha(\cdot,0)^2 = \alpha(\alpha-2)u_\alpha(\cdot,0)^2 + 2u_\alpha(\cdot,0)^3 - u_\alpha(\cdot,0)^4 \text{ in } \mathbb{R}. \tag{P3$_{\mathrm{BO}}$}$$

*iv) For every* $\alpha \in (1,2]$,

$$\sup_{\mathbb{R}} u_\alpha(\cdot,0) + \inf_{\mathbb{R}} u_\alpha(\cdot,0) = 2. \tag{P4$_{\mathrm{BO}}$}$$

**Finding solutions of** (PN$_\lambda$)**.**

We now prove Theorem 1.1.8 by using Theorem 1.3.4, Theorem 1.3.5 and the properties of the solutions of (BO) given in Proposition 1.3.7.

Let $\lambda \in \mathbb{R}$ and $f$ be a solution of (PN$_\lambda$). By Theorem 1.3.4, there exist two solutions $u$ and $v$ of (BO) that are related to $f$. Since solutions of (BO) are entirely known thanks to Theorem 1.3.5, we proceed by testing all possible cases. Each case consists in choosing $v$ (the possible choices being $u_0$, $u_1$, $u_2$ and $u_\alpha$, with $\alpha \in (1,2)$, from Theorem 1.3.5), and then testing each choice for $u$ (if needed). For each case, we will obtain either a candidate function $f$ to be a solution of (PN$_\lambda$), or a contradiction, that means that case cannot occur. Note that, because of (1.63), we also have to consider the candidate functions $f$ obtained by inverting $u$ and $v$.

We need the following lemmas for the first two cases.

**Lemma 1.3.8.** *For every $\alpha \in \{0\} \cup [1,2]$, $u_\alpha(\cdot,0)$ is integrable on $\mathbb{R}$ if and only if $\alpha \in \{0,2\}$.*

*Proof.* It suffices to prove that $u_0(\cdot,0)$ and $u_2(\cdot,0)$ are integrable on $\mathbb{R}$ and that the other options for $u_\alpha(\cdot,0)$ are not integrable on $\mathbb{R}$. Since $u_0 \equiv 0$ and $u_1 \equiv 1$, it is clear that $u_0(\cdot,0)$ is integrable on $\mathbb{R}$ and $u_1(\cdot,0)$ is not integrable on $\mathbb{R}$. For every $x_1 \in \mathbb{R}$,

$$u_2(x_1,0) = \frac{2}{x_1^2+1},$$

thus $u_2(\cdot,0)$ is clearly integrable on $\mathbb{R}$. For every $\alpha \in (1,2)$, for every $x_1 \in \mathbb{R}$,

$$u_\alpha(x_1,0) = \frac{2\sigma\gamma}{\cos^2(\sigma x_1) + \gamma^2 \sin^2(\sigma x_1)} \geqslant \frac{2\sigma\gamma}{1+\gamma^2},$$

thus $u_\alpha(\cdot,0)$ is not integrable on $\mathbb{R}$. □

**Lemma 1.3.9.** *For every $\alpha \in (1,2)$ and $x_2 \geqslant 0$,*

$$\int_0^{\pi/\sigma} u_\alpha(x_1,x_2)\ \mathrm{d}x_1 = 2\pi. \tag{1.71}$$

*Proof.* Let $\alpha \in (1,2)$ and $x_2 \geqslant 0$ be fixed. By change of variable and by the dominated convergence theorem,

$$\int_0^{\pi/\sigma} u_\alpha(x_1,x_2)\ \mathrm{d}x_1 = \lim_{\varepsilon\to 0} \int_\varepsilon^{\pi-\varepsilon} \frac{2\Gamma_\alpha(x_2)}{\cos^2(x_1) + \Gamma_\alpha(x_2)^2 \sin^2(x_1)}\ \mathrm{d}x_1.$$

Moreover, $\Gamma_\alpha(x_2) > 0$ by (1.69), and for every $\varepsilon > 0$,

$$\int_\varepsilon^{\pi-\varepsilon} \frac{2\Gamma_\alpha(x_2)}{\cos^2(x_1) + \Gamma_\alpha(x_2)^2 \sin^2(x_1)}\ \mathrm{d}x_1 = -2\int_\varepsilon^{\pi-\varepsilon} \frac{g'_{\alpha,x_2}(x_1)}{1+g_{\alpha,x_2}(x_1)^2}\ \mathrm{d}x_1$$

where $g_{\alpha,x_2}(x_1) = \frac{\cot(x_1)}{\Gamma_\alpha(x_2)}$, hence

$$\begin{aligned}\int_\varepsilon^{\pi-\varepsilon} &\frac{2\Gamma_\alpha(x_2)}{\cos^2(x_1) + \Gamma_\alpha(x_2)^2 \sin^2(x_1)}\ \mathrm{d}x_1 \\ &= -2\left[\arctan\left(\frac{\cot(\pi-\varepsilon)}{\Gamma_\alpha(x_2)}\right) - \arctan\left(\frac{\cot(\varepsilon)}{\Gamma_\alpha(x_2)}\right)\right].\end{aligned}$$

Taking the limits when $\varepsilon \to 0$, we get

$$\int_0^{\pi/\sigma} u_\alpha(x_1,x_2)\ \mathrm{d}x_1 = 2\pi.$$

□

We come back to the proof of Theorem 1.1.8.

**Case 1: $v = u_0$.**

By (1.63), $\partial_1 f = u$ in $\mathbb{R}^2_+$. Note that this equality extends to $\overline{\mathbb{R}^2_+}$ by continuity of $\partial_1 f$ and $v$.

– Subcase 1: $u = u_0$.

Then $\partial_1 f \equiv 0$ in $\overline{\mathbb{R}^2_+}$ and by (1.65), $\cos f(\cdot,0) \equiv -1$ in $\mathbb{R}$, i.e. there exists an integer $n \in \mathbb{Z}$ such that $f(\cdot,0) = (2n+1)\pi$ in $\mathbb{R}$. Since $(x_1,x_2) \mapsto f(x_1,x_2) - \lambda x_2$ and $(x_1,x_2) \mapsto (2n+1)\pi$ are bounded harmonic functions in $\mathbb{R}^2_+$ which coincide on $\mathbb{R}\times\{0\} = \partial(\mathbb{R}^2_+)\setminus\{\infty\}$, they coincide in $\overline{\mathbb{R}^2_+}$ by the Phragmén-Lindelöf principle [41, Theorem 2.3.2]. As a consequence, for every $(x_1,x_2)\in\overline{\mathbb{R}^2_+}$,

$$f(x_1,x_2) = (2n+1)\pi + \lambda x_2.$$

– Subcase 2: $u = u_1$.

Then $\partial_1 f \equiv 1$ in $\overline{\mathbb{R}^2_+}$, so that $f(\cdot,x_2)$ is not bounded in $\mathbb{R}$ for every $x_2>0$. This contradicts the fact that, for every fixed $x_2>0$, $f(\cdot,x_2) = (f(\cdot,x_2)-\lambda x_2)+\lambda x_2$ is a bounded function since $f$ is a solution of $(\mathrm{PN}_\lambda)$. Hence, the configuration $(u,v) = (u_1,u_0)$ is not possible.

– Subcase 3: $u = u_2$.

Up to a translation with respect to the variable $x_1$, we assume that for every $x_1\in\mathbb{R}$,

$$\partial_1 f(x_1,0) = \frac{2}{x_1^2+1},$$

using (1.70) in Theorem 1.3.5, from which it follows

$$f(x_1,0) = f(0,0) + 2\arctan(x_1).$$

Furthermore, by (1.65), we have $\cos f(0,0) = u_2(0,0)-1 = 1$ and thus, there exists an integer $n\in\mathbb{Z}$ such that $f(0,0) = 2n\pi$. Since $(x_1,x_2)\mapsto f(x_1,x_2)-\lambda x_2$ and $(x_1,x_2)\mapsto 2n\pi + 2\arctan\left(\frac{x_1}{1+x_2}\right)$ are bounded harmonic functions in $\mathbb{R}^2_+$ which coincide on $\mathbb{R}\times\{0\} = \partial(\mathbb{R}^2_+)\setminus\{\infty\}$, they coincide in $\overline{\mathbb{R}^2_+}$ by the Phragmén-Lindelöf principle. As a consequence, for every $(x_1,x_2)\in\overline{\mathbb{R}^2_+}$,

$$f(x_1,x_2) = 2n\pi + 2\arctan\left(\frac{x_1}{1+x_2}\right) + \lambda x_2.$$

– Subcase 4: $u = u_\alpha$ with $\alpha\in(1,2)$.

Then $u>0$ by $(\mathrm{P1_{BO}})$ and, since $u = \partial_1 f$ and $f$ is bounded on $\mathbb{R}\times\{0\}$, $u$ is integrable on $\mathbb{R}\times\{0\}$. This is a contradiction with Lemma 1.3.8, thus the configuration $(u,v) = (u_\alpha,u_0)$ with $\alpha\in(1,2)$ is not possible.

**Case 2:** $v = u_1$**.**

By (1.63), $\partial_1 f = u-1$ in $\mathbb{R}^2_+$. Note that this equality extends to $\overline{\mathbb{R}^2_+}$ by continuity of $\partial_1 f$ and $u$.

– Subcase 1: $u = u_0$.

Then $\partial_1 f \equiv -1$ in $\overline{\mathbb{R}^2_+}$, and we get a contradiction as in the Subcase 2 of Case 1. Hence, the configuration $(u,v) = (u_0,u_1)$ is not possible.

– Subcase 2: $u = u_1$.

Then $\partial_1 f \equiv 0$ in $\overline{\mathbb{R}^2_+}$ and by (1.65), $\cos f(\cdot,0)\equiv 1$ in $\mathbb{R}$, i.e. there exists an integer $n\in\mathbb{Z}$ such that $f(\cdot,0) = 2n\pi$ in $\mathbb{R}$. Proceeding as in the Subcase 1 of Case 1, we get, for every $(x_1,x_2)\in\overline{\mathbb{R}^2_+}$,

$$f(x_1,x_2) = 2n\pi + \lambda x_2.$$

– Subcase 3: $u = u_2$.

Then $u>0$ by $(\mathrm{P1_{BO}})$ and, since $u = \partial_1 f + 1$ and $f$ is bounded on $\mathbb{R}\times\{0\}$, $u$ is not integrable on $\mathbb{R}\times\{0\}$. This is a contradiction with Lemma 1.3.8, thus the configuration $(u,v) = (u_2,u_1)$ is not possible.

– Subcase 4: $u = u_\alpha$ with $\alpha \in (1, 2)$.

Let $\sigma$ be given at (1.69). Since $u = \partial_1 f + 1$ in $\overline{\mathbb{R}^2_+}$, we have

$$\int_0^{N\pi/\sigma} u(x_1, x_2)\ \mathrm{d}x_1 = \int_0^{N\pi/\sigma} (\partial_1 f(x_1, x_2) + 1)\ \mathrm{d}x_1 = f\left(\frac{N\pi}{\sigma}, x_2\right) - f\left(0, x_2\right) + \frac{N\pi}{\sigma}$$

for every $x_2 > 0$ and $N \in \mathbb{N}^*$. By Lemma 1.3.9 and since $u_\alpha$ is $\frac{\pi}{\sigma}$-periodic in $x_1$ by (P2$_{\mathrm{BO}}$), we get

$$2\pi N = N \int_0^{\pi/\sigma} u(x_1, x_2)\ \mathrm{d}x_1 = f\left(\frac{N\pi}{\sigma}, x_2\right) - f(0, x_2) + \frac{N\pi}{\sigma},$$

i.e.

$$2\pi - \frac{\pi}{\sigma} = \frac{1}{N}\left[\left(f\left(\frac{N\pi}{\sigma}, x_2\right) - \lambda x_2\right) - \left(f(0, x_2) - \lambda x_2\right)\right]$$

for every $x_2 > 0$ and $N \in \mathbb{N}^*$. But $f(\cdot, x_2) - \lambda x_2$ is bounded for every $x_2 > 0$ so that, letting $N$ tend to $+\infty$, we get

$$2\pi = \frac{\pi}{\sigma}$$

that is to say $\alpha = 1$, thanks to (1.69). This is a contradiction, since we assumed $\alpha \in (1, 2)$. Hence, the configuration $(u, v) = (u_\alpha, u_1)$ with $\alpha \in (1, 2)$ is not possible.

**Case 3: $v = u_2$ (or a translation of $u_2$ in the variable $x_1$).**

Up to a translation with respect to the variable $x_1$, we assume that for every $x_1 \in \mathbb{R}$,

$$v(x_1, 0) = \frac{2}{x_1^2 + 1}$$

using (1.70) in Theorem 1.3.5. We clearly have $u(0, 0) = 2$. By (1.65), we have

$$u(\cdot, 0) = 1 + \cos f(\cdot, 0) - v(\cdot, 0) \leqslant 2 - v(\cdot, 0) \quad \text{in } \mathbb{R}.$$

In particular, $u(0, 0) \leqslant 0$ and then, by (P1$_{\mathrm{BO}}$), we necessarily have $u = u_0$. By (1.63) and using the odd symmetry of the problem (PN$_\lambda$) as explained in Remark 1.3.3, we can proceed similarly to the Subcase 3 of Case 1. We get, for every $(x_1, x_2) \in \overline{\mathbb{R}^2_+}$,

$$f(x_1, x_2) = 2n\pi - 2\arctan\left(\frac{x_1}{1 + x_2}\right) + \lambda x_2.$$

**Case 4: $v = u_\alpha$ (or a translation of $u_\alpha$ in the variable $x_1$) with $\alpha \in (1, 2)$.**

Let $\alpha \in (1, 2)$. Up to a translation with respect to the variable $x_1$, we assume that

$$v(x_1, x_2) = \frac{2\sigma\Gamma_\alpha(x_2)}{\cos^2(\sigma x_1) + \Gamma_\alpha(x_2)^2 \sin^2(\sigma x_1)}$$

for every $(x_1, x_2) \in \overline{\mathbb{R}^2_+}$, using (1.68) in Theorem 1.3.5. By (1.63) and using the odd symmetry of the problem (PN$_\lambda$) as explained in Remark 1.3.3, we can exclude the three configurations $(u_0, u_\alpha)$, $(u_1, u_\alpha)$ and $(u_2, u_\alpha)$. Indeed, we can proceed similarly to the Subcase 4 of Case 1 (for $u = u_0$), the Subcase 4 of Case 2 (for $u = u_1$) and the Case 3 (for $u = u_2$). Hence, it remains to consider the case $u = u_{\alpha'}$ for some $\alpha' \in (1, 2)$.

**Lemma 1.3.10.** *There exists $x_0 \in \mathbb{R}$ such that $\partial_1 v(x_0, 0) \neq 0$ and*

$$v(x_0 - x_1, 0) = u(x_0 + x_1, 0) \tag{1.72}$$

*for every $x_1 \in \mathbb{R}$.*

**Lemma 1.3.11.** *Let $x_0 \in \mathbb{R}$ be given by Lemma 1.3.10. Then there exists a unique maximizer $x_M \in \mathbb{R}$ of $v(\cdot, 0)$ that is closest to $x_0$. Moreover, we have*

$$|x_0 - x_M| = \frac{\pi}{4\sigma} \tag{1.73}$$

*and*

$$\cos f(x_M, 0) = 1. \tag{1.74}$$

We refer to Toland [43] – equations (5.9) to (5.13) – for the proofs of both lemmas.

Let $x_0 \in \mathbb{R}$ be given by Lemma 1.3.10. By Lemma 1.3.11, there exists a unique maximizer $x_M$ of $u(\cdot, 0)$ such that the distance between $x_0$ and $x_M$ is $\frac{\pi}{4\sigma}$. This gives us two configurations: either $x_M < x_0$ or $x_M > x_0$. Using the translation invariance of $v(\cdot, 0)$, we can reduce this two subcases to:

– Subcase 1: $x_M = 0$ and $x_0 = \frac{\pi}{4\sigma}$,

– Subcase 2: $x_M = 0$ and $x_0 = -\frac{\pi}{4\sigma}$.

From now on, we assume $x_M = 0$ and we keep the notation $x_0$ (with $x_0 = \pm\frac{\pi}{4\sigma}$). By (1.63) and (1.72), for every $x_1 \in \mathbb{R}$,

$$\partial_1 f(x_0 + x_1, 0) = u(x_0 + x_1, 0) - v(x_0 + x_1, 0) = v(x_0 - x_1, 0) - v(x_0 + x_1, 0),$$

i.e., after a change of variable,

$$\partial_1 f(x_1, 0) = v(2x_0 - x_1, 0) - v(x_1, 0). \tag{1.75}$$

As $x_M = 0$, we deduce that, for every $x_1 \in \mathbb{R}$,

$$\begin{aligned} f(x_1, 0) &= f(x_M, 0) + \int_{x_M}^{x_1} \partial_1 f(s, 0) \mathrm{d}s \\ &= f(0, 0) + \int_0^{x_1} \left(v(2x_0 - s, 0) - v(s, 0)\right) \mathrm{d}s. \end{aligned}$$

By (1.74), there exists $n \in \mathbb{Z}$ such that $f(0, 0) = 2n\pi$, so that

$$f(x_1, 0) = 2n\pi + \int_0^{x_1} \left(v(2x_0 - s, 0) - v(s, 0)\right) \mathrm{d}s$$

for every $x_1 \in \mathbb{R}$. Hence, the function $(x_1, x_2) \mapsto f(x_1, x_2) - \lambda x_2 - 2n\pi$ is harmonic and bounded in $\mathbb{R}^2_+$ – as solution of $(\mathrm{PN}_\lambda)$ because $\sin(2n\pi) = 0$ and $\cos(2n\pi) = 1$ – and its value on $\partial(\mathbb{R}^2_+) \setminus \{\infty\} = \mathbb{R} \times \{0\}$ is

$$\int_0^{x_1} \left(v(2x_0 - s, 0) - v(s, 0)\right) \mathrm{d}s$$

for every $x_1 \in \mathbb{R}$. In order to extend the above function to $\overline{\mathbb{R}^2_+}$, we cite [43, Equation (5.16)]:

**Lemma 1.3.12.** *Assume $x_0 = \pm\frac{\pi}{4\sigma}$. The function $g\colon \mathbb{R}^2_+ \to \mathbb{R}$ defined as*

$$g(x_1,x_2) = \int_0^{x_1} \left(v\left(2x_0 - s, x_2\right) - v\left(s, x_2\right)\right) \mathrm{d}s \tag{1.76}$$

*is harmonic and bounded.*

The functions $g$ defined in Lemma 1.3.12 and $(x_1,x_2) \mapsto f(x_1,x_2) - \lambda x_2 - 2n\pi$ are bounded harmonic functions in $\mathbb{R}^2_+$ which coincide on $\mathbb{R}\times\{0\} = \partial(\mathbb{R}^2_+)\setminus\{0\}$. By the Phragmén-Lindelöf principe, they coincide in $\overline{\mathbb{R}^2_+}$ and thus, for every $(x_1,x_2)\in\overline{\mathbb{R}^2_+}$,

$$f(x_1,x_2) = 2n\pi + \int_0^{x_1} (v(2x_0 - s, x_2) - v(s,x_2))\,\mathrm{d}s + \lambda x_2.$$

We now compute the integral explicitely in order to show that $f$ has the expression given at (1.24) in Theorem 1.1.8.

**Lemma 1.3.13.** *Assume $x_0 = \pm\frac{\pi}{4\sigma}$. For every $x_1 \in \mathbb{R}\setminus\left(\frac{\pi}{2\sigma} + \frac{\pi}{\sigma}\mathbb{Z}\right)$ and $x_2 \geqslant 0$,*

$$\begin{aligned}&\int_0^{x_1} (v(2x_0 - s, x_2) - v(s,x_2))\,\mathrm{d}s \\ &\qquad = 2\left[\arctan\left(\frac{\tan(\sigma x_1)}{\Gamma_\alpha(x_2)}\right) - \arctan\left(\Gamma_\alpha(x_2)\tan(\sigma x_1)\right)\right].\end{aligned}$$

*Proof.* Let $x_2 \geqslant 0$ be fixed. By (1.69), we have $\Gamma_\alpha(x_2) > 0$. For every $x_1 \in (-\frac{\pi}{2\sigma}, \frac{\pi}{2\sigma})$,

$$\begin{aligned}&\int_0^{x_1} (v(2x_0 - s, x_2) - v(s,x_2))\,\mathrm{d}s \\ &= \int_0^{x_1}\left(\frac{2\sigma\Gamma_\alpha(x_2)}{\cos^2\left(\pm\frac{\pi}{2} - \sigma s\right) + \Gamma_\alpha(x_2)^2\sin^2\left(\pm\frac{\pi}{2} - \sigma s\right)}\right. \\ &\qquad\qquad \left. - \frac{2\sigma\Gamma_\alpha(x_2)}{\cos^2(\sigma s) + \Gamma_\alpha(x_2)^2\sin^2(\sigma s)}\right)\mathrm{d}s \\ &= \int_0^{x_1}\left(\frac{2\sigma\Gamma_\alpha(x_2)}{\sin^2(\sigma s) + \Gamma_\alpha(x_2)^2\cos^2(\sigma s)} - \frac{2\sigma\Gamma_\alpha(x_2)}{\cos^2(\sigma s) + \Gamma_\alpha(x_2)^2\sin^2(\sigma s)}\right)\mathrm{d}s \\ &= 2\int_0^{x_1}\left(\frac{\frac{\sigma}{\Gamma_\alpha(x_2)\cos^2(\sigma s)}}{1 + \left(\frac{\tan(\sigma s)}{\Gamma_\alpha(x_2)}\right)^2} - \frac{\frac{\sigma\Gamma_\alpha(x_2)}{\cos^2(\sigma s)}}{1 + (\Gamma_\alpha(x_2)\tan(\sigma s))^2}\right)\mathrm{d}s \\ &= 2\arctan\left(\frac{\tan(\sigma x_1)}{\Gamma_\alpha(x_2)}\right) - 2\arctan\left(\Gamma_\alpha(x_2)\tan(\sigma x_1)\right).\end{aligned}$$

By $\frac{\pi}{\sigma}$-periodicity of $v(\cdot, x_2)$, we extend this equalities to every $x_1 \in \mathbb{R}\setminus\left(\frac{\pi}{2\sigma} + \frac{\pi}{\sigma}\mathbb{Z}\right)$. $\square$

*Remark* 1.3.14. The function

$$(x_1,x_2) \mapsto \arctan\left(\frac{\tan(\sigma x_1)}{\Gamma_\alpha(x_2)}\right) - \arctan\left(\Gamma_\alpha(x_2)\tan(\sigma x_1)\right),$$

defined for every $x_1 \in \mathbb{R}\setminus\left(\frac{\pi}{2\sigma} + \frac{\pi}{\sigma}\mathbb{Z}\right)$ and $x_2 \geqslant 0$, extends smoothly to $\overline{\mathbb{R}^2_+}$ by taking the value zero when $x_1 \in \frac{\pi}{2\sigma} + \frac{\pi}{\sigma}\mathbb{Z}$. In the following, we thus assume the above expression to be defined in $\overline{\mathbb{R}^2_+}$, keeping in mind that it is zero when $\tan(\delta x_1)$ is not well defined.

*Remark* 1.3.15. Inverting the roles of $u$ and $v$, i.e. choosing firstly $u$ and secondly $v$, we get the same candidates $f$ in the first three cases. In Case 4, we only get a sign change in the relation (1.75). Hence, the function $f$ defined as

$$f(x_1, x_2) = 2n\pi - 2\left[\arctan\left(\frac{\tan(\sigma x_1)}{\Gamma_\alpha(x_2)}\right) - \arctan\left(\Gamma_\alpha(x_2)\tan(\sigma x_1)\right)\right] + \lambda x_2,$$

for every $(x_1, x_2) \in \overline{\mathbb{R}^2_+}$, is also a candidate to be a solution of $(\mathrm{PN}_\lambda)$.

We summarize the possible configurations $(u, v)$ in the following table, with the possible expressions for $f$.

| $(u,v)$ | $f(x_1,x_2)$ |
|---|---|
| $(u_0,u_0)$ | $(2n+1)\pi + \lambda x_2$ |
| $(u_0,u_2)$ | $2n\pi + 2\arctan\left(\frac{x_1}{1+x_2}\right) + \lambda x_2$ |
| $(u_1,u_1)$ | $2n\pi + \lambda x_2$ |
| $(u_2,u_0)$ | $2n\pi - 2\arctan\left(\frac{x_1}{1+x_2}\right) + \lambda x_2$ |
| $(u_\alpha,v_\alpha)$ | $2n\pi \pm 2\left[\arctan\left(\frac{\tan(\sigma x_1)}{\Gamma_\alpha(x_2)}\right) - \arctan\left(\Gamma_\alpha(x_2)\tan(\sigma x_1)\right)\right] + \lambda x_2$ |

*Remark* 1.3.16. Note that $(x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto f(x_1, x_2) - \lambda x_2$ takes values in an interval of length less than $2\pi$.

Now it remains to check if all possible functions $f$ in the above table are solutions of $(\mathrm{PN}_\lambda)$. Let us recall

$$\begin{cases} f \in C^\infty(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+}), \\ (x_1,x_2) \mapsto f(x_1,x_2) - \lambda x_2 \text{ is bounded in } \mathbb{R}^2_+, \\ \Delta f = 0 \text{ in } \mathbb{R}^2_+, \\ \partial_2 f - \lambda + \sin f = 0 \text{ on } \mathbb{R}\times\{0\}\,. \end{cases} \tag{$\mathrm{PN}_\lambda$}$$

For any $n \in \mathbb{Z}$, the functions $(x_1, x_2) \mapsto (2n+1)\pi + \lambda x_2$ and $(x_1, x_2) \mapsto 2n\pi + \lambda x_2$ are clearly solutions of $(\mathrm{PN}_\lambda)$, in particular because $\sin(k\pi) = 0$ for every $k \in \mathbb{Z}$. For any $n \in \mathbb{Z}$, the functions

$$f^\pm : (x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto 2n\pi \pm 2\arctan\left(\frac{x_1}{1+x_2}\right) + \lambda x_2$$

are harmonic in $\mathbb{R}^2_+$, of class $C^1$ in $\overline{\mathbb{R}^2_+}$ and the functions

$$(x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto f^\pm(x_1, x_2) - \lambda x_2$$

are bounded. Moreover, for every $x_1 \in \mathbb{R}$, $\partial_2 f^\pm(x_1, 0) - \lambda + \sin f^\pm(x_1, 0) = 0$. This shows that the functions $f^\pm$ are solutions of $(\mathrm{PN}_\lambda)$. To finish with, for $\alpha \in (1, 2)$ and $n \in \mathbb{Z}$ fixed, we consider the function $f^\pm_\alpha : \overline{\mathbb{R}^2_+} \to \mathbb{R}$ defined as

$$f^\pm_\alpha(x_1, x_2) = 2n\pi \pm 2\left[\arctan\left(\frac{\tan(\sigma x_1)}{\Gamma_\alpha(x_2)}\right) - \arctan\left(\Gamma_\alpha(x_2)\tan(\sigma x_1)\right)\right] + \lambda x_2$$

which is harmonic in $\mathbb{R}^2_+$ and of class $C^1$ in $\overline{\mathbb{R}^2_+}$ by Lemmas 1.3.12 and 1.3.13. Moreover, the function $(x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto f^\pm_\alpha(x_1, x_2) - \lambda x_2$ is bounded. It remains to show that, for every $x_1 \in \mathbb{R}$,

$$\partial_2 f^\pm_\alpha(x_1, 0) - \lambda + \sin f^\pm_\alpha(x_1, 0) = 0.$$

First, we compute $\sin f_\alpha(\cdot,0)$. For every $x_1 \in \mathbb{R}$,

$$f_\alpha^\pm(x_1,0) = 2n\pi \pm 2\left(\arctan\left(\frac{1}{\gamma}\tan(\sigma x_1)\right) - \arctan\left(\gamma\tan(\sigma x_1)\right)\right)$$

since $\Gamma_\alpha(0)=\gamma$, and using trigonometric relations, we get

$$\sin f_\alpha^\pm(x_1,0) = \pm\left(\frac{1}{\gamma}-\gamma\right)\frac{2\tan(\sigma x_1)(1+\tan^2(\sigma x_1))}{\left(1+\frac{1}{\gamma^2}\tan^2(\sigma x_1)\right)\left(1+\gamma^2\tan^2(\sigma x_1)\right)}.$$

On the other hand, computing $\partial_2 f_\alpha^\pm(x_1,\cdot)$ for $x_1$ fixed, and using the identities $\Gamma_\alpha(0)=\gamma$ and $\Gamma_\alpha'(0)=\sigma(1-\gamma^2)$, we get

$$\partial_2 f_\alpha^\pm(x_1,0)-\lambda = \mp\sigma\left(1-\gamma^2\right)\left(1+\frac{1}{\gamma^2}\right)\frac{2\tan(\sigma x_1)(1+\tan^2(\sigma x_1))}{\left(1+\frac{1}{\gamma^2}\tan^2(\sigma x_1)\right)\left(1+\gamma^2\tan^2(\sigma x_1)\right)}$$

for every $x_1\in\mathbb{R}$. Hence it remains to check that

$$\frac{1}{\gamma}-\gamma = \sigma\left(1-\gamma^2\right)\left(1+\frac{1}{\gamma^2}\right).$$

This is the following computation:

$$\frac{1}{\gamma}-\gamma-\sigma\left(1-\gamma^2\right)\left(1+\frac{1}{\gamma^2}\right) = \left(\frac{1}{\gamma}-\gamma\right)\frac{\gamma-\sigma-\sigma\gamma^2}{\gamma}$$

where, using (1.69),

$$\gamma-\sigma-\sigma\gamma^2 = \frac{\alpha}{2\sigma}-\sigma-\frac{\alpha^2}{4\sigma} = \frac{2\alpha-4\sigma^2-\alpha^2}{4\sigma} = \frac{1}{4\sigma}\left(2\alpha-\alpha(2-\alpha)-\alpha^2\right)=0.$$

It confirms that $f_\alpha^\pm$ is solution of $(\mathrm{PN}_\lambda)$. The proof of Theorem 1.1.8 is complete.

### 1.3.3 Proof of Theorem 1.1.9

We now study the link between solutions of $(\mathrm{PN}_\lambda)$, given by Theorem 1.1.8, and local minimizers of $E_\varepsilon^\delta$ in $\mathbb{R}^2_+$ in the sense of De Giorgi. More precisely, we get interested in the behaviour of the critical points $\varphi_\varepsilon$ of $E_\varepsilon^\delta$ near the boundary, in order to show the presence of boundary vortices (see [33]). To go through this, we expect $(x_1,x_2)\mapsto\varphi_\varepsilon(x_1,x_2)-\delta_2 x_2$ to be nonconstant and nonperiodic.

Let $\lambda\in\mathbb{R}$. From Theorem 1.1.8, we deduce that the only nonconstant and nonperiodic solutions of $(\mathrm{PN}_\lambda)$, up to substracting $\lambda x_2$, are the functions

$$(x_1,x_2)\in\overline{\mathbb{R}^2_+}\mapsto 2n\pi\pm 2\arctan\left(\frac{x_1+a}{x_2+1}\right)+\lambda x_2$$

where $n\in\mathbb{Z}$ and $a\in\mathbb{R}$. Using (1.60) and (1.61), from the above solutions of $(\mathrm{PN}_\lambda)$, we obtain the following corresponding solutions of (1.57) (see the proof of Proposition 1.3.18 below for more details):

$$(x_1,x_2)\in\overline{\mathbb{R}^2_+}\mapsto n\pi\pm\arctan\left(\frac{x_1+\varepsilon a}{x_2+\varepsilon}\right)+\delta_2 x_2-\frac{\pi}{2} \tag{1.77}$$

where $n\in\mathbb{Z}$ and $a\in\mathbb{R}$.

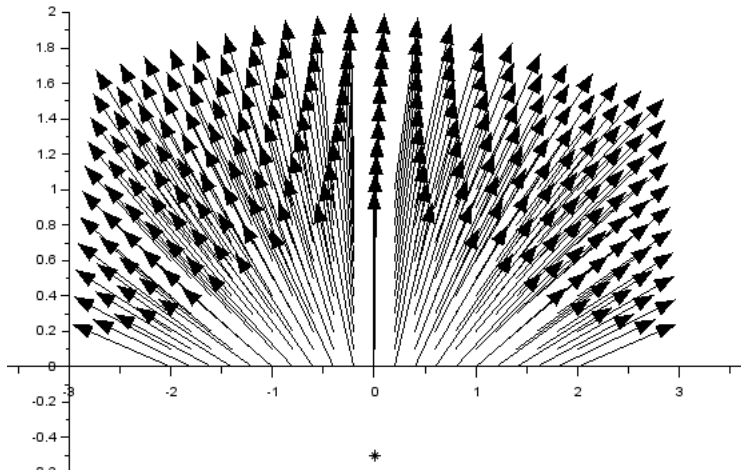


Figure 1.1: Magnetization $m = e^{i\varphi}$ when $\varphi$ is of the form (1.77) with $n = 1$, $\pm = -$, $a = 0$, $\varepsilon = 1/2$ and $\delta_2 = -1/10$.

This type of solutions is particularly relevant, because it illustrates a boundary vortex for the domain $\mathbb{R}^2_+$ at the point $(0, 0)$ when taking the limits when $\varepsilon \to 0$ (at scale $\varepsilon$, the vortex point is the point of coordinates $(-\varepsilon a, -\varepsilon)$). Indeed, assuming for a while that $\varepsilon$ and $\delta_2$ are negligible, we can test some combinations of $(n, \pm)$. The case $(n, \pm) = (1, -)$ gives a boundary vortex of degree $+1/2$ (see [7], [33] or [24] for more information on the degree of $\mathbb{S}^1$-valued maps), and the magnetization $m = e^{i\varphi}$ seems like escaping from the point of coordinates $(-\varepsilon a, -\varepsilon)$ and behaves like $e^{i\theta}$ (see Figure 1.1). The case $(n, \pm) = (0, +)$ gives a boundary vortex of degree $-1/2$, and the magnetization $m = e^{i\varphi}$ seems like converging to the point of coordinates $(-\varepsilon a, -\varepsilon)$ and behaves like $e^{-i\theta}$. Figure 1.2 shows the influence of $\delta_2$ in the interior of the domain.

The rest of this section is devoted to prove Theorem 1.1.9, i.e. we show that under the conditions (1.27), the local minimizers $\varphi_\varepsilon$ of $E^\delta_\varepsilon$ in $\mathbb{R}^2_+$ in the sense of De Giorgi are the functions in (1.77) that correspond to the case $(n, \pm) = (1, -)$, which is the case of a boundary vortex of degree $+1/2$.

To begin with, we show in Proposition 1.3.18 that any local minimizer of $E^\delta_\varepsilon$ in $\mathbb{R}^2_+$ in the sense of De Giorgi with conditions (1.27) must be of the expected form (1.26). Conversely, for proving that functions of the form (1.26) are such local minimizers, we introduce the following definition.

**Definition 1.3.17.** A function $\psi$ is a layer function associated to $E^\delta_\varepsilon$ in the sense of Cabré and Solà-Morales if it satisfies

$$\begin{cases} \Delta\psi &= 0 & \text{in } \mathbb{R}^2_+, \\ \partial_2\psi &= \frac{1}{2\varepsilon}\sin(2\psi) + \delta_2 & \text{on } \mathbb{R}\times\{0\}, \end{cases}$$

and for every $x_1 \in \mathbb{R}$,

$$\partial_1\psi(x_1, 0) < 0, \lim_{x_1\to+\infty} \psi(x_1, 0) = 0 \text{ and } \lim_{x_1\to-\infty} \psi(x_1, 0) = \pi.$$

These layer functions were studied by Cabré and Solà-Morales in [10]. In that paper, properties in dimension two are given on the half plane $(0, +\infty)\times\mathbb{R}$, while we consider here $\mathbb{R}^2_+ = \mathbb{R}\times(0, +\infty)$. The correspondence between a layer function $\psi$ in Definition 1.3.17 and a layer function $u$ in [10]

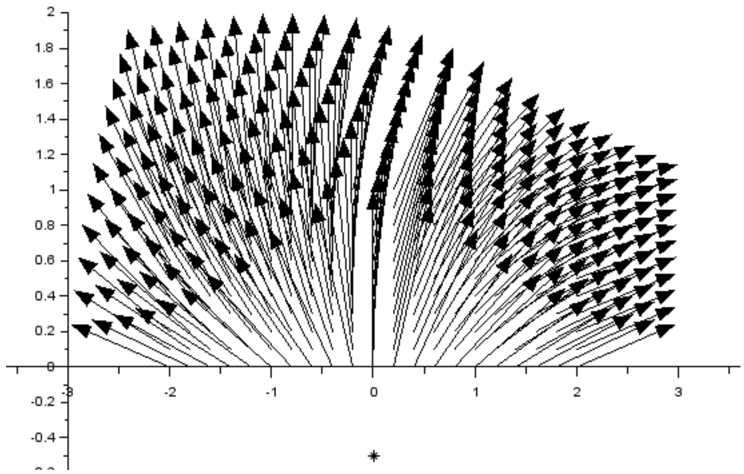


Figure 1.2: Magnetization $m = e^{i\varphi}$ when $\varphi$ is of the form (1.77) with $n = 1$, $\pm = -$, $a = 0$, $\varepsilon = 1/2$ and $\delta_2 = -1/2$.

is given by the relation

$$\psi(x_1, x_2) = \frac{\pi}{2}\left(1 - u(-x_2, x_1)\right)$$

for any $(x_1, x_2) \in \mathbb{R}^2_+$. In [10, Lemma 3.1], Cabré and Solà-Morales prove that, if $r > 0$ and $\psi$ is a layer function associated to $E^\delta_\varepsilon$, then $\psi$ is the unique weak solution of the problem

$$\begin{cases} \Delta\varphi = 0 & \text{in } B_r^+, \\ \partial_2\varphi = \frac{1}{2\varepsilon}\sin(2\varphi) + \delta_2 & \text{on } (-r, r) \times \{0\}, \\ 0 \leqslant \varphi(x_1, x_2) - \delta_2 x_2 \leqslant \pi & \text{for every } (x_1, x_2) \in B_r^+, \\ \varphi = \psi & \text{on } \partial B_r^+ \cap \mathbb{R}^2_+. \end{cases} \tag{1.78}$$

The method for proving this uniqueness property is the sliding method. This property will allow us to show that functions $\varphi_\varepsilon$ given at (1.26) are local minimizers of $E^\delta_\varepsilon$ in $\mathbb{R}^2_+$ in the sense of De Giorgi that satisfy (1.27). Our strategy of proof is the following: let $\varphi_\varepsilon$ be given at (1.26). Given $r > 0$ and a minimizer $\widetilde{\varphi}_\varepsilon$ of $E^\delta_\varepsilon(\cdot; B_r)$ such that $\widetilde{\varphi}_\varepsilon = \varphi_\varepsilon$ on $\partial B_r^+ \cap \mathbb{R}^2_+$, we will prove in Proposition 1.3.19 that there exists a minimizer $\widehat{\varphi}_\varepsilon$ of $E^\delta_\varepsilon(\cdot; B_r)$ that satisfies (1.78) with $\psi = \varphi_\varepsilon$ and $\varphi = \widehat{\varphi}_\varepsilon$. In Proposition 1.3.20, we show that $\varphi_\varepsilon$ is a layer function associated to $E^\delta_\varepsilon$ in the sense of Cabré and Solà-Morales. Hence by [10, Lemma 3.1], it follows that $\varphi_\varepsilon = \widehat{\varphi}_\varepsilon$ is a minimizer of $E^\delta_\varepsilon(\cdot; B_r)$ with the boundary condition $\varphi_\varepsilon$ on $\partial B_r^+ \cap \mathbb{R}^2_+$. We finally conclude since $r > 0$ is arbitrary.

**Proposition 1.3.18.** *Let $\varphi_\varepsilon \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ be a local minimizer of $E^\delta_\varepsilon$ in $\mathbb{R}^2_+$ in the sense of De Giorgi such that* (1.27) *is satisfied, i.e.*

$$\lim_{x_1 \to +\infty} \varphi_\varepsilon(x_1, 0) = 0, \quad \lim_{x_1 \to -\infty} \varphi_\varepsilon(x_1, 0) = \pi \text{ and } [(x_1, x_2) \mapsto \varphi_\varepsilon(x_1, x_2) - \delta_2 x_2] \in L^\infty(\mathbb{R}^2_+).$$

*Then*

$$\varphi_\varepsilon(x_1, x_2) = \frac{\pi}{2} - \arctan\left(\frac{x_1 + \varepsilon a}{x_2 + \varepsilon}\right) + \delta_2 x_2$$

*for some $a \in \mathbb{R}$.*

*Proof.* By Proposition 1.3.2, $\varphi_\varepsilon \in C^\infty(\overline{\mathbb{R}^2_+})$ and satisfies (1.57). Set

$$\phi_\varepsilon \colon (x_1, x_2) \in \overline{\mathbb{R}^2_+} \mapsto 2\varphi_\varepsilon(\varepsilon x_1, \varepsilon x_2) + \pi,$$

as in (1.60). Then $\phi_\varepsilon$ satisfies (1.61), i.e.

$$\begin{cases} \Delta\phi_\varepsilon &= 0 \quad \text{in } \mathbb{R}^2_+, \\ \partial_2\phi_\varepsilon - \lambda_\varepsilon + \sin(\phi_\varepsilon) &= 0 \quad \text{on } \mathbb{R} \times \{0\}, \end{cases}$$

with $\lambda_\varepsilon = 2\varepsilon\delta_2$. Moreover $\phi_\varepsilon \in C^\infty(\overline{\mathbb{R}^2_+})$ and $(x_1, x_2) \mapsto \phi_\varepsilon(x_1, x_2) - \lambda_\varepsilon x_2$ is bounded in $\mathbb{R}^2_+$, because

$$\phi_\varepsilon(x_1, x_2) - \lambda_\varepsilon x_2 = 2\varphi_\varepsilon(\varepsilon x_1, \varepsilon x_2) + \pi - 2\varepsilon\delta_2 x_2 = 2\underbrace{(\varphi_\varepsilon(\varepsilon x_1, \varepsilon x_2) - \delta_2\varepsilon x_2)}_{\in L^\infty(\mathbb{R}^2_+)} + \pi$$

for every $(x_1, x_2) \in \mathbb{R}^2_+$. It follows that $\phi_\varepsilon$ satisfies $(\mathrm{PN}_{\lambda_\varepsilon})$. By Theorem 1.1.8, $\phi_\varepsilon$ must be one of the three following types of functions:

– Firstly, the functions $(x_1, x_2) \mapsto n\pi + \lambda_\varepsilon x_2$ for some $n \in \mathbb{Z}$. However, on the boundary line $\mathbb{R} \times \{0\}$, these functions are constant (equal to $n\pi$). Hence, the boundary conditions $\lim_{x_1 \to +\infty} \varphi_\varepsilon(x_1, 0) = 0$ and $\lim_{x_1 \to -\infty} \varphi_\varepsilon(x_1, 0) = \pi$ cannot be satisfied, and $\phi_\varepsilon$ is not of this first form.

– Secondly, the $x_1$-periodic functions given at (1.24). However, the boundary conditions $\lim_{x_1 \to +\infty} \varphi_\varepsilon(x_1, 0) = 0$ and $\lim_{x_1 \to -\infty} \varphi_\varepsilon(x_1, 0) = \pi$ are not compatible with the periodicity in the variable $x_1$. Thus $\phi_\varepsilon$ is not of this second form.

– Thirdly, the functions

$$(x_1, x_2) \mapsto 2n\pi \pm 2\arctan\left(\frac{x_1 + a}{x_2 + 1}\right) + \lambda_\varepsilon x_2$$

for some $n \in \mathbb{Z}$ and $a \in \mathbb{R}$. Coming back to $\varphi_\varepsilon$ instead of $\phi_\varepsilon$, we get, for every $(x_1, x_2) \in \overline{\mathbb{R}^2_+}$,

$$2\varphi_\varepsilon(\varepsilon x_1, \varepsilon x_2) + \pi = 2n\pi \pm 2\arctan\left(\frac{x_1 + a}{x_2 + 1}\right) + 2\varepsilon\delta_2 x_2,$$

i.e.

$$\varphi_\varepsilon(x_1, x_2) = n\pi \pm \arctan\left(\frac{x_1 + \varepsilon a}{x_2 + \varepsilon}\right) + \delta_2 x_2 - \frac{\pi}{2}.$$

Since $\lim_{x_1 \to +\infty} \varphi_\varepsilon(x_1, 0) = 0$, we deduce that $n\pi \pm \frac{\pi}{2} - \frac{\pi}{2} = 0$. It follows that the pair $(n, \pm)$ is either $(1, -)$ or $(0, +)$. As $\lim_{x_1 \to -\infty} \varphi_\varepsilon(x_1, x_2) = \pi$, we also have $n\pi \mp \frac{\pi}{2} - \frac{\pi}{2} = \pi$, hence the only possible pair $(n, \pm)$ is $(1, -)$. As a consequence, for every $(x_1, x_2) \in \overline{\mathbb{R}^2_+}$,

$$\varphi_\varepsilon(x_1, x_2) = \frac{\pi}{2} - \arctan\left(\frac{x_1 + \varepsilon a}{x_2 + \varepsilon}\right) + \delta_2 x_2.$$

□

**Proposition 1.3.19.** *Let $r > 0$ and $\varphi_\varepsilon$ be given at* (1.26)*. Let $\widetilde{\varphi}_\varepsilon$ be a minimizer of $E^\delta_\varepsilon(\cdot; B_r)$ such that $\widetilde{\varphi}_\varepsilon = \varphi_\varepsilon$ on $\partial B^+_r \cap \mathbb{R}^2_+$. Then there exists a minimizer $\widehat{\varphi}_\varepsilon$ of $E^\delta_\varepsilon(\cdot; B_r)$ that satisfies $\widehat{\varphi}_\varepsilon = \varphi_\varepsilon$ on $\partial B^+_r \cap \mathbb{R}^2_+$ and $0 \leqslant \widehat{\varphi}_\varepsilon(x_1, x_2) - \delta_2 x_2 \leqslant \pi$ for every $(x_1, x_2) \in \overline{B^+_r}$.*

*Proof.* We define $\widehat{\varphi}_\varepsilon$ in $\overline{B_r^+}$ as follows:

$$\widehat{\varphi}_\varepsilon(x_1,x_2) = \delta_2 x_2 + \begin{cases} \widetilde{\varphi}_\varepsilon(x_1,x_2) - \delta_2 x_2 & \text{if } 0 \leqslant \widetilde{\varphi}_\varepsilon(x_1,x_2) - \delta_2 x_2 \leqslant \pi, \\ 0 & \text{if } \widetilde{\varphi}_\varepsilon(x_1,x_2) - \delta_2 x_2 \leqslant 0, \\ \pi & \text{if } \widetilde{\varphi}_\varepsilon(x_1,x_2) - \delta_2 x_2 \geqslant \pi. \end{cases} \tag{1.79}$$

It is obvious that $0 \leqslant \widehat{\varphi}_\varepsilon(x_1,x_2) - \delta_2 x_2 \leqslant \pi$ in $\overline{B_r^+}$. Moreover, on $\partial B_r^+ \cap \mathbb{R}^2_+$,

$$\widetilde{\varphi}_\varepsilon(x_1,x_2) - \delta_2 x_2 = \varphi_\varepsilon(x_1,x_2) - \delta_2 x_2 \in [0,\pi],$$

thus $\widehat{\varphi}_\varepsilon(x_1,x_2) = \delta_2 x_2 + \widetilde{\varphi}_\varepsilon(x_1,x_2) - \delta_2 x_2 = \widetilde{\varphi}_\varepsilon(x_1,x_2) = \varphi_\varepsilon(x_1,x_2)$. For proving that $\widehat{\varphi}_\varepsilon$ is a minimizer of $E_\varepsilon^\delta(\cdot;B_r)$, it suffices to show that $E_\varepsilon^\delta(\widehat{\varphi}_\varepsilon;B_r) \leqslant E_\varepsilon^\delta(\widetilde{\varphi}_\varepsilon;B_r)$.

For the boundary integral on $B_r \cap (\mathbb{R}\times\{0\}) = (-r,r)\times\{0\}$, we note that on this line segment, $\widehat{\varphi}_\varepsilon$ is equal either to $\widetilde{\varphi}_\varepsilon$, or to 0, or to $\pi$. Thus $\sin^2\widehat{\varphi}_\varepsilon \leqslant \sin^2\widetilde{\varphi}_\varepsilon$, and

$$\int_{(-r,r)\times\{0\}} \sin^2\widehat{\varphi}_\varepsilon \, \mathrm{d}\mathcal{H}^1 \leqslant \int_{(-r,r)\times\{0\}} \sin^2\widetilde{\varphi}_\varepsilon \, \mathrm{d}\mathcal{H}^1.$$

For the interior integral, we note that

$$\begin{aligned}&\int_{B_r^+} \left(|\nabla\widehat{\varphi}_\varepsilon|^2 - 2\delta\cdot\nabla\widehat{\varphi}_\varepsilon\right)\mathrm{d}x \\ &\qquad = \int_{B_r^+}\left(|\partial_1\widehat{\varphi}_\varepsilon|^2 - 2\delta_1\partial_1\widehat{\varphi}_\varepsilon\right)\mathrm{d}x + \int_{B_r^+}|\partial_2\widehat{\varphi}_\varepsilon - \delta_2|^2\,\mathrm{d}x - \delta_2^2\left|B_r^+\right|.\end{aligned}$$

On the one hand, using (1.79), we have

$$|\partial_1\widehat{\varphi}_\varepsilon| = \begin{cases} |\partial_1\widetilde{\varphi}_\varepsilon| & \text{if } 0 \leqslant \widetilde{\varphi}_\varepsilon(x_1,x_2) - \delta_2 x_2 \leqslant \pi, \\ 0 & \text{elsewhere,}\end{cases}$$

thus $|\partial_1\widehat{\varphi}_\varepsilon| \leqslant |\partial_1\widetilde{\varphi}_\varepsilon|$ in $B_r^+$, and $\int_{B_r^+}|\partial_1\widehat{\varphi}_\varepsilon|^2\,\mathrm{d}x \leqslant \int_{B_r^+}|\partial_1\widetilde{\varphi}_\varepsilon|^2\,\mathrm{d}x$. Moreover,

$$\int_{B_r^+}\partial_1\widehat{\varphi}_\varepsilon\,\mathrm{d}x = \int_{\partial B_r^+\cap\mathbb{R}^2_+}\widehat{\varphi}_\varepsilon\nu_1\,\mathrm{d}\mathcal{H}^1 = \int_{\partial B_r^+\cap\mathbb{R}^2_+}\varphi_\varepsilon\nu_1\,\mathrm{d}\mathcal{H}^1 = \int_{\partial B_r^+\cap\mathbb{R}^2_+}\widetilde{\varphi}_\varepsilon\nu_1\,\mathrm{d}\mathcal{H}^1 = \int_{B_r^+}\partial_1\widetilde{\varphi}_\varepsilon\,\mathrm{d}x,$$

using that $\nu_1 = 0$ on $B_r \cap (\mathbb{R}\times\{0\})$. On the other hand, using (1.79),

$$|\partial_2\widehat{\varphi}_\varepsilon - \delta_2| = \begin{cases} |\partial_2\widetilde{\varphi}_\varepsilon - \delta_2| & \text{if } 0 \leqslant \widetilde{\varphi}_\varepsilon(x_1,x_2) - \delta_2 x_2 \leqslant \pi, \\ 0 & \text{elsewhere,}\end{cases}$$

thus $|\partial_2\widehat{\varphi}_\varepsilon - \delta_2| \leqslant |\partial_2\widetilde{\varphi}_\varepsilon - \delta_2|$ in $B_r^+$, and $\int_{B_r^+}|\partial_2\widehat{\varphi}_\varepsilon - \delta_2|^2\,\mathrm{d}x \leqslant \int_{B_r^+}|\partial_2\widetilde{\varphi}_\varepsilon - \delta_2|^2\,\mathrm{d}x$. Combining the above inequalities comparing the boundary and interior parts of $E_\varepsilon^\delta(\widehat{\varphi}_\varepsilon;B_r)$ and $E_\varepsilon^\delta(\widetilde{\varphi}_\varepsilon;B_r)$, we deduce that $E_\varepsilon^\delta(\widehat{\varphi}_\varepsilon;B_r) \leqslant E_\varepsilon^\delta(\widetilde{\varphi}_\varepsilon;B_r)$. □

**Proposition 1.3.20.** *Let $\varphi_\varepsilon$ be given at* (1.26)*, i.e.*

$$\varphi_\varepsilon\colon (x_1,x_2)\in\overline{\mathbb{R}^2_+} \mapsto \frac{\pi}{2} - \arctan\left(\frac{x_1+\varepsilon a}{x_2+\varepsilon}\right) + \delta_2 x_2$$

*for some $a\in\mathbb{R}$. Then $\varphi_\varepsilon$ is a layer function associated to $E_\varepsilon^\delta$ in the sense of Cabré and Solà-Morales.*

*Proof.* It is clear that $\varphi_\varepsilon \in C^\infty(\overline{\mathbb{R}^2_+})$ and is a harmonic function in $\mathbb{R}^2_+$. The boundary condition

$$\partial_2\varphi_\varepsilon(x_1,0) = \frac{1}{2\varepsilon}\sin(2\varphi_\varepsilon(x_1,0)) + \delta_2$$

for every $x_1 \in \mathbb{R}$ follows from a standard calculation. Moreover, for every $x_1 \in \mathbb{R}$,

$$\partial_1\varphi_\varepsilon(x_1,0) = \frac{-\varepsilon}{(x_1+\varepsilon a)^2+\varepsilon^2} < 0,$$

and we also have $\lim_{x_1\to+\infty}\varphi_\varepsilon(x_1,0) = \frac{\pi}{2} - \frac{\pi}{2} = 0$ and $\lim_{x_1\to-\infty}\varphi_\varepsilon(x_1,0) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$. □

*Proof of Theorem 1.1.9.* Let $\varphi_\varepsilon \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ be a local minimizer of $E^\delta_\varepsilon$ in $\mathbb{R}^2_+$ in the sense of De Giorgi that satisfies (1.27). By Proposition 1.3.18, $\varphi_\varepsilon$ is of the form (1.26).

It remains to check that functions $\varphi_\varepsilon$ of the form (1.26) are local minimizers of $E^\delta_\varepsilon$ in $\mathbb{R}^2_+$ in the sense of De Giorgi that satisfy (1.27). Let $\varphi_\varepsilon$ be given at (1.26). Then conditions (1.27) are directly satisfied and by Proposition 1.3.20, $\varphi_\varepsilon$ is a layer function associated to $E^\delta_\varepsilon$ in the sense of Cabré and Solà-Morales. Let $r > 0$. By the direct method in the calculus of variations, $E^\delta_\varepsilon(\cdot; B_r)$ admits a minimizer $\widetilde{\varphi}_\varepsilon \in H_1(B^+_r)$ such that $\widetilde{\varphi}_\varepsilon = \varphi_\varepsilon$ on $\partial B^+_r \cap \mathbb{R}^2_+$. By Proposition 1.3.19, there exists a minimizer $\widehat{\varphi}_\varepsilon$ of $E^\delta_\varepsilon(\cdot; B_r)$ such that $\widehat{\varphi}_\varepsilon = \varphi_\varepsilon$ on $\partial B^+_r \cap \mathbb{R}^2_+$ and $0 \leqslant \widehat{\varphi}_\varepsilon(x_1,x_2) - \delta_2 x_2 \leqslant \pi$ for every $(x_1,x_2) \in \overline{B^+_r}$. Moreover,

$$\begin{cases} \Delta\widehat{\varphi}_\varepsilon &= 0 & \text{in } B^+_r, \\ \partial_2\widehat{\varphi}_\varepsilon &= \frac{1}{2\varepsilon}\sin 2\widehat{\varphi}_\varepsilon + \delta_2 & \text{on } (-r,r)\times\{0\}. \end{cases}$$

similarly than in Proposition 1.3.2. It follows that (1.78) is satisfied with $\psi = \varphi_\varepsilon$ and $\varphi = \widehat{\varphi}_\varepsilon$. By [10, Lemma 3.1], $\varphi_\varepsilon = \widehat{\varphi}_\varepsilon$, thus $\varphi_\varepsilon$ is a minimizer of $E^\delta_\varepsilon(\cdot; B_r)$ with the boundary condition $\varphi_\varepsilon$ on $\partial B^+_r \cap \mathbb{R}^2_+$. This fact being true for every $r > 0$, $\varphi_\varepsilon$ is a local minimizer of $E^\delta_\varepsilon$ in $\mathbb{R}^2_+$ in the sense of De Giorgi. □

*Remark* 1.3.21. In dimension greater than two, solving ($\mathrm{PN}_\lambda$) and finding local minimizers of $E^\delta_\varepsilon$ in the sense of De Giorgi is more difficult because there is no classification as for the Benjamin-Ono problem. In the case where $\delta = 0$, Cabré and Solà-Morales [10] introduce the notion of layer solution of ($\mathrm{PN}_0$) (or similar problems with a different nonlinearity instead of sin) and show that a layer solution of ($\mathrm{PN}_0$) is a local minimizer of $E^0_\varepsilon$ in the sense of De Giorgi in any dimension.

# Chapter 2

## Gamma-convergence of the micromagnetic energy in a thin-film regime for boundary vortices and renormalized energy between boundary vortices

**Abstract**

We consider the three-dimensional micromagnetic model with Dzyaloshinskii-Moriya interaction in a thin-film regime for boundary vortices. We reduce the three-dimensional model to a two-dimensional model in the considered regime, and prove the Gamma-convergence of the reduced model by computing an asymptotic expansion of the micromagnetic energy at the second order. We study the existence of minimizers and compute explicitely the renormalized energy, that represents the interaction between boundary vortices, involved at the second order of the asymptotic expansion. Finally, we prove the Gamma-convergence for the three-dimensional model and deduce the concentration of the energy around boundary vortices. This chapter is based on results of Ignat-Kurzke [24], [25] that do not take in account the Dzyaloshinskii-Moriya interaction, and contains some statements to appear in [27].

## Contents

## 2.1 Introduction

We begin this chapter with recalling the frame of the theory of micromagnetics, already mentioned in the introduction of Chapter 1.

### 2.1.1 The general three-dimensional model and a nondimensionalization in length

We consider a ferromagnetic sample of cylindrical shape

$$\Omega_t^\ell = \Omega^\ell \times (0,t) \subset \mathbb{R}^3,$$

where $\Omega^\ell \subset \mathbb{R}^2$ is a bounded, simply connected and $C^{1,1}$ smooth domain of typical length $\ell$ (for example, $\Omega^\ell$ can be assumed to be an open disk of diameter $\ell$).

The magnetization $m$ is a unitary three-dimensional vector field

$$m \colon \Omega_t^\ell \to \mathbb{S}^2,$$

where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. The constraint $|m| = 1$ yields the non-convexity of the problem. We consider the micromagnetic energy $E(m)$ given at (1.3) in the case $\Phi = 0$ and $H_{\text{ext}} = 0$, i.e.

$$E(m) = A^2 \int_{\Omega_t^\ell} |\nabla m|^2 \,\mathrm{d}x + \int_{\Omega_t^\ell} D : \nabla m \wedge m \,\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x. \tag{2.1}$$

The first term in the micromagnetic energy is the exchange energy, generated by small-distance interactions in the sample. It involves the exchange length $A > 0$ that is an intrinsic parameter of the ferromagnetic material, typically of the order of nanometers. The second term is the Dzyaloshinskii-Moriya interaction. This interaction was introduced in the 1950s [17] to describe the magnetization in some materials with few symmetry properties. We assume here that the Dzyaloshinskii-Moriya interaction density in three dimensions is defined as

$$D : \nabla m \wedge m = \sum_{j=1}^{3} D_j \cdot \partial_j m \wedge m, \tag{2.2}$$

where $D = (D_1, D_2, D_3) \in \mathbb{R}^{3\times 3}$, $\cdot$ denotes the inner product in $\mathbb{R}^3$, and $\wedge$ denotes the cross product in $\mathbb{R}^3$. The third term is called magnetostatic or demagnetizing energy. This energy is generated by the large-distance interactions in the sample. It is in fact the energy generated by the magnetic field induced by magnetization. More precisely, the demagnetizing potential $u \in H^1(\mathbb{R}^3, \mathbb{R})$ satisfies

$$\Delta u = \operatorname{div}(m \mathbb{1}_{\Omega_t^\ell}) \text{ in the distributional sense in } \mathbb{R}^3, \tag{2.3}$$

where $\mathbb{1}_{\Omega_t^\ell}(x) = 1$ if $x \in \Omega_t^\ell$, and $\mathbb{1}_{\Omega_t^\ell}(x) = 0$ elsewhere. For more details about the components of the micromagnetic energy, especially physical interpretations, we refer to [1], [20] or [17].

The multiscale aspect of the micromagnetic energy (2.1) is obvious. Indeed, beside the tensor $D$, three length paramaters of the ferromagnetic device interact together: the exchange length $A$, the planar diameter $\ell$ and the thickness $t$ of the sample. From these parameters, we introduce the dimensionless parameters

$$h = \frac{t}{\ell} \quad \text{and} \quad \eta = \frac{A}{\ell}.$$

By letting $h$ tend to zero, the relative thickness of the ferromagnetic device tends to zero: it is a thin-film limit. The consequences concerning the magnetization and the micromagnetic energy depend on the relations between $h$ and $\eta$, i.e. on the thin-film asymptotic regime.

In order to study the micromagnetic energy in a thin-film regime, it is convenient to nondimensionalize it in length ; in particular, we get from the three length parameters $A$, $\ell$ and $t$ only two dimensionless parameters $h$ and $\eta$ defined above. We set

$$\Omega_h = \frac{\Omega_t^\ell}{\ell} = \Omega \times (0, h) \subset \mathbb{R}^3,$$

where $\Omega = \frac{\Omega^\ell}{\ell} \subset \mathbb{R}^2$ is a bounded, simply connected and $C^{1,1}$ smooth domain of typical length 1 (for example, $\Omega$ can be assumed to be the unit disk in $\mathbb{R}^2$). To each $x = (x_1, x_2, x_3) \in \Omega$, we associate $\widehat{x} = \frac{x}{\ell} \in \Omega_h$ and we set $\widehat{D} = \frac{1}{\ell} D$. We also consider the maps $m_h \colon \Omega_h \to \mathbb{S}^2$ and $u_h \colon \mathbb{R}^3 \to \mathbb{R}$ such that, for every $\widehat{x} = \frac{x}{\ell} \in \Omega_h$,

$$m_h(\widehat{x}) = m(x), \quad u_h(\widehat{x}) = \frac{1}{\ell} u(x),$$

that satisfy

$$\Delta u_h = \operatorname{div}(m_h \mathbb{1}_{\Omega_h}) \text{ in the distributional sense in } \mathbb{R}^3. \tag{2.4}$$

The micromagnetic energy (2.1) can then be written in terms of $m_h$:

$$\widehat{E}(m_h) = \ell^3 \left[ \eta^2 \int_{\Omega_h} |\nabla m_h|^2 \mathrm{d}\widehat{x} + \int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \, \mathrm{d}\widehat{x} + \int_{\mathbb{R}^3} |\nabla u_h|^2 \mathrm{d}\widehat{x} \right]. \tag{2.5}$$

For simplicity of the notations, we write $x$ instead of $\widehat{x}$ in the following.

### 2.1.2 Global Jacobian

Before giving the main results of Chapter 2, we present the notion of global Jacobian introduced by Ignat and Kurzke [24], [25], that will play a central role in the sequel.

For a two-dimensional map $m' \in H^1(\Omega, \mathbb{R}^2)$ defined in a bounded and $C^{1,1}$ smooth domain $\Omega \subset \mathbb{R}^2$, we call global Jacobian of $m'$ the linear operator $\mathcal{J}(m') \colon W^{1,\infty}(\Omega) \to \mathbb{R}$ defined as

$$\langle \mathcal{J}(m'), \zeta \rangle = - \int_\Omega m' \wedge \nabla' m' \cdot \nabla'^{\perp} \zeta \, \mathrm{d}x',$$

for every Lipschitz function $\zeta \colon \Omega \to \mathbb{R}$, where $m' \wedge \nabla' m' = (m' \wedge \partial_1 m', m' \wedge \partial_2 m')$ with the usual notation $\wedge$, $\nabla'^{\perp} = (-\partial_2, \partial_1)$ and $\langle \cdot, \cdot \rangle$ stands for the algebraic dual pairing between $(W^{1,\infty}(\Omega))^*$ and $W^{1,\infty}(\Omega)$. In particular, the global Jacobian has zero average, i.e. $\langle \mathcal{J}(m'), 1 \rangle = 0$.

On the one hand, considering a test function $\zeta \in W^{1,\infty}(\Omega)$ that vanishes on the boundary $\partial\Omega$, we have, using integration by parts,

$$\langle \mathcal{J}(m'), \zeta \rangle = \int_\Omega \nabla'^{\perp} (m' \wedge \nabla' m') \zeta \, \mathrm{d}x' = \int_\Omega 2 \operatorname{jac}(m') \zeta \, \mathrm{d}x' = \langle 2 \operatorname{jac}(m'), \zeta \rangle,$$

where $\operatorname{jac}(m') = \det(\nabla' m') = \partial_1 m' \wedge \partial_2 m' \in L^1(\Omega)$ is called the interior Jacobian of $m'$. Hence, the global Jacobian carries the topological information in the interior of $\Omega$, where it coincides with twice the interior Jacobian, that detects the interior vortices.

On the other hand, the global Jacobian also carries the topological information at the boundary $\partial\Omega$ and enables us to detect boundary vortices. More precisely, we define the boundary Jacobian of $m'$ to be the linear operator $\mathcal{J}_{\mathrm{bd}}(m') \colon W^{1,\infty}(\Omega) \to \mathbb{R}$ as

$$\mathcal{J}_{\mathrm{bd}}(m') = \mathcal{J}(m') - 2 \operatorname{jac}(m').$$

The operator $\mathcal{J}_{\mathrm{bd}}(m')$ acts only on the boundary of $\partial\Omega$ (see [24, Proposition 2.2]). In particular, if $m' \in C^2(\overline{\Omega}, \mathbb{R}^2)$, then for every Lipschitz function $\zeta\colon \Omega \to \mathbb{R}$,

$$\langle \mathcal{J}_{\mathrm{bd}}(m'), \zeta\rangle = -\int_{\partial\Omega} (m' \wedge \partial_\tau m')\zeta \,\mathrm{d}\mathcal{H}^1,$$

where $\tau = \nu^\perp = (-\nu_2, \nu_1)$ is the tangent vector at $\partial\Omega$ such that $(\nu, \tau)$ forms an oriented frame, and $\partial_\tau$ denotes the derivative along the boundary.

For a $\mathbb{S}^1$-valued map $m'$ given through a smooth lifting $\varphi \in C^2(\overline{\Omega}, \mathbb{R})$, i.e. $m' = (\cos\varphi, \sin\varphi)$ in $\Omega$, the interior Jacobian $\mathrm{jac}(m')$ vanishes in $\Omega$, so that the whole topological information is carried by the tangential derivative of $\varphi$ at the boundary, i.e.

$$\mathrm{jac}(m') = 0, \quad \mathcal{J}(m') = \mathcal{J}_{\mathrm{bd}}(m') = -\partial_\tau\varphi\mathcal{H}^1\llcorner\partial\Omega \quad\text{and}\quad \langle \mathcal{J}_{\mathrm{bd}}(m'), 1\rangle = 0.$$

In the following, we will be led to consider the measure $J$ given by

$$J = -\kappa\mathcal{H}^1\llcorner\partial\Omega + \pi\sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}}$$

supported on the boundary $\partial\Omega$ of a bounded, simply connected and $C^{1,1}$ regular domain $\Omega \subset \mathbb{R}^2$, where $\kappa$ is the curvature on $\partial\Omega$, $N \geqslant 1$ and, for every $j \in \{1, ..., N\}$, $a_j \in \partial\Omega$ are distinct points and $d_j \in \mathbb{Z}\setminus\{0\}$. The zero-average relation $\langle J, 1\rangle = 0$ implies

$$-\int_{\partial\Omega} \kappa \,\mathrm{d}\mathcal{H}^1 + \pi\sum_{j=1}^N d_j = 0,$$

and by the Gauss-Bonnet formula, using that $\Omega$ is simply connected, we have $\int_{\partial\Omega}\kappa\,\mathrm{d}\mathcal{H}^1 = 2\pi$. Hence, the integers $d_1, ..., d_N$ must satisfy the constraint $\sum_{j=1}^N d_j = 2$. For more information about the global Jacobian, we refer to [24, Section 2].

### 2.1.3 Main results of Chapter 2

In this chapter, we study the micromagnetic energy $\widehat{E}(m_h)$ defined at (2.5) in a thin-film regime that is slightly different than in Chapter 1. More precisely, we consider here the regime

$$\begin{aligned} &h \ll 1, \quad \eta \ll 1, \quad \frac{1}{|\log h|} \ll \varepsilon \ll 1, \quad \frac{\widehat{D}_{13}}{\eta^2} \to 2\delta_1, \quad \frac{\widehat{D}_{23}}{\eta^2} \to 2\delta_2, \\ &\frac{1}{\eta^2}\sum_{j,k=1}^2 |\widehat{D}_{jk}| \ll 1, \quad \frac{1}{\eta^2}\sum_{k=1}^3 |\widehat{D}_{3k}| \ll 1, \end{aligned} \tag{2.6}$$

where $\delta_1, \delta_2 \in \mathbb{R}$, $\widehat{D} = (\widehat{D}_{jk})_{(j,k)\in\{1,2,3\}^2} \in \mathbb{R}^{3\times 3}$, and

$$\varepsilon = \frac{\eta^2}{h\,|\log h|}.$$

The parameters $\eta = \eta(h)$, $\varepsilon = \varepsilon(h)$ and $\widehat{D} = \widehat{D}(h)$ are assumed to be functions in $h$. As $a \ll b \ll 1$ implies $a\,|\log a| \ll b\,|\log b| \ll 1$, we deduce for $a = \frac{1}{|\log h|}$ and $b = \varepsilon$, that the regime (2.6) implies:

$$\begin{aligned} &\frac{\log|\log h|}{|\log h|} \ll \varepsilon\,|\log\varepsilon| \ll 1, \quad \frac{\widehat{D}_{13}}{\eta^2} \to 2\delta_1, \quad \frac{\widehat{D}_{23}}{\eta^2} \to 2\delta_2, \\ &\frac{1}{\eta^2}\sum_{j,k=1}^2 |\widehat{D}_{jk}| \ll 1, \quad \frac{1}{\eta^2}\sum_{k=1}^3 |\widehat{D}_{3k}| \ll 1. \end{aligned} \tag{2.7}$$

For proving Gamma-convergence statements at the second order, we will also consider the regime

$$\begin{aligned}
&\frac{\log|\log h|}{|\log h|} \ll \varepsilon \ll 1, \quad \left|\frac{\widehat{D}_{13}}{\eta^2} - 2\delta_1\right| \ll \frac{1}{|\log\varepsilon|}, \quad \left|\frac{\widehat{D}_{23}}{\eta^2} - 2\delta_2\right| \ll \frac{1}{|\log\varepsilon|}, \\
&\frac{1}{\eta^2}\sum_{j,k=1}^{2}|\widehat{D}_{jk}| \ll 1, \quad \frac{1}{\eta^2}\sum_{k=1}^{3}|\widehat{D}_{3k}| \ll \frac{1}{|\log\varepsilon|},
\end{aligned} \tag{2.8}$$

which is narrower than (2.7).

We assume that $\Omega \subset \mathbb{R}^2$ is a bounded, simply connected and $C^{1,1}$ smooth domain. We consider the three-dimensional rescaled energy

$$\mathcal{E}_h(m_h) = \frac{\widehat{E}(m_h)}{\ell^3 h\eta^2\,|\log\varepsilon|} = \frac{\widehat{E}(m_h)}{\ell^3 h^2\,|\log h|\,\varepsilon\,|\log\varepsilon|}, \tag{2.9}$$

for maps $m_h\colon \Omega_h = \Omega\times(0,h) \to \mathbb{S}^2$ with

$$\Delta u_h = \operatorname{div}(m_h \mathbb{1}_{\Omega_h}) \ \text{ in the distributional sense in } \mathbb{R}^3.$$

More precisely, the energy $\mathcal{E}_h(m_h)$ is given by

$$\mathcal{E}_h(m_h) = \frac{1}{|\log\varepsilon|}\left(\frac{1}{h}\int_{\Omega_h}|\nabla m_h|^2\,\mathrm{d}x + \frac{1}{h\eta^2}\int_{\Omega_h}\widehat{D}:\nabla m_h\wedge m_h\,\mathrm{d}x + \frac{1}{h\eta^2}\int_{\mathbb{R}^3}|\nabla u_h|^2\,\mathrm{d}x\right). \tag{2.10}$$

The energy functional $\mathcal{E}_h$ has been studied by Ignat and Kurzke [25] when the Dzyaloshinskii-Moriya interaction is negligible, i.e. $\widehat{D}=0$. As their results will be useful in the following, we use the notation $\mathcal{E}_h^0$ for the functional $\mathcal{E}_h$ when $\widehat{D}=0$, i.e.

$$\mathcal{E}_h^0(m_h) = \frac{1}{|\log\varepsilon|}\left(\frac{1}{h}\int_{\Omega_h}|\nabla m_h|^2\,\mathrm{d}x + \frac{1}{h\eta^2}\int_{\mathbb{R}^3}|\nabla u_h|^2\,\mathrm{d}x\right). \tag{2.11}$$

In contrast with Chapter 1, in which we assumed that $\varepsilon = \frac{\eta^2}{h|\log h|} \to \alpha$ where $\alpha$ is a positive constant (in Section 1.2), we consider here the case where $\varepsilon \to 0$. Our results are based on the work of Ignat and Kurzke [24], [25], who studied the energy $\mathcal{E}_h^0$ given at (2.11).

#### 2.1.3.1 Two-dimensional model for maps $m'\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$

Following the strategy of Ignat and Kurzke, we first consider a two-dimensional model (see [24]) for maps $m'\colon \Omega\subset\mathbb{R}^2\to\mathbb{R}^2$. More precisely, given a two-dimensional vector $\delta\in\mathbb{R}^2$, we consider the two-dimensional energy functional

$$\begin{aligned}
E_{\varepsilon,\eta}^{\delta}(m') = &\int_\Omega |\nabla' m'|^2\,\mathrm{d}x + 2\int_\Omega \delta\cdot\nabla' m'\wedge m'\,\mathrm{d}x \\
&+ \frac{1}{\eta^2}\int_\Omega \left(1-|m'|^2\right)^2\mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega}(m'\cdot\nu')^2\mathrm{d}\mathcal{H}^1,
\end{aligned} \tag{2.12}$$

where $\nabla' = (\partial_1,\partial_2)$ and $\nu'$ is the outer unit normal vector on $\partial\Omega$. Note that we partially recognize the energy considered in Section 1.3.1 with a boundary penalization, that favors boundary vortices of the magnetization. Moreover, there is an interior penalization here, similar to the Ginzburg-Landau model, that comes from $m_3^2$ (indeed $m'$ takes values in $\mathbb{R}^2$ here, and not $\mathbb{S}^1$). As this reduced model combines an interior penalization (as in the classical Ginzburg-Landau model [7]) and a boundary penalization (as in Chapter 1, or Kurzke [32]), topological singularities may be located in the interior or at the boundary of the two-dimensional domain. The main point of the analysis consists in detecting the topological singularities with the global Jacobian introduced in [24].

We analyze the energy functional $E^\delta_{\varepsilon,\eta}$ in the asymptotic regime

$$\eta \ll 1, \quad \varepsilon \ll 1, \quad |\log \varepsilon| \ll |\log \eta| \,. \tag{2.13}$$

Note that the thin-film regime of the three-dimensional model (2.6) implies the regime (2.13). Indeed, the regime (2.6) is equivalent with $h \ll \eta^2 \ll h\,|\log h| \ll 1$ by using the definition of $\varepsilon$, hence $|\log h| \sim |\log \eta|$. It follows from (2.6) that $\frac{1}{\varepsilon} \ll |\log \eta|$ and from (2.7) that $|\log \varepsilon| \ll \frac{1}{\varepsilon}$, hence (2.13) is satisfied.

### Gamma-convergence of $E^\delta_{\varepsilon,\eta}$

Our Gamma-convergence statements for $E^\delta_{\varepsilon,\eta}$ are based on the work of Ignat and Kurzke [24], who studied the energy functional $E^0_{\varepsilon,\eta}$ (i.e. $E^\delta_{\varepsilon,\eta}$ in the case $\delta = 0$). We begin with showing that, under the assumption $E^\delta_{\varepsilon,\eta} = O(|\log \varepsilon|)$, we also have in our regime $E^0_{\varepsilon,\eta} = O(|\log \varepsilon|)$. This latter assumption being necessary to apply the main results of Ignat-Kurzke [24], we can thus use their statements for $E^0_{\varepsilon,\eta}$. In order to obtain similar results for the Gamma-convergence of $E^\delta_{\varepsilon,\eta}$, we then use their strategy for making $E^\delta_{\varepsilon,\eta}(m')$ easier to study. It consists firstly in approximating the maps $m'$ by $\mathbb{S}^1$-valued maps $\mathfrak{m}'$, and secondly in replacing the vector-valued maps $\mathfrak{m}'$ by scalar-valued functions $\varphi$, obtained as liftings of $\mathfrak{m}'$. After having proved Gamma-convergence statements for $\varphi$, we come back to $m'$ and show the following statements.

**Theorem 2.1.1** (Compactness at the boundary and lower bound at the first order)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. Let $\delta \in \mathbb{R}^2$. Assume $\varepsilon \to 0$ and $\eta = \eta(\varepsilon) \to 0$ in the regime* (2.13)*. Let $(m'_\varepsilon)$ be a family in $H^1(\Omega, \mathbb{R}^2)$ such that*

$$\limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} E^\delta_{\varepsilon,\eta}(m'_\varepsilon) < +\infty. \tag{2.14}$$

*Then the following statements hold.*

*(i)* ***Compactness of global Jacobians and $L^p(\partial\Omega)$-compactness of $m'_\varepsilon|_{\partial\Omega}$.***
*For a subsequence, $(\mathcal{J}(m'_\varepsilon))$ converges to a measure $J$ on the closure $\overline{\Omega}$, in the sense that*

$$\lim_{\varepsilon \to 0} \left( \sup_{|\nabla' \zeta| \leqslant 1 \text{ in } \Omega} |\langle \mathcal{J}(m'_\varepsilon) - J, \zeta \rangle| \right) = 0. \tag{2.15}$$

*Moreover, $J$ is supported on $\partial\Omega$ and has the form*

$$J = -\kappa \mathcal{H}^1 \llcorner \partial\Omega + \pi \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}} \tag{2.16}$$

*for $N \geqslant 1$ distinct boundary vortices $a_j \in \partial\Omega$ carrying the multiplicities $d_j \in \mathbb{Z} \setminus \{0\}$, for $j \in \{1, ..., N\}$, such that $\sum_{j=1}^N d_j = 2$.*
*Moreover, for a subsequence, $(m'_\varepsilon|_{\partial\Omega})$ converges to $e^{i\varphi_0} \in BV(\partial\Omega, \mathbb{S}^1)$ in $L^p(\partial\Omega, \mathbb{R}^2)$, for every $p \in [1, +\infty)$, where $\varphi_0 \in BV(\partial\Omega, \pi\mathbb{Z})$ is a lifting of the tangent field $\pm\tau$ on $\partial\Omega$ determined (up to a constant in $\pi\mathbb{Z}$) by*

$$\partial_\tau \varphi_0 = \kappa \mathcal{H}^1 \llcorner \partial\Omega - \pi \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}} \quad \text{as measure on } \partial\Omega.$$

*(ii)* ***Energy lower bound at the first order.***
*If $(\mathcal{J}(m'_\varepsilon))$ satisfies the convergence assumption in (i) as $\varepsilon \to 0$, then the energy lower bound at the first order is the total mass of the measure $J + \kappa \mathcal{H}^1 \llcorner \partial\Omega$ on $\partial\Omega$:*

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} E^\delta_{\varepsilon,\eta}(m'_\varepsilon) \geqslant \pi \sum_{j=1}^N |d_j| = \left| J + \kappa \mathcal{H}^1 \llcorner \partial\Omega \right| (\partial\Omega).$$

In order to get a lower bound at the second order for the energy functional, we introduce the following renormalized energy, in the spirit of Brezis-Bethuel-Hélein [7]. The renormalized energy allows us to study the energy functional asymptotically at the second order by eliminating the first order infinite energy carried in the neighborhood of boundary vortices. In comparison with Ignat-Kurzke [24], [25], our renormalized energy includes not only the contribution of the Dirichlet energy, but also the contribution of the Dzyaloshinskii-Moriya interaction energy far away from vortices.

**Definition 2.1.2.** Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. Consider $\varphi_0 \colon \partial\Omega \to \mathbb{R}$ to be a BV function such that $e^{i\varphi_0} \cdot \nu' = 0$ in $\partial\Omega \setminus \{a_1, ..., a_N\}$, and

$$\partial_\tau \varphi_0 = \kappa \mathcal{H}^1 \llcorner \partial\Omega - \pi \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}} \quad \text{as measure on } \partial\Omega,$$

with $d_j \in \{-1, +1\}$ for every $j \in \{1, ..., N\}$ and $\sum_{j=1}^N d_j = 2$, for $N \geqslant 1$ distinct points $a_j \in \partial\Omega$ carrying the degrees $d_j$ for $j \in \{1, ..., N\}$. Let $\delta \in \mathbb{R}^2$. If $\varphi_*$ is the harmonic extension of $\varphi_0$ to $\Omega$, then the renormalized energy of $\{(a_j, d_j)\}_{j \in \{1,...,N\}}$ is defined as

$$W^\delta_\Omega(\{(a_j, d_j)\}) = \liminf_{r \to 0} \left( \int_{\Omega \setminus \bigcup_{j=1}^N B_r(a_j)} \left( |\nabla' \varphi_*|^2 - 2\delta \cdot \nabla' \varphi_* \right) \mathrm{d}x - N\pi \log \frac{1}{r} \right), \tag{2.17}$$

where $B_r(a_j)$ is the disk of center $a_j$ and radius $r > 0$.

Note that the lim inf above is in fact a limit: its existence will be proved in the following. We then have the two following theorems, that allow us to get an asymptotic expansion at the second order for the energy.

**Theorem 2.1.3** (Compactness in the interior and lower bound at the second order)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. Let $\delta \in \mathbb{R}^2$. Assume $\varepsilon \to 0$ and $\eta = \eta(\varepsilon) \to 0$ in the regime* (2.13)*. Let $(m'_\varepsilon)$ be a family in $H^1(\Omega, \mathbb{R}^2)$ satisfying* (2.14) *and the convergence at (i) in Theorem 2.1.1 to the measure $J$ given at* (2.16) *as $\varepsilon \to 0$. In addition, we assume the following more precise bound than* (2.14)*:*

$$\limsup_{\varepsilon \to 0} \left( E^\delta_{\varepsilon,\eta}(m'_\varepsilon) - \pi |\log \varepsilon| \sum_{j=1}^N |d_j| \right) < +\infty. \tag{2.18}$$

*Then the following statements hold.*

*(i)* ***Single multiplicity and second order lower bound.***
*The multiplicities satisfy $d_j \in \{-1, +1\}$ for every $j \in \{1, ..., N\}$, so we have $\sum_{j=1}^N |d_j| = N$, and there holds the following second order energy lower bound:*

$$\liminf_{\varepsilon \to 0} \left( E^\delta_{\varepsilon,\eta}(m'_\varepsilon) - N\pi |\log \varepsilon| \right) \geqslant W^\delta_\Omega(\{(a_j, d_j)\}) + N\gamma_0$$

*where $\gamma_0 = \pi \log \frac{e}{4\pi}$ and $W^\delta_\Omega(\{(a_j, d_j)\})$ is the renormalized energy defined at* (2.17)*.*

(ii) ***Penalty bound.***
*The penalty terms in the energy are of order $O(1)$, i.e.*

$$\limsup_{\varepsilon\to 0}\left(\frac{1}{\eta^2}\int_\Omega \left(1-|m'_\varepsilon|^2\right)^2 \mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega}(m'_\varepsilon\cdot\nu')^2 \mathrm{d}\mathcal{H}^1\right) < +\infty. \tag{2.19}$$

(iii) ***Lower bound for the energy near the boundary vortex core.***
*There exist $r_0>0$, $\varepsilon_0>0$ and $C>0$ such that the Dirichlet energy of $m'_\varepsilon$ near the singularities $\{a_j\}_{j\in\{1,\dots,N\}}$ satisfies, for all $\varepsilon\in(0,\varepsilon_0)$ and $r\in(0,r_0)$,*

$$\int_{\Omega\cap\bigcup_j B_r(a_j)} |\nabla m'_\varepsilon|^2 \,\mathrm{d}x - N\pi\log\frac{r}{\varepsilon} \geqslant -C. \tag{2.20}$$

(iv) $W^{1,q}(\Omega)$***-weak compactness and*** $L^p(\Omega)$***-compactness of maps*** $m'_\varepsilon$***.***
*For any $q\in[1,2)$, $(m'_\varepsilon)$ is uniformly bounded in $W^{1,q}(\Omega,\mathbb{R}^2)$. Moreover, for a subsequence, $(m'_\varepsilon)$ converges weakly in $W^{1,q}(\Omega,\mathbb{R}^2)$, for every $q\in[1,2)$, and strongly in $L^p(\Omega,\mathbb{R}^2)$, for every $p\in[1,+\infty)$, to $e^{i\widehat{\varphi}_0}$, where $\widehat{\varphi}_0\in W^{1,q}(\Omega)$ is an extension (not necessarily harmonic) to $\Omega$ of the lifting $\varphi_0\in BV(\partial\Omega,\pi\mathbb{Z})$ determined in Theorem 2.1.1(i).*

(v) ***DMI bound.***
*The Dzyaloshinskii-Moriya interaction energy is of order $O(1)$, i.e.*

$$\limsup_{\varepsilon\to 0}\int_\Omega |\delta\cdot\nabla' m'_\varepsilon\wedge m'_\varepsilon| \,\mathrm{d}x < +\infty. \tag{2.21}$$

**Theorem 2.1.4** (Upper bound)**.** *Let $\Omega\subset\mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. Let $\{a_j\}_{j\in\{1,\dots,N\}}\in(\partial\Omega)^N$ be $N\geqslant 1$ distinct points and $d_j\in\mathbb{Z}\setminus\{0\}$ be the corresponding multiplicities, for $j\in\{1,\dots,N\}$, that satisfy $\sum_{j=1}^N d_j = 2$. Let $\delta\in\mathbb{R}^2$. Assume $\varepsilon\to 0$ and $\eta=\eta(\varepsilon)\to 0$ in the regime* (2.13)*. Then we can construct a family $(m'_\varepsilon)$ in $H^1(\Omega,\mathbb{S}^1)$ such that $(\mathcal{J}(m'_\varepsilon))$ converges as in* (2.15) *to the measure*

$$J = -\kappa\mathcal{H}^1\llcorner\partial\Omega + \pi\sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}}.$$

*Furthermore, $(m'_\varepsilon)$ converges strongly to $e^{i\varphi_*}$ in $L^p(\Omega,\mathbb{R}^2)$ and in $L^p(\partial\Omega,\mathbb{R}^2)$, for every $p\in[1,+\infty)$, where $\varphi_*$ is the harmonic extension to $\Omega$ of a boundary lifting $\varphi_0$ that satisfies $e^{i\varphi_0}\cdot\nu'=0$ and $\partial_\tau\varphi_0 = \kappa\mathcal{H}^1\llcorner\partial\Omega - \pi\sum_{j=1}^N d_j\mathbb{1}_{\{a_j\}}$ as measure on $\partial\Omega$, and the energy of $m'_\varepsilon$ satisfies*

$$\lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}E^\delta_{\varepsilon,\eta}(m'_\varepsilon) = \pi\sum_{j=1}^N |d_j|.$$

*Furthermore, if $d_j\in\{-1,+1\}$ for every $j\in\{1,\dots,N\}$, then $m'_\varepsilon$ can be chosen such that*

$$\lim_{\varepsilon\to 0}\left(E^\delta_{\varepsilon,\eta}(m'_\varepsilon) - N\pi|\log\varepsilon|\right) = W^\delta_\Omega(\{(a_j,d_j)\}) + N\gamma_0$$

*where $\gamma_0 = \pi\log\frac{e}{4\pi}$ and $W^\delta_\Omega(\{(a_j,d_j)\})$ is the renormalized energy defined at* (2.17)*.*

**Minimization of the renormalized energy** $W^\delta_\Omega(\{(a_j,d_j)\})$

Using [25], we compute the renormalized energy $W^\delta_\Omega(\{(a_j,d_j)\})$ that is involved at the second order of the Gamma-expansion.

**Theorem 2.1.5.** *Let $\delta\in\mathbb{R}^2$.*

*(i) We denote by $B_1$ the unit disk in $\mathbb{R}^2$. Let $\{a_j\}_{j\in\{1,...,N\}}\in(\partial B_1)^N$ be $N\geqslant 2$ distinct points and $d_j\in\{-1,+1\}$ be the corresponding multiplicities, for $j\in\{1,...,N\}$, that satisfy $\sum_{j=1}^N d_j=2$. Then the renormalized energy of $\{(a_j,d_j)\}$ in $B_1$ satisfies*

$$W^\delta_{B_1}(\{(a_j,d_j)\})=-2\pi\sum_{1\leqslant j<k\leqslant N}d_jd_k\log|a_j-a_k|+2\pi\sum_{j=1}^N d_j\delta\cdot a_j^\perp. \tag{2.22}$$

*(ii) Let $\Omega\subset\mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain. Let $\kappa$ be the curvature on $\partial\Omega$ and $\nu$ be the outer unit normal vector on $\partial\Omega$. Let $\Phi\colon\overline{B_1}\to\overline{\Omega}$ be a $C^1$ conformal diffeomorphism with inverse $\Psi=\Phi^{-1}$. Let $\{a_j\}_{j\in\{1,...,N\}}\in(\partial\Omega)^N$ be $N\geqslant 2$ distinct points and $d_j\in\{-1,+1\}$ be the corresponding multiplicities, for $j\in\{1,...,N\}$, that satisfy $\sum_{j=1}^N d_j=2$. Then the renormalized energy of $\{(a_j,d_j)\}$ in $\Omega$ satisfies*

$$\begin{aligned}&W^\delta_\Omega(\{(a_j,d_j)\})\\&\quad=-2\pi\sum_{1\leqslant j<k\leqslant N}d_jd_k\log|\Psi(a_j)-\Psi(a_k)|+\pi\sum_{j=1}^N(d_j-1)\log|\Psi'(a_j)|\\&\qquad+\int_{\partial\Omega}(\kappa+2\delta^\perp\cdot\nu')(w)\left(\sum_{j=1}^N d_j\log|\Psi(w)-\Psi(a_j)|-\log|\Psi'(w)|\right)\mathrm{d}\mathcal{H}^1(w).\end{aligned} \tag{2.23}$$

**Corollary 2.1.6.** *Let $\Omega\subset\mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain. Let $\delta\in\mathbb{R}^2$. There exists a pair $(a_1^*,a_2^*)\in\partial\Omega\times\partial\Omega$ of distinct points such that*

$$W^\delta_\Omega(\{(a_1^*,1),(a_2^*,1)\})=\min\left\{W^\delta_\Omega(\{(\tilde a_1,1),(\tilde a_2,1)\}):\tilde a_1\neq\tilde a_2\in\partial\Omega\right\}.$$

For the model with $\delta=0$ studied by Ignat and Kurzke [24], [25], in the case of the unit disk $\Omega=B_1$, the pair of boundary vortices that minimizes the renormalized energy corresponds to two diametrically opposed points on $\partial B_1$ and is only unique up to a rotation. For the model presented here, the location of the vortices that minimizes the renormalized energy in $B_1$ is influenced by the Dzyaloshinskii-Moriya interaction. Indeed, the renormalized energy for two vortices at $a_1,a_2\in\partial B_1$ of multiplicity 1 is

$$W^\delta_{B_1}(\{(a_1,1),(a_2,1)\})=-2\pi\log|a_1-a_2|+2\pi\delta\cdot(a_1^\perp+a_2^\perp).$$

For $\delta\neq0$, the minimization of the renormalized energy is different because there is a competition between two quantities. In the next theorem, we show that the minimal configuration for the points $a_1,a_2\in\partial B_1$ is unique (up to switching $a_1$ and $a_2$) and the vortices are not necessarily diametrically opposed, but symmetric with respect to $\delta^\perp$.

**Theorem 2.1.7.** *Let $B_1$ be the unit disk in $\mathbb{R}^2$ and $\delta=|\delta|e^{i\theta}\in\mathbb{R}^2\setminus\{(0,0)\}$ for some $\theta\in\mathbb{R}$. Then the pair of distinct points $(a_1^*,a_2^*)\in\partial B_1\times\partial B_1$ that minimizes the renormalized energy $W^\delta_{B_1}(\{(\cdot,1),(\cdot,1)\})$ in Corollary 2.1.6 is unique (up to switching $a_1^*$ and $a_2^*$) and given by*

$$a_1^*=e^{i(\theta+\theta_0)}\quad\text{and}\quad a_2^*=e^{i(\theta+\pi-\theta_0)}$$

*where $\theta_0=\arcsin\left(\sqrt{1+\frac{1}{16|\delta|^2}}-\frac{1}{4|\delta|}\right)$. In particular, $a_1^*$ and $a_2^*$ are symmetric with respect to $\delta^\perp$.*

**Corollary 2.1.8.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. Let $\delta \in \mathbb{R}^2$. Assume $\varepsilon \to 0$ and $\eta = \eta(\varepsilon) \to 0$ in the regime (2.13). For every family $(m'_\varepsilon)$ of minimizers of $E^\delta_{\varepsilon,\eta}$ on $H^1(\Omega, \mathbb{R}^2)$, there exists a subsequence $\varepsilon \to 0$ such that $(\mathcal{J}(m'_\varepsilon))$ converges as in (2.15) to the measure*

$$J = -\kappa \mathcal{H}^1 \llcorner \partial\Omega + \pi(\mathbb{1}_{\{a_1\}} + \mathbb{1}_{\{a_2\}}),$$

*where $a_1$ and $a_2$ are two distinct points in $\partial\Omega$ that minimize the renormalized energy (for the multiplicities $d_1 = d_2 = 1$), i.e.*

$$W^\delta_\Omega(\{(a_1, 1), (a_2, 1)\}) = \min \left\{ W^\delta_\Omega(\{(\tilde{a}_1, 1), (\tilde{a}_2, 1)\}) : \tilde{a}_1 \neq \tilde{a}_2 \in \partial\Omega \right\}.$$

*Furthermore, $(m'_\varepsilon)$ converges weakly to $e^{i\varphi_*}$ in $W^{1,q}(\Omega, \mathbb{R}^2)$, for every $q \in [1, 2)$, where $\varphi_*$ is the harmonic extension to $\Omega$ of a boundary lifting $\varphi_0 \in BV(\partial\Omega, \pi\mathbb{Z})$ that satisfies*

$$\partial_\tau \varphi_0 = \kappa \mathcal{H}^1 \llcorner \partial\Omega - \pi(\mathbb{1}_{\{a_1\}} + \mathbb{1}_{\{a_2\}}) \quad \text{as measure on } \partial\Omega$$

*and $e^{i\varphi_0} \cdot \nu' = 0$ in $\partial\Omega \setminus \{a_1, a_2\}$.*

*Furthermore, there holds the following second order energy expansion:*

$$E^\delta_{\varepsilon,\eta}(m'_\varepsilon) = 2\pi |\log \varepsilon| + W^\delta_\Omega(\{(a_1, 1), (a_2, 1)\}) + 2\gamma_0 + o(1) \quad \text{as } \varepsilon \to 0,$$

*where $\gamma_0 = \pi \log \frac{e}{4\pi}$.*

In the case of the unit disk $\Omega = B_1$, combining Theorem 2.1.5 and Corollary 2.1.8, we deduce that

$$\lim_{\varepsilon \to 0} \left( E^\delta_{\varepsilon,\eta}(m'_\varepsilon) - 2\pi |\log \varepsilon| \right) = -2\pi \log |a_1 - a_2| + 2\pi\delta \cdot (a_1^\perp + a_2^\perp) + 2\pi \log \frac{e}{4\pi}.$$

#### 2.1.3.2 Three-dimensional model for maps $m_h \colon \Omega_h \subset \mathbb{R}^3 \to \mathbb{S}^2$

We come back to the three-dimensional model (see [25]) for maps $m_h \colon \Omega_h \subset \mathbb{R}^3 \to \mathbb{S}^2$ and study the 3D energy $\mathcal{E}_h$ given at (2.10). As in Chapter 1, since we consider here again a thin-film regime, we use the $x_3$-averaged magnetization $\overline{m}_h \colon \Omega \to \overline{B^2}$, where $B^2$ is the unit ball in $\mathbb{R}^3$, that is essential for getting rid of the nonlocal effect of the stray-field energy. Recall that

$$\overline{m}_h(x') = \frac{1}{h} \int_0^h m_h(x', x_3) \, \mathrm{d}x_3$$

for every $x' \in \Omega$, and we denote by $\overline{u}_h \colon \mathbb{R}^3 \to \mathbb{R}$ the associated stray field potential given by

$$\Delta \overline{u}_h = \operatorname{div}(\overline{m}_h \mathbb{1}_{\Omega_h}) \text{ in the distributional sense in } \mathbb{R}^3.$$

We introduce the reduced 2D energy functional

$$\begin{aligned} \overline{\mathcal{E}}_h(\overline{m}_h) = \frac{1}{|\log \varepsilon|} \Bigg( & \int_\Omega |\nabla' \overline{m}_h|^2 \, \mathrm{d}x' + \frac{1}{\eta^2} \int_\Omega \widehat{D}' : \nabla' \overline{m}_h \wedge \overline{m}_h \, \mathrm{d}x' \\ & + \frac{1}{\eta^2} \int_\Omega \left(1 - |\overline{m}'_h|^2\right) \mathrm{d}x' + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (\overline{m}'_h \cdot \nu')^2 \, \mathrm{d}\mathcal{H}^1 \Bigg) \end{aligned} \tag{2.24}$$

with $\overline{m}'_h = (\overline{m}_{h,1}, \overline{m}_{h,2})$ and $\nu'$ is the outer unit normal vector on $\partial\Omega$. For references to the energy functional studied by Ignat and Kurzke [25] when $\widehat{D} = 0$, we also set

$$\overline{\mathcal{E}}^0_h(\overline{m}_h) = \frac{1}{|\log \varepsilon|} \left( \int_\Omega |\nabla' \overline{m}_h|^2 \, \mathrm{d}x' + \frac{1}{\eta^2} \int_\Omega \left(1 - |\overline{m}'_h|^2\right) \mathrm{d}x' + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (\overline{m}'_h \cdot \nu')^2 \, \mathrm{d}\mathcal{H}^1 \right). \tag{2.25}$$

An important argument for introducing this averaged energy functional is that in the regime (2.6), the stray-field energy in (2.11) is close to the interior and boundary penalty terms in (2.25) (see [25, Lemmas 15 and 16]). Hence, the averaged energy functional $\overline{\mathcal{E}}^0_h$ allows to get rid of the nonlocality of the stray-field energy and replaces it by local terms. The ideas used by Ignat and Kurzke improve estimates of Carbou [11] and Kohn-Slastikov [32].

### Dimension reduction from the 3D energy functional to a 2D energy functional

For proving the main results of compactness and Gamma-convergence for the 3D energy $\mathcal{E}_h$, we first approximate this 3D energy by the 2D energy $E^\delta_{\varepsilon,\eta}$ studied in Section 2.2. The link between $\mathcal{E}_h$ and $E^\delta_{\varepsilon,\eta}$, with $\frac{1}{2\eta^2}(\widehat{D}_{13}, \widehat{D}_{23}) \to \delta$, is established in two steps. First, we approximate $\mathcal{E}_h(m_h)$ by the averaged energy functional $\overline{\mathcal{E}}_h(\overline{m}_h)$ defined at (2.24) ; then we reduce $\overline{\mathcal{E}}_h(\overline{m}_h)$ to $E^\delta_{\varepsilon,\eta}(\overline{m}'_h)$. The following remark is essential for extending the upper bounds obtained in the two-dimensional model.

*Remark* 2.1.9. If $m_h$ is independent of $x_3$, i.e. $m_h = \overline{m}_h$, then

$$\frac{1}{h}\int_{\Omega_h} |\nabla m_h|^2 \,\mathrm{d}x = \int_\Omega |\nabla' \overline{m}_h|^2 \,\mathrm{d}x',$$

and

$$\frac{1}{h}\int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \,\mathrm{d}x = \int_\Omega \widehat{D}' : \nabla' \overline{m}_h \wedge \overline{m}_h \,\mathrm{d}x'.$$

**Theorem 2.1.10.** *Let $\Omega_h = \Omega \times (0,h)$ with $\Omega \subset \mathbb{R}^2$ a bounded, simply connected and $C^{1,1}$ smooth domain. In the regime* (2.6)*, consider a family of magnetizations $\{m_h \colon \Omega_h \to \mathbb{S}^2\}$ that satisfies*

$$\limsup_{h\to 0} \mathcal{E}_h(m_h) < +\infty.$$

*Then*

$$\mathcal{E}_h(m_h) \geqslant \overline{\mathcal{E}}_h(\overline{m}_h) - o(1) \quad \textit{as } h \to 0.$$

*Moreover, in the more restrictive regime* (2.8)*, we have*

$$\mathcal{E}_h(m_h) \geqslant \overline{\mathcal{E}}_h(\overline{m}_h) - o\left(\frac{1}{|\log\varepsilon|}\right) \quad \textit{as } h \to 0.$$

*Furthermore, if $m_h$ is independent of $x_3$, then in the regime* (2.6)*, we have*

$$\mathcal{E}_h(m_h) = \overline{\mathcal{E}}_h(\overline{m}_h) - o(1) \quad \textit{as } h \to 0,$$

*and in the regime* (2.8)*, we have*

$$\mathcal{E}_h(m_h) = \overline{\mathcal{E}}_h(\overline{m}_h) - o\left(\frac{1}{|\log\varepsilon|}\right) \quad \textit{as } h \to 0.$$

We then show that in the regime (2.6), the two-dimensional energy $\overline{\mathcal{E}}_h$ of the three-dimensional averaged magnetization $\overline{m}_h$ is in fact close to the two-dimensional energy $E^\delta_{\varepsilon,\eta}$ (divided by $|\log\varepsilon|$) of the two-dimensional averaged magnetization $\overline{m}'_h$, so that we get the following statement.

**Corollary 2.1.11.** *Let $\Omega_h = \Omega \times (0,h)$ with $\Omega \subset \mathbb{R}^2$ a bounded, simply connected and $C^{1,1}$ smooth domain. In the regime* (2.6)*, consider a family of magnetizations $\{m_h \colon \Omega_h \to \mathbb{S}^2\}$ that satisfies*

$$\limsup_{h\to 0} \mathcal{E}_h(m_h) < +\infty.$$

*Then*

$$\mathcal{E}_h(m_h) \geqslant \frac{1}{|\log\varepsilon|} E^\delta_{\varepsilon,\eta}(\overline{m}'_h) - o(1) \quad \textit{as } h \to 0.$$

*Moreover, in the more restrictive regime* (2.8)*, we have*

$$\mathcal{E}_h(m_h) \geqslant \frac{1}{|\log\varepsilon|} E^\delta_{\varepsilon,\eta}(\overline{m}'_h) - o\left(\frac{1}{|\log\varepsilon|}\right) \quad \textit{as } h \to 0.$$

*Furthermore, if $m_h$ is independent of $x_3$ and $m_h = (m'_h, 0)$, then in the regime* (2.6)*, we have*

$$\mathcal{E}_h(m_h) = \frac{1}{|\log \varepsilon|} E^{\delta}_{\varepsilon,\eta}(\overline{m}'_h) - o(1) \quad \text{as } h \to 0,$$

*and in the regime* (2.8)*, we have*

$$\mathcal{E}_h(m_h) = \frac{1}{|\log \varepsilon|} E^{\delta}_{\varepsilon,\eta}(\overline{m}'_h) - o\left(\frac{1}{|\log \varepsilon|}\right) \quad \text{as } h \to 0.$$

**Gamma-convergence of the 3D energy $\mathcal{E}_h$**

Once the link between $\mathcal{E}_h$ and $E^{\delta}_{\varepsilon,\eta}$ has been established, we can use the 2D model studied in Section 2.2 for proving compactness of the global Jacobians and a lower bound at the first order for $\mathcal{E}_h$.

**Theorem 2.1.12.** *Let $\Omega_h = \Omega \times (0, h)$ with $\Omega \subset \mathbb{R}^2$ a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. In the regime* (2.6)*, consider a family of magnetizations $\{m_h : \Omega_h \to \mathbb{S}^2\}$ that satisfies*

$$\limsup_{h\to 0} \mathcal{E}_h(m_h) < +\infty. \tag{2.26}$$

*(i)* ***Compactness of the global Jacobians and $L^p(\partial\Omega)$-compactness of the traces $\overline{m}_h|_{\partial\Omega}$.*** *For a subsequence, $(\mathcal{J}(\overline{m}'_h))$ converges to a measure $J$ on the closure $\overline{\Omega}$, in the sense that*

$$\lim_{h\to 0}\left(\sup_{|\nabla'\zeta|\leqslant 1 \text{ in } \Omega} |\langle \mathcal{J}(\overline{m}'_h) - J, \zeta\rangle|\right) = 0. \tag{2.27}$$

*Moreover, $J$ is supported on $\partial\Omega$ and has the form*

$$J = -\kappa\mathcal{H}^1 \llcorner \partial\Omega + \pi \sum_{j=1}^{N} d_j \mathbb{1}_{\{a_j\}} \tag{2.28}$$

*for $N \geqslant 1$ distinct boundary vortices $a_j \in \partial\Omega$ carrying the multiplicities $d_j \in \mathbb{Z} \setminus \{0\}$, for $j \in \{1, ..., N\}$, such that $\sum_{j=1}^{N} d_j = 2$.*
*Furthermore, for a subsequence, $(\overline{m}_h|_{\partial\Omega})$ converges to $(e^{i\varphi_0}, 0) \in BV(\partial\Omega, \mathbb{S}^1 \times \{0\})$ in $L^p(\partial\Omega, \mathbb{R}^3)$, for every $p \in [1, +\infty)$, where $\varphi_0 \in BV(\partial\Omega, \pi\mathbb{Z})$ is a lifting of the tangent field $\pm\tau$ on $\partial\Omega$ determined (up to a constant in $\pi\mathbb{Z}$) by*

$$\partial_\tau \varphi_0 = \kappa\mathcal{H}^1 \llcorner \partial\Omega - \pi \sum_{j=1}^{N} d_j \mathbb{1}_{\{a_j\}} \quad \text{as measure on } \partial\Omega.$$

*(ii)* ***Energy lower bound at the first order.***
*If $(\mathcal{J}(\overline{m}'_h))$ satisfies the convergence assumption in (i) as $h \to 0$, then the energy lower bound at the first order is the total mass of the measure $J + \kappa\mathcal{H}^1 \llcorner \partial\Omega$ on $\partial\Omega$:*

$$\liminf_{h\to 0} \mathcal{E}_h(m_h) \geqslant \pi \sum_{j=1}^{N} |d_j| = \left|J + \kappa\mathcal{H}^1 \llcorner \partial\Omega\right|(\partial\Omega).$$

As in Theorem 2.1.3, within a more precise bound similar to (2.18), we prove the following lower bound at the second order for $\mathcal{E}_h$.

**Theorem 2.1.13.** *Let $\Omega_h = \Omega \times (0,h)$ with $\Omega \subset \mathbb{R}^2$ a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. In the regime* (2.8)*, consider a family of magnetizations $\{m_h : \Omega_h \to \mathbb{S}^2\}$ that satisfies* (2.26) *and the convergence at (i) in Theorem 2.1.12 to the measure $J$ given at* (2.16) *as $h \to 0$. In addition, we assume the following more precise bound than* (2.26)*:*

$$\limsup_{h\to 0} |\log\varepsilon| \left( \mathcal{E}_h(m_h) - \pi \sum_{j=1}^{N} |d_j| \right) < +\infty. \tag{2.29}$$

*(i)* ***Single multiplicity and second order lower bound.***
*The multiplicities satisfy $d_j \in \{-1,+1\}$ for every $j \in \{1,...,N\}$, so we have $\sum_{j=1}^N |d_j| = N$, and there holds the following second order energy lower bound:*

$$\liminf_{h\to 0} |\log\varepsilon| \left(\mathcal{E}_h(m_h) - N\pi\right) \geqslant W_\Omega^\delta(\{(a_j,d_j)\}) + N\gamma_0$$

*where $\gamma_0 = \pi \log \frac{e}{4\pi}$ and $W_\Omega^\delta(\{(a_j,d_j)\})$ is the renormalized energy defined at* (2.17)*.*

*(ii)* ***$L^p(\Omega)$-compactness of the rescaled magnetizations.***
*For a subsequence, the family $\{\widetilde{m}_h : \Omega_1 \to \mathbb{S}^2\}$, defined as $\widetilde{m}_h(x', x_3) = m_h(x', hx_3)$, converges strongly in $L^p(\Omega_1, \mathbb{R}^3)$, for every $p \in [1,+\infty)$, to a map $\widetilde{m} = (\widetilde{m}', 0) \in W^{1,q}(\Omega_1, \mathbb{R}^3)$, for every $q \in [1,2)$, such that $|\widetilde{m}| = |\widetilde{m}'| = 1$ and $\partial_3 \widetilde{m} = 0$, i.e. $\widetilde{m} = \widetilde{m}(x') \in W^{1,q}(\Omega, \mathbb{S}^1 \times \{0\})$. Furthermore, the global Jacobian $\mathcal{J}(\overline{\widetilde{m}}')$ coincides with the measure $J$ on $\overline{\Omega}$ given at* (2.28)*.*

**Theorem 2.1.14.** *Let $\Omega_h = \Omega \times (0,h)$ with $\Omega \subset \mathbb{R}^2$ a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. Let $\{a_j\}_{j\in\{1,...,N\}} \in (\partial\Omega)^N$ be $N \geqslant 1$ distinct points and $d_j \in \mathbb{Z} \setminus \{0\}$ be the corresponding multiplicities, for $j \in \{1,...,N\}$, that satisfy $\sum_{j=1}^N d_j = 2$. Then in the regime* (2.6)*, we can construct a family $\{m_h = (m_h', 0)\}$ of $H^1(\Omega_h, \mathbb{S}^1 \times \{0\})$ functions with the following properties.*

*(i) For every $h > 0$, $m_h$ is independent of $x_3$.*

*(ii) $(\mathcal{J}(m_h'))$ converges to $J = -\kappa\mathcal{H}^1 \llcorner \partial\Omega + \pi \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}}$ as in* (2.27)*.*

*(iii) We have*

$$\lim_{h\to 0} \mathcal{E}_h(m_h) = \pi \sum_{j=1}^{N} |d_j| \,.$$

*Furthermore, if $d_j \in \{-1,+1\}$ for every $j \in \{1,...,N\}$ and the regime* (2.8) *holds, then $(m_h)$ can be chosen such that*

$$\lim_{h\to 0} |\log\varepsilon| \left(\mathcal{E}_h(m_h) - N\pi\right) = W_\Omega^\delta(\{(a_j,d_j)\}) + N\gamma_0$$

*where $\gamma_0 = \pi \log \frac{e}{4\pi}$ and $W_\Omega^\delta(\{(a_j,d_j)\})$ is the renormalized energy defined at* (2.17)*.*

**Corollary 2.1.15.** *Let $\Omega_h = \Omega \times (0,h)$ with $\Omega \subset \mathbb{R}^2$ a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. In the regime* (2.6)*, for every family $(m_h)$ of minimizers of $\mathcal{E}_h$ on $H^1(\Omega_h, \mathbb{R}^3)$, there exists a subsequence $h \to 0$ such that the global Jacobians $\mathcal{J}(\overline{m}_h')$ of the in-plane averages $\overline{m}_h'$ converge as in* (2.27) *to the measure*

$$J = -\kappa\mathcal{H}^1 \llcorner \partial\Omega + \pi(\mathbb{1}_{\{a_1\}} + \mathbb{1}_{\{a_2\}})$$

*where $a_1$ and $a_2$ are two points on $\partial\Omega$. Moreover, we have*

$$\lim_{h\to 0} \mathcal{E}_h(m_h) = 2\pi.$$

*Moreover, in the regime (2.8), $a_1 \neq a_2$, the pair $(a_1, a_2)$ minimizes the renormalized energy (for the multiplicities $d_1 = d_2 = 1$) over the set $\{(\tilde{a}_1, \tilde{a}_2) \in \partial\Omega \times \partial\Omega : \tilde{a}_1 \neq \tilde{a}_2\}$, i.e.*

$$W_\Omega^\delta(\{(a_1,1),(a_2,1)\}) = \min\left\{W_\Omega^\delta(\{(\tilde{a}_1,1),(\tilde{a}_2,1)\}) : \tilde{a}_1 \neq \tilde{a}_2 \in \partial\Omega\right\},$$

*and*

$$\lim_{h\to 0} |\log\varepsilon| \left(\mathcal{E}_h(m_h) - 2\pi\right) = W_\Omega^\delta(\{(a_1,1),(a_2,1)\}) + 2\gamma_0$$

*where $\gamma_0 = \pi \log \frac{e}{4\pi}$.*

In the case of the unit disk $\Omega = B_1$, combining Theorem 2.1.5 and Corollary 2.1.15, we deduce that

$$\lim_{h\to 0} |\log\varepsilon| \left(\mathcal{E}_h(m_h) - 2\pi\right) = -2\pi \log|a_1 - a_2| + 2\pi\delta \cdot (a_1^\perp + a_2^\perp) + 2\pi \log \frac{e}{4\pi}.$$

## 2.2 Two-dimensional model for maps $m'\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$

**Since all quantities in this section are two-dimensional quantities, we drop the primes $'$ in the notations.**

### 2.2.1 Approximation by $\mathbb{S}^1$-valued maps

Given a map $m\colon \Omega \to \mathbb{R}^2$ such that $E_{\varepsilon,\eta}^\delta(m) = O(|\log\varepsilon|)$, we show in this section that $m$ can be approximated by a $\mathbb{S}^1$-valued map $\mathfrak{m}\colon \Omega \to \mathbb{S}^1$ in the regime (2.13). This approximation in the regime (2.13) has already been proved by Ignat and Kurzke [24, Theorem 3.1] in the case $\delta = 0$. In order to use their result, we first show that under the assumption $E_{\varepsilon,\eta}^\delta(m) = O(|\log\varepsilon|)$ in the regime (2.13), we also have $E_{\varepsilon,\eta}^0(m) = O(|\log\varepsilon|)$.

**Lemma 2.2.1.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain. Let $\delta \in \mathbb{R}^2$. Assume $\varepsilon \to 0$ and $\eta = \eta(\varepsilon) \to 0$ in the regime (2.13). For every $m = m_\varepsilon\colon \Omega \to \mathbb{R}^2$ satisfying $E_{\varepsilon,\eta}^\delta(m) = O(|\log\varepsilon|)$, we have $E_{\varepsilon,\eta}^0(m) = O(|\log\varepsilon|)$.*

*Proof.* Let $m = m_\varepsilon\colon \Omega \to \mathbb{R}^2$ be such that $E_{\varepsilon,\eta}^\delta(m) = O(|\log\varepsilon|)$. We have

$$\begin{aligned}\left|E_{\varepsilon,\eta}^\delta(m) - E_{\varepsilon,\eta}^0(m)\right| = 2\left|\int_\Omega \delta\cdot\nabla m \wedge m \,\mathrm{d}x\right| &\leqslant 2\int_\Omega |\delta\cdot\nabla m\wedge m|\,\mathrm{d}x \\ &\leqslant 2\int_\Omega |\delta|\,|\nabla m|\,|m|\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\int_\Omega \left(|\nabla m|^2 + 4\,|\delta|^2\,|m|^2\right)\mathrm{d}x,\end{aligned}$$

by Young's inequality. Setting $S = \{x \in \Omega : |m(x)|^2 \geqslant 2\}$, it follows that

$$\begin{aligned}\left|E_{\varepsilon,\eta}^\delta(m) - E_{\varepsilon,\eta}^0(m)\right| &\leqslant \frac{1}{2}\int_\Omega |\nabla m|^2\,\mathrm{d}x + 2\,|\delta|^2\int_S |m|^2\mathrm{d}x + 2\,|\delta|^2\int_{\Omega\setminus S}|m|^2\mathrm{d}x \\ &\leqslant \frac{1}{2}\int_\Omega |\nabla m|^2\,\mathrm{d}x + 2\,|\delta|^2\int_S |m|^2\mathrm{d}x + 4\,|\delta|^2\,|\Omega\setminus S|\,.\end{aligned}$$

Moreover, for every $x \in S$, $|m(x)|^2 - 1 \geqslant \frac{1}{\sqrt{2}}|m(x)|$, and thus

$$\begin{aligned}\left|E^\delta_{\varepsilon,\eta}(m) - E^0_{\varepsilon,\eta}(m)\right| &\leqslant \frac{1}{2}\int_\Omega |\nabla m|^2 \,\mathrm{d}x + 4|\delta|^2 \int_S \left(1-|m|^2\right)^2 \mathrm{d}x + 4|\delta|^2 |\Omega\setminus S| \\ &\leqslant \frac{1}{2}\int_\Omega |\nabla m|^2 \,\mathrm{d}x + 4|\delta|^2 \int_\Omega \left(1-|m|^2\right)^2 \mathrm{d}x + 4|\delta|^2 |\Omega| .\end{aligned}$$

Since $8|\delta|^2 = O(1) \ll \frac{1}{\eta^2}$ and $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, we have, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}\left|E^\delta_{\varepsilon,\eta}(m) - E^0_{\varepsilon,\eta}(m)\right| &\leqslant \frac{1}{2}\int_\Omega |\nabla m|^2 \,\mathrm{d}x + \frac{1}{2\eta^2}\int_\Omega \left(1-|m|^2\right)^2 \mathrm{d}x + 4|\delta|^2 |\Omega| \\ &\leqslant \frac{1}{2}E^0_{\varepsilon,\eta}(m) + 4|\delta|^2|\Omega| .\end{aligned}$$

It follows that

$$E^\delta_{\varepsilon,\eta}(m) \geqslant \frac{1}{2}E^0_{\varepsilon,\eta}(m) - 4|\delta|^2|\Omega| ,$$

hence

$$E^0_{\varepsilon,\eta}(m) \leqslant 2E^\delta_{\varepsilon,\eta}(m) + 8|\delta|^2|\Omega| = O(|\log\varepsilon|),$$

since $E^\delta_{\varepsilon,\eta}(m) = O(|\log\varepsilon|)$, $\delta$ is constant and $\Omega$ is bounded. ☐

*Remark* 2.2.2. We assumed in the above lemma that $\delta \in \mathbb{R}^2$ is constant. However, this assumption may be weakened: indeed, since $\Omega$ is bounded, the estimate $|\delta| = O(|\log\varepsilon|^{1/2})$ is sufficient to get the expected result from the last inequality in the proof above.

*Notation* 2.2.3. Let $\delta \in \mathbb{R}^2$, $\varepsilon > 0$ and $\eta = \eta(\varepsilon) > 0$. For any open set $G \subset \Omega$ and $m\colon \Omega \to \mathbb{R}^2$, we define the localized functional

$$\begin{aligned}E^\delta_{\varepsilon,\eta}(m;G) = &\int_G |\nabla m|^2 \,\mathrm{d}x + 2\int_G \delta\cdot\nabla m \wedge m \,\mathrm{d}x \\ &+ \frac{1}{\eta^2}\int_G \left(1-|m|^2\right)^2 \mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\overline{G}\cap\partial\Omega} (m\cdot\nu)^2 \mathrm{d}\mathcal{H}^1.\end{aligned}$$

The next theorem gives the approximation of $m\colon \Omega \to \mathbb{R}^2$ as a $\mathbb{S}^1$-valued map $\mathfrak{m}\colon \Omega \to \mathbb{S}^1$. These maps and their global Jacobians are close.

**Theorem 2.2.4.** *Let $\beta \in (\frac{1}{2}, 1)$, $C > 0$ and $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain. Let $\delta \in \mathbb{R}^2$. Assume $\varepsilon \to 0$ and $\eta = \eta(\varepsilon) \to 0$ in the regime (2.13). There exist $\varepsilon_0 > 0$, $c_0 > 0$, $\widetilde{C} > 0$ and $\widehat{C} > 0$, depending only on $\beta$, $C$ and $\Omega$, and $\widetilde{\beta} \in \left(0, \frac{1-\beta}{6}\right)$ such that for every $\varepsilon \in (0,\varepsilon_0)$ and every $m = m_\varepsilon\colon \Omega \to \mathbb{R}^2$ satisfying $E^\delta_{\varepsilon,\eta}(m) \leqslant C|\log\varepsilon|$, we can construct a unit-length map $\mathfrak{m} = \mathfrak{m}_\varepsilon\colon \Omega \to \mathbb{S}^1$ that satisfies the following relations:*

$$\int_\Omega |\mathfrak{m} - m|^2 \,\mathrm{d}x \lesssim \eta^{2\beta} E^0_{\varepsilon,\eta}(m), \tag{2.30}$$

$$\int_\Omega \left(|\nabla\mathfrak{m}|^2 + |\nabla m|^2\right) \mathrm{d}x \lesssim E^0_{\varepsilon,\eta}(m), \tag{2.31}$$

$$\left|\int_\Omega \delta\cdot(\nabla\mathfrak{m}\wedge\mathfrak{m} - \nabla m\wedge m)\,\mathrm{d}x\right| \lesssim \eta^\beta E^0_{\varepsilon,\eta}(m) + \eta^{\beta/2}\sqrt{E^0_{\varepsilon,\eta}(m)}, \tag{2.32}$$

$$\int_{\partial\Omega} |\mathfrak{m} - m|^2 \,\mathrm{d}\mathcal{H}^1 \lesssim \eta^\beta E^0_{\varepsilon,\eta}(m), \tag{2.33}$$

$$E^0_{\varepsilon,\eta}(\mathfrak{m}) \leqslant E^0_{\varepsilon,c_0\eta}(m) + \widetilde{C}\eta^{\widetilde{\beta}} \left(E^0_{\varepsilon,c_0\eta}(m) + \sqrt{E^0_{\varepsilon,c_0\eta}(m)}\right), \tag{2.34}$$

*and*

$$\begin{aligned} E^\delta_{\varepsilon,\eta}(\mathfrak{m}) \leqslant E^\delta_{\varepsilon,c_0\eta}(m) &+ \widetilde{C}\eta^{\widetilde{\beta}} \left(E^0_{\varepsilon,c_0\eta}(m) + \sqrt{E^0_{\varepsilon,c_0\eta}(m)}\right) \\ &+ \widehat{C}\left(\eta^\beta E^0_{\varepsilon,\eta}(m) + \eta^{\beta/2}\sqrt{E^0_{\varepsilon,\eta}(m)}\right). \end{aligned} \tag{2.35}$$

*As a consequence, for every* $p \in [1, +\infty)$,

$$\lim_{\varepsilon\to 0} \|\mathfrak{m} - m\|_{L^p(\partial\Omega)} = 0, \tag{2.36}$$

$$\lim_{\varepsilon\to 0} \|\mathfrak{m} - m\|_{L^p(\Omega)} = 0, \tag{2.37}$$

$$\|\mathrm{jac}(m)\|_{(W^{1,\infty}_0)^*(\Omega)} \lesssim \eta^\beta E^0_{\varepsilon,\eta}(m), \tag{2.38}$$

*and*

$$\|\mathcal{J}(\mathfrak{m}) - \mathcal{J}(m)\|_{(\mathrm{Lip}(\Omega))^*} \lesssim \sqrt{\eta^\beta E^0_{\varepsilon,\eta}(m)}. \tag{2.39}$$

*The map* $\mathfrak{m}$ *also satisfies the following local estimate. For any open set* $G \subset \Omega$ *independent of* $\varepsilon$*, there exists a constant* $\widetilde{C}_G > 0$ *such that*

$$E^0_{\varepsilon,\eta}(\mathfrak{m}; G_\eta) \leqslant E^0_{\varepsilon,c_0\eta}(m; G) + \widetilde{C}_G\eta^{\widetilde{\beta}} \left(E^0_{\varepsilon,c_0\eta}(m; G) + \sqrt{E^0_{\varepsilon,c_0\eta}(m; G)}\right) \tag{2.40}$$

*where*

$$G_\eta = \left\{x \in G : \mathrm{dist}(x, \Omega \cap \partial G) > 3\eta^\beta\right\}.$$

*Proof.* This theorem is an extension of [24, Theorem 3.1]. The ideas used in the proof come from [31] combined with [26]. The relations (2.30), (2.31), (2.33), (2.34), (2.36), (2.37), (2.38), (2.39) and (2.40) have been proved in [24, Theorem 3.1] in the case $\delta = 0$. Using Lemma 2.2.1, we can extend these relations to an arbitrary $\delta$.

Let us prove (2.32). Using integration by parts, we have

$$\begin{aligned} \int_\Omega \delta \cdot (\nabla\mathfrak{m} \wedge \mathfrak{m} - \nabla m \wedge m)\,\mathrm{d}x &= \int_\Omega \delta \cdot ((\nabla\mathfrak{m} - \nabla m) \wedge \mathfrak{m} - \nabla m \wedge (m - \mathfrak{m}))\,\mathrm{d}x \\ &= \int_{\partial\Omega} (\delta\cdot\nu)(\mathfrak{m} - m) \wedge \mathfrak{m}\,\mathrm{d}\mathcal{H}^1 \\ &\quad - \int_\Omega \delta \cdot ((\mathfrak{m} - m) \wedge \nabla\mathfrak{m} + \nabla m \wedge (m - \mathfrak{m}))\,\mathrm{d}x. \end{aligned}$$

We deduce that

$$\begin{aligned} \left|\int_\Omega \delta \cdot (\nabla\mathfrak{m} \wedge \mathfrak{m} - \nabla m \wedge m)\,\mathrm{d}x\right| &\leqslant \int_{\partial\Omega} |\delta\cdot\nu|\,|\mathfrak{m} - m|\,\mathrm{d}\mathcal{H}^1 \\ &\quad + \int_\Omega |\delta|\,(|\nabla\mathfrak{m}| + |\nabla m|)\,|\mathfrak{m} - m|\,\mathrm{d}x \\ &\leqslant C_\delta\left[\int_{\partial\Omega} |\mathfrak{m} - m|^2\,\mathrm{d}\mathcal{H}^1\right. \\ &\quad \left. + \left(\int_\Omega \left(|\nabla\mathfrak{m}|^2 + |\nabla m|^2\right)\mathrm{d}x\right)\left(\int_\Omega |\mathfrak{m} - m|^2\,\mathrm{d}x\right)\right], \end{aligned}$$

where $C_\delta > 0$ is a constant depending only on $\delta$. Using (2.33), (2.30) and (2.31), we deduce

$$\left|\int_\Omega \delta \cdot (\nabla \mathfrak{m} \wedge \mathfrak{m} - \nabla m \wedge m)\, \mathrm{d}x\right| \lesssim \eta^{\beta/2}\sqrt{E^0_{\varepsilon,\eta}(m)} + \eta^\beta E^0_{\varepsilon,\eta}(m),$$

which is (2.32).

We can now deduce (2.35) from (2.34) and (2.32). More precisely, by (2.34),

$$\begin{aligned} E^\delta_{\varepsilon,\eta}(\mathfrak{m}) &= E^0_{\varepsilon,\eta}(\mathfrak{m}) + 2\int_\Omega \delta \cdot \nabla \mathfrak{m} \wedge \mathfrak{m}\ \mathrm{d}x \\ &\leqslant E^0_{\varepsilon,c_0\eta}(m) + \widetilde{C}\eta^{\widetilde{\beta}}\left(E^0_{\varepsilon,c_0\eta}(m) + \sqrt{E^0_{\varepsilon,c_0\eta}(m)}\right) \\ &\quad + 2\int_\Omega \delta \cdot (\nabla \mathfrak{m} \wedge \mathfrak{m} - \nabla m \wedge m)\, \mathrm{d}x + 2\int_\Omega \delta \cdot \nabla m \wedge m\ \mathrm{d}x \end{aligned}$$

and, by (2.32),

$$\begin{aligned} E^\delta_{\varepsilon,\eta}(\mathfrak{m}) &\leqslant E^\delta_{\varepsilon,c_0\eta}(m) + \widetilde{C}\eta^{\widetilde{\beta}}\left(E^0_{\varepsilon,c_0\eta}(m) + \sqrt{E^0_{\varepsilon,c_0\eta}(m)}\right) \\ &\quad + \widehat{C}\left(\eta^\beta E^0_{\varepsilon,\eta}(m) + \eta^{\beta/2}\sqrt{E^0_{\varepsilon,\eta}(m)}\right). \end{aligned}$$

□

### 2.2.2 Approximation by lifting and localization lemma

The following lemma, dued to Bethuel-Zheng [8], is useful in order to simplify the analyse of the energy functional $E^\delta_{\varepsilon,\eta}$ of $\mathbb{S}^1$-valued maps $\mathfrak{m}$.

**Lemma 2.2.5.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain. Let $\delta \in \mathbb{R}^2$ and $\mathfrak{m} \in H^1(\Omega, \mathbb{S}^1)$. There exists a lifting $\varphi \in H^1(\Omega, \mathbb{R})$ such that $\mathfrak{m} = e^{i\varphi}$ and $\varphi$ is unique up to an additive constant in $2\pi\mathbb{Z}$. Furthermore, for every $\varepsilon > 0$ and $\eta > 0$ sufficiently small,*

$$E^\delta_{\varepsilon,\eta}(\mathfrak{m}) = \int_\Omega \left(|\nabla\varphi|^2 - 2\delta\cdot\nabla\varphi\right)\mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} \sin^2(\varphi - g)\ \mathrm{d}\mathcal{H}^1 =: \mathcal{G}^\delta_\varepsilon(\varphi), \tag{2.41}$$

*where $g$ is a lifting of the unit tangent vector field $\tau$ on $\partial\Omega$, i.e.*

$$e^{ig} = \tau = i\nu \quad \text{on } \partial\Omega, \tag{2.42}$$

*and $g$ is continuous except at one point of $\partial\Omega$.*

*Proof.* Existence and uniqueness of the lifting $\varphi$ of $\mathfrak{m}$ in $\Omega$ come from a well-known theorem of Bethuel and Zheng [8]. For the existence of $g$, we note that $\tau$ has winding number 1 around $\partial\Omega$ as $\Omega$ is simply connected, hence no continuous $g\colon \partial\Omega \to \mathbb{R}$ with $e^{ig} = \tau$ can exist. However, if $\partial\Omega$ is $C^{1,1}$, we can choose $g$ to be locally Lipschitz except at one point of $\partial\Omega$ where it jumps by $2\pi$. Clearly, the curvature $\kappa$ of $\partial\Omega$ is given by the absolutely continuous part of the derivative of $g$ (as a BV function), i.e. $\kappa = (\partial_\tau g)_{ac}$ and $\int_{\partial\Omega} \kappa\ \mathrm{d}\mathcal{H}^1 = 2\pi$, which is the Gauss-Bonnet formula for the boundary of a simply connected domain. As $|\nabla\mathfrak{m}| = |\nabla\varphi|$ and $\nabla\mathfrak{m}\wedge\mathfrak{m} = \mathfrak{Im}(-i\nabla\varphi e^{-i\varphi}e^{i\varphi}) = -\nabla\varphi$ in $\Omega$, and $\mathfrak{m}\cdot\nu = -\sin(\varphi - g)$ on $\partial\Omega$, we deduce the equality between $E^\delta_{\varepsilon,\eta}(\mathfrak{m})$ and $\mathcal{G}^\delta_\varepsilon(\varphi)$. □

The functional $\mathcal{G}^\delta_\varepsilon$ in the above lemma has been studied by Kurzke [33], [34]. In the following, we will prove Gamma-convergence for $\mathcal{G}^\delta_\varepsilon$ and use these result for proving Gamma-convergence for $E^\delta_{\varepsilon,\eta}$. Note that, up to a constant, $\mathcal{G}^\delta_\varepsilon$ is exactly the energy $E^\delta_\varepsilon(\varphi;\Omega)$ that we studied in Section 1.3 in the case where $\nu = -e_2$, i.e. $g = 0$. In fact, in order to obtain the asymptotic expansion by Gamma-convergence, we need to get rid of $g$. To do so, we will use the following localization lemma that comes from [24, Lemma 4.3]. The assumption of $C^{1,1}$ smoothness is required here.

**Lemma 2.2.6** ([24], Lemma 4.3)**.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected and $C^{1,1}$ smooth domain. There exist constants $c_1 = c_1(\Omega) > 0$ and $r_0 = r_0(\Omega) \in (0,1)$ such that, for any $a \in \partial\Omega$, we can find a $C^1$ map $\Psi_a \colon \overline{B^+_{2r_0}} \to \overline{\Omega}$ with the following properties.*

*(i)* $\Psi_a \colon \overline{B^+_{r_0(1+c_1 r_0 \log \frac{1}{r_0})}} \to \Psi_a(\overline{B^+_{r_0(1+c_1 r_0 \log \frac{1}{r_0})}})$ *is a conformal diffeomorphism with* $\Psi_a(0) = a$.

*(ii) For any $\phi \in H^1(\Omega, \mathbb{R})$, setting $\psi = \phi \circ \Psi_a$, then for any $r \in (0, r_0)$,*

$$\int_{B^+_{r(1-c_1 r \log \frac{1}{r})}} |\nabla\psi|^2 \, \mathrm{d}x \leqslant \int_{\Omega \cap B_r(a)} |\nabla\phi|^2 \, \mathrm{d}x \leqslant \int_{B^+_{r(1+c_1 r \log \frac{1}{r})}} |\nabla\psi|^2 \, \mathrm{d}x,$$

*and*

$$\begin{aligned}\left(1 - c_1 r \log \frac{1}{r}\right) \int_{I_{r(1-c_1 r \log \frac{1}{r})}} \sin^2 \psi(\cdot, 0) \, \mathrm{d}\mathcal{H}^1 &\leqslant \int_{\partial\Omega \cap B_r(a)} \sin^2 \phi \, \mathrm{d}\mathcal{H}^1 \\ &\leqslant \left(1 + c_1 r \log \frac{1}{r}\right) \int_{I_{r(1+c_1 r \log \frac{1}{r})}} \sin^2 \psi(\cdot, 0) \, \mathrm{d}\mathcal{H}^1.\end{aligned}$$

*where $I_\rho = (-\rho, \rho)$ for any $\rho > 0$.*

**Lemma 2.2.7.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected and $C^{1,1}$ smooth domain. Let $a \in \partial\Omega$. The map $\Psi_a$ in [24, Lemma 4.3] satisfies, for $r > 0$ sufficiently small,*

$$\Omega \cap B_{r(1-c_1 r \log \frac{1}{r})}(a) \subset \Psi_a(B^+_r) \subset \Omega \cap B_{r(1+c_1 r \log \frac{1}{r})}(a).$$

*Proof.* Using the proof of [24, Lemma 4.3], we have

$$|\Psi_a(z) - a - \tau_a z| \leqslant C \, |z|^2 \log \frac{1}{|z|},$$

for $|z|$ sufficiently small, where $\tau_a$ denotes the unit tangent vector at the point $a \in \partial\Omega$.
Let $y \in \Psi_a(B^+_r)$ for some $r > 0$ small. Then, there exists $z \in B^+_r$ such that $\Psi_a(z) = y$ and

$$|y - a| = |\Psi_a(z) - a| \leqslant |\Psi_a(z) - a - \tau_a z| + |z| \leqslant C \, |z|^2 \log \frac{1}{|z|} + |z| \leqslant r \left(1 + C r \log \frac{1}{r}\right),$$

because $t \mapsto t \log \frac{1}{t}$ increases for $t > 0$ small enough. Hence, $y \in \Omega \cap B_{r(1+c_1 r \log \frac{1}{r})}(a)$ for $r > 0$ sufficiently small. The other inclusion is obtained similarly (and up to taking a smaller $r > 0$ than above), using that

$$\left|\Psi_a^{-1}(y) - \gamma(y - a)\right| \leqslant C \, |y - a|^2 \log \frac{1}{|y - a|},$$

for $|y|$ sufficiently small, where $\gamma = (\Psi_a^{-1})'(a)$ has modulus less than 1. □

### 2.2.3 Estimates for the energy near boundary vortices

For the second order term in the asymptotic expansion of the energy by Gamma-convergence, we need to estimate the energy $\mathcal{G}^\delta_\varepsilon$ defined at (2.41) in a neighborhood of the boundary vortices. Up to use a conformal map and make a blow-up near a boundary vortex, we can consider the following localized functional.

*Notation* 2.2.8. Let $\delta \in \mathbb{R}^2$, $\varepsilon > 0$ and $\eta = \eta(\varepsilon) > 0$. For any open set $G \subset \mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty)$ and $\psi \colon G \to \mathbb{R}$, we define the localized functional

$$F^\delta_\varepsilon(\psi; G) = \int_G \left(|\nabla\psi|^2 - 2\delta \cdot \nabla\psi\right) \mathrm{d}x + \frac{1}{2\pi\varepsilon} \int_{\overline{G} \cap (\mathbb{R} \times \{0\})} \sin^2 \psi(x_1, 0) \, \mathrm{d}x_1.$$

*Notation* 2.2.9. For every $(x_1, x_2) \in \mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty)$, we set

$$\phi^*(x_1, x_2) = \arg(x_1 + ix_2) = \frac{\pi}{2} - \arctan\left(\frac{x_1}{x_2}\right),$$

and

$$\phi^*_\varepsilon(x_1, x_2) = \arg(x_1 + i(x_2 + 2\pi\varepsilon)) = \frac{\pi}{2} - \arctan\left(\frac{x_1}{x_2 + 2\pi\varepsilon}\right).$$

**Lemma 2.2.10.** *Let $\delta = (\delta_1, \delta_2) \in \mathbb{R}^2$. For any $r \in (0,1)$, set $I_r = (-r, r)$. Then the quantity*

$$\gamma_2 = \liminf_{r\to 0} \liminf_{\varepsilon\to 0} \left( \inf_{\psi=\phi^*_\varepsilon \text{ on } \partial B^+_r \setminus I_r} F^\delta_\varepsilon(\psi; B^+_r) - \pi \log \frac{r}{\varepsilon} \right) \tag{2.43}$$

*is a limit as $r \to 0$. Moreover, we have $\gamma_2 = \pi \log \frac{e}{4\pi}$.*

*Proof.* In the case $\delta = 0$, Cabré and Sola-Moralès showed [10, Lemma 3.1] that, for $r \in (0,1)$,

$$\inf_{\psi=\phi^*_\varepsilon \text{ on } \partial B^+_r \setminus I_r} F^0_\varepsilon(\psi; B^+_r) = F^0_\varepsilon(\phi^*_\varepsilon; B^+_r). \tag{2.44}$$

Furthermore, Ignat and Kurzke [24, Lemma 4.14] proved that

$$\lim_{r\to 0} \liminf_{\varepsilon\to 0} \left( F^0_\varepsilon(\phi^*_\varepsilon; B^+_r) - \pi \log \frac{r}{\varepsilon} \right) = \pi \log \frac{e}{4\pi}. \tag{2.45}$$

We will use both this statements to prove that $\gamma_2$ is a limit as $r \to 0$ and is equal to $\pi \log \frac{e}{4\pi}$.

*Step 1 :* We first prove that $\gamma_2 \leqslant \pi \log \frac{e}{4\pi}$.

To do so, we observe that $\inf \left\{ F^\delta_\varepsilon(\psi; B^+_r) \colon \psi = \phi^*_\varepsilon \text{ on } \partial B^+_r \setminus I_r \right\} \leqslant F^\delta_\varepsilon(\phi^*_\varepsilon; B^+_r)$, hence

$$\gamma_2 \leqslant \liminf_{r\to 0} \liminf_{\varepsilon\to 0} \left( F^\delta_\varepsilon(\phi^*_\varepsilon; B^+_r) - \pi \log \frac{r}{\varepsilon} \right). \tag{2.46}$$

Let $r \in (0,1)$. By Green's formula, we have

$$F^\delta_\varepsilon(\phi^*_\varepsilon; B^+_r) = F^0_\varepsilon(\phi^*_\varepsilon; B^+_r) - 2 \int_{B^+_r} \delta \cdot \nabla \phi^*_\varepsilon \, \mathrm{d}x = F^0_\varepsilon(\phi^*_\varepsilon; B^+_r) - 2 \int_{\partial B^+_r} \phi^*_\varepsilon (\delta \cdot \nu) \, \mathrm{d}\mathcal{H}^1,$$

where $\nu = (\nu_1, \nu_2)$ is the outer unit normal vector on $\partial B^+_r$. Moreover,

$$\left| \int_{\partial B^+_r} \phi^*_\varepsilon (\delta \cdot \nu) \, \mathrm{d}\mathcal{H}^1 \right| \leqslant |\delta| \, \|\phi^*_\varepsilon\|_{L^\infty} \left| \partial B^+_r \right| \leqslant Cr$$

for some $C > 0$ independent of $\varepsilon$ and $r$, since $\phi^*_\varepsilon$ is bounded independently of $\varepsilon$ (by definition of $\phi^*_\varepsilon$, we have $\phi^*_\varepsilon \in [0, \pi]$) and $|\partial B^+_r| = (\pi + 2)r$. We deduce that

$$F^\delta_\varepsilon(\phi^*_\varepsilon; B^+_r) - \pi \log \frac{r}{\varepsilon} = F^0_\varepsilon(\phi^*_\varepsilon; B^+_r) - \pi \log \frac{r}{\varepsilon} - O(r).$$

Using (2.45), it follows that

$$\liminf_{r\to 0} \liminf_{\varepsilon\to 0} \left( F^\delta_\varepsilon(\phi^*_\varepsilon; B^+_r) - \pi \log \frac{r}{\varepsilon} \right) = \pi \log \frac{e}{4\pi}. \tag{2.47}$$

Combining (2.46) and (2.47), we get $\gamma_2 \leqslant \pi \log \frac{e}{4\pi}$.

*Step 2 :* We prove that $\gamma_2 \geqslant \pi \log \frac{e}{4\pi}$.

Let $r \in (0,1)$ and $\psi\colon \mathbb{R}^2_+ \to \mathbb{R}$ such that $\psi = \phi^*_\varepsilon$ on $\partial B_r^+ \setminus I_r$. As $\phi^*_\varepsilon \in [0,\pi]$ in $\mathbb{R}^2_+$, then on $\partial B_r^+ \setminus I_r$, we have

$$-\left|\delta_2\right| r \leqslant \phi^*_\varepsilon(x_1,x_2) - \delta_2 x_2 \leqslant \pi + \left|\delta_2\right| r.$$

Since $r \in (0,1)$, there exists an integer $N \geqslant 0$ such that on $\partial B_r^+ \setminus I_r$,

$$-N\pi \leqslant \phi^*_\varepsilon(x_1,x_2) - \delta_2 x_2 \leqslant N\pi.$$

We define $\widetilde{\psi}$ in $\overline{B_r^+}$ as follows:

$$\widetilde{\psi}(x_1,x_2) = \delta_2 x_2 + \begin{cases} \psi(x_1,x_2) - \delta_2 x_2 & \text{if } -N\pi \leqslant \psi(x_1,x_2) - \delta_2 x_2 \leqslant N\pi, \\ -N\pi & \text{if } \psi(x_1,x_2) \leqslant -N\pi, \\ N\pi & \text{if } \psi(x_1,x_2) \geqslant N\pi. \end{cases} \tag{2.48}$$

Note that $\widetilde{\psi}(\cdot,0)$ is bounded with values in $[-N\pi, N\pi]$ on $I_r$, and $\widetilde{\psi} = \phi^*_\varepsilon$ on $\partial B_r^+ \setminus I_r$. Let us show that $F^\delta_\varepsilon(\widetilde{\psi}; B_r^+) \leqslant F^\delta_\varepsilon(\psi; B_r^+)$. First, we note that on $I_r$, $\widetilde{\psi}(\cdot,0)$ is equal either to $\psi(\cdot,0)$, or to $-N\pi$, or to $N\pi$. Thus $\sin^2 \widetilde{\psi}(\cdot,0) \leqslant \sin^2 \psi(\cdot,0)$, and

$$\int_{I_r} \sin^2 \widetilde{\psi}(x_1,0)\ \mathrm{d}x_1 \leqslant \int_{I_r} \sin^2 \psi(x_1,0)\ \mathrm{d}x_1.$$

For the interior integral, we note that

$$\int_{B_r^+} \left(|\nabla\widetilde{\psi}|^2 - 2\delta\cdot\nabla\widetilde{\psi}\right)\mathrm{d}x = \int_{B_r^+} \left(|\partial_1\widetilde{\psi}|^2 - 2\delta_1\partial_1\widetilde{\psi}\right)\mathrm{d}x + \int_{B_r^+} |\partial_2\widetilde{\psi} - \delta_2|^2 \mathrm{d}x - \delta_2^2\left|B_r^+\right|.$$

On the one hand, using (2.48), we have

$$|\partial_1\widetilde{\psi}| = \begin{cases} |\partial_1\psi| & \text{if } -N\pi \leqslant \psi(x_1,x_2) - \delta_2 x_2 \leqslant N\pi, \\ 0 & \text{elsewhere,} \end{cases}$$

thus $|\partial_1\widetilde{\psi}| \leqslant |\partial_1\psi|$ in $B_r^+$, and $\int_{B_r^+} |\partial_1\widetilde{\psi}|^2\mathrm{d}x \leqslant \int_{B_r^+} |\partial_1\psi|^2\mathrm{d}x$. Moreover,

$$\int_{B_r^+} \partial_1\widetilde{\psi}\ \mathrm{d}x = \int_{\partial B_r^+\setminus I_r} \widetilde{\psi}\nu_1\ \mathrm{d}\mathcal{H}^1 = \int_{\partial B_r^+\setminus I_r} \phi^*_\varepsilon\nu_1\ \mathrm{d}\mathcal{H}^1 = \int_{B_r^+} \partial_1\psi\ \mathrm{d}x$$

using that $\nu_1 = 0$ on $I_r$. On the other hand, using (2.48),

$$|\partial_2\widetilde{\psi} - \delta_2| = \begin{cases} |\partial_2\psi - \delta_2| & \text{if } -N\pi \leqslant \psi(x_1,x_2) - \delta_2 x_2 \leqslant N\pi, \\ 0 & \text{elsewhere,} \end{cases}$$

thus $|\partial_2\widetilde{\psi} - \delta_2| \leqslant |\partial_2\psi - \delta_2|$ in $B_r^+$, and $\int_{B_r^+} |\partial_2\widetilde{\psi} - \delta_2|^2\mathrm{d}x \leqslant \int_{B_r^+} |\partial_2\psi - \delta_2|^2\mathrm{d}x$. Combining the above inequalities, comparing the boundary and interior parts of $F^\delta_\varepsilon(\widetilde{\psi}; B_r^+)$ and $F^\delta_\varepsilon(\psi; B_r^+)$, we deduce that $F^\delta_\varepsilon(\widetilde{\psi}; B_r^+) \leqslant F^\delta_\varepsilon(\psi; B_r^+)$. By definition of $F^\delta_\varepsilon(\widetilde{\psi}; B_r^+)$ and using Green's formula,

$$\begin{aligned} F^\delta_\varepsilon(\widetilde{\psi}; B_r^+) &= F^0_\varepsilon(\widetilde{\psi}; B_r^+) - \int_{B_r^+} 2\delta\cdot\nabla\widetilde{\psi}\ \mathrm{d}x \\ &= F^0_\varepsilon(\widetilde{\psi}; B_r^+) - \int_{\partial B_r^+\setminus I_r} 2(\delta\cdot\nu)\widetilde{\psi}\ \mathrm{d}\mathcal{H}^1 + \int_{I_r} 2\delta_2\nu_2\widetilde{\psi}(x_1,0)\ \mathrm{d}x_1. \end{aligned}$$

Since $\widetilde{\psi} = \phi_\varepsilon^*$ on $\partial B_r^+ \setminus I_r$, we can use (2.44) to get

$$F_\varepsilon^\delta(\widetilde{\psi}; B_r^+) \geqslant F_\varepsilon^0(\phi_\varepsilon^*; B_r^+) - \int_{\partial B_r^+ \setminus I_r} 2(\delta \cdot \nu)\phi_\varepsilon^* \, \mathrm{d}\mathcal{H}^1 + \int_{I_r} 2\delta_2 \nu_2 \widetilde{\psi}(x_1, 0) \, \mathrm{d}x_1.$$

Using that $\phi_\varepsilon^* \in [0, \pi]$ and $\widetilde{\psi}(\cdot, 0) \in [-N\pi, N\pi]$ on $I_r$, we get

$$F_\varepsilon^\delta(\widetilde{\psi}; B_r^+) \geqslant F_\varepsilon^0(\phi_\varepsilon^*; B_r^+) - Cr$$

for some $C > 0$ independent of $\varepsilon$ and $r$. Hence,

$$\begin{aligned}
\gamma_2 &= \liminf_{r \to 0} \liminf_{\varepsilon \to 0} \left( \inf_{\widetilde{\psi} = \phi_\varepsilon^* \text{ on } \partial B_r^+ \setminus I_r} F_\varepsilon^\delta(\widetilde{\psi}; B_r^+) - \pi \log \frac{r}{\varepsilon} \right) \\
&\geqslant \liminf_{r \to 0} \left( \liminf_{\varepsilon \to 0} \left( F_\varepsilon^0(\phi_\varepsilon^*; B_r^+) - \pi \log \frac{r}{\varepsilon} \right) - Cr \right) \\
&= \liminf_{r \to 0} \liminf_{\varepsilon \to 0} \left( F_\varepsilon^0(\phi_\varepsilon^*; B_r^+) - \pi \log \frac{r}{\varepsilon} \right) \\
&= \pi \log \frac{e}{4\pi},
\end{aligned}$$

the last equality coming from (2.45).

Combining Step 1 and Step 2, we get $\gamma_2 = \pi \log \frac{e}{4\pi}$ and the $\liminf$ as $r \to 0$ in the definition of $\gamma_2$ is a limit as $r \to 0$. $\square$

**Lemma 2.2.11.** *Let $\delta = (\delta_1, \delta_2) \in \mathbb{R}^2$. For any $r \in (0, 1)$, set $I_r = (-r, r)$. Then the quantity*

$$\gamma_1 = \liminf_{r \to 0} \liminf_{\varepsilon \to 0} \left( \inf_{\psi = \phi^* \text{ on } \partial B_r^+ \setminus I_r} F_\varepsilon^\delta(\psi; B_r^+) - \pi \log \frac{r}{\varepsilon} \right), \tag{2.49}$$

*is a limit as $r \to 0$. Moreover, we have $\gamma_1 = \gamma_2 = \pi \log \frac{e}{4\pi}$, where $\gamma_2$ is defined in* (2.43)*.*

*Proof.* By Lemma 2.2.10, $\gamma_2$ is well-defined as a limit as $r \to 0$ and $\gamma_2 = \pi \log \frac{e}{4\pi}$.

*Step 1.* Let us show that $\gamma_2 \leqslant \gamma_1$.

Let $r \in (0, 1)$. Consider the family of functions $(\phi_\varepsilon)_{\varepsilon > 0}$ defined in $B_{r(1+r)}^+ \setminus B_r$ as

$$\phi_\varepsilon(x_1, x_2) = \arg \left( x_1 + i \left( x_2 + 2\pi\varepsilon \frac{\sqrt{x_1^2 + x_2^2} - r}{r^2} \right) \right)$$

for every $(x_1, x_2) \in B_{r(1+r)}^+ \setminus B_r$, and that satisfies, for every $\varepsilon > 0$, $\phi_\varepsilon = \phi_\varepsilon^*$ on the half-circle $\partial B_{r(1+r)}^+ \setminus I_{r(1+r)}$ and $\phi_\varepsilon = \phi^*$ on the half-circle $\partial B_r^+ \setminus I_r$. We observe that

$$\begin{aligned}
F_\varepsilon^\delta(\phi_\varepsilon; B_{r(1+r)}^+) - \pi \log \frac{r(1+r)}{\varepsilon} &= F_\varepsilon^\delta(\phi_\varepsilon; B_r^+) - \pi \log \frac{r}{\varepsilon} - \pi \log(1+r) \\
&\quad + \int_{B_{r(1+r)}^+ \setminus B_r} \left( |\nabla \phi_\varepsilon|^2 - 2\delta \cdot \nabla \phi_\varepsilon \right) \mathrm{d}x \\
&\quad + \frac{1}{2\pi\varepsilon} \int_{I_{r(1+r)} \setminus I_r} \sin^2 \phi_\varepsilon(x_1, 0) \, \mathrm{d}x_1.
\end{aligned} \tag{2.50}$$

Since both the argument function and the function multiplied by $\varepsilon$ are smooth away from zero, we can use the dominated convergence theorem to get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{B^+_{r(1+r)}\setminus B_r} &\left(|\nabla\phi_\varepsilon|^2 - 2\delta\cdot\nabla\phi_\varepsilon\right)\mathrm{d}x\\ &= \int_{B^+_{r(1+r)}\setminus B_r}\left(|\nabla\phi^*|^2 - 2\delta\cdot\nabla\phi^*\right)\mathrm{d}x\\ &= \int_{B^+_{r(1+r)}\setminus B_r}\left(\frac{1}{x_1^2+x_2^2} - \frac{2(\delta_2 x_1 - \delta_1 x_2)}{x_1^2+x_2^2}\right)\mathrm{d}x_1\mathrm{d}x_2.\end{aligned}$$

Using polar coordinates and Fubini's theorem, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{B^+_{r(1+r)}\setminus B_r} &\left(|\nabla\phi_\varepsilon|^2 - 2\delta\cdot\nabla\phi_\varepsilon\right)\mathrm{d}x\\ &= \int_{\rho=r}^{r(1+r)}\int_{\theta=0}^{\pi}\left(\frac{1}{\rho^2} - \frac{2\rho(\delta_2\cos\theta - \delta_1\sin\theta)}{\rho^2}\right)\rho\ \mathrm{d}\theta\mathrm{d}\rho\\ &= \pi\int_r^{r(1+r)}\frac{1}{\rho}\ \mathrm{d}\rho - 2\int_0^{\pi}(\delta_2\cos\theta - \delta_1\sin\theta)\ \mathrm{d}\theta\int_r^{r(1+r)}\mathrm{d}\rho\\ &= \pi\log(1+r) + 4\delta_1 r^2.\end{aligned}$$

Moreover, for $\varepsilon > 0$ sufficiently small and for every $x_1 \in I_{r(1+r)}\setminus I_r$,

$$\sin^2\phi_\varepsilon(x_1,0) = \sin^2\left(\arg\left(x_1 + 2i\pi\varepsilon\frac{|x_1|-r}{r^2}\right)\right) \leqslant \sin^2\left(\arg\left(r(1+r)+2i\pi\varepsilon\right)\right),$$

and

$$\sin^2\left(\arg\left(r(1+r)+2i\pi\varepsilon\right)\right) = \sin^2\left(\arctan\frac{2\pi\varepsilon}{r(1+r)}\right) \leqslant \sin^2\left(\arctan\frac{2\pi\varepsilon}{r}\right) \leqslant \left(\frac{2\pi\varepsilon}{r}\right)^2,$$

since $\sin^2 \leqslant \tan^2$, thus

$$\frac{1}{2\pi\varepsilon}\int_{I_{r(1+r)}\setminus I_r}\sin^2\phi_\varepsilon(x_1,0)\ \mathrm{d}x_1 \leqslant \frac{2\pi\varepsilon}{r^2}\left|I_{r(1+r)}\setminus I_r\right| = 4\pi\varepsilon.$$

Combining the above inequalities, (2.50) becomes

$$\begin{aligned}F_\varepsilon^\delta(\phi_\varepsilon; B^+_{r(1+r)}) - \pi\log\frac{r(1+r)}{\varepsilon} \leqslant{}& F_\varepsilon^\delta(\phi_\varepsilon; B^+_r) - \pi\log\frac{r}{\varepsilon} - \pi\log(1+r)\\ &+ \pi\log(1+r) + 4\delta_1 r^2 + 4\pi\varepsilon,\end{aligned}$$

i.e.

$$F_\varepsilon^\delta(\phi_\varepsilon; B^+_{r(1+r)}) - \pi\log\frac{r(1+r)}{\varepsilon} \leqslant F_\varepsilon^\delta(\phi_\varepsilon; B^+_r) - \pi\log\frac{r}{\varepsilon} + 4\delta_1 r^2 + 4\pi\varepsilon. \tag{2.51}$$

The left-hand side in (2.51) is greater than

$$\inf_{\psi=\phi_\varepsilon^* \text{ on } \partial B^+_{r(1+r)}\setminus I_{r(1+r)}} F_\varepsilon^\delta(\psi; B^+_{r(1+r)}) - \pi\log\frac{r(1+r)}{\varepsilon},$$

so that, taking then the infimum of $F_\varepsilon^\delta(\psi; B_r^+)$ over functions $\psi$ such that $\psi = \phi^*$ on $\partial B_r^+ \setminus I_r$ in the right-hand side in (2.51), taking the lim inf as $\varepsilon \to 0$, and taking finally the lim inf as $r \to 0$, we get

$$\gamma_2 \leqslant \liminf_{r\to 0} \liminf_{\varepsilon\to 0} \left( \inf_{\psi=\phi^* \text{ on } \partial B_r^+\setminus I_r} F_\varepsilon^\delta(\psi; B_r^+) - \pi \log \frac{r}{\varepsilon} + 4\delta_1 r^2 + 4\pi\varepsilon \right) \leqslant \gamma_1.$$

*Step 2.* Considering $r(1-r)$ instead of $r(1+r)$ and proceeding as in Step 1, we show similarly that

$$\limsup_{r\to 0} \liminf_{\varepsilon\to 0} \left( \inf_{\psi=\phi^* \text{ on } \partial B_r^+\setminus I_r} F_\varepsilon^\delta(\psi; B_r^+) - \pi \log \frac{r}{\varepsilon} \right) \leqslant \gamma_2.$$

*Step 3.* Combining Step 1 and Step 2, we deduce that

$$\begin{aligned}\gamma_2 \leqslant \gamma_1 &= \liminf_{r\to 0} \liminf_{\varepsilon\to 0} \left( \inf_{\psi=\phi^* \text{ on } \partial B_r^+\setminus I_r} F_\varepsilon^\delta(\psi; B_r^+) - \pi \log \frac{r}{\varepsilon} \right) \\ &\leqslant \limsup_{r\to 0} \liminf_{\varepsilon\to 0} \left( \inf_{\psi=\phi^* \text{ on } \partial B_r^+\setminus I_r} F_\varepsilon^\delta(\psi; B_r^+) - \pi \log \frac{r}{\varepsilon} \right) \\ &\leqslant \gamma_2.\end{aligned}$$

It follows that

$$\gamma_1 = \lim_{r\to 0} \liminf_{\varepsilon\to 0} \left( \inf_{\psi=\phi^* \text{ on } \partial B_r^+\setminus I_r} F_\varepsilon^\delta(\psi; B_r^+) - \pi \log \frac{r}{\varepsilon} \right) = \gamma_2 = \pi \log \frac{e}{4\pi}.$$

□

**Lemma 2.2.12.** *Let $\delta = (\delta_1, \delta_2) \in \mathbb{R}^2$ and $d \in \mathbb{N}^*$. For every $(x_1, x_2) \in \mathbb{R}_+^2 = \mathbb{R} \times (0, +\infty)$, set*

$$\phi_d^*(x_1, x_2) = d \arg(x_1 + ix_2).$$

*For every $r \in (0,1)$ and $\varepsilon \in (0, e^{-1/r^2})$, there exists a function $\phi_{d,\varepsilon} \colon B_r^+ \to \mathbb{R}$ such that $\phi_{d,\varepsilon} = \phi_d^*$ on $\partial B_r \cap \mathbb{R}_+^2$ and*

$$F_\varepsilon^\delta(\phi_{d,\varepsilon}; B_r^+) \leqslant \pi d \log \frac{r}{\varepsilon} + C d^2 (1 + |\log r| + \log|\log \varepsilon|),$$

*where $C > 0$ is independent of $r$ and $\varepsilon$.*

*Proof.* We use the same arguments as in [24, Lemma 4.15]. Let $r \in (0,1)$ and $\varepsilon \in (0, e^{-1/r^2})$. Set $a_\varepsilon = \frac{1}{|\log \varepsilon|}$ and, for $j \in \{1, ..., d\}$, $x_\varepsilon^j = j a_\varepsilon$. Consider the interpolation function

$$f \colon (x_1, x_2) \in \mathbb{R}_+^2 \mapsto \begin{cases} 1 & \text{if } \sqrt{x_1^2 + x_2^2} < r(1-r), \\ \frac{r - \sqrt{x_1^2+x_2^2}}{r^2} & \text{if } r(1-r) \leqslant \sqrt{x_1^2 + x_2^2} \leqslant r, \\ 0 & \text{if } \sqrt{x_1^2 + x_2^2} > r, \end{cases}$$

and set

$$\phi_{d,\varepsilon} \colon (x_1, x_2) \in \mathbb{R}_+^2 \mapsto \sum_{j=1}^{d} \arg\left(x_1 - f(x_1, x_2) x_\varepsilon^j + i(x_2 + 2\pi\varepsilon f(x_1, x_2))\right).$$

We have

$$F_\varepsilon^\delta(\phi_{d,\varepsilon}; B_r^+) = \int_{B_r^+} |\nabla \phi_{d,\varepsilon}|^2 \, \mathrm{d}x + \frac{1}{2\pi\varepsilon} \int_{I_r} \sin^2 \phi_{d,\varepsilon}(x_1, 0) \, \mathrm{d}x_1 - 2 \int_{B_r^+} (\delta_1 \partial_1 \phi_{d,\varepsilon} + \delta_2 \partial_2 \phi_{d,\varepsilon}) \, \mathrm{d}x.$$

From [24, Lemma 4.15], we have

$$\int_{B_r^+} |\nabla \phi_{d,\varepsilon}|^2 \,\mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{I_r} \sin^2 \phi_{d,\varepsilon}(x_1,0)\ \mathrm{d}x_1 \leqslant \pi d \log\frac{r}{\varepsilon} + Cd^2(1+|\log r| + \log|\log\varepsilon|),$$

where $C>0$ is independent of $\varepsilon$ and $r$. Moreover, using Green's formula,

$$\begin{aligned}\left|\int_{B_r^+}(\delta_1\partial_1\phi_{d,\varepsilon}+\delta_2\partial_2\phi_{d,\varepsilon})\ \mathrm{d}x\right| &= \left|\int_{\partial B_r^+}\phi_{d,\varepsilon}(\delta_1\nu_1+\delta_2\nu_2)\mathrm{d}\mathcal{H}^1\right|\\ &\leqslant \|\phi_{d,\varepsilon}\|_{L^\infty(\partial B_r^+)}\left(|\delta_1|+|\delta_2|\right)\left|\partial B_r^+\right|\\ &\leqslant Cd\left(|\delta_1|+|\delta_2|\right),\end{aligned}$$

where $\nu=(\nu_1,\nu_2)$ is the outer unit normal vector on $\partial B_r^+$, since $\phi_{d,\varepsilon}$ is bounded independently of $\varepsilon$ (recall that by definition, $\phi_\varepsilon^*\in[0,d\pi]$), $|\partial B_r^+|=(\pi+2)r$ and $r\in(0,1)$. We finally get

$$\begin{aligned}F_\varepsilon^\delta(\widehat{\phi}_{d,\varepsilon};B_r^+) &\leqslant \pi d\log\frac{r}{\varepsilon}+Cd^2(1+|\log r|+\log|\log\varepsilon|)+Cd\left(|\delta_1|+|\delta_2|\right)\\ &\leqslant \pi d\log\frac{r}{\varepsilon}+C'd^2(1+|\log r|+\log|\log\varepsilon|),\end{aligned}$$

where $C'>0$ is independent of $\varepsilon$ and $r$. □

### 2.2.4 Gamma-convergence for scalar-valued functions

We now present and prove three theorems showing the Gamma-convergence for $\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon)$ defined at (2.41). These results will be useful to deduce similar statements for $E_{\varepsilon,\eta}^\delta(m_\varepsilon)$ in the next section. The first statement establishes the $L^p(\partial\Omega)$-compactness of $\varphi_\varepsilon$ and a lower bound for $\mathcal{G}_\varepsilon^\delta$ at the first order.

**Theorem 2.2.13.** *Let $\Omega\subset\mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. Let $\delta\in\mathbb{R}^2$. Let $(\varphi_\varepsilon)$ be a family in $H^1(\Omega)$ such that*

$$\limsup_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon)<+\infty. \tag{2.52}$$

*There exists a family $(z_\varepsilon)$ of integers such that $(\varphi_\varepsilon-\pi z_\varepsilon)$ is bounded in $L^p(\partial\Omega)$, for every $p\in[1,+\infty)$. Moreover, for a subsequence, $(\varphi_\varepsilon-\pi z_\varepsilon)$ converges strongly in $L^p(\partial\Omega)$ to a limit $\varphi_0$ such that $\varphi_0-g\in BV(\partial\Omega,\pi\mathbb{Z})$, with $g$ given at (2.42), and*

$$\partial_\tau\varphi_0=\kappa\mathcal{H}^1 \llcorner \partial\Omega-\pi\sum_{j=1}^N d_j\mathbb{1}_{\{a_j\}}\quad \text{as measure on } \partial\Omega$$

*where, for $j\in\{1,...,N\}$ with $N\geqslant 1$, $a_j\in\partial\Omega$ are distinct points, $d_j\in\mathbb{Z}\setminus\{0\}$ and $\sum_{j=1}^N d_j=2$, and $(\partial_\tau\varphi_\varepsilon)$ converges to $\partial_\tau\varphi_0$ in $W^{-1,p}(\partial\Omega)$ for every $p\in[1,+\infty)$. Furthermore, we have the following first order lower bound:*

$$\liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon)\geqslant\left|\partial_\tau\varphi_0-\kappa\mathcal{H}^1\llcorner\partial\Omega\right|(\partial\Omega)=\pi\sum_{j=1}^N|d_j|. \tag{2.53}$$

*Proof.* We start by observing that, for any $\varepsilon > 0$, using Young's inequality, for any $\sigma > 0$, we have

$$\int_\Omega |\nabla\varphi_\varepsilon|^2 \,\mathrm{d}x = \int_\Omega |(\nabla\varphi_\varepsilon - \delta) + \delta|^2 \,\mathrm{d}x \leqslant (1+\sigma)\int_\Omega |\nabla\varphi_\varepsilon - \delta|^2 \,\mathrm{d}x + \left(1+\frac{1}{\sigma}\right)|\delta|^2\,|\Omega|\,,$$

since $(a+b)^2 = a^2 + b^2 + 2ab \leqslant a^2 + b^2 + (\sqrt{\sigma}a)^2 + \left(\frac{1}{\sqrt{\sigma}}b\right)^2$ for every $a, b \in \mathbb{R}$. Hence,

$$\begin{aligned}
\mathcal{G}^0_\varepsilon(\varphi_\varepsilon) &= \int_\Omega |\nabla\varphi_\varepsilon|^2 \,\mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} \sin^2(\varphi_\varepsilon - g)\,\mathrm{d}\mathcal{H}^1 \\
&\leqslant (1+\sigma)\int_\Omega |\nabla\varphi_\varepsilon - \delta|^2 \,\mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} \sin^2(\varphi_\varepsilon - g)\,\mathrm{d}\mathcal{H}^1 + \left(1+\frac{1}{\sigma}\right)|\delta|^2\,|\Omega| \\
&\leqslant (1+\sigma)\int_\Omega \left(|\nabla\varphi_\varepsilon|^2 - 2\delta\cdot\nabla\varphi_\varepsilon\right)\mathrm{d}x + (1+\sigma)\,|\delta|^2\,|\Omega| \\
&\quad + \frac{1+\sigma}{2\pi\varepsilon}\int_{\partial\Omega} \sin^2(\varphi_\varepsilon - g)\,\mathrm{d}\mathcal{H}^1 + \left(1+\frac{1}{\sigma}\right)|\delta|^2\,|\Omega| \\
&\leqslant (1+\sigma)\mathcal{G}^\delta_\varepsilon(\varphi_\varepsilon) + \left(2+\sigma+\frac{1}{\sigma}\right)|\delta|^2\,|\Omega|\,.
\end{aligned}$$

By (2.52), we deduce that $\limsup\limits_{\varepsilon\to 0} \frac{\mathcal{G}^0_\varepsilon(\varphi_\varepsilon)}{|\log\varepsilon|} < +\infty$. Hence, we can apply [24, Theorem 4.2] and deduce all the announced compactness results except the lower bound (2.53): it remains to show that the Dzyaloshinskii-Moriya interaction does not change the lower bound in [24]. By [24, Theorem 4.2], we have

$$\liminf_{\varepsilon\to 0} \frac{\mathcal{G}^0_\varepsilon(\varphi_\varepsilon)}{|\log\varepsilon|} \geqslant \pi\sum_{j=1}^N |d_j|\,,$$

so that, using the above inequalities,

$$\liminf_{\varepsilon\to 0}(1+\sigma)\frac{\mathcal{G}^\delta_\varepsilon(\varphi_\varepsilon)}{|\log\varepsilon|} \geqslant \pi\sum_{j=1}^N |d_j|\,.$$

Letting $\sigma \to 0$, we get (2.53). $\square$

The following theorem gives a second order lower bound in the asymptotic expansion of $\mathcal{G}^\delta_\varepsilon$.

**Theorem 2.2.14.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. Let $\delta \in \mathbb{R}^2$. Let $(\varphi_\varepsilon)$ be a family in $H^1(\Omega)$ satisfying the convergence in Theorem 2.2.13 with the limit $\varphi_0$ on $\partial\Omega$ as $\varepsilon \to 0$. Assume additionally that*

$$\limsup_{\varepsilon\to 0}\left(\mathcal{G}^\delta_\varepsilon(\varphi_\varepsilon) - \pi\,|\log\varepsilon|\sum_{j=1}^N |d_j|\right) < +\infty. \tag{2.54}$$

*Then $d_j \in \{-1, +1\}$ for every $j \in \{1, ..., N\}$ and, for a subsequence, $(\nabla\varphi_\varepsilon)$ converges weakly in $L^q(\Omega, \mathbb{R}^2)$ for any $q \in [1, 2)$ to $\nabla\widehat{\varphi}_0$, where $\widehat{\varphi}_0 \in W^{1,q}(\Omega)$ is an extension (not necessarily harmonic) of $\varphi_0$ to $\Omega$. Furthermore, we have the following second order lower bound:*

$$\liminf_{\varepsilon\to 0}\left(\mathcal{G}^\delta_\varepsilon(\varphi_\varepsilon) - N\pi\,|\log\varepsilon|\right) \geqslant W^\delta_\Omega(\{(a_j, d_j)\}) + N\gamma_0, \tag{2.55}$$

*where $\gamma_0 = \pi\log\frac{e}{4\pi}$ and $W^\delta_\Omega(\{(a_j, d_j)\})$ is the renormalized energy defined at (2.17).*

*Proof.* Since $\Omega$ is bounded, then for $r>0$ sufficiently small, we can find using [24, Lemma 4.3] (see Lemma 2.2.6 above) a finite number $J \in \mathbb{N}^*$ of points $p_1, ..., p_J \in \partial\Omega$ such that the union of the sets

$$A_j = \Psi_{p_j}(B^+_{r(1-c_1 r \log \frac{1}{r})} \cup I_{r(1-c_1 r\log\frac{1}{r})}) \subset B_r(p_j) \cap \overline{\Omega},$$

for $j \in \{1, ..., J\}$, covers a neighborhood of $\partial\Omega$ and is relatively open in $\overline{\Omega}$.

Let $j \in \{1, ..., J\}$ be fixed. Set $\psi_\varepsilon^{(j)} = \varphi_\varepsilon \circ \Psi_{p_j}$, $g^{(j)} = g \circ \Psi_{p_j}$ and $w_\varepsilon^{(j)} = \psi_\varepsilon^{(j)} - \widehat{g^{(j)}}_\rho$, where $\rho = r(1-c_1 r \log\frac{1}{r})$ and $\widehat{g^{(j)}}_\rho$ is the unique bounded harmonic extension of

$$g_\rho^{(j)} \colon x_1 \in \mathbb{R} \mapsto \begin{cases} g^{(j)}(x_1) & \text{if } |x_1| \leqslant \rho, \\ g^{(j)}\left(\frac{\rho^2}{x_1}\right) & \text{if } |x_1| > \rho. \end{cases}$$

*Step 1:* We prove that $d_j \in \{-1, +1\}$ for every $j \in \{1, ..., N\}$.

Let $\varepsilon > 0$ be small and $\widetilde{\varepsilon} = \frac{\varepsilon}{1-c_1 r\log\frac{1}{r}}$. Using [24, Equation (82)], we have

$$\begin{aligned}
\sum_{j=1}^N & \left( \int_{B_\rho^+} |\nabla w_\varepsilon^{(j)}|^2 \mathrm{d}x + \frac{1}{2\pi\widetilde{\varepsilon}} \int_{I_\rho} \sin^2 w_\varepsilon^{(j)}(\cdot, 0)\, \mathrm{d}\mathcal{H}^1 \right) \\
&\leqslant \int_\Omega |\nabla\varphi_\varepsilon|^2 \mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega} \sin^2(\varphi_\varepsilon - g)\, \mathrm{d}\mathcal{H}^1 + CNr^{1/2} \\
&= \mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon) + 2\int_\Omega \delta\cdot\nabla\varphi_\varepsilon\, \mathrm{d}x + CNr^{1/2} \\
&\leqslant \mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon) + C,
\end{aligned}$$

for some constant $C > 0$, because $r \in (0,1)$ and by Theorem 2.2.13,

$$\int_\Omega \delta\cdot\nabla\varphi_\varepsilon\, \mathrm{d}x = \int_\Omega \delta\cdot\nabla(\varphi_\varepsilon - \pi z_\varepsilon)\, \mathrm{d}x = \int_{\partial\Omega} (\varphi_\varepsilon - \pi z_\varepsilon)\delta\cdot\nu\, \mathrm{d}\mathcal{H}^1 \leqslant |\delta|\, \|\varphi_\varepsilon - \pi z_\varepsilon\|_{L^1(\partial\Omega)} \leqslant C.$$

Using (2.54), we deduce that there exists a constant $K_0 = K_0(\rho) = K_0(r)$, independent of $\varepsilon$, such that

$$\sum_{j=1}^N \left( \int_{B_\rho^+} |\nabla w_\varepsilon^{(j)}|^2 \mathrm{d}x + \frac{1}{2\pi\widetilde{\varepsilon}} \int_{I_\rho} \sin^2 w_\varepsilon^{(j)}(\cdot, 0)\, \mathrm{d}\mathcal{H}^1 \right) \leqslant \pi\left(\log\frac{\rho}{\widetilde{\varepsilon}}\right) \sum_{j=1}^N |d_j| + K_0. \tag{2.56}$$

Let $\sigma \in (0,\rho)$ and $I_\sigma = (-\sigma,\sigma)\times\{0\}$. For every $j \in \{1, ..., N\}$, $(w_\varepsilon^{(j)}(\cdot, 0))$ converges in $L^1(I_\sigma)$ to a locally constant function with one single jump of height $d_j\pi$. We can thus apply, up to subtract a suitable constant, [24, Lemma 4.5] and [24, Corollary 4.12] to get

$$\sum_{j=1}^N \left( \int_{B_\sigma^+} |\nabla w_\varepsilon^{(j)}|^2 \mathrm{d}x + \frac{1}{2\pi\widetilde{\varepsilon}} \int_{I_\sigma} \sin^2 w_\varepsilon^{(j)}(\cdot, 0)\, \mathrm{d}\mathcal{H}^1 \right) \geqslant \left(\pi\log\frac{\sigma}{\widetilde{\varepsilon}} - M_2\right) \sum_{j=1}^N |d_j|. \tag{2.57}$$

Subtracting (2.57) to (2.56) and taking the lim sup as $\varepsilon \to 0$, we get

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \sum_{j=1}^N & \left( \int_{B_\rho^+ \setminus B_\sigma} |\nabla w_\varepsilon^{(j)}|^2 \mathrm{d}x + \frac{1}{2\pi\widetilde{\varepsilon}} \int_{I_\rho\setminus I_\sigma} \sin^2 w_\varepsilon^{(j)}(\cdot, 0)\, \mathrm{d}\mathcal{H}^1 \right) \\
&\leqslant \pi\left(\log\frac{\rho}{\sigma}\right)\sum_{j=1}^N |d_j| + K_0 + M_2 \sum_{j=1}^N |d_j|.
\end{aligned} \tag{2.58}$$

Moreover, using again that $(w_\varepsilon^{(j)}(\cdot,0))$ converges in $L^1(I_\rho)$ to a locally constant function with one single jump of height $d_j\pi$, for every $j \in \{1, ..., N\}$, we deduce from [24, Equation (81)] applied with $\widetilde{\varepsilon}$ (and denoting $\widetilde{w}_\varepsilon = w_{\widetilde{\varepsilon}}$) that

$$\liminf_{\varepsilon\to 0} \sum_{j=1}^{N} \left( \int_{B_\rho^+ \setminus B_\sigma} |\nabla \widetilde{w}_\varepsilon^{(j)}|^2 \mathrm{d}x + \frac{1}{2\pi\widetilde{\varepsilon}} \int_{I_\rho \setminus I_\sigma} \sin^2 \widetilde{w}_\varepsilon^{(j)}(\cdot,0)\ \mathrm{d}\mathcal{H}^1 \right) \geqslant \pi \left( \log \frac{\rho}{\sigma} \right) \sum_{j=1}^{N} d_j^2. \tag{2.59}$$

Combining (2.58) and (2.59), we deduce

$$\pi \left( \log \frac{\rho}{\sigma} \right) \sum_{j=1}^{N} \left( d_j^2 - |d_j| \right) \leqslant K_0 + M_2 \sum_{j=1}^{N} |d_j|.$$

Letting $\sigma \to 0$, it follows necessarily that

$$\sum_{j=1}^{N} \left( d_j^2 - |d_j| \right) = 0,$$

i.e. $d_j \in \{-1, +1\}$ for every $j \in \{1, ..., N\}$.

*Step 2:* We prove the weak convergence of $(\nabla\varphi_\varepsilon)$.

Set $\Omega^r = \Omega \setminus \bigcup_{j=1}^{N} B_r(a_j)$ for any small $r > 0$. We have, for $r > 0$ sufficiently small,

$$\begin{aligned} \int_{\Omega^r} |\nabla\varphi_\varepsilon|^2 \mathrm{d}x &= \int_\Omega |\nabla\varphi_\varepsilon|^2 \mathrm{d}x - \sum_{j=1}^{N} \int_{\Omega\cap B_r(a_j)} |\nabla\varphi_\varepsilon|^2 \mathrm{d}x \\ &\leqslant \int_\Omega |\nabla\varphi_\varepsilon|^2 \mathrm{d}x + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} \sin^2(\varphi_\varepsilon - g)\ \mathrm{d}\mathcal{H}^1 \\ &\quad - \sum_{j=1}^{N} \left( \int_{\Omega\cap B_r(a_j)} |\nabla\varphi_\varepsilon|^2 \mathrm{d}x + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega\cap B_r(a_j)} \sin^2(\varphi_\varepsilon - g)\ \mathrm{d}\mathcal{H}^1 \right). \end{aligned}$$

By definition of $\mathcal{G}_\varepsilon^\delta$ and using [24, Equations (82)–(83)], we deduce that

$$\begin{aligned} \int_{\Omega^r} |\nabla\varphi_\varepsilon|^2 \mathrm{d}x \leqslant \mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon) + 2 \int_\Omega \delta \cdot \nabla\varphi_\varepsilon\ \mathrm{d}x - N\pi \log \frac{r}{\varepsilon} \\ + NM_2 - 2\pi N \log \left( 1 - c_1 r \log \frac{1}{r} \right) + CNr^{1/2}. \end{aligned}$$

By (2.54), we have $\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon) \leqslant N\pi \log \frac{1}{\varepsilon} + O(1)$. Moreover, $\int_\Omega \delta \cdot \nabla\varphi_\varepsilon\ \mathrm{d}x$ is bounded independently of $\varepsilon$, because

$$\int_\Omega \delta \cdot \nabla\varphi_\varepsilon\ \mathrm{d}x = \int_\Omega \delta \cdot \nabla(\varphi_\varepsilon - \pi z_\varepsilon)\ \mathrm{d}x = \int_{\partial\Omega} (\varphi_\varepsilon - \pi z_\varepsilon)\delta \cdot \nu\ \mathrm{d}\mathcal{H}^1,$$

and $(\varphi_\varepsilon - \pi z_\varepsilon)$ is bounded in $L^1(\partial\Omega)$. Hence, up to taking $r > 0$ so that $\log\left(1 - c_1 r \log \frac{1}{r}\right) < 2$ for example, we have

$$\int_{\Omega^r} |\nabla\varphi_\varepsilon|^2 \mathrm{d}x \leqslant N\pi \log \frac{1}{r} + C,$$

for some $C > 0$ independent of $\varepsilon$ and $r$. By [24, Lemma 4.17], the functions $\nabla\varphi_\varepsilon$ are uniformly bounded in $L^q(\Omega, \mathbb{R}^2)$ for every $q \in [1, 2)$. It follows that there exists $\widehat{\varphi}_0 \in W^{1,q}(\Omega)$ for every $q \in [1,2)$ such that, for a subsequence, $(\varphi_\varepsilon)$ converges weakly to $\widehat{\varphi}_0$ in $W^{1,q}(\Omega)$ and in $H^1(\omega)$ for any open set $\omega$ such that $\overline{\omega} \subset \overline{\Omega} \setminus \{a_1, ..., a_N\}$. By the trace theorem, we deduce that $\widehat{\varphi}_0$ is an extension (to $\Omega$) of the boundary limit $\varphi_0$ found in Theorem 2.2.13.

*Step 3:* We prove the second order lower bound (2.55).

To do so, we replace $\varphi_\varepsilon$ by the harmonic extension of $\varphi_\varepsilon|_{\partial\Omega}$, denoted by $\varphi_\varepsilon^*$. More precisely, $\varphi_\varepsilon^*$ is the minimizer of the Dirichlet energy in $\Omega$ under the Dirichlet boundary condition $\varphi_\varepsilon|_{\partial\Omega}$. Since

$$\int_\Omega \delta \cdot \nabla\varphi_\varepsilon \; \mathrm{d}x = \int_{\partial\Omega} \varphi_\varepsilon \; \delta \cdot \nu \; \mathrm{d}\mathcal{H}^1 = \int_{\partial\Omega} \varphi_\varepsilon^* \; \delta \cdot \nu \; \mathrm{d}\mathcal{H}^1 = \int_\Omega \delta \cdot \nabla\varphi_\varepsilon^* \; \mathrm{d}x,$$

we deduce that $\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon) \geqslant \mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon^*)$, thus it suffices to prove (2.55) for $\varphi_\varepsilon^*$. Using the same argument as before for the convergence of $(\varphi_\varepsilon)$, we know that $(\varphi_\varepsilon^*)$ converges weakly in $W^{1,q}(\Omega)$, for every $q \in [1,2)$, and weakly in $H^1(\omega)$, for any open set $\omega$ such that $\overline{\omega} \subset \overline{\Omega} \setminus \{a_1, ..., a_N\}$, to the harmonic extension $\varphi_*$ of $\varphi_0$ to $\Omega$. Let $r > 0$ be small. By weak lower semicontinuity of the Dirichlet integral, we have

$$\int_{\Omega^r} |\nabla\varphi_*|^2 \mathrm{d}x \leqslant \liminf_{\varepsilon\to 0} \int_{\Omega^r} |\nabla\varphi_\varepsilon^*|^2 \mathrm{d}x. \tag{2.60}$$

By weak convergence of $(\nabla\varphi_\varepsilon^*)$ to $\nabla\varphi_*$ in $L^2(\Omega^r, \mathbb{R}^2)$, we have

$$\liminf_{\varepsilon\to 0} \int_{\Omega^r} \delta \cdot \nabla\varphi_\varepsilon^* \; \mathrm{d}x = \int_{\Omega^r} \delta \cdot \nabla\varphi_* \; \mathrm{d}x. \tag{2.61}$$

Up to taking a smaller $r > 0$, we have

$$\int_{\Omega\setminus\Omega^r} \delta \cdot \nabla\varphi_\varepsilon^* \; \mathrm{d}x = \int_{\bigcup_{j=1}^N \Omega\cap B_r(a_j)} \delta \cdot \nabla\varphi_\varepsilon^* \; \mathrm{d}x = \sum_{j=1}^N \int_{\Omega\cap B_r(a_j)} \delta \cdot \nabla\varphi_\varepsilon^* \; \mathrm{d}x.$$

But, for every $j \in \{1, ..., N\}$, by Hölder's inequality,

$$\begin{aligned} \left| \int_{\Omega\cap B_r(a_j)} \delta \cdot \nabla\varphi_\varepsilon^* \; \mathrm{d}x \right| &\leqslant \left( \int_{\Omega\cap B_r(a_j)} |\delta|^3 \, \mathrm{d}x \right)^{1/3} \left( \int_{\Omega\cap B_r(a_j)} |\nabla\varphi_\varepsilon^*|^{3/2} \, \mathrm{d}x \right)^{2/3} \\ &\leqslant Cr^{2/3} \, \|\nabla\varphi_\varepsilon^*\|_{L^{3/2}(\Omega)} \\ &\leqslant Cr^{2/3}, \end{aligned}$$

for some constant $C > 0$, because $\delta$ is constant and $(\nabla\varphi_\varepsilon^*)$ is bounded (independently of $\varepsilon$), in $L^{3/2}(\Omega, \mathbb{R}^2)$. We deduce that, for every $j \in \{1, ..., N\}$,

$$\int_{\Omega\cap B_r(a_j)} \delta \cdot \nabla\varphi_\varepsilon^* \; \mathrm{d}x = O(r^{2/3}). \tag{2.62}$$

Applying [24, Equation (82)] to $\widetilde{\varphi}_\varepsilon^* = \varphi_{\widetilde{\varepsilon}}^*$ with the corresponding functions $\widetilde{w}_\varepsilon^{*(j)} = w_{\widetilde{\varepsilon}}^{*(j)}$, we have, for $r > 0$ sufficiently small,

$$\begin{aligned} &\sum_{j=1}^N \left( \int_{\Omega\cap B_r(a_j)} |\nabla\widetilde{\varphi}_\varepsilon^*|^2 \mathrm{d}x + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega\cap B_r(a_j)} \sin^2(\widetilde{\varphi}_\varepsilon^* - g) \; \mathrm{d}\mathcal{H}^1 \right) - N \left( \pi \log \frac{r}{\varepsilon} + \gamma_0 \right) \\ &\quad \geqslant \sum_{j=1}^N \left[ \int_{B_\rho^+} |\nabla\widetilde{w}_\varepsilon^{*(j)}|^2 \mathrm{d}x + \frac{1}{2\pi\widetilde{\varepsilon}} \int_{I_\rho} \sin^2 \widetilde{w}_\varepsilon^{*(j)}(\cdot, 0) \; \mathrm{d}\mathcal{H}^1 - \left( \pi \log \frac{r}{\varepsilon} + \gamma_0 \right) \right] - CNr^{1/2}. \end{aligned}$$

Setting $I_\rho = (-\rho,\rho)\times\{0\}$ and using that $(\widetilde{w}_\varepsilon^{*(j)}(\cdot,0))$ converges in $L^1(I_\rho)$ to a locally constant function with one single jump of height $d_j\pi$, for every $j\in\{1,...,N\}$, we deduce from [24, Proposition 4.16] that

$$\liminf_{\varepsilon\to 0}\left[\sum_{j=1}^N\left(\int_{\Omega\cap B_r(a_j)}|\nabla\widetilde{\varphi}_\varepsilon^*|^2\mathrm{d}x+\frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap B_r(a_j)}\sin^2(\widetilde{\varphi}_\varepsilon^*-g)\ \mathrm{d}\mathcal{H}^1\right)\right.$$
$$\left.-N\left(\pi\log\frac{r}{\varepsilon}+\gamma_0\right)\right]\geqslant -CNr^{1/2}. \tag{2.63}$$

We note that, for $r>0$ sufficiently small,

$$\begin{aligned}
&\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon^*)-N\pi\,|\log\varepsilon|-W_\Omega^\delta(\{(a_j,d_j)\})-N\gamma_0\\
&\quad\geqslant\sum_{j=1}^N\left(\int_{\Omega\cap B_r(a_j)}|\nabla\varphi_\varepsilon^*|^2\mathrm{d}x+\frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap B_r(a_j)}\sin^2(\varphi_\varepsilon^*-g)\ \mathrm{d}\mathcal{H}^1\right)-N\left(\pi\log\frac{r}{\varepsilon}+\gamma_0\right)\\
&\quad\quad+\int_{\Omega^r}|\nabla\varphi_\varepsilon^*|^2\mathrm{d}x-2\int_{\Omega^r}\delta\cdot\nabla\varphi_\varepsilon^*\ \mathrm{d}x-N\pi\log\frac{1}{r}-W_\Omega^\delta(\{(a_j,d_j)\})\\
&\quad\quad-2\sum_{j=1}^N\int_{\Omega\cap B_r(a_j)}\delta\cdot\nabla\varphi_\varepsilon^*\ \mathrm{d}x,
\end{aligned}$$

the inequality coming from the nonnegativity of $\sin^2$ on $\partial\Omega^r$. Using (2.62) and taking then the $\liminf$ as $\varepsilon\to 0$ in the above inequality, we deduce from (2.60), (2.61) and (2.63) that, as $r\to 0$,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\left(\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon^*)-N\pi\,|\log\varepsilon|\right)-W_\Omega^\delta(\{(a_j,d_j)\})-N\gamma_0\\
&\quad\geqslant -CNr^{1/2}\\
&\quad\quad+\int_{\Omega^r}\left(|\nabla\varphi_*|^2-2\delta\cdot\nabla\varphi_*\right)\mathrm{d}x-N\pi\log\frac{1}{r}-W_\Omega^\delta(\{(a_j,d_j)\})\\
&\quad\quad-O(r^{2/3}).
\end{aligned}$$

By (2.17), we know that, as $r\to 0$,

$$\int_{\Omega^r}\left(|\nabla\varphi_*|^2-2\delta\cdot\nabla\varphi_*\right)\mathrm{d}x=N\pi\log\frac{1}{r}+W_\Omega^\delta(\{(a_j,d_j)\})+o(1),$$

hence

$$\liminf_{\varepsilon\to 0}\left(\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon^*)-N\pi\,|\log\varepsilon|\right)-W_\Omega^\delta(\{(a_j,d_j)\})-N\gamma_0\geqslant -CNr^{1/2}-O(r^{2/3})+o(1)=o(1).$$

Taking the limits as $r\to 0$, we get (2.55) for $\varphi_\varepsilon^*$. □

*Remark* 2.2.15. In the previous proof, the inequality (2.63) holds true by replacing $\varphi_\varepsilon^*$ by $\varphi_\varepsilon$ (and the corresponding inequality is obtained by replacing the functions $w_\varepsilon^{*(j)}$ by the functions $w_\varepsilon^{(j)}$).

**Theorem 2.2.16.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. Let $\delta \in \mathbb{R}^2$. Let $\varphi_0 \colon \partial\Omega \to \mathbb{R}$ be such that*

$$\partial_\tau \varphi_0 = \kappa \mathcal{H}^1 \llcorner \partial\Omega - \pi \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}} \quad \text{as measure on } \partial\Omega,$$

*where $d_j \in \mathbb{Z} \setminus \{0\}$ for every $j \in \{1, ..., N\}$, $\sum_{j=1}^N d_j = 2$ and $e^{i\varphi_0} \cdot \nu = 0$ in $\partial\Omega \setminus \{a_1, ..., a_N\}$ for $N \geqslant 1$ distinct points $a_1, ..., a_N \in \partial\Omega$. There exists a family $(\varphi_\varepsilon)$ in $H^1(\Omega)$ such that $(\varphi_\varepsilon)$ converges to $\varphi_0$ in $L^p(\partial\Omega)$ and to $\varphi_*$ in $L^p(\Omega)$, for every $p \in [1, +\infty)$, where $\varphi_*$ is the harmonic extension of $\varphi_0$ to $\Omega$, and we have*

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon) = \pi \sum_{j=1}^N |d_j| \,. \tag{2.64}$$

*Furthermore, if $d_j \in \{-1, +1\}$ for every $j \in \{1, ..., N\}$, then*

$$\lim_{\varepsilon \to 0} \left( \mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon) - N\pi |\log \varepsilon| \right) = W_\Omega^\delta(\{(a_j, d_j)\}) + N\gamma_0, \tag{2.65}$$

*where $\gamma_0 = \pi \log \frac{e}{4\pi}$ and $W_\Omega^\delta(\{(a_j, d_j)\})$ is the renormalized energy defined at (2.17).*

*Proof.* Let $\varphi_*$ be the harmonic extension of $\varphi_0$ to $\Omega$, that satisfies (2.17). We begin with constructing a family of functions $(\widehat{\psi}_\varepsilon)$ using the ideas in [24]. By [24, Lemma 4.3] (see Lemma 2.2.6 above), there exist $r_0 \in (0,1)$ and $c = c(\Omega) > 0$ such that, for every $j \in \{1, ..., N\}$, there exists a map

$$\Psi_{a_j} : \overline{B_{2r_0}^+} \to \overline{\Omega}$$

such that $\Psi_{a_j}$ is a conformal diffeomorphism from $\overline{B^+_{r_0(1+cr_0 \log \frac{1}{r_0})}}$ onto $\Psi_{a_j}(\overline{B^+_{r_0(1+cr_0 \log \frac{1}{r_0})}})$ with $\Psi_{a_j}(0) = a_j$ and, for any $\phi \in H^1(\Omega, \mathbb{R})$, setting $\psi = \phi \circ \Psi_{a_j}$, we have, for every $r \in (0, r_0)$,

$$\int_{B^+_{r(1-cr\log\frac{1}{r})}} |\nabla\psi|^2 \, \mathrm{d}x \leqslant \int_{\Omega \cap B_r(a_j)} |\nabla\phi|^2 \, \mathrm{d}x \leqslant \int_{B^+_{r(1+cr\log\frac{1}{r})}} |\nabla\psi|^2 \, \mathrm{d}x,$$

and

$$\begin{aligned}\left(1 - cr\log\frac{1}{r}\right) \int_{I_{r(1-cr\log\frac{1}{r})}} \sin^2\psi(\cdot, 0) \, \mathrm{d}\mathcal{H}^1 &\leqslant \int_{\partial\Omega \cap B_r(a_j)} \sin^2\phi \, \mathrm{d}\mathcal{H}^1 \\ &\leqslant \left(1 + cr\log\frac{1}{r}\right) \int_{I_{r(1+cr\log\frac{1}{r})}} \sin^2\psi(\cdot, 0) \, \mathrm{d}\mathcal{H}^1.\end{aligned}$$

Let $j \in \{1, ..., N\}$ be fixed. For a suitable choice of the argument function, there exists a function $h = h_j$ defined in a neighborhood of $a_j$ and bounded in $W^{1,p}$, for every $p \in [1, +\infty)$, with bounds depending only on $p$, $\partial\Omega$ and $\{(a_j, d_j)\}$, such that

$$\varphi_*(z) = d_j \arg(z - a_j) + h(z)$$

in that neighborhood of $a_j$. Using that, on $\partial\Omega$, $\varphi_* = \varphi_0$, $e^{i\varphi_0} \cdot \nu = 0$ and $e^{ig} = \tau = i\nu$, we deduce that $\sin(h - g) = e^{ig} \cdot (e^{ih})^\perp = 0$.

Set $\widehat{\varphi}_* = \varphi_* \circ \Psi_{a_j}$. Then $\widehat{\varphi}_*(z) = d_j \arg(z) + \widehat{h}(z)$ in a neighborhood of zero, where $\widehat{h} = h \circ \Psi_{a_j}$ is bounded in $W^{1,p}$ (in that neighborhood of zero), for every $p \in [1, +\infty)$, with bounds depending only on $p$, $\partial\Omega$ and $\{(a_j, d_j)\}$ as above.

Let $\varepsilon > 0$. Let $r \in (0, r_0)$ be fixed. We construct a function $\widehat{\psi}_\varepsilon$ defined in $\Omega$ as follows: let $z \in \Omega$.

– If $|\Psi_{a_j}^{-1}(z)| > r$ for every $j \in \{1, ..., N\}$, then we set

$$\widehat{\psi}_\varepsilon(z) = \varphi_*(z).$$

– Otherwise, if $|\Psi_{a_j}^{-1}(z)| \leqslant r$ for some $j \in \{1, ..., N\}$, then we set

$$\widehat{\psi}_\varepsilon(z) = \phi_\varepsilon^j(\Psi_{a_j}^{-1}(z)) + h(z),$$

where $\phi_\varepsilon^j$ is defined in $\mathbb{R}_+^2$ as

$$\phi_\varepsilon^j(x_1, x_2) = \sum_{k=0}^{|d_j|-1} \arg\left(x_1 - \frac{k}{|\log\varepsilon|} f(x_1, x_2) + i\left(x_2 + 2\pi\varepsilon f(x_1, x_2)\right)\right)$$

where

$$f(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in B^+_{r(1-r)}, \\ \frac{r - \sqrt{x_1^2 + x_2^2}}{r^2} & \text{if } (x_1, x_2) \in B_r^+ \setminus B_{r(1-r)}, \\ 0 & \text{otherwise.} \end{cases}$$

From our construction, $\widehat{\psi}_\varepsilon$ is continuous in $\overline{\Omega}$, and the family $(\widehat{\psi}_\varepsilon)$ converges to $\varphi_0$ in $L^p(\partial\Omega)$, for every $p \in [1, +\infty)$.

*Step 1: Upper bounds for $\mathcal{G}_\varepsilon^\delta$.*

Recall that

$$\mathcal{G}_\varepsilon^\delta(\widehat{\psi}_\varepsilon) = \mathcal{G}_\varepsilon^0(\widehat{\psi}_\varepsilon) - 2\int_\Omega \delta \cdot \nabla\widehat{\psi}_\varepsilon \, \mathrm{d}x. \tag{2.66}$$

Assume first that $|d_j| = 1$ for every $j \in \{1, ..., N\}$. By the definition of $W_\Omega^0(\{(a_j, d_j)\})$ and Equation (50) in [24, Theorem 4.2.3],

$$\mathcal{G}_\varepsilon^0(\widehat{\psi}_\varepsilon) \leqslant N\pi|\log\varepsilon| + \left(\int_{\Omega\setminus\bigcup_{j=1}^N B_r(a_j)} |\nabla\varphi_*|^2 \, \mathrm{d}x - N\pi\log\frac{1}{r}\right) - o(1) + N\gamma_0 + o(1) \quad \text{as } \varepsilon, r \to 0. \tag{2.67}$$

Moreover, we have

$$\int_\Omega \delta\cdot\nabla\widehat{\psi}_\varepsilon \, \mathrm{d}x = \int_{\Omega\setminus\bigcup_j B_r(a_j)} \delta\cdot\nabla\widehat{\psi}_\varepsilon \, \mathrm{d}x + \int_{\Omega\cap\bigcup_j B_r(a_j)} \delta\cdot\nabla\widehat{\psi}_\varepsilon \, \mathrm{d}x. \tag{2.68}$$

Let us estimate the last integral above. For $r > 0$ sufficiently small,

$$\int_{\Omega\cap\bigcup_j B_r(a_j)} \delta\cdot\nabla\widehat{\psi}_\varepsilon \, \mathrm{d}x = \sum_{j=1}^N \int_{\Omega\cap B_r(a_j)} \delta\cdot\nabla\widehat{\psi}_\varepsilon \, \mathrm{d}x.$$

By [24, Theorem 4.2.2], for every $q \in [1, 2)$, there exists a constant $C_q > 0$ such that

$$\|\nabla\widehat{\psi}_\varepsilon\|_{L^q(\Omega)} \leqslant C_q.$$

Hence, for every $j \in \{1, ..., N\}$, by Hölder's inequality,

$$\left|\int_{\Omega\cap B_r(a_j)} \delta\cdot\nabla\widehat{\psi}_\varepsilon \, \mathrm{d}x\right| \leqslant |\delta|\, C r^{1/3}\|\nabla\widehat{\psi}_\varepsilon\|_{L^{3/2}(\Omega)} = O(r^{1/3}) \quad \text{as } r \to 0.$$

It follows that

$$\lim_{\varepsilon\to 0}\int_{\Omega\cap\bigcup_j B_r(a_j)} \delta\cdot\nabla\widehat{\psi}_\varepsilon \,\mathrm{d}x = 0. \tag{2.69}$$

Moreover, by [24, Theorem 4.2.2] again, $(\nabla\widehat{\psi}_\varepsilon)$ converges weakly to $\nabla\varphi_*$ in $L^1(\Omega,\mathbb{R}^2)$, thus

$$\lim_{\varepsilon\to 0}\int_{\Omega\setminus\bigcup_j B_r(a_j)} \delta\cdot\nabla\widehat{\psi}_\varepsilon \,\mathrm{d}x = \int_{\Omega\setminus\bigcup_j B_r(a_j)} \delta\cdot\nabla\varphi_* \,\mathrm{d}x \tag{2.70}$$

Combining (2.66), (2.67) and (2.68), and taking the lim sup as $\varepsilon\to 0$, we deduce using (2.69) and (2.70) that, as $r\to 0$,

$$\limsup_{\varepsilon\to 0}\left(\mathcal{G}^\delta_\varepsilon(\widehat{\psi}_\varepsilon) - N\pi\left|\log\varepsilon\right|\right) \leqslant \left(\int_{\Omega\setminus\bigcup_{j=1}^N B_r(a_j)}\left(\left|\nabla\varphi_*\right|^2 - 2\delta\cdot\nabla\varphi_*\right)\mathrm{d}x - N\pi\log\frac{1}{r}\right) - o(1) + N\gamma_0 - O(r^{1/3}).$$

Taking the lim inf as $r\to 0$, we get

$$\limsup_{\varepsilon\to 0}\left(\mathcal{G}^\delta_\varepsilon(\widehat{\psi}_\varepsilon) - N\pi\left|\log\varepsilon\right|\right) \leqslant W^\delta_\Omega(\{(a_j,d_j)\}) + N\gamma_0.$$

It remains to treat the case of arbitrary integers $d_j$. By Equation (49) in [24, Theorem 4.2.3],

$$\mathcal{G}^0_\varepsilon(\widehat{\psi}_\varepsilon) \leqslant \pi\left|\log\varepsilon\right|\sum_{j=1}^N |d_j| + o(1) \quad \text{as } \varepsilon\to 0.$$

Moreover, we know again by [24, Theorem 4.2.2] that, for every $q\in[1,2)$, there exists a constant $C_q>0$ such that

$$\|\nabla\widehat{\psi}_\varepsilon\|_{L^q(\Omega)} \leqslant C_q,$$

and that $(\nabla\widehat{\psi}_\varepsilon)$ converges weakly to $\nabla\varphi_*$ in $L^1(\Omega,\mathbb{R}^2)$. Hence, we deduce similarly that

$$\limsup_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\mathcal{G}^\delta_\varepsilon(\widehat{\psi}_\varepsilon) \leqslant \pi\sum_{j=1}^N |d_j|\,.$$

*Step 2: Lower bounds for $\mathcal{G}^\delta_\varepsilon$ and convergence to $\varphi_*$ in $L^p(\Omega)$.*

For any $\varepsilon>0$, we denote by $\widehat{\psi}^*_\varepsilon$ the harmonic extension of $\widehat{\psi}_\varepsilon|_{\partial\Omega}$ to $\Omega$, i.e. $\widehat{\psi}^*_\varepsilon$ is the minimizer of the Dirichlet energy in $\Omega$ under the boundary condition $\widehat{\psi}_\varepsilon|_{\partial\Omega}$. First, the family $(\widehat{\psi}^*_\varepsilon)$ satisfies the convergence statements for $\mathcal{G}^\delta_\varepsilon$. Indeed, since $\widehat{\psi}^*_\varepsilon = \widehat{\psi}_\varepsilon$ on $\partial\Omega$, we have

$$\int_\Omega \delta\cdot\nabla\widehat{\psi}_\varepsilon\,\mathrm{d}x = \int_{\partial\Omega}\widehat{\psi}_\varepsilon\delta\cdot\nu\,\mathrm{d}\mathcal{H}^1 = \int_{\partial\Omega}\widehat{\psi}^*_\varepsilon\delta\cdot\nu\,\mathrm{d}\mathcal{H}^1 = \int_\Omega \delta\cdot\nabla\widehat{\psi}^*_\varepsilon\,\mathrm{d}x,$$

and thus $\mathcal{G}^\delta_\varepsilon(\widehat{\psi}^*_\varepsilon)\leqslant\mathcal{G}^\delta_\varepsilon(\widehat{\psi}_\varepsilon)$. If $d_j\in\{-1,+1\}$ for every $j\in\{1,...,N\}$, it follows by Step 1 that

$$\limsup_{\varepsilon\to 0}\left(\mathcal{G}^\delta_\varepsilon(\widehat{\psi}^*_\varepsilon) - N\pi\left|\log\varepsilon\right|\right) \leqslant W^\delta_\Omega(\{(a_j,d_j)\}) + N\gamma_0,$$

and by Theorem 2.2.14, we have more precisely

$$\lim_{\varepsilon\to 0}\left(\mathcal{G}^\delta_\varepsilon(\widehat{\psi}^*_\varepsilon) - N\pi\left|\log\varepsilon\right|\right) = W^\delta_\Omega(\{(a_j,d_j)\}) + N\gamma_0.$$

In the case of arbitrary integers $d_j \in \mathbb{Z}$, we have by Step 1

$$\limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \mathcal{G}_\varepsilon^\delta(\widehat{\psi}_\varepsilon^*) \leqslant \pi \sum_{j=1}^N |d_j| \,,$$

and by Theorem 2.2.13, we have more precisely

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \mathcal{G}_\varepsilon^\delta(\widehat{\psi}_\varepsilon^*) = \pi \sum_{j=1}^N |d_j| \,.$$

It remains to show the convergence statements for the family $(\widehat{\psi}_\varepsilon^*)$. Since $\widehat{\psi}_\varepsilon^* = \widehat{\psi}_\varepsilon$ on $\partial\Omega$, then $(\widehat{\psi}_\varepsilon^*)$ converges to $\varphi_0$ in $W^{1,p}(\partial\Omega)$ for every $p \in [1, +\infty)$. Moreover, since harmonic functions satisfy the inequality $\|\cdot\|_{L^2(\Omega)} \leqslant C\|\cdot\|_{L^2(\partial\Omega)}$, we deduce that $(\widehat{\psi}_\varepsilon^*)$ converges to $\varphi_*$ in $L^2(\Omega)$. Furthermore, as $W^{1,p}(\partial\Omega) \subset L^\infty(\partial\Omega)$ for some $p \in [1, +\infty)$, then the sequence $(\widehat{\psi}_\varepsilon^* - \varphi_*)$ is uniformly bounded on $\partial\Omega$. Moreover for every $\varepsilon > 0$, the functions $\widehat{\psi}_\varepsilon^* - \varphi_*$ are harmonic in $\Omega$, hence by the weak maximum principle, we have

$$\sup_\Omega |\widehat{\psi}_\varepsilon^* - \varphi_*| = \sup_{\partial\Omega} |\widehat{\psi}_\varepsilon^* - \varphi_*|,$$

thus taking the supremum on $\varepsilon > 0$ in the above equality, we deduce that the sequence $(\widehat{\psi}_\varepsilon^* - \varphi_*)$ is uniformly bounded in $\Omega$. Then there exists a constant $C > 0$ such that, for every $p \in (2, +\infty)$,

$$|\widehat{\psi}_\varepsilon^* - \varphi_*|^p \leqslant C|\widehat{\psi}_\varepsilon^* - \varphi_*|^2.$$

Integrating over $\Omega$ and using that $(\widehat{\psi}_\varepsilon^*)$ converges to $\varphi_*$ in $L^2(\Omega)$, we get the convergence of $(\widehat{\psi}_\varepsilon^*)$ to $\varphi_*$ in $L^p(\Omega)$ for every $p \in (2, +\infty)$. Besides, by Hölder's inequality,

$$\int_\Omega |\widehat{\psi}_\varepsilon^* - \varphi_*|^p \mathrm{d}x \leqslant \left( \int_\Omega |\widehat{\psi}_\varepsilon^* - \varphi_*|^2 \mathrm{d}x \right)^{p/2} |\Omega|^{(2-p)/2},$$

for every $p \in [1, 2]$. Since $\Omega$ is bounded and $(\widehat{\psi}_\varepsilon^*)$ converges to $\varphi_*$ in $L^2(\Omega)$, we get the convergence of $(\widehat{\psi}_\varepsilon^*)$ to $\varphi_*$ in $L^p(\Omega)$ for every $p \in [1, 2]$. □

### 2.2.5 Gamma-convergence for vector-valued maps

We now prove Theorems 2.1.1, 2.1.3 and 2.1.4.

*Proof of Theorem 2.1.1.* Let $(m_\varepsilon)$ be a family in $H^1(\Omega, \mathbb{R}^2)$ such that $E_{\varepsilon,\eta}^\delta(m_\varepsilon) \leqslant C\,|\log \varepsilon|$, for some constant $C > 0$. Using Theorem 2.2.4 and Lemma 2.2.1, we can construct a family $(\mathfrak{m}_\varepsilon)$ in $H^1(\Omega, \mathbb{S}^1)$ such that

$$E_{\varepsilon,\eta}^\delta(\mathfrak{m}_\varepsilon) \leqslant E_{\varepsilon,\eta}^\delta(m_\varepsilon) + o(1) \quad \text{as } \varepsilon \to 0, \tag{2.71}$$

$$\lim_{\varepsilon \to 0} \|\mathfrak{m}_\varepsilon - m_\varepsilon\|_{L^p(\partial\Omega)} = 0, \quad \text{for every } p \in [1, +\infty), \tag{2.72}$$

and

$$\lim_{\varepsilon \to 0} \|\mathcal{J}(\mathfrak{m}_\varepsilon) - \mathcal{J}(m_\varepsilon)\|_{(\mathrm{Lip}(\Omega))^*} = 0. \tag{2.73}$$

Using Lemma 2.2.5, for every $\varepsilon > 0$, there exists a function $\varphi_\varepsilon \in H^1(\Omega)$ such that $\mathfrak{m}_\varepsilon = e^{i\varphi_\varepsilon}$ and $E_{\varepsilon,\eta}^\delta(\mathfrak{m}_\varepsilon) = \mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon)$. Moreover, for every $\varepsilon > 0$, the global Jacobian of $\mathfrak{m}_\varepsilon$ is given by

$$\mathcal{J}(\mathfrak{m}_\varepsilon) = \mathcal{J}_{\mathrm{bd}}(\mathfrak{m}_\varepsilon) = -\partial_\tau \varphi_\varepsilon \mathcal{H}^1 \llcorner \partial\Omega$$

as a distribution in $H^{-1/2}(\partial\Omega)$. Using (2.71) and $\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon) = E_{\varepsilon,\eta}^\delta(\mathfrak{m}_\varepsilon)$, we deduce from Theorem 2.2.13 that, for a subsequence, there exists a family of integers $(z_\varepsilon)$ – that we can all assume to be either even or odd, up to taking a further subsequence – such that $(\varphi_\varepsilon - \pi z_\varepsilon)$ converges strongly in $L^p(\partial\Omega)$, for every $p \in [1, +\infty)$, to a limit $\phi_0$ that satisfies $\phi_0 - g \in BV(\partial\Omega, \pi\mathbb{Z})$ with $g$ such that $e^{ig} = \tau = i\nu$ on $\partial\Omega$. Let $\varphi_0$ be such that $\varphi_0 = \phi_0$ if the integers $z_\varepsilon$ are all even, and $\varphi_0 = \phi_0 - \pi$ if the integers $z_\varepsilon$ are all odd. By definition of $\phi_0$ and $g$, $\varphi_0$ is a BV lifting of $\pm\tau$. Moreover, as $\left|e^{is} - e^{it}\right| \leqslant \frac{\pi}{2}|s-t|$ for every $s, t \in \mathbb{R}$, we have, for every $\varepsilon > 0$,

$$\begin{aligned}
\left|\mathfrak{m}_\varepsilon - e^{i\varphi_0}\right| &= \left|e^{i\varphi_\varepsilon} - e^{i\varphi_0}\right| \\
&= \left|e^{i(\varphi_\varepsilon - \pi z_\varepsilon)} e^{i\pi z_\varepsilon} - e^{i\phi_0}(-1)^{z_\varepsilon}\right| \\
&= \left|e^{i(\varphi_\varepsilon - \pi z_\varepsilon)} - e^{i\phi_0}\right| \\
&\leqslant \frac{\pi}{2}\left|(\varphi_\varepsilon - \pi z_\varepsilon) - \phi_0\right|.
\end{aligned}$$

It follows that $(\mathfrak{m}_\varepsilon)$ converges strongly to $e^{i\varphi_0}$ in $L^p(\partial\Omega)$, for every $p \in [1, +\infty)$. Combining this with (2.72), we deduce that $(m_\varepsilon)$ converges strongly to $e^{i\varphi_0}$ in $L^p(\partial\Omega)$, for every $p \in [1, +\infty)$. By Theorem 2.2.13, we also have

$$\partial_\tau \varphi_0 = \partial_\tau \phi_0 = \kappa \mathcal{H}^1 \llcorner \partial\Omega - \pi \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}} \quad \text{as measure on } \partial\Omega$$

where, for every $j \in \{1, ..., N\}$, $a_j \in \partial\Omega$ are distinct points, $d_j \in \mathbb{Z} \setminus \{0\}$ and $\sum_{j=1}^N d_j = 2$. We also get, again by Theorem 2.2.13, the convergence of $(\partial_\tau \varphi_\varepsilon)$ to $\partial_\tau \phi_0$ in $W^{-1,p}(\partial\Omega)$ for every $p \in (1, +\infty)$. For every Lipschitz function $\zeta \in W^{1,\infty}(\Omega)$, using that $\mathfrak{m}_\varepsilon = e^{i\varphi_\varepsilon}$ and integrating by parts, we have

$$\langle \mathcal{J}(\mathfrak{m}_\varepsilon), \zeta\rangle = -\int_\Omega \mathfrak{m}_\varepsilon \wedge \nabla' \mathfrak{m}_\varepsilon \cdot \nabla'^\perp \zeta \, \mathrm{d}x = -\int_\Omega \nabla' \varphi_\varepsilon \cdot \nabla'^\perp \zeta \, \mathrm{d}x = -\int_{\partial\Omega} \partial_\tau \varphi_\varepsilon \zeta \, \mathrm{d}\mathcal{H}^1$$

Given $p \in (1, +\infty)$ and $\varepsilon > 0$ such that $\frac{1}{p} + \frac{1}{\varepsilon} = 1$, using that $\partial_\tau \varphi_\varepsilon \in W^{-1,p}(\partial\Omega)$ and $\zeta \in W^{1,\varepsilon}(\Omega)$, we deduce by letting $\varepsilon \to 0$ that

$$\langle \mathcal{J}(\mathfrak{m}_\varepsilon), \zeta\rangle \to -\int_{\partial\Omega} \partial_\tau \varphi_0 \zeta \, \mathrm{d}\mathcal{H}^1.$$

Hence, $(\mathcal{J}(\mathfrak{m}_\varepsilon))$ converges in $(W^{1,\infty}(\Omega))^*$ to the measure $J$ on $\overline{\Omega}$ given by

$$J = -\partial_\tau \phi_0 \mathcal{H}^1 \llcorner \partial\Omega = -\kappa \mathcal{H}^1 \llcorner \partial\Omega + \pi \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}}.$$

Combining this with (2.73), we deduce that $(\mathcal{J}(m_\varepsilon))$ converges to $J$ in $(W^{1,\infty}(\Omega))^*$. Finally, since

$$E_{\varepsilon,\eta}^\delta(m_\varepsilon) \geqslant E_{\varepsilon,\eta}^\delta(\mathfrak{m}_\varepsilon) - o(1) = \mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon) - o(1) \quad \text{as } \varepsilon \to 0$$

thanks to (2.71), the lower bound (2.53) for $\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon)$ given by Theorem 2.2.13 gives the expected lower bound for $E_{\varepsilon,\eta}^\delta(m_\varepsilon)$ at the first order. □

*Proof of Theorem 2.1.3.* Continuing as in the proof of Theorem 2.1.1 (with the same notations), we now assume the stronger condition (2.18). By definition of $\mathfrak{m}_\varepsilon$, Theorem 2.2.4 and Lemma 2.2.1,

$$\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon) = E_{\varepsilon,\eta}^\delta(\mathfrak{m}_\varepsilon) \leqslant E_{\varepsilon,\eta}^\delta(m_\varepsilon) + o(1) \quad \text{as } \varepsilon \to 0,$$

hence by (2.18),

$$\limsup_{\varepsilon\to 0}\left(\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon)-\pi\,|\log\varepsilon|\sum_{j=1}^N|d_j|\right)<+\infty. \tag{2.74}$$

*Step 1: Proof of (i).*

We deduce immediately from the above assumption and Theorem 2.2.14 that $d_j\in\{-1,+1\}$ for every $j\in\{1,...,N\}$, and that

$$\liminf_{\varepsilon\to 0}\left(\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon)-N\pi\,|\log\varepsilon|\right)\geqslant W_\Omega^\delta(\{(a_j,d_j)\})+N\gamma_0.$$

Since $E_{\varepsilon,\eta}^\delta(m_\varepsilon)\geqslant\mathcal{G}_\varepsilon^\delta(\varphi_\varepsilon)-o(1)$ as $\varepsilon\to 0$, it follows that

$$\liminf_{\varepsilon\to 0}\left(E_{\varepsilon,\eta}^\delta(m_\varepsilon)-N\pi\,|\log\varepsilon|\right)\geqslant W_\Omega^\delta(\{(a_j,d_j)\})+N\gamma_0.$$

*Step 2: Upper bound for the DMI term.*

We clearly have

$$\left|\int_\Omega\delta\cdot\nabla m_\varepsilon\wedge m_\varepsilon\;\mathrm{d}x\right|\leqslant\left|\int_\Omega\delta\cdot\nabla\mathfrak{m}_\varepsilon\wedge\mathfrak{m}_\varepsilon\;\mathrm{d}x\right|+\left|\int_\Omega\delta\cdot(\nabla\mathfrak{m}_\varepsilon\wedge\mathfrak{m}_\varepsilon-\nabla m_\varepsilon\wedge m_\varepsilon)\,\mathrm{d}x\right|.$$

Using (2.18), Lemma 2.2.1 and Theorem 2.2.4, it follows that, as $\varepsilon\to 0$,

$$\left|\int_\Omega\delta\cdot\nabla m_\varepsilon\wedge m_\varepsilon\;\mathrm{d}x\right|\leqslant\left|\int_\Omega\delta\cdot\nabla\mathfrak{m}_\varepsilon\wedge\mathfrak{m}_\varepsilon\;\mathrm{d}x\right|+o(1). \tag{2.75}$$

Moreover, by definition of $\mathfrak{m}_\varepsilon$, we have

$$\left|\int_\Omega\delta\cdot\nabla\mathfrak{m}_\varepsilon\wedge\mathfrak{m}_\varepsilon\;\mathrm{d}x\right|=\left|\int_\Omega\delta\cdot\nabla\varphi_\varepsilon\;\mathrm{d}x\right|.$$

By Theorem 2.2.14, $(\nabla\varphi_\varepsilon)$ is bounded in $L^{3/2}(\Omega)$. By Hölder's inequality, we deduce that

$$\left|\int_\Omega\delta\cdot\nabla\mathfrak{m}_\varepsilon\wedge\mathfrak{m}_\varepsilon\;\mathrm{d}x\right|\leqslant\left(\int_\Omega|\delta|^3\,\mathrm{d}x\right)^{1/3}\left(\int_\Omega|\nabla\varphi_\varepsilon|^{3/2}\,\mathrm{d}x\right)^{2/3}=|\delta|\,|\Omega|^{1/3}\,\|\nabla\varphi_\varepsilon\|_{L^{3/2}(\Omega)}\leqslant C,$$

for some constant $C>0$ independent of $\varepsilon$. Combining this with (2.75), we deduce that

$$\limsup_{\varepsilon\to 0}\left|\int_\Omega\delta\cdot\nabla m_\varepsilon\wedge m_\varepsilon\;\mathrm{d}x\right|<+\infty. \tag{2.76}$$

*Step 3: Lower bound for $E_{\varepsilon,\eta}^0(m_\varepsilon;\Omega\cap\bigcup_{j=1}^N B_r(a_j))$.*

We apply (2.40) with $G=\Omega\cap\bigcup_{j=1}^N B_{\rho(1+\rho)}(a_j)$, so that $\Omega\cap\bigcup_{j=1}^N B_\rho(a_j)\subset G_{\eta/c_0}$, for some $\rho>0$ sufficiently small. We get, up to taking a smaller $\rho>0$,

$$\begin{aligned}\int_{\Omega\cap\bigcup_j B_{\rho(1+\rho)}(a_j)}&\left(|\nabla m_\varepsilon|^2+\frac{1}{\eta^2}(1-|m_\varepsilon|^2)^2\right)\mathrm{d}x+\frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap\bigcup_j B_{\rho(1+\rho)}(a_j)}(m_\varepsilon\cdot\nu)^2\mathrm{d}\mathcal{H}^1\\&\geqslant E_{\varepsilon,\eta/c_0}^0(\mathfrak{m}_\varepsilon;\Omega\cap\bigcup_j B_\rho(a_j))-o(1)\\&=\int_{\Omega\cap\bigcup_j B_\rho(a_j)}|\nabla\mathfrak{m}_\varepsilon|^2\,\mathrm{d}x+\frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap\bigcup_j B_\rho(a_j)}(\mathfrak{m}_\varepsilon\cdot\nu)^2\mathrm{d}\mathcal{H}^1-o(1)\end{aligned}$$

as $\varepsilon \to 0$, hence

$$\begin{aligned}
&\int_{\Omega\cap\bigcup_j B_{\rho(1+\rho)}(a_j)} \left(|\nabla m_\varepsilon|^2 + \frac{1}{\eta^2}(1-|m_\varepsilon|^2)^2\right) \mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap\bigcup_j B_{\rho(1+\rho)}(a_j)} (m_\varepsilon\cdot\nu)^2 \mathrm{d}\mathcal{H}^1 \\
&\quad\geqslant \int_{\Omega\cap\bigcup_j B_\rho(a_j)} |\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap\bigcup_j B_\rho(a_j)} \sin^2(\varphi_\varepsilon - g)\;\mathrm{d}\mathcal{H}^1 - o(1) \\
&\quad= \sum_{j=1}^N \left(\int_{\Omega\cap B_\rho(a_j)} |\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap B_\rho(a_j)} \sin^2(\varphi_\varepsilon - g)\;\mathrm{d}\mathcal{H}^1\right) - o(1).
\end{aligned}$$

Let $j \in \{1,...,N\}$. By Theorem 2.2.13, there exists a family of integers $(z_\varepsilon)$ such that the family $(\varphi_\varepsilon - \pi z_\varepsilon - g)$ converges in $L^1(\partial\Omega\cap B_\rho(a_j))$ to a locally constant function with one single jump of height $\pm\pi$, with $g$ given at (2.42). We deduce from [24, Corollary 4.12] that there exist $\varepsilon_{0,j} > 0$, $\rho_{0,j} > 0$ and $C_j > 0$ such that, for every $\varepsilon \in (0,\varepsilon_{0,j})$ and $\rho \in (0,\rho_{0,j})$,

$$\int_{\Omega\cap B_\rho(a_j)} |\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap B_\rho(a_j)} \sin^2(\varphi_\varepsilon - g)\;\mathrm{d}\mathcal{H}^1 \geqslant \pi\log\frac{\rho}{\varepsilon} - C_j.$$

Setting $\widetilde{\varepsilon}_0 = \min_{j\in\{1,\dots,N\}}\{\varepsilon_{0,j}\}$ and $\widetilde{\rho}_0 = \min_{j\in\{1,\dots,N\}}\{\rho_{0,j}\}$, it follows that for every $\varepsilon \in (0,\widetilde{\varepsilon}_0)$ and $\rho \in (0,\widetilde{\rho}_0)$,

$$\begin{aligned}
&\int_{\Omega\cap\bigcup_j B_{\rho(1+\rho)}(a_j)} \left(|\nabla m_\varepsilon|^2 + \frac{1}{\eta^2}(1-|m_\varepsilon|^2)^2\right) \mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap\bigcup_j B_{\rho(1+\rho)}(a_j)} (m_\varepsilon\cdot\nu)^2 \mathrm{d}\mathcal{H}^1 \\
&\quad\geqslant N\pi\log\frac{\rho}{\varepsilon} - C,
\end{aligned}$$

for some constant $C > 0$ independent of $\varepsilon$ and $\rho$. In order to bound $E^0_{\varepsilon,\eta}(m_\varepsilon;\Omega\cap\bigcup_{j=1}^N B_r(a_j))$ by below, we set $r = \rho(1+\rho)$, so that $\rho = \frac{1}{2}(\sqrt{1+4r}-1)$, i.e. $\rho = r + o(r)$. Hence, the above estimate gives

$$\begin{aligned}
&\int_{\Omega\cap\bigcup_j B_r(a_j)} \left(|\nabla m_\varepsilon|^2 + \frac{1}{\eta^2}(1-|m_\varepsilon|^2)^2\right) \mathrm{d}x + \frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap\bigcup_j B_r(a_j)} (m_\varepsilon\cdot\nu)^2 \mathrm{d}\mathcal{H}^1 \\
&\quad\geqslant N\pi\log\frac{r}{\varepsilon} - C,
\end{aligned} \tag{2.77}$$

for $\varepsilon > 0$ and $r > 0$ sufficiently small, where $C > 0$ is independent of $\varepsilon$ and $r$.

*Step 4: Proof of (ii).*

Setting $\widetilde{m}_{2\varepsilon} = m_\varepsilon$ and $\widetilde{\mathfrak{m}}_{2\varepsilon} = \mathfrak{m}_\varepsilon$, we can apply the same arguments as in Step 3 to $\widetilde{m}_{2\varepsilon}$ and $\widetilde{\mathfrak{m}}_{2\varepsilon}$. We get, for $\varepsilon > 0$ and $r > 0$ sufficiently small, and using the nonnegativity of the integrated functions,

$$E^0_{2\varepsilon,2\eta}(m_\varepsilon) = E^0_{2\varepsilon,2\eta}(\widetilde{m}_{2\varepsilon}) \geqslant N\pi\log\frac{r}{2\varepsilon} - C,$$

for some constant $C > 0$. Let $r > 0$ as above be fixed. By (2.18), for $\varepsilon > 0$ sufficiently small, we have $E^\delta_{\varepsilon,\eta}(m_\varepsilon) \leqslant C' + N\pi\log\frac{1}{\varepsilon}$ for some constant $C' > 0$. Hence we get, for $\varepsilon > 0$ sufficiently small,

$$E^\delta_{\varepsilon,\eta}(m_\varepsilon) - E^0_{2\varepsilon,2\eta}(m_\varepsilon) \leqslant N\pi\log\frac{r}{2} + C + C' = \widetilde{C}$$

where $\widetilde{C} > 0$ is independent of $\varepsilon$, i.e.

$$\frac{3}{4\eta^2}\int_\Omega (1-|m_\varepsilon|^2)^2\mathrm{d}x + \frac{1}{4\pi\varepsilon}\int_{\partial\Omega}(m_\varepsilon\cdot\nu)^2\mathrm{d}\mathcal{H}^1 + 2\int_\Omega \delta\cdot\nabla m_\varepsilon\wedge m_\varepsilon\;\mathrm{d}x \leqslant \widetilde{C}.$$

Using (2.76), we deduce that

$$\limsup_{\varepsilon\to 0}\left(\frac{1}{\eta^2}\int_\Omega(1-|m_\varepsilon|^2)^2\mathrm{d}x+\frac{1}{2\pi\varepsilon}(m_\varepsilon\cdot\nu)^2\mathrm{d}\mathcal{H}^1\right)<+\infty.$$

*Step 5: Proof of (iii).*

By (2.77), for $\varepsilon>0$ and $r>0$ sufficiently small,

$$\begin{aligned}&\int_{\Omega\cap\bigcup_j B_r(a_j)}|\nabla m_\varepsilon|^2\,\mathrm{d}x-N\pi\log\frac{r}{\varepsilon}\\&\quad\geqslant -C-\left(\frac{1}{\eta^2}\int_{\Omega\cap\bigcup_j B_r(a_j)}(1-|m_\varepsilon|^2)^2\mathrm{d}x+\frac{1}{2\pi\varepsilon}\int_{\partial\Omega\cap\bigcup_j B_r(a_j)}(m_\varepsilon\cdot\nu)^2\mathrm{d}\mathcal{H}^1\right)\\&\quad\geqslant -C-\left(\frac{1}{\eta^2}\int_\Omega(1-|m_\varepsilon|^2)^2\mathrm{d}x+\frac{1}{2\pi\varepsilon}\int_{\partial\Omega}(m_\varepsilon\cdot\nu)^2\mathrm{d}\mathcal{H}^1\right).\end{aligned}$$

Using *(ii)*, we deduce that there exist $\varepsilon_0>0$, $r_0>0$ and $C>0$ such that, for $\varepsilon\in(0,\varepsilon_0)$ and $r\in(0,r_0)$,

$$\int_{\Omega\cap\bigcup_j B_r(a_j)}|\nabla m_\varepsilon|^2\,\mathrm{d}x-N\pi\log\frac{r}{\varepsilon}\geqslant -C.$$

*Step 6: Proof of (iv).*

For every $\varepsilon>0$ and $r>0$, we have

$$\begin{aligned}\int_{\Omega\setminus\bigcup_j B_r(a_j)}|\nabla m_\varepsilon|^2\,\mathrm{d}x&=\int_\Omega|\nabla m_\varepsilon|^2\,\mathrm{d}x-\int_{\Omega\cap\bigcup_j B_r(a_j)}|\nabla m_\varepsilon|^2\,\mathrm{d}x\\&\leqslant E^0_{\varepsilon,\eta}(m_\varepsilon)-\int_{\Omega\cap\bigcup_j B_r(a_j)}|\nabla m_\varepsilon|^2\,\mathrm{d}x\\&=E^\delta_{\varepsilon,\eta}(m_\varepsilon)-2\int_\Omega\delta\cdot\nabla m_\varepsilon\wedge m_\varepsilon\;\mathrm{d}x-\int_{\Omega\cap\bigcup_j B_r(a_j)}|\nabla m_\varepsilon|^2\,\mathrm{d}x.\end{aligned}$$

Using (2.18), (2.76) and *(iii)*, we deduce that for $\varepsilon>0$ and $r>0$ sufficiently small,

$$\int_{\Omega\setminus\bigcup_j B_r(a_j)}|\nabla m_\varepsilon|^2\,\mathrm{d}x\leqslant N\pi\log\frac{1}{r}+C,\tag{2.78}$$

for some $C>0$ independent of $\varepsilon$ and $r$. By [24, Lemma 4.17], we have $\limsup_{\varepsilon\to 0}\|\nabla m_\varepsilon\|_{L^q(\Omega)}<+\infty$ for every $q\in[1,2)$, thus $(m_\varepsilon)$ is bounded in $W^{1,q}(\Omega,\mathbb{R}^2)$. As we proved that (2.74) holds, we can apply Theorem 2.2.14, so that for every $q\in[1,2)$, $(\varphi_\varepsilon)$ converges weakly in $W^{1,q}(\Omega)$ to an extension (not necessarily harmonic) $\widehat{\varphi}_0\in W^{1,q}(\Omega)$ of $\varphi_0$ to $\Omega$. Using that $\mathfrak{m}_\varepsilon=e^{i\varphi_\varepsilon}$ for every $\varepsilon>0$, we deduce that $(\mathfrak{m}_\varepsilon)$ converges weakly to $e^{i\widehat{\varphi_0}}$ in $W^{1,q}(\Omega,\mathbb{R}^2)$, for every $q\in[1,2)$. By Sobolev compact embedding of $W^{1,q}(\Omega)$ in $L^p(\Omega)$ for every $p\in[1,+\infty)$, we deduce that up to a new subsequence, $(\mathfrak{m}_\varepsilon)$ converges strongly to $e^{i\widehat{\varphi_0}}$ in $L^p(\Omega,\mathbb{R}^2)$, for every $p\in[1,+\infty)$. Since $(m_\varepsilon)$ is bounded in $W^{1,q}(\Omega,\mathbb{R}^2)$, for every $q\in[1,2)$, then we can assume that, up to a subsequence, $(m_\varepsilon)$ converges weakly in $W^{1,q}(\Omega,\mathbb{R}^2)$, for every $q\in[1,2)$. Moreover, by Theorem 2.2.4, for every $p\in[1,+\infty)$,

$$\lim_{\varepsilon\to 0}\|\mathfrak{m}_\varepsilon-m_\varepsilon\|_{L^p(\Omega)}=0,$$

hence $(m_\varepsilon)$ converges weakly to $e^{i\widehat{\varphi_0}}$ in $W^{1,q}(\Omega,\mathbb{R}^2)$, for every $q \in [1,2)$, and $(m_\varepsilon)$ converges strongly to $e^{i\widehat{\varphi_0}}$ in $L^p(\Omega,\mathbb{R}^2)$, for every $p \in [1,+\infty)$.

*Step 7: Proof of (v).*

We finally note that (2.76) can be refined by using *(iv)*. Indeed $(\nabla m_\varepsilon)$ is bounded in $L^{3/2}(\Omega)$ and $(m_\varepsilon)$ is bounded in $L^3(\Omega)$, thus by Hölder's inequality,

$$\int_\Omega |\delta \cdot \nabla m_\varepsilon \wedge m_\varepsilon| \,\mathrm{d}x \leqslant |\delta| \, \|\nabla m_\varepsilon\|_{L^{3/2}(\Omega)} \, \|m_\varepsilon\|_{L^3(\Omega)} \leqslant C$$

for some constant $C > 0$ independent of $\varepsilon$. □

*Remark* 2.2.17. With the notations of the previous proof, we have

$$\liminf_{r\to 0} \liminf_{\varepsilon \to 0} \left( \int_{\Omega \cap \bigcup_j B_r(a_j)} \left( |\nabla m_\varepsilon|^2 + \frac{1}{\eta^2}(1-|m_\varepsilon|^2)^2 \right) \mathrm{d}x \right. \\ \left. + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega \cap \bigcup_j B_r(a_j)} (m_\varepsilon \cdot \nu)^2 \, \mathrm{d}\mathcal{H}^1 - N\pi \log \frac{r}{\varepsilon} - N\gamma_0 \right) \geqslant 0.$$

that is to say

$$\liminf_{r\to 0} \liminf_{\varepsilon \to 0} \left( E^0_{\varepsilon,\eta}(m_\varepsilon; \Omega \cap \bigcup_j B_r(a_j)) - N\pi \log \frac{r}{\varepsilon} - N\gamma_0 \right) \geqslant 0. \tag{2.79}$$

Indeed, applying (2.40) in Theorem 2.2.4 as in Step 3 of the previous proof, we get, for $\rho > 0$ sufficiently small, we have, as $\varepsilon \to 0$,

$$\begin{aligned} &\int_{\Omega \cap \bigcup_j B_{\rho(1+\rho)}(a_j)} \left( |\nabla m_\varepsilon|^2 + \frac{1}{\eta^2}(1-|m_\varepsilon|^2)^2 \right) \mathrm{d}x + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega \cap \bigcup_j B_{\rho(1+\rho)}(a_j)} (m_\varepsilon \cdot \nu)^2 \mathrm{d}\mathcal{H}^1 \\ &- N\pi \log \frac{\rho(1+\rho)}{\varepsilon} - N\gamma_0 \\ &\geqslant \sum_{j=1}^N \left( \int_{\Omega \cap B_\rho(a_j)} |\nabla \varphi_\varepsilon|^2 \, \mathrm{d}x + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega \cap B_\rho(a_j)} \sin^2(\varphi_\varepsilon - g) \, \mathrm{d}\mathcal{H}^1 \right) - N\pi \log \frac{\rho}{\varepsilon} - N\gamma_0 \\ &\quad - N\pi \log(1+\rho) - o(1). \end{aligned}$$

Using (2.63), that holds by Remark 2.2.15, we deduce by taking the lim inf as $\varepsilon \to 0$ that

$$\begin{aligned} \liminf_{\varepsilon \to 0} &\left( \int_{\Omega \cap \bigcup_j B_{\rho(1+\rho)}(a_j)} \left( |\nabla m_\varepsilon|^2 + \frac{1}{\eta^2}(1-|m_\varepsilon|^2)^2 \right) \mathrm{d}x \right. \\ &\left. + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega \cap \bigcup_j B_{\rho(1+\rho)}(a_j)} (m_\varepsilon \cdot \nu)^2 \, \mathrm{d}\mathcal{H}^1 - N\pi \log \frac{\rho(1+\rho)}{\varepsilon} - N\gamma_0 \right) \\ &\geqslant -CN\rho^{1/2} - N\pi \log(1+\rho). \end{aligned}$$

Setting $r = \rho(1+\rho)$, so that $\rho = \frac{1}{2}(\sqrt{1+4r} - 1)$, and noting that $\rho \to 0$ as $r \to 0$, we obtain (2.79) by taking the lim inf as $r \to 0$ in the above inequality (after having replaced $\rho$ by its expression in function of $r$).

*Proof of Theorem 2.1.4.* Let $\varphi_0 \colon \partial\Omega \to \mathbb{R}$ be such that $\partial_\tau \varphi_0 = \kappa \mathcal{H}^1 \llcorner \partial\Omega - \pi \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}}$ as measure on $\partial\Omega$ and $e^{i\varphi_0} \cdot \nu = 0$ in $\partial\Omega \setminus \{a_1, ..., a_N\}$. Let $\varphi_*$ be the harmonic extension of $\varphi_0$ to $\Omega$. For any $\varepsilon > 0$, we consider $\widehat{\psi}_\varepsilon^*$ as in the proof of Theorem 2.2.16 and we set $m_\varepsilon = e^{i\widehat{\psi}_\varepsilon^*}$. Then, for every $\varepsilon > 0$, $m_\varepsilon \in H^1(\Omega, \mathbb{S}^1)$ and $\mathcal{J}(m_\varepsilon) = -\partial_\tau \widehat{\psi}_\varepsilon^* \mathcal{H}^1 \llcorner \partial\Omega$. Since, for every $\varepsilon > 0$,

$$|m_\varepsilon - e^{i\varphi_*}| = |e^{i\widehat{\psi}_\varepsilon^*} - e^{i\varphi_*}| \leqslant \frac{\pi}{2} |\widehat{\psi}_\varepsilon^* - \varphi_*|,$$

it follows from Theorem 2.2.16 that $(m_\varepsilon)$ converges strongly to $\varphi_*$ in $L^p(\Omega)$ and in $L^p(\partial\Omega)$ for every $p \in [1, +\infty)$, and that $(\mathcal{J}(m_\varepsilon))$ converges to

$$-\partial_\tau \varphi_0 = -\kappa \mathcal{H}^1 \llcorner \partial\Omega + \pi \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}}$$

in $(W^{1,\infty}(\Omega))^*$. Finally, the expected limits at first and second order for $E_{\varepsilon,\eta}^\delta(m_\varepsilon)$ follow from (2.64) for the first order, from (2.65) for the second order, combined with the equality $E_{\varepsilon,\eta}^\delta(m_\varepsilon) = \mathcal{G}_\varepsilon^\delta(\widehat{\psi}_\varepsilon^*)$. □

### 2.2.6 Minimizers of the renormalized energy

For proving Corollary 2.1.8, we firstly need to prove that $E_{\varepsilon,\eta}^\delta$ admits minimizers in $H^1(\Omega, \mathbb{R}^2)$, and secondly that the renormalized energy (2.17) admits minimizers corresponding to two boundary vortices of multiplicities 1, i.e. $N = 2$ and $d_1 = d_2 = 1$ (this is the Corollary 2.1.6).

**Lemma 2.2.18.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected and $C^{1,1}$ smooth domain. Let $\delta \in \mathbb{R}^2$. Assume $\varepsilon \to 0$ and $\eta = \eta(\varepsilon) \to 0$ in the regime (2.13). There exists a minimizer of $E_{\varepsilon,\eta}^\delta$ over the set $H^1(\Omega, \mathbb{R}^2)$.*

*Proof.* We apply the direct method in the calculus of variations. Let $\delta \in \mathbb{R}^2$. Since $|\delta|$ is constant and $\eta = \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, we can assume that $(\varepsilon, \eta) \in \mathbb{R}_+^* \times (0, \frac{1}{2})$ satisfies $4|\delta|^2 \leqslant \frac{1}{2\eta^2}$. For every $m = m_\varepsilon \in H^1(\Omega, \mathbb{R}^2)$, we have

$$\left| 2 \int_\Omega \delta \cdot \nabla m \wedge m \, \mathrm{d}x \right| \leqslant \frac{1}{2} \int_\Omega |\nabla m|^2 \, \mathrm{d}x + 4 |\delta|^2 \int_\Omega (1 - |m|^2)^2 \mathrm{d}x + 4 |\delta|^2 |\Omega|,$$

as in the proof of Lemma 2.2.1. It follows that

$$\begin{aligned} E_{\varepsilon,\eta}^\delta(m) &= \int_\Omega \left( |\nabla m|^2 + 2\delta \cdot \nabla m \wedge m + \frac{1}{\eta^2}(1 - |m|^2)^2 \right) \mathrm{d}x + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (m \cdot \nu)^2 \mathrm{d}\mathcal{H}^1 \\ &\geqslant \frac{1}{2} \int_\Omega \left( |\nabla m|^2 + \frac{1}{\eta^2}(1 - |m|^2)^2 \right) \mathrm{d}x + \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (m \cdot \nu)^2 \mathrm{d}\mathcal{H}^1 - 4 |\delta|^2 |\Omega| \\ &\geqslant -4 |\delta|^2 |\Omega| > -\infty. \end{aligned}$$

Moreover, for $m_0 = e_1$ (or another constant vector on $\mathbb{S}^1$), we have $E_{\varepsilon,\eta}^\delta(m_0) \leqslant \frac{|\partial\Omega|}{2\pi\varepsilon} < +\infty$. Hence, there exists a minimizing sequence $(m_n)_{n \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^2)$, i.e.

$$\lim_{n \to +\infty} E_{\varepsilon,\eta}^\delta(m_n) = \inf_{H^1(\Omega, \mathbb{R}^2)} E_{\varepsilon,\eta}^\delta \geqslant -4 |\delta|^2 |\Omega|.$$

It follows that there exists a constant $C = C(\varepsilon) > 0$ (that may depend on $\varepsilon$) such that for every $n \geqslant 0$, $E_{\varepsilon,\eta}^\delta(m_n) \leqslant C(\varepsilon)$. In particular, for every $n \geqslant 0$,

$$\int_\Omega |\nabla m_n|^2 \, \mathrm{d}x + \frac{1}{\eta^2} \int_\Omega (1 - |m_n|^2)^2 \mathrm{d}x \leqslant 2 \left( E_{\varepsilon,\eta}^\delta(m_n) + 4 |\delta|^2 |\Omega| \right) \leqslant C(\varepsilon). \tag{2.80}$$

Moreover, for every $n \geqslant 0$, setting $S = \{x \in \Omega \colon |m_n(x)|^2 \geqslant 2\}$, we have

$$\int_\Omega |m_n|^2 \,\mathrm{d}x = \int_S |m_n|^2 \,\mathrm{d}x + \int_{\Omega\setminus S} |m_n|^2 \,\mathrm{d}x \leqslant 2\int_S \left(1-|m_n|^2\right)^2 \mathrm{d}x + 2\,|\Omega\setminus S|\,,$$

since for every $x \in S$, $|m_n(x)|^2 - 1 \geqslant \frac{|m_n(x)|}{\sqrt{2}}$. As $\eta \to 0$, we can assume (up to taking $\varepsilon$ and $\eta$ smaller than before) that

$$\int_\Omega |m_n|^2 \,\mathrm{d}x \leqslant \frac{1}{\eta^2}\int_S \left(1-|m_n|^2\right)^2 \mathrm{d}x + 2\,|\Omega\setminus S| \leqslant \frac{1}{\eta^2}\int_\Omega \left(1-|m_n|^2\right)^2 \mathrm{d}x + 2\,|\Omega|\,.$$

Combining this with (2.80), we deduce that

$$\int_\Omega |m_n|^2 \,\mathrm{d}x + \int_\Omega |\nabla m_n|^2 \,\mathrm{d}x \leqslant C(\varepsilon),$$

so that $(m_n)$ is bounded in $H^1(\Omega,\mathbb{R}^2)$. By the Banach-Alaoglu theorem, there exist a subsequence (we do not relabel) $(m_n) \subset H^1(\Omega,\mathbb{R}^2)$ and $m_\infty \in H^1(\Omega,\mathbb{R}^2)$ such that $(m_n)$ converges weakly to $m_\infty$ in $H^1(\Omega,\mathbb{R}^2)$. By Sobolev compact embedding, up to a further subsequence, $(m_n)$ converges strongly to $m_\infty$ in $L^p(\Omega,\mathbb{R}^2)$ for every $p \in [1,4]$. By weak lowersemicontinuity of the Dirichlet integral, we have

$$\int_\Omega |\nabla m_\infty|^2 \,\mathrm{d}x \leqslant \liminf_{n\to+\infty} \int_\Omega |\nabla m_n|^2 \,\mathrm{d}x.$$

As $(\nabla m_n)$ converges to $\nabla m_\infty$ weakly in $L^2(\Omega,\mathbb{R}^2)$ and $(m_n)$ converges to $m_\infty$ strongly in $L^2(\Omega,\mathbb{R}^2)$, we have

$$\lim_{n\to+\infty} \int_\Omega 2\delta\cdot\nabla m_n \wedge m_n \;\mathrm{d}x = \int_\Omega 2\delta\cdot\nabla m_\infty \wedge m_\infty \;\mathrm{d}x,$$

and as $(m_n)$ converges to $m_\infty$ strongly in $L^4(\Omega,\mathbb{R}^2)$,

$$\lim_{n\to+\infty} \int_\Omega (1-|m_n|^2)^2 \mathrm{d}x = \int_\Omega (1-|m_\infty|^2)^2 \mathrm{d}x.$$

Finally, since $\Omega$ is a bounded $C^{1,1}$ smooth domain, the trace operator $H^1(\Omega,\mathbb{R}^2) \to H^{1/2}(\partial\Omega,\mathbb{R}^2)$ is linear and continuous. By weak convergence of $(m_n)$ to $m_\infty$ in $H^1(\Omega,\mathbb{R}^2)$ and using the compact embedding of $H^{1/2}(\partial\Omega,\mathbb{R}^2)$ in $L^2(\partial\Omega,\mathbb{R}^2)$, we get the strong convergence of $(m_n)$ to $m_\infty$ in $L^2(\partial\Omega,\mathbb{R}^2)$, thus

$$\lim_{n\to+\infty} \int_{\partial\Omega} (m_n\cdot\nu)^2 \mathrm{d}\mathcal{H}^1 = \int_{\partial\Omega} (m_\infty\cdot\nu)^2 \mathrm{d}\mathcal{H}^1.$$

It follows that

$$E^\delta_{\varepsilon,\eta}(m_\infty) \leqslant \liminf_{n\to+\infty} E^\delta_{\varepsilon,\eta}(m_n) = \inf_{H^1(\Omega,\mathbb{R}^2)} E^\delta_{\varepsilon,\eta},$$

and thus

$$E^\delta_{\varepsilon,\eta}(m_\infty) = \inf_{H^1(\Omega,\mathbb{R}^2)} E^\delta_{\varepsilon,\eta},$$

i.e. $m_\infty$ is a minimizer of $E^\delta_{\varepsilon,\eta}$ in $H^1(\Omega,\mathbb{R}^2)$. □

The following statement extends [25, Proposition 20] by giving a formula for the renormalized energy $W^\delta_\Omega(\{(a_j,d_j)\})$ defined at (2.17). In particular, it shows that the limit in (2.17) exists. The renormalized energy is computed using the solution of a Neumann problem as in [7].

**Proposition 2.2.19.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected, and $C^{1,1}$ smooth domain, and $\kappa$ be the curvature on $\partial\Omega$. Let $\{a_j\}_{j\in\{1,...,N\}} \in (\partial\Omega)^N$ be $N \geqslant 2$ distinct points and $d_j \in \{-1,+1\}$ be the corresponding multiplicities, for $j \in \{1,...,N\}$, that satisfy $\sum_{j=1}^N d_j = 2$. Let $\delta \in \mathbb{R}^2$. Then the limit in* (2.17) *exists and the renormalized energy of $\{(a_j, d_j)\}$ satisfies*

$$\begin{aligned} W_\Omega^\delta(\{(a_j,d_j)\}) = -2\pi &\sum_{1\leqslant j<k\leqslant N} d_j d_k \log|a_j - a_k| \\ &- \int_{\partial\Omega} \psi(\kappa + 2\delta^\perp \cdot \nu)\ \mathrm{d}\mathcal{H}^1 + \pi \sum_{j=1}^N d_j R(a_j), \end{aligned} \tag{2.81}$$

*where $\nu$ is the outer unit normal vector on $\partial\Omega$ and $\psi$ denotes the unique solution (up to an additive constant) in $W^{1,q}(\Omega)$, for every $q \in [1,2)$, of the inhomogeneous Neumann problem*

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega, \\ \frac{\partial\psi}{\partial\nu} = -\kappa + \pi\sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}} & \text{on } \partial\Omega, \end{cases} \tag{2.82}$$

*and $R$ is the harmonic function given by*

$$R(z) = \psi(z) + \sum_{j=1}^N d_j \log|z - a_j|, \tag{2.83}$$

*for every $z \in \Omega$. Moreover, we have $R \in C^{0,\alpha}(\overline{\Omega}) \cap W^{s,p}(\Omega)$ for every $\alpha \in (0,1)$, $p \in [1,+\infty)$ and $s = s(p) \in [1, 1+\frac{1}{p})$.*

*Proof.* By the definition of $W_\Omega^\delta(\{(a_j,d_j)\})$ in (2.17), we have

$$W_\Omega^\delta(\{(a_j,d_j)\}) = W_\Omega^0(\{(a_j,d_j)\}) - \lim_{r\to 0}\int_{\Omega^r} 2\delta\cdot\nabla\varphi_*\ \mathrm{d}x,$$

where, for $r > 0$, $\Omega^r = \Omega \setminus \bigcup_{j=1}^N B_r(a_j)$ and $\varphi_*$ is the harmonic extension of $\varphi_0$ (defined in Definition 2.1.2) to $\Omega$. First, by [25, Proposition 20], we have

$$W_\Omega^0(\{(a_j,d_j)\}) = -2\pi \sum_{1\leqslant j<k\leqslant N} d_j d_k \log|a_j - a_k| - \int_{\partial\Omega} \psi\kappa\ \mathrm{d}\mathcal{H}^1 + \pi\sum_{j=1}^N d_j R(a_j), \tag{2.84}$$

with $\psi \in W^{1,1}(\Omega)$ as in (2.82) and $R$ as in (2.83). Moreover, any solution $\psi$ of (2.82) is clearly a harmonic conjugate of $\varphi_*$, hence we have $\nabla\varphi_* = -\nabla^\perp\psi$. In particular, since $\varphi_* \in W^{1,1}(\Omega)$, we have $\nabla\varphi_* \in L^1(\Omega)$. It follows by dominated convergence theorem that

$$\begin{aligned} \lim_{r\to 0}\int_{\Omega^r} 2\delta\cdot\nabla\varphi_*\ \mathrm{d}x = 2\int_\Omega \delta\cdot\nabla\varphi_*\ \mathrm{d}x &= -2\int_\Omega \delta\cdot\nabla^\perp\psi\ \mathrm{d}x \\ &= -2\int_\Omega (-\delta_1\partial_2\psi + \delta_2\partial_1\psi)\,\mathrm{d}x \\ &= -2\int_{\partial\Omega} \psi\,(-\delta_1\nu_2 + \delta_2\nu_1)\,\mathrm{d}\mathcal{H}^1 \\ &= 2\int_{\partial\Omega} \psi(\delta^\perp\cdot\nu)\ \mathrm{d}\mathcal{H}^1. \end{aligned}$$

Hence,

$$\lim_{r\to 0}\int_{\Omega^r} 2\delta\cdot\nabla\varphi_*\ \mathrm{d}x = 2\int_{\partial\Omega} \psi(\delta^\perp\cdot\nu)\ \mathrm{d}\mathcal{H}^1, \tag{2.85}$$

and we obtain (2.81) by subtracting (2.85) to (2.84). ☐

We now prove Theorem 2.1.5 by using the above Proposition 2.2.19, [25, Theorem 6] and a lemma of complex analysis.

*Proof of Theorem 2.1.5.* By Proposition 2.2.19, we have

$$W_\Omega^\delta(\{(a_j,d_j)\}) = W_\Omega^0(\{(a_j,d_j)\}) - 2\int_{\partial\Omega} \psi(\delta^\perp\cdot\nu)\ \mathrm{d}\mathcal{H}^1, \tag{2.86}$$

for any bounded, simply connected and $C^{1,1}$ smooth domain $\Omega\subset\mathbb{R}^2$, with boundary curvature $\kappa$.

*(i)* We assume that $\Omega = B_1$. By [25, Theorem 6], we have

$$W_{B_1}^0(\{(a_j,d_j)\}) = -2\pi\sum_{1\leqslant j<k\leqslant N} d_jd_k\log|a_j-a_k|, \tag{2.87}$$

and, for every $z\in B_1$,

$$\psi(z) = -\sum_{j=1}^N d_j\log|z-a_j|. \tag{2.88}$$

By Green's formula,

$$\int_{\partial B_1}\psi(w)\delta^\perp\cdot\nu(w)\ \mathrm{d}\mathcal{H}^1(w) = \int_{B_1}\mathrm{div}(\psi(z)\delta^\perp)\ \mathrm{d}z = -\sum_{j=1}^N d_j\int_{B_1}\mathrm{div}(\log|z-a_j|)\delta^\perp)\ \mathrm{d}z.$$

For any $j\in\{1,...,N\}$ and $z\in B_1$,

$$\mathrm{div}(\log|z-a_j|)\delta^\perp) = \delta^\perp\cdot\frac{z-a_j}{|z-a_j|^2} = \mathfrak{Re}\left(\frac{\overline{z-a_j}}{|z-a_j|^2}\delta^\perp\right) = \mathfrak{Re}\left(\frac{1}{z-a_j}\delta^\perp\right)$$

with the identification $\delta^\perp = \binom{-\delta_2}{\delta_1} = -\delta_2+i\delta_1$, so that

$$\int_{B_1}\mathrm{div}(\log|z-a_j|)\delta^\perp)\ \mathrm{d}z = \mathfrak{Re}\left(\delta^\perp\int_{B_1}\frac{1}{z-a_j}\ \mathrm{d}z\right).$$

Using the next Lemma 2.2.20, for every $j\in\{1,...,N\}$, we get

$$\int_{B_1}\mathrm{div}(\log|z-a_j|)\delta^\perp)\ \mathrm{d}z = -\pi\mathfrak{Re}\left(\delta^\perp\overline{a_j}\right) = -\pi\delta^\perp\cdot a_j = \pi\delta\cdot a_j^\perp$$

so that

$$\int_{\partial B_1}\psi(w)\delta^\perp\cdot\nu(w)\ \mathrm{d}\mathcal{H}^1(w) = -\pi\sum_{j=1}^N d_j\delta\cdot a_j^\perp \tag{2.89}$$

and we deduce (2.22) combining (2.87) and (2.89) in (2.86).

*(ii)* By [25, Theorem 6], we have

$$\begin{aligned} &W_\Omega^0(\{(a_j,d_j)\}) \\ &\quad= -2\pi\sum_{1\leqslant j<k\leqslant N} d_jd_k\log|\Psi(a_j)-\Psi(a_k)| + \pi\sum_{j=1}^N(d_j-1)\log|\Psi'(a_j)| \\ &\quad\quad+\int_{\partial\Omega}\kappa(w)\left(\sum_{j=1}^N d_j\log|\Psi(w)-\Psi(a_j)| - \log|\Psi'(w)|\right)\mathrm{d}\mathcal{H}^1(w), \end{aligned}$$

and, for every $w \in \Omega$,

$$\psi(w) = -\sum_{j=1}^{N} d_j \log|\Psi(w) - \Psi(a_j)| + \log|\Psi'(w)|.$$

The expected identity (2.23) is a direct consequence of (2.86) and the above identities.

□

**Lemma 2.2.20.** *Let $B_1 \subset \mathbb{R}^2$ be the unit disk. For any $a = (a_1, a_2) \in \partial B_1$,*

$$\int_{B_1} \frac{1}{z-a}\,\mathrm{d}z = -\pi\overline{a}$$

*where $\overline{a} = (a_1, -a_2) = a_1 - ia_2$ stands for the complex conjugate of $a$.*

*Proof.* Using polar coordinates, we get

$$\int_{B_1} \frac{1}{z-a}\,\mathrm{d}z = \int_{r=0}^{1}\int_{t=0}^{2\pi} \frac{r}{re^{it}-a}\,\mathrm{d}t\mathrm{d}r. \tag{2.90}$$

For any $r \in (0,1)$, we compute the integral $\int_{|z|=r} \frac{1}{z(z-a)}\,\mathrm{d}z$ in two ways. On the one hand, using polar coordinates,

$$\int_{|z|=r} \frac{1}{z(z-a)}\,\mathrm{d}z = i\int_{0}^{2\pi} \frac{1}{re^{it}-a}\,\mathrm{d}t.$$

On the other hand, $z \mapsto \frac{1}{z(z-a)}$ admits a simple pole at $z=0$ inside the contour $\{|z| = r\}$, thus by the residue theorem,

$$\int_{|z|=r} \frac{1}{z(z-a)}\,\mathrm{d}z = 2i\pi \left.\frac{1}{z-a}\right|_{z=0} = -\frac{2i\pi}{a} = -2i\pi\overline{a}$$

since $|a|^2 = a\overline{a} = 1$. Combining both expressions of $\int_{|z|=r} \frac{1}{z(z-a)}\,\mathrm{d}z$, we get

$$\int_{0}^{2\pi} \frac{1}{re^{it}-a}\,\mathrm{d}t = -2\pi\overline{a},$$

and inserting this in (2.90),

$$\int_{B_1} \frac{1}{z-a}\,\mathrm{d}z = -2\pi\overline{a}\int_{0}^{1} r\,\mathrm{d}r = -\pi\overline{a}.$$

□

We now prove Corollary 2.1.6, that gives the existence of a minimizing pair $(a_1^*, a_2^*)$ for the renormalized energy when $N = 2$ and $d_1 = d_2 = 1$.

*Proof of Corollary 2.1.6.* Let $\Phi\colon \overline{B_1} \to \overline{\Omega}$ be a $C^1$ conformal diffeomorphism with inverse $\Psi = \Phi^{-1}$. Let $\mathfrak{D} = (\partial\Omega \times \partial\Omega) \setminus \{(a,a) : a \in \partial\Omega\}$. By Theorem 2.1.5, for every $(a_1, a_2) \in \mathfrak{D}$,

$$W_\Omega^\delta(\{(a_1,1),(a_2,1)\}) = -2\pi\log|\Psi(a_1) - \Psi(a_2)| + F(a_1,a_2)$$

where

$$F\colon (a_1,a_2)\in\mathfrak{D}\mapsto \int_{\partial\Omega}(\kappa(w)+2\delta^\perp\cdot\nu(w))(\log|\Psi(w)-\Psi(a_1)|+\log|\Psi(w)-\Psi(a_2)| \\ -\log|\Psi'(w)|)\mathrm{d}\mathcal{H}^1(w).$$

Setting $b_1=\Psi(a_1)$ and $b_2=\Psi(a_2)$, we get, after changing variables,

$$F(a_1,a_2)=\int_{\partial B_1}(\kappa(\Phi(z))+2\delta^\perp\cdot\nu(\Phi(z)))(\log|z-b_1|+\log|z-b_2| \\ +\log|\Phi'(z)|)\,|\Phi'(z)|\,\mathrm{d}\mathcal{H}^1(z).$$

We deduce that $F$ is bounded in $\mathfrak{D}$, since $\kappa+2\delta^\perp\cdot\nu\in L^\infty(\partial\Omega)$ and the functions $z\mapsto\log|z-b_1|$ and $z\mapsto\log|z-b_2|$ are in $L^1(\partial B_1)$. Moreover, the function $(a_1,a_2)\in\mathfrak{D}\mapsto W_\Omega^\delta(\{(a_1,1),(a_2,1)\})$ is continuous. Let $(a_1^{(n)},a_2^{(n)})\subset\mathfrak{D}$ be a minimizing sequence for $W_\Omega^\delta(\{(\cdot,1),(\cdot,1)\})$, i.e.

$$\lim_{n\to+\infty}W_\Omega^\delta(\{(a_1^{(n)},1),(a_2^{(n)},1)\})=\inf_{\mathfrak{D}}W_\Omega^\delta(\{(\cdot,1),(\cdot,1)\}).$$

Note that such a sequence exists because $F$ is bounded in $\mathfrak{D}$. As $\partial\Omega$ is compact, so is $\partial\Omega\times\partial\Omega$, and then we can assume (up to a subsequence) that $(a_1^{(n)},a_2^{(n)})$ converges to some $(a_1^*,a_2^*)\in\partial\Omega\times\partial\Omega$. Since $(W_\Omega^\delta(\{(a_1^{(n)},1),(a_2^{(n)},1)\}))$ is bounded, then using the boundedness of $F$, we deduce that the sequence $(\log|\Psi(a_1^{(n)})-\Psi(a_2^{(n)})|)$ is bounded. By taking the limits as $n\to+\infty$, we deduce that $\Psi(a_1^*)\neq\Psi(a_2^*)$, and since $\Psi$ is injective, $a_1^*\neq a_2^*$, i.e. $(a_1^*,a_2^*)\in\mathfrak{D}$. Finally, by continuity of $W_\Omega^\delta(\{(\cdot,1),(\cdot,1)\})$ over $\mathfrak{D}$, we get

$$W_\Omega^\delta(\{(a_1^*,1),(a_2^*,1)\})=\inf_{\mathfrak{D}}W_\Omega^\delta(\{(\cdot,1),(\cdot,1)\}).$$

□

We prove Theorem 2.1.7, that gives the configuration of the vortices that minimizes the renormalized energy in the unit disk $B_1$ in $\mathbb{R}^2$.

*Proof of Theorem 2.1.7.* By Theorem 2.1.5 and Corollary 2.1.6, there exists a pair of distinct points $(a_1^*,a_2^*)\in\partial B_1\times\partial B_1$ that minimizes the renormalized energy

$$W_{B_1}^\delta(\{(a_1,1),(a_2,1)\})=-2\pi\log|a_1-a_2|+2\pi\delta\cdot(a_1^\perp+a_2^\perp)$$

defined for $(a_1,a_2)\in\partial B_1\times\partial B_1$ such that $a_1\neq a_2$. There exist $\varphi_1,\varphi_2\in\mathbb{R}$ such that $a_1^*=e^{i\varphi_1}$ and $a_2^*=e^{i\varphi_2}$. Since we must have $a_1^*\neq a_2^*$, we can assume $\varphi_1-\varphi_2\notin 2\pi\mathbb{Z}$ in the sequel. Computing the renormalized energy above in terms of $\varphi_1$, $\varphi_2$ and $\delta=(\delta_1,\delta_2)$, we deduce that

$$W_{B_1}^\delta(\{(a_1^*,1),(a_2^*,1)\})=2\pi f(\varphi_1,\varphi_2)$$

where

$$f(\varphi_1,\varphi_2)=-\frac{1}{2}\log 2-\frac{1}{2}\log(1-\cos(\varphi_1-\varphi_2))-\delta_1(\sin\varphi_1+\sin\varphi_2)+\delta_2(\cos\varphi_1+\cos\varphi_2)$$

is $C^\infty(\mathbb{R}^2\setminus S)$ where $S=\{(\varphi_1,\varphi_2)\in\mathbb{R}^2:\varphi_1-\varphi_2\in 2\pi\mathbb{Z}\}$. Hence, the problem of minimizing the renormalized energy turns in minimizing $f$.

*Step 1 :* We compute the critical points of $f$.
For every $(\varphi_1, \varphi_2) \in \mathbb{R}^2 \setminus S$,

$$\nabla f(\varphi_1, \varphi_2) = \begin{pmatrix} -\frac{\sin(\varphi_1 - \varphi_2)}{2(1-\cos(\varphi_1-\varphi_2))} - a_1^* \cdot \delta \\ \frac{\sin(\varphi_1 - \varphi_2)}{2(1-\cos(\varphi_1-\varphi_2))} - a_2^* \cdot \delta \end{pmatrix}.$$

It follows that, for $(\varphi_1, \varphi_2) \in \mathbb{R}^2 \setminus S$,

$$\nabla f(\varphi_1, \varphi_2) = 0 \iff \begin{cases} \frac{\sin(\varphi_1 - \varphi_2)}{1-\cos(\varphi_1-\varphi_2)} &= -2a_1^* \cdot \delta, \\ (a_1^* + a_2^*) \cdot \delta &= 0. \end{cases} \tag{2.91}$$

By the second equation in (2.91):

$$\begin{aligned} (a_1^* + a_2^*) \cdot \delta = 0 &\iff \delta_1(\cos\varphi_1 + \cos\varphi_2) + \delta_2(\sin\varphi_1 + \sin\varphi_2) = 0 \\ &\iff 2\cos\left(\frac{\varphi_1 - \varphi_2}{2}\right)\left(\delta_1 \cos\frac{\varphi_1+\varphi_2}{2} + \delta_2 \sin\frac{\varphi_1+\varphi_2}{2}\right) = 0 \\ &\iff \cos\frac{\varphi_1 - \varphi_2}{2} = 0 \quad \text{or} \quad \delta \cdot b = 0 \text{ where } b = e^{i(\varphi_1+\varphi_2)/2} \end{aligned}$$

*Case 1:* $\cos\frac{\varphi_1-\varphi_2}{2} = 0$. Then we have

$$\cos\frac{\varphi_1 - \varphi_2}{2} = 0 \iff \varphi_1 - \varphi_2 = \pi \mod 2\pi,$$

thus $a_1^*$ and $a_2^*$ are diametrically opposed (i.e. $a_2^* = -a_1^*$). By the first equation in (2.91) and since $\delta = |\delta| e^{i\theta}$, we deduce that

$$a_1^* \cdot \delta = 0 \iff \cos(\varphi_1 - \theta) = 0 \iff \varphi_1 = \theta + \frac{\pi}{2} \mod \pi.$$

Hence, $a_1^* = \pm e^{i\varphi_1} = \pm i e^{i\theta} = \pm \frac{1}{|\delta|} i\delta = \pm\frac{1}{|\delta|}\delta^\perp$.

*Case 2:* $\delta \cdot b = 0$ where $b = e^{i(\varphi_1+\varphi_2)/2}$. Since $\delta = |\delta| e^{i\theta}$, we have

$$\delta \cdot b = 0 \iff \delta \perp b \iff \frac{\varphi_1 + \varphi_2}{2} = \theta + \frac{\pi}{2} \mod \pi \iff \varphi_1 + \varphi_2 = 2\theta + \pi \mod 2\pi,$$

thus $a_1^*$ and $a_2^*$ are symmetric with respect to $\delta^\perp$. By the first equation in (2.91), we have

$$\begin{aligned} -2|\delta| a_1^* \cdot e^{i\theta} = \frac{\sin(\varphi_1 - (2\theta + \pi - \varphi_1))}{1 - \cos(\varphi_1 - (2\theta+\pi-\varphi_1))} &\iff -2|\delta|\cos(\varphi_1 - \theta) = \frac{-\sin(\varphi_1-\theta)}{\cos(\varphi_1 - \theta)} \\ &\iff \frac{2|\delta|\sin^2(\varphi_1-\theta) + \sin(\varphi_1 - \theta) - 2|\delta|}{\cos(\varphi_1-\theta)} = 0 \\ &\iff \begin{cases} 2|\delta| X^2 - X + 2|\delta| = 0 \\ X = \sin(\varphi_1 - \theta) \\ \cos(\varphi_1 - \theta) \neq 0 \end{cases} \end{aligned}$$

The solutions of the first equation above are

$$X = \frac{-1 \pm \sqrt{1 + 16|\delta|^2}}{4|\delta|} = \pm\sqrt{1 + \frac{1}{16|\delta|^2}} - \frac{1}{4|\delta|}$$

but the solution for $\pm = -$ cannot satisfy $X = \sin(\varphi_1 - \theta)$ because it is strictly less than $-1$. As a consequence, the second equation gives

$$\sin(\varphi_1 - \theta) = \sqrt{1 + \frac{1}{16\,|\delta|^2}} - \frac{1}{4\,|\delta|} \in [-1, 1],$$

thus $\varphi_1 = \theta + \theta_0$ and $\varphi_2 = 2\theta + \pi - \varphi_1 = \theta + \pi - \theta_0$, where $\theta_0 = \arcsin\left(\sqrt{1 + \frac{1}{16|\delta|^2}} - \frac{1}{4|\delta|}\right)$.

*Step 2 :* We study the nature of the critical points of $f$.
For every $(\varphi_1, \varphi_2) \in \mathbb{R}^2 \setminus S$, the Hessian matrix of $f$ is

$$Hess(f)(\varphi_1, \varphi_2) = \begin{pmatrix} \frac{1}{2(1-\cos(\varphi_1-\varphi_2))^2} + a_1^* \cdot \delta^\perp & \frac{-1}{2(1-\cos(\varphi_1-\varphi_2))^2} \\ \frac{-1}{2(1-\cos(\varphi_1-\varphi_2))^2} & \frac{1}{2(1-\cos(\varphi_1-\varphi_2))^2} + a_2^* \cdot \delta^\perp \end{pmatrix}$$

and its determinant is

$$h(\varphi_1, \varphi_2) = \frac{1}{2(1 - \cos(\varphi_1 - \varphi_2))^2}(a_1^* + a_2^*) \cdot \delta^\perp + (a_1^* \cdot \delta^\perp)(a_2^* \cdot \delta^\perp).$$

– For the case of diametrically opposed points $a_2^* = -a_1^*$ with $a_1^* = \pm\frac{1}{|\delta|}\delta^\perp$, we have

$$h(\varphi_1, \varphi_2) = (a_1^* \cdot \delta^\perp)(a_2^* \cdot \delta^\perp) = -(a_1^* \cdot \delta^\perp)^2 = -|\delta|^2 < 0.$$

Hence, $f(\varphi_1, \varphi_2)$ is a saddle point, but neither a minimum, nor a maximum.
– Consider the case of $a_1^*$ and $a_2^*$ symmetric with respect to $\delta^\perp$, with $\varphi_1 = \theta + \theta_0$ and $\varphi_2 = \theta + \pi - \theta_0$ where $\theta_0$ is given above. Then $h(\varphi_1, \varphi_2) > 0$, because

$$h(\varphi_1, \varphi_2) = \left(1 + \frac{1}{2(1 - \cos(\varphi_1 - \varphi_2))^2}\right)\left(a_1^* \cdot \delta^\perp + a_2^* \cdot \delta^\perp\right)$$

with

$$\begin{aligned} a_1^* \cdot \delta^\perp = e^{i\varphi_1} \cdot |\delta|\, e^{i(\theta+\pi/2)} &= |\delta| \cos\left(\varphi_1 - \theta - \frac{\pi}{2}\right) \\ &= |\delta| \sin(\varphi_1 - \theta) = |\delta| \left(\sqrt{1 + \frac{1}{16}\,|\delta|^2} - \frac{1}{4\,|\delta|}\right) > 0, \end{aligned}$$

and $a_2^* \cdot \delta^\perp = a_1^* \cdot \delta^\perp > 0$, as $a_2^*$ and $a_1^*$ are symmetric with respect to $\delta^\perp$. Moreover, the first component of $Hess(f)(\varphi_1, \varphi_2)$ is positive, hence $(\varphi_1, \varphi_2) = (\theta + \theta_0, \theta + \pi - \theta_0)$ minimizes $f$. ☐

To conclude this section, we prove Corollary 2.1.8.

*Proof of Corollary 2.1.8.* By Lemma 2.2.18, there exists a minimizer $m_\varepsilon$ of $E^\delta_{\varepsilon,\eta}$ on $H^1(\Omega, \mathbb{R}^2)$. By Corollary 2.1.6, there exist two points $a_1^* \neq a_2^* \in \partial\Omega$ such that

$$W^\delta_\Omega(\{(a_1^*, 1), (a_2^*, 1)\}) = \min\left\{W^\delta_\Omega(\{(\tilde{a}_1, 1), (\tilde{a}_2, 1)\}) : \tilde{a}_1 \neq \tilde{a}_2 \in \partial\Omega\right\}. \tag{2.92}$$

By Theorem 2.1.4 applied to $\{(a_1^*, 1), (a_2^*, 1)\}$, the minimizers $m_\varepsilon$ must satisfy

$$E^\delta_{\varepsilon,\eta}(m_\varepsilon) \leqslant 2\pi\,|\log\varepsilon| + W^\delta_\Omega(\{(a_1^*, 1), (a_2^*, 1)\}) + 2\gamma_0 + o(1) \quad \text{as } \varepsilon \to 0. \tag{2.93}$$

By Theorem 2.1.1*(i)*, for a subsequence, $(\mathcal{J}(m_\varepsilon))_{\varepsilon>0}$ converges as in (2.15) to

$$J = -\kappa\mathcal{H}^1 \llcorner \partial\Omega + \pi \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}}$$

for $N \geqslant 2$ distinct boundary points $a_1, ..., a_N \in \partial\Omega$, with $d_1, ..., d_N \in \mathbb{Z}\backslash\{0\}$ such that $\sum_{j=1}^N d_j = 2$. By Theorem 2.1.1*(ii)*, we also have

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} E^\delta_{\varepsilon,\eta}(m_\varepsilon) \geqslant \pi \sum_{j=1}^N |d_j| \,. \tag{2.94}$$

Combining (2.93) and (2.94), we get $\sum_{j=1}^N |d_j| \leqslant 2$, hence $\sum_{j=1}^N (|d_j| - d_j) \leqslant 0$ so that

$$\sum_{j=1}^N |d_j| = \sum_{j=1}^N d_j = 2.$$

By Theorem 2.1.3*(i)*, we deduce that $N = 2$, $d_1 = d_2 = 1$ and

$$E^\delta_{\varepsilon,\eta}(m_\varepsilon) \geqslant 2\pi\,|\log\varepsilon| + W^\delta_\Omega(\{(a_1,1),(a_2,1)\}) + 2\gamma_0 + o(1) \quad \text{as } \varepsilon \to 0. \tag{2.95}$$

Combining (2.93) and (2.95), and letting $\varepsilon \to 0$, we get

$$W^\delta_\Omega(\{(a_1,1),(a_2,1)\}) \leqslant W^\delta_\Omega(\{(a_1^*,1),(a_2^*,1)\}),$$

so by definition of $W^\delta_\Omega(\{(a_1^*,1),(a_2^*,1)\})$, we deduce that $(a_1, a_2)$ is also a minimizer in (2.92). It follows then from (2.93) and (2.95) that

$$E^\delta_{\varepsilon,\eta}(m_\varepsilon) = 2\pi\,|\log\varepsilon| + W^\delta_\Omega(\{(a_1,1),(a_2,1)\}) + 2\gamma_0 + o(1) \quad \text{as } \varepsilon \to 0. \tag{2.96}$$

By Theorem 2.1.3*(v)*, for a subsequence, $(m_\varepsilon)$ converges weakly in $W^{1,q}(\Omega,\mathbb{R}^2)$ for every $q \in [1,2)$, and strongly in $L^p(\Omega,\mathbb{R}^2)$ for every $p \in [1,+\infty)$, to $e^{i\widehat{\varphi}_0}$, where $\widehat{\varphi}_0$ is an extension to $\Omega$ of a function $\varphi_0 \in BV(\partial\Omega, \pi\mathbb{Z})$ that satisfies

$$\partial_\tau \varphi_0 = \kappa \mathcal{H}^1 \llcorner \partial\Omega - \pi(\mathbb{1}_{\{a_1\}} + \mathbb{1}_{\{a_2\}}) \quad \text{as measure on } \partial\Omega$$

and $e^{i\varphi_0} \cdot \nu = 0$ in $\partial\Omega \setminus \{a_1, a_2\}$. It remains to prove that $\widehat{\varphi}_0$ is harmonic in $\Omega$. Let $r > 0$ be small. Using (2.78), we see that $(\nabla m_\varepsilon)$ converges weakly to $\nabla e^{i\widehat{\varphi}_0}$ in $L^2(\Omega \setminus (B_r(a_1) \cup B_r(a_2))$, thus

$$\int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} |\nabla\widehat{\varphi}_0|^2 \mathrm{d}x \leqslant \liminf_{\varepsilon\to 0} \int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} |\nabla m_\varepsilon|^2 \mathrm{d}x,$$

since $|\nabla(e^{i\widehat{\varphi}_0})| = |\nabla\widehat{\varphi}_0|$. Moreover, using that $(m_\varepsilon)$ converges strongly to $e^{i\widehat{\varphi}_0}$ in $L^2(\Omega)$ and $(\nabla m_\varepsilon)$ converges weakly to $\nabla e^{i\widehat{\varphi}_0}$ in $L^2(\Omega \setminus (B_r(a_1) \cup B_r(a_2))$, we get

$$\int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} -\delta \cdot \nabla\widehat{\varphi}_0 \;\mathrm{d}x = \lim_{\varepsilon\to 0} \int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} \delta \cdot \nabla m_\varepsilon \wedge m_\varepsilon \;\mathrm{d}x,$$

since $\nabla(e^{i\widehat{\varphi}_0}) \wedge e^{i\widehat{\varphi}_0} = -\nabla\widehat{\varphi}_0$. From both these observations, we deduce

$$\begin{aligned} &\liminf_{r\to 0} \left( \int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} \left(|\nabla\widehat{\varphi}_0|^2 - 2\delta\cdot\nabla\widehat{\varphi}_0\right) \mathrm{d}x - 2\pi \log\frac{1}{r} \right) \\ &\leqslant \liminf_{r\to 0} \liminf_{\varepsilon\to 0} \left( \int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} \left(|\nabla m_\varepsilon|^2 + 2\delta\cdot\nabla m_\varepsilon \wedge m_\varepsilon\right) \mathrm{d}x - 2\pi\log\frac{1}{r} \right). \end{aligned} \tag{2.97}$$

By Hölder's inequality, as $(\nabla m_\varepsilon)$ converges weakly in $L^{3/2}(\Omega)$, we have

$$\left|\int_{\Omega\cap(B_r(a_1)\cup B_r(a_2))} 2\delta\cdot\nabla m_\varepsilon\wedge m_\varepsilon\ \mathrm{d}x\right| \leqslant 2\,|\delta|\,|\Omega\cap(B_r(a_1)\cup B_r(a_2))|^{1/3}\,\|\nabla m_\varepsilon\|_{L^{3/2}(\Omega)} \leqslant Cr^{2/3},$$

for some constant $C>0$ independent of $\varepsilon$ and $r$, and thus

$$\liminf_{r\to 0}\liminf_{\varepsilon\to 0}\int_{\Omega\cap(B_r(a_1)\cup B_r(a_2))} 2\delta\cdot\nabla m_\varepsilon\wedge m_\varepsilon\ \mathrm{d}x = 0. \tag{2.98}$$

Moreover, by Remark 2.2.17, we have

$$\liminf_{r\to 0}\liminf_{\varepsilon\to 0}\left(E^0_{\varepsilon,\eta}(m_\varepsilon;\Omega\cap(B_r(a_1)\cup B_r(a_2))) - 2\pi\log\frac{r}{\varepsilon} - 2\gamma_0\right) \geqslant 0. \tag{2.99}$$

Note that, for every small $r>0$,

$$\begin{aligned}
&E^\delta_{\varepsilon,\eta}(m_\varepsilon) - 2\pi\,|\log\varepsilon| - 2\gamma_0\\
&\quad= E^0_{\varepsilon,\eta}(m_\varepsilon;\Omega\cap(B_r(a_1)\cup B_r(a_2))) - 2\pi\log\frac{r}{\varepsilon} - 2\gamma_0\\
&\qquad+\int_{\Omega\cap(B_r(a_1)\cup B_r(a_2))} 2\delta\cdot\nabla m_\varepsilon\wedge m_\varepsilon\ \mathrm{d}x\\
&\qquad+E^\delta_{\varepsilon,\eta}(m_\varepsilon;\Omega\setminus(B_r(a_1)\cup B_r(a_2))) - 2\pi\log\frac{1}{r}\\
&\quad\geqslant E^0_{\varepsilon,\eta}(m_\varepsilon;\Omega\cap(B_r(a_1)\cup B_r(a_2))) - 2\pi\log\frac{r}{\varepsilon} - 2\gamma_0\\
&\qquad+\int_{\Omega\cap(B_r(a_1)\cup B_r(a_2))} 2\delta\cdot\nabla m_\varepsilon\wedge m_\varepsilon\ \mathrm{d}x\\
&\qquad+\int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))}\left(|\nabla m_\varepsilon|^2 + 2\delta\cdot\nabla m_\varepsilon\wedge m_\varepsilon\right)\mathrm{d}x - 2\pi\log\frac{1}{r}.
\end{aligned}$$

By (2.97), (2.98) and (2.99), it follows that

$$\begin{aligned}
&\liminf_{r\to 0}\liminf_{\varepsilon\to 0}\left(E^\delta_{\varepsilon,\eta}(m_\varepsilon) - 2\pi\,|\log\varepsilon| - 2\gamma_0\right)\\
&\qquad\geqslant \liminf_{r\to 0}\left(\int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))}\left(|\nabla\widehat{\varphi}_0|^2 - 2\delta\cdot\nabla\widehat{\varphi}_0\right)\mathrm{d}x - 2\pi\log\frac{1}{r}\right),
\end{aligned}$$

i.e., using (2.96),

$$\begin{aligned}
&W^\delta_\Omega(\{(a_1,1),(a_2,1)\})\\
&\qquad\geqslant \liminf_{r\to 0}\left(\int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))}\left(|\nabla\widehat{\varphi}_0|^2 - 2\delta\cdot\nabla\widehat{\varphi}_0\right)\mathrm{d}x - 2\pi\log\frac{1}{r}\right).
\end{aligned} \tag{2.100}$$

Let $\varphi_*$ be the harmonic extension of $\varphi_0$ to $\Omega$. Then $\widehat{\varphi}_0-\varphi_*\in W^{1,q}_0(\Omega)$, for every $q\in[1,2)$. For every small $r>0$,

$$\begin{aligned}
\int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} 2\delta\cdot(\nabla\widehat{\varphi}_0-\nabla\varphi_*)\ \mathrm{d}x &= \int_\Omega 2\delta\cdot\nabla(\widehat{\varphi}_0-\varphi_*)\ \mathrm{d}x\\
&\quad-2\int_{\Omega\cap(B_r(a_1)\cup B_r(a_2))} 2\delta\cdot\nabla(\widehat{\varphi}_0-\varphi_*)\ \mathrm{d}x,
\end{aligned}$$

with

$$\int_\Omega 2\delta \cdot \nabla(\widehat{\varphi}_0 - \varphi_*)\ \mathrm{d}x = \int_{\partial\Omega} 2(\widehat{\varphi}_0 - \varphi_*)\delta \cdot \nu\ \mathrm{d}x = 0,$$

by Green's formula and using that $\widehat{\varphi}_0 = \varphi_*$ on $\partial\Omega$, and

$$\left| -2 \int_{\Omega\cap(B_r(a_1)\cup B_r(a_2))} 2\delta \cdot \nabla(\widehat{\varphi}_0 - \varphi_*)\ \mathrm{d}x \right| \leqslant Cr^{2/3} \left\| \nabla(\widehat{\varphi}_0 - \varphi_*) \right\|_{L^{3/2}(\Omega)} \leqslant Cr^{2/3},$$

for some constant $C > 0$ independent of $\varepsilon$ and $r$, by Hölder's inequality, since $\widehat{\varphi}_0$ and $\varphi_*$ are in $W_0^{1,3/2}(\Omega)$. Hence,

$$\liminf_{r\to 0} \int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} 2\delta \cdot (\nabla\widehat{\varphi}_0 - \nabla\varphi_*)\ \mathrm{d}x = 0.$$

Combining this observation with the definition of $W_\Omega^\delta(\{(a_1,1),(a_2,1)\})$, we deduce that

$$\begin{aligned}
&W_\Omega^\delta(\{(a_1,1),(a_2,1)\}) \\
&\quad \geqslant \liminf_{r\to 0} \int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} \left(|\nabla\widehat{\varphi}_0|^2 - |\nabla\varphi_*|^2\right) \mathrm{d}x \\
&\qquad + \liminf_{r\to 0} \left( \int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} \left(|\nabla\varphi_*|^2 - 2\delta\cdot\nabla\varphi_*\right) \mathrm{d}x - 2\pi \log\frac{1}{r} \right) \\
&\quad = \liminf_{r\to 0} \int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} \left(|\nabla\widehat{\varphi}_0|^2 - |\nabla\varphi_*|^2\right) \mathrm{d}x \\
&\qquad + W_\Omega^\delta(\{(a_1,1),(a_2,1)\}),
\end{aligned}$$

i.e.

$$\chi = \liminf_{r\to 0} \int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} \left(|\nabla\widehat{\varphi}_0|^2 - |\nabla\varphi_*|^2\right) \mathrm{d}x \leqslant 0. \tag{2.101}$$

From the Claim 2.2.21 below, we deduce that $\widehat{\varphi}_0 = \varphi_*$, thus $\widehat{\varphi}_0$ is harmonic in $\Omega$. □

*Claim* 2.2.21. With the notations in the previous proof, if $\varphi_*$ is the harmonic extension of $\varphi_0$ to $\Omega$, then $\widehat{\varphi}_0 = \varphi_*$.

*Proof of Claim 2.2.21.* We have

$$|\nabla\widehat{\varphi}_0|^2 - |\nabla\varphi_*|^2 = |\nabla(\widehat{\varphi}_0 - \varphi_*)|^2 + 2\nabla\varphi_* \cdot \nabla(\widehat{\varphi}_0 - \varphi_*).$$

Since $\varphi_*$ behaves (up to a conformal map as in [24, Lemma 4.3]) as the sum of an angular function around $a_1$ and $a_2$ and a harmonic function $h \in W^{1,p}(\Omega)$ for every $p \in [1,+\infty)$ (as in the proof of Theorem 2.2.16), then using integration by parts, for every $r > 0$ small enough,

$$\begin{aligned}
\int_{\Omega\setminus(B_r(a_1)\cup B_r(a_2))} \nabla\varphi_* \cdot \nabla(\widehat{\varphi}_0 - \varphi_*)\ \mathrm{d}x &= \int_{\partial(\Omega\setminus(B_r(a_1)\cup B_r(a_2)))} \partial_\nu\varphi_*(\widehat{\varphi}_0 - \varphi_*)\ \mathrm{d}\mathcal{H}^1 \\
&= \int_{\Omega\cap\partial(B_r(a_1)\cup B_r(a_2))} \partial_\nu h(\widehat{\varphi}_0 - \varphi_*)\ \mathrm{d}\mathcal{H}^1 \\
&= \int_{\partial(\Omega\cap(B_r(a_1)\cup B_r(a_2)))} \partial_\nu h(\widehat{\varphi}_0 - \varphi_*)\ \mathrm{d}\mathcal{H}^1 \\
&= \int_{\Omega\cap(B_r(a_1)\cup B_r(a_2))} \nabla h \cdot \nabla(\widehat{\varphi}_0 - \varphi_*)\ \mathrm{d}x.
\end{aligned}$$

By Hölder's inequality, we deduce that

$$
\begin{aligned}
&\left| \int_{\Omega \setminus (B_r(a_1) \cup B_r(a_2))} \nabla \varphi_* \cdot \nabla (\widehat{\varphi}_0 - \varphi_*) \, \mathrm{d}x \right| \\
&\qquad \leqslant \|\nabla h\|_{L^3(\Omega \cap (B_r(a_1) \cup B_r(a_2)))} \|\nabla (\widehat{\varphi}_0 - \varphi_*)\|_{L^{3/2}(\Omega \cap (B_r(a_1) \cup B_r(a_2)))} \\
&\qquad \leqslant Cr \|\nabla h\|_{L^3(\Omega)} \|\nabla (\widehat{\varphi}_0 - \varphi_*)\|_{L^{3/2}(\Omega)},
\end{aligned}
$$

for some constant $C > 0$ independent of $\varepsilon$ and $r$, since $h \in W^{1,3}(\Omega)$ and $\widehat{\varphi}_0$, $\varphi_* \in W_0^{3/2}(\Omega)$. As a consequence,

$$
\lim_{r \to 0} \int_{\Omega \setminus (B_r(a_1) \cup B_r(a_2))} \nabla \varphi_* \cdot \nabla (\widehat{\varphi}_0 - \varphi_*) \, \mathrm{d}x = 0,
$$

and

$$
0 \geqslant \chi = \liminf_{r \to 0} \int_{\Omega \setminus (B_r(a_1) \cup B_r(a_2))} |\nabla (\widehat{\varphi}_0 - \varphi_*)|^2 \mathrm{d}x.
$$

We deduce that $\chi = 0$ and $\widehat{\varphi}_0 = \varphi_* + c$ for some constant $c \in \mathbb{R}$, but $\widehat{\varphi}_0 = \varphi_*$ on $\partial\Omega$, hence $c = 0$ and $\widehat{\varphi}_0 = \varphi_*$ is harmonic in $\Omega$. □

## 2.3 Three-dimensional model for maps $m_h \colon \Omega_h \subset \mathbb{R}^3 \to \mathbb{S}^2$

### 2.3.1 Reduction from the three-dimensional model to a reduced two-dimensional model

We begin with stating a lemma that enables us to get rid of the Dzyaloshinskii-Moriya interaction term. More precisely, under the assumption that $\mathcal{E}_h(m_h)$ is uniformly bounded, then it is also the case for $\mathcal{E}_h^0(m_h)$. This latter energy has been studied by Ignat and Kurzke [25], hence this lemma allows us to use the statements they obtained on $\mathcal{E}_h^0(m_h)$.

**Lemma 2.3.1.** *Let $\Omega_h = \Omega \times (0, h)$ with $\Omega \subset \mathbb{R}^2$ a bounded, simply connected and $C^{1,1}$ smooth domain. Assume that $\frac{1}{\eta^2}|\widehat{D}| = O_h(1)$ and consider a family of magnetizations $\{m_h \colon \Omega_h \to \mathbb{S}^2\}$ that satisfies*

$$
\limsup_{h \to 0} \mathcal{E}_h(m_h) < +\infty.
$$

*Then, we have $\limsup_{h \to 0} \mathcal{E}_h^0(m_h) < +\infty$.*

*Proof.* Assume that there exists a constant $C > 0$ such that $\sup_{h>0} \mathcal{E}_h(m_h) \leqslant C$. Then, for $h > 0$ sufficiently small,

$$
\begin{aligned}
|\log \varepsilon| \left| \mathcal{E}_h(m_h) - \mathcal{E}_h^0(m_h) \right| &= \frac{1}{h\eta^2} \left| \int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \, \mathrm{d}x \right| \\
&\leqslant \frac{1}{h\eta^2} \int_{\Omega_h} |\widehat{D}| \, |\nabla m_h| \, |m_h| \, \mathrm{d}x \\
&= \frac{1}{h\eta^2} \int_{\Omega_h} |\widehat{D}| \, |\nabla m_h| \, \mathrm{d}x \\
&\leqslant \frac{1}{2h} \int_{\Omega_h} \left( |\nabla m_h|^2 + \left( \frac{|\widehat{D}|}{\eta^2} \right)^2 \right) \mathrm{d}x \\
&= \frac{1}{2} \mathcal{E}_h^0(m_h) + \frac{|\Omega|}{2} \left( \frac{|\widehat{D}|}{\eta^2} \right)^2
\end{aligned}
$$

using $|m_h| = 1$ and Young's inequality. Dividing by $|\log\varepsilon|$, we deduce that, for $h > 0$ sufficiently small,

$$\left|\mathcal{E}_h(m_h) - \mathcal{E}_h^0(m_h)\right| \leqslant \frac{1}{2}\mathcal{E}_h^0(m_h) + \frac{C'}{|\log\varepsilon|}\left(\frac{|\widehat{D}|}{\eta^2}\right)^2$$

where $C' > 0$ depends only on $\Omega$. Hence, for $h > 0$ sufficiently small,

$$\mathcal{E}_h^0(m_h) \leqslant 2\left(\mathcal{E}_h(m_h) + \frac{C'}{|\log\varepsilon|}\left(\frac{|\widehat{D}|}{\eta^2}\right)^2\right) \leqslant C'',$$

because $\mathcal{E}_h(m_h) \leqslant C$, $\frac{|\widehat{D}|}{\eta^2} = O(1)$ and $\varepsilon = \varepsilon(h) \to 0$ as $h \to 0$. $\square$

**Lemma 2.3.2.** *Let $\Omega_h = \Omega \times (0,h)$ with $\Omega \subset \mathbb{R}^2$ a bounded, simply connected and $C^{1,1}$ smooth domain. In the regime (2.6), consider a family of magnetizations $\{m_h \colon \Omega_h \to \mathbb{S}^2\}$ that satisfies*

$$\limsup_{h\to 0} \mathcal{E}_h(m_h) < +\infty.$$

*Then, we have*

$$\frac{1}{|\log\varepsilon|}\frac{1}{h\eta^2}\int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \,\mathrm{d}x = \frac{1}{|\log\varepsilon|}\frac{1}{\eta^2}\int_{\Omega} \widehat{D}' : \nabla'\overline{m}_h \wedge \overline{m}_h \,\mathrm{d}x' + O(R_2(h)) \quad \text{as } h \to 0, \tag{2.102}$$

*where*

$$R_2(h) = h\frac{|\widehat{D}_1| + |\widehat{D}_2|}{\eta^2} + \frac{|\widehat{D}_3|}{\eta^2}. \tag{2.103}$$

*Proof.* Let $h > 0$ be small. By observing that

$$\widehat{D} : \nabla m_h \wedge m_h = \widehat{D}' : \nabla' m_h \wedge \overline{m}_h + \widehat{D}' : \nabla' m_h \wedge (m_h - \overline{m}_h) + \widehat{D}_3 \cdot \partial_3 m_h \wedge m_h,$$

we use the decomposition

$$\int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \,\mathrm{d}x = I_1 + I_2 + I_3 \tag{2.104}$$

with

$$I_1 = \int_{\Omega_h} \widehat{D}' : \nabla' m_h \wedge \overline{m}_h \,\mathrm{d}x = \sum_{j=1}^{2}\int_{\Omega_h} \widehat{D}_j \cdot \partial_j m_h \wedge \overline{m}_h \,\mathrm{d}x,$$

$$I_2 = \int_{\Omega_h} \widehat{D}' : \nabla' m_h \wedge (m_h - \overline{m}_h) \,\mathrm{d}x = \sum_{j=1}^{2}\int_{\Omega_h} \widehat{D}_j \cdot \partial_j m_h \wedge (m_h - \overline{m}_h) \,\mathrm{d}x,$$

and

$$I_3 = \int_{\Omega_h} \widehat{D}_3 \cdot \partial_3 m_h \wedge m_h \,\mathrm{d}x.$$

For calculating $I_1$, we use Fubini's theorem and the fact that $j \in \{1,2\}$, to get

$$
\begin{aligned}
I_1 = \sum_{j=1}^{2} \int_{\Omega_h} \widehat{D}_j \cdot \partial_j m_h \wedge \overline{m}_h \, \mathrm{d}x &= \sum_{j=1}^{2} \int_{\Omega} \int_0^h \left( \widehat{D}_j \cdot \partial_j m_h(x', x_3) \wedge \overline{m}_h(x') \right) \mathrm{d}x_3 \mathrm{d}x' \\
&= \sum_{j=1}^{2} \int_{\Omega} \widehat{D}_j \cdot \left( \int_0^h \partial_j \overline{m}_h(x', x_3) \mathrm{d}x_3 \right) \wedge \overline{m}_h(x') \mathrm{d}x' \\
&= h \sum_{j=1}^{2} \int_{\Omega} \widehat{D}_j \cdot \partial_j \left( \frac{1}{h} \int_0^h \overline{m}_h(x', x_3) \mathrm{d}x_3 \right) \wedge \overline{m}_h(x') \mathrm{d}x' \\
&= h \sum_{j=1}^{2} \int_{\Omega} \widehat{D}_j \cdot \partial_j \overline{m}_h \wedge \overline{m}_h \, \mathrm{d}x',
\end{aligned}
$$

i.e.

$$
I_1 = h \int_{\Omega} \widehat{D}' : \nabla' \overline{m}_h \wedge \overline{m}_h \, \mathrm{d}x'. \tag{2.105}
$$

Besides, by Cauchy-Schwarz inequality,

$$
\begin{aligned}
|I_2| &\leqslant \sum_{j=1}^{2} \int_{\Omega_h} \left| \widehat{D}_j \cdot \partial_j m_h \wedge (m_h - \overline{m}_h) \right| \mathrm{d}x \\
&\leqslant \sum_{j=1}^{2} |\widehat{D}_j| \left( \int_{\Omega_h} |\partial_j m_h|^2 \, \mathrm{d}x \right)^{1/2} \left( \int_{\Omega_h} |m_h - \overline{m}_h|^2 \, \mathrm{d}x \right)^{1/2}.
\end{aligned}
$$

Using Poincaré-Wirtinger inequality for $m_h = (m_{h,1}, m_{h,2}, m_{h,3})$, we have

$$
\begin{aligned}
\int_{\Omega_h} |m_h - \overline{m}_h|^2 \, \mathrm{d}x &= \sum_{k=1}^{3} \int_{\Omega_h} |m_{h,k} - \overline{m}_{h,k}|^2 \, \mathrm{d}x \\
&= \sum_{k=1}^{3} \int_{\Omega} \left( \int_0^h |m_{h,k}(x', x_3) - \overline{m}_{h,k}(x')|^2 \, \mathrm{d}x_3 \right) \mathrm{d}x' \\
&\leqslant Ch^2 \sum_{k=1}^{3} \int_{\Omega} \int_0^h |\partial_3 m_{h,k}(x', x_3)|^2 \, \mathrm{d}x_3 \mathrm{d}x' \\
&= Ch^2 \int_{\Omega_h} |\partial_3 m_h|^2 \, \mathrm{d}x
\end{aligned}
$$

for some constant $C > 0$ independent of $h$. We deduce that

$$
|I_2| \leqslant Ch \sum_{j=1}^{2} |\widehat{D}_j| \left( \int_{\Omega_h} |\partial_j m_h|^2 \, \mathrm{d}x \right)^{1/2} \left( \int_{\Omega_h} |\partial_3 m_h|^2 \, \mathrm{d}x \right)^{1/2} \leqslant Ch \sum_{j=1}^{2} |\widehat{D}_j| \int_{\Omega_h} |\nabla m_h|^2 \, \mathrm{d}x,
$$

thus

$$
|I_2| \leqslant Ch^2 \left( |\widehat{D}_1| + |\widehat{D}_2| \right) |\log \varepsilon| \, \mathcal{E}_h^0(m_h) \tag{2.106}
$$

with $C > 0$ independent of $h$. Finally, since $|m_h| = 1$ and using Cauchy-Schwarz inequality, we have

$$
|I_3| \leqslant \int_{\Omega_h} |\widehat{D}_3 \cdot \partial_3 m_h \wedge m_h| \mathrm{d}x \leqslant |\widehat{D}_3| \int_{\Omega_h} |\partial_3 m_h| \, \mathrm{d}x \leqslant C |\widehat{D}_3| \sqrt{h} \left( \int_{\Omega_h} |\partial_3 m_h|^2 \, \mathrm{d}x \right)^{1/2},
$$

but $\int_{\Omega_h} |\partial_3 m_h|^2 \,\mathrm{d}x \leqslant \int_{\Omega_h} |\nabla m_h|^2 \,\mathrm{d}x \leqslant h\,|\log\varepsilon|\,\mathcal{E}_h^0(m_h)$, thus

$$|I_3| \leqslant C|\widehat{D}_3| h\sqrt{|\log\varepsilon|\,\mathcal{E}_h^0(m_h)}. \tag{2.107}$$

Dividing by $|\log\varepsilon|\,h\eta^2$, the three above estimates (2.105), (2.106) and (2.107) combined with (2.104) lead to

$$\begin{aligned}\frac{1}{|\log\varepsilon|}\frac{1}{h\eta^2}\int_{\Omega_h}\widehat{D}:\nabla m_h\wedge m_h\,\mathrm{d}x \geqslant \frac{1}{|\log\varepsilon|}\frac{1}{\eta^2}\int_{\Omega}\widehat{D}':\nabla'\overline{m}_h\wedge\overline{m}_h\,\mathrm{d}x' \\ - Ch\frac{|\widehat{D}_1|+|\widehat{D}_2|}{\eta^2}\mathcal{E}_h^0(m_h) - C\frac{|\widehat{D}_3|}{\eta^2}\sqrt{\frac{\mathcal{E}_h^0(m_h)}{|\log\varepsilon|}}.\end{aligned}$$

Since $\limsup\limits_{h\to 0}\mathcal{E}_h(m_h) < +\infty$, it follows by Lemma 2.3.1 that $\mathcal{E}_h^0(m_h) \leqslant C$ for $h > 0$ sufficiently small. Hence, we deduce (2.102) as $h \to 0$. □

*Remark* 2.3.3. In the sequel, we consider the quantity

$$R_1(h) = \frac{h}{\eta^2}\left(1 + \frac{\log\frac{\eta^2}{h}}{|\log\varepsilon|}\right). \tag{2.108}$$

Using the relation $\varepsilon = \frac{\eta^2}{h|\log h|}$, we deduce

$$R_1(h) = \frac{1}{\varepsilon\,|\log h|}\left(1 + \frac{\log(\varepsilon\,|\log h|)}{|\log\varepsilon|}\right) = \frac{1}{\varepsilon\,|\log\varepsilon|}\frac{\log|\log h|}{|\log h|}.$$

As a consequence, in the regime (2.7), and also in the regime (2.6), we have $R_1(h) = o(1)$. Furthermore, in the regime (2.7), we have $R_1(h) = o\left(\frac{1}{|\log\varepsilon|}\right)$

The following lemma is a new version of [25, Lemma 15] for $\mathcal{E}_h(m_h)$, that takes into account the Dzyaloshinskii-Moriya interaction.

**Lemma 2.3.4.** *Let $\Omega_h = \Omega \times (0,h)$ with $\Omega \subset \mathbb{R}^2$ a bounded, simply connected and $C^{1,1}$ smooth domain. In the regime* (2.6), *consider a family of magnetizations $\{m_h\colon \Omega_h \to \mathbb{S}^2\}$ that satisfies*

$$\limsup_{h\to 0}\mathcal{E}_h(m_h) < +\infty.$$

*Then, we have*

$$\mathcal{E}_h(m_h) \geqslant \overline{\mathcal{E}}_h(\overline{m}_h) - \left(\overline{\mathcal{E}}_h(\overline{m}_h) + 1\right) O(R(h)) \quad \text{as } h \to 0, \tag{2.109}$$

*where*

$$R(h) = R_1(h) + R_2(h) = \frac{h}{\eta^2}\left(1 + \frac{\log\frac{\eta^2}{h}}{|\log\varepsilon|}\right) + h\frac{|\widehat{D}_1|+|\widehat{D}_2|}{\eta^2} + \frac{|\widehat{D}_3|}{\eta^2}. \tag{2.110}$$

*Moreover, there exists a constant $C > 0$ such that, for $h > 0$ sufficiently small,*

$$\overline{\mathcal{E}}_h^0(\overline{m}_h) \leqslant \overline{\mathcal{E}}_h(\overline{m}_h) + C. \tag{2.111}$$

*Proof.* We divide the proof in two steps. Firstly, in the spirit of [25, Lemma 15], we show that

$$\mathcal{E}_h(m_h) \geqslant \overline{\mathcal{E}}_h(\overline{m}_h) - \left(\overline{\mathcal{E}}_h^0(\overline{m}_h) + 1\right) O(R(h)) \quad \text{as } h \to 0. \tag{2.112}$$

Secondly, in order to prove the next Theorem 2.1.10, we need to have $\overline{\mathcal{E}}_h(\overline{m}_h)$ instead of $\overline{\mathcal{E}}_h^0(\overline{m}_h)$ in the parentheses of the above inequality. To do so, we show that for small $h > 0$, (2.111) holds. Finally, combining (2.111) and (2.112), we deduce the expected estimate (2.109).

*Step 1:* Let us prove (2.112).

For every $h > 0$,

$$\mathcal{E}_h(m_h) = \mathcal{E}_h^0(m_h) + \frac{1}{|\log\varepsilon|}\frac{1}{h\eta^2}\int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \,\mathrm{d}x.$$

By Lemma 2.3.2, for small $h > 0$,

$$\mathcal{E}_h(m_h) \geqslant \mathcal{E}_h^0(m_h) + \frac{1}{|\log\varepsilon|}\frac{1}{\eta^2}\int_{\Omega} \widehat{D}' : \nabla'\overline{m}_h \wedge \overline{m}_h \,\mathrm{d}x' - CR_2(h) \tag{2.113}$$

for some constant $C > 0$. In [25, Lemma 15], Ignat and Kurzke proved that

$$\mathcal{E}_h^0(m_h) \geqslant \overline{\mathcal{E}}_h^0(\overline{m}_h) - \left(\overline{\mathcal{E}}_h^0(\overline{m}_h) + \sqrt{\frac{\mathcal{E}_h^0(m_h)}{|\log\varepsilon|}}\right) O(R_1(h)) \quad \text{as } h \to 0.$$

Since $\limsup_{h\to 0} \mathcal{E}_h(m_h) < +\infty$, it follows by Lemma 2.3.1 that $\mathcal{E}_h^0(m_h) \leqslant C$ for $h > 0$ sufficiently small. Hence, the above inequality becomes

$$\mathcal{E}_h^0(m_h) \geqslant \overline{\mathcal{E}}_h^0(\overline{m}_h) - \left(\overline{\mathcal{E}}_h^0(\overline{m}_h) + 1\right) O(R_1(h)) \quad \text{as } h \to 0. \tag{2.114}$$

Combining (2.113) and (2.114), by definition of $\overline{\mathcal{E}}_h(\overline{m}_h)$ and by positivity of $\overline{\mathcal{E}}_h^0(\overline{m}_h)$, $R_1(h)$ and $R_2(h)$, we get (2.112).

*Step 2:* Let us prove (2.111).

For every $h > 0$,

$$\overline{\mathcal{E}}_h^0(\overline{m}_h) = \overline{\mathcal{E}}_h(\overline{m}_h) - \frac{1}{|\log\varepsilon|}\frac{1}{\eta^2}\int_{\Omega} \widehat{D}' : \nabla'\overline{m}_h \wedge \overline{m}_h \,\mathrm{d}x'.$$

Using Fubini's theorem as for $I_1$ in the proof of Lemma 2.3.4, we see that, for every $h > 0$,

$$\left|\frac{1}{\eta^2}\int_{\Omega} \widehat{D}' : \nabla'\overline{m}_h \wedge \overline{m}_h \,\mathrm{d}x'\right| = \left|\frac{1}{h\eta^2}\int_{\Omega_h} \widehat{D}' : \nabla' m_h \wedge \overline{m}_h \,\mathrm{d}x\right|.$$

Using $|\overline{m}_h| \leqslant 1$ and Young's inequality, we then get, for every $h > 0$,

$$\begin{aligned}\left|\frac{1}{\eta^2}\int_{\Omega} \widehat{D}' : \nabla'\overline{m}_h \wedge \overline{m}_h \,\mathrm{d}x'\right| &\leqslant \frac{1}{h\eta^2}\int_{\Omega_h} |\widehat{D}'|\,|\nabla' m_h|\,|\overline{m}_h|\,\mathrm{d}x \\ &\leqslant C\left(\frac{1}{h}\int_{\Omega_h} |\nabla' m_h|^2\,\mathrm{d}x + \left(\frac{|\widehat{D}'|}{\eta^2}\right)^2\right) \\ &\leqslant C\,|\log\varepsilon|\,\mathcal{E}_h^0(m_h) + C\left(\frac{|\widehat{D}'|}{\eta^2}\right)^2\end{aligned}$$

for some constant $C > 0$. It follows that

$$\overline{\mathcal{E}}_h^0(\overline{m}_h) \leqslant \overline{\mathcal{E}}_h(\overline{m}_h) + C\mathcal{E}_h^0(m_h) + \frac{C}{|\log\varepsilon|}\left(\frac{|\widehat{D}'|}{\eta^2}\right)^2.$$

Once again by Lemma 2.3.1, $\mathcal{E}_h^0(m_h) \leqslant C$ for $h > 0$ sufficiently small. Moreover, $\frac{|\widehat{D'}|}{\eta^2} = O(1)$ in the regime (2.6). As a consequence, for $h > 0$ sufficiently small,

$$\overline{\mathcal{E}}_h^0(\overline{m}_h) \leqslant \overline{\mathcal{E}}_h(\overline{m}_h) + C + \frac{C}{|\log \varepsilon|} = \overline{\mathcal{E}}_h(\overline{m}_h) + C$$

which is (2.111). □

We are now in position to prove Theorem 2.1.10, that links the general three-dimensional $\mathcal{E}_h(m_h)$ to $\overline{\mathcal{E}}_h(\overline{m}_h)$. This is the first step in the dimension reduction from three-dimensional quantities to two-dimensional quantities, as $\overline{m}_h$ takes values in $\Omega \subset \mathbb{R}^2$.

*Proof of Theorem 2.1.10.* Assume that there exists a constant $C > 0$ such that $\sup\limits_{h>0} \mathcal{E}_h(m_h) \leqslant C$. Applying Lemma 2.3.4, we deduce that, for $h > 0$ sufficiently small,

$$C \geqslant \mathcal{E}_h(m_h) \geqslant \overline{\mathcal{E}}_h(\overline{m}_h) - C' \left(\overline{\mathcal{E}}_h(\overline{m}_h) + 1\right) R(h)$$

for some constant $C' > 0$. By Remark 2.3.3 and by (2.103), we see that $R(h) = o(1)$ in the regime (2.6), so that we necessarily have

$$\limsup_{h \to 0} \overline{\mathcal{E}}_h(\overline{m}_h) \leqslant C'',$$

for some constant $C'' > 0$. Hence,

$$\mathcal{E}_h(m_h) \geqslant \overline{\mathcal{E}}_h(\overline{m}_h) - C'(C''+1)R(h) = \overline{\mathcal{E}}_h(\overline{m}_h) - o(1) \quad \text{as } h \to 0,$$

as expected. Moreover, in the regime (2.8), we have $R(h) = o\left(\frac{1}{|\log \varepsilon|}\right)$, so that

$$\mathcal{E}_h(m_h) \geqslant \overline{\mathcal{E}}_h(\overline{m}_h) - C'(C''+1)R(h) = \overline{\mathcal{E}}_h(\overline{m}_h) - o\left(\frac{1}{|\log \varepsilon|}\right) \quad \text{as } h \to 0,$$

with the same reasoning as above. Finally, if $m_h$ is independent of $x_3$, we see by Remark 2.1.9 that the terms of exchange energy and DMI energy in $\mathcal{E}_h(m_h)$, and $\overline{\mathcal{E}}_h(\overline{m}_h)$ respectively, are equal. By [25, Lemmas 15 and 16], the remaining term in $|\mathcal{E}_h(m_h) - \overline{\mathcal{E}}_h(\overline{m}_h)|$ is estimated, for $h > 0$ sufficiently small, as:

$$\begin{aligned} &\frac{1}{|\log \varepsilon|} \left| \frac{1}{h\eta^2} \int_{\mathbb{R}^3} |\nabla u_h|^2 \, \mathrm{d}x - \frac{1}{\eta^2} \int_\Omega \left(1 - |\overline{m}_h'|^2\right) \mathrm{d}x' - \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (\overline{m}_h' \cdot \nu')^2 \, \mathrm{d}\mathcal{H}^1 \right| \\ &\qquad \leqslant C \left( \overline{\mathcal{E}}_h^0(\overline{m}_h) + \sqrt{\frac{\mathcal{E}_h^0(m_h)}{|\log \varepsilon|}} \right) R_1(h), \end{aligned}$$

where $u_h \colon \mathbb{R}^3 \to \mathbb{R}$ denotes the stray field potential associated to $m_h$ and $\nu'$ is the outer unit normal vector on $\partial\Omega$. By Lemma 2.3.1, for $h > 0$ sufficiently small, $\mathcal{E}_h^0(m_h) \leqslant C$. As

$$\limsup_{h \to 0} \overline{\mathcal{E}}_h(\overline{m}_h) \leqslant C''$$

in the regime (2.6) (and thus also in the regime (2.8)), then by (2.111) in Lemma 2.3.4, we get $\overline{\mathcal{E}}_h^0(\overline{m}_h) \leqslant C$ for $h > 0$ sufficiently small. We deduce that, for $h > 0$ sufficiently small,

$$\begin{aligned} &\frac{1}{|\log \varepsilon|} \left| \frac{1}{h\eta^2} \int_{\mathbb{R}^3} |\nabla u_h|^2 \, \mathrm{d}x - \frac{1}{\eta^2} \int_\Omega \left(1 - |\overline{m}_h'|^2\right) \mathrm{d}x' - \frac{1}{2\pi\varepsilon} \int_{\partial\Omega} (\overline{m}_h' \cdot \nu')^2 \, \mathrm{d}\mathcal{H}^1 \right| \\ &\qquad \leqslant C \left( C + \sqrt{\frac{C}{|\log \varepsilon|}} \right) R_1(h) \leqslant C R_1(h). \end{aligned}$$

As a consequence, $\mathcal{E}_h(m_h) = \overline{\mathcal{E}}_h(\overline{m}_h) - O(R_1(h))$ as $h \to 0$, and both expected estimates follow from the behaviour of $R_1(h)$ in regimes (2.6) and (2.8) respectively (see Remark 2.3.3). □

The following proposition reduces the Dzyaloshinskii-Moriya interaction density $\widehat{D} \in \mathbb{R}^{3\times 3}$ to the two-dimensional vector $\delta \in \mathbb{R}^2$, and cancels the third component of the averaged magnetization.

**Proposition 2.3.5.** *Let $\Omega_h = \Omega\times(0,h)$ with $\Omega \subset \mathbb{R}^2$ a bounded, simply connected and $C^{1,1}$ smooth domain. In the regime (2.6), consider a family of magnetizations $\{m_h\colon \Omega_h \to \mathbb{S}^2\}$ that satisfies*

$$\limsup_{h\to 0} \mathcal{E}_h(m_h) < +\infty.$$

*Then, we have*

$$\overline{\mathcal{E}}_h(\overline{m}_h) \geqslant \frac{1}{|\log\varepsilon|} E^{\delta}_{\varepsilon,\eta}(\overline{m}'_h) - o(1) \quad \text{as } h \to 0.$$

*Moreover, in the more restrictive regime (2.8), we have*

$$\overline{\mathcal{E}}_h(\overline{m}_h) \geqslant \frac{1}{|\log\varepsilon|} E^{\delta}_{\varepsilon,\eta}(\overline{m}'_h) - o\left(\frac{1}{|\log\varepsilon|}\right) \quad \text{as } h \to 0.$$

*Furthermore, if $m_h = (m'_h, 0)$, then in the regime (2.6), we have*

$$\overline{\mathcal{E}}_h(\overline{m}_h) = \frac{1}{|\log\varepsilon|} E^{\delta}_{\varepsilon,\eta}(\overline{m}'_h) - o(1) \quad \text{as } h \to 0,$$

*and in the regime (2.8), we have*

$$\overline{\mathcal{E}}_h(\overline{m}_h) = \frac{1}{|\log\varepsilon|} E^{\delta}_{\varepsilon,\eta}(\overline{m}'_h) - o\left(\frac{1}{|\log\varepsilon|}\right) \quad \text{as } h \to 0.$$

*Proof.* Let a small $h > 0$ be fixed for a moment. Note that the boundary penalty terms in $\overline{\mathcal{E}}_h(\overline{m}_h)$, and $\frac{1}{|\log\varepsilon|} E^{\delta}_{\varepsilon,\eta}(\overline{m}'_h)$ respectively, are equal. We then get

$$\begin{aligned} |\log\varepsilon|\,\overline{\mathcal{E}}_h(\overline{m}_h) - E^{\delta}_{\varepsilon,\eta}(\overline{m}'_h) &= \int_\Omega \left(|\nabla'\overline{m}_h|^2 - |\nabla'\overline{m}'_h|^2\right) \mathrm{d}x \\ &+ \int_\Omega \left(\frac{1}{\eta^2}\widehat{D}' : \nabla'\overline{m}_h \wedge \overline{m}_h - 2\delta\cdot\nabla'\overline{m}'_h \wedge \overline{m}'_h\right) \mathrm{d}x \qquad (2.115) \\ &+ \frac{1}{\eta^2}\int_\Omega \left((1-|\overline{m}'_h|^2) - (1-|\overline{m}'_h|^2)^2\right) \mathrm{d}x. \end{aligned}$$

Note that $|\nabla'\overline{m}_h|^2 - |\nabla'\overline{m}'_h|^2 = |\nabla'\overline{m}_{h,3}|^2$ and that, since $|\overline{m}'_h| \leqslant 1$, we have $(1-|\overline{m}'_h|^2)^2 \leqslant 1-|\overline{m}'_h|^2$. We deduce that

$$\begin{aligned} |\log\varepsilon|\,\overline{\mathcal{E}}_h(\overline{m}_h) - E^{\delta}_{\varepsilon,\eta}(\overline{m}'_h) &\geqslant \int_\Omega |\nabla'\overline{m}_{h,3}|^2 \,\mathrm{d}x \\ &+ \int_\Omega \left(\frac{1}{\eta^2}\widehat{D}' : \nabla'\overline{m}_h \wedge \overline{m}_h - 2\delta\cdot\nabla'\overline{m}'_h \wedge \overline{m}'_h\right) \mathrm{d}x. \end{aligned}$$

The next step consists in absorbing the DMI terms containing $\widehat{D}_{11}$, $\widehat{D}_{12}$, $\widehat{D}_{21}$ and $\widehat{D}_{22}$ inside the positive Dirichlet term $\int_\Omega |\nabla'\overline{m}_{h,3}|^2 \,\mathrm{d}x$. The scalar component of the DMI containing $\widehat{D}_{11}$ is

$$\frac{\widehat{D}_{11}}{\eta^2}\int_\Omega (\overline{m}_{h,3}\partial_1\overline{m}_{h,2} - \overline{m}_{h,2}\partial_1\overline{m}_{h,3})\,\mathrm{d}x' = \frac{\widehat{D}_{11}}{\eta^2}\left(-2\int_\Omega \overline{m}_{h,2}\partial_1\overline{m}_{h,3}\,\mathrm{d}x' + \int_{\partial\Omega} \overline{m}_{h,2}\overline{m}_{h,3}\nu'_1\,\mathrm{d}\mathcal{H}^1\right).$$

By Young's inequality and using $|\overline{m}_h| \leqslant 1$, we deduce that

$$\left|\frac{\widehat{D}_{11}}{\eta^2}\int_\Omega (\overline{m}_{h,3}\partial_1\overline{m}_{h,2} - \overline{m}_{h,2}\partial_1\overline{m}_{h,3})\,\mathrm{d}x'\right| \leqslant \frac{|\widehat{D}_{11}|}{\eta^2}\left(\int_\Omega |\nabla'\overline{m}_{h,3}|^2\,\mathrm{d}x' + |\Omega| + |\partial\Omega|\right).$$

Similar inequalities happen for the DMI terms involving $\widehat{D}_{12}$, $\widehat{D}_{21}$ and $\widehat{D}_{22}$, i.e. the terms involving $\overline{m}_{h,3}$. We also observe for the terms involving $\widehat{D}_{13}$ and $\widehat{D}_{23}$ that

$$\frac{\widehat{D}_{13}}{\eta^2}\int_\Omega (\overline{m}_{h,2}\partial_1\overline{m}_{h,1} - \overline{m}_{h,1}\partial_1\overline{m}_{h,2})\,\mathrm{d}x' = \frac{\widehat{D}_{13}}{\eta^2}\int_\Omega \partial_1\overline{m}'_h \wedge \overline{m}'_h\,\mathrm{d}x',$$

with the analogous relation for $\widehat{D}_{23}$. We deduce that

$$\begin{aligned}|\log\varepsilon|\,\overline{\mathcal{E}}_h(\overline{m}_h) - E^\delta_{\varepsilon,\eta}(\overline{m}'_h) \geqslant{}& \left(1 - \frac{|\widehat{D}_{11}| + |\widehat{D}_{12}| + |\widehat{D}_{21}| + |\widehat{D}_{22}|}{\eta^2}\right)\int_\Omega |\nabla'\overline{m}_{h,3}|^2\,\mathrm{d}x' \\ &- \frac{|\widehat{D}_{11}| + |\widehat{D}_{12}| + |\widehat{D}_{21}| + |\widehat{D}_{22}|}{\eta^2}\left(|\Omega| + |\partial\Omega|\right) \\ &+ \left(\frac{\widehat{D}_{13}}{\eta^2} - 2\delta_1\right)\int_\Omega \partial_1\overline{m}'_h \wedge \overline{m}'_h\,\mathrm{d}x' \\ &+ \left(\frac{\widehat{D}_{23}}{\eta^2} - 2\delta_2\right)\int_\Omega \partial_2\overline{m}'_h \wedge \overline{m}'_h\,\mathrm{d}x'.\end{aligned}$$

In the regime (2.6), we have $\frac{|\widehat{D}_{11}| + |\widehat{D}_{12}| + |\widehat{D}_{21}| + |\widehat{D}_{22}|}{\eta^2} = o(1)$, hence for $h > 0$ sufficiently small, we can assume that

$$1 - \frac{|\widehat{D}_{11}| + |\widehat{D}_{12}| + |\widehat{D}_{21}| + |\widehat{D}_{22}|}{\eta^2} \geqslant \frac{1}{2},$$

thus, as $h \to 0$,

$$\begin{aligned}|\log\varepsilon|\,\overline{\mathcal{E}}_h(\overline{m}_h) - E^\delta_{\varepsilon,\eta}(\overline{m}'_h) \geqslant{}& \frac{1}{2}\int_\Omega |\nabla'\overline{m}_{h,3}|^2\,\mathrm{d}x' - o(1) \\ &+ \left(\frac{\widehat{D}_{13}}{\eta^2} - 2\delta_1\right)\int_\Omega \partial_1\overline{m}'_h \wedge \overline{m}'_h\,\mathrm{d}x' \\ &+ \left(\frac{\widehat{D}_{23}}{\eta^2} - 2\delta_2\right)\int_\Omega \partial_2\overline{m}'_h \wedge \overline{m}'_h\,\mathrm{d}x'.\end{aligned}$$

For $j \in \{1, 2\}$, using $|\overline{m}'_h| \leqslant 1$ and Cauchy-Schwarz inequality, we have

$$\left|\int_\Omega \partial_j\overline{m}'_h \wedge \overline{m}'_h\,\mathrm{d}x'\right| \leqslant \int_\Omega |\partial_j\overline{m}'_h|\,\mathrm{d}x' \leqslant C\left(\int_\Omega |\partial_j\overline{m}'_h|^2\,\mathrm{d}x'\right)^{1/2} \leqslant C\left(|\log\varepsilon|\,\overline{\mathcal{E}}^0_h(\overline{m}_h)\right)^{1/2}$$

where $C > 0$ depends only on $\Omega$. We deduce that, as $h \to 0$,

$$|\log\varepsilon|\,\overline{\mathcal{E}}_h(\overline{m}_h) - E^\delta_{\varepsilon,\eta}(\overline{m}'_h) \geqslant -C\left(|\log\varepsilon|\,\overline{\mathcal{E}}^0_h(\overline{m}_h)\right)^{1/2}\left(\frac{\widehat{D}_{13}}{\eta^2} - 2\delta_1 + \frac{\widehat{D}_{23}}{\eta^2} - 2\delta_2\right) - o(1),$$

i.e.

$$\overline{\mathcal{E}}_h(\overline{m}_h) - \frac{1}{|\log\varepsilon|}E^\delta_{\varepsilon,\eta}(\overline{m}'_h) \geqslant -C\left(\frac{\overline{\mathcal{E}}^0_h(\overline{m}_h)}{|\log\varepsilon|}\right)^{1/2}\left(\frac{\widehat{D}_{13}}{\eta^2} - 2\delta_1 + \frac{\widehat{D}_{23}}{\eta^2} - 2\delta_2\right) - o\left(\frac{1}{|\log\varepsilon|}\right).$$

In the regime (2.6), we have $\frac{\widehat{D}_{13}}{\eta^2} - 2\delta_1 + \frac{\widehat{D}_{23}}{\eta^2} - 2\delta_2 = o(1)$. Moreover, since $\limsup_{h\to 0} \mathcal{E}_h(m_h) < +\infty$, it follows by (2.109) in Lemma 2.3.4 that, for $h > 0$ sufficiently small,

$$C \geqslant \mathcal{E}_h(m_h) \geqslant \overline{\mathcal{E}}_h(\overline{m}_h) - C\left(\overline{\mathcal{E}}_h(\overline{m}_h) + 1\right) R(h).$$

By Remark 2.3.3 and by (2.103), we see that $R(h) = o(1)$ in the regime (2.6) (and also in the regime (2.8)), so that we necessarily have

$$\limsup_{h\to 0} \overline{\mathcal{E}}_h(\overline{m}_h) \leqslant C.$$

Hence, using (2.111) in Lemma 2.3.4, we get $\overline{\mathcal{E}}^0_h(\overline{m}_h) \leqslant C$. It follows that

$$\overline{\mathcal{E}}_h(\overline{m}_h) - \frac{1}{|\log\varepsilon|} E^\delta_{\varepsilon,\eta}(\overline{m}'_h) \geqslant -o\left(\frac{1}{|\log\varepsilon|^{1/2}}\right) - o\left(\frac{1}{|\log\varepsilon|}\right) = -o(1) \quad \text{as } h \to 0.$$

Moreover, in the regime (2.8), we have $\frac{\widehat{D}_{13}}{\eta^2} - 2\delta_1 + \frac{\widehat{D}_{23}}{\eta^2} - 2\delta_2 = o\left(\frac{1}{|\log\varepsilon|}\right)$. Hence, we get

$$\overline{\mathcal{E}}_h(\overline{m}_h) - \frac{1}{|\log\varepsilon|} E^\delta_{\varepsilon,\eta}(\overline{m}'_h) \geqslant -o\left(\frac{1}{|\log\varepsilon|^{3/2}}\right) - o\left(\frac{1}{|\log\varepsilon|}\right) = -o\left(\frac{1}{|\log\varepsilon|}\right) \quad \text{as } h \to 0.$$

Finally, if $m_h = (m'_h, 0)$, i.e. $m_{h,3} = 0$, then $\left|\nabla' \overline{m}'_{h,3}\right| = 0$, the DMI terms involving $\widehat{D}_{11}$, $\widehat{D}_{12}$, $\widehat{D}_{21}$ and $\widehat{D}_{22}$ are zero, and $|\overline{m}'_h| = 1$. Hence, (2.115) reduces to

$$|\log\varepsilon|\, \overline{\mathcal{E}}_h(\overline{m}_h) - E^\delta_{\varepsilon,\eta}(\overline{m}'_h) = \left(\frac{\widehat{D}_{13}}{\eta^2} - 2\delta_1\right) \int_\Omega \partial_1 \overline{m}'_h \wedge \overline{m}'_h \,\mathrm{d}x' + \left(\frac{\widehat{D}_{23}}{\eta^2} - 2\delta_2\right) \int_\Omega \partial_2 \overline{m}'_h \wedge \overline{m}'_h \,\mathrm{d}x'.$$

The above estimates of the right-hand side of this inequality remain true in regimes (2.6) and (2.8) respectively, hence we obtain the expected asymptotic equalities. □

*Proof of Corollary 2.1.11.* These estimates are a direct consequence of Theorem 2.1.10 and Proposition 2.3.5. □

### 2.3.2 Gamma-convergence of the three-dimensional energy

In this section, we prove the Gamma-convergence for $\mathcal{E}_h(m_h)$, i.e. Theorems 2.1.12, 2.1.13, 2.1.14 and Corollary 2.1.15.

*Remark* 2.3.6. Recall that the regime (2.6) is equivalent with $h \ll \eta^2 \ll h\,|\log h| \ll 1$ by using the definition of $\varepsilon$, hence $|\log h| \sim |\log\eta|$. It follows from (2.6) that $\frac{1}{\varepsilon} \ll |\log\eta|$ and from (2.7) that $|\log\varepsilon| \ll \frac{1}{\varepsilon}$, thus $|\log\varepsilon| \ll \frac{1}{\varepsilon} \ll |\log\eta|$. Hence, regime (2.6) (and also (2.8)) implies regime (2.13).

*Proof of Theorem 2.1.12.* Let $\left\{m_h \colon \Omega_h \to \mathbb{S}^2\right\}$ be a family of magnetizations such that, for some constant $C > 0$, $\sup_{h\to 0} \mathcal{E}_h(m_h) \leqslant C$. In the regime (2.6), we have by Corollary 2.1.11,

$$\limsup_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} E^\delta_{\varepsilon,\eta}(\overline{m}'_h) \leqslant \limsup_{h\to 0} \mathcal{E}_h(m_h) \leqslant C.$$

Moreover, in the regime (2.8), $|\log \varepsilon| \ll |\log \eta|$ (see Remark 2.3.6), hence we can apply Theorem 2.1.1 to $m_\varepsilon = \overline{m}'_h$. By Theorem 2.1.1 *(i)*, for a subsequence, $(\mathcal{J}(\overline{m}'_h))$ converges to a measure $J$ on the closure $\overline{\Omega}$, in the sense that

$$\lim_{h\to 0}\left(\sup_{|\nabla'\zeta|\leqslant 1 \text{ in } \Omega} |\langle \mathcal{J}(\overline{m}'_h) - J, \zeta\rangle|\right) = 0.$$

Besides, $J$ is supported on $\partial\Omega$ and has the form

$$J = -\kappa \mathcal{H}^1 \llcorner \partial\Omega + \pi \sum_{j=1}^{N} d_j \mathbb{1}_{\{a_j\}}$$

for $N \geqslant 1$ distinct boundary vortices $a_j \in \partial\Omega$ carrying the multiplicities $d_j \in \mathbb{Z}\setminus\{0\}$, for $j \in \{1, ..., N\}$, such that $\sum_{j=1}^N d_j = 2$. Moreover, for a subsequence, $(\overline{m}'_h|_{\partial\Omega})$ converges to $e^{i\varphi_0} \in BV(\partial\Omega, \mathbb{S}^1)$ in $L^p(\partial\Omega)$, for every $p \in [1, +\infty)$, where $\varphi_0 \in BV(\partial\Omega, \pi\mathbb{Z})$ is a lifting of the tangent field $\pm\tau$ on $\partial\Omega$ determined (up to a constant in $\pi\mathbb{Z}$) by

$$\partial_\tau \varphi_0 = \kappa \mathcal{H}^1 \llcorner \partial\Omega - \pi \sum_{j=1}^{N} d_j \mathbb{1}_{\{a_j\}} \quad \text{as measure on } \partial\Omega.$$

Since $\overline{m}_{h,3}^2 \leqslant 1 - |\overline{m}'_h|^2$ and $(\overline{m}'_h)$ converges to $e^{i\varphi_0}$ in $L^2(\Omega, \mathbb{R}^2)$ with $|e^{i\varphi_0}| = 1$, we deduce that $(\overline{m}_{h,3})$ converges to zero in $L^2(\partial\Omega)$ and almost everywhere on $\partial\Omega$ (up to a subsequence). As $|\overline{m}_{h,3}| \leqslant 1$, by dominated convergence theorem, we get that $(\overline{m}_{h,3})$ converges to zero in $L^p(\partial\Omega)$ for every $p \in [1, +\infty)$, and we deduce that $(\overline{m}_h|_{\partial\Omega})$ converges to $(e^{i\varphi_0}, 0)$ in $L^p(\partial\Omega)$ for every $p \in [1, +\infty)$. For proving *(ii)*, we apply Theorem 2.1.1 *(ii)* and Corollary 2.1.11 to get

$$\pi \sum_{j=1}^{N} |d_j| \leqslant \liminf_{h\to 0} \frac{1}{|\log \varepsilon|} E^\delta_{\varepsilon,\eta}(\overline{m}'_h) \leqslant \liminf_{h\to 0} \mathcal{E}_h(m_h).$$

□

*Proof of Theorem 2.1.13.* Let $\{m_h \colon \Omega_h \to \mathbb{S}^2\}$ be a family of magnetizations such that, for some constant $C > 0$, $\sup_{h\to 0} \mathcal{E}_h(m_h) \leqslant C$. By (2.29) and Corollary 2.1.11 in the regime (2.8), we have

$$C \geqslant \limsup_{h\to 0} |\log \varepsilon| \left(\mathcal{E}_h(m_h) - \pi \sum_{j=1}^{N} |d_j|\right) \geqslant \limsup_{h\to 0} \left(E^\delta_{\varepsilon,\eta}(\overline{m}'_h) - \pi |\log \varepsilon| \sum_{j=1}^{N} |d_j|\right).$$

Moreover, in the regime (2.8), $|\log \varepsilon| \ll |\log \eta|$ (see Remark 2.3.6), hence we can apply Theorem 2.1.3 to $m_\varepsilon = \overline{m}'_h$. By Theorem 2.1.3 *(i)* and Corollary 2.1.11 in the regime (2.8), we have $d_j \in \{-1, +1\}$ for every $j \in \{1, ..., N\}$, so that $\sum_{j=1}^N |d_j| = N$, and

$$W^\delta_\Omega(\{(a_j, d_j)\}) + N\gamma_0 \leqslant \liminf_{h\to 0} \left(E^\delta_{\varepsilon,\eta}(\overline{m}'_h) - N\pi |\log \varepsilon|\right) \leqslant \liminf_{h\to 0} |\log \varepsilon| \left(\mathcal{E}_h(m_h) - N\pi\right).$$

Let us prove *(ii)*. Let $h > 0$ be fixed for a moment. Using Lemma 2.3.2, we have

$$\begin{aligned}
|\log \varepsilon| \left|\mathcal{E}_h(m_h) - \mathcal{E}^0_h(m_h)\right| &= \left|\frac{1}{h\eta^2} \int_{\Omega_h} \widehat{D} : \nabla m_h \wedge m_h \,\mathrm{d}x\right| \\
&\leqslant \left|\frac{1}{\eta^2} \int_{\Omega} \widehat{D}' : \nabla' \overline{m}_h \wedge \overline{m}_h \,\mathrm{d}x'\right| + C |\log \varepsilon| R_2(h)
\end{aligned}$$

for some constant $C > 0$. We can decompose the quantity $\left|\frac{1}{\eta^2}\int_\Omega \widehat{D}' : \nabla'\overline{m}_h \wedge \overline{m}_h \,\mathrm{d}x'\right|$ in six scalar components involving $\widehat{D}_{11}$, $\widehat{D}_{12}$, $\widehat{D}_{13}$, $\widehat{D}_{21}$, $\widehat{D}_{22}$ and $\widehat{D}_{23}$ respectively. Using integration by parts in order to eliminate the derivatives of $\overline{m}_{h,3}$ and since $|\overline{m}_h| \leqslant 1$, we deduce that

$$\left|\frac{1}{\eta^2}\int_\Omega \widehat{D}' : \nabla'\overline{m}_h \wedge \overline{m}_h \,\mathrm{d}x'\right| \leqslant C\frac{|\widehat{D}'|}{\eta^2}\left(1+\int_\Omega |\nabla'\overline{m}_h'|\,\mathrm{d}x'\right) = C\frac{|\widehat{D}'|}{\eta^2}\left(1+\|\nabla'\overline{m}_h'\|_{L^1(\Omega)}\right)$$

for some constant $C > 0$ depending only on $\Omega$. By Theorem 2.1.3 *(v)*, $(\overline{m}_h')$ converges weakly in $L^1(\Omega, \mathbb{R}^2)$, thus

$$\left|\frac{1}{\eta^2}\int_\Omega \widehat{D}' : \nabla'\overline{m}_h \wedge \overline{m}_h \,\mathrm{d}x'\right| \leqslant C\frac{|\widehat{D}'|}{\eta^2}$$

and

$$|\log\varepsilon|\left|\mathcal{E}_h(m_h) - \mathcal{E}_h^0(m_h)\right| \leqslant C\frac{|\widehat{D}'|}{\eta^2} + C|\log\varepsilon|\,R_2(h).$$

In the regime (2.8), $\frac{|\widehat{D}'|}{\eta^2} = O(1)$ and $R_2(h) = o\left(\frac{1}{|\log\varepsilon|}\right)$, thus

$$|\log\varepsilon|\left|\mathcal{E}_h(m_h) - \mathcal{E}_h^0(m_h)\right| = O(1) \quad \text{as } h \to 0.$$

It follows that

$$\limsup_{h\to 0}\left(\mathcal{E}_h^0(m_h) - N\pi\right) \leqslant \limsup_{h\to 0}\left(\mathcal{E}_h(m_h) - N\pi\right) + C \leqslant C,$$

for constants $C > 0$, hence we can apply [25, Theorem 9 *(iv)*] and we obtain the expected compactness of $(m_h)$ and $(\mathcal{J}(\overline{m}_h'))$. □

*Proof of Theorem 2.1.14.* Let $\{a_j\}_{j\in\{1,\dots,N\}} \in (\partial\Omega)^N$ be $N \geqslant 1$ distinct points and $d_j \in \mathbb{Z}\setminus\{0\}$ be the corresponding multiplicities, for $j \in \{1,\dots,N\}$, that satisfy $\sum_{j=1}^N d_j = 2$. In the regime (2.6), we have $|\log\varepsilon| \ll |\log\eta|$ (see Remark 2.3.6), hence we can apply Theorem 2.1.4. Let $\{m_\varepsilon : \Omega \to \mathbb{S}^1\}$ be chosen as in Theorem 2.1.4. Set

$$m_h : \left|\begin{array}{ccc} \Omega_h & \to & \mathbb{S}^1\times\{0\} \\ (x', x_3) & \mapsto & (m_\varepsilon(x'), 0). \end{array}\right.$$

For every $h > 0$, $m_h$ is clearly independent of $x_3$, thus $m_h = \overline{m}_h$. By Theorem 2.1.4 applied to $m_\varepsilon = \overline{m}_h'$, it follows that $(\mathcal{J}(m_h(\cdot, x_3))) = (\mathcal{J}(m_h')) = (\mathcal{J}(\overline{m}_h')) = (\mathcal{J}(m_\varepsilon))$ converges to

$$J = -\kappa\mathcal{H}^1 \llcorner \partial\Omega + \sum_{j=1}^N d_j \mathbb{1}_{\{a_j\}}$$

in the sense that

$$\lim_{h\to 0}\left(\sup_{|\nabla'\zeta|\leqslant 1 \text{ in } \Omega} |\langle \mathcal{J}(\overline{m}_h') - J, \zeta\rangle|\right) = 0.$$

By Corollary 2.1.11 in the regime (2.6),

$$\mathcal{E}_h(m_h) = \frac{1}{|\log\varepsilon|}E^\delta_{\varepsilon,\eta}(m_\varepsilon) - o(1),$$

so that by Theorem 2.1.4,

$$\lim_{h\to 0}\mathcal{E}_h(m_h) = \lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}E^\delta_{\varepsilon,\eta}(m_\varepsilon) = \pi\sum_{j=1}^N |d_j|\,.$$

Furthermore, if $d_j \in \{-1,+1\}$ for every $j \in \{1,...,N\}$ and the regime (2.8) holds, then by Corollary 2.1.11 in the regime (2.8),

$$\begin{aligned}|\log\varepsilon|\left(\mathcal{E}_h(m_h)-N\pi\right) &= |\log\varepsilon|\left(\frac{1}{|\log\varepsilon|}E^{\delta}_{\varepsilon,\eta}(m_\varepsilon)-N\pi-o\left(\frac{1}{|\log\varepsilon|}\right)\right)\\ &= E^{\delta}_{\varepsilon,\eta}(m_\varepsilon)-N\pi\,|\log\varepsilon|-o(1).\end{aligned}$$

Taking the limits when $h\to 0$ and using Theorem 2.1.4, we deduce that

$$\lim_{h\to 0}|\log\varepsilon|\left(\mathcal{E}_h(m_h)-N\pi\right)=\lim_{\varepsilon\to 0}\left(E^{\delta}_{\varepsilon,\eta}(m_\varepsilon)-N\pi\,|\log\varepsilon|\right)=W^{\delta}_{\Omega}(\{(a_j,d_j)\})+N\gamma_0.$$

□

*Proof of Corollary 2.1.15.* By Corollary 2.1.6, there exist two points $a_1^*\neq a_2^*\in\partial\Omega$ such that

$$W^{\delta}_{\Omega}(\{(a_1^*,1),(a_2^*,1)\})=\min\left\{W^{\delta}_{\Omega}(\{(\tilde a_1,1),(\tilde a_2,1)\}):\tilde a_1\neq\tilde a_2\in\partial\Omega\right\}.$$

Let $(m_h)$ be a family of minimizers of $\mathcal{E}_h$ on $H^1(\Omega_h,\mathbb{R}^3)$ (such a family exists in the regime (2.6), by the direct method in the calculus of variations, similarly to the 2D case in Lemma 2.2.18). By Theorem 2.1.14 applied to $\{(a_1^*,1),(a_2^*,1)\}$, the minimizers $m_h$ must satisfy

$$\liminf_{h\to 0}\mathcal{E}_h(m_h)\leqslant 2\pi. \tag{2.116}$$

Hence, we can apply Theorem 2.1.12 *(i)* and we deduce that, for a subsequence, $(\mathcal{J}(\overline{m}'_h))$ converges as in (2.27) to

$$J=-\kappa\mathcal{H}^1\llcorner\partial\Omega+\sum_{j=1}^{N}d_j\mathbb{1}_{\{a_j\}}$$

for $N\geqslant 1$ distinct boundary points $a_1,...,a_N\in\partial\Omega$, with $d_1,...,d_N\in\mathbb{Z}\backslash\{0\}$ such that $\sum_{j=1}^N d_j=2$. Moreover, by Theorem 2.1.12 *(ii)*, we also have

$$\liminf_{h\to 0}\mathcal{E}_h(m_h)\geqslant\pi\sum_{j=1}^{N}|d_j|\,. \tag{2.117}$$

Combining (2.116) and (2.117), we get $\sum_{j=1}^N|d_j|\leqslant 2=\sum_{j=1}^N d_j$, hence $\sum_{j=1}^N(|d_j|-d_j)\leqslant 0$ so that for every $j\in\{1,...,N\}$, $|d_j|=d_j$. It follows that

$$\lim_{h\to 0}\mathcal{E}_h(m_h)=2\pi,$$

and two cases can occur: either $N=1$ and $d_1=2$, or $N=2$ and $d_1=d_2=1$. Hence, there are two boundary points $a_1,a_2\in\partial\Omega$ such that

$$J=-\kappa\mathcal{H}^1\llcorner\partial\Omega+\pi(\mathbb{1}_{\{a_1\}}+\mathbb{1}_{\{a_2\}}).$$

We now assume that the regime (2.8) holds. By Theorem 2.1.14, we necessarily have

$$|\log\varepsilon|\left(\mathcal{E}_h(m_h)-2\pi\right)\leqslant W^{\delta}_{\Omega}(\{(a_1^*,1),(a_2^*,1)\})+2\gamma_0+o(1)\quad\text{as } h\to 0. \tag{2.118}$$

Hence, we can apply Theorem 2.1.13 *(i)* and we deduce that $d_1=d_2=1$, $N=2$ (thus $a_1\neq a_2$) and

$$|\log\varepsilon|\left(\mathcal{E}_h(m_h)-2\pi\right)\geqslant W^{\delta}_{\Omega}(\{(a_1,1),(a_2,1)\})+2\gamma_0+o(1)\quad\text{as } h\to 0. \tag{2.119}$$

Combining (2.118) and (2.119), we deduce that

$$W_\Omega^\delta(\{(a_1,1),(a_2,1)\}) \leqslant W_\Omega^\delta(\{(a_1^*,1),(a_2^*,1)\}).$$

By definition of $W_\Omega^\delta(\{(a_1^*,1),(a_2^*,1)\})$, we have

$$W_\Omega^\delta(\{(a_1,1),(a_2,1)\}) = W_\Omega^\delta(\{(a_1^*,1),(a_2^*,1)\}),$$

and thus

$$\lim_{\varepsilon\to 0} |\log\varepsilon| \left(\mathcal{E}_h(m_h) - 2\pi\right) = W_\Omega^\delta(\{(a_1,1),(a_2,1)\}) + 2\gamma_0.$$

□

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