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Stability analysis of linear ODE-PDE interconnected systems

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Résumé

Titre : Analyse de stabilité de systèmes linéaires EDO-EDP interconnectés.

Les systèmes de dimension infinie permettent de modéliser un large spectre de phénomènes physiques pour lesquels les variables d'état évoluent temporellement et spatialement. Ce manuscrit s'intéresse à l'évaluation de la stabilité de leur point d'équilibre. Deux études de cas seront en particulier traitées : l'analyse de stabilité des systèmes interconnectés à une équation de transport, et à une équation de réaction-diffusion.

Des outils théoriques existent pour l'analyse de stabilité de ces systèmes linéaires de dimension infinie et s'appuient sur une algèbre d'opérateurs plutôt que matricielle. Cependant, ces résultats d'existence soulèvent un problème de constructibilité numérique. Lors de l'implémentation, une approximation est réalisée et les résultats sont conservatifs. La conception d'outils numériques menant à des garanties de stabilité pour lesquelles le degré de conservatisme est évalué et maîtrisé est alors un enjeu majeur. Comment développer des critères numériques fiables permettant de statuer sur la stabilité ou l'instabilité des systèmes linéaires de dimension infinie?

Afin de répondre à cette question, nous proposons ici une nouvelle méthode générique qui se décompose en deux temps. D'abord, sous l'angle de l'approximation sur les polynômes de Legendre, des modèles augmentés sont construits et découpent le système original en deux blocs : d'une part, un système de dimension finie approximant est isolé, d'autre part, l'erreur de troncature de dimension infinie est conservée et modélisée. Ensuite, des outils fréquentiels et temporels de dimension finie sont déployés afin de proposer des critères de stabilité plus ou moins coûteux numériquement en fonction de l'ordre d'approximation choisi. En fréquentiel, à l'aide du théorème du petit gain, des conditions suffisantes de stabilité sont obtenues. En temporel, à l'aide du théorème de Lyapunov, une sous-estimation des régions de stabilité est proposée sous forme d'inégalité matricielle linéaire et une sur-estimation sous forme de test de positivité.

Nos deux études de cas ont ainsi été traitées à l'aide de cette méthodologie générale. Le principal résultat obtenu concerne le cas des systèmes EDO-transport interconnectés, pour lequel l'approximation et l'analyse de stabilité à l'aide des polynômes de Legendre mène à des estimations des régions de stabilité qui convergent exponentiellement vite. La méthode développée dans ce manuscrit peut être adaptée à d'autres types d'approximations et exportée à d'autres systèmes linéaires de dimension infinie. Ce travail ouvre ainsi la voie à l'obtention de conditions nécessaires et suffisantes de stabilité de dimension finie pour les systèmes de dimension infinie.

Mots-clefs : Systèmes de dimension infinie, Couplage EDO-EDP, Stabilité de Lyapunov, Inégalités matricielles linéaires, Approximation polynomiale.

Abstract

Title : Stability analysis of linear ODE-PDE interconnected systems.

Infinite dimensional systems allow to model a large panel of physical phenomena for which the state variables evolve both temporally and spatially. This manuscript deals with the evaluation of the stability of their equilibrium point. Two case studies are treated in particular: the stability analysis of ODE-transport, and ODE-reaction-diffusion interconnected systems.

Theoretical tools exist for the stability analysis of these infinite-dimensional linear systems and are based on an operator algebra rather than a matrix algebra. However, these existence results raise a problem of numerical constructibility. During implementation, an approximation is performed and the results are conservative. The design of numerical tools leading to stability guarantees for which the degree of conservatism is evaluated and controlled is then a major issue. How can we develop reliable numerical criteria to rule on the stability or instability of infinite-dimensional linear systems?

In order to answer this question, one proposes here a new generic method, which is decomposed in two steps. First, from the perspective of Legendre polynomials approximation, augmented models are built and split the original system into two blocks: on the one hand, a finite-dimensional approximated system is isolated, on the other hand, the infinite-dimensional truncation error is preserved and modeled. Then, frequency and time tools of finite dimension are deployed in order to propose stability criteria that have high or low numerical load depending on the approximated order. In frequencies, with the aid of the small gain theorem, sufficient stability conditions are obtained. In temporal, with the aid of the Lyapunov theorem, an under estimate of the stability regions is proposed as a linear matrix inequality and an over estimate as a positivity test.

Our two case studies have been treated with this general methodology. The main result concerns the case of ODE-transport interconnected systems, for which the approximation and stability analysis using Legendre polynomials leads to exponentially fast converging estimates of stability regions. The method developed in this manuscript can be adapted to other types of approximations and exported to other infinite-dimensional linear systems. Thus, this work opens the way to obtain necessary and sufficient finite-dimensional conditions of stability for infinite-dimensional systems.

Mots-clefs : Infinite dimensional systems, ODE-PDE coupling, Lyapunov stability, Linear matrix inequalities, Polynomial approximation.

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Notations and acronyms

"The Book of Nature is written in mathematical language; without these, one is wandering in a dark labyrinth." The Assayer, G. Galilei.

This section provides the main notations and acronyms used in this manuscript.

For linear algebra, the following notations were selected.

$\mathbb{N}$	The set of non-negative integers.
$\mathbb{N}^*$	The set positive integers.
$\mathbb{Z}$	The set of integers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of non-negative real numbers.
$\mathbb{C}$	The set of complex numbers. For any z in $\mathbb{C}$, we denote $z = \mathcal{R}(z) + \imath\mathcal{I}(z)$ where $\mathcal{R}(z)$ and $\mathcal{I}(z)$ are respectively the real and imaginary parts of z. The conjugate of such complex number z is denoted $\bar{z} = \mathcal{R}(z) - \imath\mathcal{I}(z)$.
$\mathbb{C}_+$	The set of complex numbers with non-negative real parts.
$\mathbb{K}$	The field $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$.
$\mathbb{K}^n$	The n-dimensional euclidian space.
$\mathbb{K}^{n\times m}$	The set of matrices with n rows and m columns (of size $n \times m$).
$\mathbb{S}^n$	The set of symmetric matrices in $\mathbb{R}^{n\times n}$.
$\mathbb{S}^n_+$	The set of symmetric definite positive matrices in $\mathbb{R}^{n\times n}$. We also use the notation $M \succ 0$ which means that $M \in \mathbb{S}^n_+$.
$M^\top$	The transpose matrix of M.
0_{nm}	The zero matrix of size $n \times m$.
I_n	The identity matrix of size $n \times n$.
$\mathcal{H}(M)$	That stands for $M + M^\top$, for any square matrix $M \in \mathbb{R}^{n\times n}$.
$\mathrm{diag}(d_1, \dots, d_n)$	The diagonal matrix with diagonal coefficients $d_1, \dots, d_n$.
$\mathrm{tril}(M)$	The triangular lower part of matrix $M \in \mathbb{R}^{n\times n}$.
$\det(M)$	The determinant of matrix $M \in \mathbb{R}^{n\times n}$.
$\mathrm{rk}(M)$	The rank of matrix $M \in \mathbb{R}^{n\times n}$.
$\mathrm{vec}(M)$	The vector in $\mathbb{R}^{mn}$ stacking successively the columns of matrices in $\mathbb{R}^{n\times m}$.
$\bar{\sigma}(M)$	The maximal norm of the complex eigenvalues of matrix $M \in \mathbb{K}^{n\times n}$.
$\underline{\sigma}(M)$	The minimal norm of the complex eigenvalues of matrix $M \in \mathbb{K}^{n\times n}$.
$\left[\begin{smallmatrix} M_1 & M_2 \\ * & M_3 \end{smallmatrix}\right]$	That stands for $\begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix}$ matrix.
$\left[\begin{smallmatrix} M_1 & M_2 \\ M_3 & 0 \end{smallmatrix}\right]$	That stands for $\begin{bmatrix} M_1 & M_2 \\ M_3 & 0_{n_3 m_2} \end{bmatrix}$ matrix, for any $M_i \in \mathbb{R}^{n_i \times m_i}$ for $i \in \{1, 2, 3\}$.
δ_{jk}	The Kronecker delta such that $\delta_{jk} = \begin{cases} 1 \text{ if } j = k, \\ 0 \text{ otherwise.} \end{cases}$
$\otimes$	The Kronecker product.
$\star$	The Redheffer star product gives $\left(\begin{smallmatrix} M_{11} & M_{12} \\ M_{13} & 0 \end{smallmatrix}\right) \star \left(\begin{smallmatrix} M_{21} & M_{22} \\ M_{23} & M_{24} \end{smallmatrix}\right) = \left(\begin{smallmatrix} M_{11}+M_{12}M_{24}M_{23} & M_{12}M_{23} \\ M_{22}M_{13} & M_{21} \end{smallmatrix}\right)$.
$\lvert\cdot\rvert$	The Shatten's ∞-norm of matrix $M \in \mathbb{K}^{n\times m}$ is given by $\lvert M\rvert^2 = \bar{\sigma}(M^\top M)$.

For mathematical analysis, the following notations were chosen.

$L^2(a,b;\mathbb{K}^{n\times m})$	The set of square-integrable functions from (a,b) to $\mathbb{K}^{n\times m}$.
$H^1(a,b;\mathbb{K}^{n\times m})$	The set of functions f from $[a,b]$ to $\mathbb{K}^{n\times m}$ such that f and its derivatives f' are in $L^2(a,b;\mathbb{K}^{n\times m})$.
$H^2(a,b;\mathbb{K}^{n\times m})$	The set of functions z from $[a,b]$ to $\mathbb{K}^{n\times m}$ such that f, f' and f'' are in $L^2(a,b;\mathbb{K}^{n\times m})$.
$C(a,b;\mathbb{K}^{n\times m})$	The set of continuous functions from $[a,b]$ to $\mathbb{K}^{n\times m}$.
$C_{pw}(a,b;\mathbb{K}^{n\times m})$	The set of piece-wise continuous functions from $[a,b]$ to $\mathbb{K}^{n\times m}$ with a finite number of discontinuities of the first kind.
$C_\infty(a,b;\mathbb{K}^{n\times m})$	The set of smooth functions from $[a,b]$ to $\mathbb{K}^{n\times m}$.
$H_0^1(a,b;\mathbb{K}^{n\times m})$	The closure of bump functions $C_\infty(a,b;\mathbb{K}^{n\times m})$ in $H^1(a,b;\mathbb{K}^{n\times m})$.
$H_0^2(a,b;\mathbb{K}^{n\times m})$	The closure of bump functions $C_\infty(a,b;\mathbb{K}^{n\times m})$ in $H^2(a,b;\mathbb{K}^{n\times m})$.
$\lVert\cdot\rVert$	The L^2 norm in $L_2(a,b;\mathbb{K}^{n\times m})$ is defined by $\lVert f\rVert = \sqrt{\int_a^b \lvert f(\tau)\rvert^2 \,\mathrm{d}\tau}$.
$\lVert\cdot\rVert_\infty$	The sup norm in $C_{pw}(a,b;\mathbb{K}^{n\times m})$ is defined by $\lVert f\rVert_\infty = \sup\limits_{[a,b]} \lvert f(\tau)\rvert$.
$\langle\cdot\vert\cdot\rangle$	The scalar product is given by $\langle f_1\vert f_2\rangle = \int_a^b \bar{\sigma}(f_1^\top(\tau)f_2(\tau))\mathrm{d}\tau$.
$\langle\cdot\vert\cdot\rangle_w$	This scalar product is given $\langle f_1\vert f_2\rangle = \int_a^b \bar{\sigma}(f_1^\top(\tau)f_2(\tau))w(\tau)\mathrm{d}\tau$.
$\lvert(\cdot,\cdot)\rvert$	This norm is given by $\lvert(x,z)\rvert = \sqrt{\lvert x\rvert^2 + \lVert z\rVert^2}$.
$\lvert(\cdot,\cdot)\rvert_\infty$	This norm is given by $\lvert(x,z)\rvert_\infty = \max(\lvert x\rvert, \lVert z\rVert_\infty)$.
$\langle\cdot\vert\cdot\rangle_\mu$	This scalar product is given by $\langle [\begin{smallmatrix} x_1 \\ z_1\end{smallmatrix}] \vert [\begin{smallmatrix} x_2 \\ z_2\end{smallmatrix}]\rangle_\mu = x_1^\top x_2 + \langle z_1\vert z_2\rangle$.
$\lvert H\rvert_{\mathcal{H}_\infty}$	The $\mathcal{H}_\infty$ norm is equal to the L^2-L^2 norm maximal ratio between the output and the input. For a stable transfer H, it is given by $\sup\limits_{\omega\in\mathbb{R}} \lvert H(\imath\omega)\rvert$.
$\mathcal{B}(\lambda,R)$	The open ball in the complex plane of radius R in $\mathbb{R}$ and center λ in $\mathbb{C}$.
$\underset{n\to\infty}{O}(\cdot)$	The notation $f_{1,n} = \underset{n\to\infty}{O}(f_{2,n})$ means that $\lim\limits_{n\to\infty}\left(\frac{\lvert f_{1,n}\rvert}{\lvert f_{2,n}\rvert}\right)$ is finite.
$\underset{s\to\lambda}{O}(\cdot)$	The notation $f_1(s) = \underset{s\to\lambda}{O}(f_2(s))$ means that $\lim\limits_{s\to\lambda}\left(\frac{\lvert f_1(s)\rvert}{\lvert f_2(s)\rvert}\right)$ is finite.
$f_1(s) \propto f_2(s)$	There exists a scalar $c>0$ such that $f_1(s) = cf_2(s)$.
$\mathcal{W}(\cdot)$	Lambert function defined as $\mathcal{W}: \begin{cases} \mathbb{R}_+ \to \mathbb{R}_+, \\ x \mapsto \mathcal{W}(x) = y, \end{cases}$ where y is uniquely defined by the relation $\mathrm{e}^y\, y = x$.
$\cosh(x)$, $\sinh(x)$, $\tanh(x)$	The hyperbolic functions given by $\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$, $\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$ and $\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}$, respectively.
$\mathrm{sign}(x)$	The sign function returns the sign of scalar x and takes values in $\{-1,1\}$.
$\lceil x\rceil$	The ceiling function maps x to the least integer greater than or equal to x.
$\lfloor x\rfloor$	The flooring function maps x to the greatest integer less than or equal to x.
$\binom{k}{i}$	The binomial coefficient is given by $\frac{\Gamma(k+1)}{\Gamma(i+1)\Gamma(k-i+1)}$.
$\Gamma(\cdot)$	The Gamma function $\Gamma(k)$ is given by $(k-1)\,!$, for all k in $\mathbb{N}$. The Gamma function $\Gamma(x)$ is given by $\int_0^\infty \tau^{x-1}\,\mathrm{e}^{-\tau}\,\mathrm{d}\tau$, for all x in $\mathbb{R}$.

Let also give the following abbreviations.

ODE	Ordinary differential equation.
PDE	Partial differential equation.
TDS	Time-delay system.
LMI(s)	Linear matrix inequality(ies).
GES	Globally exponentially stable.
s.t.	such that.
i.e.	that is (id est).
resp.	respectively.

Part I

Introduction

Chapter 1

Formulation of the problem

"Remind us how hard it is for our finite minds to grasp a concept as large as infinity."
The infinite hotel paradox, J. Dekofsky.

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Focusing on dynamical systems, one of the task of an automation engineer is to assess the stability of the equilibrium points. With an initial condition near the equilibrium, the goal is to determine if the trajectory converges and to characterize the asymptotic behavior of the system. Many tools have been proposed to study the stability of finite-dimensional systems in the frequency or time domains. Nevertheless, once the system becomes infinite-dimensional, the implementation of these tools fails or is limited by computation loads and sizes. To face or bypass these limitations, the stability analysis question is often incorrectly formulated through model approximation or simplification. The construction of numerical certificates for the stability of many infinite-dimensional systems remains therefore a significant research issue. The proposed research project is deeply embedded into this context. The main objective of my thesis is to develop dedicated tools to take the benefits of the accuracy of some infinite-dimensional models and are summarize by the following questions.

- *How to develop a numerical stability test for a system interconnected with the transport equation?*
- *What about a system interconnected with the reaction-diffusion equation?*

These two problems are introduced hereafter.

In the first section, infinite-dimensional systems are presented. In this wide class of systems, a focus is made on interconnections between linear ordinary and partial differential equations. In the second section, two work cases are highlighted, and two stability questions are formulated, to be treated in Part II and III of this thesis. In the third section, the main challenges raised by the infinite dimension and the stability analysis are recalled to emphasize our study goals.

1.1 A subclass of linear infinite-dimensional systems

1.1.1 Linear infinite-dimensional systems

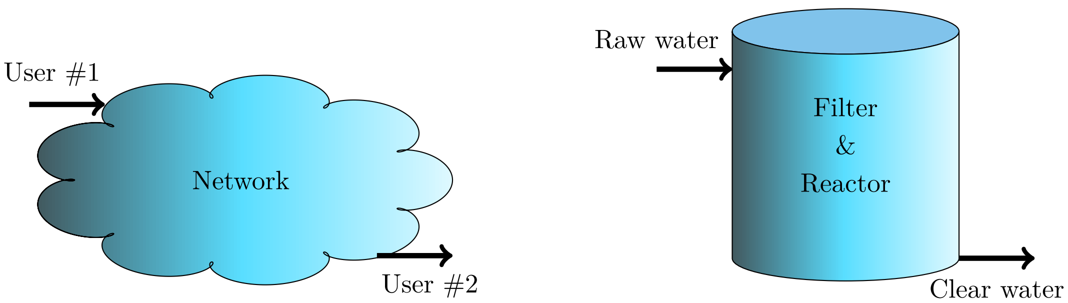
Physical phenomena

The emergence of new technologies has profoundly affected the world of science and research. The fast development of robotic manipulators, autonomous cars, or power grids, which starts to surround us and will become increasingly present in the future, goes along with research interests. In that context, cyber-physical systems [11, 190] have received a boost for the last decades. This large class of systems merges computation and physical processes. It results in complex interconnections of a wide variety of components of different natures. Such complexity arise therefore from essential features present in or between the subsystems.

The complexity of connected objects is sometimes due to the dynamics of the object itself. In mechanics, the drilling process can be modeled by string equations with dynamical boundary conditions [24]. In biology, reaction and diffusion equations and Lotka–Volterra equations (i.e. predator-prey equations) are used to describe pharmacology [73] or epidemiology [194] phenomena. The model complexity can also be impacted by interconnections. In computer science, from the transmitter to the recipient, information is conveyed by a transport equation, see Figure 1.1a or [7] for a network with transmission control protocol (TCP). In electronics, telegraph equations describe the voltage and current on long transmission lines, for instance in between power converters [55]. In physics, the electronic temperature in the plasma of nuclear fusion or fission reactors is governed by the heat equation. In fluid mechanics, the Navier-Stokes equation can be simplified to transport-diffusion-like equations, where the diffusion part is proportional to the fluid viscosity. These simplifications are used in practice to model wastewater treatment into pipes, see Figure 1.1a or [109]. The main issue in the series of systems listed previously lies in the representation and the handling of complex dynamical phenomena. The study becomes even more complicated since it covers many different fields.

The objective of automatic control is to develop common tools to analyze, control and observe such large panel systems. The idea is to encompass a wide variety of systems while keeping into consideration their particularities. Hereafter, two basic tools dedicated to such systems with significant application results are presented. The traffic congestion during TCP data transfer has been regulated by a controlled router through structured state feedback and quadratic separation framework [7]. After climate investigations and surface temperature measurements, the design of observers via backstepping techniques has led to Arctic sea ice thickness estimations [134]. In both cases, lumped quantities (number of packets or temperature) driven by a physical phenomenon (transport or diffusion) make the problem complicated and have to be taken into consideration for the design.

The core of the difficulty raised by the above dynamical processes is essentially its infinite-dimensional nature. Indeed, physical phenomena occur and traduce dynamics of infinite many quantities distributed on an interval. The corresponding systems are called distributed parameter systems [53] and belong to the wide class of infinite-dimensional systems [54]. More particularly, they are governed by partial differential equations (PDE) such as the transport, diffusion, reaction or wave equations [76]. These equations are described by distributed parameters z normalized in this manuscript in the interval $[0, 1]$. Our goal is to propose an overarching framework and methodology to model and analyze such a class of systems.



(a) Data transfer process. (b) Biofiltration process.

Figure 1.1: Physical systems.

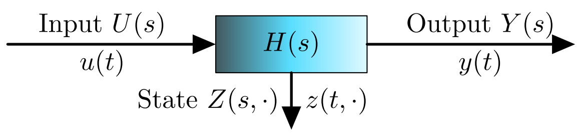


Figure 1.2: Block diagram of an infinite-dimensional system.

A global framework

All these processes are linearized around an equilibrium point to be studied. The resulting linear system belongs to the wide class of linear infinite-dimensional systems [76], where the infinite-dimensional state $z(t)$ is a function in $L^2(0,1;\mathbb{R}^{n_z})$ for all $t \in \mathbb{R}_+$. An appropriate framework for such a class of linear systems is the semigroup theory [178]. Indeed, it is possible to define a strongly continuous semigroup in the Hilbert space $L^2(0,1;\mathbb{R}^{n_z})$ which generates a linear infinitesimal generator $\mathcal{A}$ defined on $\mathcal{D} \subset L^2(0,1;\mathbb{R}^{n_z})$. In that context, formal results on stability and control have been developed in [199]. It extends finite-dimensional tools to infinite-dimension by means of operators, grammians and invertible semigroups. However, note that the analytical aspect of such machinery are often not suitable numerically. Moreover, in order to fit with applications, these models are often subject to boundary conditions that depend on input $u(t)$. In most cases, the input is neither null nor periodic and leads to heterogeneous PDE [146]. The associated operators $\mathcal{A}$ are then unbounded [110] and generally hard to deal with.

Another perspective is to also model these linear equations as an input-output relation using Laplace transformation (with the convention of using capital letters U, Z and Y to denote the Laplace transform of input u, state z and output y, respectively). In the Laplace domain, looking at Figure 1.2, linear infinite-dimensional systems can rather be modeled by irrational transfer functions [53, 54]

$$G(s,\cdot) = \frac{Z(s)}{U(s)} \in L^2(0,1;\mathbb{C}^{n_z \times n_z}), \quad H(s) = \frac{Y(s)}{U(s)} \in \mathbb{C}^{n_z \times n_z}, \quad \forall s \in \mathbb{C}. \tag{1.1}$$

The separation between finite and infinite-dimensional linear systems is finally the capacity to express analytically the transfer function $H(s)$ as a rational fraction (i.e. both numerator and denominator are polynomials) or not.

Focusing on the infinite-dimensional part of the system modeled by a PDE with a localized input and output, it can be seen as a dynamical equation, or as the transfer function $H(s)$. Throughout our work, the up-and-down between time and Laplace domains is fundamental. Since linear systems are considered, Laplace transform is well defined, and both transfer functions $H(s)$ and $G(s)$ are analytic. Notice also that since the state z takes its values in $\mathbb{R}^{n_z}$, the transfer functions $H(s)$ and $G(s)$ are real. In addition, most of the time, the transfer function $H(s)$ is proper and is bounded in $\mathbb{C}_+$. More generally, the proposed methodology developed in this manuscript is dedicated to Callier-Desoer class of real systems. The properties of this class are summarized in [44] or [54, §A.7.4] and are reformulated and exposed as our main assumption which gives an insight to the large panel of systems which could be regarded.

Assumption 1.1. Assume that G is smooth and real and that H is proper, which means that the following equations hold:

$$G(s) \in \mathcal{S}_G := \left\{ C_\infty(0,1;\mathbb{C}^{n_z \times n_z}) \text{ s.t. } \left\| G^{(d)}(s) \right\|_\infty \leq \rho_d(s) \right\}, \quad \forall s \in \mathbb{C}_+, \tag{1.2}$$

$$\overline{G(s)} = G(\overline{s}), \quad \forall s \in \mathbb{C}, \tag{1.3}$$

$$\exists r \in \mathbb{R} \text{ s.t. } \sup_{s \in \mathbb{C}_+ \cap |s| \geq r} |H(s)| \leq 1, \tag{1.4}$$

where $\{\rho_d\}_{d \in \mathbb{N}}$ is a sequence of $C_\infty(\mathbb{C}, \mathbb{C})$ functions.

To better understand these assumptions, counterexamples which get out of this class can be suggested. Remind us that these hypotheses are a guarantee to have a pointwise spectrum symmetric with respect to the real axis and that there is no characteristic root of the system with a norm which tends to infinity. The first assumption is simply the smoothness property. For simplicity, PDEs with positive characteristic roots are excluded but this assumption can be relaxed considering $\mathbb{C}_+ \backslash \sigma_d$ instead of $\mathbb{C}_+$ where σ_d denotes the point spectrum of G [167]. Concerning the second one, if the state z takes its values in $\mathbb{C}^{n_z}$ the transfer function $G(s)$ is not real and does not satisfy (1.3). It is the Shrödinger's equation case [178, Section 7.5]. Lastly, if $H(s) = \mathrm{e}^s I_{n_z}$, then $H(s)$ is not proper and $|H(s)|$ is unbounded when $\mathcal{R}(s)$ tends to infinity and (1.4) does not hold. It is the case of non-causal systems and does not correspond to any physical phenomenon. The third assumption also lies if $|H(s)| \to \infty$ when $\mathcal{I}(s)$ tends to infinity. It can be the case of neutral time-delay systems or non damped wave phenomena.

1.1.2 Partial differential equations modelling

As introduced before, many physical applications need to deal with spatial derivatives in addition to time derivatives. The corresponding and adequate models, at least around an equilibrium point, can then be described by linear PDE [76].

For instance, a homogeneous second order linear PDE with constant coefficients and two variables is given by

$$(a\partial_{tt} + b\partial_{t\theta} + c\partial_{\theta\theta} + d\partial_t + e\partial_\theta + f)\, z(t,\theta) = 0, \quad \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \tag{1.5}$$

where a, b, c, d, e, f are in $\mathbb{R}$. If at least one of the coefficients a, b, c, d, e, f is in $\mathbb{C}$, then assumption (1.3) becomes false. Note also that boundary conditions depending on non periodic input $u(t)$, for all $t \in \mathbb{R}_+$ will be added and make (1.5) a heterogeneous PDE [146].

These equations reflect many different behaviors in space and time, which can be classified into three categories [76, Chapter 2] depending on the sign of

$$\delta = b^2 - 4ac. \tag{1.6}$$

Among them, simple cases extensively used in [137] have been exhibited.

Hyperbolic equations

<u>Case $\delta > 0$:</u> At the price of variable changes, it is always possible to study the so-called wave equation

$$\partial_{tt} z(t,\theta) = \gamma^2 \partial_{\theta\theta} z(t,\theta), \quad \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \tag{1.7}$$

where the coefficient $\gamma > 0$ is the velocity and with two boundary conditons.

For simplicity reasons, a hyperbolic PDE of the first order is often considered in stabilization studies [28] and will be used in the sequel.

Definition 1.1. The transport or advection equation can be written as

$$\left\{ \begin{array}{rcll} h\partial_t z(t,\theta) & = & \partial_\theta z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \\ z(t,1) & = & u(t), & \forall t \in \mathbb{R}_+, \\ y(t) & = & z(t,0), & \forall t \in \mathbb{R}_+, \\ z(0,\theta) & = & z_0(\theta), & \forall \theta \in [0,1], \end{array} \right. \tag{$\mathcal{S}_{1\infty}$}$$

where the coefficient $h = \frac{1}{\gamma} > 0$ is the delay and where $z_0 \in L^2(0,1;\mathbb{R}^{n_z})$ is the initial condition.

In the Laplace domain, the transfer function G from the input U to the state Z is

$$G(s,\theta) = \mathrm{e}^{h(\theta-1)s}\, I_{n_z}, \quad \forall (s,\theta) \in \mathbb{C} \times [0,1]. \tag{1.8}$$

Noticing that the output $Y = Z(0)$, the input-output transfer function H from U to Y is

$$H(s) = G(s,0) = \mathrm{e}^{-hs}\, I_{n_z}, \quad \forall s \in \mathbb{C}, \tag{1.9}$$

assuming that the initial condition is null.
One can recognize the time-shift property of the Laplace transform. Indeed, this transfer function is usually found when there is a delay element that creates lag time or dwell time. In the literature on time-delay systems [86], this delay traduces a processing time or a transport of materials or information. It is also very often used to study the effect of zero-order hold function in digital-to-analog converter [75]. It is worth noticing that these transfer functions verify Assumption 1.1. Indeed, the property of smoothness given in (1.2) holds with $\rho_d(s) = |hs|^d$ as well as property of realness (1.3) and boundedness property (1.4) with $r = 0$.

Parabolic equations

<u>Case $\delta = 0$:</u> At the price of variable changes and simplifications, it is possible to study the reaction-diffusion equation given below.

Definition 1.2. The reaction-diffusion equation can be written as

$$\begin{cases} \partial_t z(t,\theta) = (\lambda + \nu\partial_{\theta\theta}) z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \\ z(t,\tfrac{1}{2}) = 0, & \forall t \in \mathbb{R}_+, \\ \partial_\theta z(t,1) = u(t), \quad \partial_\theta z(t,0) = u(t), & \forall t \in \mathbb{R}_+, \\ y(t) = z(t,1), & \forall t \in \mathbb{R}_+, \\ z(0,\theta) = z_0(\theta), & \forall \theta \in [0,1], \end{cases} \tag{$\mathcal{S}_{2\infty}$}$$

where $\nu > 0$ is the diffusion coefficient and $\lambda \in \mathbb{R}$ is the reaction coefficient and where $z_0 \in L^2(0,1;\mathbb{R}^{n_z})$ is the initial condition.

Most of the time, the reaction-diffusion with mixed boundary conditions is given by

$$\begin{cases} \partial_t z(t,\theta) = (\lambda + \nu\partial_{\theta\theta}) z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_+ \times [\tfrac{1}{2},1], \\ z(t,\tfrac{1}{2}) = 0, & \forall t \in \mathbb{R}_+, \\ \partial_\theta z(t,1) = u(t), & \forall t \in \mathbb{R}_+, \\ y(t) = z(t,1), & \forall t \in \mathbb{R}_+. \end{cases} \tag{1.10}$$

For technical reasons, without any change on the interval $[\frac{1}{2},1]$, the reaction-diffusion equation is also regarded on an artificial interval $[0,\frac{1}{2}]$. By the antisymmetry relation $z(\theta) = -z(1-\theta)$, for all $\theta \in [0,\frac{1}{2}]$, we have

$$\begin{cases} \partial_t z(t,\theta) = (\lambda + \nu\partial_{\theta\theta}) z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,\tfrac{1}{2}], \\ z(t,\tfrac{1}{2}) = 0, & \forall t \in \mathbb{R}_+, \\ \partial_\theta z(t,0) = u(t), & \forall t \in \mathbb{R}_+, \\ y(t) = -z(t,0), & \forall t \in \mathbb{R}_+. \end{cases} \tag{1.11}$$

Throughout this manuscript, we take into consideration the original reaction-diffusion equation (1.10) as well as its symmetric representation (1.11) in a same model written as $(\mathcal{S}_{2\infty})$.

The stabilizability, controllability, or observability of such equation has been investigated at great length (see [78] and reference therein). In the following, the reaction-diffusion equation is mixed up with an ordinary differential equation, which modifies and complicates the properties of the system and will need to be regarded more in detail. However, focusing only on system $(\mathcal{S}_{2\infty})$, the spectrum of the reaction-diffusion operator with mixed boundary conditions can be characterized. It is a point spectrum given by

$$\sigma_p = \{\lambda - \nu(2k+1)^2\pi^2\}_{k\in\mathbb{N}^*}.$$

Therefore, we already know that system $(\mathcal{S}_{2\infty})$ in open-loop becomes unstable if $\lambda > \pi^2$.
Note that, in the Laplace domain, the transfer function G from the input U to the state Z is given by

$$G(s,\theta) = \frac{\sinh\left(\sqrt{\frac{s-\lambda}{\nu}}(\theta - \frac{1}{2})\right)}{\sqrt{\frac{s-\lambda}{\nu}}\cosh\left(\frac{1}{2}\sqrt{\frac{s-\lambda}{\nu}}\right)} I_{n_z}, \quad \forall (s,\theta) \in \mathbb{C}\backslash\sigma_p \times [0,1], \tag{1.12}$$

assuming that the initial condition is null.
Noticing that the output $Y = Z(0)$, the input-output transfer function H from U to Y is

$$H(s) = G(s,1) = \frac{\tanh\left(\frac{1}{2}\sqrt{\frac{s-\lambda}{\nu}}\right)}{\sqrt{\frac{s-\lambda}{\nu}}} I_{n_z}, \quad \forall s \in \mathbb{C}\backslash\sigma_p. \tag{1.13}$$

It is worth noticing that both functions G and H are not defined for $s \in \sigma_p$. Note also that different boundary conditions would lead to different transfer functions, another set σ_p, and other stability properties.
Lastly, the reaction-diffusion equation satisfies Assumption 1.1. Indeed, the first hypothesis (1.2) is satisfied with $\rho_d(s) = \left|\frac{s-\lambda}{\nu}\right|^{\frac{d-1}{2}}$ and the last hypothesis (1.4) holds with $r = \lambda$.

Elliptic equations

Case $\delta < 0$: At the price of variable changes, it is always possible to study the so-called Laplace equation

$$\partial_{tt} z(t,\theta) + \gamma^2 \partial_{\theta\theta} z(t,\theta) = 0, \quad \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \tag{1.14}$$

Otherwise, for a PDE at the first order, it comes down to

$$\partial_t z(t,\theta) + h\partial_\theta z(t,\theta) = 0, \quad \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \tag{1.15}$$

with $h > 0$. This last equation has no sense physically, is non-causal and does not satisfy hypothesis (1.4). Indeed, we obtain $H(s) = \mathrm{e}^{hs}$ which is unbounded as s tends to infinity. This case is discarded and not treated in our manuscript.

1.1.3 Interconnections of ordinary-partial differential equations

The above PDEs have been widely studied by themselves and gave rise to a rich and helpful foundation (see [76] and reference therein). They have also already been put in series with ordinary differential equations (ODE) [137]. This coupling in cascade does not really affect the stability properties since the spectrum is the union of each spectrum set but adds a degree of complexity to do controller synthesis [5, 124].
In this manuscript, following the literature [125, Chapter 8], one focuses on interconnections of an ODE with a PDE depicted in Figure 1.3. This class of systems is worth interesting since it can model cyber-physical systems as described in the introduction. It can even be seen as a closed-loop between any physical system (upper part of Figure 1.3) and a finite-dimensional controller (lower part of Figure 1.3).

Definition 1.3. Define the interconnected system between two subsystems.

- A real and proper infinite-dimensional system represented by a linear PDE with boundary input u, state z and, boundary output y. In the Laplace domain, transfer functions G, H satisfy Assumption 1.1 and write

$$\begin{cases} Z(s,\theta) = G(s,\theta)U(s), & \forall (s,\theta) \in \mathbb{C} \times [0,1], \\ Y(s) = H(s)U(s), & \forall s \in \mathbb{C}. \end{cases} \tag{1.16}$$

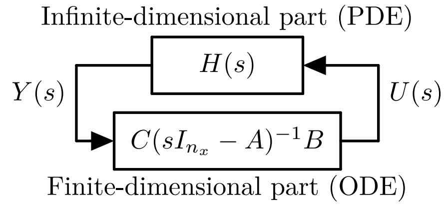


Figure 1.3: Block diagram of an interconnected ODE-PDE.

- A strictly causal linear finite-dimensional system with constant and real coefficients $\left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$ where A is the state matrix in $\mathbb{R}^{n_x \times n_x}$, B is the input matrix in $\mathbb{R}^{n_x \times n_z}$ and C is the output matrix in $\mathbb{R}^{n_z \times n_x}$. In the time domain, the state representation is

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t), \\ u(t) = Cx(t), \end{cases} \quad \forall t \in \mathbb{R}_+, \tag{1.17}$$

where y, x and u are respectively the input, state and output of the finite-dimensional system. Without loss of generality, output matrix C is normalized so that $|C| = 1$ all along the manuscript.

The interconnection puts into question many theoretical aspects from the existence and uniqueness of solutions to the controllability and observability, which need to be studied again [110, 146]. It also changes the spectrum location and the stability properties [23]. Recently, many works have tried to export and extend existing tools. For stability analysis, small-gain theorem can be used to obtain input-to-state stability properties [60] or tackle robustness issues. The Lyapunov method can also be extended as explained in Appendix A and even deal with non linearity [111, 153]. For control synthesis or stabilization problems, finite-dimensional tools such that the state or dynamical feedback synthesis [61, 167, 195] and infinite-dimensional tools such that the backstepping method [9, 1, 32] have been examined.
The way of of implementing these theoretical tools is also crucial. Applying the Lyapunov method before or after numerical approximation leads to two types of sufficient stability conditions for ODE-PDE interconnections solved by semi-definite programming. From one side, integral inequalities constraints are obtained and sum of squares are used [180, 191, 200]. From the other side, it directly leads to linear matrix inequalities [39, 183]. The distinction also occurs in control theory and is named early or late lumping approaches.

Before presenting the two systems, which have been studied, an important and common theorem has to be highlighted. In the following, it allows to unlock the development of converse conditions of stability for ODE-PDE coupled systems. It gives then an overview of the large set of systems that could be treated by our methodology. To obtain this theorem, the PDE transfer functions must satisfy Assumption 1.1.

Theorem 1.1. *Assume that there exists a quadratic, continuous and differentiable functional $V(x,z)$ such that its time derivatives verifies $\dot{V}(x,z) = -|x|^2$ for any (x,z) along the trajectories of the interconnected system* (1.16)-(1.17)*.*
If s^* is an unstable characteristic root (i.e. $\mathcal{R}(s^*) > 0$) of system (1.16)-(1.17)*, then there exists*

$$(x_0, z_0) \in \mathcal{S} := \left\{ (x,z) \in \mathbb{R}^{n_x} \times C_\infty(0,1;\mathbb{R}^{n_z}) \ s.t. \ |x| = 1, \ \left\| z^{(d)} \right\|_\infty \leq \rho_d, \ \forall d \in \mathbb{N} \right\}, \tag{1.18}$$

such that the following inequality

$$V(x_0, z_0) \leq -\frac{1}{2\mathcal{R}(s^*)} \leq -\frac{1}{2r} < 0, \tag{1.19}$$

holds where $\{\rho_d\}_{d\in\mathbb{N}}$ *is a sequence of positive integers and where*

$$r = \max(r_1, r_2) \text{ with } \begin{cases} r_1 &= \min_{r\in\mathbb{R}} r \text{ s.t. } \sup_{s\in\mathbb{C}_+\cap|s|\geq r} |H(s)| \leq 1, \\ r_2 &= |A| + |B|. \end{cases} \tag{1.20}$$

Proof. The proof is divided into three parts. Firstly, assuming that s^* is a characteristic root of system (1.16)-(1.17), we prove $\overline{s^*}$ is also a characteristic root of system (1.16)-(1.17). Moreover, there exists (u_1, u_2) in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ such that $|u_1| = 1$, $|u_2| \leq 1$, $u_1^\top u_2 = 0$ and that the corresponding characteristic vectors associated to $(s^*, \overline{s^*})$ are $\begin{bmatrix} x^*(t;s^*) \\ z^*(t;s^*) \end{bmatrix}, \begin{bmatrix} \overline{x^*(t;s^*)} \\ \overline{z^*(t;s^*)} \end{bmatrix}$ given by

$$\begin{bmatrix} x^*(t;s^*) \\ z^*(t,\theta;s^*) \end{bmatrix} = \mathrm{e}^{s^*t} \begin{bmatrix} I_{n_x} \\ G(s^*,\theta)C \end{bmatrix} (u_1 + \imath u_2), \quad \forall (t,\theta) \in \mathbb{R}_+ \times [0,1].$$

Secondly, any unstable characteristic root s^* satisfies $|s^*| \leq r$ where r is given by (1.20). Finally, assuming that $\dot{V}(x,z) = -|x|^2$ along the trajectories of the system (1.16)-(1.17), we prove that $V(x_0, z_0) \leq -\frac{1}{2r}$ with an adequate function (x_0, z_0) related to $(x^*(0), z^*(0))$ in the set $\mathcal{S}$.

<u>Step 1</u>*: Characteristic values and vectors of system* (1.16)-(1.17).
Assume that s^* satisfies $\det(s^* I_{n_x} - A - BH(s^*)C) = 0$. Then, according to Lemma B.1 given in Appendix B, there exists (u_1, u_2) in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ such that $|u_1| = 1$, $|u_2| \leq 1$, $u_1^\top u_2 = 0$ and

$$(s^* I_{n_x} - A - BH(s^*)C)(u_1 + \imath u_2) = 0. \tag{1.21}$$

Therefore, $\begin{bmatrix} x^*(t;s^*) \\ z^*(t,\theta;s^*) \end{bmatrix} = \mathrm{e}^{s^*t}\begin{bmatrix} I_{n_x} \\ G(s^*,\theta)C \end{bmatrix}(u_1 + \imath u_2)$ is a characteristic vector associated to s^*. Furthermore, according to the second item (1.3) of Assumption 1.1, we also have

$$\begin{aligned} (\overline{s^*} I_{n_x} - A - BH(\overline{s^*})C)(u_1 - \imath u_2) &= (\overline{s^*} I_{n_x} - A - B\overline{H(s^*)}C)(u_1 - \imath u_2), \\ &= \overline{(s^* I_{n_x} - A - BH(s^*)C)}(u_1 - \imath u_2), \\ &= 0, \end{aligned}$$

which means that $\overline{s^*}$ is also a characteristic root of system (1.16)-(1.17). Noticing that $G(\overline{s^*}, \theta) = \overline{G(s^*, \theta)}$, for $\theta \in [0,1]$, the corresponding characteristic vector is then given by

$$\begin{bmatrix} x^*(t;\overline{s^*}) \\ z^*(t,\theta;\overline{s^*}) \end{bmatrix} = \mathrm{e}^{\overline{s^*}t} \begin{bmatrix} I_{n_x} \\ G(\overline{s^*},\theta)C \end{bmatrix} (u_1 - \imath u_2) = \begin{bmatrix} \overline{x^*(t;s^*)} \\ \overline{z^*(t,\theta;s^*)} \end{bmatrix},$$

and belongs to $\mathbb{R}^{n_x} \times C_\infty(0,1;\mathbb{R}^{n_z})$ since function G is smooth according to Assumption 1.1.

<u>Step 2</u>*: Boundedness property of the modulus of unstable characteristic roots.*
Assume that $s^* = \mathcal{R}(s^*) + \imath\mathcal{I}(s^*)$ is an unstable characteristic root of system (1.16)-(1.17) (i.e. $\mathcal{R}(s^*) > 0$ holds, which implies instability). Take r_1 given by (1.20) which exists and is finite according to the third item (1.4) of Assumption 1.1. Now, to prove that $|s^*| \leq r$, two parts of the right half complex planes are isolated. If $|s^*| \leq r_1$, the inequality holds. Otherwise if $|s^*| > r_1$, there exists $u \in \mathbb{C}^{n_x}\backslash\{0\}$ such that $(s^* I_{n_x} - A - BH(s^*)C)u = 0$, since s^* is solution of $\det(s^* I_{n_x} - A - BH(s^*)C) = 0$. Passing through the norm, the following inequality holds

$$|s^* I_{n_x}| = |A + BH(s^*)C| \leq |A| + |B| \, |H(s^*)| \, |C| \leq |A| + |B| \, |C| = |A| + |B|. \tag{1.22}$$

Therefore, in that case, we have $|s^*| \leq r_2 := |A| + |B|$, which concludes Step 2.

<u>Step 3</u>*: Negative upper bound of the functional* V *for unstable systems.*

Assume that $s^* = \mathcal{R}(s^*)+\imath\mathcal{I}(s^*)$ is a characteristic root of system (1.16)-(1.17). According to Step 1, there exists a non trivial trajectory with values in $\mathbb{R}^{n_x} \times C_\infty(0,1;\mathbb{R}^{n_z})$ given by

$$\begin{bmatrix} \hat{x}(t;s^*) \\ \hat{z}(t,\theta;s^*) \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} x^*(t;s^*) \\ z^*(t,\theta;s^*) \end{bmatrix} + \begin{bmatrix} \overline{x^*}(t;s^*) \\ \overline{z^*}(t,\theta;s^*) \end{bmatrix}\right), \quad \forall (t,\theta) \in \mathbb{R}_+ \times [0,1]. \tag{1.23}$$

Since $\dot{V}(x,z) = -|x|^2$ along the trajectories of system (1.16)-(1.17), an integration from $t=0$ to $t=T>0$ leads to

$$V(x(T),z(T)) - V(x(0),z(0)) = -\int_0^T |x(t)|^2 \,\mathrm{d}t.$$

Considering the particular trajectory given by (1.23), we know that

$$\hat{x}(t;s^*) = \mathrm{e}^{\mathcal{R}(s^*)t}\left(\cos\big(\mathcal{I}(s^*)t\big)u_1 - \sin\big(\mathcal{I}(s^*)t\big)u_2\right),$$

with $|u_1| = 1$, $|u_2| \le 1$, $u_1^\top u_2 = 0$. Based on this expression and because V is quadratic, we obtain

$$\begin{aligned}
\left(\mathrm{e}^{2\mathcal{R}(s^*)T} - 1\right) V(\hat{x}(0;s^*),\hat{z}(0;s^*)) &= -\int_0^T \mathrm{e}^{2\mathcal{R}(s^*)t}\left|\cos\big(\mathcal{I}(s^*)t\big)u_1 - \sin\big(\mathcal{I}(s^*)t\big)u_2\right|^2 \mathrm{d}t,\\
&= -\int_0^T \mathrm{e}^{2\mathcal{R}(s^*)t}\left(\cos^2\big(\mathcal{I}(s^*)t\big) + \sin\big(\mathcal{I}(s^*)t\big)|u_2|^2\right)\mathrm{d}t,\\
&\le -\int_0^T \mathrm{e}^{2\mathcal{R}(s^*)t}\,\mathrm{d}t = -\frac{\mathrm{e}^{2\mathcal{R}(s^*)T} - 1}{2\mathcal{R}(s^*)}.
\end{aligned}$$

To close the proof, assume that s^* is an unstable characteristic root (i.e. $\mathcal{R}(s^*) > 0$) which means that the following inequality holds

$$V(\hat{x}(0;s^*),\hat{z}(0;s^*)) \le -\frac{1}{2\mathcal{R}(s^*)}. \tag{1.24}$$

Then, *Step* 2 ensures that $|s^*| \le r$. (i.e. $\mathcal{R}(s^*) \le |s^*| \le r$). Therefore, inequality (1.24) leads to

$$V(\hat{x}(0;s^*),\hat{z}(0;s^*)) \le -\frac{1}{2r} < 0, \tag{1.25}$$

Moreover, it is worth noticing that such exhibited function $(x_0,z_0) := (\hat{x}(0;s^*),\hat{z}(0;s^*))$ belongs to $\mathbb{R}^{n_x} \times C_\infty(0,1;\mathbb{R}^{n_z})$ and is given by

$$\begin{bmatrix} x_0 \\ z_0(\theta) \end{bmatrix} = \begin{bmatrix} \hat{x}(0;s^*) \\ \hat{z}(0,\theta;s^*) \end{bmatrix} = \begin{bmatrix} \cos\big(\mathcal{I}(s^*)\big)u_1 \\ \mathcal{R}\big(G(s^*,\theta)\big)C\cos\big(\mathcal{I}(s^*)\big)u_1 - \mathcal{I}\big(G(s^*,\theta)\big)C\cos\big(\mathcal{I}(s^*)\big)u_2 \end{bmatrix}. \tag{1.26}$$

It satisfies $|x_0| = 1$ and $\left\|z_0^{(d)}\right\|_\infty \le \left\|G^{(d)}(s^*)\right\|_\infty \le \sup_{|s|\le r} \rho_d(s) = \rho_d$. The sequence $\{\rho_d\}_{d\in\mathbb{N}}$ exists according to the first item (1.2) of Assumption 1.1. To sum up, for unstable systems, we have found a smooth function (x_0,z_0) such that the functional V is negative. □

Remark 1.1. Note that under the Lyapunov condition (i.e. the spectrum is not symmetric with respect to the real axis), this so-called converse Lyapunov functional V exists and is unique [58]. For a system interconnected with a transport equation or a reaction-diffusion equation, a methodology is provided to find such a functional in Appendix A.2.

This converse theorem based on converse functionals is used in the literature to give rise to the necessity of sufficient conditions of instability (see [68, 95, 155] for the case of time-delay systems).

Remark 1.2. Note that for time-delay systems, such a theorem is rather based on V such that the time derivative satisfies $\dot{V}(x,z) = -|x(t-h)|^2 = -z^\top(0)C^\top Cz(0)$. The proof is then similar to the one proposed in [15] which extends [98, Appendix A] to C_∞ class of functions.

When the infinite-dimensional part comes from a transport equation $(\mathcal{S}_{1\infty})$ we explicitly know the expression of parameters ρ_d and maximal radius r. They are given by

$$\begin{aligned} \rho_d &= (hr)^d, \quad \forall d \in \mathbb{N}, \\ r &= |A| + |B|. \end{aligned} \tag{1.27}$$

Similarly, for a reaction-diffusion equation ($\mathcal{S}_{2\infty}$), the following expressions have been obtained

$$\begin{aligned} \rho_d &= \left(\frac{r}{\nu}\right)^{\frac{d-1}{2}}, \quad \forall d \in \mathbb{N}, \\ r &= \max\left(\lambda, |A| + |B|\right). \end{aligned} \tag{1.28}$$

Remark 1.3. Note that these bounds have been given with respect to r, which is the maximal spectral radius. However, these bounds are not optimal and could be improved providing separately an upper bound for the real and imaginary part of the unstable characteristic roots (see [196] for time-delay systems case).

In the next section, a focus on these two systems is provided and the corresponding stability analysis problems are formulated.

1.2 Problem statement for two case studies

1.2.1 First case: System coupled with a transport equation

Historically, time-delay systems are often modeled by functional-differential equations [112, 135, 136]. It is the most intuitive and common way to describe the dynamical behavior of the finite-dimensional instantaneous state x, which is subject to delays. In the following, a single delay h is considered but the proposed representations encompass systems with multiple commensurate delays or with distributed delays where the distributed function can be seen as an impulse response of a linear finite-dimensional system [131].

Definition 1.4. Let a non null matrix A_d in $\mathbb{R}^{n_x \times n_x}$ decomposed into a product BC, with $B, C^\top$ being full column rank matrices, $n_z = \mathrm{rk}(A_d)$ and $|C| = 1$ and consider a time-delay system

$$\begin{cases} \dot{x}(t) = Ax(t) + BCx(t-h), \quad \forall t \in \mathbb{R}_+, & (1.29a) \\ \begin{bmatrix} x(0) \\ Cx(h(\theta-1)) \end{bmatrix} = \begin{bmatrix} x_0 \\ z_0(\theta) \end{bmatrix} \in \mathcal{D}_1 := \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{n_x} \times H^1(0,1;\mathbb{R}^{n_z}) \text{ s.t. } z(1) = Cx \right\}, & (1.29b) \end{cases}$$

where the delay $h > 0$ and matrices (A, B, C) in $\mathbb{R}^{n_x \times n_x} \times \mathbb{R}^{n_x \times n_z} \times \mathbb{R}^{n_z \times n_x}$ are constant and known.

The whole state $\big(x(t), Cx(t + h(\theta - 1))\big)$ of system (1.29) is composed of an instantaneous vector state $x(t)$ and a distributed function state $Cx(t + h(\theta - 1))$ with $\theta \in (0, 1)$ that accounts for the post values of the state $Cx(t)$.

Remark 1.4. Note that system (1.29) can also be initialized with $\varphi \in C_{pw}(-h, 0; \mathbb{R}^{n_x})$ by

$$x(\tau) = \varphi(\tau), \qquad \forall \tau \in [-h, 0]. \tag{1.30}$$

The corresponding solution is $x_t(\varphi) : \begin{cases} [-h, 0] \to \mathbb{R}^{n_x} \\ \tau \mapsto x(t+\tau) \end{cases}$ and belongs to $C_{pw}([-h, 0]; \mathbb{R}^{n_x})$.

Remark 1.5. Notice that the decomposition $A_d = BC$ is only aesthetic and has the virtue of reducing the number of involved variables. It will not infer well-posedness and stability properties.

System (1.29) is sometimes presented and used in the literature as an interconnection of ordinary and partial differential equations [138, 181]. This model enhances the infinite-dimensional nature of time-delay systems via the transport equation ($\mathcal{S}_{1\infty}$). Moreover, to work on the control and observation of time-delay systems, this new ODE-transport representation facilitates many calculations, like the application of the backstepping method [137].

Definition 1.5. Consider an ordinary differential equation interconnected with a transport equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bz(t,0), & \forall t \in \mathbb{R}_+, & (\mathcal{S}_1\text{a}) \\ h\partial_t z(t,\theta) = \partial_\theta z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], & (\mathcal{S}_1\text{b}) \\ z(t,1) = Cx(t), & \forall t \in \mathbb{R}_+, & (\mathcal{S}_1\text{c}) \\ \left[\begin{smallmatrix} x(0) \\ z(0,\theta) \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_0 \\ z_0(\theta) \end{smallmatrix}\right] \in \mathcal{D}_1 := \left\{ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{R}^{n_x} \times H^1(0,1;\mathbb{R}^{n_z}) \text{ s.t. } z(1) = Cx \right\}, & & (\mathcal{S}_1\text{d}) \end{cases}$$

where, without loss of generality, the rank of matrices $B, C^\top$ is full and C is normalized to $|C| = 1$.

Compared to the previous representation, this last model is more appropriate in the sense that the decoupling of the state (x,z) into a finite and an infinite-dimensional part appears clearly. The choice of initial conditions in the Hilbert space $\mathcal{D}_1 \subset \mathcal{D}_0 := \mathbb{R}^{n_x} \times L^2(0,1;\mathbb{R}^{n_z})$ instead of the Banach space $C_{pw}(-h,0;\mathbb{R}^{n_x})$ relies on the following subsection, where the proof of existence is confined to the state space $\mathcal{D}_1$. Nonetheless, taking care of some derivatives definitions, note that it would also be possible to deal with initial conditions $x(0) = \varphi(0)$ and $z(0,\theta) = C\varphi(h(\theta-1))$, for $\theta \in (0,1)$, with $\varphi \in C_{pw}(-h,0;\mathbb{R}^{n_x})$ similarly to initial conditions (1.30).
Finally, there exists a last modelling using a semigroup approach [54, 178]. Indeed, time-delay systems can be seen as a linear infinite-dimensional system [31] where the infinitesimal generator is given by

$$\mathcal{A} : \begin{cases} \mathcal{D}_1 \subset \mathcal{D}_0 & \to \mathcal{D}_0 := \mathbb{R}^{n_x} \times L^2(0,1;\mathbb{R}^{n_z}), \\ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] & \mapsto \mathcal{A}\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] := \begin{bmatrix} Ax+Bz(0) \\ \frac{1}{h}\partial_\theta z \end{bmatrix}, \end{cases} \tag{1.31}$$

and where the boundary condition $z(1) = Cx$ is embedded into the domain $\mathcal{D}_1$ of operator $\mathcal{A}$.
In the last decades, many studies have been devoted to ODE-transport systems and have tried to tackle problems brought by the infinite dimension. One needs to look at the Cauchy problem for the well-posedness [135], at the stability properties [86, 193] and at the control synthesis issues [124].

Existence and uniqueness of solutions

As a first step and before studying the stability of such a class of systems, one verifies the existence and uniqueness of solutions for system $(\mathcal{S}_1)$ and its consistency with the solution to system (1.29).
Let us start to prove the well-posedness of system $(\mathcal{S}_1)$.

Proposition 1.1. *System* $(\mathcal{S}_1\text{a})$-$(\mathcal{S}_1\text{c})$ *with initial condition* $(\mathcal{S}_1\text{d})$ *admits a unique and continuous solution* $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$ *from* $\mathbb{R}_+$ *to* $\mathcal{D}_1$.

Proof. The proof resumes the key elements enlightened in [79]. First, one easily shows that $\mathcal{A}$ is closed in $\mathcal{D}_0 \times \mathcal{D}_0$ and that $\mathcal{D}_1$ is dense in the Hilbert space $\mathcal{D}_0$. Moreover, integration by parts gives

$$\begin{aligned} \left\langle \mathcal{A}\left[\begin{smallmatrix} x(t) \\ z(t) \end{smallmatrix}\right] \middle| \left[\begin{smallmatrix} x(t) \\ z(t) \end{smallmatrix}\right] \right\rangle_\mu &= 2x^\top(t)\dot{x}(t) + 2\int_0^1 z^\top(t,\theta)\partial_t z(t,\theta)\mathrm{d}\theta, \\ &= 2x^\top(t)\big(Ax(t) + Bz(t,0)\big) + \frac{1}{h}\left[z^\top(t,\theta)z(t,\theta)\right]_0^1, \\ &= \begin{bmatrix} x(t) \\ z(t,0) \end{bmatrix}^\top \begin{bmatrix} \mathcal{H}(A) + \frac{1}{h}C^\top C & B \\ * & -\frac{1}{h}I_{n_z} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t,0) \end{bmatrix}, \end{aligned}$$

which ensures from appropriate Young inequality that

$$\langle \mathcal{A}\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] | \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \rangle_\mu \leq c_1 \left|(x,z)\right|^2, \quad \forall \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathcal{D}_1,$$

holds for some $c_1 > 0$. Similar calculations found in [79] would lead to $\langle \mathcal{A}^* \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] | \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \rangle_\mu \leq c_2 \left|(x,z)\right|^2$, where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$. According to [54, Corrolary 2.2.3], $\mathcal{A}$ is an infinitesimal generator of a strongly continuous semigroup $\mathcal{T}$ on $\mathcal{D}_0$. Applying [199, Proposition 2.3.5], one concludes on the existence of a continuous and unique solution $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] : t \mapsto \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \mathcal{T}(t)\left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right]$ from $\mathbb{R}_+$ to $\mathcal{D}_1$ with respect to the graph norm associated to the graph $\mathcal{D}_0 \times \mathcal{D}_0$. $\square$

Then, we prove that the representations ($\mathcal{S}_1$) and (1.29) are equivalent since there are two ways of modelling the same system.

Proposition 1.2. *For all* $\left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right] \in \mathcal{D}_1$, *the solution* $\big(x(t), Cx(t + h(\theta - 1))\big)$ *of system* (1.29) *and the solution* $\big(x(t), z(t, \theta)\big)$ *of system* ($\mathcal{S}_1$) *are equal, for all* (t, θ) *in* $\mathbb{R}_+ \in [0, 1]$.

Proof. Both solutions satisfy the same dynamical equations and initial conditions. □

It is worth noticing that both propositions can be enlarged to larger initial condition domains, such that the set $\mathbb{R}^{n_x} \times C_{pw}(-h, 0; \mathbb{R}^{n_z})$, see [114] for more details in this direction.

Equilibrium

The second step consists in characterizing the equilibrium of ($\mathcal{S}_1$). More particularly, in order to deal with global stability, one has to understand under which condition, system ($\mathcal{S}_1$) admits a unique equilibrium. This is formulated in the next proposition.

Proposition 1.3. *System* ($\mathcal{S}_1$) *admits a unique trivial equilibrium* $(x_e, z_e) = (0, 0)$ *if and only if* $\Omega = A + BC$ *is non singular.*

Proof. Let (x_e, z_e) be an equilibrium of system ($\mathcal{S}_1$), meaning that the following relations hold

$$\begin{cases} Ax_e + Bz_e(0) = 0, & \text{(1.32a)} \\ \partial_\theta z_e(\theta) = 0, & \text{(1.32b)} \\ z_e(1) = Cx_e. & \text{(1.32c)} \end{cases}$$

Equations (1.32b),(1.32c) lead to $z_e(\theta) = Cx_e$, for all θ in $[0, 1]$. Then, the last constraint (1.32a) involves $(A+BC)x_e = 0$. Hence, system (1.32) admits a unique solution $(x_e, z_e) = (0, 0)$ if and only if $\det(A + BC) \neq 0$. □

The characteristic polynomial of system ($\mathcal{S}_1$) is given by $\chi(s) = \det(sI_{n_x} - A - B\,\mathrm{e}^{-hs}\,C)$. The condition imposed by Proposition 1.3 corresponds to $\chi(0) = 0$ which means that 0 is a characteristic root of system ($\mathcal{S}_1$). In that case, which can directly be detected by a test on the singularity of matrix $A + BC$, the system is not asymptotically stable, because the trivial solution is not attractive.

Stability definitions

The last step requests to agree on the stability definition [136]. Recall the global exponential stability with the usual norm

$$\big|(\cdot,\cdot)\big| : \begin{cases} \mathcal{D}_0 \to \mathbb{R}_+, \\ (x, z) \mapsto \big|(x, z)\big| = \sqrt{|x|^2 + \displaystyle\int_0^1 |z(\theta)|^2 \,\mathrm{d}\theta}, \end{cases} \tag{1.33}$$

in the Hilbert space $\mathcal{D}_0 := \mathbb{R}^{n_x} \times L^2(0, 1; \mathbb{R}^{n_z})$.

Definition 1.6. The trivial solution to system ($\mathcal{S}_1$) is globally exponentially stable (GES) if, there exist $\mu > 0$ and $\kappa \geq 1$ such that for every initial function $\left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right] \in \mathcal{D}_1$, inequality $\big|\big(x(t), z(t)\big)\big| \leq \kappa e^{-\mu t}\big|\big(x_0, z_0\big)\big|$ holds, for all $t \in \mathbb{R}_+$.

Another way to deal with the stability of ODE-transport systems is to consider the norm

$$\big|(\cdot,\cdot)\big|_\infty : \begin{cases} \tilde{\mathcal{D}}_0 \to \mathbb{R}, \\ (x, z) \mapsto \big|(x, z)\big|_\infty = \max\left(|x|, \sup\limits_{[0,1]} |z(\theta)|\right), \end{cases} \tag{1.34}$$

in the Banach space $\tilde{\mathcal{D}}_0 := \mathbb{R}^{n_x} \times C_{pw}(0, 1; \mathbb{R}^{n_x})$ for the initial condition and, the finite-dimensional Euclidian norm $|\cdot|$ for the trajectory. The corresponding definition of global exponential stability at the origin is given below.

Definition 1.7. The trivial solution to system ($\mathcal{S}_1$) is globally exponentially stable (GES) if, there exist $\mu > 0$ and $\kappa \geq 1$ such that for every initial function $\left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right] \in \mathcal{D}_1$, inequality $|x(t)| \leq \kappa\, \mathrm{e}^{-\mu t} \left|\left(x_0, z_0\right)\right|_\infty$ holds, for all $t \in \mathbb{R}_+$.

Hereafter, it is shown that global exponential stability definitions with both norms are equivalent. Indeed, as formulated in the following proposition, the definitions overlap.

Proposition 1.4. *Global exponential stability within the meaning of Definition 1.6 and 1.7 are equivalent.*

Proof. Firstly, noticing that

$$\left|\left(x, z\right)\right| \leq \sqrt{2}\left|\left(x, z\right)\right|_\infty, \quad \forall \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathcal{D}_1, \tag{1.35}$$

the Definition 1.6 can indifferently be written as

$$\exists \mu > 0,\ \kappa \geq 1; \quad \forall \left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right] \in \mathcal{D}_1, \quad \left|\left(x(t), z(t)\right)\right| \leq \kappa\, \mathrm{e}^{-\mu t} \left|\left(x_0, z_0\right)\right|, \quad \forall t \in \mathbb{R}_+, \tag{1.36}$$

or as

$$\exists \mu > 0,\ \kappa \geq 1; \quad \forall \left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right] \in \mathcal{D}_1, \quad \left|\left(x(t), z(t)\right)\right| \leq \kappa\, \mathrm{e}^{-\mu t} \left|\left(x_0, z_0\right)\right|_\infty, \quad \forall t \in \mathbb{R}_+. \tag{1.37}$$

From (1.36) to (1.37), the proof is trivial and, from (1.37) to (1.36), the proof is conducted by contraposition.
Secondly, according to Proposition 1.2, the assertion (1.37) amounts to

$$\exists \mu > 0,\ \kappa \geq 1; \quad \forall \left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right] \in \mathcal{D}_1, \quad \left|\left(x(t), Cx(t + h(\theta - 1))\right)\right| \leq \kappa\, \mathrm{e}^{-\mu t} \left|\left(x_0, z_0\right)\right|_\infty, \quad \forall t \in \mathbb{R}_+, \tag{1.38}$$

where x is the solution to system (1.29). Thirdly, it is possible to state that (1.38) is equivalent to

$$\exists \mu > 0,\ \kappa' \geq 1; \quad \forall \left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right] \in \mathcal{D}_1, \quad |x(t)| \leq \kappa'\, \mathrm{e}^{-\mu t} \left|\left(x_0, z_0\right)\right|_\infty, \quad \forall t \in \mathbb{R}_+. \tag{1.39}$$

Indeed, assertion (1.38) directly implies (1.39) with $\kappa' = \kappa$.
Reciprocally, inequality (1.39) leads to

$$\left|\left(x(t), Cx(t + h(\cdot - 1))\right)\right|_\infty \leq \max(1, |C|)\kappa'\, \mathrm{e}^{-\mu t} \left|\left(x_0, z_0\right)\right|_\infty$$

which implies (1.38) with $\kappa = \sqrt{2}\max(1, |C|)\kappa'$. □

Remark 1.6. In the case of linear retarded time-delay systems, notice that the asymptotic and exponential stability are equivalent. Since the system's parameters are time-invariant, the stability is said to be uniform even if it is not notified, which means that scalars κ, μ do not depend on the initial time $t_0 = 0$.

We are now in the capacity to formulate the stability problem for ODE-transport systems.

Problem statement 1.1. How to assess global exponential stability of system ($\mathcal{S}_1$) at the origin using tractable numerical tools?

This problem has been widely regarded in the literature through different angles [86]. In the time domain, Lyapunov analysis can be pursued [107, 131, 186]. In the frequency domain, the characteristic root locus can be analyzed [37, 171]. Both approaches are considered and detailed in the sequel and Part II of the manuscript is dedicated to this problem statement.

To understand and point out the difficulty raised by the problem, two examples which are used throughout this thesis, are presented.

Example 1.1. Consider ($\mathcal{S}_1$) with $A = 1$, $B = -2$ and $C = 1$.

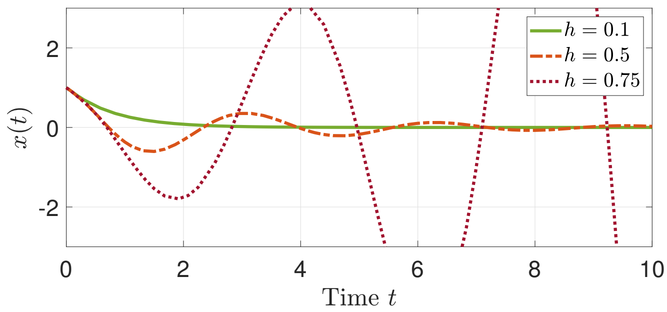


Figure 1.4: Trajectories of Example 1.1 for $h \in \{0.1, 0.5, 0.75\}$.

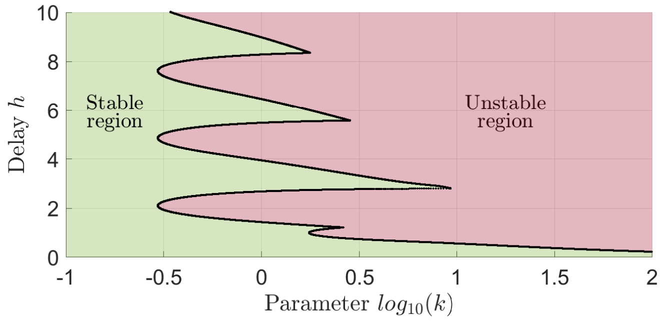


Figure 1.5: Stability regions for Example 1.2 in the plane (k, h) using D-partition method.

Remark 1.7. Note that the first example is a toy example since in the scalar case the solution of the stability question is directly swept by the sign of the real part of the maximal characteristic root $s^* = \frac{1}{h}\mathcal{W}(Ah\,\mathrm{e}^{-Bt}) + B$ [151], where $\mathcal{W}$ is the Lambert function.

Example 1.2. Consider $(\mathcal{S}_1)$ with $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10-k & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ k \\ 0 \end{bmatrix}$ and $C^\top = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, issued from a vibrating system [84].

For Example 1.1, in Figure 1.4, the behavior of the one-dimensional state x is depicted for delays $h \in \{0.1, 0.5, 0.75\}$. Depending on the delay h, the system can be stable or unstable. For Example 1.2, on Figure 1.5, the limitation between stable or unstable regions is also represented in the plane (k, h). For low values of the control gain k, the system seems to be unconditionally stable and, for large values of the control gain k, a maximal allowable delay h_{max} can be identified. For medium values of control gain k, pockets of stability can even be detected. Thus, the stability property of ODE-transport systems is delay-dependent. The development of tools to certify that system $(\mathcal{S}_1)$ is stable or unstable for a given delay h is fundamental.

1.2.2 Second case: System coupled with a reaction-diffusion equation

Replacing the transport equation by the reaction-diffusion equation $(\mathcal{S}_{2\infty})$, the problem statement formulated previously can be extended to other classes of systems.

Definition 1.8. Consider an ordinary differential equation interconnected with a reaction-diffusion equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bz(t,1), & \forall t \in \mathbb{R}_+, & (\mathcal{S}_2\text{a}) \\ \partial_t z(t,\theta) = (\nu\partial_{\theta\theta} + \lambda)z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], & (\mathcal{S}_2\text{b}) \\ z(t,\tfrac{1}{2}) = 0, & \forall t \in \mathbb{R}_+, & (\mathcal{S}_2\text{c}) \\ \partial_\theta z(t,1) = Cx(t), \quad \partial_\theta z(t,0) = Cx(t), & \forall t \in \mathbb{R}_+, & (\mathcal{S}_2\text{d}) \\ \left[\begin{smallmatrix} x(0) \\ z(0,\theta) \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_0 \\ z_0(\theta) \end{smallmatrix}\right] \in \mathcal{D}_2 := \left\{ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{R}^{n_x} \times H^2(0,1;\mathbb{R}^{n_z}) \text{ s.t. } \left[\begin{smallmatrix} \partial_\theta z(1) \\ z(\theta) \end{smallmatrix}\right] = \left[\begin{smallmatrix} Cx \\ -z(1-\theta) \end{smallmatrix}\right] \right\}, & & (\mathcal{S}_2\text{e}) \end{cases}$$

where, without lost of generality, the rank of matrices $B, C^\top$ is full and C is normalized to $|C| = 1$.

This interconnected system ($\mathcal{S}_2$a)-($\mathcal{S}_2$d) can also be modeled using the framework of semigroups and operators [54, 178] as follows

$$\mathcal{A} : \begin{cases} \mathcal{D}_2 \subset \mathcal{D}_0 & \to \mathcal{D}_0 := \mathbb{R}^{n_x} \times L^2(0,1;\mathbb{R}^{n_z}), \\ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] & \mapsto \mathcal{A}\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] := \left[\begin{smallmatrix} Ax+Bz(1) \\ (\lambda+\nu\partial_{\theta\theta})z \end{smallmatrix}\right], \end{cases} \tag{1.40}$$

where the domain $\mathcal{D}_2 := \left\{ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{R}^{n_x} \times H^2(0,1;\mathbb{R}^{n_z}) \text{ s.t. } \left[\begin{smallmatrix} \partial_\theta z(1) \\ z(\theta) \end{smallmatrix}\right] = \left[\begin{smallmatrix} Cx \\ -z(1-\theta) \end{smallmatrix}\right] \right\}$.

Remark 1.8. Here, mixed boundary conditions have been selected. Note that other boundary conditions could also be taken into consideration. It would lead to different solutions and need dedicated studies.

In the literature, ordinary differential equation interconnected with parabolic equation has been regarded in many ways. Mention for instance stabilization problems [126], state feedback synthesis [195], or dynamical output feedback synthesis [128]. In this manuscript, we only focus on stability analysis.

Existence and uniqueness of solutions

As a first step and before studying the stability of such a class of systems, one verifies the existence and uniqueness of solutions for ODE-reaction-diffusion system ($\mathcal{S}_2$). This issue can be tackled thanks to operator approaches and the application of Lumer-Phillips's theorem [54, 110, 199]. The proof is based on the quasi-dissipativity of operator $\mathcal{A}$ as in [28] and the well-posedness condition is stated in the next proposition.

Proposition 1.5. *System ($\mathcal{S}_2$a)-($\mathcal{S}_2$d) with initial condition ($\mathcal{S}_2$e) admits a unique continuous solution $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$ from $\mathbb{R}_+$ to $\mathcal{D}_2$.*

Proof. One easily shows that $\mathcal{A}$ is closed in $\mathcal{D}_0 \times \mathcal{D}_0$ and that $\mathcal{D}_2$ is dense in the Hilbert space $\mathcal{D}_0$. Moreover, integration by parts gives

$$\langle \mathcal{A}\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] | \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \rangle_\mu = \left[\begin{smallmatrix} x \\ z(1) \end{smallmatrix}\right]^\top \left[\begin{smallmatrix} \mathcal{H}(A) & B+2\nu C^\top \\ * & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ z(1) \end{smallmatrix}\right] + 2\lambda \|z\|^2 - 2\nu \|\partial_\theta z\|^2, \quad \forall \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathcal{D}_2. \tag{1.41}$$

Then, Wirtinger and Cauchy-Schwarz inequalities give

$$\int_{\frac{1}{2}}^1 |\partial_\theta z(\theta)|^2 \, \mathrm{d}\theta \geq \pi^2 \int_{\frac{1}{2}}^1 |z(\theta) - z(1)|^2 \, \mathrm{d}\theta,$$
$$\frac{1}{2} \|\partial_\theta z\|^2 \geq \pi^2 \left(\frac{1}{2} \|z\|^2 + \frac{1}{2} |z(1)|^2 - \|z\| \, |z(1)| \right),$$

which ensures that inequality

$$\langle \mathcal{A}\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] | \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \rangle_\mu \leq \begin{bmatrix} x \\ z(1) \\ \|z\| \end{bmatrix}^\top \begin{bmatrix} \mathcal{H}(A) & B+2\nu C^\top & 0 \\ * & -2\nu\pi^2 I_{n_z} & 2\nu\pi^2 \mathrm{sign}(z(1)) I_{n_z} \\ * & * & 2(\lambda-\nu\pi^2) I_{n_z} \end{bmatrix} \begin{bmatrix} x \\ z(1) \\ \|z\| \end{bmatrix}, \tag{1.42}$$

holds. Therefore, for sufficiently large constant c_1 in $\mathbb{R}$, we have

$$\langle \mathcal{A}[\begin{smallmatrix} x \\ z \end{smallmatrix}] | [\begin{smallmatrix} x \\ z \end{smallmatrix}] \rangle_\mu \leq c_1 \left| (x, z) \right|^2 .$$

Similar calculations lead to $\langle \mathcal{A}^*[\begin{smallmatrix} x \\ z \end{smallmatrix}] | [\begin{smallmatrix} x \\ z \end{smallmatrix}] \rangle_\mu = \langle [\begin{smallmatrix} x \\ z \end{smallmatrix}] | \mathcal{A}[\begin{smallmatrix} x \\ z \end{smallmatrix}] \rangle_\mu \leq c_2 \left| (x, z) \right|^2$, for some scalar c_2 in $\mathbb{R}$. Applying [54, Corrolary 2.2.3], $\mathcal{A}$ is an infinitesimal generator of a strongly continuous semigroup $\mathcal{T}$ on $\mathcal{D}_0$. Applying [199, Proposition 2.3.5], one concludes on the existence of a continuous and unique solution $[\begin{smallmatrix} x \\ z \end{smallmatrix}] : t \mapsto \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \mathcal{T}(t)[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}]$ from $\mathbb{R}_+$ to $\mathcal{D}_2$ with respect to the graph norm associated to the graph $\mathcal{D}_0 \times \mathcal{D}_0$. □

Equilibrium

The second step consists in characterizing the equilibrium of ODE-reaction-diffusion system $(\mathcal{S}_2)$.

Proposition 1.6. *System $(\mathcal{S}_2)$ admits a unique trivial equilibrium $(x_e, z_e) = (0, 0)$ if and only if matrix $\Omega = A + BH(0)C$ is non singular, with the transfer function H given by* (1.13).

Proof. Let (x_e, z_e) be an equilibrium of system $(\mathcal{S}_2)$, meaning that the following relations hold

$$\begin{cases} Ax_e + Bz_e(1) = 0, & (1.43a) \\ (\lambda + \nu\partial_{\theta\theta})z_e(\theta) = 0, & (1.43b) \\ z_e(\tfrac{1}{2}) = 0, & (1.43c) \\ \partial_\theta z_e(1) = Cx_e. & (1.43d) \end{cases}$$

Integrating the differential equation (1.43b) with condition (1.43c), one obtains that

$$z_e(\theta) = \begin{cases} \sinh\left(\tilde{\lambda}(\theta - \tfrac{1}{2})\right)\eta_e, & \text{if } \lambda < 0, \\ \theta\eta, & \text{if } \lambda = 0, \\ \sin\left(\tilde{\lambda}(\theta - \tfrac{1}{2})\right)\eta_e, & \text{if } \lambda > 0, \end{cases}$$

with $\tilde{\lambda} = \sqrt{\frac{|\lambda|}{\nu}}$ and where η_e in $\mathbb{R}^{n_z}$ to be fixed. Computing $\partial_\theta \bar{z}$ and re-injecting this expression into (1.43d) yields

$$\eta_e = \begin{cases} \frac{1}{\tilde{\lambda}\cosh(\frac{\tilde{\lambda}}{2})}Cx_e, & \text{if } \lambda < 0, \\ Cx_e, & \text{if } \lambda = 0, \\ \frac{1}{\tilde{\lambda}\cos(\frac{\tilde{\lambda}}{2})}Cx_e, & \text{if } \lambda > 0, \end{cases}$$

The last constraint (1.43a) can then be written as $\Omega x_e = 0$. Hence, system (1.43) admits a unique solution leading to the trivial equilibrium $(\bar{x}, \bar{z}) = (0, 0)$ if and only if $\det(\Omega) \neq 0$. □

If condition $\det(\Omega) \neq 0$ does not hold, then $s = 0$ is solution of $\det(sI_{n_x} - A - BH(s)C) = 0$ and there is several constant steady states for system $(\mathcal{S}_2)$. Therefore, the trivial equilibirum of system $(\mathcal{S}_2)$ cannot be globally stable. Such a particular case can directly be discarded by a simple test on the singularity of matrix Ω.

Stability definition

Since the solution exists and is unique in $\mathcal{D}_2 \subset \mathcal{D}_0 := \mathbb{R}^{n_x} \times L^2(0, 1; \mathbb{R}^{n_z})$ and since the equilibrium point is only $(0, 0)$, the question of global exponential stability can be raised in the sense of the $\mathcal{D}_0$ norm.

Definition 1.9. The equilibrium point of system $(\mathcal{S}_2)$ is said to be globally exponentially stable if there exists $\mu > 0$ and $\kappa \geq 1$ such that for every initial function $[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}] \in \mathcal{D}_2$, inequality $\left|(x, z)\right| \leq \kappa\, \mathrm{e}^{-\mu t} \left|(x_0, z_0)\right|$ holds, for all $t \in \mathbb{R}_+$.

The problem statement for ODE-reaction-diffusion systems considered in Part III of the manuscript is enunciated.

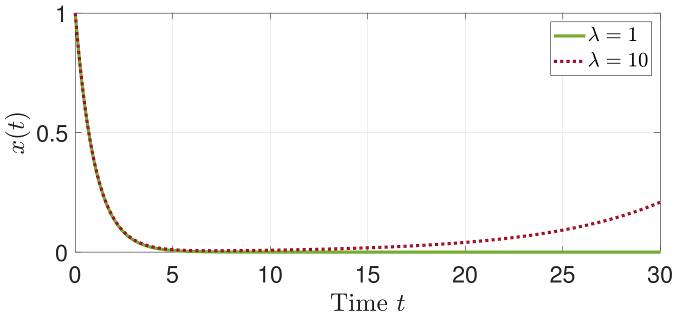


Figure 1.6: Trajectories of Example 1.3 for $k = \frac{1}{4}$ and $\lambda \in \{1, 10\}$.

Problem statement 1.2. How to assess global exponential stability of system ($\mathcal{S}_2$) at the origin using tractable numerical tools?

The stability issue has been investigated through Lyapunov methods [30, 144] or input to state properties [57]. In Part III of the manuscript, the aim is to deal with the stability properties returning to these two techniques.
The following example is used to illustrate our results.

Example 1.3. Consider ($\mathcal{S}_2$) with $A = -1$, $B = k$, $C = 1$ and $\nu = 1$.

Depending on the parameter λ such a system can be stable or unstable as depicted in Figure 1.6.

1.3 Scopes and objectives

1.3.1 Issues and challenges

The notions of equilibrium and stability have been introduced by Lagrange and Dirichlet in the earlier nineteenth century. Initially, the goal was to describe and characterize the asymptotic behavior of mechanical systems. With an initial condition next to the equilibrium and taking into account the internal and external energies of the system, the idea was to determine if the trajectory is bounded or not. Since the second half of the twentieth century, these stability concepts have been formalized [10, 130] thanks to Lyapunov definition. The state-space representation also provides a common framework to study the stability of any kind of system. In the finite dimension, two main techniques can be counted. The first one is spectral analysis [42]. After linearization around an equilibrium point, the eigenvalues of the Jacobian matrix traduces the local stability property thanks to the sign of the real part. The second one is the Lyapunov analysis detailed in Appendix A. The problem can be posed in the linear matrix inequality (LMI) framework [6, 35, 88], which can be solved by convex optimization and semi-definite programming solvers.

The stability of coupled systems has also been investigated in the literature [40, 60]. For two systems in series, the task is easy because it suffices to merge the two spectrums or to combine the two Lyapunov functionals. Nevertheless, for two feedback closed-loop interconnected systems, the stability study is slightly more complicated. Indeed, two stable systems may become unstable once interconnected. Dissipativity [40] or input-to-state properties [60] need then to be regarded. The corresponding results, such as the one obtained by the small-gain theorem or μ-analysis [197], are conservative. Therefore, in the finite dimension, the best way to analyze the stability of interconnected systems is to rewrite the whole interconnection as a new finite-dimensional system and to use the tools proposed in the finite-dimensional field in the previous paragraph.

Actually, the difficulty lodges in the infinite dimension nature of the interconnected system. From one side, the spectral analysis does not resume to eigenvalues of a matrix as in the finite dimension. From the other side, the use of Lyapunov necessary and sufficient theorems (given in Lemmas A.2 and A.4, respectively) cannot be tested with a numerical setup (LMI or Sum Of Squares tools for instance).

In other words, the use of necessary and sufficient conditions of stability dedicated to linear finite-dimensional systems can be translated to infinite dimension thanks to the semigroup theory [54, 199] but amounts to rush headlong into computational issues. For instance, a formal generalization of the Lyapunov theorem is provided in [58] but does not provide solutions to test numerically the proposed conditions.

To tackle Problem Statements 1.1 and 1.2, further analysis on the infinite-dimensional part is then needed by taking into consideration its specifications and particularities. Two main approaches have been borrowed. From one side, the frequency method consists in splitting the system in well-known blocks and using input-output properties of each interconnected elements [57, 212]. The most standard way to proceed with is to characterize the $\mathcal{H}_2$ norm [22, 121] (induced $\sup - L^2$ norm) or the $\mathcal{H}_\infty$ norm [80, 157] (induced $L^2 - L^2$ norm) of each subsystems. On the other side, the time method consists in finding candidate Lyapunov functionals and using standard inequalities such that Wirtinger or Bessel inequalities to obtain sufficient conditions of stability [153, 158, 183]. One can also use quadratic separation approaches [67, 103, 179]. Note that frequency and time methods can be related in the light of Kalman-Yacubovith-Popov lemma [120, 185]. These two techniques are known to be conservative. According to the literature, only Lyapunov functionals have been built and lead to converse Lyapunov theorems (see [159, 160]).

In that direction, in order to reduce the conservatism, more general Lyapunov functionals have been constructed. By early-lumping approach, the Lyapunov functional is based on a state extended with approximated coefficients (see the discretization procedure in [108] or the consideration of Legendre polynomials coefficients in [188, 189]). By late-lumping approach, staying as long as possible in infinite-dimension with operators theory, Lyapunov theorem gives sum of squares constraints that finally need to be solved via semi-definite programming by approximation [180, 182, 200]. However, few studies have discussed, investigated and expressed the error made during the approximation step. It is in this niche that we will place ourselves.

Our topic can then be spread in a series of open problems:

- *How to assess the stability of interconnected ordinary-partial differential systems?*
- *How to design numerical tools which guarantee stability or instability?*
- *How to quantify the conservatism of the proposed approaches?*
- *How to reduce such a conservatism?*

The main objective of this manuscript is to provide answers to these questions.

1.3.2 Study aims and expected results

The aim of our study is to answer Problem Statements 1.1 and 1.2, respectively in Part II and III of this manuscript. We propose a numerical certificate on stability for systems interconnected with a transport or a reaction-diffusion equation keeping in mind that the methodology could be also followed for other hyperbolic or parabolic differential equations. The main idea is to draw inspiration from finite-dimensional methods. Applying approximation techniques depending on an order n performs on the infinite-dimensional part [201]. However, contrary to previous works, our main contribution is to keep track of the infinite-dimensional residual part and to identify, qualify or quantify the accuracy and the loss of conservatism. This approach is pursued using several strategies.

From one side, in the frequency domain, multiple ways to analyze the stability of a system interconnected with an infinite-dimensional system are conceivable.
Through spectral analysis, the characteristic roots of the whole operator $\mathcal{A}$ need to be determined. For that, a finite-dimensional test on the eigenvalues of an approximated matrix $\mathbf{A}_n$ can be performed. The root locus can then be approximated [36, 208]. Even if stability cannot be directly determined, taking into account the error part allows us to evaluate the accuracy of different approximations.
Through $\mathcal{H}_\infty$ analysis, the small-gain theorem combined with approximation leads to efficient and tractable conditions for stability [80]. It can be presented as frequency-sweeping tests on the approximated rational transfer function H_n, which is supposed to be interconnected with the error. The

Domain	Method	Finite dimension	Infinite dimension
Frequency	Spectral	A Hurwitz	"$\mathbf{A}_n$ Hurwitz"
	$\mathcal{H}_\infty$ analysis	$\lvert H\rvert_{\mathcal{H}_\infty} \leq \gamma$	$\lvert H_n\rvert_{\mathcal{H}_\infty} \leq \gamma_n$
Time	Lyapunov (Sufficiency)	$\exists P \succ 0$; $\mathcal{H}(PA) \prec 0$	$\exists \mathbf{P}_n \succ 0$; $\Xi_n \prec 0$
	Lyapunov (Necessity)	P definite positive	$\mathbf{P}_n$ definite positive

Table 1.1: Finite-dimensional methods for stability analysis of infinite-dimensional systems.

conservatism of the small-gain theorem is then balanced by the accuracy of the approximation.

On the other side, in the time domain, we use the Lyapunov necessary and sufficient theorem.
Thanks to the sufficient side, approximated Lyapunov functionals $\mathcal{V}_n$ of complete Lyapunov functionals $\mathcal{V}$ can be constructed based on an extended state [108, 188]. For the case of Legendre coefficients extension, the work has been studied for systems interconnected with transport [186], wave [25] or diffusion [30] equations for many years. They are presented as LMI conditions and have been proven to be hierarchical with respect to the approximation order n [189]. Here, by performing appropriate manipulations on the error part, we are now in position to prove that these tools converge towards the expected sets of stability and to estimate the rate of convergence.
Thanks to a necessary side, a test of positivity on the complete Lyapunov functional is presented. In the finite dimension, it is straightforward and not very relevant but becomes interesting in infinite dimension. Hence, a simple test of the eigenvalues of a matrix is proposed to certify the instability of the infinite-dimensional system [71, 72, 164]. Moreover, by performing appropriate manipulations on the error part, we can prove that it converges towards the expected sets of stability and estimate the rate of convergence [96, 98].

These techniques to assess the stability of infinite-dimensional systems numerically have been summed up in Table 1.1.

1.3.3 Outline of the manuscript

The manuscript is organized as follows.

In Chapter 2, a brief survey on approximation methods of infinite-dimensional state spaces is conducted. This state of the art is directed in order to justify and explain the choice of Legendre polynomials to realize approximation. In view of Legendre approximation, the convergence rates properties and Wirtinger and Bessel inequalities will be presented. It lays the foundations of the manuscript.

Part II is dedicated to ODE-transport interconnected system ($\mathcal{S}_1$). In Chapter 3, two models based on Legendre approximation are constructed. Links with existing models are enlightened thanks to frequency-time up-and-down. In Chapter 4, answers to problem statement 1.1 are provided in light of frequency or time methods. In the frequency domain, only sufficient conditions of stability are derived [13, 16]. In the time domain, the existence of complete Lyapunov functionals leads to necessary and sufficient conditions of stability [14, 15, 18].

Part III is dedicated to ODE-reaction-diffusion interconnected system ($\mathcal{S}_2$). In Chapter 5, models based on trigonometric or polynomial approximation are constructed. The one based on Legendre approximation turns out to be a Padé approximant. In Chapter 6, taking support on both models, the problem statement 1.2 is tackled and sufficient conditions of stability are obtained [17, 19].

List of publications related to Part II

[13] M. Bajodek, F. Gouaisbaut, and A. Seuret. "Frequency delay-dependent stability criterion for time-delay systems thanks to Fourier-Legendre remainders". In: *International Journal of Robust and Nonlinear Control* 31.12 (2021), pp. 5813–5831.

[14] M. Bajodek, F. Gouaisbaut, and A. Seuret. *Estimation of the necessary order of Legendre-LMI conditions to assess stability of time-delay systems.* submitted to 17th IFAC Workshop on Time Delay Systems (TDS). 2022.

[15] M. Bajodek, F. Gouaisbaut, and A. Seuret. *Necessary and sufficient stability condition for time-delay systems arising from Legendre approximation.* hal-03435028. submitted to IEEE Transactions on Automatic Control (second round). 2022.

[16] M. Bajodek, A. Seuret, and F. Gouaisbaut. "Insight into stability analysis of time-delay systems using Legendre polynomials". In: *Proceedings of the 21st IFAC World Congress.* Vol. 53. 2. 2020, pp. 4816–4821.

[18] M. Bajodek, A. Seuret, and F. Gouaisbaut. *On the necessity of sufficient LMI conditions for time-delay systems arising from Legendre approximation.* hal-03435008v2. Provisionally accepted for publication in Automatica. 2022.

List of related publications related to Part III

[17] M. Bajodek, A. Seuret, and F. Gouaisbaut. "Insight into the stability analysis of the reaction-diffusion equation interconnected with a finite-dimensional system taking support on Legendre orthogonal basis". In: *Advances in Distributed Parameter Systems.* Ed. by J. Auriol, J. Deutscher, G. Mazanti, and G. Valmorbida. Vol. 14. 2021. Chap. 5.

[19] M. Bajodek, A. Seuret, and F. Gouaisbaut. *Stability analysis of an ordinary differential equation interconnected with the reaction-diffusion equation.* hal-03150194. Accepted for publication in Automatica. 2022.

Other publications

[132] C. Kitsos, M. Bajodek, and L. Baudouin. "Estimation of the potential in a 1D wave equation via exponential observers"". In: *Proceedings of the 60th IEEE Conference on Decision and Control (CDC).* 2021.

[133] C. Kitsos, M. Bajodek, and L. Baudouin. *Joint Coefficient and Solution Estimation for the 1D Wave Equation: An Observer-Based Solution to Inverse Problems.* hal-03447113. Provisionally accepted for publication in IEEE Transactions on Automatic Control. 2021.

Chapter 2

Approximation methods for stability analysis purpose

"I have no satisfaction in formulas unless I feel their numerical magnitude."
From lecture at Johns Hopkins University, W. Thomson.

Contents

In a world surrounded by machines, computing science has taken a prominent place. This prominence is justified by the need to reach numerical solutions and to certify their accuracy [38, 63]. Since computers can only carry out a finite number of operations with a finite number of digits, the leitmotif of computer scientists is to approximate appropriately to reduce the complexity of the tasks to be achieved. For instance, an irrational function has to be seen as rational with a reasonable error [20, 21]. Similarly, any function must be seen as a truncated series to be displayed [81]. To sum up, these problems are related to the following questions.

- *How to operate such approximations?*
- *How to qualify and quantify the approximation's accuracy and choose the best approximant in terms of convergence and maneuverability?*

These two questions issued from numerical analysis [198] and approximation theory [48] are the centerpiece of this chapter.

The first section introduces different approximation basis used to approximate functions in $L^2(a, b; \mathbb{K}^{m\times p})$. A comparison between trigonometric and polynomial methods is performed to outline their benefits and drawbacks. Legendre polynomials are finally chosen due to the conditions of use for stability analysis of interconnected ODE-PDE systems and the convergence performances.

The second section presents the convergence properties and standard inequalities verified by the Legendre truncated error function. From one part, Bessel and Wirtinger inequalities are used to propose sufficient stability conditions for infinite-dimensional systems developed in the chapters hereafter. From the other part, the convergence properties are the key for unlocking the proof of the asymptotic necessity of these numerical stability conditions.

2.1 Basics on approximation

The study aim is to approximate functions in the Hilbert space $L^2(0,1;\mathbb{K}^{m\times p})$ in the context pre-established in Chapter 1. Indeed, the idea is to look at state functions, which belong to $\mathcal{D}_1$ for system $(\mathcal{S}_1)$ and to $\mathcal{D}_2$ for system $(\mathcal{S}_2)$. They are issued from infinite-dimensional systems and satisfy partial differential equations (PDE). Our approximation issues are then correlated to the way to manipulate and represent an infinite-dimensional state [51, 141]. The problem of spatial approximation of PDE has been raised.

In the literature, finite element methods are used most of the time to pave multi-dimensional spaces (lines, surface, or volumes) and solve static or dynamical problems [202]. Such methods cover up discretization and interpolation underlying techniques and require a correct selection of the mesh and of the interpolated polynomials. Nowadays, for numerous PDEs, consistent and stable spatial discretization schema are well-known [52] and are related to the time discretization schema through Courant–Friedrichs–Lewy conditions. For one-dimensional linear PDE with homogeneous boundary conditions, it is also possible to perform spectral methods [102, 115] based on Ritz-Galerkin approximation [81]. Indeed, for Strum-Liouville problems, the adequate inner product (i.e. $\langle\cdot|\cdot\rangle_w$ with a weight function w) as well as the privileged basis (i.e. the one generated by the characteristic vectors $\{\varphi_k\}_{k\in\mathbb{N}}$) can be determined. Note that, for large-scale ODE, it is sometimes easier to take approximated models typically intended for standard PDE [29].

Nevertheless, when the PDE is interconnected with an ODE via the boundary conditions, the PDE becomes heterogeneous. It is then more complicated to choose the suitable discretization scheme or search for the spectral basis. Pseudo-spectral approaches such as collocation [101] or *tau* methods [173] have been then derived. Behind these methods, the choice of the approximated support basis is crucial even if a polynomial approximation is often recommended [34, 83] and Legendre polynomials are often considered [149, 163]. This selection of Legendre polynomials approximated basis is explained and argued in this section.

2.1.1 Orthogonal basis of a Hilbert space

Consider a Hilbert space $(\mathcal{D},\langle\cdot|\cdot\rangle_{\mathcal{D}})$, i.e. a complete vector space with respect the symmetric, bilinear and positive definite inner product $\langle\cdot|\cdot\rangle_{\mathcal{D}}$. Orthogonal complete basis of $\mathcal{D}$ can be then constructed as follows.

Definition 2.1. The sequence of functions $\{\varphi_k\}_{k\in\mathbb{N}}$ is orthogonal if the following relation holds

$$\langle\varphi_j|\varphi_k\rangle_{\mathcal{D}} = \begin{cases} \|\varphi_k\|^2_{\mathcal{D}} > 0 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \qquad \forall (j,k) \in \mathbb{N}^2. \tag{2.1}$$

Denoting $\mathcal{D}^\flat$ the closure of the smallest linear subspace of $\mathcal{D}$ containing $\{\varphi_k\}_{k\in\mathbb{N}}$, the sequence of functions $\{\varphi_k\}_{k\in\mathbb{N}}$ forms an orthogonal basis of $\mathcal{D}^\flat \subseteq \mathcal{D}$. Moreover, if $\mathcal{D}^\flat = \mathcal{D}$, the sequence of functions $\{\varphi_k\}_{k\in\mathbb{N}}$ forms a complete orthogonal basis of $\mathcal{D}$.

In other words, most of the time, the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ satisfying (2.1) generates a subspace of $\mathcal{D}$ and is not complete in $\mathcal{D}$. When linear combinations of $\{\varphi_k\}_{k\in\mathbb{N}}$ allow spanning a set, which is dense in $\mathcal{D}$, the completeness is satisfied and the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is said to be complete in $\mathcal{D}$.

Hereafter, several complete orthogonal sequences of the Hilbert space $(L^2(0,1;\mathbb{R}), \langle\cdot|\cdot\rangle)$, where

$$\langle\cdot|\cdot\rangle : \begin{cases} L^2(0,1;\mathbb{R}) \times L^2(0,1;\mathbb{R}) \to \mathbb{R}, \\ \qquad\qquad (f_1, f_2) \mapsto \langle f_1|f_2\rangle = \displaystyle\int_0^1 f_1(\theta) f_2(\theta) \mathrm{d}\theta, \end{cases}$$

will be presented. They are potential candidates for the representation of state functions in $L^2(0,1;\mathbb{R})$ [51]. These sequences allow projecting distributed states and decomposing them via sorted coefficients. Indeed, in view of Galerkin-like approximations, a complete orthogonal basis gives the structure to describe any functions in $L^2(0,1;\mathbb{R})$ [141].

Fourier trigonometric basis

Fourier trigonometric basis and series have been introduced in the eighteenth century to propose analytic solutions for the wave equation (d'Alembert's vibrating string) or for the heat equation (Fourier's theory [82]). In fact, the spectral basis of Sturm-Liouville systems is often formed by trigonometric functions. Through Galerkin approximation, trigonometric basis has then been used to describe the infinite-dimensional state behavior and solve many PDEs with homogeneous boundary conditions [168].

Briefly, the Fourier functions on the interval $[0,1]$ are defined below. One can refer to [90] for further details.

Definition 2.2. Fourier functions q_k are given by

$$q_k(\theta) = \begin{cases} \cos(k\pi\theta) & \text{if } k \in \mathbb{N} \text{ even}, \\ \sin((k+1)\pi\theta) & \text{if } k \in \mathbb{N} \text{ odd}, \end{cases} \qquad \forall \theta \in [0,1]. \tag{2.2}$$

The first Fourier functions are depicted in Figure 2.1a, they are equal to

$$q_0(\theta) = 1, \quad q_1(\theta) = \sin(2\pi\theta), \quad q_2(\theta) = \cos(2\pi\theta), \quad q_3(\theta) = \sin(4\pi\theta).$$

The sequence $\{q_k\}_{k\in\mathbb{N}}$ generates the set of trigonometric polynomials (1-periodic functions), which is dense in $L^2(0,1;\mathbb{R})$. Furthermore, the orthogonality condition holds

$$\langle q_j|q_k\rangle = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k = 0, \\ \frac{1}{2} & \text{if } j = k \neq 0, \end{cases} \qquad \forall (j,k) \in \mathbb{N}^2. \tag{2.3}$$

Then, the sequence of functions $\{q_k\}_{k\in\mathbb{N}}$ forms a complete orthogonal basis of $L^2(0,1;\mathbb{R})$.
It is worth noticing that $q_k(0) = q_k(1)$, for all k in $\mathbb{N}$. Therefore, the corresponding Fourier approximation requires the periodicity of the function to be approximated everywhere, including at the boundaries.

Remark 2.1. Notice that the sequence of functions $\{q_k^c(\theta) := \cos(k\pi\theta)\}_{k\in\mathbb{N}}$ or $\{q_k^s(\theta) := \sin(k\pi\theta)\}_{k\in\mathbb{N}^*}$ also forms a complete orthogonal basis of $L^2(0,1;\mathbb{R})$. These functions $\{q_k^c\}_{k\in\mathbb{N}}$ and $\{q_k^s\}_{k\in\mathbb{N}^*}$ correspond to the characteristic vectors of the reaction-diffusion equation with Neumann and Dirichlet boundary conditions, respectively.

Trigonometric basis, being eigenbasis of standard PDEs with homogeneous boundary conditions, are usually emphasized to represent the corresponding infinite-dimensional states. In order to approximate state functions that do not fall within this framework, other complete orthogonal basis are proposed.

Legendre polynomials basis

Legendre polynomials basis have been introduced as an analytical tool in the nineteenth-century [143]. In physics, they are for instance used to solve the Laplace equation in spherical coordinates. Getting over standard problems and considering PDEs with heterogeneous boundary conditions, Legendre polynomials basis gains interest. Through pseudo-spectral methods, they have been used to approximate the state of infinite-dimensional Hamiltonian system [149], of fractional PDEs [163] or of neural networks [203].

Legendre polynomials are often defined by Rodrigue's formula and are fully described in [139]. Here, an explicit expression of Legendre polynomials is provided.

Definition 2.3. Legendre polynomials l_k are defined as

$$l_k : \begin{cases} [0,1] \to \mathbb{R}, \\ \qquad \theta \mapsto l_k(\theta) = \displaystyle\sum_{i=0}^{k} \binom{k}{i}\binom{k+i}{i}(\theta-1)^i = \sum_{i=0}^{k} \binom{k}{i}^2 (\theta-1)^i \theta^{k-i}. \end{cases} \tag{2.4}$$

The first polynomials are depicted in Figure 2.1b and are equal to

$$l_0(\theta) = 1, \quad l_1(\theta) = 2\theta - 1, \quad l_2(\theta) = 6\theta^2 - 6\theta + 1, \quad l_3(\theta) = 20\theta^3 - 30\theta^2 + 12\theta - 1.$$

The sequence $\{l_k\}_{k\in\mathbb{N}}$ generates the set of polynomials on $[0,1]$, which is dense in $L^2(0,1;\mathbb{R})$. Furthermore, the orthogonality condition holds

$$\langle l_j | l_k \rangle = \begin{cases} 0 & \text{if } j \neq k, \\ \frac{1}{2k+1} & \text{if } j = k, \end{cases} \qquad \forall (j,k) \in \mathbb{N}^2. \tag{2.5}$$

Then, the sequence of polynomials $\{l_k\}_{k\in\mathbb{N}}$ forms a complete orthogonal basis of $L^2(0,1;\mathbb{R})$.
From Bonnet's formula, these polynomials can also be defined recursively as follows.

Definition 2.4. Legendre polynomials l_k are given recursively by

$$\begin{cases} l_0(\theta) = 1, \qquad l_1(\theta) = 2\theta - 1, \\ (k+1)l_{k+1}(\theta) = (2k+1)l_1(\theta)l_k(\theta) - k l_{k-1}(\theta), \quad \forall k \in \mathbb{N}^*. \end{cases} \tag{2.6}$$

Here, it is worth noticing that $l_k(0) \neq l_k(1)$, for all odd integer k. Therefore, there is no restriction a priori on the function to be approximated. It is then possible to approximate non-periodic functions, including at the boundaries.
Contrary to trigonometric functions, Legendre polynomials are model-free. They are not related to specific boundary conditions, can be adapted to deal with heterogeneous PDEs such as ODE-PDE interconnections, and can model a larger range of infinite-dimensional systems. When there is no assumption on the state to be approximated, it is then often relevant to select Legendre polynomials basis.

Other polynomials basis

Other polynomials basis can also be taken into consideration to solve non standard PDEs, where the state belongs to the Hilbert space $(L^2(a,b;\mathbb{R}), \langle\cdot|\cdot\rangle_w)$ with not finite support $[a,b]$ or not-unitary weight w [65]. The inner product is then defined by

$$\langle\cdot|\cdot\rangle_w : \begin{cases} L^2(a,b;\mathbb{R}) \times L^2(a,b;\mathbb{R}) \to \mathbb{R}, \\ \qquad\qquad (f_1, f_2) \mapsto \langle f_1 | f_2 \rangle = \displaystyle\int_a^b f_1(\tau) f_2(\tau) w(\tau) \mathrm{d}\tau. \end{cases}$$

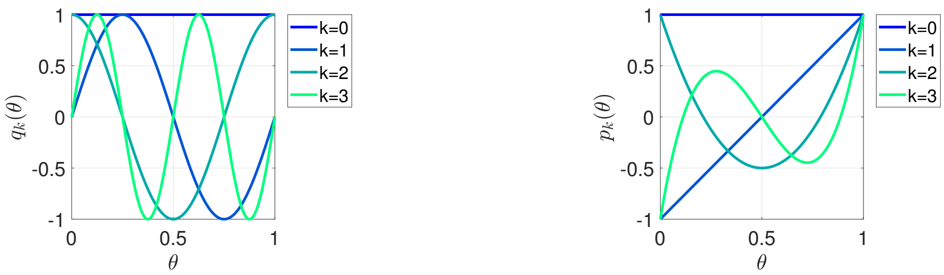
In the case of the space $(L^2(0,1;\mathbb{R}), \langle\cdot|\cdot\rangle_w)$ with weight $w(\theta) = \frac{1}{\sqrt{\theta(1-\theta)}}$, for all θ in $[0,1]$, the appropriate polynomials are given below.

Definition 2.5. Chebyshev polynomials of the first kind t_k are given recursively by

$$\begin{cases} t_0(\theta) = 1, \quad t_1(\theta) = 2\theta - 1, \\ t_{k+1}(\theta) = 2t_1(\theta)t_k(\theta) - t_{k-1}(\theta), \ \forall k \in \mathbb{N}^*, \end{cases} \qquad \forall \theta \in [0,1].$$

The relations and properties verified by Chebyshev polynomials can be found in [83].
Similarly to Legendre polynomials, the sequence $\{t_k\}_{k\in\mathbb{N}}$ generates the set of polynomials on $[0,1]$, which is dense in $L^2(0,1;\mathbb{R})$. Furthermore, the orthogonality condition holds

$$\langle t_j | t_k \rangle_w = \begin{cases} 0 & \text{if } j \neq k, \\ \frac{\pi}{2} & \text{if } j = k \neq 0, \\ \pi & \text{if } j = k = 0, \end{cases} \qquad \forall (j,k) \in \mathbb{N}^2. \tag{2.7}$$



(a) Fourier trigonometric functions. (b) Legendre polynomials.

Figure 2.1: Representation of the first Legendre and Fourier functions.

Then, the sequence of functions $\{t_k\}_{k\in\mathbb{N}}$ forms a complete orthogonal basis of $(L^2(0,1;\mathbb{R}), \langle\cdot|\cdot\rangle_w)$ with weight $w(\theta) = \frac{1}{\sqrt{\theta(1-\theta)}}$. As recommended in [34], to approximate a function in $L^2(0,1;\mathbb{R})$ when there is absolutely no clue, the best choice is Chebyshev polynomials via the use of Chefun library [62].

More generally, according to [2, Chapter 22], Jacobi polynomials are defined by

$$p_k^{\alpha,\beta}(\theta) = \sum_{i=0}^{k} \binom{k+\alpha}{k-i}\binom{k+\beta}{i}(\theta-1)^i\theta^{k-i}, \quad \forall\theta\in[0,1], \quad \forall k\in\mathbb{N}, \tag{2.8}$$

and satisfy the orthogonality condition

$$\left\langle p_j^{\alpha,\beta} \middle| p_k^{\alpha,\beta} \right\rangle_w = \begin{cases} 0 & \text{if } j\neq k, \\ \frac{1}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!} & \text{if } j=k, \end{cases} \quad \forall (j,k)\in\mathbb{N}^2, \tag{2.9}$$

with weight $w(\theta) = 2^{\alpha+\beta}(1-\theta)^\alpha\theta^\beta$. The sequence of functions $\{p_k^{\alpha,\beta}\}_{k\in\mathbb{N}}$ forms a complete orthogonal basis of $(L^2(0,1;\mathbb{R}), \langle\cdot|\cdot\rangle_w)$.

Remark 2.2. It is worth noticing that both Legendre and Chebyshev polynomials are particular cases of Jacobi polynomials, with respectively $(\alpha,\beta)=(0,0)$ and $(\alpha,\beta)=(-\frac{1}{2},-\frac{1}{2})$. Indeed, we can identify $l_k = p_k^{0,0}$ and $t_k = \frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}p_k^{-\frac{1}{2},-\frac{1}{2}}$, for all integer k.

Considering now the interval $[0,\infty)$ with the weight $w(\theta)=\mathrm{e}^{-\theta}$, Laguerre polynomials are orthogonal. They are then often selected to approximate infinite-dimensional states on semi-infinite intervals [152]. On infinite intervals $(-\infty,\infty)$ with the weight $w(\theta)=\mathrm{e}^{-\theta^2}$, the choice of Hermite polynomials is done. Lastly, in some cases [192], one also uses Bernoulli or Euler polynomials.

2.1.2 Definition of approximated and truncated error functions

By Galerkin-like methods [81], the sequences $(\varphi_k)_{k\in\mathbb{N}}$ introduced above allow to build approximations. Indeed, given a truncated order n, the orthogonal projection of any function f in $L^2(0,1;\mathbb{K}^{m\times p})$ on the set generated by the n first functions $(\varphi_k)_{k\in\{0,\dots,n-1\}}$ can be represented by a finite number n of coefficients. These n coefficients are issued from the projection of f on φ_k, for k in $\{0,\dots,n-1\}$, and can be stored and manipulated numerically as a matrix.
These techniques are used to represent infinite-dimensional states as finite-dimensional ones. If the support basis corresponds to the eigenbasis, one talks about spectral approximation [102]. If not, for any other Riesz basis, one talks about pseudo-spectral *tau* approximation [173]. In practice, these approximations can be used to solve PDEs [118, 119] or to synthesize finite-dimensional controllers [165, 166] or observers [128, 129] for infinite-dimensional systems.

Consider $(\varphi_k)_{k\in\mathbb{N}}$ a complete orthogonal basis of $L^2(0,1;\mathbb{R})$ and an order n in $\mathbb{N}$. For writing convenience, one introduces the following notations:

- the vector ϕ_n in $\mathbb{R}^{nm\times m}$ collocates the n first functions of the sequence $(\varphi_k)_{k\in\mathbb{N}}$ and is given by

$$\phi_n(\theta) = \begin{bmatrix}\varphi_0(\theta)I_m & \dots & \varphi_{n-1}(\theta)I_m\end{bmatrix}^\top, \quad \forall\theta\in[0,1], \tag{2.10}$$

- the vector $\mathcal{F}_n$ in $\mathbb{K}^{nm\times p}$ collocates the n first coefficients of any function f in $L^2(0,1;\mathbb{K}^{m\times p})$ on the n first functions of the family $(\varphi_k)_{k\in\mathbb{N}}$ and is given by

$$\mathcal{F}_n = \left(\int_0^1 \phi_n(\theta)\phi_n^\top(\theta)\mathrm{d}\theta\right)^{-1}\left(\int_0^1 \phi_n(\theta)f(\theta)\mathrm{d}\theta\right). \tag{2.11}$$

In the sequel, one denotes with calligraphic letters such collection of coefficients.

Based on these n first functions and components, it is possible to split any function f in $L^2(0,1;\mathbb{K}^{m\times p})$ into two parts. From one part, the approximated function at order n is composed of the n first basis functions weighted by the corresponding coefficients. It can be seen as a finite sum, i.e. as the linear combination of n elements. From the other part, the truncation error at order n is the difference between the original function and its approximation at order n. It can be seen as an infinite series.

Definition 2.6. For any function f in $L^2(0,1;\mathbb{K}^{m\times p})$, define the approximated function at order n as

$$f_n : \left\{\begin{array}{ccl} [0,1] & \to & \mathbb{K}^{m\times p}, \\ \theta & \mapsto & f_n(\theta) = \phi_n^\top(\theta)\mathcal{F}_n. \end{array}\right. \tag{2.12}$$

where ϕ_n and $\mathcal{F}_n$ are respectively given by (2.10), (2.11) and the truncated error function at order n as

$$\tilde{f}_n : \left\{\begin{array}{ccl} [0,1] & \to & \mathbb{K}^{m\times p}, \\ \tau & \mapsto & \tilde{f}_n(\theta) = f(\theta) - f_n(\theta). \end{array}\right. \tag{2.13}$$

Remark 2.3. Note that, up to a normalization preprocessing, any function in $L^2(a,b;\mathbb{K}^{m\times p})$ can be approximated likewise.

Thanks to the completeness, when the order n increases, the following proposition ensures the convergence of the approximated function f_n towards the original function f.

Proposition 2.1. *Assume that the sequence $(\varphi_k)_{k\in\mathbb{N}}$ forms a complete orthogonal basis of $L^2(0,1;\mathbb{R})$. For any function f in $L^2(0,1;\mathbb{K}^{m\times p})$, the L^2 norm (i) and simple (ii) convergences of the approximated function f_n given by* (2.12) *towards f are satisfied.*

(i) The $L^2(0,1;\mathbb{K}^{m\times p})$ norm of the truncated error function $\tilde{f}_n$ given by (2.13) *tends to zero as n tends to infinity, i.e. $\|\tilde{f}_n\| \underset{n\to\infty}{\longrightarrow} 0$.*

(ii) The truncated error function $\tilde{f}_n$ given by (2.13) *converges almost everywhere to zero as n tends to infinity, i.e. $|\tilde{f}_n(\theta)| \underset{n\to\infty}{\longrightarrow} 0$ for almost all $\theta \in (0,1)$.*

Proof. The proof can be found in [48, Section 6.1]. It is based on Stone-Weierstrass's theorem and density arguments. □

For both trigonometric or polynomial basis, L^2 norm and simple convergences are obtained. They are both appropriate basis to process approximation. However, not enough information is given by this statement to choose the best approximation basis. Apart from the convergence of the approximation towards the original function, several questions naturally arise.

- *Is the convergence can be uniform?*
- *How fast is the convergence rate?*

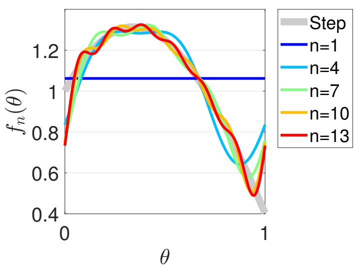
These questions are tackled in the sequel.

2.1.3 Selection of Legendre approximation

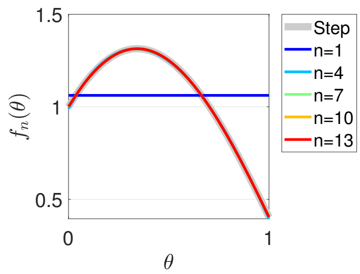
According to the recommendation of [34], summarized in Table 2.1, we propose to use Legendre polynomials. However, this choice is also motivated by uniform convergence property and convergence rates arguments [205].

Support	Periodic $[0,1]$	Finite $[0,1]$		$[0,\infty)$	$(-\infty,\infty)$
Weight	1	1	$\frac{1}{\sqrt{\theta(1-\theta)}}$	$\mathrm{e}^{-\theta}$	$\mathrm{e}^{-\theta^2}$
Basis	Fourier	Legendre	Chebyshev	Laguerre	Hermite

Table 2.1: Privileged basis with respect to supports and measures.



(a) Fourier approximation.



(b) Legendre approximation.

Figure 2.2: Approximations at orders $n \in \{1,4,7,10,13\}$ of a smooth function.

Firstly, the simple convergence ensured by Proposition 2.1 (ii) does not rule the convergence at the endpoints $\theta \in \{0,1\}$. The approximation may suffer Gibb's phenomenon with the presence of permanent oscillations in the boundaries of the interval $[0,1]$. Such Gibb's phenomenon can be avoided thanks to uniform convergence proposition stated below.

Proposition 2.2. *For any function f in $H^1(0,1;\mathbb{K}^{m\times p})$ and under requirements depending on the approximation method, the approximated function f_n given by* (2.12) *converges uniformly towards f means that the sup-norm of the truncated error function $\tilde{f}_n$ given by* (2.13) *converges to zero as n tends to infinity, i.e.* $\|\tilde{f}_n\|_\infty \underset{n\to\infty}{\longrightarrow} 0$.

For instance, the requirements needed for Fourier approximation is the fact that function f needs additionally to be periodic, i.e. $f^{(d)}(0) = f^{(d)}(1)$ for all d in $\mathbb{N}$, or at least to verify $f(0) = f(1)$. For polynomial approximation, the periodicity is not required.

Example 2.1. Consider a smooth but non-periodic function

$$f(\theta) = (\cos(2\theta) + \sin(2\theta))\, e^{-0.2\theta}, \quad \forall\theta \in [0,1]. \tag{2.14}$$

To illustrate these theoretical results, a comparison between Fourier and Legendre approximations has been performed in Figure 2.2 on Example 2.1. At order $n = 1$, Legendre and Fourier approximation give the same result. Next, increasing the order n, Figure 2.2b shows that Legendre approximation converge everywhere on $[0,1]$. It is not the case of Fourier approximation on Figure 2.2a, which lies at the endpoints $\{0,1\}$. Indeed, Fourier approximation fails because $f_n(0) = f_n(1)$ for all n in $\mathbb{N}$ while there is a mismatch $f(0) \neq f(1)$.

Remark 2.4. To overpass these limitations, there are some guidelines. For non periodicity, a change of variable can be performed to recover the condition $f(0) = f(1)$. It amounts to use the first polynomials and to perform combined trigonometric-polynomial approximation. Note that imposing conditions $f^{(d)}(0) = f^{(d)}(1)$, for all d in $\mathbb{N}$, falls back to polynomial approximation. For nonregularity, other approximation methods need to be considered such as wavelets with discontinuous mother wavelets [46]. Discretization procedures and piece-wise polynomial approximations could also be explored.

Secondly, in order to reduce the computational load in terms of the number of variables, the objective is to minimize the order n (i.e. the size of matrix $\mathcal{F}_n$) for a given upper bound on error $\|\tilde{f}_n\|_\infty$. Therefore, a focus on the convergence rates is made.

Definition 2.7. Define algebraic and exponential convergences rates.
The approximation converges algebraically at order d, if $\|\tilde{f}_n\|_\infty \underset{n\to\infty}{=} O(\frac{1}{n^d})$.
The approximation converges exponentially with index ι, if $\|\tilde{f}_n\|_\infty \underset{n\to\infty}{=} O(\mathrm{e}^{-qn^\iota})$.
Moreover, the exponential convergence is said to be

- subgeometric if $\iota < 1$, i.e. $\lim\limits_{n\to\infty} \frac{\log\|\tilde{f}_n\|}{n} = 0$,
- geometric if $\iota = 1$, i.e. $\lim\limits_{n\to\infty} \frac{\log\|\tilde{f}_n\|}{n} = -q$ with a positive constant q,
- supergeometric if $\iota > 1$, i.e. $\lim\limits_{n\to\infty} \frac{\log\|\tilde{f}_n\|}{n} = -\infty$.

Before everything else, according to Darboux's principle, the rate of convergence is impacted by the gravest singularity of the function. Indeed, for functions in $H^1(0,1;\mathbb{K}^{m\times p})$ the convergence rate is slower than for smooth functions in $C_\infty(0,1;\mathbb{K}^{m\times p})$. However, the choice of the support basis also has a significant impact. Considering smooth functions f, polynomial approximations converge exponentially fast with a supergeometric convergence rate [205] whereas Fourier approximation achieves the same result if and only if the periodicity is respected, i.e. $f^{(d)}(0) = f^{(d)}(1)$ for all d in $\mathbb{N}$.

On Figure 2.2b, the efficiency and fast convergence rate of Legendre approximation are enlightened. From order $n = 4$, Legendre approximated function f_n already fit the original function f.

To go further into details, the Legendre approximation satisfies the following property [205].

Proposition 2.3. *Let $d \geq 2$. For any function f such that $f, \dots, f^{(d-1)}$ are absolutely continuous in $[0,1]$ and that $f^{(d)}$ belongs to $H^1(0,1;\mathbb{K}^{m\times p})$, the Legendre approximation of function f given by Definition 2.6 satisfies $\|\tilde{f}_n\|_\infty \underset{n\to\infty}{=} O\left(\frac{1}{n^{d-\frac{1}{2}}}\right)$.*

Proof. The proof can be found in [205, Theorem 2.5]. Some elements of the proof are also presented in the next section. $\square$

This proposition highlights the fact that the regularity of each successive derivatives of the function f allows to increase unitary the algebraic convergence rate. As an underlying result, for smooth functions f, we recover the supergeometric convergence of Legendre approximation because $\|\tilde{f}_n\|_\infty = \underset{n\to\infty}{O}\left(\frac{1}{n!}\right) = \underset{n\to\infty}{O}\left(\frac{1}{n^n}\right) = \underset{n\to\infty}{O}\left(\mathrm{e}^{-n\log(n)}\right)$ applying Stirling formula.

Remark 2.5. Notice that, the convergence rate of Legendre approximation is better by a factor $\frac{1}{\sqrt{n}}$ in the open interval $(0,1)$ than in the closed interval $[0,1]$. In that vein, Chebychev approximation has been introduced and has led to $\|\tilde{f}_n\|_\infty \underset{n\to\infty}{=} O\left(\frac{1}{n^d}\right)$ on the closed interval $[0,1]$ for any function f such that $f, \dots, f^{(d-1)}$ are absolutely continuous in $[0,1]$ and that $f^{(d)}$ belongs to $H^1(0,1;\mathbb{K}^{m\times p})$. The uniform convergence rate of Chebychev approximation is then slightly faster than Legendre approximation by a factor $\frac{1}{\sqrt{n}}$ [83], which justifies its fame in numerical analysis field [62].

Remark 2.6. Notice also that, for periodic functions, the fast Fourier transform stays the most efficient and tractable numerical tool to realize approximation [207].

Lastly, regardless of the convergence rate, the approximation scheme needs to be easy to manipulate for stability analysis purposes. In order to ease the calculations, the unitary uniform weight $w(\theta) = 1$, for all $\theta \in [0,1]$ has been selected. Moreover, Lyapunov functionals are often provided with such a weight. Chebychev approximation has then been discarded.
Balancing all these arguments, Legendre polynomials were selected in order to use the most accurate and convenient basis to deal with apriori non-periodic function in $L^2(0,1;\mathbb{K}^{m\times p})$. In the following, the convergence properties raised in this paragraph are detailed for the Legendre approximation. Through the prism of Legendre polynomials, a focus on inequalities used for stability analysis is also performed.

2.2 Properties of Legendre approximation

After a detailed discussion and comparison, the Legendre approximation has been selected to represent in a finite dimension space the infinite-dimensional state of dynamical systems. However, the approximation error should not be overlooked. To fill this gap between finite and infinite dimension, quantitative relations satisfied by the truncated Legendre error function needs to be constructed.

- *What are the convergence guarantees satisfied by the Legendre approximation?*
- *Do Bessel and Wirtinger inequalities can be adapted to Legendre approximation?*

2.2.1 Pointwise and derivation properties of Legendre polynomials

In this section, some elementary properties satisfied by Legendre polynomials are recalled hereafter to be used in the proofs of the main results.

Property 2.1. *The Legendre polynomials verify the following properties.*

- ***Point wise values [89]:*** *Legendre polynomials are evaluated point wisely by*

$$l_k(0) = (-1)^k, \quad l_k(1) = 1, \qquad \forall k \in \mathbb{N}, \tag{2.15a}$$
$$l'_k(1) = k(k+1), \qquad \forall k \in \mathbb{N}, \tag{2.15b}$$
$$\tag{2.15c}$$

- ***Derivation [89] [102, Chapter 2]:*** *The following differentiation rule is satisfied*

$$l'_{k+1}(\theta) - l'_{k-1}(\theta) = 2(2k+1)l_k(\theta), \quad \forall \theta \in [0,1], \quad \forall k \in \mathbb{N}^*. \tag{2.16}$$

At the first order, Legendre polynomials verify

$$l'_k(\theta) = \sum_{i=0}^{k-1}(2i+1)(1-(-1)^{k+i})l_i(\theta), \quad \forall \theta \in [0,1], \tag{2.17}$$

and, for any f in $H_0^1(0,1;\mathbb{K}^{m\times p})$, the projections satisfy

$$\int_0^1 l_k(\theta)f'(\theta)\mathrm{d}\theta = \sum_{i=k+1}^{\infty}(2i+1)(1-(-1)^{k+i})\int_0^1 l_i(\theta)f(\theta)\mathrm{d}\theta. \tag{2.18}$$

At the second order, Legendre polynomials verify

$$l''_k(\theta) = \sum_{i=0}^{k-1}(2i+1)(1-(-1)^{k+i})\sum_{j=0}^{i-1}(2j+1)(1-(-1)^{i+j})l_j(\theta), \quad \forall \theta \in [0,1], \tag{2.19}$$

and, for any f in $H_0^2(0,1;\mathbb{K}^{m\times p})$, the projections satisfy

$$\int_0^1 l_k(\theta)f''(\theta)\mathrm{d}\theta = \sum_{i=k+1}^{\infty}(2i+1)(1-(-1)^{k+i})\sum_{j=i+1}^{\infty}(2j+1)(1-(-1)^{i+j})\int_0^1 l_j(\theta)f(\theta)\mathrm{d}\theta. \tag{2.20}$$

- ***Boundedness [89] [184, Theorem 61]:*** *Legendre polynomials are bounded by*

$$|l_k(\theta)| \leq 1, \qquad \forall \theta \in [0,1], \qquad \forall k \in \mathbb{N}, \tag{2.21a}$$
$$|l'_k(\theta)| \leq k(k+1), \qquad \forall \theta \in [0,1], \qquad \forall k \in \mathbb{N}, \tag{2.21b}$$
$$|l_k(\theta)| \leq \frac{1}{2}\sqrt{\frac{\pi}{2k\theta(1-\theta)}}, \qquad \forall \theta \in (0,1), \qquad \forall k \in \mathbb{N}^*. \tag{2.21c}$$

Since Legendre polynomials are used in the rest of the manuscript and for writing comfort, notations (2.10), (2.11) are adapted to $\varphi_k := l_k$, the Legendre polynomials. These properties will be widely used to design models and derive stability properties based on the first Legendre polynomials in the following chapters.

2.2.2 Convergence properties of Legendre truncated error function

As mentioned in Section 2.1.3, for particular sets of functions, the Legendre truncated error function converges uniformly to zero as n tend to infinity on the closed interval $[0, 1]$. Based on the results presented in [205], we quantify here such convergence rates satisfied by Legendre approximation. These properties are really helpful and have a significant impact on the main development of the manuscript. Indeed, it is the key for conducting the proof of convergence for some stability criteria.

Uniform convergence for continuously differentiable functions

Firstly, we focus on a set of functions for which the convergence is simple and limited to $\underset{n\to\infty}{O}(\frac{1}{\sqrt{n}})$. Let $c = \frac{1}{2}$ and consider, for simplicity and consistency reasons, the following set

$$\mathcal{C} := C_0(0, 1; \mathbb{K}^{m\times p}) \cap C_2(0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1); \mathbb{K}^{m\times p}). \tag{2.22}$$

For f in $\mathcal{C}$, the discontinuity point concerns the derivatives of f and is located at $c = \frac{1}{2}$.

Lemma 2.1. *Consider a function $f \in \mathcal{C}$, given by (2.22), such that $\left|f'(\frac{1}{2}^+) - f'(\frac{1}{2}^-)\right| = 2\bar{f}$ at $c = \frac{1}{2}$ and that $\|f''\|_\infty \leq \rho_2$ holds on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. The approximated Legendre function $f_n(\theta)$ defined by (2.12) converges uniformly to $f(\theta)$ on the closed interval $[0, 1]$. More precisely, for all $n \geq 4$, the following inequality holds*

$$\left|\tilde{f}_n(\theta)\right| \leq \frac{\sqrt{\frac{\pi}{2}}(\pi\rho_2 + \bar{f})}{2\sqrt{n-3}}. \tag{2.23}$$

Proof. The objective of the proof is to demonstrate the uniform convergence towards zero of

$$\tilde{f}_n(\theta) = f(\theta) - \phi_n^\top(\theta)\mathcal{F}_n := f(\theta) - \sum_{k=0}^{n-1} l_k(\theta)a_k,$$

with

$$a_k = (2k + 1)\int_0^1 l_k(\theta)f(\theta)\mathrm{d}\theta, \quad \forall k \in \mathbb{N}. \tag{2.24}$$

One aims at showing that the series $\sum_{k=n}^{\infty} l_k(\theta)a_k$ exists and converges uniformly to zero with respect to θ as n tends to infinity. Recalling from (2.21a) that Legendre polynomials satisfy $|l_k(\theta)| \leq 1$, for all $\theta \in [0, 1]$, we have

$$\left|\sum_{k=n}^{N} l_k(\theta)a_k\right| \leq \sum_{k=n}^{N} |a_k|, \quad \forall\tau \in [a, b].$$

Hence, let us now find an upper bound of a_k given in (2.24). Using the relation (2.16) verified by Legendre polynomials, an integration by parts yields

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^1 \left(l'_{k+1}(\theta) - l'_{k-1}(\theta)\right) f(\theta)\mathrm{d}\theta, \\ &= \frac{1}{2}\left(\int_0^1 (l_{k+1}(\theta) - l_{k-1}(\theta))\, f'(\theta)\mathrm{d}\theta + \left[(l_{k-1}(\theta) - l_{k+1}(\theta))\, f(\theta)\right]_0^1\right), \\ &= \frac{1}{2}\int_0^1 (l_{k-1}(\theta) - l_{k+1}(\theta))\, f'(\theta)\mathrm{d}\theta, \end{aligned} \tag{2.25}$$

where the boundary terms vanish insofar as $l_{k+1}(0) = l_{k-1}(0)$ and $l_{k+1}(1) = l_{k-1}(1)$. Repeating this operation, coefficient a_k can be rewritten as, for all $k \geq 2$,

$$\begin{aligned} a_k = & \frac{1}{4(2k-1)} \int_0^1 (l_{k-2}(\theta) - l_k(\theta)) \, f''(\theta) \mathrm{d}\theta - \frac{1}{4(2k+3)} \int_0^1 (l_k(\theta) - l_{k+2}(\theta)) f''(\theta) \mathrm{d}\theta, \\ & + \frac{1}{4(2k-1)} \left([(l_k(\theta) - l_{k-2}(\theta)) \, f'(\theta)]_0^{\frac{1}{2}} + [(l_k(\theta) - l_{k-2}(\theta)) \, f'(\theta)]_{\frac{1}{2}}^1 \right), \\ & - \frac{1}{4(2k+3)} \left([(l_{k+2}(\theta) - l_k(\theta)) \, f'(\theta)]_0^{\frac{1}{2}} - [(l_{k+2}(\theta) - l_k(\theta)) \, f'(\theta)]_{\frac{1}{2}}^1 \right), \\ = & \frac{1}{4(2k-1)} \int_0^1 (l_{k-2}(\theta) - l_k(\theta)) \, f''(\theta) \mathrm{d}\theta - \frac{1}{4(2k+3)} \int_0^1 (l_k(\theta) - l_{k+2}(\theta)) f''(\theta) \mathrm{d}\theta, \\ & - \frac{1}{2(2k-1)} \left(l_k(1/2) - l_{k-2}(1/2) \right) \bar{f} + \frac{1}{2(2k+3)} \left(l_{k+2}(1/2) - l_k(1/2) \right) \bar{f}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} |a_k| \leq & \frac{1}{2k-1} \int_0^1 \min\left(|l_{k-2}(\theta)|, |l_k(\theta)|, |l_{k+2}(\theta)| \right) |f''(\theta)| \, \mathrm{d}\theta \\ & + \frac{1}{2(2k-1)} \min\left(|l_k(1/2)|, |l_{k-2}(1/2)|, |l_{k+2}(1/2)|, |l_k(1/2)| \right) \bar{f}. \end{aligned}$$

By applying (2.21c), one obtains the following upper bound

$$\begin{aligned} |a_k| & \leq \frac{1}{2(2k-1)} \sqrt{\frac{\pi}{2(k-2)}} \left(\int_0^1 \frac{|f''(\theta)|}{\sqrt{\theta(\theta-1)}} \mathrm{d}\theta + \bar{f} \right), \\ & \leq \frac{\sqrt{\frac{\pi}{2}}}{4(k-\frac{1}{2})\sqrt{k-2}} (\pi \rho_2 + \bar{f}) \leq \frac{\sqrt{\frac{\pi}{2}}}{4(k-2)^{\frac{3}{2}}} (\pi \rho_2 + \bar{f}), \quad \forall k \geq 3. \end{aligned}$$

Then, by integral-series comparison, the following inequality holds for any $n \geq 4$,

$$\begin{aligned} \sum_{k=n}^{N} |a_k| & \leq \int_{n-1}^{N} \frac{\sqrt{\frac{\pi}{2}}}{4(k-2)^{\frac{3}{2}}} (\pi \rho_2 + \bar{f}), \\ & = \left(\frac{1}{\sqrt{n-3}} - \frac{1}{\sqrt{N-2}} \right) \frac{\sqrt{\frac{\pi}{2}}}{2} (\pi \rho_2 + \bar{f}). \end{aligned}$$

This sum is bounded as N tends to infinity. Thus, it is possible to define the sequence of functions $\tilde{f}_n(\theta) = \lim_{N\to\infty} \sum_{k=n}^{N} l_k(\theta) a_k$ and to identify an uniform upper bound

$$\left| \tilde{f}_n(\theta) \right| \leq \frac{\sqrt{\frac{\pi}{2}}(\pi \rho_2 + \bar{f})}{2\sqrt{n-3}}, \quad \forall \tau \in [a, b],$$

which concludes the proof. □

From the uniform upper bound given by Lemma 2.1, the following theorem gives an estimation of the order from which $\left\| \tilde{f}_n \right\|_\infty \leq \varepsilon$ is satisfied, for any $\varepsilon > 0$.

Theorem 2.1. *Consider $f \in \mathcal{C}$, given by (2.22), such that $\left| f'(\frac{1}{2}^+) - f'(\frac{1}{2}^-) \right| = 2\bar{f}$ at $c = \frac{1}{2}$ and that $\|f''\|_\infty \leq (2\mu)^2 \rho_0$ holds on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. The truncated error function $\tilde{f}_n$ given by (2.13) verifies, for any $\varepsilon > 0$,*

$$\left\| \tilde{f}_n \right\|_\infty \leq \varepsilon, \quad \forall n \geq \bar{\mathcal{N}}(\varepsilon), \tag{2.26}$$

where the function $\bar{\mathcal{N}}$ is given by

$$\bar{\mathcal{N}}(\varepsilon) = 3 + \left\lceil \frac{\pi(\pi(2\mu)^2 \rho_0 + \bar{f})^2}{8\varepsilon^2} \right\rceil. \tag{2.27}$$

Proof. According to (2.23), the Legendre truncated error function satisfies

$$\left\|\tilde{f}_n\right\|_\infty \leq \frac{\sqrt{\frac{\pi}{2}}(\pi(2\mu)^2\rho_0 + \bar{f})}{2\sqrt{n-3}}, \quad \forall n \geq 4.$$

Therefore, taking $\varepsilon > 0$, one easily checks that, for all $n \geq \bar{N}(\varepsilon)$, inequality $\left\|\tilde{f}_n\right\|_\infty \leq \varepsilon$ holds. □

In the case of functions in $\mathcal{C}$, given by (2.22), the convergence rate is really slow. The relation between the maximal bound ε of $\left\|\tilde{f}_n\right\|_\infty$ and the order n calculated by (2.27) depends on the factor $(\frac{1}{\varepsilon})^2$.

Estimation of the uniform convergence rate for smooth functions

For smooth functions, the Legendre truncated error function converges uniformly and exponentially fast to zero as n tend to infinity on the closed interval $[0,1]$. The following lemma expresses analytically such a uniform upper bound for the Legendre truncated error function.

Lemma 2.2. *Consider f in $C_\infty(0,1;\mathbb{K}^{m\times p})$ such that $\left\|f^{(d)}\right\|_\infty \leq \rho_d$ holds for all $d \in \mathbb{N}$. The approximated Legendre function $f_n(\theta)$ defined by* (2.12) *converges uniformly to $f(\theta)$ on the closed interval* $[0,1]$. *More precisely, for all $d \geq 2$ and $n \geq d+1$, the Legendre truncated error function defined by* (2.13) *satisfies the following inequality*

$$\left\|\tilde{f}_n\right\|_\infty \leq \frac{\rho_{d+1}}{2^d(d-1)(n-\frac{3}{2})\dots(n-d+\frac{1}{2})}. \tag{2.28}$$

Proof. The objective of the proof is to estimate convergent uniform upper bounds depending on ρ_d for the Legendre truncated error function at order n given by

$$\tilde{f}_n(\theta) = f(\theta) - \phi_n^\top(\theta)\mathcal{F}_n := f(\theta) - \sum_{k=0}^{n-1} l_k(\theta)a_k, \quad \forall\theta \in [0,1],$$

with coefficients a_k described by (2.24). To do so, let us show that the series $\sum\limits_{k=n}^{\infty} l_k(\theta)a_k$ exists, is bounded on the closed interval $[0,1]$ and converges uniformly to zero as n tends to infinity. As Legendre polynomials satisfy (2.21a), i.e. $|l_k(\theta)| \leq 1$ on $[0,1]$, we obtain

$$\left|\sum_{k=n}^{N} l_k(\theta)a_k\right| \leq \sum_{k=n}^{N} |a_k|, \quad \forall\theta \in [0,1]. \tag{2.29}$$

Hence, let us now find an upper bound of a_k given in (2.24) similarly to [205]. By integration by parts and repeating operation (2.25) d times, coefficient a_k can be rewritten as, for all $k \geq d+1$,

$$\begin{aligned}
a_k &= \frac{1}{4(2k-1)}\left(\int_0^1 l_{k-2}(\theta)f''(\theta)\mathrm{d}\theta - \int_0^1 \left(1+\tfrac{2k-1}{2k+3}\right)l_k(\theta)f''(\theta)\mathrm{d}\theta + \int_0^1 \left(\tfrac{2k-1}{2k+3}\right)l_{k+2}(\theta)f''(\theta)\mathrm{d}\theta\right), \\
&= \frac{1}{4(2k-1)(2k-3)}\left(\begin{array}{c}
\int_0^1 l_{k-3}(\theta)f'''(\theta)\mathrm{d}\theta \\
-\int_0^1 \left(1+\tfrac{2k-3}{2k+1}+\tfrac{(2k-1)(2k-3)}{(2k+3)(2k+1)}\right)l_{k-1}(\theta)f'''(\theta)\mathrm{d}\theta \\
+\int_0^1 \left(\tfrac{2k-3}{2k+1}+\tfrac{(2k-1)(2k-3)}{(2k+3)(2k+1)}+\tfrac{(2k-1)(2k-3)}{(2k+3)(2k+5)}\right)l_{k+1}(\theta)f'''(\theta)\mathrm{d}\theta \\
-\int_0^1 \left(\tfrac{(2k-1)(2k-3)}{(2k+3)(2k+5)}\right)l_{k+3}(\theta)f'''(\theta)\mathrm{d}\theta
\end{array}\right), \\
&= \frac{1}{2^{d+1}(2k-1)\dots(2(k-d)+1)}\sum_{i=0}^{d+1}\binom{d+1}{i}\alpha_{k,i}\int_0^1 l_{k-1-d+2i}(\theta)f^{(d+1)}(\theta)\mathrm{d}\theta,
\end{aligned}$$

where $\alpha_{k,i}$ are positive coefficients whose expression is omitted for simplicity but which verify $|\alpha_{k,i}| \leq$

1. Hence, using again that $|l_k(\theta)| \leq 1$ is satisfied for all $\theta \in [0,1]$, the following inequalities hold

$$|a_k| \leq \frac{1}{(2k-1)\dots(2(k-d)+1)} \int_0^1 \left| f^{(d+1)}(\theta) \right| \mathrm{d}\theta \leq \frac{\rho_{d+1}}{2^d(k-\frac{1}{2})\dots(k-d+\frac{1}{2})}, \tag{2.30}$$

under the assumption $\left\| f^{(d)} \right\|_\infty \leq \rho_d$. Then, for all $d \geq 2$ and $n \geq d+1$, it yields

$$\begin{aligned}
\sum_{k=n}^{N} |a_k| &\leq \frac{\rho_{d+1}}{2^d} \sum_{k=n}^{N} \left(\frac{1}{(k-\frac{1}{2})\dots(k-d+\frac{1}{2})} \right), \\
&= \frac{\rho_{d+1}}{2^d(d-1)} \sum_{k=n}^{N} \left(\frac{1}{(k-\frac{3}{2})\dots(k-d+\frac{1}{2})} - \frac{1}{(k-\frac{1}{2})\dots(k-d+\frac{3}{2})} \right), \\
&= \frac{\rho_{d+1}}{2^d(d-1)} \left(\frac{1}{(n-\frac{3}{2})\dots(n-d+\frac{1}{2})} - \frac{1}{(N-\frac{1}{2})\dots(N-d+\frac{3}{2})} \right).
\end{aligned}$$

This sum is bounded as N tends to infinity. Consequently, it is possible to define the sequence of functions $\tilde{f}_n(\theta) = \lim\limits_{N\to\infty} \sum\limits_{k=n}^{N} l_k(\theta) a_k$ and to identify the uniform upper bound

$$\left| \tilde{f}_n(\theta) \right| \leq \frac{\rho_{d+1}}{2^d(d-1)(n-\frac{3}{2})\dots(n-d+\frac{1}{2})}, \quad \forall \theta \in [0,1],$$

which concludes the proof. □

As formulated in Section 2.1.3, when the first derivatives of function f are bounded up to order $d+1$, this lemma ensures that $\left| \tilde{f}_n \right| = \underset{n\to\infty}{O} \left(\frac{1}{n^{d-1}} \right)$ and that the convergence of Legendre approximation is algebraic at order $d-1$. Moreover, for smooth functions f, we retrieve that the convergence is supergeometric since $\left\| \tilde{f}_n \right\|_\infty = \underset{n\to\infty}{O} \left(\frac{1}{n!} \right) = \underset{n\to\infty}{O} \left(\frac{1}{n^n} \right)$ from Stirling formula.

Compared to the expectations stated in Section 2.1.3, a small gap can be identified. Indeed, compared to Proposition 2.3, the convergence seems here to be slower by a factor $\frac{1}{\sqrt{n}}$. Actually, refinements could be brought in the proof. By the use of (2.21c) instead of (2.21a) at stage (2.30), we obtain

$$\left\| \tilde{f}_n \right\|_\infty \leq \frac{(\frac{\pi}{2})^{\frac{3}{2}} \rho_{d+1}}{2^d(d-1)(n-\frac{3}{2})\dots(n-d+\frac{1}{2})\sqrt{n-d-1}} = \underset{n\to\infty}{O} \left(\frac{1}{n^{d-\frac{1}{2}}} \right).$$

Note that by the use of (2.21c) also at stage (2.29), we retrieve in L^2 norm

$$\left\| \tilde{f}_n \right\| \leq \frac{(\frac{\pi}{2})^{3} \rho_{d+1}}{2^d(d-1)(n-\frac{3}{2})\dots(n-d+\frac{1}{2})(n-d-1)} = \underset{n\to\infty}{O} \left(\frac{1}{n^{d}} \right).$$

These extensions help us to get back on our feet.

Focusing on smooth functions, we take the advantage of the bounds proposed in Lemma 2.2 to obtain an estimation of the smallest order from which $\left\| \tilde{f}_n \right\|_\infty \leq \varepsilon$ holds, for any $\varepsilon > 0$.

Theorem 2.2. *Consider f in $C_\infty(0,1;\mathbb{K}^{m\times p})$ such that $\left\| f^{(d)} \right\|_\infty \leq (2\mu)^d \rho_0$ for all $d \in \mathbb{N}$. The truncated error function $\tilde{f}_n$ given by (2.13) verifies, for any $\varepsilon > 0$,*

$$\left\| \tilde{f}_n \right\|_\infty \leq \varepsilon, \quad \forall n \geq \mathcal{N}_0(\varepsilon), \tag{2.31}$$

where the function $\mathcal{N}_0$ is given by

$$\mathcal{N}_0(\varepsilon) = 2 + \left\lceil \mu\, \mathrm{e}^{1+\mathcal{W}\left((\mu\, \mathrm{e})^{-1} \log\left(\frac{2\mu^2 \rho_0}{\mathrm{e}^{\lfloor \mu \rfloor} \varepsilon} \right) \right)} \right\rceil, \tag{2.32}$$

and where $\mathcal{W}$ is the Lambert function [50] defined from $\mathbb{R}_+$ to $\mathbb{R}_+$ by $\mathcal{W}(z) = y$ where y is uniquely defined by the relation $\mathrm{e}^y\, y = z$.

Proof. According to (2.28), the Legendre truncated error function $\tilde{f}_n$ is bounded by

$$\|\tilde{f}_n\|_\infty \leq \frac{2\mu^{d+1}\rho_0}{(n-2)\dots(n-d)} = (2\mu^2\rho_0)\prod_{k=2}^{d}\left(\frac{\mu}{n-k}\right), \tag{2.33}$$

for any $d \geq 2$ and $n \geq d+1$. First, note that the order n has to be greater than 3, a condition which will be checked at the end of the proof. Then, in order to obtain the tightest upper bound, the optimal value of d is selected for a given order n. We use the argument of the minima "argmin" function and get to

$$d_m = \underset{2\leq d\leq n-1}{\operatorname{argmin}} \prod_{k=2}^{d}\left(\frac{\mu}{n-k}\right) = \begin{cases} 2 & \text{if } \mu \geq n-2, \\ n-1 & \text{if } \mu < 1, \\ n-1-\lfloor\mu\rfloor & \text{otherwise.}\end{cases} \tag{2.34}$$

It is the maximal order d which satisfies $\frac{\mu}{n-d} \leq 1$. The first case never occurs to the matter of fact that the order n will be sufficiently large. The second case corresponds to small value of μ in $[0,1)$, for which the best order is limited to $d_m = n-1$. Bringing together the two last cases, d_m is equal to $n-1-\lfloor\mu\rfloor$. For such an order d_m, let us apply the logarithm function "log" to (2.33) to obtain

$$\log\left(\|\tilde{f}_n\|_\infty\right) \leq \log(2\mu^2\rho_0) + (d_m-1)\log(\mu) - \sum_{k=2}^{d_m}\log(n-k). \tag{2.35}$$

Applying Maclaurin–Cauchy integral test gives

$$\sum_{k=2}^{d_m}\log(n-k) = \sum_{k=n-d_m}^{n-2}\log(k) \geq \int_{n-d_m-1}^{n-2}\log(x)\mathrm{d}x = \left[x\log\left(\frac{x}{\mathrm{e}}\right)\right]_{n-d_m-1}^{n-2}, \tag{2.36}$$

where e denotes the exponential of 1. Then, inequality (2.35) leads to

$$\log\left(\|\tilde{f}_n\|_\infty\right) \leq \log(2\mu^2\rho_0) + (d_m-1)\log(\mu) - \left[x\log\left(\frac{x}{\mathrm{e}}\right)\right]_{n-1-d_m}^{n-2}, \tag{2.37}$$

$$= -(n-2)\log\left(\frac{n-2}{\mathrm{e}\,\mu}\right) + (n-1-d_m)\log\left(\frac{n-1-d_m}{\mathrm{e}\,\mu}\right) + \log(2\mu^2\rho_0). \tag{2.38}$$

Using relations $n-1-d_m = \lfloor\mu\rfloor$ and $\frac{\lfloor\mu\rfloor}{\mu} \leq 1$, the previous inequality is rewritten as

$$\log\left(\|\tilde{f}_n\|_\infty\right) \leq -(n-2)\log\left(\frac{n-2}{\mu\,\mathrm{e}}\right) + \log\left(\frac{2\mu^2\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right) + \log(\varepsilon). \tag{2.39}$$

Moreover, for any $n \geq \mathcal{N}_0(\varepsilon)$, the following inequality

$$\log\left(\frac{n-2}{\mu\,\mathrm{e}}\right) \geq \mathcal{W}\left((\mu\,\mathrm{e})^{-1}\log\left(\frac{2\mu^2\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right)\right),$$

holds and implies that

$$\left(\frac{n-2}{\mu\,\mathrm{e}}\right)\log\left(\frac{n-2}{\mu\,\mathrm{e}}\right) \geq \mathrm{e}^{\mathcal{W}\left((\mu\,\mathrm{e})^{-1}\log\left(\frac{2\mu^2\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right)\right)}\mathcal{W}\left((\mu\,\mathrm{e})^{-1}\log\left(\frac{2\mu^2\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right)\right) = \frac{\log\left(\frac{2\mu^2\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right)}{\mu\,\mathrm{e}}, \tag{2.40}$$

holds by definition of Lambert's function [50]. To conclude, (2.39) and (2.40) lead to $\|\tilde{f}_n\|_\infty \leq \varepsilon$. □

Dealing with a function f which is the state of a dynamical system, this theorem will be used to estimate the orders from which some stability results must be necessarily satisfied.

Estimation of the uniform convergence rate of the first derivative

For smooth functions, the first derivative of Legendre approximation also converges uniformly and exponentially fast as n tends to infinity. Extending [205], the following lemma expresses analytically a uniform upper bound for the derivatives of the Legendre truncated error function.

Lemma 2.3. *Consider f in $C_\infty(0,1;\mathbb{K}^{m\times p})$ such that $\left\|f^{(d)}\right\|_\infty \leq \rho_d$ holds for all $d\in\mathbb{N}$. The derivatives of the approximated Legendre function $f_n'(\theta)$ defined by* (2.12) *converges uniformly to $f'(\theta)$ on the closed interval $[0,1]$. More precisely, for all $d\geq 4$ and $n\geq d+1$, the derivatives of the Legendre truncated error function defined by* (2.13) *satisfies the following inequality*

$$\left\|\tilde{f}_n'\right\|_\infty \leq \frac{\rho_{d+1}}{2^{d-1}(d-3)(n-\frac{7}{2})\dots(n-d+\frac{1}{2})}, \tag{2.41}$$

Proof. Recall that the first derivative of the Legendre truncated error function at order n is given by

$$\tilde{f}_n'(\theta) = f'(\theta) - \phi_n'^\top(\theta)\mathcal{F}_n := f'(\theta) - \sum_{k=0}^{n-1} l_k'(\theta)a_k, \quad \forall\theta\in[0,1],$$

with coefficients a_k described by (2.24). Again, let us show that the series $\frac{\mathrm{d}}{\mathrm{d}\tau}\sum\limits_{k=n}^{\infty} l_k'(\theta)a_k$ exists, is bounded and converges uniformly to zero with respect to τ as n tends to infinity. As Legendre polynomials satisfy (2.21b), i.e. $l_k'(\theta)\leq k(k+1)$, for all $\theta\in[0,1]$, we have

$$\left|\sum_{k=n}^{N} l_k'(\theta)a_k\right| \leq \sum_{k=n}^{N} k(k+1)\left|a_k\right|, \quad \forall\theta\in[0,1].$$

Hence, by the use of the upper bound (2.30) demonstrated in the proof of Lemma 2.2, the following inequality holds

$$\sum_{k=n}^{N}|a_k| \leq \frac{\rho_{d+1}}{2^d}\sum_{k=n}^{N}\left(\frac{k(k+1)}{(k-\frac{1}{2})(k-\frac{3}{2})(k-\frac{5}{2})\dots(k-d+\frac{1}{2})}\right), \quad \forall n\geq d+1.$$

Noticing that $\frac{k(k+1)}{(k-\frac{1}{2})(k-\frac{3}{2})}$ is equal to $\frac{40}{21}<2$ for $k=5$ and decreases as k increases, we can apply the following developments

$$\begin{aligned}
\sum_{k=n}^{N}|a_k| &\leq \frac{\rho_{d+1}}{2^{d-1}}\sum_{k=n}^{N}\left(\frac{1}{(k-\frac{5}{2})\dots(k-d+\frac{1}{2})}\right),\\
&= \frac{\rho_{d+1}}{2^{d-1}(d-3)}\sum_{k=n}^{N}\left(\frac{1}{(k-\frac{7}{2})\dots(k-d+\frac{1}{2})} - \frac{1}{(k-\frac{5}{2})\dots(k-d+\frac{3}{2})}\right),\\
&= \frac{\rho_{d+1}}{2^{d-1}(d-3)}\left(\frac{1}{(n-\frac{7}{2})\dots(n-d+\frac{1}{2})} - \frac{1}{(N-\frac{5}{2})\dots(N-d+\frac{3}{2})}\right).
\end{aligned}$$

This sum is bounded as N tends to infinity. Consequently, it is possible to define the sequence of functions $\tilde{f}_n'(\tau) = \lim\limits_{N\to\infty}\frac{\mathrm{d}}{\mathrm{d}\tau}\sum\limits_{k=n}^{N} l_k(\frac{\tau-a}{b-a})a_k$ and to identify the uniform upper bound

$$\left|\tilde{f}_n'(\theta)\right| \leq \frac{\rho_{d+1}}{2^{d-1}(d-3)(n-\frac{7}{2})\dots(n-d+\frac{1}{2})}, \quad \forall\theta\in[0,1],$$

which concludes the proof. □

As previously, when the first derivatives of function f are bounded up to order $d+1$, this lemma ensures that $\left|\tilde{f}_n'\right| = \underset{n\to\infty}{O}\left(\frac{1}{n^{d-3}}\right)$ and that the convergence of the first derivative of Legendre approximation is algebraic at order $d-3$. Moreover, for smooth functions, the convergence rate is supergeometric because $\left\|\tilde{f}_n\right\|_\infty = \underset{n\to\infty}{O}\left(\frac{1}{n!}\right) = \underset{n\to\infty}{O}\left(\frac{1}{n^n}\right)$ from Stirling formula.

It could also be possible to extend these convergence rates by a factor $\frac{1}{\sqrt{n}}$ by the use of (2.21c) instead of (2.21b) and to push the upper bound to

$$\left\|\tilde{f}_n'\right\|_\infty \leq \frac{(\frac{\pi}{2})^{\frac{3}{2}}\rho_{d+1}}{2^{d-1}(d-3)(n-\frac{7}{2})\dots(n-d+\frac{1}{2})\sqrt{n-d-1}} = \underset{n\to\infty}{O}\left(\frac{1}{n^{d-\frac{5}{2}}}\right).$$

Focusing on smooth functions, we take the advantage of the bounds proposed in Lemma 2.3 to obtain an estimation of the smallest order from which $\|\tilde{f}'_n\|_\infty \leq \varepsilon$ holds, for any $\varepsilon > 0$.

Theorem 2.3. *Consider f in $C_\infty(0,1;\mathbb{K}^{m\times p})$ such that $\|f^{(d)}\|_\infty \leq (2\mu)^d\rho_0$ for all $d \in \mathbb{N}$. The derivatives of the truncated error function $\tilde{f}_n$ given by (2.13) verifies, for any $\varepsilon > 0$,*

$$\|\tilde{f}'_n\|_\infty \leq \varepsilon, \quad \forall n \geq \mathcal{N}_1(\varepsilon), \tag{2.42}$$

where the function $\mathcal{N}_1$ is given by

$$\mathcal{N}_1(\varepsilon) = 4 + \left\lceil \mu\,\mathrm{e}^{1+\mathcal{W}\left((\mu\,\mathrm{e})^{-1}\log\left(\frac{4\mu^4\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right)\right)} \right\rceil \geq \mathcal{N}_0(\varepsilon), \tag{2.43}$$

and where $\mathcal{W}$ is the Lambert function [50] defined from $\mathbb{R}_+$ to $\mathbb{R}_+$ by $\mathcal{W}(z) = y$ where y is uniquely defined by the relation $\mathrm{e}^y\,y = z$.

Proof. According to (2.41), the derivatives of the Legendre truncated error function $\tilde{f}'_n$ is bounded by

$$\|\tilde{f}'_n\|_\infty \leq \frac{4\mu^{d+1}\rho_0}{(n-4)\dots(n-d)} = (4\mu^4\rho_0)\prod_{k=4}^{d}\left(\frac{\mu}{n-k}\right), \tag{2.44}$$

for any $d \geq 4$ and $n \geq d+1$. First, note that the order n has to be greater than 5, a condition which will be checked at the end of the proof. Then, in order to obtain the tightest upper bound, the optimal value of d is selected for a given order n. We use the argument of the minima "argmin" function and get to

$$d_m = \underset{4\leq d\leq n-1}{\operatorname{argmin}} \prod_{k=4}^{d}\left(\frac{\mu}{n-k}\right) = \begin{cases} 4 & \text{if } \mu \geq n-4, \\ n-1 & \text{if } \mu < 1, \\ n-1-\lfloor\mu\rfloor & \text{otherwise.} \end{cases} \tag{2.45}$$

It is the maximal order d which satisfies $\frac{\mu}{n-d} \leq 1$. The first case never occur to the matter of fact that the order n will be sufficiently large. Bringing together the two other cases, we have $d_m = n-1-\lfloor\mu\rfloor$. For such an order d_m, let us apply the logarithm function "log" to (2.44) to obtain

$$\log\left(\|\tilde{f}_n\|_\infty\right) \leq \log(4\mu^4\rho_0) + (d_m-3)\log(\mu) - \sum_{k=4}^{d_m}\log(n-k). \tag{2.46}$$

Applying Maclaurin–Cauchy integral test gives

$$\sum_{k=4}^{d_m}\log(n-k) = \sum_{k=n-d_m}^{n-4}\log(k) \geq \int_{n-d_m-1}^{n-4}\log(x)\mathrm{d}x = \left[x\log\left(\frac{x}{\mathrm{e}}\right)\right]_{n-d_m-1}^{n-4}. \tag{2.47}$$

Then, inequality (2.35) leads to

$$\log\left(\|\tilde{f}_n\|_\infty\right) \leq \log(4\mu^4\rho_0) + (d_m-3)\log(\mu) - \left[x\log\left(\frac{x}{\mathrm{e}}\right)\right]_{n-1-d_m}^{n-4}, \tag{2.48}$$

$$= -(n-4)\log\left(\frac{n-4}{\mathrm{e}\,\mu}\right) + (n-1-d_m)\log\left(\frac{n-1-d_m}{\mathrm{e}\,\mu}\right) + \log(4\mu^4\rho_0). \tag{2.49}$$

For the same reasons mentioned in (2.39), using relations $n-1-d_m = \lfloor\mu\rfloor$ and $\frac{\lfloor\mu\rfloor}{\mu} \leq 1$, the previous inequality boils down to

$$\log\left(\|\tilde{f}_n\|_\infty\right) \leq -(n-4)\log\left(\frac{n-4}{\mu\,\mathrm{e}}\right) + \log\left(\frac{4\mu^4\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right) + \log(\varepsilon). \tag{2.50}$$

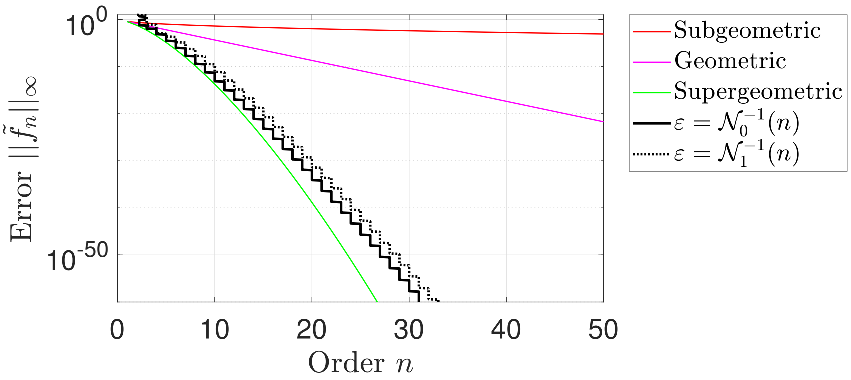


Figure 2.3: Maximal error with respect to approximation orders.

For any $n \geq \mathcal{N}_1(\varepsilon)$, the following inequality

$$\log\left(\frac{n-4}{\mu\,\mathrm{e}}\right) \geq \mathcal{W}\left((\mu\,\mathrm{e})^{-1}\log\left(\frac{4\mu^4\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right)\right),$$

holds and implies that

$$\left(\frac{n-4}{\mu\,\mathrm{e}}\right)\log\left(\frac{n-4}{\mu\,\mathrm{e}}\right) \geq \mathrm{e}^{\mathcal{W}\left((\mu\,\mathrm{e})^{-1}\log\left(\frac{4\mu^2\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right)\right)}\,\mathcal{W}\left((\mu\,\mathrm{e})^{-1}\log\left(\frac{4\mu^2\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right)\right) = \frac{\log\left(\frac{4\mu^2\rho_0}{\mathrm{e}^{\lfloor\mu\rfloor}\,\varepsilon}\right)}{\mu\,\mathrm{e}}, \quad (2.51)$$

holds by definition of Lambert's function [50]. To conclude, (2.50) and (2.51) lead to $\left\|\tilde{f}'_n\right\|_\infty \leq \varepsilon$. Lastly, relation $\mathcal{N}_0(\varepsilon) \leq \mathcal{N}_1(\varepsilon)$ is satisfied for any $\varepsilon > 0$, because Lambert function $\mathcal{W}$ is an increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$. □

For any $\varepsilon > 0$, Theorems 2.2 and 2.3 give respectively an estimation of the minimal orders $\mathcal{N}_0(\varepsilon)$ and $\mathcal{N}_1(\varepsilon)$ such that $\left\|\tilde{f}_n\right\|_\infty \leq \varepsilon$ and $\left\|\tilde{f}'_n\right\|_\infty \leq \varepsilon$ hold. Taking support on Example 2.1 which satisfies $\left\|f^{(d)}\right\|_\infty \leq (2\mu)^d\rho_0$ with $\mu = 0.1$ and $\rho_0 = 2$ and for several errors $\varepsilon \in [10^{-60}, 10]$, estimated orders $\mathcal{N}_0(\varepsilon)$ and $\mathcal{N}_1(\varepsilon)$ are depicted on Fig 2.3. In addition, the subgeometric $\underset{n\to\infty}{O}(e^{-\sqrt{n}})$, geometric $\underset{n\to\infty}{O}(e^{-n})$ and supergeometric $\underset{n\to\infty}{O}(e^{-n^{\frac{3}{2}}})$ behaviors are represented. One can see that both functions $\mathcal{N}_0$ and $\mathcal{N}_1$ are related to supergeometric convergence rates. Indeed, for smooth functions, we demonstrated that $\left\|\tilde{f}_n\right\|_\infty$ and $\left\|\tilde{f}'_n\right\|_\infty$ are in $\underset{n\to\infty}{O}\left(e^{-n\log(n)}\right)$. Thanks to such extremely fast convergence rate, it is worth noticing that the precision 10^{-60} can be reached from order 30. Lastly, based on the expression of functions $\mathcal{N}_0$ (2.32) and of $\mathcal{N}_1$ (2.43), note also that these estimated orders increase as μ or ρ_0 increase.

In the main developments of this manuscript, we will use these functions to estimate orders from which stability conditions holds.Notice that the use of these theorems is not limited to the results presented in the next chapters but could cover many convergence results.

2.2.3 Inequalities on the norm of the truncated error function

Before going any further, Bessel and Wirtinger inequalities are presented in this section. These two inequalities are a milestone to answer to Problem Statements 1.1 and 1.2. As highlighted in the study's aims, considering interconnections between finite and infinite-dimensional systems, the main issue is to manage the infinite-dimensional part. These inequalities, though pessimistic, are crucial to deduce information from the L^2 or H^1 norm of the infinite-dimension state. By realizing the Legendre approximation, the objective is to keep track of this information hidden in the L^2 and H^1 norm of the Legendre truncated error function. Modified Bessel and Wirtinger inequalities are then stated and will be used in the main development of this manuscript.

Modified Bessel inequality

To analyze the stability properties of a system, Lyapunov arguments can be followed as mentioned in Appendix A. However, in an infinite-dimensional field, the conditions required by the Lyapunov theorem cannot be expressed as linear matrix inequalities. For instance, in the derivatives of quadratic Lyapunov functionals, the $L^2(0,1;\mathbb{R}^m)$ norm of the infinite-dimension state appears. Bessel's inequality, widely used in the literature in that context, makes a link between the L^2 norm of the original function $\|f\|$ and the L^2 norm of the approximated function $\|f_n\|$. Thanks to this tool, inequalities involving L^2 norms of the state can be converted to matrix inequalities, of finite dimension.

Let us first recall the generalized Bessel's inequality.

Lemma 2.4. *For any function $f \in L^2(0,1;\mathbb{R}^m)$ and for any integer n in $\mathbb{N}$, the Bessel inequality states that the following inequality holds for any integer n in $\mathbb{N}$*

$$\|f\|^2 \geq \mathcal{F}_n^\top \left(\int_0^1 \phi_n(\theta)\phi_n^\top(\theta)\mathrm{d}\theta\right)\mathcal{F}_n, \tag{2.52}$$

where ϕ_n and $\mathcal{F}_n$ collocate the n first Legendre polynomials and coefficients as given in (2.10) *and* (2.11)*, respectively. They are recalled below*

$$\begin{cases} \phi_n(\theta) = \begin{bmatrix} l_0(\theta)I_m & \dots & l_{n-1}(\theta)I_m \end{bmatrix}^\top \in \mathbb{R}^{nm\times m}, & \forall \theta \in [0,1], \\ \mathcal{F}_n = \left(\int_0^1 \phi_n(\theta)\phi_n^\top(\theta)\mathrm{d}\theta\right)^{-1}\left(\int_0^1 \phi_n(\theta)f(\theta)\mathrm{d}\theta\right) \in \mathbb{R}^{nm}. \end{cases}$$

Proof. The proof is directly derived from the truncation at order n introduced in Definition 2.6. Using the orthogonality property (2.1) of Legendre polynomials, the following equality

$$\|f\|^2 = \mathcal{F}_n^\top \left(\int_0^1 \phi_n(\theta)\phi_n^\top(\theta)\mathrm{d}\theta\right)\mathcal{F}_n + \left\|\tilde{f}_n\right\|^2, \tag{2.53}$$

holds. Then, since $\left\|\tilde{f}_n\right\|^2$ is positive, inequality (2.52) is satisfied. □

When Φ_n collocate Legendre polynomials, (2.52) is Bessel-Legendre inequality used in [30].
This inequality can be also rewritten in order to balance the L^2 norm of the truncated error function with boundary terms.

Theorem 2.4. *For any function f in $H^1(0,1;\mathbb{R}^m)$ and any n in $\mathbb{N}$, the Legendre truncated error function $\tilde{f}_n$ defined by* (2.13) *of f verifies*

$$\left\|\tilde{f}_n'\right\|^2 \geq \begin{bmatrix} \frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2} \\ \frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2} \end{bmatrix}^\top \Upsilon_{n+1}^B \begin{bmatrix} \frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2} \\ \frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2} \end{bmatrix}, \tag{2.54}$$

where, for all k in $\mathbb{N}^*$, matrix Υ_k^B is given by

$$\Upsilon_k^B = \begin{bmatrix} 2k(k+1)I_m & 0 \\ 0 & 2(k-1)kI_m \end{bmatrix}. \tag{2.55}$$

Proof. Thanks to the Bessel-Legendre inequality (2.52) at order $n+1$, the following inequality holds

$$\left\|\tilde{f}_n'\right\|^2 \geq \left(\int_0^1 \phi_{n+1}(\theta)\tilde{f}_n'(\theta)\mathrm{d}\theta\right)^\top \left(\int_0^1 \phi_{n+1}(\theta)\phi_{n+1}^\top(\theta)\mathrm{d}\theta\right)^{-1}\left(\int_0^1 \phi_{n+1}(\theta)\tilde{f}_n'(\theta)\mathrm{d}\theta\right).$$

In addition, performing integration by parts yields

$$\begin{aligned}\int_0^1 \phi_{n+1}(\theta)\tilde{f}_n'(\theta)\mathrm{d}\theta &= -\int_0^1 \phi_{n+1}'(\theta)\tilde{f}_n(\theta)\mathrm{d}\theta + \phi_{n+1}(1)\tilde{f}_n(1) - \phi_{n+1}(0)\tilde{f}_n(0), \\ &= -\int_0^1 \phi_{n+1}'(\theta)\tilde{f}_n(\theta)\mathrm{d}\theta \\ &\quad + (\phi_{n+1}(1) + (-1)^n\phi_{n+1}(0))\left(\frac{\tilde{f}_n(1) - (-1)^n\tilde{f}_n(0)}{2}\right) \\ &\quad + (\phi_{n+1}(1) - (-1)^n\phi_{n+1}(0))\left(\frac{\tilde{f}_n(1) + (-1)^n\tilde{f}_n(0)}{2}\right).\end{aligned} \tag{2.56}$$

Then, we recall that $\tilde{f}_n$ is the Legendre truncated error function, which is consequently orthogonal to the n first Legendre polynomials. Therefore, the first term of the previous equality is zero so that

$$\left\|\tilde{f}_n'\right\|^2 \geq \begin{bmatrix}\frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2} \\ \frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2}\end{bmatrix}^\top \begin{bmatrix}\tilde{\Phi}_{n+1} & \begin{bmatrix}\tilde{\Phi}_n \\ 0\end{bmatrix}\end{bmatrix}^\top \mathcal{I}_{n+1} \begin{bmatrix}\tilde{\Phi}_{n+1} & \begin{bmatrix}\tilde{\Phi}_n \\ 0\end{bmatrix}\end{bmatrix} \begin{bmatrix}\frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2} \\ \frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2}\end{bmatrix},$$

where matrices $\tilde{\Phi}_n = \phi_n(1) - (-1)^n\phi_n(0)$ and $\mathcal{I}_n = \left(\int_0^1 \phi_n(\theta)\phi_n^\top(\theta)\mathrm{d}\theta\right)^{-1}$. To complete the proof, it remains to compute the components of the matrix. By the use of the orthogonality (2.5) of Legendre polynomials we know that $\mathcal{I}_n$ is block diagonal and is equal to $\mathrm{diag}\big(I_m, \ldots, (2n-1)I_m\big)$. By the use of boundaries values of Legendre polynomials given by (2.15a), we also know that $\tilde{\Phi}_n = \left[\,(1-(-1)^n)I_m \;\cdots\; 2I_m\,\right]^\top$. Performing the matrix multiplication, non diagonal coefficients are null. On the diagonal, calculations yields

$$\tilde{\Phi}_n^\top \mathcal{I}_n \tilde{\Phi}_n = \sum_{k=0}^{n-1}(2k+1)(1-(-1)^{n+k})^2 = 2n(n+1),$$

and lead to the result. □

This modified Bessel inequality is used in Chapter 6 in the case $\tilde{f}_n(1) = -\tilde{f}_n(0)$ under the following form

$$\left\|\tilde{f}_{2n}'\right\|^2 \geq \upsilon_n^B \left|\tilde{f}_{2n}(1)\right|^2, \tag{2.57}$$

with $\upsilon_n^B = 4(n+1)(2n+1)$.

Modified Wirtinger inequality

To analyze the stability of PDE at order 2 such that the reaction-diffusion system of Problem Statement 1.2, Lyapunov approaches involve the H^1 norm of the infinite-dimensional state. Wirtinger's inequality, as a particular case of Poincaré inequalities, is often used in that context. It makes the missing connection between H^1 and L^2 norms. Combined with Bessel's inequality, a test involving H^1 norms can then be converted into a matrix inequality, in a pessimistic way.
Let us first recall the Wirtinger inequalities of the first and second kinds [66].

Lemma 2.5. *For any function f in $H^1(0,1;\mathbb{R}^m)$, satisfying $f(0) = f(1) = 0$, inequality $\|f'\| \geq \pi\|f\|$ holds. If, in addition, the mean value of f over $(0,1)$ is null, then inequality $\|f'\| \geq 2\pi\|f\|$ holds.*

Proof. The proof is omitted but can be found in [161]. □

The following theorem extends Wirtinger's inequality, in the situation where no assumptions on the boundary values of the function f nor its truncated error function are needed.

Theorem 2.5. *For any function f in $H^1(0,1;\mathbb{R}^m)$ and for any $n \geq 2$, the Legendre truncated error function $\tilde{f}_n$ defined by* (2.13) *of f verifies*

$$\left\|\tilde{f}_n'\right\|^2 - (\kappa_n\pi)^2 \left\|\tilde{f}_n\right\|^2 \geq \begin{bmatrix} \frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2} \\ \frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2} \end{bmatrix}^\top \Upsilon_n^W \begin{bmatrix} \frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2} \\ \frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2} \end{bmatrix}, \tag{2.58}$$

where

$$\kappa_n = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{otherwise}, \end{cases} \tag{2.59}$$

and where

$$\Upsilon_n^W = \Upsilon_{n-1}^B + (\kappa_n\pi)^2 \begin{bmatrix} \frac{1}{2n-1} & 0 \\ 0 & \frac{1}{2n-3} \end{bmatrix}, \tag{2.60}$$

with Υ_n^B defined in (2.55).

Proof. Let first introduce function $\tilde{g}_n$, defined for all $\theta \in [0,1]$ such that

$$\tilde{g}_n'(\theta) = \tilde{f}_n'(\theta) - \phi_{n-1}(\theta)\left(\int_0^1 \phi_{n-1}(\theta)\phi_{n-1}^\top(\theta)\mathrm{d}\theta\right)^{-1}\left(\int_0^1 \phi_{n-1}(\theta)\tilde{f}_n'(\theta)\mathrm{d}\theta\right).$$

Integrating this function and performing several simplifications, function $\tilde{g}_n$ is given by

$$\tilde{g}_n(\theta) = \tilde{f}_n(\theta) - l_{n-1}(\theta)\left(\frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2}\right) - l_{n-2}(\theta)\left(\frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2}\right). \tag{2.61}$$

First, one has to verify the assumptions of the Wirtinger inequality in Lemma 2.5, that is $\tilde{g}_n(0) = \tilde{g}_n(1) = 0$. From (2.15a), $l_{n-2}(1) = l_{n-1}(1) = 1$, evaluating $\tilde{g}_n(1)$ writes

$$\tilde{g}_n(1) = \tilde{f}_n(1) - \frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2} - \frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2} = \tilde{f}_n(1) - \tilde{f}_n(1) = 0.$$

Similarly, from (2.15a), recalling that $l_{n-2}(0) = -l_{n-1}(0) = (-1)^n$, we have $\tilde{g}_n(0) = \tilde{f}_n(0) - \tilde{f}_n(0) = 0$. Moreover, we also have $\int_0^1 \tilde{g}_n(\theta)\mathrm{d}\theta = 0$ for $n \geq 3$. Therefore, under the previous conditions, Lemma 2.5 states that the inequality $\|\tilde{g}_n'\| \geq (\kappa_n\pi)\|\tilde{g}_n\|$ holds. It remains to compute $\|\tilde{g}_n'\|$ and $\|\tilde{g}_n\|$. On the first-hand side, using the orthogonality, we note that

$$\|\tilde{g}_n'\|^2 = \left\|\tilde{f}_n'\right\|^2 - \left(\int_0^1 \phi_{n-1}(\theta)\tilde{f}_n'(\theta)\mathrm{d}\theta\right)^\top\left(\int_0^1 \phi_{n-1}(\theta)\phi_{n-1}^\top(\theta)\mathrm{d}\theta\right)^{-1}\left(\int_0^1 \phi_{n-1}(\theta)\tilde{f}_n'(\theta)\mathrm{d}\theta\right).$$

The last term of the previous expression has already been computed in (2.56), and we have

$$\|\tilde{g}_n'\|^2 = \left\|\tilde{f}_n'\right\|^2 - \begin{bmatrix} \frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2} \\ \frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2} \end{bmatrix}^\top \Upsilon_{n-1}^B \begin{bmatrix} \frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2} \\ \frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2} \end{bmatrix}. \tag{2.62}$$

On the other hand side, due to orthogonality (2.5), the norm of $\tilde{g}_n$ can be computed as

$$\|\tilde{g}_n\|^2 = \left\|\tilde{f}_n\right\|^2 + \frac{1}{2n-1}\left|\frac{\tilde{f}_n(1)-(-1)^n\tilde{f}_n(0)}{2}\right|^2 + \frac{1}{2n-3}\left|\frac{\tilde{f}_n(1)+(-1)^n\tilde{f}_n(0)}{2}\right|^2. \tag{2.63}$$

The two expressions given in (2.62) and (2.63) merged into $\|\tilde{g}_n'\|^2 \geq (\kappa_n\pi)^2\|\tilde{g}_n\|^2$ simplify to the final equation (2.58). ☐

This theorem is used in Chapter 6 in the case $\tilde{f}_n(1) = -\tilde{f}_n(0)$ under the following form

$$\left\|\tilde{f}_{2n}'\right\|^2 - (2\pi)^2\left\|\tilde{f}_{2n}\right\|^2 \geq \upsilon_n^W\left|\tilde{f}_{2n}(1)\right|^2, \tag{2.64}$$

with $\upsilon_n^W = 4n(2n-1) + \frac{(2\pi)^2}{4n-1}$.

Remark 2.7. With a different approach developed in [77] based on the consideration of the Legendre approximation of function f', it is also possible to obtain values of κ_n which increase with the order n.

Conclusion

This chapter has emphasized the relevance of Legendre polynomial basis for studying stability of infinite-dimensional systems. The capacities of Legendre polynomial approximation lie in its supergeometric convergence rate as well as in the simple reformulation of Bessel and Wirtinger inequalities. In the following, we choose Legendre approximations to build models and explore its potential to analyze the stability of ODE-PDE interconnected systems.

Part II

System interconnected with the transport equation

Chapter 3

Modelling of ODE-transport systems through approximation

"Approximation by polynomials and rational functions is important because ultimately computers can only carry out polynomial and rational operations." About Chebyshev's method, P. Kirchberger.

Contents

NUMEROUS approximated models allow to approximate ODE-transport systems solutions [93]. Beyond such simulation results, numerical models can also be used for root locus [27], stability analysis [107] or control design [165] purposes. However, choosing the best rational or pseudo-spectral approximation scheme is a fundamental question. Besides the interest of being more accurate or less time-consuming, the choice of model also depends on the application field.

- *Can we exhibit links between the existing approximated models?*
- *How to select an appropriate model for stability analysis purposes?*

The first section presents an overview of commonly used techniques to approximate the transport equation. In the second section, leaning on the previous chapter, two models based on Legendre polynomials approximation are introduced. Their interest is compared to those of other techniques proposed in the literature. Relations with existing models will also be highlighted. In the third section, the models based on Legendre approximation will be applied to the case of ODE-transport systems. Finally, the relevance of such models in the context of our Problem Statement 1.1 will be explained.

3.1 Existing models for the transport equation

Recall the transport equation introducted in Chapter 1

$$\begin{cases} h\partial_t z(t,\theta) = \partial_\theta z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \\ z(t,1) = u(t), & \forall t \in \mathbb{R}_+, \\ y(t) = z(t,0), & \forall t \in \mathbb{R}_+, \\ z(0,\theta) = z_0(\theta), & \forall \theta \in (0,1), \end{cases} \tag{$\mathcal{S}_{1\infty}$}$$

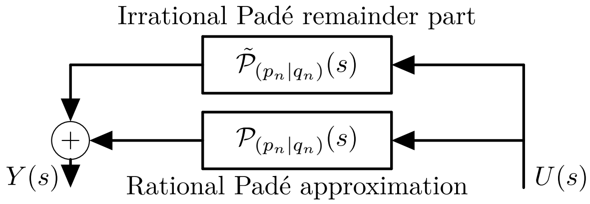


Figure 3.1: Padé modelling of the transport transfer function $H(s) = \mathrm{e}^{-hs}$.

Looking at the literature [27, 118], to approximate the transport equation several approximation methods have been proposed and are recalled in the sequel. However, the forthcoming list of methods is far from being exhaustive and we will focus from one side on frequency methods [27, 177] based on Padé approximation and from the other side on time-domain methods [118, 119] based on Galerkin-like approximation.

3.1.1 Padé methods

Padé methods have been introduced in [21, 175] to approximate irrational functions. These methods can be seen as Diophantine approximation or as a generalization of Taylor expansion with a ratio of two polynomials given as power series.

Let an analytic function $H \in C_\infty(\mathbb{C}, \mathbb{C})$ be expanded in a Maclaurin series.

Definition 3.1. The rational approximation with numerator $N_p(s) = \sum_{i=0}^{p} a_i s^i$ at order p and denominator $D_q(s) = \sum_{i=0}^{q} b_i s^i$ at order q is called $(p|q)$ Padé approximant of function $H(s)$ if

$$H(s) - \frac{N_p(s)}{D_q(s)} = \underset{s\to 0}{O}(s^{p+q+1}). \tag{3.1}$$

Actually, Padé approximations are often used to solve numerically nonlinear fractional partial differential equations, and some extensions such as Padé-Chebyshev [122] or Padé-Legendre [47] approximations have been proposed to improve the solution.

Considering system $(\mathcal{S}_{1\infty})$ in Laplace domain, in the light of Definition 3.1, Padé rational approximations $\mathcal{P}_{(p_n|q_n)}(s) = \frac{N_{p_n}(s)}{D_{q_n}(s)}$ of the transfer function of the transport equation $H(s) = \mathrm{e}^{-hs} I_{n_z}$ can be selected. Indices p_n and q_n are positive integers, which are given as functions of n in $\mathbb{N}$.

Proposition 3.1. *For any order n in $\mathbb{N}$, the delay transfer function $H(s) = \mathrm{e}^{-hs} I_{n_z}$ of system $(\mathcal{S}_{1\infty})$ can be split into two parts*

$$H(s) = \mathcal{P}_{(p_n|q_n)}(s) + \big(H(s) - \mathcal{P}_{(p_n|q_n)}(s)\big), \tag{3.2}$$

where $\tilde{\mathcal{P}}_{(p_n|q_n)}(s) = H(s) - \mathcal{P}_{(p_n|q_n)}(s)$ is the $(p_n|q_n)$ Padé remainder.

This decomposition is depicted in Figure 3.1.

Remark 3.1. Note that the numerical schema [12, 43, 154] for the computation of $\mathcal{P}_{(p_n|q_n)}$ by induction is well-known. The matrix case has also been regarded in [8].

According to [21], the rational approximated part is as accurate as required on any compact subset of $\mathbb{C}$ if the limit of $\frac{p_n}{q_n}$ is finite, as n tends to infinity. This is more formally stated in the next lemma.

Lemma 3.1. *Let sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ satisfy $\lim\limits_{n\to\infty} \frac{p_n}{q_n} = \mu$ and $\mathcal{P}_{(p_n|q_n)}(s) = \frac{N_{p_n}(s)}{D_{q_n}(s)}$ the Padé approximation of $H(s) = \mathrm{e}^{-hs} I_{n_z}$ given by Definition 3.1.*
For any $r > 0$, functions $N_{p_n}(s)$ and $D_{q_n}(s)$ uniformly converge to $\mathrm{e}^{-\frac{\mu}{1+\mu}hs} I_{n_z}$ and $\mathrm{e}^{-\frac{1}{1+\mu}hs} I_{n_z}$

respectively on the open ball $\mathcal{B}(0,r)$.
In other words, for any integers m, n and scalar $r>0$, the following inequalities hold

$$\begin{aligned}\left|N_{p_n}(s)-e^{-\frac{\mu}{1+\mu}hs}I_{n_x}\right| &\le 2\left(\frac{r\,\mathrm{e}}{m}\right)^m+\sum_{k=0}^{m}\left(\frac{r^k}{k!}\left|\prod_{i=0}^{k-1}\frac{p_n-i}{p_n+q_n-i}-\left(\frac{\mu}{1+\mu}\right)^k\right|\right),\\ \left|D_{q_n}(s)-e^{\frac{1}{1+\mu}hs}I_{n_x}\right| &\le 2\left(\frac{r\,\mathrm{e}}{m}\right)^m+\sum_{k=0}^{m}\left(\frac{r^k}{k!}\left|\prod_{i=0}^{k-1}\frac{q_n-i}{p_n+q_n-i}-\left(\frac{1}{1+\mu}\right)^k\right|\right).\end{aligned} \tag{3.3}$$

The remainder of the manuscript will be focused on the $(n-1|n)$ and $(n|n)$ Padé approximants of the transfer function $H(s)=\mathrm{e}^{-hs}\,I_{n_x}$ and the following theorem presents a particular case of Lemma 3.1 in the case $\mu=1$.

Theorem 3.1. *Consider the $(n-1|n)$ and $(n|n)$ Padé approximations of $H(s)=\mathrm{e}^{-hs}\,I_{n_z}$ given by Definition 3.1. For any $\varepsilon>0$ and $r>0$, the numerator N_{p_n} and denominator D_{q_n} of Padé approximation verify*

$$\begin{cases}\left|N_{p_n}(s)-e^{-\frac{hs}{2}}I_{n_x}\right|\le\varepsilon,\\ \left|D_{q_n}(s)-e^{\frac{hs}{2}}I_{n_x}\right|\le\varepsilon,\end{cases}\qquad \forall n\ge\mathcal{M}(\varepsilon), \tag{3.4}$$

where the function $\mathcal{M}$ is given by

$$\begin{aligned}\mathcal{M}(\varepsilon)&=\left\lceil\frac{\mathfrak{m}(\varepsilon)}{2}+\frac{3\,\mathrm{e}^{\max(1,\frac{r\mathfrak{m}(\varepsilon)}{2})}}{\varepsilon}\right\rceil,\\ \mathfrak{m}(\varepsilon)&=\left\lceil 1+r\,\mathrm{e}^{1+\mathcal{W}(r\,\mathrm{e}\log(\frac{3}{\varepsilon}))}\right\rceil.\end{aligned} \tag{3.5}$$

Proof. Consider (3.3) with $\mu=1$, $p_n\in\{n-1,n\}$ and $q_n=n$. First, choose the minimal integer $\mathfrak{m}(\varepsilon)$ such that $\left(\frac{r\,\mathrm{e}}{\mathfrak{m}(\varepsilon)}\right)^{\mathfrak{m}(\varepsilon)}\le\frac{\varepsilon}{3}$ holds. Then, in both inequalities and for any $n>\frac{\mathfrak{m}(\varepsilon)}{2}$, the second term can be upper bounded by the following expression

$$\begin{aligned}\sum_{k=0}^{\mathfrak{m}(\varepsilon)}\left(\frac{r^k}{2^kk!}\left|\prod_{i=0}^{k-1}\left(1+\frac{\left|\frac{q_n-p_n}{2}\right|-\frac{i}{2}}{\frac{p_n+q_n}{2}-\frac{i}{2}}\right)-1\right|\right)&\le\sum_{k=0}^{\mathfrak{m}(\varepsilon)}\left(\frac{r^k}{2^kk!}\left|\left(1+\frac{\frac{1}{2}+\frac{\mathfrak{m}(\varepsilon)-1}{2}}{\frac{2n-1}{2}-\frac{\mathfrak{m}(\varepsilon)-1}{2}}\right)^k-1\right|\right),\\ &\le\sum_{k=0}^{\mathfrak{m}(\varepsilon)}\left(\frac{r^k}{2^kk!}\sum_{i=1}^{k}\binom{k}{i}\left(\frac{\frac{\mathfrak{m}(\varepsilon)}{2}}{n-\frac{\mathfrak{m}(\varepsilon)}{2}}\right)^i\right),\\ &\le\left(\frac{1}{n-\frac{\mathfrak{m}(\varepsilon)}{2}}\right)\sum_{k=0}^{\mathfrak{m}(\varepsilon)}\left(\frac{\max\left(1,\frac{r\mathfrak{m}(\varepsilon)}{2}\right)^k}{k!}\right),\\ &\le\left(\frac{\mathrm{e}^{\max(1,\frac{r\mathfrak{m}(\varepsilon)}{2})}}{n-\frac{\mathfrak{m}(\varepsilon)}{2}}\right).\end{aligned}$$

Therefore, when $n\ge\mathcal{M}(\varepsilon)$, this bound is smaller than $\frac{\varepsilon}{3}$ which yields the result. □

This theorem will be used in the next chapter to derive a stability criterion through quasi-spectral approximation.

Notice that, the finite-dimensional part of Padé modelling is used in the delay blocks on Matlab [93] allowing a classical simulation. Apart from simulation of solutions, the use of the whole Padé modelling to analyze the stability of ODE-transport systems (or indifferently time-delay systems) has been investigated via robust [140] or μ-analysis [210, 211]. In the sequel, the attention is made to study and report the error into the modelling.

3.1.2 Pseudo-spectral methods

Spectral methods based on Galerkin approximation are usually performed to solve linear partial differential equations with homogeneous boundary conditions [115]. But, due to non-homogeneous boundary conditions in system ($\mathcal{S}_{1\infty}$), spectral decomposition is no longer applicable and pseudo-spectral methods are recommended [117]. Two options stand out in the literature: collocation [36] and *tau* [173] methods. Particular attention is paid to the latter.

As an extension of the Galerkin method, *tau* methods were introduced in [141]. It consists in solving the problem satisfied by the truncated series, solution to the differential equations, by putting aside the truncated error [173]. As the solution to system ($\mathcal{S}_{1\infty}$) belongs to the Hilbert space $H^1(0,1;\mathbb{R}^{n_z}) \subset L^2(0,1;\mathbb{R}^{n_z})$, any complete orthogonal basis of L^2 can be selected. As mentioned in the previous chapter, it is the case of Legendre polynomials.

For writing comfort, notations (2.10), (2.11) are then adapted to $\varphi_k := l_k$, the Legendre polynomials, and to dimension n_z, the dimension of the state z which is taken into consideration. Legendre approximation introduced in Definition 2.6 is hereafter applied to the state z in $L^2(0,1;\mathbb{R}^{n_z})$.

Definition 3.2. For any order n in $\mathbb{N}$, the state z can be split on Legendre polynomials basis into an approximated function and a truncated error $\tilde{z}_n$ as follows

$$z(t,\theta) = \ell_n^\top(\theta)\mathcal{Z}_n(t) + \tilde{z}_n(t,\theta), \quad \forall (t,\theta) \in \mathbb{R}_+ \times [0,1]. \tag{3.6}$$

where the vector $\mathcal{Z}_n$ in $\mathbb{R}^{nn_z}$, which collocates the n first Legendre coefficients of the state z, is given by

$$\mathcal{Z}_n(t) = \left(\int_0^1 \ell_n(\theta)\ell_n^\top(\theta)\mathrm{d}\theta\right)^{-1} \left(\int_0^1 \ell_n(\theta) z(t,\theta)\mathrm{d}\theta\right), \quad \forall t \in \mathbb{R}_+. \tag{3.7}$$

and where the matrix ℓ_n in $\mathbb{R}^{nn_z \times n_z}$, which collocates the n first Legendre polynomials, is given by

$$\ell_n(\theta) = \begin{bmatrix} l_0(\theta)I_{n_z} & \dots & l_{n-1}(\theta)I_{n_z}\end{bmatrix}^\top, \quad \forall \theta \in [0,1]. \tag{3.8}$$

Recall also the properties of Legendre polynomials and associated notations which are used all along.

Property 3.1. *In light of Property 2.1 of Legendre polynomials, the following properties are derived.*

- ***Orthogonality*** (2.5)***:*** *Legendre polynomials are orthogonal to each other and, for any $S \in \mathbb{S}_+^n$, they satisfy the orthogonality relation*

$$\mathcal{I}_n^{I_{n_z}} \left(\int_0^1 \ell_n(\theta) S \ell_n^\top(\theta)\mathrm{d}\theta\right)\mathcal{I}_n^{I_{n_z}} = \mathcal{I}_n^S \ \ \textit{with}\ \ \mathcal{I}_n^S = \mathrm{diag}(S, 3S, \dots, (2n-1)S) \in \mathbb{R}^{nn_z \times nn_z}. \tag{3.9}$$

  With a light abuse of notations, $\mathcal{I}_n$ will stand for $\mathcal{I}_n^{I_{n_z}}$.

- ***Point-wise values*** (2.15)***:*** *Legendre polynomials are evaluated point wisely by*

$$\begin{aligned}
\ell_{n,0} &= \ell_n(0) = \begin{bmatrix} l_0(0)I_{n_z} \\ \vdots \\ l_{n-1}(0)I_{n_z}\end{bmatrix} = \begin{bmatrix} I_{n_z} \\ \vdots \\ (-1)^{n-1}I_{n_z}\end{bmatrix} \in \mathbb{R}^{nn_z \times n_z}, \\
\ell_{n,1} &= \ell_n(1) = \begin{bmatrix} l_0(1)I_{n_z} \\ \vdots \\ l_{n-1}(1)I_{n_z}\end{bmatrix} = \begin{bmatrix} I_{n_z} \\ \vdots \\ I_{n_z}\end{bmatrix} \in \mathbb{R}^{nn_z \times n_z}, \\
\ell_{n,2} &= \ell_n(1) - (-1)^n\ell_n(0) = \ell_{n,1} - (-1)^n\ell_{n,0} = \begin{bmatrix} (1-(-1)^n)I_{n_z} \\ \vdots \\ 2I_{n_z}\end{bmatrix} \in \mathbb{R}^{nn_z \times n_z},
\end{aligned} \tag{3.10}$$

- ***Derivation*** (2.17)-(2.18)***:*** *Legendre polynomials verify the following differentiation rule*

$$\ell_n'(\theta) = \mathcal{L}_n \mathcal{I}_n \ell_n(\theta),\ \forall \theta \in [0,1] \quad \textit{with } \mathcal{L}_n = \mathrm{tril}(\ell_{n,1}\ell_{n,1}^\top - \ell_{n,0}\ell_{n,0}^\top) \in \mathbb{R}^{nn_z \times nn_z}, \tag{3.11}$$

and, for any $z \in H_0^1(0,1;\mathbb{R}^{n_z})$, the projections satisfy

$$\int_0^1 \ell_n(\theta)\partial_\theta z(\theta)\mathrm{d}\theta = \mathcal{L}_n^\top \mathcal{I}_n \int_0^1 \ell_n(\theta) z(\theta)\mathrm{d}\theta + \sum_{i=n}^{\infty}(2i+1)(\ell_{n,1} - (-1)^i \ell_{n,0}) \int_0^1 l_i(\theta) f(\theta)\mathrm{d}\theta. \tag{3.12}$$

Notice that we also have the following relation

$$\mathcal{L}_n + \mathcal{L}_n^\top = \ell_{n,1}\ell_{n,1}^\top - \ell_{n,0}\ell_{n,0}^\top. \tag{3.13}$$

Then, the Legendre-*tau* model is derived in the following proposition.

Proposition 3.2. *For any order n in $\mathbb{N}$, system $(\mathcal{S}_{1\infty})$ with an initial condition z_0 in $H_0^1(0,1;\mathbb{R}^{n_z})$ is modeled by the Legendre-tau method as*

$$\begin{cases} h\dot{\mathcal{Z}}_n(t) = \mathcal{I}_n(\mathcal{L}_n^\top - \ell_{n,2}\ell_{n,1}^\top)\mathcal{Z}_n(t) + \mathcal{I}_n \ell_{n,2} u(t) - \mathcal{I}_n \ell_{n,0}\tau_n(t), & \forall t \in \mathbb{R}_+, \\ y(t) = (-1)^{n-1}\ell_{n,2}^\top \mathcal{Z}_n(t) + (-1)^n u(t) + \tau_n(t), & \forall t \in \mathbb{R}_+, \\ \mathcal{Z}_n(0) = \mathcal{I}_n \int_0^1 \ell_n(\theta) z_0(\theta)\mathrm{d}\theta, \end{cases} \tag{3.14}$$

where the error is equal to

$$\tau_n(t) = \sum_{i=n+1}^{\infty} (2i+1)\left((-1)^i - (-1)^n\right) \int_0^1 l_i(\theta) z(t,\theta)\mathrm{d}\theta, \quad \forall t \in \mathbb{R}_+. \tag{3.15}$$

Proof. If z is solution to system $(\mathcal{S}_{1\infty})$, then the n first Legendre coefficients dynamics satisfy

$$h\mathcal{I}_n^{-1}\dot{\mathcal{Z}}_n(t) = h\int_0^1 \ell_n(\theta)\partial_t z(t,\theta)\mathrm{d}\theta = \int_0^1 \ell_n(\theta)\partial_\theta z(t,\theta)\mathrm{d}\theta, \tag{3.16}$$

Based on Galerkin approximation applied to Legendre polynomials (3.12), we obtain

$$h\mathcal{I}_n^{-1}\dot{\mathcal{Z}}_n(t) = \mathcal{L}_n^\top \mathcal{Z}_n + \sum_{i=n}^{\infty}(2i+1)(\ell_{n,1} - (-1)^i\ell_{n,0}) \int_0^1 l_i(\theta) z(t,\theta)\mathrm{d}\theta. \tag{3.17}$$

Moreover, since z belongs to $H_0^1(0,1;\mathbb{R}^{n_z})$, point-wise convergence of Legendre approximation is satisfied on the closed interval $[0,1]$. Then, Definition 3.2 evaluated at $\theta \in \{0,1\}$ yields

$$u(t) = z(t,1) = \ell_{n,1}^\top \mathcal{Z}_n(t) + \sum_{i=n}^{\infty}(2i+1)\int_0^1 l_i(\theta) z(t,\theta)\mathrm{d}\theta, \tag{3.18}$$

$$y(t) = z(t,0) = \ell_{n,0}^\top \mathcal{Z}_n(t) + \sum_{i=n}^{\infty}(2i+1)(-1)^i \int_0^1 l_i(\theta) z(t,\theta)\mathrm{d}\theta. \tag{3.19}$$

From (3.18), it is worth noticing that

$$(2n+1)\int_0^1 l_n(\theta) z(t,\theta)\mathrm{d}\theta = u(t) - \ell_{n,1}^\top \mathcal{Z}_n(t) - \sum_{i=n+1}^{\infty}(2i+1)\int_0^1 l_i(\theta) z(t,\theta)\mathrm{d}\theta.$$

Then, the dynamics (3.17) are given by

$$\begin{aligned}
h\mathcal{I}_n^{-1}\dot{\mathcal{Z}}_n(t) &= \mathcal{L}_n^\top \mathcal{Z}_n(t) + \ell_{n,2}\left(u(t) - \ell_{n,1}^\top \mathcal{Z}_n(t) - \sum_{i=n+1}^{\infty}(2i+1)\int_0^1 l_i(\theta)z(t,\theta)\mathrm{d}\theta\right) \\
&\quad + \sum_{i=n+1}^{\infty}(2i+1)(\ell_{n,1} - (-1)^i\ell_{n,0})\int_0^1 l_i(\theta)z(t,\theta)\mathrm{d}\theta, \\
&= (\mathcal{L}_n^\top - \ell_{n,2}\ell_{n,1}^\top)\mathcal{Z}_n(t) + \ell_{n,2}u(t) \\
&\quad -\ell_{n,0}\sum_{i=n+1}^{\infty}(2i+1)\left((-1)^i - (-1)^n\right)\int_0^1 l_i(\theta)z(t,\theta)\mathrm{d}\theta,
\end{aligned}$$

and the output defined by (3.19) is given by

$$\begin{aligned}
y(t) &= \ell_{n,0}^\top \mathcal{Z}_n(t) + (-1)^n\left(u(t) - \ell_{n,1}^\top \mathcal{Z}_n(t) - \sum_{i=n+1}^{\infty}(2i+1)\int_0^1 l_i(\theta)z(t,\theta)\mathrm{d}\theta\right) \\
&\quad + \sum_{i=n+1}^{\infty}(2i+1)(-1)^i\int_0^1 l_i(\theta)z(t,\theta)\mathrm{d}\theta, \\
&= -(-1)^n\ell_{n,2}^\top \mathcal{Z}_n(t) + (-1)^n u(t) \\
&\quad + \sum_{i=n+1}^{\infty}(2i+1)\left((-1)^i - (-1)^n\right)\int_0^1 l_i(\theta)z(t,\theta)\mathrm{d}\theta,
\end{aligned}$$

which concludes the proof. □

From the finite-dimensional part of such modelling and given an initial condition $\mathcal{Z}_n(0)$, the components $\mathcal{Z}_n(t)$ can be approximated. Then, the Legendre truncated series $\ell_n^\top(\theta)\mathcal{Z}_n(t)$ at order n in $\mathbb{N}$ introduced in Definition 3.2 can be computed.

As aforementioned, we can also take support on another support basis to obtain other models. For the case of ODE-transport systems, a comparison in terms of accuracy of the root locus has been performed in [204]. The Legendre-*tau* model presented in the previous proposition is revealed to be the most appropriate. Moreover, for a given order, *tau* methods turn out to be much more precise than discretization procedures or least-square methods [204].

The *tau* method based on Legendre polynomials is then illustrated in Figure 3.2 with

$$\begin{cases}
\bar{\mathcal{A}}_n = h^{-1}\mathcal{I}_n(\mathcal{L}_n^\top - \ell_{n,2}\ell_{n,1}^\top), & \bar{\mathcal{B}}_n = h^{-1}\mathcal{I}_n\ell_{n,2}, \quad \bar{\mathcal{B}}_n^* = -h^{-1}\mathcal{I}_n\ell_{n,0}, \\
\bar{\mathcal{C}}_n = -(-1)^n\ell_{n,2}^\top, & \bar{\mathcal{D}}_n = (-1)^n.
\end{cases} \tag{3.20}$$

and with an unknown irrational transfer function related to an infinite-dimensional residual system $R_n^\tau(s)$ from $U(s)$ to $\mathcal{T}_n(s)$, which is depicted in Figure 3.2.

Putting aside the *tau* error, matrix $\mathbf{\bar{A}}_n = \left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right) \star \left(\begin{smallmatrix} \bar{\mathcal{A}}_n & \bar{\mathcal{B}}_n \\ \bar{\mathcal{C}}_n & \bar{\mathcal{D}}_n \end{smallmatrix}\right)$ can be constructed, where $\star$ is the Redheffer star product defined in the notations. Such a matrix is used to approximate the solutions to ODE-transport systems (i.e. time-delay systems) and the proof of convergence is provided in [118]. Taking now into account the *tau* error, it lacks information on the transfer function from U to $\mathcal{T}_n$ to perform stability analysis. Indeed, the maximal amount of information on the remainder needs to be kept to obtain certificates on the stability. That is why, in the next section, two new models will be proposed to describe completely the remainder part.

3.2 Legendre modelling for the transport equation

In Part I, we have introduced the transport equation and presented Legendre polynomials as an efficient manner to perform approximation. In this section, we propose two models for the transport equation $(\mathcal{S}_{1\infty})$, i.e. for the the delay transfer function $H(s) = \mathrm{e}^{-hs} I_{n_z}$. As an extension of the previous paragraphs, the idea is to straddle temporal and frequency domains to benefit from their respective advantages. Then, two near models have been developed to realize the dynamics of the first Legendre coefficients of the state (3.7) while keeping a frequency interpretation of block R_n^τ. To keep track of

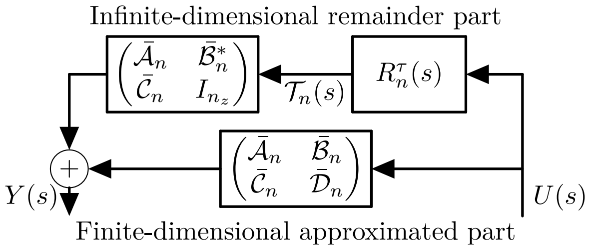


Figure 3.2: Legendre-*tau* modelling of the transport equation $(\mathcal{S}_{1\infty})$.

such a frequency representation of the remaining part, the transfer function from $U(s)$ to $Z(s,\theta)$, given by $G(s,\theta) = \mathrm{e}^{h(\theta-1)s} I_{n_z}$, is split at order n in $\mathbb{N}$ into two parts thanks to Legendre approximation framework given by Definition 2.6 and recalled hereafter.

Definition 3.3. For any order n in $\mathbb{N}$, transfer function G can be split on Legendre polynomials basis into an approximated function and a truncated error $\tilde{G}_n$ as follows

$$G(s,\theta) = \ell_n^\top(\theta)\mathcal{G}_n(s) + \tilde{G}_n(s,\theta), \quad \forall (s,\theta) \in \mathbb{C} \times [0,1], \tag{3.21}$$

where the matrix $\mathcal{G}_n$ in $\mathbb{C}^{nn_z \times n_z}$ collocates the n first Legendre coefficients of the transfer function G and is given by

$$\mathcal{G}_n(s) = \left(\int_0^1 \ell_n(\theta)\ell_n^\top(\theta)\mathrm{d}\theta\right)^{-1} \left(\int_0^1 \ell_n(\theta)G(s,\theta)\mathrm{d}\theta\right), \quad \forall s \in \mathbb{C}, \tag{3.22}$$

and where ℓ_n collocates the n first Legendre polynomials as defined in (3.8).

The milestones will be similar to the ones used for *tau* modelling. Two new features are introduced. The first technical difference comes from the fact that the proof does not require a state in $H_0^1(0,1;\mathbb{R}^{n_z})$ but in $H^1(0,1;\mathbb{R}^{n_z})$. The second philosophical difference is that a permanent cross-check between frequency and time is done along the approximation process.

3.2.1 Model I: a complete realization of $(n-1|n)$ Padé model

From the first part, we derive a strictly causal model based on the n first Legendre polynomials. Applying Legendre approximation at order n, one obtains the following model.

Proposition 3.3. *For any order n in $\mathbb{N}$, system $(\mathcal{S}_{1\infty})$ with an initial condition z_0 in $H^1(0,1;\mathbb{R}^{n_z})$ can be modeled as follows*

$$\left\{\begin{aligned} h\dot{\mathcal{Z}}_n(t) &= -\mathcal{I}_n(\mathcal{L}_n + \ell_{n,0}\ell_{n,0}^\top)\mathcal{Z}_n(t) + \mathcal{I}_n\ell_{n,1}u(t) - \mathcal{I}_n\ell_{n,0}e_n(t), && (3.23a)\\ h\partial_t\tilde{z}_n(t,\theta) &= \partial_\theta\tilde{z}_n(t,\theta) - \ell_n^\top(\theta)\mathcal{I}_n\left(-\ell_{n,1}\ell_{n,1}^\top\mathcal{Z}_n(t) + \ell_{n,1}u(t) - \ell_{n,0}e_n(t)\right), && (3.23b)\\ \tilde{z}_n(t,1) &= u(t) - \ell_{n,1}^\top\mathcal{Z}_n(t), && (3.23c)\\ y(t) &= \ell_{n,0}^\top\mathcal{Z}_n(t) + e_n(t), && (3.23d)\\ e_n(t) &= \tilde{z}_n(t,0), && (3.23e) \end{aligned}\right.$$

for all (t,θ) in $\mathbb{R}_+ \times [0,1]$, with the initial condition

$$\left\{\begin{aligned} \mathcal{Z}_n(0) &= \mathcal{I}_n\int_0^1 \ell_n(\theta)z_0(\theta)\mathrm{d}\theta,\\ \tilde{z}_n(0,\theta) &= z_0(\theta) - \ell_n^\top(\theta)\mathcal{I}_n\int_0^1 \ell_n(\theta)z_0(\theta)\mathrm{d}\theta, \quad \forall\theta \in [0,1]. \end{aligned}\right. \tag{3.24}$$

The error $e_n(t)$ is the boundary output of (3.23b) *and can be seen in the Laplace domain as*

$$E_n(s) = \tilde{G}_n(s,0)U(s), \tag{3.25}$$

where $\tilde{G}_n(s,0)$ is the Legendre truncated error of function $G(s,\theta)$ at order n given by (3.21) *evaluated at $\theta = 0$.*

Proof. Assume that z is solution to system $(\mathcal{S}_{1\infty})$. Firstly, applying an integration by parts to (3.16), the following equation holds

$$h\mathcal{I}_n^{-1}\dot{\mathcal{Z}}_n(t) = -\int_0^1 \ell_n'(\theta)z(t,\theta)\mathrm{d}\theta + \ell_n(1)z(t,1) - \ell_n(0)z(t,0),$$

and can be rewritten using the differentiation property (3.11) satisfied by Legendre polynomials as

$$h\mathcal{I}_n^{-1}\dot{\mathcal{Z}}_n(t) = -\mathcal{L}_n\mathcal{Z}_n(t) + \ell_{n,1}z(t,1) - \ell_{n,0}z(t,0), \tag{3.26}$$

Furthermore, applying the Legendre approximation to state z in light of Definition 3.2 in time or Definition 3.3 in frequencies, allows decomposing the output into two parts

$$\begin{cases} y(t) = z(t,0) = \ell_{n,0}^\top \mathcal{Z}_n(t) + \tilde{z}_n(t,0), \\ Y(s) = Z(s,0) = \left(\ell_{n,0}^\top \mathcal{G}_n(s) + \tilde{G}_n(s,0)\right)U(s). \end{cases} \tag{3.27}$$

The truncated error $e_n(t) = \tilde{z}_n(t,0)$ in Laplace domain is given by $E_n(s) = \tilde{G}_n(s,0)U(s)$. Then, replacing $z(t,0)$ by the expression given by (3.27) yields the first piece of the result

$$h\mathcal{I}_n^{-1}\dot{\mathcal{Z}}_n(t) = -(\mathcal{L}_n + \ell_{n,0}\ell_{n,0}^\top)\mathcal{Z}_n(t) + \ell_{n,1}u(t) - \ell_{n,0}\tilde{z}_n(t,0). \tag{3.28}$$

The second piece of the result concerns the PDE satisfied by the Legendre truncated error. From one side, the boundary condition is issued from the following relation

$$u(t) = z(t,1) = \ell_{n,1}^\top \mathcal{Z}_n(t) + \tilde{z}_n(t,1). \tag{3.29}$$

From the other side, the Legendre truncated error function $\tilde{z}_n$ of the state z at order n given by (3.6) satisfies the following dynamics

$$\begin{aligned} h\partial_t \tilde{z}_n(t,\theta) &= h\partial_t\left(z(t,\theta) - \ell_n^\top(\theta)\mathcal{Z}_n(t)\right), \\ &= \partial_\theta z(t,\theta) - h\ell_n^\top(\theta)\dot{\mathcal{Z}}_n(t), \\ &= \partial_\theta \tilde{z}_n(t,\theta) + \ell_n^{\top\prime}(\theta)\mathcal{Z}_n(t) - h\ell_n^\top(\theta)\dot{\mathcal{Z}}_n(t), \\ &= \partial_\theta \tilde{z}_n(t,\theta) + \ell_n^\top(\theta)\mathcal{I}_n\mathcal{L}_n^\top\mathcal{Z}_n(t) - h\ell_n^\top(\theta)\dot{\mathcal{Z}}_n(t), \end{aligned}$$

by the use of the differential rule (3.11). Then, from (3.28), the last term can be expressed in terms of $u, \mathcal{Z}_n, \tilde{z}_n(t,0)$ as

$$\begin{aligned} h\ell_n^\top(\theta)\dot{\mathcal{Z}}_n(t) \;&= \ell_n^\top(\theta)\mathcal{I}_n(-\mathcal{L}_n + \ell_{n,1}\ell_{n,1}^\top - \ell_{n,0}\ell_{n,0}^\top)\mathcal{Z}_n(t) \\ &\quad + \ell_n^\top(\theta)\mathcal{I}_n\left(-\ell_{n,1}\ell_{n,1}^\top\mathcal{Z}_n(t) + \ell_{n,1}u(t) - \ell_{n,0}\tilde{z}_n(t,0)\right). \end{aligned}$$

Finally, noticing that $(-\mathcal{L}_n + \ell_{n,1}\ell_{n,1}^\top - \ell_{n,0}\ell_{n,0}^\top) = \mathcal{L}_n^\top$ from (3.13), the last piece of the result is obtained

$$\begin{aligned} h\partial_t\tilde{z}_n(t,\theta) \;&= \partial_\theta\tilde{z}_n(t,\theta) + \cancel{\ell_n^\top(\theta)\mathcal{I}_n\mathcal{L}_n^\top\mathcal{Z}_n(t)} - \cancel{\ell_n^\top(\theta)\mathcal{I}_n\mathcal{L}_n^\top\mathcal{Z}_n(t)} \\ &\quad -\ell_n^\top(\theta)\mathcal{I}_n\left(-\ell_{n,1}\ell_{n,1}^\top\mathcal{Z}_n(t) + \ell_{n,1}u(t) - \ell_{n,0}\tilde{z}_n(t,0)\right). \end{aligned} \tag{3.30}$$

To summarize, equations (3.28), (3.30), (3.29) and (3.27) respectively lead to equations (3.23a), (3.23b), (3.23c) and (3.23d). □

The choice of this model to pursue stability analysis makes sense for two reasons. From one side, it is a way to realize Padé models, which are extensively used in the literature on time-delay systems [140]. From the other side, it goes beyond the limitation of finding Padé filters W_n to grab information from the remainder part. These two characteristics are explained in the following property.

Property 3.2. *System* $\begin{pmatrix} \mathcal{A}_n & \mathcal{B}_n \\ \mathcal{C}_n & 0 \end{pmatrix}$ *is a realization of the* $(n-1|n)$ *Padé approximant of* $H(s) = \mathrm{e}^{-hs} I_{n_z}$ *and the corresponding Padé remainder is given by the following expression*

$$\tilde{\mathcal{P}}_{(n-1|n)}(s) = \underbrace{\left(1 + \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n^*\right)}_{W_n(s)} \underbrace{\tilde{G}_n(s,0)}_{R_n(s)}. \quad (3.31)$$

where matrices are given by

$$\mathcal{A}_n = -h^{-1}\mathcal{I}_n(\mathcal{L}_n + \ell_{n,0}\ell_{n,0}^\top), \quad \mathcal{B}_n = h^{-1}\mathcal{I}_n\ell_{n,1}, \quad \mathcal{C}_n = \ell_{n,0}^\top, \quad \mathcal{B}_n^* = -h^{-1}\mathcal{I}_n\ell_{n,0}, \quad (3.32)$$

and where $\tilde{G}_n(s,0)$ *is the Legendre truncated error of function* $G(s,\theta)$ *at order* n *evaluated at* $\theta = 0$.

Proof. Swiping our linear system (3.23) to the Laplace domain, we obtain

$$Y(s) = \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n U(s) + \left(I_{n_z} + \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n^*\right) E_n(s)$$

Moreover, according to (3.25), we have $E_n(s) = \tilde{G}_n(s,0)U(s)$. Therefore, the input-output transfer function of our first Legendre model can be decomposed into two parts:

$$H(s) = \underbrace{\mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n}_{H_n(s)} + \underbrace{\left(I_{n_z} + \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n^*\right)\tilde{G}_n(s,0)}_{\tilde{H}_n(s)}$$

A technical lemma, postpone to Appendix B.2 helps us to ensure that

$$I_{n_z} + \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n^* = I_{n_z} - \ell_{n,0}^\top(hsI_{nn_z} + \mathcal{I}_n\mathcal{L}_n + \mathcal{I}_n\ell_{n,0}\ell_{n,0}^\top)^{-1}\mathcal{I}_n\ell_{n,0} = \underset{s\to 0}{O}(s^n). \quad (3.33)$$

It corresponds to Lemma B.4 with $L = \mathcal{I}_n\mathcal{L}_n$, $u = \mathcal{I}_n\ell_{n,0}$ and $v = \ell_{n,0}$, whose proof is provided in the appendix. In addition, noticing that $\left\|\partial_\theta^{(n)} G(s)\right\|_\infty = \sup_{\theta\in[0,1]} \left|(hs)^n \mathrm{e}^{h(\theta-1)s} I_{n_z}\right| \leq (hs)^n$, the application of Lemma 2.2 at order $d = n-1$ leads to

$$\left|\tilde{G}_n(s,0)\right| \leq \frac{(hs)^n}{2^{n-1}(n-2)(n-\frac{3}{2})\dots(\frac{3}{2})}.$$

Noticing that $\left|\partial_\theta^{(n)}\tilde{G}_n(0,0)\right| \neq 0$, we have $\tilde{G}_n(s,0) = \underset{s\to 0}{O}(s^n)$. Therefore, since $\tilde{H}_n(s) = \underset{s\to 0}{O}(s^{2n})$, Definition 3.1 allows us to identify $H_n(s) := \mathcal{P}_{(n-1|n)}(s)$ and $\tilde{H}_n(s) := \tilde{\mathcal{P}}_{(n-1|n)}(s)$. □

Remark 3.2. As a complement, two different proofs have been provided in [13] by Taylor's expansion or in [16] by induction.

Remark 3.3. Based on (3.33), it is worth noticing that $W_n(s) = \underset{s\to 0}{O}(s^n)$ is a high-pass filter. From a retro analysis in the Laplace domain, such a filter design is issued from the following relation

$$\tilde{P}_{(n-1|n)}(s) = \left(\mathrm{e}^{-hs} - \frac{N_{n-1}(s)}{D_n(s)}\right) I_{n_z} = \underbrace{\left(\frac{s^n}{D_n(s)} I_{n_z}\right)}_{\propto W_n(s)} \underbrace{\left(\frac{\mathrm{e}^{-hs} D_n(s) - N_{n-1}(s)}{s^n} I_{n_z}\right)}_{\propto R_n(s)}.$$

The tricky part comes out from the realization of denominator $D_n(s)$ with the first Legendre polynomials coefficients as state vectors and needs case-by-case studies.

This first modelling is illustrated in Figure 3.3. This new model (3.23) can then be connected to a linear system $\left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$. The whole interconnection is used in the next chapter to analyze the stability properties of ODE-transport systems. It is a strictly causal model, which eases the analysis in the time domain. From the link with Padé rational approximation, convergence properties of the finite-dimensional part with respect to the order n could be deduced. From the determination of a filter W_n expressed with the same state matrix as the finite-dimensional part, frequency analysis could be facilitated.

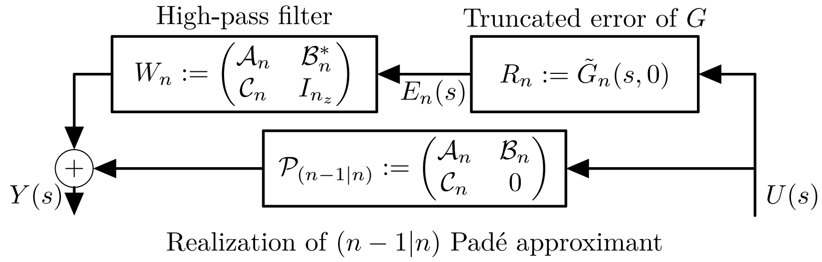


Figure 3.3: Legendre-modelling I of the transport equation ($\mathcal{S}_{1\infty}$).

3.2.2 Model II: a complete realization of $(n|n)$ Padé model

From the first part, we derive a causal model based on the n first Legendre polynomials. The Legendre approximation at order $n+1$ yields the following model.

Proposition 3.4. *For any order n in $\mathbb{N}$, system ($\mathcal{S}_{1\infty}$) with an initial condition z_0 in $H^1(0,1;\mathbb{R}^{n_z})$ can be modeled as follows*

$$\begin{cases} h\dot{\mathcal{Z}}_n(t) = \mathcal{I}_n(-\mathcal{L}_n + (-1)^n \ell_{n,0}\ell_{n,2}^\top)\mathcal{Z}_n(t) + \mathcal{I}_n \ell_{n,2} u(t) - \mathcal{I}_n \ell_{n,0} e_n^\flat(t), & (3.34a) \\ h\partial_t \tilde{z}_n(t,\theta) = \partial_\theta \tilde{z}_n(t,\theta) - \ell_n^\top(\theta)\mathcal{I}_n\left(-\ell_{n,2}\ell_{n,1}^\top \mathcal{Z}_n(t) + \ell_{n,2}u(t) - \ell_{n,0}e_n^\flat(t)\right), & (3.34b) \\ \tilde{z}_n(t,1) = u(t) - \ell_{n,1}^\top \mathcal{Z}_n(t), & (3.34c) \\ y(t) = (-1)^{n-1}\ell_{n,2}^\top \mathcal{Z}_n + (-1)^n u(t) + e_n^\flat(t), & (3.34d) \\ e_n^\flat(t) = \tilde{z}_n(t,0) - (-1)^n \tilde{z}_n(t,1), & (3.34e) \end{cases}$$

for all (t,θ) in $\mathbb{R}_+ \times [0,1]$, with the initial condition given by (3.24)*.*
The error $e_n^\flat$ is given in the Laplace domain by

$$E_n^\flat(s) = \left(\tilde{G}_{n+1}(s,0) - (-1)^n \tilde{G}_{n+1}(s,1)\right) U(s) = \left(\tilde{G}_n(t,0) - (-1)^n \tilde{G}_n(t,1)\right) U(s). \tag{3.35}$$

Proof. Assume that z is solution to system ($\mathcal{S}_{1\infty}$). Using the boundary condition $u(t) = z(t,1)$, the Legendre approximation (3.6) at order $n+1$ evaluated at $\theta = 1$ gives

$$(2n+1)\langle l_n | z\rangle = u(t) - \ell_{n,1}^\top \mathcal{Z}_n(t) - \tilde{z}_{n+1}(t,1).$$

Contrary to the previous modelling, the output $y(t) = z(t,0)$ is decomposed up to the order $n+1$ thanks to (3.6) evaluated at $\theta = 0$ as follows

$$\begin{aligned} y(t) = z(t,0) &= \ell_{n,0}^\top \mathcal{Z}_n(t) + (-1)^n(2n+1)\langle l_n | z(t)\rangle + \tilde{z}_{n+1}(t,0), \\ &= -(-1)^n \ell_{n,2}^\top \mathcal{Z}_n(t) + (-1)^n u(t) + \tilde{z}_{n+1}(t,0) - (-1)^n \tilde{z}_{n+1}(t,1). \end{aligned} \tag{3.36}$$

From one side, the relation (3.26) where $z(t,0)$ is replaced by expression (3.36) gives

$$h\mathcal{I}_n^{-1}\dot{\mathcal{Z}}_n(t) = (-\mathcal{L}_n + (-1)^n \ell_{n,0}\ell_{n,2}^\top)\mathcal{Z}_n(t) + \ell_{n,2}u(t) - \ell_{n,0}\underbrace{\left(\tilde{z}_{n+1}(t,0) - (-1)^n \tilde{z}_{n+1}(t,1)\right)}_{e_n^\flat(t)}. \tag{3.37}$$

Likewise, using Definition 3.3 in the Laplace domain, the error denoted here $E_n^\flat(s)$, is given by

$$E_n^\flat(s) = \left(\tilde{G}_{n+1}(s,0) - (-1)^n \tilde{G}_{n+1}(s,1)\right) U(s). \tag{3.38}$$

From the other side, (3.30) can be rewritten as

$$h\partial_t \tilde{z}_n(t,\theta) = \partial_\theta \tilde{z}_n(t,\theta) - \ell_n^\top(\theta)\mathcal{I}_n \begin{bmatrix} -\ell_{n,2}\ell_{n,1}^\top & \ell_{n,2} & -\ell_{n,0} \end{bmatrix} \begin{bmatrix} \mathcal{Z}_n(t) \\ u(t) \\ \tilde{z}_{n+1}(t,0) - (-1)^n \tilde{z}_{n+1}(t,1) \end{bmatrix}, \tag{3.39}$$

which concludes the proof. □

Here, Legendre modelling II given by (3.34) has pushed the approximation up to order $n+1$ and included one more polynomial coefficient through the boundary conditions than Legendre modelling I given by (3.23). The finite-dimensional part of model II is more accurate but is no more strictly causal. The infinite-dimensional part has also been modified to put aside a Legendre truncated error at order $n+1$, which can also be seen as a combination of Legendre truncated errors at order n.

Before going any further, it is worth mentioning that model II given by (3.34) corresponds exactly to the Legendre-*tau* model.

Property 3.3. *Input-output models* (3.14) *and* (3.34) *are identical.*

Proof. From (3.13), we have $\mathcal{L}_n^\top = -\mathcal{L}_n + \ell_{n,1}\ell_{n,1}^\top - \ell_{n,0}\ell_{n,0}^\top$ and identify

$$-\mathcal{L}_n^\top + (-1)^n \ell_{n,0}\ell_{n,2}^\top = \mathcal{L}_n - \ell_{n,1}\ell_{n,1}^\top + \cancel{\ell_{n,0}\ell_{n,0}^\top} + (-1)^n\ell_{n,0}\ell_{n,1}^\top - \cancel{\ell_{n,0}\ell_{n,0}^\top} = \mathcal{L}_n - \ell_{n,2}\ell_{n,1}^\top,$$

and ensure that $\bar{\mathcal{A}}_n = \mathcal{A}_n^\flat$ holds. The end of the proof is trivial since the other matrices and signals already match, i.e. equalities

$$\bar{\mathcal{B}}_n = \mathcal{B}_n^\flat, \quad \bar{\mathcal{C}}_n = \mathcal{C}_n^\flat, \quad \bar{\mathcal{D}}_n = \mathcal{D}_n^\flat, \quad \bar{\mathcal{B}}_n^* = \mathcal{B}_n^*. \tag{3.40}$$

hold. To conclude, both representation models $\left(\begin{smallmatrix} \bar{\mathcal{A}}_n & \bar{\mathcal{B}}_n & \bar{\mathcal{B}}_n^* \\ \bar{\mathcal{C}}_n & \bar{\mathcal{D}}_n & I_{n_z}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathcal{A}_n^\flat & \mathcal{B}_n^\flat & \mathcal{B}_n^* \\ \mathcal{C}_n^\flat & \mathcal{D}_n & I_{n_z}\end{smallmatrix}\right)$ are equivalent, where the matrices are given by (3.20) and (3.42), respectively. □

At this stage, there are then at least two feasible methodologies to build *tau* models. The classical way, exposed in the previous section, consists in approximating directly the derivatives of the state (using (3.12) for Legendre polynomials). The reverse manipulation has been proposed in this section. An integration by part is performed on the state variables and the approximation of the derivatives of the basis functions (using (3.11) for Legendre polynomials) is done a posteriori.

It is also worth mentioning that the finite-dimensional part of model II is also a realization of Padé's rational approximation. Compared to model I, it leads to a $(n|n)$ model and gives another way to cut the Padé remainder part. A new filter $W_n^\flat$ and bounded function $R_n^\flat$ are then introduced.

Property 3.4. *System* $\begin{pmatrix} \mathcal{A}_n^\flat & \mathcal{B}_n^\flat \\ \mathcal{C}_n^\flat & \mathcal{D}_n^\flat \end{pmatrix}$ *is a realization of the* $(n|n)$ *Padé approximant of* $H(s) = \mathrm{e}^{-hs} I_{n_z}$ *and the corresponding Padé remainder is given by the following expression*

$$\tilde{\mathcal{P}}_{(n|n)}(s) = \underbrace{\left(I_{n_z} + \mathcal{C}_n^\flat(sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^*\right)}_{W_n^\flat(s)} \underbrace{\left(\tilde{G}_{n+1}(s,0) - (-1)^n\tilde{G}_{n+1}(s,1)\right)}_{R_n^\flat(s)}. \tag{3.41}$$

where matrices are given by

$$\mathcal{A}_n^\flat = h^{-1}\mathcal{I}_n(-\mathcal{L}_n + (-1)^n\ell_{n,0}\ell_{n,2}^\top),\ \mathcal{B}_n^\flat = h^{-1}\mathcal{I}_n\ell_{n,2},\ \mathcal{C}_n^\flat = (-1)^{n-1}\ell_{n,2}^\top,\ \mathcal{D}_n^\flat = (-1)^n, \tag{3.42}$$

and where $\tilde{G}_{n+1}(s)$ *is the Legendre truncated error of function* $G(s)$ *at order* $n+1$.

Proof. Swiping our linear system (3.34) to the Laplace domain, we obtain

$$Y(s) = \left(\mathcal{C}_n^\flat(sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^\flat + \mathcal{D}_n^\flat\right)U(s) + \left(I_{n_z} + \mathcal{C}_n^\flat(sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^*\right)E_n^\flat(s).$$

Moreover, according to (3.35), we have $E_n(s) = \left(\tilde{G}_{n+1}(s,0) - (-1)^n\tilde{G}_{n+1}(s,1)\right)U(s)$. Therefore,

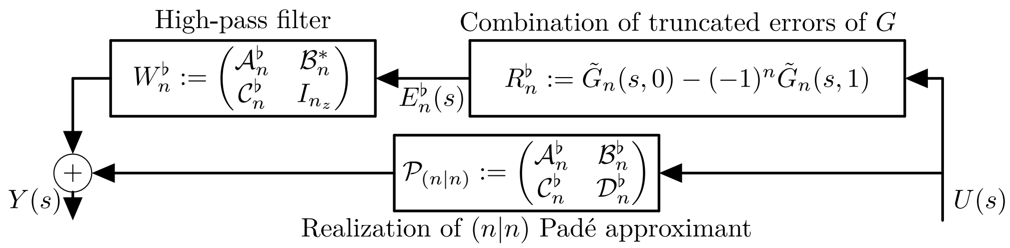


Figure 3.4: Legendre-modelling II of the transport equation $(\mathcal{S}_{1\infty})$.

the transfer function from $U(s)$ to $Y(s)$ of our second Legendre model can be decomposed as

$$\begin{aligned} H(s) \;&=\; \underbrace{\mathcal{C}_n^\flat (sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^\flat + \mathcal{D}_n^\flat}_{H_n^\flat(s)} \\ &\quad + \underbrace{\left(I_{n_z} + \mathcal{C}_n^\flat (sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^*\right)\left(\tilde{G}_{n+1}(s,0) - (-1)^n \tilde{G}_{n+1}(s,1)\right)}_{\tilde{H}_n^\flat(s)}. \end{aligned}$$

Working intensively on the remainder part $\tilde{H}_n^\flat(s)$, a technical lemma helps us to ensure that

$$I_{n_z} + \mathcal{C}_n^\flat (sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^{\flat *} = 1 - (-1)^{n-1}\ell_{n,2}^\top \left(hsI_{nn_z} + \mathcal{I}_n\mathcal{L}_n + (-1)^{n-1}\mathcal{I}_n\ell_{n,0}\ell_{n,2}^\top\right)^{-1}\mathcal{I}_n\ell_{n,0}, = \underset{s\to 0}{O}(s^n), \tag{3.43}$$

It corresponds to Lemma B.4 with $L = \mathcal{I}_n\mathcal{L}_n$, $u = \mathcal{I}_n\ell_{n,0}$ and $v = (-1)^{n-1}\ell_{n,2}$, whose proof is provided in Appendix B.2. In addition, noticing that $\left\|G^{(n+1)}\right\|_\infty \leq (hs)^{n+1}$, the application of Lemma 2.2 at order $d = n$ leads to

$$\left|\tilde{G}_{n+1}(s,\theta)\right| \leq \frac{(hs)^{n+1}}{2^n(n-1)(n-\frac{3}{2})\dots(\frac{1}{2})}, \quad \forall \theta \in [0,1],$$

Noticing that $\left|\tilde{G}_n^{(n+1)}(0,\theta)\right| \neq 0$, we have $\tilde{G}_n(s,\theta) = \underset{s\to 0}{O}(s^{n+1})$ for $\theta \in \{0,1\}$. Therefore, since $\tilde{H}_n(s) = \underset{s\to 0}{O}(s^{2n+1})$, Definition 3.1 allows us to identify $H_n^\flat(s) := \mathcal{P}_{(n|n)}(s)$ and $\tilde{H}_n^\flat(s) := \tilde{\mathcal{P}}_{(n|n)}(s)$ and to conclude the proof. □

Remark 3.4. As a complement, two different proofs have been provided in [13] by Taylor's expansion or in [16] by induction. Notice that the link between the Legendre-*tau* model and $(n|n)$ Padé approximant has also been exposed in [3].

Remark 3.5. Once again, based on (3.43), it is worth noticing that $W_n^\flat(s) = \underset{s\to 0}{O}(s^n)$ is a high-pass filter. The determination of such a filter can be induced in the Laplace domain by the following relation

$$\tilde{P}_{(n|n)}(s) = \left(e^{-hs} - \frac{N_n(s)}{D_n(s)}\right) I_{n_z} = \underbrace{\left(\frac{s^n}{D_n(s)} I_{n_z}\right)}_{\propto W_n^\flat(s)} \underbrace{\left(\frac{e^{-hs}D_n(s) - N_n(s)}{s^n} I_{n_z}\right)}_{\propto R_n^\flat(s)}.$$

However, its state representation presented above is much harder to intuit.

This second modelling is illustrated in Figure 3.4. Once connected to a linear system $\left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$, we compare in the next chapter models I and II in terms of interest in a stability analysis context. From Lemma 3.1, the finite-dimensional parts converge on closed sets in both cases.

3.3 Proposed models for ODE-transport systems

From models I and II introduced in the previous section for the transport equation $(\mathcal{S}_{1\infty})$, i.e. the exponential transfer function $H(s) = \mathrm{e}^{-hs} I_{n_z}$, a Redheffer star product with a finite-dimensional system $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ allows us to propose new models for ODE-transport systems. This opens the way to new representations of ODE-transport systems and to new manners of applying system analysis techniques.

Indeed, the main interest of these models is to develop new criteria of stability or to design new controllers or observers for ODE-transport systems.

3.3.1 Extension of model I to ODE-transport systems

From the first part, let us extend model I given by Proposition 3.3 for transport system ($\mathcal{S}_{1\infty}$) to ODE-transport system ($\mathcal{S}_1$).

Proposition 3.5. *For any order n in $\mathbb{N}$, system ($\mathcal{S}_1$) can be modeled as follows*

$$\begin{cases} \dot{\xi}_n(t) = \mathbf{A}_n \xi_n(t) + \mathbf{B}_n e_n(t), & \forall t \in \mathbb{R}_+, \\ h\partial_t \tilde{z}_n(t,\theta) = \partial_\theta \tilde{z}_n(t,\theta) - \ell_n^\top(\theta)\big(\ell_{n,1}\tilde{\mathbf{C}}_n \xi_n(t) - \ell_{n,0} e_n(t)\big), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \\ \tilde{z}_n(t,1) = \tilde{\mathbf{C}}_n \xi_n(t), & \forall t \in \mathbb{R}_+, \\ u(t) = \mathbf{C}_n \xi_n(t), & \forall t \in \mathbb{R}_+, \\ e_n(t) = \tilde{z}_n(t,0), & \forall t \in \mathbb{R}_+. \end{cases} \tag{$\mathcal{S}_{1n}$}$$

where the matrices are given by

$$\mathbf{A}_n = \begin{bmatrix} A & B\mathcal{C}_n \\ \mathcal{B}_n C & \mathcal{A}_n \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} B \\ \mathcal{B}_n^* \end{bmatrix}, \quad \mathbf{C}_n = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \tilde{\mathbf{C}}_n = \begin{bmatrix} C & -\ell_{n,1} \end{bmatrix}. \tag{3.44}$$

The initial condition is given by

$$\begin{cases} \xi_n(0) = \begin{bmatrix} x_0 \\ \mathcal{I}_n \int_0^1 \ell_n(\theta) z_0(\theta) \mathrm{d}\theta \end{bmatrix}, \\ \tilde{z}_n(0,\theta) = z_0(\theta) - \ell_n^\top(\theta)\mathcal{I}_n \displaystyle\int_0^1 \ell_n(\theta) z_0(\theta)\mathrm{d}\theta, \quad \forall \theta \in [0,1], \end{cases} \tag{3.45}$$

where $\left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right]$ belongs to the set $\mathcal{D}_1 = \left\{ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{R}^{n_x} \times H^1(0,1;\mathbb{R}^{n_z}) \text{ s.t. } z(1) = Cx \right\}$.

Proof. The proof is simply a Redheffer star product of the finite-dimensional part ($\mathcal{S}_1$a) with the model of the transport equation ($\mathcal{S}_{1\infty}$) proposed in Proposition 3.3. □

This modelling is depicted in Figure 3.5. The lower part can be seen as an approximated finite-dimensional system, which is often used in early-lumping approaches. The upper part, which is the novelty brought by our models, is an expression of the leftover infinite-dimensional part. Contrary to early-lumping, it is then possible to pursue a direct analysis or synthesis for the original system ($\mathcal{S}_1$). Indeed, system ($\mathcal{S}_{1n}$) with initial conditions (3.45) is equivalent to system ($\mathcal{S}_1$). They have the same stability properties. Then, to analyze the stability, both models can then be used indifferently.
Moreover, the stability analysis can be simplified and enhanced with this modelling. First, we are now in position to use finite-dimensional tools for stability analysis on delay-dependent models. Even better, we have incremental finite-dimensional models depending on order n which are supposed to converge to the original system as n increases. Lastly, the leftover infinite-dimensional part is simply a Legendre remainder at orders n and can be given by

$$R_n(s) = \tilde{G}_n(s,0) = \mathrm{e}^{-hs} I_{n_z} - \ell_{n,0}^\top (hsI_{nn_z} - \mathcal{I}_n \mathcal{L}_n)^{-1}(\ell_{n,1} - \ell_{n,0}\,\mathrm{e}^{-hs}). \tag{3.46}$$

This error can be calculated and analyzed easily.

Remark 3.6. It is also worth noticing that, the partial differential equation part of the extended model ($\mathcal{S}_{1n}$) satisfied by the Legendre truncated error $\tilde{z}_n$ of state z can be seen as a modified transport equation where the additional term is a polynomial at order $n-1$ and is orthogonal to the error $\tilde{z}_n$. It will ease stability analysis developments.

3.3.2 Extension of model II to ODE-transport systems

From the second part, let us extend model II given by Proposition 3.4 for transport system ($\mathcal{S}_{1\infty}$) to ODE-transport system ($\mathcal{S}_1$).

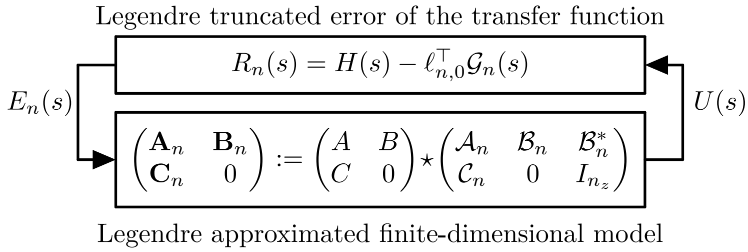


Figure 3.5: Legendre-modelling I of ODE-transport system $(\mathcal{S}_1)$.

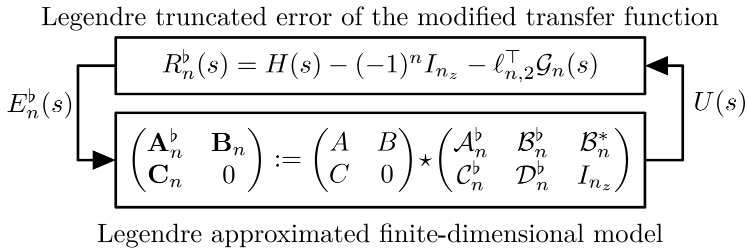


Figure 3.6: Legendre-modelling II of ODE-transport system $(\mathcal{S}_1)$.

Proposition 3.6. *The dynamics of system* $(\mathcal{S}_1)$ *can be modeled as follows*

$$\begin{cases} \dot{\xi}_n(t) = \mathbf{A}_n^\flat \xi_n(t) + \mathbf{B}_n e_n^\flat(t), & \forall t \in \mathbb{R}_+, \\ h\partial_t \tilde{z}_n(t,\theta) = \partial_\theta \tilde{z}_n(t,\theta) - \ell_n^\top(\theta)\big(\ell_{n,2}\xi_n(t) - \ell_{n,0} e_n^\flat(t)\big), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \\ \tilde{z}_n(t,1) = \tilde{\mathbf{C}}_n \xi_n(t), & \forall t \in \mathbb{R}_+, \\ u(t) = \mathbf{C}_n \xi_n(t), & \forall t \in \mathbb{R}_+, \\ e_n^\flat(t) = \tilde{z}_n(t,0) - (-1)^n \tilde{z}_n(t,1), & \forall t \in \mathbb{R}_+, \end{cases} \tag{$\mathcal{S}_{1n}^\flat$}$$

where matrices $\mathbf{B}_n, \mathbf{C}_n, \bar{\mathbf{C}}_n$ *are given by* (3.44) *and where*

$$\mathbf{A}_n^\flat = \begin{bmatrix} A + B\mathcal{D}_n^\flat C & B\mathcal{C}_n^\flat \\ \mathcal{B}_n^\flat C & \mathcal{A}_n^\flat \end{bmatrix}. \tag{3.47}$$

The initial condition is given by (3.45).

Proof. The proof is simply a Redheffer star product of the finite-dimensional part $(\mathcal{S}_1\text{a})$ with the model of the transport equation proposed in Proposition 3.4. □

This modelling is depicted in Figure 3.6. The same intuition as for extended model I about the advantages and interest of such modelling for stability analysis is maintained. Here, the leftover infinite-dimensional part is simply a Legendre remainder at orders $n+1$ and can be given by the following closed-form expression

$$\begin{aligned} R_n^\flat(s) &= \tilde{G}_{n+1}(s,0) - (-1)^n \tilde{G}_{n+1}(s,1) = \tilde{G}_n(s,0) - (-1)^n \tilde{G}_n(s,1), \\ &= \left(\mathrm{e}^{-hs} - (-1)^n\right) I_{n_z} - \ell_{n,2}^\top (hsI_{nn_z} - \mathcal{I}_n \mathcal{L}_n)^{-1}(\ell_{n,1} - \ell_{n,0}\,\mathrm{e}^{-hs}). \end{aligned} \tag{3.48}$$

Once again, such a structure will ease the stability analysis of system $(\mathcal{S}_1)$.

Conclusion

This chapter has presented two augmented models for ODE-transport system $(\mathcal{S}_1)$. Based on Legendre approximation, the resulting models have the interest of having a finite-dimensional part linked with Padé realizations and a structured infinite-dimensional remained part expressed analytically. We are now in position to apply finite-dimensional tools to analyze its stability properties.

In the next chapter, both extended models I and II of ODE-transport systems are exploited to propose stability tests through input-output arguments or for Lyapunov analysis. Moreover, Legendre polynomials approximation helps us to better understand the effectiveness of forthcoming stability criteria.

Chapter 4

Stability analysis of ODE-transport systems

"One of the key points of the stability approach is the notion of decomposition."
Model theory and the philosophy of mathematical practice, J.T. Baldwin.

Contents

IN the late thirties, many studies have been conducted to analyze the stability of ODE-transport systems (i.e. time-delay systems) [107]. There have been related to frequency [171] or time domain [86] approaches and particular attention has been paid to three theoretical frameworks: spectrum, robust and Lyapunov analysis. In that sense, root locus characterization methods [37], frequency sweeping tests [211], construction of linear matrix inequalities [189] and formulation of positivity criteria [98] have been provided to obtain delay-dependent stability conditions for ODE-transport systems. However, the proposed methodologies are often conservative and it lacks elements to characterize, evaluate, or reduce it. The main goal of this chapter is to obtain delay-dependent necessary and sufficient conditions of stability for ODE-transport systems. The subsequent objective is also to better understand the role played by the approximation and the model on the degree of conservatism and to answer the following questions.

- *Is it possible to obtain necessary and sufficient stability conditions taking support on the models obtained by Legendre approximation?*
- *How to construct a bridge between different stability approaches employing the same modelling?*
- *Can we rule on the effectiveness of each criterion in terms of numerical complexity?*

In the sequel, the common denominator is the model issued from the Legendre approximation designed in the previous chapter. Based on these models, each stability analysis technique is then treated separately. Section 4.1 is devoted to spectral analysis and the effectiveness of the finite-dimensional parts of our models to approximate the root locus. Section 4.2 concerns frequency analysis and the application of the small gain theorem. Section 4.3 and 4.4 are finally focused on manipulating a necessary and sufficient Lyapunov condition, presented in Appendix A for the finite-dimensional case.

Respectively, inner and outer estimates of the stability regions, which converge with respect to the approximation order, are provided. The convergence results and degree of conservatism arising from our approximation are commented and quantified along this chapter.

4.1 Characteristic roots approximation

Initially, the so-called pseudo-spectral methods were used to prove the convergence of numerical simulations of PDE systems. For instance, *tau*-models are well-known to approximate ODE-transport systems solutions (or indifferently time-delay systems solutions) and have nice convergence properties [118, 119]. Indeed, these methods allow approximating the spectrum of ODE-transport systems, at least a finite part [36]. This assertion is also true for both collocation and approximation techniques. Note that dedicated tools have been developed such that DDEbiftool package [74]. In addition, it provides a first estimation of the stability properties [37] and this is what has been done on augmented models I and II.

From the finite-dimensional part of the two models proposed in the previous chapter, we can draw information from the eigenvalues of matrices $\mathbf{A}_n$ and $\mathbf{A}_n^\flat$, i.e. characteristic roots of the following polynomials

$$\chi_n(s) = \det(sI_{n_x+nn_z} - \mathbf{A}_n) = \det\left((sI_{n_x} - A)D_{q_n}(s) - BN_{p_n}(s)C\right), \tag{4.1}$$

$$\chi_n^\flat(s) = \det(sI_{n_x+nn_z} - \mathbf{A}_n^\flat) = \det\Big((sI_{n_x} - A)D_{q_n^\flat}(s) - BN_{p_n^\flat}(s)C\Big), \tag{4.2}$$

where $(N_{p_n}, D_{q_n}) = (N_{n-1}, D_n)$ (resp. $(N_{p_n^\flat}, D_{q_n^\flat}) = (N_n, D_n)$) correspond to the numerator and denominator of $(n-1|n)$ (resp. $(n|n)$) Padé approximants of transfer function $H(s) = \mathrm{e}^{-hs}\, I_{n_z}$. Recall that the numerator and denominator (N_{p_n}, D_{q_n}) are under the form $\sum_{i=0}^{p_n} a_i s_i I_{n_z}$ and $\sum_{i=0}^{q_n} b_i s_i I_{n_z}$ with coefficients (a_i, b_i) verifying Definition 3.1, respectively.
Depending on matrices A, B, C and on the delay h, the location of the eigenvalues of matrices $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$) already gives an answer on the stability of ODE-transport systems.

4.1.1 Convergence of the eigenvalues towards the characteristic roots

The following theorem establishes that the zeros of χ_n are near some zeros of χ (i.e. characteristic roots of the original system), for sufficiently large orders.

Theorem 4.1. *If system $(\mathcal{S}_1)$ contains K characteristic roots s_k^*, with multiplicities μ_k^*, for k in $\{1,\dots,K\}$, inside the open ball $\mathcal{B}(0,r)$ for some radius $r>0$, then $\sum_{k=1}^{K}\mu_k^*$ eigenvalues $\{s_{k,i}^n\}_{\substack{k\in\{1,\dots,K\}\\ i\in\{\mu_1^*,\dots,\mu_K^*\}}}$ of matrix $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$) converges towards them.*
More precisely, for any $\varepsilon>0$,

$$\max_{\substack{k\in\{1,\dots,K\}\\ i\in\{\mu_1^*,\dots,\mu_k^*\}}} \left|s_{k,i}^n - s_k^*\right| \le \varepsilon, \quad \forall n \ge \bar{\mathcal{M}}(\varepsilon), \tag{4.3}$$

where the function $\bar{\mathcal{M}}$ is given by

$$\bar{\mathcal{M}}(\varepsilon) = \max_{k\in\{1,\dots,K\}} \left\{\mathcal{M}\left(\frac{1}{|A|+|B|}\right), \mathcal{M}\left(\frac{C_k (r_k^*)^{\mu_k^*}}{C_0}\right), \mathcal{M}\left(\frac{C_k \varepsilon^{\mu_k^*}}{C_0}\right)\right\}, \tag{4.4}$$

with respect to function $\mathcal{M}$ described in (3.5) and parameters

$$\begin{cases} C_0 = \displaystyle\max_{\substack{s\in\mathcal{B}(0,r)\\ \Gamma\in\mathbb{C}^{n_x\times n_x};|\Gamma|<1}} \left|\det'\big(\Delta(s)+\Gamma\big)\right|(|A|+|B|), & r_0 = \displaystyle\min_{i\neq j}(\left|s_i^* - s_j^*\right|), \\ C_k = \dfrac{\left|\chi^{(\mu_k^*)}(s_k^*)\right|}{2\mu_k^*!}, & r_k^* = \min\left(r_0, \dfrac{1}{2}\dfrac{(\mu_k^*+1)\left|\chi^{(\mu_k^*)}(s_k^*)\right|}{\displaystyle\max_{s\in\mathcal{B}(s_k^*,r_0)}\left|\chi^{(\mu_k^*+1)}(s)\right|}\right). \end{cases}$$

Proof. In [37], the uniform convergence of the eigenvalues obtained by pseudospectral method on first Chebyshev polynomials towards the characteristic roots is proven by Rouché arguments. Here, the demonstration follows a similar methodology.

Step 1: Uniform convergence of χ_n towards χ.
First, recall Theorem 3.1 which traduces the nice convergence properties of Padé approximations of the delay transfer function e^{-hs} given in [20]. The ratio $\frac{p_n}{q_n}$ tends to 1 when n tends to ∞ as in both studied cases, $N_{p_n}(s)$ and $D_{q_n}(s)$ uniformly converge towards $\mathrm{e}^{-\frac{hs}{2}}$ and $\mathrm{e}^{\frac{hs}{2}}$, respectively. More precisely, Theorem 3.1 gives an estimation of the convergence rate. On any open ball $\mathcal{B}(0,r)$ and for any $\varepsilon > 0$, we have

$$\max_{s\in\mathcal{B}(0,r)}\left(\left|N_{p_n}(s)-\mathrm{e}^{-\frac{hs}{2}}\right|,\left|D_{q_n}(s)-\mathrm{e}^{\frac{hs}{2}}\right|\right)\leq\varepsilon,\quad\forall n\geq\mathcal{M}(\varepsilon). \tag{4.5}$$

Consequently, noticing that

$$|\Delta_n(s)-\Delta(s)|\leq(|A|+|B|)\max\left(\left|N_{p_n}(s)-\mathrm{e}^{-\frac{hs}{2}}\right|,\left|D_{q_n}(s)-\mathrm{e}^{\frac{hs}{2}}\right|\right), \tag{4.6}$$

holds, we known that the matrix $\Delta_n(s)=(sI_{n_x}-A)D_{q_n}(s)-BN_{p_n}(s)C$ converges uniformly towards $\Delta(s)=(sI_{n_x}-A)\,\mathrm{e}^{\frac{hs}{2}}-B\,\mathrm{e}^{-\frac{hs}{2}}C$ on open balls $\mathcal{B}(0,r)$. Then, taking the determinant, an upper bound dependent of $|(\Delta_n-\Delta)(s)|$ is found:

$$\begin{aligned}|\chi_n(s)-\chi(s)| &= \left|\det\big(\Delta_n(s)\big)-\det\big(\Delta(s)\big)\right|,\\ &= \left|\int_0^1\det{}'\big(\Delta(s)+\sigma(\Delta_n-\Delta)(s)\big)(\Delta_n-\Delta)(s)\mathrm{d}\sigma\right|,\\ &\leq \max_{\sigma\in[0,1]}\left|\det{}'\big(\Delta(s)+\sigma(\Delta_n-\Delta)(s)\right|.\left|(\Delta_n-\Delta)(s)\right|,\end{aligned}$$

where $\det'(M)$ is the derivative of $\det(M)$ given by Jacobi's formula for any square matrix $M\in\mathbb{C}^{n_x\times n_x}$. Therefore, according to (4.5),(4.6) and taking $n_0^*=\mathcal{M}\left(\frac{1}{|A|+|B|}\right)$, we obtain for all $n\geq n_0^*$ and $s\in\mathcal{B}(0,r)$,

$$|\chi_n(s)-\chi(s)|\leq C_0\max\left(\left|N_{p_n}(s)-\mathrm{e}^{-\frac{hs}{2}}\right|,\left|D_{q_n}(s)-\mathrm{e}^{\frac{hs}{2}}\right|\right). \tag{4.7}$$

From (4.5), the uniform convergence of χ_n towards χ on open balls $\mathcal{B}(0,r)$ is verified.

Step 2: Convergence of some zeros of χ_n towards a zero of χ.
Around a root $s^*\in\mathcal{B}(0,r)$ of multiplicity μ^*, the Taylor's expansion gives

$$\left|\chi(s)-\frac{\left|\chi^{(\mu^*)}(s^*)\right|}{\mu^*!}\,|s-s^*|^{\mu^*}\right|\leq\frac{1}{(\mu^*+1)!}\max_{s\in\mathcal{B}(0,r_0)}\left|\chi^{(\mu^*+1)}(s)\right|\,|s-s^*|^{\mu^*},$$

with r_0 the smallest radius between s^* and other zeros of χ. By choosing

$$r^*=\min\left(r_0,\frac{1}{2}\frac{(\mu^*+1)\left|\chi^{(\mu^*)}(s^*)\right|}{\max\limits_{s\in\mathcal{B}(s^*,r_0)}\left|\chi^{(\mu^*+1)}(s)\right|}\right),$$

we obtain

$$\forall s\in\mathcal{B}(s^*,r^*)\backslash\{s^*\},\quad|\chi(s)|>\frac{1}{2}\frac{\left|\chi^{(\mu^*)}(s^*)\right|}{\mu^*\,!}\,|s-s^*|^{\mu^*}=C_k\,|s-s^*|^{\mu^*}. \tag{4.8}$$

For any $\varepsilon > 0$, let $n_1^* = \max\left\{\mathcal{M}\left(\frac{C_k (r^*)^{\mu^*}}{C_0}\right), \mathcal{M}\left(\frac{C_k \varepsilon^{\mu^*}}{C_0}\right)\right\}$. According to (4.5), we obtain and $n \geq n_1^*$ the following inequality

$$r_1 = \left(\frac{C_0 \max\limits_{s \in \mathcal{B}(0,r)} \left(\left| N_{p_n}(s) - \mathrm{e}^{-\frac{hs}{2}} \right|, \left| D_{q_n}(s) - \mathrm{e}^{\frac{hs}{2}} \right| \right)}{C_k} \right)^{\frac{1}{\mu^*}} \leq \min(r^*, \varepsilon) \leq r^*. \tag{4.9}$$

From (4.7), (4.8), we conclude that, for any $n \geq \max(n_0^*, n_1^*)$,

$$\forall s \in \mathcal{B}(s^*, r_1) \backslash \{s^*\}, \quad |\chi_n(s) - \chi(s)| \leq C_k (r^*)^{\mu^*} < |\chi(s)|. \tag{4.10}$$

Applying Rouché's theorem, the characteristic equation $\chi_n(s) = 0$ has μ^* roots denoted $(s_i^n)_{i \in \{1,\dots,\mu^*\}}$ in $\mathcal{B}(s^*, r_1)$ each counted with its multiplicities. This implies that, for any $n \geq \max(n_0^*, n_1^*)$ and $i \in \{1, \dots, \mu^*\}$,

$$\forall i \in \{1, \dots, \mu^*\}, \qquad |s_i^n - s^*| \leq r_1 \leq \varepsilon.$$

This step can also be resumed to the application of Hurwitz's theorem [49].

<u>*Step* 3</u>*: Convergence of some zeros of χ_n towards those of χ.*
Assume that the open ball $\mathcal{B}(0, r)$ contains K zeros of χ with multiplicities $\mu^*_{k \in \{1,\dots,K\}}$. Repeating the previous *Step* 2 on each ball $\mathcal{B}(s_k^*, r_k)$ for each root s_k^* and there exists an order $n_\infty^* = \max\limits_{\{0,\dots,K\}} n_k^*$ such that for any $n \geq n_\infty^*$ we have

$$\forall k \in \{1, \dots, K\}, \quad \forall i \in \{1, \dots, \mu_k^*\}, \quad \left| s_{k,i}^n - s_k^* \right| \leq \max_{k \in \{1,\dots,K\}} r_k \leq \varepsilon.$$

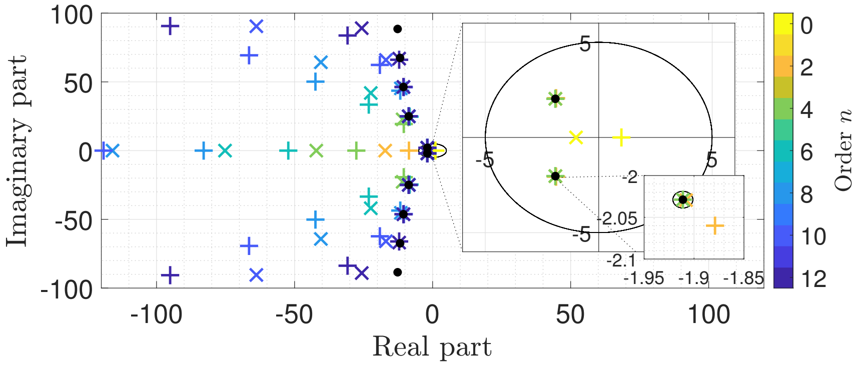
The condition (4.3) is finally obtained. □

This theorem highlights the accuracy of some eigenvalues of matrix $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$) as the order n increases. However, function $\mathcal{M}$ limits the convergence rate to $\underset{n \to \infty}{O}(\frac{1}{n})$.

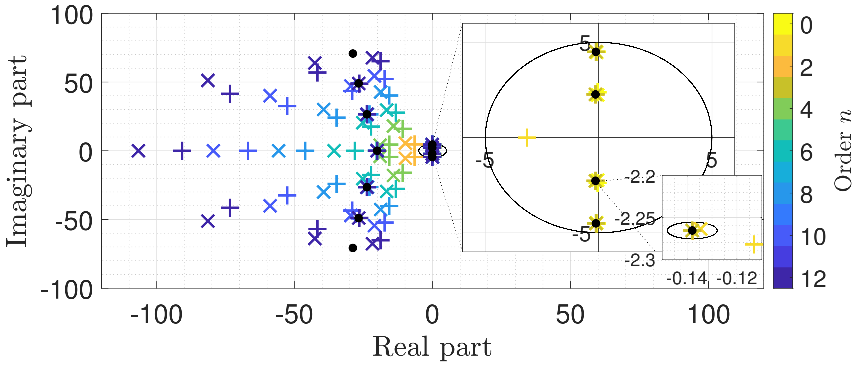
Remark 4.1. Notice that there are other methods to approximate the spectral operator. For instance, collocation methods based on Chebyshev polynomials can be used [37]. The use of Hankel singular values [91] also allows to improve the convergence rate [92].

Applications to the Examples 1.1 and 1.2 introduced in Chapter 1 also accompany this theorem. On Figure 4.1a for Example 1.1 and Figure 4.1b for Example 1.2 with $k = 1$, the eigenvalues of $\mathbf{A}_n$ (i.e. $(n-1|n)$ Padé approximant) and $\mathbf{A}_n^\flat$ (i.e. $(n|n)$ Padé approximant) are respectively represented with $+$ and $\times$ markers for different orders n and compared with the expected ones (dark points) computed with a precision of 10^{-15}. The error done on the location of the characteristic roots in norm is also depicted on Figure 4.2a for Example 1.1 and Figure 4.2b for Example 1.2 with respect to the norm of the expected eigenvalues itself.

First, as expected, the approximated eigenvalues are getting closer to the expected ones as the order n increases and that the eigenvalues are computed with $\mathbf{A}_n^\flat$ are more accurate than the ones computed with $\mathbf{A}_n$. Indeed, $(n|n)$ Padé approximation is more precise than $(n-1|n)$ Padé approximation. We have also proved in Property 3.3 that matrix $\mathbf{A}_n^\flat$ can also be obtained by the Legendre-*tau* method. Such methods are well-known to have a reliable numerical precision [119] and to be better than Fourier-*tau*, Chebychev-*tau*, or least-squares alternative methods [204]. Moreover, it is also worth noticing that the eigenvalues close to 0 in norm are approximated with smaller orders than those which are far from the origin. On Figure 4.2, for a given precision $\varepsilon = 10^{-5}$ and looking at characteristic roots such that $|s^*| \leq 5$, the error done on the eigenvalues is reached from order $n^* = 4$ for both examples. Looking at the other characteristic roots (i.e. $|s^*| > 5$), much larger orders are required. This makes sense since $\bar{\mathcal{M}}(\varepsilon)$ increases as r increases. In the same way, zooming on balls $\mathcal{B}(0, r)$ with $r = 5$ and $\mathcal{B}(s^*, \varepsilon)$ with $\varepsilon = 0.01$ on the right of Figure 4.1, one can find a sufficiently large order n^* from which the approximated eigenvalues reached this precision. Hence, choosing radius r and ε in a relevant way, one can derive stability criteria based on the eigenvalues of $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$).

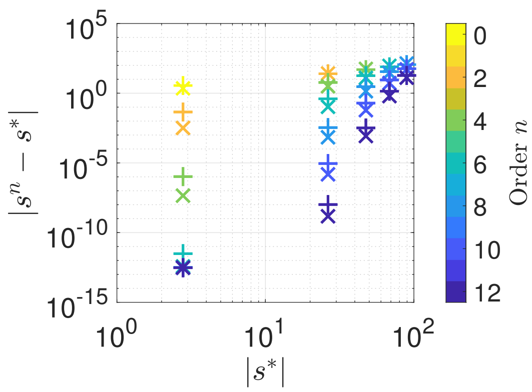


(a) Example 1.1.

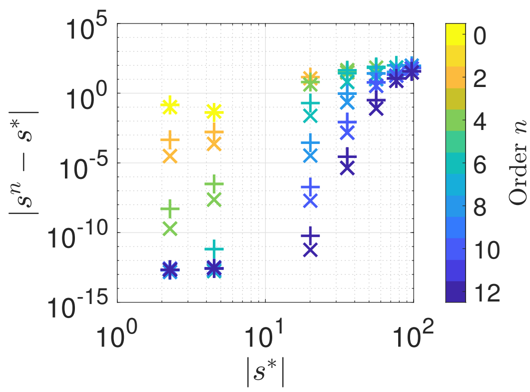


(b) Example 1.2 for $k = 1$.

Figure 4.1: Eigenvalues of matrices $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$) with respect to the order n.
(Markers $+$, $\times$ and $\cdot$ denote the eigenvalues of $\mathbf{A}_n$, $\mathbf{A}_n^\flat$ and the expected characteristic roots, respectively.)



(a) Example 1.1.



(b) Example 1.2 for $k = 1$.

Figure 4.2: Accuracy of the eigenvalues with respect to the order n.
(Markers $+$, $\times$ denote the error between the eigenvalues of $\mathbf{A}_n$, $\mathbf{A}_n^\flat$ and the expected characteristic roots.)

4.1.2 Sufficient stability condition based on the eigenvalues location

For finite-dimensional systems, the stability is ensured if and only if the state matrix is Hurwitz. Extending this spectral approach to ODE-transport systems, a sufficient condition of stability based on approximate state matrices $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$) can be formulated.

Recall that, ODE-transport systems (i.e. retarded time-delay systems) have a point spectrum with a finite number of characteristic roots with positive real parts [112]. In addition, the characteristic roots with positive real parts are encapsulated in a ball $\mathcal{B}(0,r)$.

Property 4.1. *For any characteristic root $s^* \in \mathbb{C}_+$ of system $(\mathcal{S}_1)$ with a positive real part, inequality $|s^*| < r$ holds with $r = |A| + |B|$.*

Proof. Since the delay element satisfies Assumtion 1.1 with $r = |A| + |B|$, Theorem 1.1 put forward in Chapter 1 can be applied and leads to the property. □

Then, applying Theorem 4.1 with radius $r = |A| + |B|$, one obtains the following stability result.

Theorem 4.2. *Let scalar $\varepsilon > 0$ and integer $n^* = \bar{\mathcal{M}}(\varepsilon)$ where $\bar{\mathcal{M}}$ is given by (4.4) with $r = |A|+|B|$. If $\mathbf{A}_{n^*}$ (resp. $\mathbf{A}_{n^*}^\flat$) is ε-stable, then system $(\mathcal{S}_1)$ is GES.*

Proof. Assume that $\mathbf{A}_{n^*} - \varepsilon I_{n_x+n^*n_z}$ (resp. $\mathbf{A}_{n^*}^\flat - \varepsilon I_{n_x+n^*n_z}$) are Hurwitz. Applying Theorem 4.1, there is no characteristic roots of system $(\mathcal{S}_1)$ in the intersection of the right half-plane with $\mathcal{B}(0,r)$. According to Property 4.1, we conclude that all the characteristic roots of system $(\mathcal{S}_1)$ are in the left-hand side of the imaginary axis and that the trivial solution of system $(\mathcal{S}_1)$ is GES. □

Computational load. The eigenvalues of $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$) are computed with the Matlab routine "eig(M)". The Householder transformation is performed to obtain a Hessenberg matrix and is followed by the QR algorithm to determine the eigenvalues of $M \in \mathbb{R}^{n\times n}$. The cost of such an algorithm is given by $\underset{n\to\infty}{O}(n^5)$ [116]. Iterative algorithms in $\underset{n\to\infty}{O}(n_I n^2)$, where n_I is the number of iterations, are able to determine the roots of the characteristic polynomial χ_n (resp. $\chi_n^\flat$) and could reduce the numerical cost.

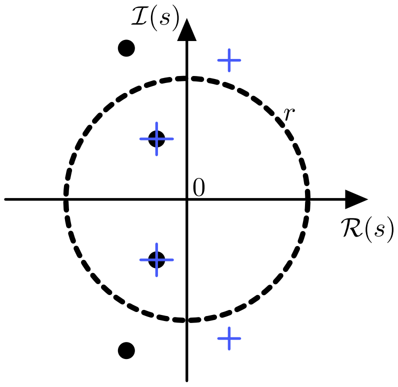
Such spectral approaches allow discarding systems with positive characteristic roots [208]. Nevertheless, three main shortcomings seem to undermine Theorem 4.2.

First, as illustrated by Figure 4.3a, many stable systems cannot be detected. For instance, some eigenvalues of matrix $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$), being far from the origin and outside of ball $\mathcal{B}(0,r)$, could be unstable, i.e. located on the right-hand plane. To reach the necessary condition, a certificate on the location of the eigenvalues in the left-half plane would be helpful [33].

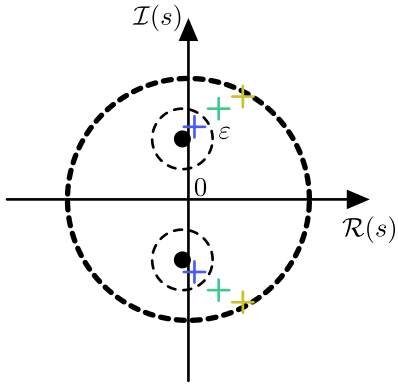
Second, the choice of the positive scalar ε is ambiguous and unknown. A user of Theorem 4.2 would be well advised to select ε small to detect a maximum of stable systems but would end up matrix $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$) in $\mathbb{R}^{n^*\times n^*}$ of large size. In addition, as illustrated by Figure 4.3b, even for small scalar $\varepsilon > 0$, if the approximated eigenvalues go to the wrong side of the imaginary axis, some stable systems remain undetectable. Therefore, the converse theorem of stability seems not reachable due to the characteristic roots located on the imaginary axis and for which alternative methods based on matrix pencils and quasi-polynomials approaches need to be used [147, 148, 171].

Finally, the proposed condition is difficult to check due to large orders n^*. Indeed, its exploitation is limited to computational issues. Here, the use of Padé's approximation to prove the convergence of the root locus leads to 1-algebraic convergence and to really large dimensions n^*. It would be better to rely on the supergeometric convergence of the Legendre approximation. In that perspective, other approximation methods have been proposed in [92], to reduce the dimension.

Putting all these arguments together, we can see that Theorem 4.2 is not satisfactory. Moreover, only the finite-dimensional approximated part has been considered in this section. All the efforts made to preserve and put aside the Legendre truncated error have not been used. In order to take into account maximum information for stability analysis purposes, the whole interconnection is considered and new sufficient stability conditions are proposed.



(a) First counterexample.



(b) Second counterexample.

Figure 4.3: System $(\mathcal{S}_1)$ stable and $\mathbf{A}_n$ unstable.

4.2 Frequency-sweeping test for stability

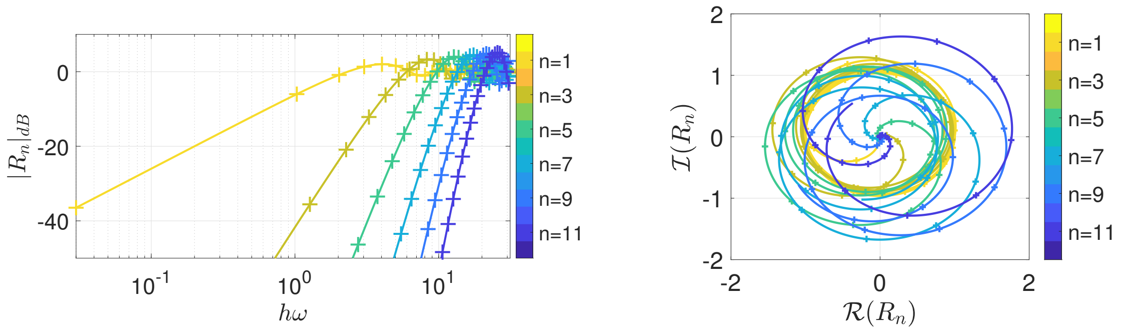
In the robust analysis framework [60, 197], when a finite-dimensional system is interconnected with an uncertainty, the application of the small gain theorem leads to stability properties. Indeed, if two interconnected systems are GES and satisfy the small gain condition, then the interconnection is input-output stable, then for linear systems the interconnection is GES. In this section, we try to export such a finite-dimensional method to an infinite-dimension field.

4.2.1 Encapsulation of the uncertainty

In the case of ODE-transport systems, the transport equation part $H(s) = \mathrm{e}^{-hs}$ can be encapsulated into an uncertainty $|\Delta|_{\mathcal{H}_\infty} \leq |H|_{\mathcal{H}_\infty} = 1$. However, the small gain condition leads to very restrictive delay-independent stability conditions. To add delay-dependency, as discussed in Chapter 3, the system has been decomposed into a finite-dimensional system (i.e. Padé approximants) interconnected with an infinite-dimensional residual part (i.e. Padé error). Noticing that $\left|\tilde{\mathcal{P}}_{(n|n)}\right|_{\mathcal{H}_\infty} = \frac{1}{2}$, the application of the small gain on these interconnection is possible but still lead to conservative results [140]. Indeed, Padé error $\tilde{\mathcal{P}}_{(n|n)}(s) = \underset{s\to 0}{O}(s^{2n+1})$ and one needs to take the benefits of the slope $20(2n+1)$dB by decade in low frequencies. That is why high-pass filters are designed [211, 210]. However, such candidate filters W_n, which satisfy $\tilde{P}_{(n|n)} = W_n R_n$ for some bounded function R_n intended to be encapsulated into an uncertainty Δ_n, are difficult to find.

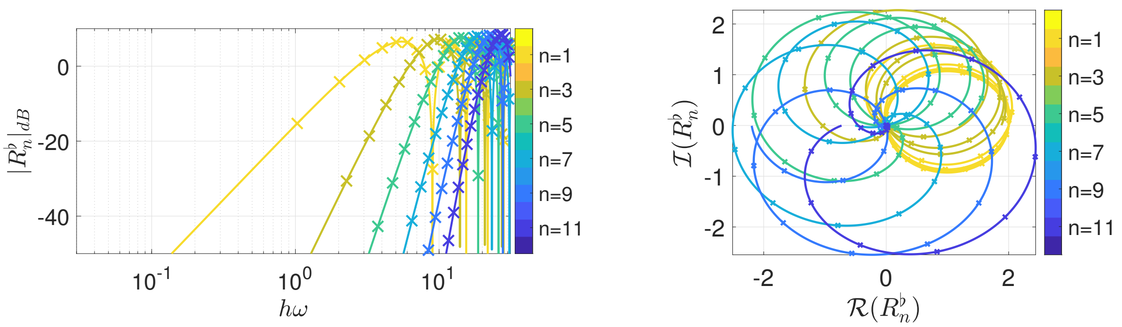
Thanks to the Legendre approximation, extended models $(\mathcal{S}_{1n})$ and $(\mathcal{S}_{1n}^\flat)$ of system $(\mathcal{S}_1)$ have been presented in Propositions 3.5 and 3.6. These new models give a realization of $(n-1|n)$ (resp. $(n|n)$) Padé approximation on the n first Legendre coefficients and identify some high-pass filter W_n (resp. $W_n^\flat$) and remainder R_n (resp. $R_n^\flat$). For more details, the reader is invited to go back to Properties 3.2 and 3.4. As explained previously, these new representations extract information from Padé errors by designing potential candidate filters W_n (resp. $W_n^\flat$) and describing the left-over infinite-dimensional part R_n (resp. $R_n^\flat$) as Legendre truncated errors. It remains for instance to encompass functions R_n (resp. $R_n^\flat$) into uncertainties, to apply the small-gain theorem and to evaluate the effectiveness of such robust approach on examples.

To understand this process, both errors R_n (resp. $R_n^\flat$) are depicted in Nyquist and Bode diagrams in Fig 4.4 (resp. Fig 4.5). On Bode diagram, we confirm that we succeed extract information of $\tilde{H}_n(s) = W_n(s)R_n(s) = \underset{s\to 0}{O}(s^{2n})$ (resp. $\tilde{H}_n^\flat(s) = W_n^\flat(s)R_n^\flat(s) = \underset{s\to 0}{O}(s^{2n+1})$) into the finite-dimensional model thanks to the high-pass filter $W_n(s) = \underset{s\to 0}{O}(s^n)$ (resp. $W_n^\flat(s) = \underset{k\to\infty}{O}(s^n)$). For low frequencies, we have $R_n(s) = \underset{s\to 0}{O}(s^n)$ (resp. $R_n(s)^\flat = \underset{s\to 0}{O}(s^{n+1})$) so that the slope has been reduced to $20n$dB (resp. $20(n+1)$dB) by decade. For sufficiently high frequencies, the behavior of $H(s) = \mathrm{e}^{-hs}$ (resp. $H^\flat(s) = H(s) - (-1)^n I_{n_z}$) is recovered. On Nyquist diagram, one can see that it will be possible to encapsulate the R_n (resp. $R_n^\flat$) into a ball $\mathcal{B}(0, \frac{1}{\gamma_n})$ (resp. $\mathcal{B}(0, \frac{1}{\gamma_n^\flat})$) and to consider it as an uncertainty



(a) Bode diagram (b) Nyquist diagram

Figure 4.4: Representation of the error transfer function R_n of modelling $(\mathcal{S}_{1n})$.



(a) Bode diagram (b) Nyquist diagram

Figure 4.5: Representation of the error transfer function $R_n^\flat$ of modelling $(\mathcal{S}_{1n}^\flat)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
γ_n	1.00	0.80	0.73	0.68	0.66	0.63	0.62	0.60	0.59	0.58	0.57	0.56	0.55
$\gamma_n^\flat$	0.50	0.47	0.45	0.44	0.43	0.42	0.41	0.40	0.40	0.39	0.38	0.38	0.37

Table 4.1: Lower bounds of $|R_n|_{\mathcal{H}_\infty}^{-1}$ and $\left|R_n^\flat\right|_{\mathcal{H}_\infty}^{-1}$ with respect to n.

Δ_n (resp. $\Delta_n^\flat$). The following lemma proves the existence of these bounds.

Lemma 4.1. *For any order n in $\mathbb{N}$, the $\mathcal{H}_\infty$ norms of R_n and $R_n^\flat$ exist.*

Proof. First, R_n and $R_n^\flat$ recalled in (3.46) and (3.48) are causal transfer functions with no poles in the right-half-plane. By confining now to the imaginary axis with frequencies denoted ω, functions $|R_n(\omega)|$ and $\left|R_n^\flat(\omega)\right|$ are smooth, null at zero and have a bounded behavior as $\omega \to \infty$ ($\lim_{s\to\infty}(R_n(s)) = 1$ and $\lim_{s\to\infty}(R_n^\flat(s)) \leq 2$). From the extremum value theorem, both errors are upper bounded. $\square$

Define error bounds γ_n and $\gamma_n^\flat$ such as

$$\begin{cases} \gamma_n \left|R_n\right|_{\mathcal{H}_\infty} < 1, \\ \gamma_n^\flat \left|R_n^\flat\right|_{\mathcal{H}_\infty} < 1. \end{cases} \tag{4.11}$$

These lower bounds are computed with a precision 10^{-2} thanks to derivative-free optimization such as Nelder-Mead algorithm [169] applied to $|R_n|_{\mathcal{H}_\infty}^{-1}$ (resp. $\left|R_n^\flat\right|_{\mathcal{H}_\infty}^{-1}$) with an initial point at low frequencies. A posteriori, we notice that $\gamma_n < 1$ and $\gamma_n^\flat < 0.5$, which justifies that the minimal bounds are not reached as ω tends to ∞.

Remark 4.2. As both errors can be given in function of hs, the bounds γ_n and $\gamma_n^\flat$ are independent of the delay h and can directly be saved and shown on Table 4.1. Consequently, a sufficient delay-dependent stability condition based on the small-gain theorem is applied to the augmented systems introduced in ($\mathcal{S}_{1n}$) and ($\mathcal{S}_{1n}^\flat$).

Based on these bounds γ_n (resp. $\gamma_n^\flat$), we are now in position to apply the small gain theorem for any integer n and to reduce the conservatism of classic robust approaches as n increases.

4.2.2 Sufficient stability condition based on the small gain theorem

Gathering the delay-dependent information from the approximated matrix $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$) and from the filter W_n (resp. $W_n^\flat$) the application of the small gain theorem leads to the following delay-dependent sufficient stability condition for system ($\mathcal{S}_1$).

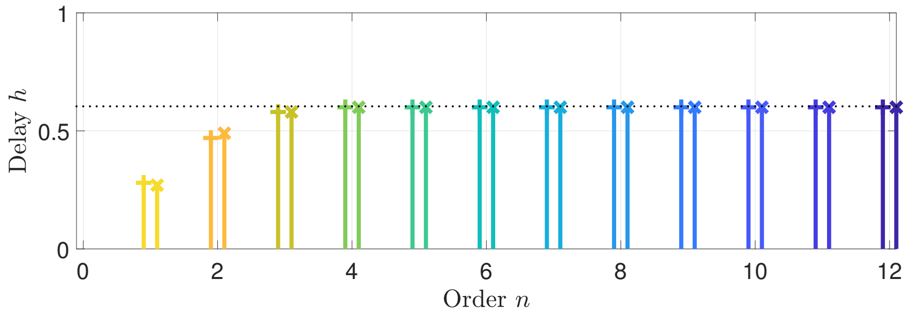
Theorem 4.3. *For any order n in $\mathbb{N}$, if the $\mathcal{H}_\infty$ norm of system $\left(\begin{smallmatrix}\mathbf{A}_n & \mathbf{B}_n\\ \mathbf{C}_n & 0\end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix}\mathbf{A}_n^\flat & \mathbf{B}_n\\ \mathbf{C}_n & 0\end{smallmatrix}\right)$) is lower than γ_n (resp. $\gamma_n^\flat$) then system ($\mathcal{S}_1$) is GES.*

Proof. Applying the small-gain theorem on the augmented system ($\mathcal{S}_{1n}$), we directly obtain the sufficient condition stability. Indeed, the inequality $|\left(\begin{smallmatrix}\mathbf{A}_n & \mathbf{B}_n\\ \mathbf{C}_n & 0\end{smallmatrix}\right)|_{\mathcal{H}_\infty} < \gamma_n$ implies, thanks to (4.11), $|\left(\begin{smallmatrix}\mathbf{A}_n & \mathbf{B}_n\\ \mathbf{C}_n & 0\end{smallmatrix}\right)|_{\mathcal{H}_\infty} |R_n|_{\mathcal{H}_\infty} < 1$. The proof works similarly for system ($\mathcal{S}_{1n}^\flat$) by replacing R_n, $\mathbf{A}_n$ and γ_n by $R_n^\flat$, $\mathbf{A}_n^\flat$ and $\gamma_n^\flat$, respectively. □

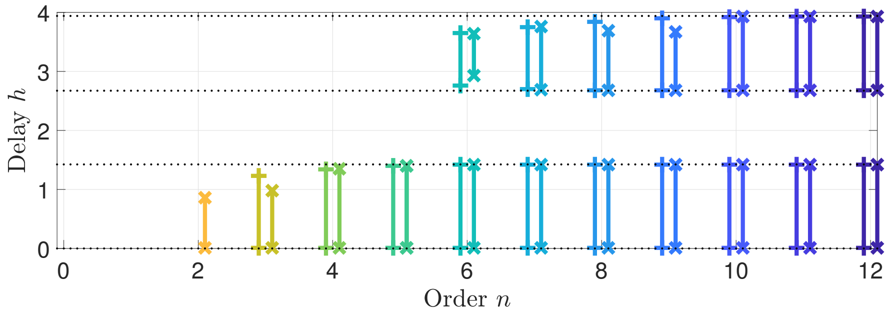
Computational load. Concerning the computation, the frequency-sweeping test is performed in two stages [41]. First, the eigenvalues of $\mathbf{A}_n$ and $\mathbf{A}_n^\flat$ are computed with the algorithms presented in the previous section. Then, if the matrix is Hurwitz, the $\mathcal{H}_\infty$ norm can be computed by bisection algorithm with a tolerance of 10^{-2}. We basically use Matlab routine "hinfnorm" in $\underset{n\to\infty}{O}(n_I n^5)$, where n_I is the number of iterations which depend on the tolerance of the $\mathcal{H}_\infty$ bounds [93]. Contrary to the spectral test presented in Theorem 4.2, the frequency test in Theorem 4.3 can be done even for low orders n, which makes the computation really fast.

This theorem is applied to several orders n and for given delays h on Examples 1.1 and 1.2. On Figure 4.6a for Example 1.1 with $h \in (0, 1]$. and Figure 4.6b for Example 1.2 with $k = 1$ and $h \in (0, 4]$, the intervals of stability with respect to the delay ensured by Theorem 4.3 are colored and the expected ones are recalled in horizontal dotted lines. Overall, Figure 4.6 underlines that the expected regions of stability are well recovered as the order increases.
For $n = 0$, both tests are never verified since it corresponds to a delay-independent frequency test. Then, for $n = 1$, the delay-dependent frequency test ensures stability of Example 1.1 for point-wise delays in $(0, 0.280]$. For Example 1.2, we have $\mathbf{A}_1$ and $\mathbf{A}_1^\flat$ for any delay h and the test fails. Then, increasing the order, the precision on the maximal allowable delays is getting better for both Examples. Moreover, for Example 1.2, the first pocket of stability $[0, 1.424]$ is detected for sufficiently large orders and the second pocket of stability $[2.673, 3.940]$ is detected for even larger orders. High values of the order n are required to evaluate the stability for larger delays h.
For any order n, the test with extended model ($\mathcal{S}_{1n}$) (+ sign) and ($\mathcal{S}_{1n}^\flat$) ($\times$ sign) are compared. Indeed, we know that the finite-dimensional part of ($\mathcal{S}_{1n}^\flat$) is more accurate than the strictly causal finite-dimensional part of ($\mathcal{S}_{1n}$) and we wonder if such model certifies stability for smaller orders than the other. Nevertheless, for stability analysis in frequencies, nothing can be conjectured since model ($\mathcal{S}_{1n}^\flat$) seems better for Example 1.1 and worse for Example 1.2 (see $n = 3$). It is also worth mentioning that the hierarchy with respect to the order n is not satisfied either.
Similarly to the spectral test, it seems difficult to prove the converse theorem, i.e. the convergence of the interval towards the expected ones with respect to the order n. The condition is only sufficient and limited to the inner estimation of the stability regions. Indeed, the approximation might fail for large values of frequencies. As illustrated in Figure 4.3, even if the system is stable, it is possible to have some characteristic roots of matrix $\mathbf{A}_n$ (resp. $\mathbf{A}_n^\flat$) on the right-half-plane. To overcome this difficulty, matrix pencil approaches [171, 170] could say what happens in high frequencies and allow them to be ignored. Then, the converse theorem could be proven. Other ways to reduce the conservatism due to the embedding of R_n (resp. $R_n^\flat$) into uncertainties are possible. For example, robust analysis via quadratic separation [67, 103] or integral quadratic constraints [26] frameworks can be conducted.

To conclude, a frequency-sweeping delay-dependent stability condition is provided by applying the small-gain theorem on a well-chosen infinite-dimensional Legendre remainder part. However, this condition is pessimistic and still poorly understood.



(a) Example 1.1.



(b) Example 1.2 for $k = 1$.

Figure 4.6: Allowable sets of stability given by Theorem 4.3 with respect to the order n. (Markers $+$ on the left and $\times$ on the right refer to the frequency-sweeping test on $(\mathcal{S}_{1n})$ and $(\mathcal{S}_{1n}^{\flat})$.)

Research on the calculation of the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the ODE-transport system in order to better understand these elements and to use them for synthesis issues could be explored [99, 121, 157]. However, for the moment, we have tried other stability analysis techniques.

- *Can we propose sufficient conditions of stability turning out to be necessary as the order increases?*
- *Is it possible to estimate the necessary order?*

In the following, time-domain methods based on complete Lyapunov functionals are investigated to obtain necessary and sufficient and stability conditions for ODE-transport systems.

4.3 Linear matrix inequality test

The stability analysis of ODE-transport systems can also be pursued in the time domain by using a Lyapunov approach. As explained in Appendix A, this approach aims at keeping track of the energy of the system evaluated through a Lyapunov functional. The methodology consists in selecting a quadratic Lyapunov candidate functional $\mathcal{V}$ and to deal with the Lyapunov conditions to derive necessary and sufficient stability conditions.

4.3.1 Sufficient stability condition based on LMI framework

The Lyapunov-Razumikhin, or Lyapunov-Krasovskii theorems [56] have led to numerous sufficient conditions of stability usually expressed in terms of LMI [86, 209]. The analysis is based on quadratic Lyapunov candidate functionals dedicated to ODE-transport systems [85, 193] and on Jensen, Bessel or other inequalities. Here the choice of the model is no more worthy of interest but the selection of tight inequalities has been investigated to reduce the conservatism [142, 176]. Behind these inequalities, approximation techniques and approximated states stay at the heart of the problem [16]. Indeed, the Lyapunov candidate functional is constructed by delay partitioning [108, 113], related to collocation methods, or by polynomial decomposition [188], related to pseudo-spectral methods.

Following the latter method with the n first Legendre polynomials coefficients, introduce the following Lyapunov functional [189]

$$\mathcal{V}_n(x,z) = \begin{bmatrix} x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix} + h\int_0^1 z^\top(\theta)(\theta R + S)z(\theta)\mathrm{d}\theta, \tag{4.12}$$

which underlies the extended model $(\mathcal{S}_{1n})$ presented in the previous chapter.
For any order $n \in \mathbb{N}$, a sufficient condition of stability based on Bessel-Legendre inequality can be then formulated in terms of LMI.

Theorem 4.4. *For any order n in $\mathbb{N}$, if there exist $(\mathbf{P}_n, S, R)$ in $\mathbb{S}^{n_x+nn_z} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ such that the following LMI hold*

$$\begin{cases} \Xi_n^+ = \mathbf{P}_n + \mathbf{I}_n^{h(S+\frac{R}{2})} + \mathbf{J}_N^{h\frac{R}{2}} \succ 0, & (4.13a) \\ \Xi_n^- = \begin{bmatrix} \mathcal{H}(\mathbf{P}_n\mathbf{A}_n) + \mathbf{C}_n^\top(S+R)\mathbf{C}_n - \mathbf{C}_n^{0\top} S\mathbf{C}_n^0 - \mathbf{I}_n^R & \mathbf{P}_n\mathbf{B}_n + \frac{1}{2}\mathbf{C}_n^{0\top}S \\ * & -S \end{bmatrix} \prec 0. & (4.13b) \end{cases}$$

where matrices $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{C}_n^0$ given by (3.44) are recalled hereafter

$$\mathbf{A}_n = \begin{bmatrix} A & B\mathcal{C}_n \\ \mathcal{B}_n C & \mathcal{A}_n \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} B \\ \mathcal{B}_n^* \end{bmatrix}, \quad \mathbf{C}_n = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \mathbf{C}_n^0 = \begin{bmatrix} 0 & -\ell_{n,0}^\top \end{bmatrix},$$
$$\mathbf{I}_n^S = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{I}_n^{-1}\mathcal{I}_n^S\mathcal{I}_n^{-1} \end{bmatrix}, \quad \mathcal{I}_n^S = \mathrm{diag}(S, 3S, \dots, (2n-1)S), \quad \forall S \in \mathbb{S}^{n_z},$$

then system $(\mathcal{S}_1)$ is GES.
In addition, this condition is hierarchic with respect to n.

Proof. For the sake of simplicity, the time argument will be omitted in the following proof. In the first part of the proof, we prove that there exist under (4.13a) for a given order n scalar $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \left|(x,z)\right|^2 \le \mathcal{V}_n(x,z) \le \alpha_2 \left|(x,z)\right|^2, \quad \forall (x,z) \in \mathcal{D}_1. \tag{4.14}$$

In the second part of the proof, we prove that there exists under (4.13b) for the same order n scalar $\alpha_3 > 0$ such that

$$\dot{\mathcal{V}}_n(x,z) \le -\alpha_3 \left|(x,z)\right|^2, \quad \forall (x,z) \in \mathcal{D}_1. \tag{4.15}$$

where $\dot{\mathcal{V}}_n$ denoted the time derivative of $\mathcal{V}_n$ along the trajectories of system $(\mathcal{S}_1)$.

Step 1*: Positivity of the Lyapunov functional.*
Consider n in $\mathbb{N}$. From Lemma 2.4, the Bessel-Legendre equality (2.53) to the state z ensures

$$h\int_0^1 z^\top(\theta)Sz(\theta)\mathrm{d}\theta = \left(\int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta\right)^\top \mathcal{I}_n^{hS}\left(\int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta\right) + h\int_0^1 \tilde{z}_n^\top(\theta)S\tilde{z}_n(\theta)\mathrm{d}\theta. \tag{4.16}$$

Then, the two previous equations are merged to obtain

$$\mathcal{V}_n(x,z) \ge \begin{bmatrix} x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix}^\top \Xi_n^+ \begin{bmatrix} x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix} + h\int_0^1 \tilde{z}_n^\top(\theta)S\tilde{z}_n(\theta)\mathrm{d}\theta + h\int_0^1 z^\top(\theta)\theta Rz(\theta)\mathrm{d}\theta.$$

The condition (4.13a) thus ensures the existence of $\alpha_1 = \min(\underline{\sigma}(\Xi_n^+), \underline{\sigma}(hS)) > 0$, such that the Lyapunov functional given by (4.12) satisfies $\mathcal{V}_n(x,z) \ge \alpha_1 \left|(x,z)\right|^2$. Moreover, since $\mathcal{V}_n$ is quadratic with respect to (x,z), selecting $\alpha_2 = \bar{\sigma}(\mathbf{P}_n) + h(\bar{\sigma}(S) + \bar{\sigma}(R)) > 0$ ensures that inequality $\mathcal{V}_n(x,z) \le \alpha_2 \left|(x,z)\right|^2$ holds.

Step 2*: Negativity of the derivative of the Lyapunov functional along $(\mathcal{S}_1)$.*

The time derivative of the Lyapunov functional at order n is written as

$$\begin{array}{rcl}\dot{\mathcal{V}}_n(x,z) & = & \begin{bmatrix}\begin{bmatrix}x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta\end{bmatrix} \\ \tilde{z}_n(0)\end{bmatrix}^\top \begin{bmatrix}\mathcal{H}(\mathbf{P}_n\mathbf{A}_n) & \mathbf{P}_n\mathbf{B}_n \\ * & 0\end{bmatrix} \begin{bmatrix}\begin{bmatrix}x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta\end{bmatrix} \\ \tilde{z}_n(0)\end{bmatrix} \\ & & +z^\top(1)(S+R)z(1) - z^\top(0)Sz(0) - \int_0^1 z(\theta)Rz^\top(\theta)\mathrm{d}\theta,\end{array}$$

along the trajectories of system $(\mathcal{S}_1)$. Applying Bessel-Legendre equality (2.53) to the state z yields

$$\dot{\mathcal{V}}_n(x,z) = \begin{bmatrix}x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta \\ \tilde{z}_n(0)\end{bmatrix}^\top \Xi_n^- \begin{bmatrix}x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta \\ \tilde{z}_n(0)\end{bmatrix} - \int_0^1 \tilde{z}_n^\top(\theta)R\tilde{z}_n(\theta)\mathrm{d}\theta. \qquad (4.17)$$

The condition (4.13b) ensures the existence of $\alpha_3 = \underline{\sigma}(\Xi_n^-, R) > 0$, such that the Lyapunov functional given by (4.12) satisfies $\dot{\mathcal{V}}_n(x,z) \leq -\alpha_3 \left|(x,z)\right|^2$. The Lyapunov theorem concludes on the exponential stability of the origin.

Step 3: Hierarchy.
Concerning the hierarchy, the proof is provided in [189, Theorem 7]. A glimpse of the proof consists in introducing $\mathbf{P}_{n+1}$ so that $\mathcal{V}_{n+1}(x,z) = \mathcal{V}_n(x,z)$ and so that one can exhibit a solution to the LMI problem at order $n+1$ based on the solution at order n. The details of the proof are omitted but strongly rely on the structure of Ξ_n^+ and Ξ_n^-. □

Remark 4.3. Note that there are many ways to express this Lyapunov functional and to obtain many equivalent LMI conditions. For instance, consider a Lyapunov functional modeled on system $(\mathcal{S}_{1n})$ and based on the extended state ξ_n and the residual error $\tilde{z}_n$

$$\mathcal{V}_n^\star(\xi_n, \tilde{z}_n) = \xi_n^\top \mathbf{P}_n \xi_n + h\int_0^1 \tilde{z}_n^\top(\theta)\left(S + \frac{R}{2}\right)\tilde{z}_n(\theta)\mathrm{d}\theta + h\int_0^1 \frac{l_1(\theta)}{2}\tilde{z}_{n-1}^\top(\theta)R\tilde{z}_{n-1}(\theta)\mathrm{d}\theta. \qquad (4.18)$$

With this functional, the Lyapunov conditions lead to the following LMI

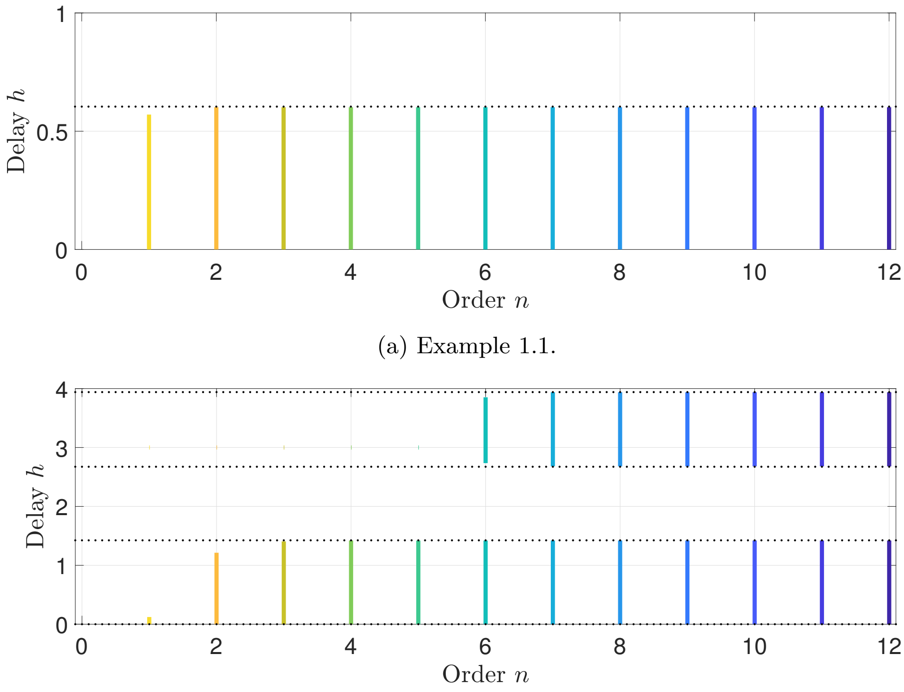
$$\begin{cases} \mathbf{P}_n \succ 0, \\ \begin{bmatrix}\mathcal{H}(\mathbf{P}_n\mathbf{A}_n) + \tilde{\mathbf{C}}_n^\top(S+\frac{R}{2})\tilde{\mathbf{C}}_n & \mathbf{P}_n\mathbf{B}_n \\ * & -(S+\frac{R}{2})\end{bmatrix} + \begin{bmatrix}\begin{bmatrix}\tilde{\mathbf{C}}_{n-1}^\top \frac{R}{2}\tilde{\mathbf{C}}_{n-1} & -\frac{n-1}{2n-1}\tilde{\mathbf{C}}_{n-1}^\top\frac{R}{2} \\ * & -\frac{1}{2n-1}\frac{R}{2}\end{bmatrix} & \begin{bmatrix}0 \\ (-1)^{n-1}\frac{n}{2n-1}\frac{R}{2}\end{bmatrix} \\ * & \frac{R}{2}\end{bmatrix} \prec 0. \end{cases} \qquad (4.19)$$

which are equivalent to (4.13). Up to congruence, it is also possible to rearrange the terms in order to bring out diagonal terms in $\mathbf{A}_n$ [77] using the Legendre approximation of the spacial derivative of $\partial_\theta z$ instead of z. Imposing a structure to matrix $\mathbf{P}_n$ would reduce the number of decision variables.

Computational load. Here, the number of LMI variables is equal to $N_{var} = \frac{(n_x+nn_z)(n_x+nn_z+1)}{2} + n_z(n_z+1) = \underset{n\to\infty}{O}(n^2)$. The feasibility convex problem is solved by the interior point method and then require $\underset{n\to\infty}{O}(n_I n^7)$ operations, where n_I is the number of iterations as explained in Appendix A.3.

Theorem 4.4 provides inner approximations of the stability regions. If LMIs (4.13) are true, then the origin of system $(\mathcal{S}_1)$ is GES. Compared to the frequency condition given by Theorem 4.3, these LMIs can be solved for any order $n \in \mathbb{N}$ but are also hierarchic property with respect to n as demonstrated in [189]. Thanks to the hierarchy, for a given precision on the delay or other system parameters, we can set up incremental tests. Lastly, compared to exact methods such as matrix pencils [148], LMI can be easily exportable to uncertain or time-varying parameters to assess robustness issues [87, 142]. Such scalability is often appreciated, but the price to pay is the computational complexity and time.

For Examples 1.1 and 1.2, Figure 4.7 reports the allowable delays, for which LMIs (4.13) are satisfied, for several orders $n \in \{1, \ldots, 12\}$. On both examples, one can see the efficiency of Theorem 4.4 even for very low orders n. On Example 1.1, the precision 10^{-3} on the upper bound $h = 0.604$ is reached from order $n = 3$. On Example 1.2, increasing the order n, the pockets of stability $[0, 1.424] \cup [2.673, 3.940]$ are also recovered. Furthermore, notice that larger orders are required as the delay increases. Lastly, on Figure 4.8, the LMIs (4.13) are solved on Example 1.2 for a panel of values $(k, h) \in [0.1, 100] \times (0, 10]$. Observe that the required orders also grow with respect to parameter k. In all cases, the regions of stability stretch progressively with respect to n, which confirms the hierarchical property. Moreover,



(a) Example 1.1.

(b) Example 1.2 for $k = 1$.

Figure 4.7: Allowable sets of stability given by Theorem 4.4 with respect to the order n.

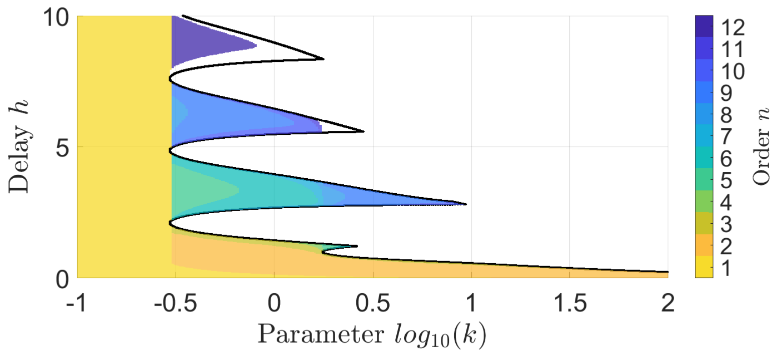


Figure 4.8: Inner approximation of the stability regions for Example 1.2 in (k, h) plane.

it seems that the regions approach fill the expected interval of stability.

In the sequel, the convergence speculation of the inner estimates of the stability regions towards the expected ones with respect to the order n will be confirmed and proven.

4.3.2 Convergence of these LMI stability conditions

The potential of Lyapunov-type approaches is strengthen when considering the necessary side of the Lyapunov theorem. Indeed, as highlighted in Appendix A.1 [58], if a linear system is GES then there exists a quadratic positive definite Lyapunov functional that satisfies the Lyapunov equation. Even more interestingly, focusing on a system interconnected with a transport equation and assuming that there is no characteristic roots s_1, s_2 such that $s_1 + s_2 = 0$, the unique converse Lyapunov functional satisfying the Lyapunov equation can be constructed as done in Appendix A.2 [131].

For any matrix W in $\mathbb{S}^{n_x}$, such a converse Lyapunov functional satisfying $\dot{V}_W(x,z) = -x^\top W x$ along the trajectories of system $(\mathcal{S}_1)$ is given by

$$\begin{aligned} V_W(x,z) = &\underbrace{x^\top U_W(0)x}_{V_a(x,z)} + \underbrace{2hx^\top \int_0^1 U_W(\theta)Bz(\theta)B\mathrm{d}\theta}_{V_b(x,z)} \\ &+ \underbrace{h^2 \int_0^1 \int_0^1 z^\top(\theta_1)B^\top U_W(\theta_2-\theta_1)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2}_{V_c(x,z)}, \end{aligned} \tag{4.20}$$

and depends on the Lyapunov matrix U_W defined from $[-1,1]$ to $\mathbb{R}^{n_x \times n_x}$ as

$$U_W(\theta) = \begin{cases} \mathrm{vec}^{-1}\left(\left[\begin{smallmatrix} I_{n_x^2} & 0 \end{smallmatrix}\right] e^{\theta M} N^{-1} \left[\begin{smallmatrix} -\mathrm{vec}(W) \\ 0 \end{smallmatrix}\right]\right) & \text{if} \quad \theta \geq 0, \\ U_W^\top(-\theta) & \text{if} \quad \theta < 0, \end{cases} \tag{4.21}$$

where matrices M, N have been given in (A.19) and are recalled hereafter

$$\begin{aligned} M &= h\left[\begin{smallmatrix} I_{n_x}\otimes A^\top & I_{n_x}\otimes A_d^\top \\ -A_d^\top\otimes I_{n_x} & -A^\top\otimes I_{n_x} \end{smallmatrix}\right], \\ N &= \left[\begin{smallmatrix} A^\top\otimes I_{n_x} + I_{n_x}\otimes A^\top & I_{n_x}\otimes A_d^\top \\ I_{n_x^2} & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} A_d^\top\otimes I_{n_x} & 0 \\ 0 & -I_{n_x^2} \end{smallmatrix}\right] e^M. \end{aligned} \tag{4.22}$$

It is the unique quadratic, continuous, and differentiable functional that satisfies

$$\dot{V}_W(x,z) = -x^\top W x, \quad \forall (x,z) \in \mathcal{D}_1, \tag{4.23}$$

along the trajectories of system $(\mathcal{S}_1)$.

Remark 4.4. It is worth noticing that such a closed-form expression of the converse functional has been found in time-delay systems context [131]. It is mainly due to the fact that ODE-transport interconnected systems can be seen as time-delay systems and that their fundamental solution is known analytically. Extensions have been made for time-delay systems with multiple commensurate [131, Chapter 3] or incommensurate [64] delays and with polynomial or piece-wise constant distributed delays [131, Chapter 4]. In more twisted cases, approximations of the function U_W are required.

A modified Lyapunov converse condition writes therefore as follows.

Lemma 4.2. *For any matrices (W_1, W_2, W_3) in $\mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$, define the so-called complete Lyapunov functional*

$$\mathcal{V}(x,z) = V_{W_1 + C^\top(W_2+W_3)C}(x,z) + h\int_0^1 z^\top(\theta)(\theta W_2 + W_3)z(\theta)\mathrm{d}\theta. \tag{4.24}$$

If system $(\mathcal{S}_1)$ is GES, then there exist $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\mathcal{V}$ satisfies

$$\alpha_1 \left|(x,z)\right|^2 + h\int_0^1 z^\top(\theta)\theta W_2 z(\theta)\mathrm{d}\theta \leq \mathcal{V}(x,z) \leq \alpha_2 \left|(x,z)\right|^2, \qquad \forall (x,z) \in \mathcal{D}_1, \tag{4.25a}$$

$$\dot{\mathcal{V}}(x,z) \leq -\alpha_3 \left|(x,z)\right|^2, \qquad \forall (x,z) \in \mathcal{D}_1. \tag{4.25b}$$

Proof. For simplicity reasons, the time argument has been removed from the proof. Firstly, for linear systems and quadratic functionals, the condition on the upper bound of $\mathcal{V}$ using α_2 can be omitted, since is a consequence of the quadratic structure of the functional. The lower bound condition needs more details, even if a part of the answer is given by [107, Theorem 5.19]. For any $\alpha_1 > 0$, introduce the functional $\mathcal{W}$ as

$$\mathcal{W}(x,z) = \mathcal{V}(x,z) - \alpha_1 \left|(x,z)\right|^2 - h\int_0^1 z^\top(\theta)\theta W_2 z(\theta)\mathrm{d}\theta.$$

According to (4.23), differentiating V along the trajectories of system $(\mathcal{S}_1)$, we obtain

$$\dot{\mathcal{V}}(x,z) = -x^\top W_1 x - \int_0^1 z^\top(\theta) W_2 z(\theta) \mathrm{d}\theta - z^\top(0) W_3 z(0). \tag{4.26}$$

Therefore, differentiating $\mathcal{W}$ along the trajectories of $(\mathcal{S}_1)$ leads to

$$\dot{\mathcal{W}}(x,z) = -\begin{bmatrix} x \\ z(0) \end{bmatrix}^\top \left(\begin{bmatrix} W_1 + C^\top W_2 C & 0 \\ 0 & W_3 \end{bmatrix} + \alpha_1 \begin{bmatrix} \mathcal{H}(A) + C^\top C & B \\ * & -I_{n_z} \end{bmatrix} \right) \begin{bmatrix} x \\ z(0) \end{bmatrix}.$$

Then, there exists a sufficiently small positive number α_1, which depend on (W_1, W_2, W_3), such that $\dot{\mathcal{W}}(x,z) \leq 0$ holds. Pursuing an integration in time from 0 to ∞, we obtain

$$\lim_{t\to\infty} \mathcal{W}\big(x(t), z(t)\big) - \mathcal{W}\big(x(0), z(0)\big) \leq 0, \quad \forall \big(x(0), z(0)\big) \in \mathcal{D}_1.$$

From the assumption that system $(\mathcal{S}_1)$ is GES, we have $\big(x(t), z(t)\big) \underset{t\to\infty}{\longrightarrow} (0,0)$. Then, the inequality $\mathcal{W}\big(x(0), z(0)\big) \geq 0$ holds for all $\big(x(0), z(0)\big) \in \mathcal{D}_1$ and leads to the first inequality of (4.25a). Based on (4.26), the last inequality (4.25b) is finally obtained taking $\alpha_3 = \min\big(\underline{\sigma}(W_1), \underline{\sigma}(W_2)\big) > 0$. □

Such a condition is not implementable since inequalities (4.25) cannot be verified numerically. At this stage, it remains a theoretical result [131].

Approximation techniques are then used to make this result tractable numerically. The philosophy is to approximate the complete Lyapunov functional. To do so, approximated Lyapunov functionals are constructed based on an approximation of the state z [72, 155] or of the Lyapunov matrix U_W. By collocation methods, the Lyapunov matrix U_W is discretized via piece-wise constant [106, 108], piece-wise linear [172], or splines [123] functions. These discretization methods have the merit to be exact of a finite number of points but make errors in between these points. By Galerkin-like methods, the Lyapunov matrix U_W is approximated on a trigonometric or polynomial basis. In particular, we will focus on the approximation on Legendre polynomials [19] in order to use the potential convergence rates discussed in Chapter 2.

Definition 4.1. For any order n in $\mathbb{N}$ and $W \in \mathbb{S}_+^{n_x}$, define the Legendre approximated functions $U_{1,n}$ and $U_{2,n}$ of the Lyapunov function U_W given by (4.21) on the interval $[0,1]$ and $[-1,1]$ respectively as follows

$$\begin{array}{rcll} U_{1,n}(\theta) & = & \ell_n^{n_x\top}(\theta)\mathcal{U}_{1,n}, & \forall\theta \in [0,1], \\ U_{2,n}(\theta) & = & \ell_n^{n_x\top}\left(\frac{\theta+1}{2}\right)\mathcal{U}_{2,n}, & \forall\theta \in [-1,1]. \end{array} \tag{4.27}$$

where matrices $\mathcal{U}_{1,n}$ and $\mathcal{U}_{2,n}$ in $\mathbb{R}^{nn_x \times n_x}$, which collocates the n first Legendre coefficients of the Lyapunov matrix U_W, are given by

$$\begin{array}{rcl} \mathcal{U}_{1,n} & = & \left(\int_0^1 \ell_n^{n_x}(\theta)\ell_n^{n_x\top}(\theta)\mathrm{d}\theta\right)^{-1} \left(\int_0^1 \ell_n^{n_x}(\theta) U_W(\theta)\mathrm{d}\theta\right), \\ \mathcal{U}_{2,n} & = & \frac{1}{2}\left(\int_0^1 \ell_n^{n_x}(\theta)\ell_n^{n_x\top}(\theta)\mathrm{d}\theta\right)^{-1} \left(\int_{-1}^1 \ell_n^{n_x}\left(\frac{\theta+1}{2}\right) U_W(\theta)\mathrm{d}\theta\right), \end{array} \tag{4.28}$$

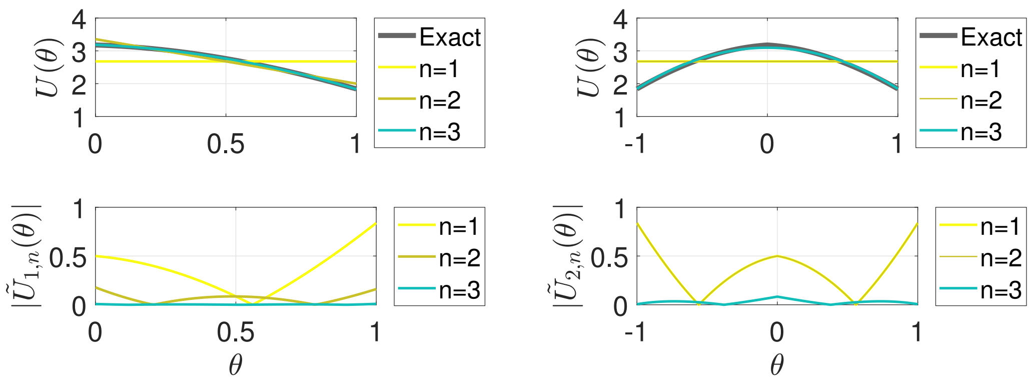
and where matrix $\ell_n^{n_x}$ in $\mathbb{R}^{nn_x \times n_x}$, which collocates the n first Legendre polynomials, is given by

$$\ell_n^{n_x}(\theta) = \begin{bmatrix} l_0(\theta) I_{n_x} & \dots & l_{n-1}(\theta) I_{n_x} \end{bmatrix}^\top, \quad \forall\theta \in [0,1]. \tag{4.29}$$

The Legendre truncated errors $\tilde{U}_{1,n}$ and $\tilde{U}_{2,n}$ of U_W at order n are defined by

$$\begin{array}{rcll} \tilde{U}_{1,n}(\theta) & = & U_W(\theta) - \ell_n^{n_x\top}(\theta)\mathcal{U}_{1,n}, & \forall\theta \in [0,1], \\ \tilde{U}_{2,n}(\theta) & = & U_W(\theta) - \ell_n^{n_x\top}\left(\frac{\theta+1}{2}\right)\mathcal{U}_{2,n}, & \forall\theta \in [-1,1]. \end{array} \tag{4.30}$$

On the interval $[0,1]$ and $[-1,1]$, the Legendre approximation of $U_{I_{n_x}}$ is performed at order n. For the scalar Example 1.1, Figure 4.9 shows the performances of this approximation. Notice that the approximation shown in Figure 4.9a is better than the one shown in Figure 4.9b due to smooth prop-



(a) Approximation $U_{1,n}(\theta)$.

(b) Approximation $U_{2,n}(\theta)$.

Figure 4.9: Legendre approximated Lyapunov function for Example 1.1 with $h = 0.5$.

erties of U_W on the interval $[0, 1]$. Referring to Chapter 2, the first convergence rate is supergeometric whereas the second one is algebraic. In the following, the proof arguments will only retain the slowest convergence, which is limiting. It is also worth noticing that the approximation on $[-1, 1]$ could be improved using the symmetric property of U_W and defining the following approximated function

$$U_n(\theta) = \begin{cases} \ell_n^{n_x \top}(\theta)\mathcal{U}_n, & \text{if} \quad \theta \geq 0, \\ U_n^\top(-\theta), & \text{if} \quad \theta < 0, \end{cases} \tag{4.31}$$

Nevertheless, such a decomposition does not satisfied the separability condition which will be needed to obtain linear matrix inequality conditions for stability. Thus, the terms V_a, V_b and V_c of the converse Lyapunov functional can be approximated with Legendre polynomials approximation defined by Definition 4.1.

Definition 4.2. For any order n in $\mathbb{N}$ and for any matrices (W_1, W_2, W_3) in $\mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$, define the approximated Lyapunov functional $\mathcal{V}_n$ given by

$$\begin{aligned} \mathcal{V}_n(x, z) = & \begin{bmatrix} x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix}^\top \begin{bmatrix} P & \mathcal{Q}_n \\ * & \mathcal{T}_n \end{bmatrix} \begin{bmatrix} x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix} \\ & + h\int_0^1 z^\top(\theta)(\theta W_2 + W_3)z(\theta)\mathrm{d}\theta. \end{aligned} \tag{4.32}$$

where matrices $(P, \mathcal{Q}_n, \mathcal{T}_n)$ are given by

$$\begin{aligned} P &= U(0), \\ \mathcal{Q}_n &= h\int_0^1 U_{1,n}(\theta)B\ell_n^\top(\theta)\mathrm{d}\theta, \\ \mathcal{T}_n &= h^2\int_0^1\int_0^1 \ell_n(\theta_1)B^\top U_{2,n}(\theta_2 - \theta_1)B\ell_n^\top(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2, \end{aligned} \tag{4.33}$$

and where $U_{1,n}$ and $U_{2,n}$ are the Legendre approximation of the Lyapunov function $U_{W_1+C^\top(W_2+W_3)C}$ at order n given by (4.31).

The approximated Lyapunov functional $\mathcal{V}_n$ given by (4.32) corresponds to the Legendre approximation at order n of the complete Lyapunov functional $\mathcal{V}$ given by (4.24). More precisely, the Lyapunov matrix $U_{W_1+C^\top(W_2+W_3)C}$ has been replaced approximated at order n. The following lemma can be expressed.

Lemma 4.3. *For any matrices (W_1, W_2, W_3) in $\mathbb{S}_+^{n_x} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$, assume that the following inequality holds*

$$\Psi_n(\theta) = \begin{bmatrix} W_1 + \Psi_{n,0} & \Psi_{n,1}(\theta) & \Psi_{n,2} \\ * & W_2 & \Psi_{n,3}(\theta) \\ * & * & W_3 \end{bmatrix} \succ 0, \quad \forall \theta \in [0,1]. \tag{4.34}$$

where matrices

$$\begin{array}{rcl} \Psi_{n,0} & = & \mathcal{H}(\tilde{U}_{1,n}(1)A_d), \quad \Psi_{n,2} = -\tilde{U}_{1,n}(0)B, \\ \Psi_{n,1}(\theta) & = & \big(hA^\top \tilde{U}_{1,n}(\theta) + hA_d^\top \tilde{U}_{2,n}^\top(1-\theta) - \tilde{U}_{1,n}'(\theta)\big)B, \\ \Psi_{n,3}(\theta) & = & hB^\top \big(\tilde{U}_{1,n}^\top(\theta)B + \tilde{U}_{2,n}^\top(\theta)A_d\big). \end{array} \tag{4.35}$$

where the truncated errors $\tilde{U}_{1,n}$ and $\tilde{U}_{2,n}$ are given by (4.30), with $W = W_1 + C^\top(W_2 + W_3)C$. Then, there exists $\alpha > 0$ such that the approximated Lyapunov functional $\mathcal{V}_n$ given by (4.32) satisfies

$$\dot{\mathcal{V}}_n(x,z) \leq -\alpha \left|(x,z)\right|^2, \quad \forall (x,z) \in \mathcal{D}_1. \tag{4.36}$$

Proof. The relation between Lyapunov functionals $\mathcal{V}_n$ and $\mathcal{V}$ comes from Definition 4.1 of the Legendre approximation of function U_W with $W = W_1 + C^\top(W_2 + W_3)C$. Splitting $U_W = U_n + \tilde{U}_n$ yields

$$\begin{aligned} \mathcal{V}(x,z) &= \underbrace{x^\top U(0)x}_{V_{na}(x,z)} + \underbrace{2hx^\top \int_0^1 U_{1,n}(\theta)Bz(\theta)\mathrm{d}\theta}_{V_{nb}(x,z)} + \underbrace{h^2 \int_0^1\int_0^1 z^\top(\theta_1)B^\top U_{2,n}(\theta_2-\theta_1)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2}_{V_{nc}(x,z)} \\ &\quad + \underbrace{2hx^\top \int_0^1 \tilde{U}_{1,n}(\theta)Bz(\theta)\mathrm{d}\theta}_{\tilde{V}_{nb}(x,z)} + \underbrace{h^2 \int_0^1\int_0^1 z^\top(\theta_1)B^\top \tilde{U}_{2,n}(\theta_2-\theta_1)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2}_{\tilde{V}_{nc}(x,z)} \\ &\quad + h\int_0^1 z^\top(\theta)(\theta W_2 + W_3)z(\theta)\mathrm{d}\theta, \\ &= \mathcal{V}_n(x,z) + \tilde{V}_{na}(x,z) + \tilde{V}_{nb}(x,z) + \tilde{V}_{nc}(x,z), \end{aligned} \tag{4.37}$$

where $\mathcal{V}$ and $\mathcal{V}_n$ are defined by (4.24) and (4.32) respectively and the errors $\tilde{U}_{1,n}$ and $\tilde{U}_{2,n}$ are given by (4.30). To be convinced, one can check each approximated term separately

$$\begin{aligned} V_{na}(x,z) &= x^\top U(0)x = x^\top Px, \\ V_{nb}(x,z) &= 2hx^\top \int_0^1 U_{1,n}(\theta)Bz(\theta)\mathrm{d}\theta = 2hx^\top \int_0^1 \left(\int_0^1 U_{1,n}(\tau)\ell_n^{n_x\top}(\tau)\mathrm{d}\tau\right)\mathcal{I}_n^{I_{n_x}}\ell_n^{n_x}(\theta)Bz(\theta)\mathrm{d}\theta \\ &= 2x^\top \mathcal{Q}_n\mathcal{I}_n\int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta, \\ V_{nc}(x,z) &= h^2\int_0^1\int_0^1 z^\top(\theta_1)B^\top U_{2,n}(\theta_2-\theta_1)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2, \\ &= h^2\int_0^1\!\!\int_0^1 z^\top(\theta_1)B^\top\ell_n^{n_x\top}(\theta_1)\mathcal{I}_n^{I_{n_x}}\left(\int_0^1\!\!\int_0^1 \ell_n^{n_x}(\tau_1)U_{2,n}(\tau_2-\tau_1)\ell_n^{n_x\top}(\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2\right)\mathcal{I}_n^{I_{n_x}}\ell_n^{n_x}(\theta_2)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2, \\ &= \left(\mathcal{I}_n\int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta\right)^\top \mathcal{T}_n\left(\mathcal{I}_n\int_0^1 \ell_n(\theta)z(\theta)\mathrm{d}\theta\right), \end{aligned}$$

using the Kronecker product properties and the expression of matrices P, $\mathcal{Q}_n$ and $\mathcal{T}_n$ given by (4.33). Let now focus on the time derivative of $\mathcal{V} - \mathcal{V}_n$. The time derivative of the first error term along the trajectories of system $(\mathcal{S}_1)$ lead to the following expression after performing an integration by parts

$$\begin{array}{rcl} \dot{\tilde{V}}_{nb}(x,z) & = & 2h\big(Ax + Bz(0)\big)^\top \displaystyle\int_0^1 \tilde{U}_{1,n}(\theta)Bz(\theta)\mathrm{d}\theta \\ & & +2x^\top\tilde{U}_{1,n}(1)Bz(1) - 2x^\top\tilde{U}_{1,n}(0)Bz(0) - 2x^\top\displaystyle\int_0^1 \tilde{U}_{1,n}'(\theta)Bz(\theta)\mathrm{d}\theta. \end{array} \tag{4.38}$$

Concerning the second error term, along the trajectories of system $(\mathcal{S}_1)$, we have

$$\begin{aligned}\dot{\tilde{V}}_{nc}(x,z) &= h\int_0^1\int_0^1 \partial_{\theta_1} z^\top(\theta_1)B^\top \tilde{U}_{2,n}(\theta_2-\theta_1)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2 \\ &\quad + h\int_0^1\int_0^1 z^\top(\theta_1)B^\top \tilde{U}_{2,n}(\theta_2-\theta_1)B\partial_{\theta_2} z(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2.\end{aligned} \tag{4.39}$$

Since $\tilde{U}_{2,n}(\theta_2-\theta_1)$ is smooth, it follows

$$\begin{aligned}\dot{\tilde{V}}_{nc}(x,z) &= -hz^\top(0)B^\top\int_0^1 \tilde{U}_{2,n}(\theta)Bz(\theta)\mathrm{d}\theta + hz^\top(1)B^\top\int_0^1 \tilde{U}_{2,n}^\top(1-\theta)Bz(\theta)\mathrm{d}\theta \\ &\quad -h\int_0^1 z^\top(\theta)B^\top\tilde{U}_{2,n}^\top(\theta)\mathrm{d}\theta Bz(0)\mathrm{d}\theta + h\int_0^1 z^\top(\theta)B^\top\tilde{U}_{2,n}(1-\theta)\mathrm{d}\theta Bz(1)\mathrm{d}\theta \\ &\quad \cancel{-h\int_0^1\int_0^1 z^\top(\theta_1)B^\top(\partial_{\theta_1}+\partial_{\theta_2})\tilde{U}_{2,n}(\theta_2-\theta_1)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2}.\end{aligned}$$

Simplifying terms and regrouping identical terms, the time derivative of the second error term along the trajectories of system $(\mathcal{S}_1)$ satisfies

$$\dot{\tilde{V}}_{nc}(x,z) = -2hz^\top(0)B^\top\int_0^1 \tilde{U}_{2,n}(\theta)Bz(\theta)\mathrm{d}\theta + 2h\int_0^1 z^\top(\theta)B^\top\tilde{U}_{2,n}(1-\theta)\mathrm{d}\theta Bz(1)\mathrm{d}\theta. \tag{4.40}$$

The reasoning follows then similar arguments like the ones used in the proof of Lemma 4.2. Therefore, the time derivative of $\mathcal{V}_n$ given by (4.32) along the trajectories of system $(\mathcal{S}_1)$ leads thanks to (4.26), (4.38), (4.40) to the following expression

$$\begin{aligned}\dot{\mathcal{V}}_n(x,z) &= -x^\top W_1 x - \int_0^1 z^\top(\theta)W_2 z(\theta)\mathrm{d}\theta - z^\top(0)W_3 z(0) \\ &\quad -2h\int_0^1 \begin{bmatrix} x \\ z(0)\end{bmatrix}^\top \begin{bmatrix} A^\top \\ B^\top\end{bmatrix}\tilde{U}_{1,n}(\theta)Bz(\theta)\mathrm{d}\theta \\ &\quad -2x^\top\int_0^1 \begin{bmatrix}\tilde{U}_{1,n}(1)A_d & -\tilde{U}'_{1,n}(\theta)B & -\tilde{U}_{1,n}(0)B\end{bmatrix}\begin{bmatrix} x \\ z(\theta) \\ z(0)\end{bmatrix}\mathrm{d}\theta \\ &\quad -2h\int_0^1 \begin{bmatrix} x \\ z(0)\end{bmatrix}^\top \begin{bmatrix} A_d^\top\tilde{U}_{2,n}^\top(1-\theta) \\ -B^\top\tilde{U}_{2,n}(\theta)\end{bmatrix}Bz(\theta)\mathrm{d}\theta.\end{aligned}$$

The terms with respect to $z(0)$ vanish and simplifies to

$$\dot{\mathcal{V}}_n(x,z) = -\int_0^1 \begin{bmatrix} x \\ z(\theta) \\ z(0)\end{bmatrix}^\top \Psi_n(\theta)\begin{bmatrix} x \\ z(\theta) \\ z(0)\end{bmatrix}\mathrm{d}\theta. \tag{4.41}$$

Thus, condition $\Psi_n(\theta) \succ 0$ for all $\theta \in [0,1]$ ensures the existence of a sufficiently small scalar $\alpha_3 > 0$ such that inequality (4.25b) holds. □

Contrary to inequality (4.25b), the evaluation or assessment of inequality (4.36) is now numerically tractable through the LMI framework. It is also worth noticing that the corresponding LMI conditions are the ones given by (4.13) since approximated Lyapunov functional $\mathcal{V}_n$ matches with the one described in (4.12) for the corresponding matrices

$$\mathbf{P}_n = \begin{bmatrix} P & \mathcal{Q}_n \\ * & \mathcal{T}_n\end{bmatrix}, \quad R = W_2, \quad S = W_3, \tag{4.42}$$

where matrices $(P, \mathcal{Q}_n, \mathcal{T}_n)$ are given by (4.33).
The condition of application of our converse theorem, makes appear the positivity of the matrix Ψ_n. It is worth noticing that this matrix Ψ_n is written as the sum of the matrix $\begin{bmatrix} W_1 & 0 & 0 \\ * & W_2 & 0 \\ * & * & W_3\end{bmatrix}$ fixed a priori and independent of the order n and $\begin{bmatrix}\Psi_{n,0} & \Psi_{n,1} & \Psi_{n,2} \\ * & 0 & \Psi_{n,3} \\ * & * & 0\end{bmatrix}$ depending only on the error of approximation of the function U_W by $U_{1,n}$ and $U_{2,n}$. Intuitively, we have $\tilde{U}_{1,n} = U_W - U_{1,n}$ and $\tilde{U}_{2,n} = U_W - U_{2,n}$ which

tends to zero as n goes to infinity, then matrix Ψ_n will be positive for sufficiently large orders. In the following, we must quantify this intuition.

Remark 4.5. The previous lemma stands for any functional $\hat{\mathcal{V}}$ constructed as a copy of the Lyapunov functional $\mathcal{V}$ given by (4.24) where Lyapunov matrix U_W is replaced by $\hat{U}_W$. Legendre approximation (4.31) can then be seen as a specific application but many other approximations could be considered. It is the case of interpolation schema with piece-wise constant [106, 108], piece-wise linear [172], or other interpolated functions [70]. Whether for collocation or Galerkin-like methods, the convergence of $\hat{\mathcal{V}}$ towards $\mathcal{V}$ could be proven. For instance, approximating U_W by piece-wise linear functions, the convergence of the approximated functional and its derivative towards the complete has been proven in [105], when the discretization step tends to zero.

In the sequel, we aim at proving and quantifying the convergence of our Legendre approximation, when the order n tends to infinity. The main issue is then to prove that inequality (4.34) is satisfied for any θ in $[0,1]$. The solution comes from the fact that U_W has nice regularity properties on the interval $[0,1]$. Indeed, the Lyapunov matrix function satisfies the following properties.

Property 4.2. *For any $W \in \mathbb{S}_+^{n_x}$, the Lyapunov matrix U_W defined in* (4.21) *satisfies*

$$\left\| U_W^{(d)} \right\|_\infty \leq (2\mu)^d \rho_0, \tag{4.43}$$

with parameters μ, ρ_0 given by

$$\mu = \frac{|M|}{2}, \quad \rho_0 = \sqrt{n_x}\, \mathrm{e}^{h|M|} \left|N^{-1}\right| |W|, \tag{4.44}$$

and matrices M, N given by (4.22).

Proof. Thanks to the equivalence of matrix norms, inequalities $\left|M^\top\right| = |M| \leq |\mathrm{vec}(M)| \leq \sqrt{p}\,|M|$ hold, for any square M of dimension p. Then, for all θ in $[0,1]$, we have

$$\begin{aligned}
\left|U_W^{(d)}(\theta)\right| &\leq \left|\left[\begin{smallmatrix} I_{n_x^2} & 0\end{smallmatrix}\right] \mathrm{e}^{\theta M} M^d N^{-1} \left[\begin{smallmatrix} -\mathrm{vec}(W) \\ 0 \end{smallmatrix}\right]\right|, \\
&\leq \left|\mathrm{e}^{\theta M}\right| \left|M^d N^{-1}\right| |\mathrm{vec}(W)|, \\
&\leq \sqrt{n_x} \left|\mathrm{e}^{\theta M}\right| |M|^d \left|N^{-1}\right| |W|.
\end{aligned}$$

Moreover, recalling the definition of exponential matrices, i.e. $\mathrm{e}^{\theta M} = \sum_{k=0}^{\infty} \frac{(\theta M)^k}{k!}$, an upper bound of $|e^{\theta \mathcal{M}}|$ is obtained as follows

$$\left|\mathrm{e}^{\theta M}\right| \leq \sum_{k=0}^{\infty} \left|\frac{(\theta M)^k}{k!}\right| \leq \sum_{k=0}^{\infty} \frac{|\theta|^k |M|^k}{k!} \leq \sum_{k=0}^{\infty} \frac{h^k |M|^k}{k!} = \mathrm{e}^{h|M|},$$

which yields the result (4.43). □

Property 4.3. *For any $W \in \mathbb{S}_+^{n_x}$, the Lyapunov matrix U_W defined in* (4.21) *satisfies*

$$2\bar{U} := \lim_{\epsilon \to 0} \left|U_W'(\epsilon) - U_W'(-\epsilon)\right| = |W|. \tag{4.45}$$

The existence of complete Lyapunov functional with nice regularity properties enlightened above enables to apply Theorem 2.1, 2.2 and 2.3 which ensures convergence of the Legendre approximation. Assuming that system $(\mathcal{S}_1)$ is GES, we are then in position to formulate a new converse theorem that concatenates two underlying results:

- the convergence of the sufficient LMI conditions (4.13) as n tends to infinity,
- the estimation of the order n^* from which the LMI conditions (4.13) are true.

Theorem 4.5. *If system* $(\mathcal{S}_1)$ *is GES, then there exists* n^* *such that for all* $n \geq n^*$*, there exists* $(\mathbf{P}_n, S, R)$ *in* $\mathbf{S}^{n_x+nn_z} \times \mathbb{S}_+^{n_z} \times \mathbb{S}_+^{n_z}$ *verifying LMI conditions* (4.13)*. Moreover, this order can be calculated by the following formula*

$$n^* = \bar{\mathcal{N}}\left(\frac{1}{6(1+hr)|B|}\right) = 3 + \left\lceil \frac{9\pi\left(\pi(2\mu)^2\rho_0 + \frac{1}{2}\right)^2}{2}(1+hr)^2|B|^2 \right\rceil, \tag{4.46}$$

where parameters μ*,* ρ_0 *and* r *are given by*

$$\mu = \frac{|M|}{2}, \quad \rho_0 = 3\sqrt{n_x}\,\mathrm{e}^{h|M|}\left|N^{-1}\right|, \quad r = |A| + |B|. \tag{4.47}$$

Proof. The first part of the proof is dedicated to the estimation of an order n^* from which the assumption (4.34) is true (i.e. $\Psi_n(\theta) \succ 0$ is satisfied for any $\theta \in [0,1]$). The second part proves that the satisfaction of Lyapunov inequalities (4.25a) and (4.36) ensures the verification of the LMI (4.13). The bottom of Figure 4.10 explains this demonstration process.

Step 1: Verification of assumption (4.34) *from order* n^* *given by* (4.46)*.*
From Shur complement, the following lower bounds of Ψ_n is derived

$$\Psi_n(\theta) \succeq \begin{bmatrix} \beta_{1,n} I_{n_x} & 0 & 0 \\ 0 & \beta_{2,n} I_{n_z} & 0 \\ 0 & 0 & \beta_{3,n} I_{n_z} \end{bmatrix},$$

with

$$\begin{aligned} \beta_{1,n} &= \underline{\sigma}(W_1) - \left((1+h|A|+2)\left\|\tilde{U}_{1,n}\right\|_\infty + h|B|\left\|\tilde{U}_{2,n}\right\|_\infty + \left\|\tilde{U}'_{1,n}\right\|_\infty\right)|B|, \\ \beta_{2,n} &= \underline{\sigma}(W_2) - \left((h|A|+h|B|)\left\|\tilde{U}_{1,n}\right\|_\infty + 2h|B|\left\|\tilde{U}_{2,n}\right\|_\infty + \left\|\tilde{U}'_{1,n}\right\|_\infty\right)|B|, \\ \beta_{3,n} &= \underline{\sigma}(W_3) - \left((1+h\,|B|)\left\|\tilde{U}_{1,n}\right\|_\infty + h\,|B|\left\|\tilde{U}_{2,n}\right\|_\infty\right)|B|, \end{aligned} \tag{4.48}$$

Consider the worst error given by

$$\varepsilon_n = \max\left(\left\|\tilde{U}_{1,n}\right\|_\infty, \left\|\tilde{U}_{2,n}\right\|_\infty, \left\|\tilde{U}'_{1,n}\right\|_\infty\right). \tag{4.49}$$

In light of approximation theory developed in Chapter 2, error ε_n converge to zero when n tends to infinity and inequality (4.34) ended up being verified. Then, assumption (4.34) is satisfied for sufficiently large orders n. Let us now estimate such minimal orders n^*. and matrices $W_1 = \frac{\sigma}{3} I_{n_x}$, $W_2 = \frac{\sigma}{3} I_{n_z}$ and $W_3 = \frac{\sigma}{3} I_{n_z}$, for any $\sigma > 0$. Assumption (4.34) is a forciori true when the following inequality holds $1 - 6(1+hr)|B|\varepsilon_n \geq 0$. Thanks to the convergence properties of Legendre polynomials provided in Chapter 2 and since function U_W belongs to $\mathcal{C} = C_0(0,1;\mathbb{R}^{n_x \times n_x}) \cap C_2(0,\frac{1}{2}) \cup (\frac{1}{2},1);\mathbb{R}^{m\times p})$ and satisfies (4.43) (4.45), the worst error ε_n converges with an algebraic convergence rate. More precisely, according to Theorems 2.1, 2.2 and 2.3, the estimation of the order n^* is made possible. If $n \geq n^* = \bar{\mathcal{N}}(\frac{1}{6(1+hr)|B|})$, the assumption (4.34) (i.e. $\Psi_n(\theta) \succ 0$ for any $\theta \in [0,1]$) is true at least from order n^* given by (4.46).

Step 2: Equivalence between Lyapunov inequalities (4.25a)-(4.36) *and LMI* (4.13)*.*
Assume that system $(\mathcal{S}_1)$ is GES and consider matrices $\mathbf{P}_n$, S and R given by (4.42). Form one part, applying Lemma 4.2, Lyapunov inequality (4.25a) holds, for any order n in $\mathbb{N}$ and any (x,z) in $\mathcal{D}_1$. In particular, for any vector $\left[\begin{smallmatrix} x_0 \\ \zeta_n \\ z_0 \end{smallmatrix}\right] \in \mathbb{R}^{n_x+n_z(n+1)}$, consider (x,z) in $\mathcal{D}_1$ expressed as follows

$$x = x_0, \quad z(\theta) = \begin{cases} Cx_0 & \text{if } \theta = 1, \\ \ell_n^\top(\theta)\zeta_n & \text{if } \theta \in (0,1), \\ z_0 & \text{if } \theta = 0. \end{cases} \tag{4.50}$$

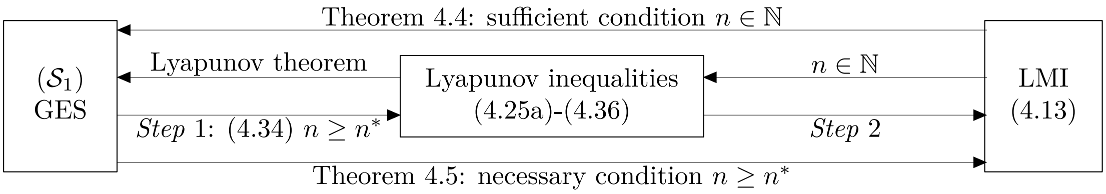


Figure 4.10: Necessary and sufficient LMI condition for stability.

Re-injecting this expression into the definition of $\mathcal{V}$ in (4.12) yields

$$\mathcal{V}(x,z) - \int_0^1 z^\top(\theta)\theta R z(\theta)\mathrm{d}\theta = \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix}^\top \Xi_n^+ \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix}, \tag{4.51}$$

In parallel, re-injecting expression (4.50) into inequality (4.25a) leads to the existence of a sufficiently small scalar $\alpha_1 > 0$ such that

$$\mathcal{V}(x,z) - \int_0^1 z^\top(\theta)\theta R z(\theta)\mathrm{d}\theta \geq \alpha_1 \left| \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix} \right|^2, \tag{4.52}$$

On the other part, according to *Step* 1 and Lemma 4.3, Lyapunov inequality (4.36) holds, for any $n \geq n^*$ given by (4.46) and any (x,z) in $\mathcal{D}_1$. In particular, for any vector $\begin{bmatrix} x_0 \\ \zeta_n \\ z_0 \end{bmatrix} \in \mathbb{R}^{n_x + n_z(n+1)}$, consider (x,z) in $\mathcal{D}_1$ given by expression (4.50). Re-injecting this expression into the definition of the time derivative of $\mathcal{V}_n$ in (4.17) yields

$$\dot{\mathcal{V}}_n(x,z) = \begin{bmatrix} x_0 \\ \zeta_n \\ z_0 \end{bmatrix}^\top \Xi_n^- \begin{bmatrix} x_0 \\ \zeta_n \\ z_0 \end{bmatrix}. \tag{4.53}$$

In parallel, re-injecting expression (4.50) into inequalities (4.36) leads to the existence of a sufficiently small scalar $\alpha_2 > 0$ such that

$$\dot{\mathcal{V}}_n(x,z) \leq -\alpha_2 \left| \begin{bmatrix} x_0 \\ \zeta_n \\ z_0 \end{bmatrix} \right|^2 \tag{4.54}$$

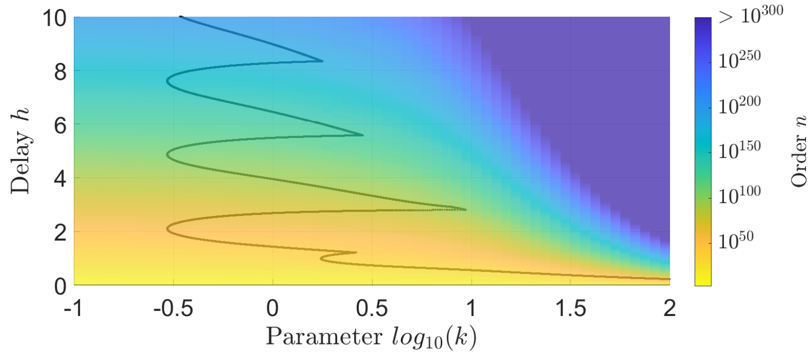
Hence, matrices Ξ_n^+ and Ξ_n^- are necessarily positive and negative definite, respectively. □

Thanks to Theorem 4.5 and as illustrated on Figure 4.10, the sufficient LMI conditions (4.13) become true for sufficiently large orders n. Moreover, in the case of stable systems, an estimation of the order n^* for which the LMI conditions (4.13) are true can be provided. A necessary and sufficient test of stability could be then proposed. First, order n^* given by (4.46) is calculated with respect to system parameters (delay h and matrices A, B, C). Then, LMI conditions (4.13) are solved at order n^*. If the test is true, then system $(\mathcal{S}_1)$ is GES. If the test if false, then system $(\mathcal{S}_1)$ is unstable.

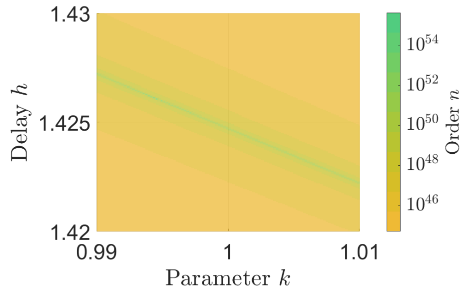
Figure 4.11 shows the values of n^* given by (4.46) for Example 1.2 and various values of $(k,h) \in [0.1, 100] \times [0, 10]$. Clearly, the values of n^* are too large to propose a tractable test of instability due to the computational load of LMI conditions (4.13) which is polynomial with respect to order n. Nonetheless, Theorem 4.5 provides a theoretical overestimation of n^*, which is already very satisfactory. Interestingly, in Figure 4.11a, we notice that increasing both k and h makes orders n^* bigger. Indeed, parameters μ, ρ_0 and r seem to grow as the delay h or the norm $|B|$ increase. On Figure 4.11b, it is also worth noticing that when k and h are getting closer to the black lines (i.e. near $(k,h) = (1, 1.425)$), estimated order n^* becomes large. These black lines correspond to the situation where some characteristic roots of system $(\mathcal{S}_1)$ crosses the imaginary axis. Indeed, it implies that matrix N tends to a singular matrix as proved in [131, Theorem 2.10] and then $\left|N^{-1}\right|$ as well as parameter ρ_0 tend to infinity. Altogether, this ascertainment can be correlated with Figure 4.8 and can explain why some stable regions are difficult to reach with LMI conditions (4.13).

In this section, a sufficient LMI condition of stability which becomes necessary for sufficiently large orders has been provided. In the following section, we reverse the stability question.

- *How to assess instability of system $(\mathcal{S}_1)$ using tractable numerical tools?*
- *Can we propose finite-dimensional sufficient conditions of instability, which turn out to be nec-*



(a) Example 2 for $(k, h) \in [0.1, 100] \times [0, 10]$.



(b) Example 2 for $(k, h) \simeq (1, 1.425)$.

Figure 4.11: Estimation of the necessary order n^* for the satisfaction of LMI conditions.

essary as the order increases?

- *Is it possible to estimate the necessary order?*

4.4 Positivity test

The instability analysis of ODE-transport systems can be performed using a converse Lyapunov approach. As explained in Appendix A, this approach deals with the converse Lyapunov functional constructed under the negativity constraint of its time derivative along the trajectories of system $(\mathcal{S}_1)$. The non-satisfaction of the positivity of such a functional provides an instability criterion.

4.4.1 Sufficient condition of instability

This problem has been tackled in [72, 155] where tractable necessary conditions have been presented as a by-product of the converse Lyapunov-Krasovskii theorem. In [71, 72], the guideline is to obtain a condition based on point-wise evaluations of the Lyapunov matrix $U = U_{I_{n_x}}$ given by (4.21). If a certain matrix which depend on U is not positive definite, then system $(\mathcal{S}_1)$ is unstable. This is made possible thanks to an approximation schema where the discretization points are evenly spaced and where exponential kernels are chosen on each subinterval. Such methods have been also extended to time-delay systems with distributed [69], neutral [94], multiple [4, 97], integral [45, 174] or periodic [100] delays. In [155, 156], similar results are obtained leaning on the piece-wise linear schema. We wonder if we can extend the methodology to another support basis and if we can reduce the computational load. Taking a closer look to the approximated functional introduced in previously, sufficient conditions of instability based on Legendre approximation can also be given.

Consider the approximated Lyapunov functional V_n of the form (4.12) and written as

$$V_n(x, z) = \begin{bmatrix} x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta) z(\theta) \mathrm{d}\theta \end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} x \\ \mathcal{I}_n \int_0^1 \ell_n(\theta) z(\theta) \mathrm{d}\theta \end{bmatrix}. \tag{4.55}$$

Matrices (R, S) have been fixed to 0 and matrix $\mathbf{P}_n$ is fixed to

$$\mathbf{P}_n = \begin{bmatrix} P & \mathcal{Q}_n \\ * & \mathcal{T}_n \end{bmatrix}, \tag{4.56}$$

where matrices $(P, \mathcal{Q}_n, \mathcal{T}_n)$ are given by

$$\begin{aligned} P &= U(0), \\ \mathcal{Q}_n &= h \int_0^1 U(\theta) B \ell_n^\top(\theta) \mathrm{d}\theta, \\ \mathcal{T}_n &= h^2 \int_0^1 \int_0^1 \ell_n(\theta_1) B^\top U(\theta_2 - \theta_1) B \ell_n^\top(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2, \end{aligned} \tag{4.57}$$

where U is the Lyapunov matrix $U_{I_{n_x}}$ expressed in (4.21) and recalled below

$$U(\theta) = \begin{cases} \mathrm{vec}^{-1}\left(\left[\begin{smallmatrix} I_{n_x^2} & 0 \end{smallmatrix}\right] e^{\theta M} N^{-1} \left[\begin{smallmatrix} -\mathrm{vec}(I_{n_x}) \\ 0 \end{smallmatrix}\right]\right) & \text{if} \quad \theta \geq 0, \\ U^\top(-\theta) & \text{if} \quad \theta < 0, \end{cases}$$

and where matrices M, N are given in (4.22) and recalled below

$$\begin{aligned} M &= h \left[\begin{smallmatrix} I_{n_x} \otimes A^\top & I_{n_x} \otimes A_d^\top \\ -A_d^\top \otimes I_{n_x} & -A^\top \otimes I_{n_x} \end{smallmatrix}\right], \\ N &= \left[\begin{smallmatrix} A^\top \otimes I_{n_x} + I_{n_x} \otimes A^\top & I_{n_x} \otimes A_d^\top \\ I_{n_x^2} & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} A_d^\top \otimes I_{n_x} & 0 \\ 0 & -I_{n_x^2} \end{smallmatrix}\right] e^M. \end{aligned}$$

Here, the approximated Lyapunov V_n given by (4.55) is related to an approximation of the distributed state z. Indeed, it has been constructed as a copy of the complete Lyapunov functional given by (4.24) where $W_1 = I_{n_x}$, $W_2 = 0$ and $W_3 = 0$ and where the state z is a polynomial of degree $n - 1$.

Remark 4.6. Note that we recognize the Legendre approximation of the converse Lyapunov functional $V_{I_{n_x}}$ given by (4.20) introduced in the previous section. Going back to Definition 4.2, thanks to the orthogonality satisfied by Legendre polynomials, matrices $\mathcal{Q}_n$ and $\mathcal{T}_n$ given by (4.57) corresponds to (4.33). The approximated Lyapunov functional $\mathcal{V}_n$ given by (4.55) corresponds then to (4.32) where $W_1 = I_{n_x}$, $W_2 = 0$ and $W_3 = 0$ have been fixed and where a slight modification has been brought on the term in P. With the Legendre approximation, approximating the state or the Lyapunov matrix amounts to the same.

Remark 4.7. It is also worth noticing that in [15], the work is rather done on the functional $V_n(x, z) = V_n(x, z) + \int_0^1 z^\top(\theta)(C^\top C)z(\theta)\mathrm{d}\theta$ which amounts to take $W_1 = 0$, $W_2 = I_{n_z}$ and $W_3 = 0$. It is the approximation of the functional $V^\flat(x, z) = V(x, z) + \int_0^1 z^\top(\theta)(C^\top C)z(\theta)\mathrm{d}\theta$ satisfying $\dot{V}^\flat(x, z) = -|x(t - h)|^2$, and which has been used in [69, 98].

Remark 4.8. In a natural manner, notice finally that the proposed methodology can even be enlarged to other approximation techniques [72, 155], based on interval slicing to approximate the state z on $[0, 1]$.

Theorem 4.6. *For any order n in $\mathbb{N}$, if matrix $\mathbf{P}_n$ given by (4.56) is not positive definite, then system $(\mathcal{S}_1)$ is not GES. In addition, this condition is hierarchic with respect to order n, this means that if $\mathbf{P}_{n_0}$ is not positive definite then $\mathbf{P}_n$ is not positive definite for all $n \geq n_0$.*

Proof. The proof of the proposition and hierarchy are separated below.

Step 1: Negativity of the converse Lyapunov functional.
Assume that matrix $\mathbf{P}_n$ is not positive definite. Then, there exist a positive scalar $\alpha > 0$ and a non null vector $\left[\begin{smallmatrix} x_0 \\ \zeta_n \end{smallmatrix}\right] \in \mathbb{R}^{n_x + n_z(n+1)} \backslash \{0\}$ such that

$$\begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix} \leq -\alpha \left| \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix} \right|^2. \tag{4.58}$$

Considering (x, z) in $\mathcal{D}_1$ expressed as follows

$$x = x_0, \quad z(\theta) = \begin{cases} Cx_0 & \text{if } \theta = 1, \\ \ell_n^\top(\theta)\zeta_n & \text{if } \theta \in (0,1), \end{cases} \tag{4.59}$$

we obtain

$$V_n(x,z) = \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix} = V_{I_{n_x}}(x,z), \text{ and } \left|\begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix}\right|^2 \geq \left|(x,z)\right|^2. \tag{4.60}$$

Therefore, there exist a positive scalar $\alpha > 0$ and a non null state $(x,z) \in \mathcal{D}_1 \backslash \{(0,0)\}$ given by (4.59) such that

$$V_{I_{n_x}}(x,z) \leq -\alpha \left|(x,z)\right|^2. \tag{4.61}$$

Step 2: Conclusion by contradiction.
Assuming that system $(\mathcal{S}_1)$ is GES and applying the converse Lyapunov theorem provided by Lemma A.4 in Appendix A.1, we will reach a contradiction and conclude that system $(\mathcal{S}_1)$ cannot be GES. Indeed, for any $\alpha_1 > 0$, introduce the functional W as

$$W(x,z) = V_{I_{n_x}}(x,z) + \alpha_1 \left|(x,z)\right|^2.$$

According to (4.23), differentiating W along the trajectories of system $(\mathcal{S}_1)$, we obtain

$$\dot{W}(x,z) = \begin{bmatrix} x \\ z(0) \end{bmatrix}^\top \left(\begin{bmatrix} -I_{n_x} & 0 \\ 0 & -\alpha_1 I_{n_z} \end{bmatrix} + \alpha_1 \begin{bmatrix} \mathcal{H}(A) + C^\top C & B \\ * & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ z(0) \end{bmatrix}. \tag{4.62}$$

Then, there exists a sufficiently small positive number $\alpha_1 < \alpha$, such that $\dot{W}(x,z) \leq 0$ holds. Pursuing an integration in time from 0 to ∞, we obtain

$$\lim_{t\to\infty} W\big(x(t), z(t)\big) - W\big(x(0), z(0)\big) \leq 0, \quad \forall \big(x(0), z(0)\big) \in \mathcal{D}_1.$$

From the assumption that system $(\mathcal{S}_1)$ is GES, we have $\big(x(t), z(t)\big) \underset{t\to\infty}{\longrightarrow} (0,0)$. Then, the inequality $W\big(x(0), z(0)\big) \geq 0$ holds for all $\big(x(0), z(0)\big) \in \mathcal{D}_1$ and leads to

$$V_{I_{n_x}}(x,z) \geq -\alpha_1 \left|(x,z)\right|^2 > -\alpha \left|(x,z)\right|^2, \quad \forall (x,z) \in \mathcal{D}_1. \tag{4.63}$$

We end up with a contradiction: assuming that system $(\mathcal{S}_1)$ is GES, the inequality (4.61) cannot be verified. Thus, matrix $\mathbf{P}_n$ not positive definite implies that system $(\mathcal{S}_1)$ is not GES.

Step 3: Hierarchy.
The hierarchy can then be proven because matrix $\mathbf{P}_n$ at order $n \geq n_0$ can be written as

$$\mathbf{P}_n = \begin{bmatrix} \mathbf{P}_{n_0} & \bar{\mathcal{Q}}_{n_0:n} \\ * & \bar{\mathcal{T}}_{n_0:n} \end{bmatrix}, \tag{4.64}$$

where matrices

$$\begin{aligned} \bar{\mathcal{Q}}_{n_0:n} &= h \int_0^1 U(\theta) B \bar{\ell}_{n_0:n}(\theta) \mathrm{d}\theta, \\ \bar{\mathcal{T}}_{n_0:n} &= h^2 \int_0^1 \int_0^1 \bar{\ell}_{n_0:n}(\theta_1) B^\top U(\theta_2 - \theta_1) B \bar{\ell}^\top_{n_0:n}(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2, \\ \bar{\ell}_{n_0:n} &= \begin{bmatrix} l_{n_0} I_{n_z} & \dots & l_{n-1} I_{n_z} \end{bmatrix}^\top. \end{aligned}$$

If $\mathbf{P}_{n_0}$ is not positive definite, then $\mathbf{P}_n$ cannot be positive definite for all $n \geq n_0$. □

Numerical calculus. To perform the numerical test presented above, each coefficient of matrix $\mathbf{P}_n$ given by (4.56) needs to be evaluated numerically. This amounts to calculate the integral terms $\mathcal{Q}_n$ and $\mathcal{T}_n$ given by (4.57). Such computation can be done analytically by computer algebra systems but may turn out to be a tough task, especially for large orders n. Some alternatives have been discussed in

Appendix B.3. While it would deserve some details and explanations [15], we obtain at the end

$$
\begin{aligned}
\mathcal{Q}_n &= \begin{bmatrix} Q_0 & \cdots & Q_{n-1} \end{bmatrix}, \\
\mathcal{T}_n &= \begin{bmatrix} T_{00} + T_{00}^\top & T_{01} - T_{01}^\top & \cdots & T_{0n-1} + (-1)^{n-1} T_{0n-1}^\top \\ * & T_{11} + T_{11}^\top & \ddots & \vdots \\ * & * & \ddots & \vdots \\ * & * & * & T_{nn} + T_{nn}^\top \end{bmatrix},
\end{aligned}
\tag{4.65}
$$

where matrices Q_k and T_{jk} are given by

$$
\begin{aligned}
Q_k &= h\mathrm{vec}^{-1}\Big(\big[\begin{smallmatrix} I_{n_x^2} & 0 \end{smallmatrix}\big]\Gamma_k N^{-1}\big[\begin{smallmatrix} -\mathrm{vec}(I_{n_x}) \\ 0 \end{smallmatrix}\big]\Big) B, \quad \forall k \in \mathbb{N}, \\
T_{jk} &= h^2 B^\top \mathrm{vec}^{-1}\left(\big[\begin{smallmatrix} I_{n_x^2} & 0 \end{smallmatrix}\big]\bar{\Gamma}_{jk} N^{-1}\big[\begin{smallmatrix} -\mathrm{vec}(I_{n_x}) \\ 0 \end{smallmatrix}\big]\right)^\top B, \quad \forall (j,k) \in \mathbb{N}^2,
\end{aligned}
\tag{4.66}
$$

and where Γ_k and $\bar{\Gamma}_{jk}$ are given by

$$
\Gamma_k = \int_0^1 l_k(\theta)\, \mathrm{e}^{M\theta}\, \mathrm{d}\theta, \quad k \in \mathbb{N}, \tag{4.67}
$$

$$
\bar{\Gamma}_{jk} = \iint_{\mathbf{T}} l_j(\theta_1) l_k(\theta_2)\, \mathrm{e}^{(\theta_1-\theta_2)M}\, \mathrm{d}\theta_1 \mathrm{d}\theta_2, \tag{4.68}
$$

on the triangle $\mathbf{T} = \{(\theta_1, \theta_2) \in [0,1]^2 \text{ s.t. } \theta_1 \geq \theta_2\}$ and can be given recursively by (B.6) and (B.9) with respect to M and N. Such an iterative method may avoid numerical burden.

Computational load. This non-positivity test resumes to an eigenvalues test $\underline{\sigma}(\mathbf{P}_n) \leq 0$. It needs only $\underset{n\to\infty}{O}(n^5)$ operations to be computed.

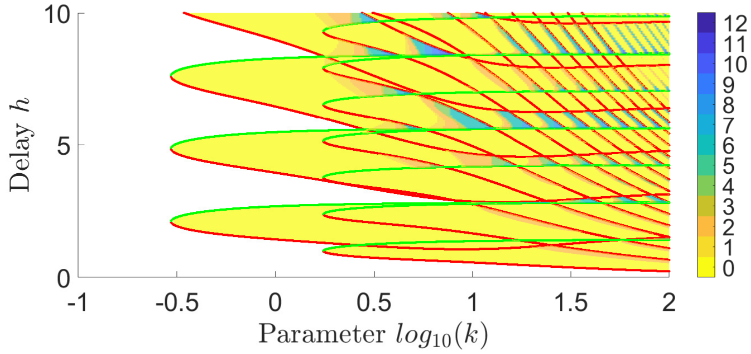
The sufficient condition of instability provided in Theorem 4.6 provides an outer approximation of the stability regions. For any given order $n \in \mathbb{N}$, if $\mathbf{P}_n$ given by (4.56) is not positive definite, then the origin of system $(\mathcal{S}_1)$ is not GES. Compared to eigenvalues, frequency, or LMI conditions given by Theorem 4.2, Theorem 4.3, and Theorem 4.4, the positivity condition ensures instability.

On Fig 4.12a, Theorem 4.6 is applied to numerical Example 1.2 for $(k,h) \in [0.1, 100] \times [0, 10]$ and for several orders $n \in \{1, \ldots, 12\}$. From order $n = 1$, considering only the mean value of the state z, one can see that the positivity condition already gives an accurate estimation of the instability region. Increasing the order n, the whole instability region is spanned. More interestingly, the hard-to-reach areas are located when the eigenvalues cross the imaginary axis from the right-half-plane to the left-half-plane (see green lines) or from the left-half-plane to the right-half-plane (see red lines).

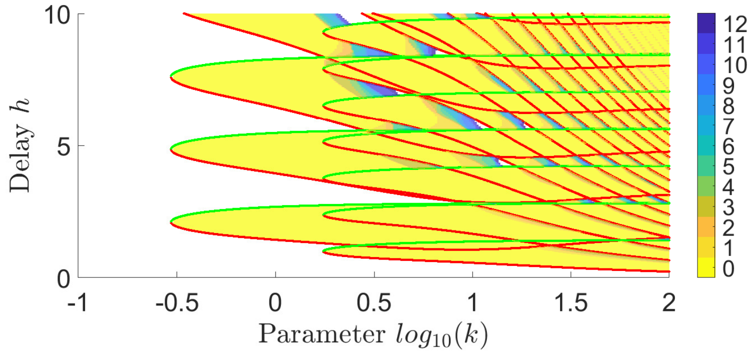
Our positivity test given by Theorem 4.6 is compared by the one provided in [72, Theorem 9]. From one side, in [72, Theorem 9], the matrix under study is denoted $\mathbf{K}_n$ and is given by

$$
\mathbf{K}_n = \begin{bmatrix} U(0) & U(\frac{1}{n}) & \cdots & \cdots & U(1) \\ * & U(0) & U(\frac{1}{n}) & \cdots & U(\frac{n-1}{n}) \\ * & * & \ddots & \ddots & \vdots \\ * & * & * & U(0) & U(\frac{1}{n}) \\ * & * & * & * & U(0) \end{bmatrix}. \tag{4.69}
$$

It is issued from a uniform discretization and a specific choice of connecting functions based on exponential kernels of the system itself. From the other side, the test is done on matrix $\mathbf{P}_n = \left[\begin{smallmatrix} P & \mathcal{Q}_n \\ * & \mathcal{T}_n \end{smallmatrix}\right]$ where $\mathcal{Q}_n$ and $\mathcal{T}_n$ given by (4.57) are calculated by induction. First of all, the tests give very similar hierarchical results. It is worth noticing that $\mathbf{P}_n$ gives slightly better estimations for low orders n. In return, Legendre approximation add numerical complexity by the recursive calculation of matrices $\mathcal{Q}_n$ and $\mathcal{T}_n$. The recursive relations (B.6) and (B.9) given in Appendix B.3 cost $\underset{n\to\infty}{O}(n^2)$ and are sensitive to numerical errors. A bad initial condition propagates and creates a growing error as the order n increases. Lastly, in both cases, the outer estimation of the stability regions seems to converge towards the expected ones and this is proven in the sequel.



(a) Application of Theorem 4.6.



(b) Comparison with [72, Theorem 9].

Figure 4.12: Outer approximation of the stability regions for Example 1.2 in (k, h) plane.

4.4.2 Convergence of the positivity condition of instability

Once again, we are able to prove the convergence of the contrapositive of Theorem 4.6. This time, Theorem 1.1 put forward in Chapter 1 is fundamental. By exploiting the converse Lyapunov theorem, it consists in exhibiting a state such that the converse Lyapunov functional V associated to $W = I_{n_x}$ is negative [68, 155]. In [15], we have proven that such an exhibited state belongs to the smooth set of functions $\mathcal{S} \subset \mathcal{D}_1$. The corresponding property is recalled below [15, 95].

Property 4.4. *If system* $(\mathcal{S}_1)$ *is not GES, then there exists*

$$(x_0, z_0) \in \mathcal{S} := \left\{(x, z) \in \mathbb{R}^{n_x} \times C_\infty(0, 1; \mathbb{R}^{n_z}) \ s.t. \ z(1) = Cx, \ |x| = 1, \ \left\|z^{(d)}\right\|_\infty \leq (hr)^d, \ \forall d \in \mathbb{N}\right\}, \tag{4.70}$$

such that the following inequality

$$V_{I_{n_x}}(x_0, z_0) \leq -\frac{1}{2r} \qquad \text{with } r = |A| + |B|, \tag{4.71}$$

holds.

Proof. The proof corresponds to the one given in Theorem 1.1 in Chapter 1 for the case ODE-transport system $(\mathcal{S}_1)$. In the sequel, assume $s^* = \mathcal{R}(s^*) + \imath\mathcal{I}(s^*)$, with $\mathcal{R}(s^*) > 0$, an unstable characteristic root of system $(\mathcal{S}_1)$, and proceed step y step.

Step 1: Verification of inequality $|s^*| \leq r := |A| + |B|$.
Let us denote $r = |A| + |B|$. By contraposition, if $|s^*| > r$, there exists $u \in \mathbb{C}^{n_x} \backslash \{0\}$ such that $(s^* I_{n_x} - A - BH(s^*)C)u = 0$, since s^* is solution of $\det(s^* I_{n_x} - A - BH(s^*)C) = 0$. Passing

through the norm, the following inequality holds

$$|s^*| = |s^* I_{n_x}| = |A + BH(s^*)C| \leq |A| + |B| \left| \mathrm{e}^{-hs^*} \right| |C| \leq |A| + |B| \, |C| = |A| + |B| \, . \tag{4.72}$$

Thus, we have $|s^*| \leq r := |A| + |B|$ and *Step* 1 is completed.

<u>Step 2:</u> *Existence of a characteristic vector* (x_0, z_0) *associated to* s^* *in* $\mathcal{S}$.
According to Lemma B.1 given in Appendix B, there exists (u_1, u_2) in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ such that $|u_1| = 1$, $|u_2| \leq 1$, $u_1^\top u_2 = 0$ and

$$(s^* I_{n_x} - A - BH(s^*)C)(u_1 + \imath u_2) = 0. \tag{4.73}$$

Therefore, the following vector

$$x^*(t; s^*) = \mathrm{e}^{s^* t}(u_1 + \imath u_2), \quad \forall t \in \mathbb{R}_+,$$

is a characteristic vector associated to s^*. Then, there exists a non trivial trajectory with values in $\mathbb{R}^{n_x} \times C_\infty(0, 1; \mathbb{R}^{n_z})$ given by

$$\begin{aligned} \begin{bmatrix} \hat{x}(t; s^*) \\ \hat{z}(t, \theta; s^*) \end{bmatrix} &= \frac{1}{2} \left(\begin{bmatrix} x^*(t; s^*) \\ Cx^*(t + h(\theta - 1); s^*) \end{bmatrix} + \begin{bmatrix} \overline{x^*}(t; s^*) \\ C\overline{x^*}(t + h(\theta - 1)s^*) \end{bmatrix} \right), \ \forall (t, \theta) \in \mathbb{R}_+ \times [0, 1], \\ &= \begin{bmatrix} \mathrm{e}^{\mathcal{R}(s^*)t} \left(\cos\left(\mathcal{I}(s^*)t\right) u_1 - \imath \sin\left(\mathcal{I}(s^*)t\right) u_2 \right) \\ C \, \mathrm{e}^{\mathcal{R}(s^*)(t + h(\theta - 1))} \left(\cos\left(\mathcal{I}(s^*)(t + h(\theta - 1))\right) u_1 - \imath \sin\left(\mathcal{I}(s^*)(t + h(\theta - 1))\right) u_2 \right) \end{bmatrix}. \end{aligned} \tag{4.74}$$

Considering $t = 0$, the characteristic vector (4.74) is equal to

$$\begin{bmatrix} x_0 \\ z_0(\theta) \end{bmatrix} = \begin{bmatrix} \hat{x}(0; s^*) \\ \hat{z}(0, \theta; s^*) \end{bmatrix} = \begin{bmatrix} u_1 \\ C \, \mathrm{e}^{\mathcal{R}(s^*)h(\theta - 1)} \cos\left(\mathcal{I}(s^*)h(\theta - 1)\right) u_1 \end{bmatrix}. \tag{4.75}$$

It is worth noticing that $|x_0| = 1$. Then, we prove that $\left\| z^{(d)} \right\|_\infty \leq (hr)^d$. Indeed, the d^{th} derivatives of z_0 is given by

$$z_0^{(d)}(\theta) = \mathcal{R}\left(C (hs^*)^d \, \mathrm{e}^{hs^*(\theta - 1)} \, u_1 \right), \quad \forall d \in \mathbb{N}. \tag{4.76}$$

Then, we directly have

$$\left\| z_0^{(d)} \right\|_\infty \leq |C| \left\| \mathcal{R}\left((hs^*)^d \, \mathrm{e}^{hs^*(\theta - 1)} \right) \right\|_\infty |u_1| \leq |C| \, |hs^*|^d \sup_{\theta \in [0, 1]} \left| \mathrm{e}^{s^* h(\theta - 1)} \right| |u_1| = |hs^*|^d, \quad \forall d \in \mathbb{N}.$$

According to *Step* 1, we prove that $\left\| z_0^{(d)} \right\|_\infty \leq (hr)^d$ and that (x_0, z_0) belongs to the set $\mathcal{S}$.

<u>Step 3:</u> *Negative upper bound of the functional* V *given by* (4.71).
Consider now $V = V_{I_{n_x}}$ given by (4.20). Since $\dot{V}(x, z) = -|x|^2$ along the trajectories of system $(\mathcal{S}_1)$, an integration from $t = 0$ to $t = T > 0$ leads to

$$V\left(x(T), z(T)\right) - V\left(x(0), z(0)\right) = -\int_0^T |x(t)|^2 \, \mathrm{d}t.$$

Along the particular trajectory $(\hat{x}(t; s^*), \hat{z}(t, \theta; s^*))$ of system $(\mathcal{S}_1)$ given by (4.74), we obtain

$$\begin{aligned} \left(\mathrm{e}^{2\mathcal{R}(s^*)T} - 1 \right) V\left(\hat{x}(0; s^*), \hat{z}(0; s^*)\right) &= -\int_0^T \mathrm{e}^{2\mathcal{R}(s^*)t} \left| \cos\big(\mathcal{I}(s^*)t\big) u_1 - \sin\big(\mathcal{I}(s^*)t\big) u_2 \right|^2 \mathrm{d}t, \\ &= -\int_0^T \mathrm{e}^{2\mathcal{R}(s^*)t} \left(\cos^2\big(\mathcal{I}(s^*)t\big) + \sin\big(\mathcal{I}(s^*)t\big) |u_2|^2 \right) \mathrm{d}t, \\ &\leq -\int_0^T \mathrm{e}^{2\mathcal{R}(s^*)t} \, \mathrm{d}t = -\frac{\mathrm{e}^{2\mathcal{R}(s^*)T} - 1}{2\mathcal{R}(s^*)}. \end{aligned}$$

Finally, *Step* 2 ensures that $\mathcal{R}(s^*) \leq |s^*| \leq r$. In sum, the following inequality holds

$$V(x_0, z_0) = V\left(\hat{x}(0; s^*), \hat{z}(0; s^*)\right)) \leq -\frac{1}{2\mathcal{R}(s^*)} \leq -\frac{1}{2r}, \tag{4.77}$$

and concludes the proof. □

Using the negative bound $-\frac{1}{2r}$, we can then expect the approximated converse functional V_n given by (4.55) will also be negative for sufficiently large orders n. To do this, the error made by approximation of the Lyapunov functional has to converge. In reality, whether it is for pseudo-spectral methods based on specific kernels [96], on interpolation [156] or on projection [15], we can defer this convergence to the approximation of the state. As long as the uniform convergence of the approximated state z_n towards z is satisfied on the interval $[0,1]$ for any z in $C_\infty(0,1;\mathbb{R}^{n_z})$, an asymptotic converse theorem is obtained.

In addition, the exhibition of a state z with nice regularity properties enables to apply Theorem 2.2, 2.3 provided in Chapter 2 which ensures exponential convergence of the Legendre approximation. In the following, we carry out the proof of the asymptotic convergence by preserving the supergeometric convergence rate of our approximation.

Assuming that system $(\mathcal{S}_1)$ is not GES, we are then in position to formulate a new converse theorem that concatenates two underlying results:

- the non positive definiteness of matrix $\mathbf{P}_n$ as n tends to infinity,
- the estimation of the order n^* from which the positivity condition $\underline{\sigma}(\mathbf{P}_n) \geq 0$ is false.

Theorem 4.7. *If system $(\mathcal{S}_1)$ is not GES, then there exists an order n^* such that for all $n \geq n^*$ matrix $\mathbf{P}_n$ is not positive definite. Moreover, this order can be calculated by the following formula*

$$n^* = \mathcal{N}_0\left(\frac{\mathcal{E}(\beta_1, \beta_2)}{h\,|B|}\right). \tag{4.78}$$

where scalars β_1, β_2 are given by

$$\beta_1 = 1 + h\,|B|\,, \quad \beta_2 = \frac{1}{2r\,\|U\|_\infty}, \quad r = |A| + |B|\,, \tag{4.79}$$

and where functions $\mathcal{N}_0$ and $\mathcal{E}$ are defined by

$$\begin{aligned} \mathcal{N}_0(\varepsilon) &= 2 + \left\lceil \frac{hr}{2}\,\mathrm{e}^{1+\mathcal{W}\left(\left(\frac{hr\,\mathrm{e}}{2}\right)^{-1}\log\left(\frac{(hr)^2}{2\,\mathrm{e}^{\lfloor \frac{hr}{2} \rfloor}\varepsilon}\right)\right)} \right\rceil, \\ \mathcal{E}(\beta_1, \beta_2) &= -\beta_1 + \sqrt{\beta_1^2 + \beta_2}. \end{aligned} \tag{4.80}$$

Proof. The link between the approximated functional V_n and the converse Lyapunov functional $V_{I_{n_x}}$ is forged. Then, convergence arguments are used to conclude.

<u>Step 1</u>*: Error between the approximated Lyapunov functional V_n and the original one V.*
The connection between Lyapunov functionals V_n (4.55) and V (4.20) comes from Legendre approximation of the state z. Similarly to (4.37), we have

$$\begin{aligned} V_n(x,z) &= V(x,z) - 2hx^\top\int_0^1 U(\theta)B\tilde{z}_n(\theta)\mathrm{d}\theta - h^2\int_0^1\int_0^1 \tilde{z}_n^\top(\theta_1)B^\top U(\theta_2-\theta_1)B\tilde{z}_n(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2, \\ &- h^2\int_0^1\int_0^1 z^\top(\theta_1)B^\top U(\theta_2-\theta_1)B\tilde{z}_n(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2 - h^2\int_0^1\int_0^1 \tilde{z}_n^\top(\theta_1)B^\top U(\theta_2-\theta_1)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2, \end{aligned} \tag{4.81}$$

where $\tilde{z}_n$ is the Legendre truncated error of z at order n defined by

$$\tilde{z}_n(\theta) = z(\theta) - \ell_n^\top(\theta)\mathcal{Z}_n, \quad \forall(t,\theta) \in \mathbb{R}_+ \times [0,1], \tag{4.82}$$

in Definition 3.2. Roughly bounding the error terms, we obtain

$$V_n(x,z) \leq V(x,z) + 2h\,|B|\,(|x| + h\,|B|\,|z|)\,\|U\|_\infty\,\|\tilde{z}_n\|_\infty + h^2\,|B|^2\,\|U\|_\infty\,\|\tilde{z}_n\|_\infty^2. \tag{4.83}$$

In the next step, assuming that the system is unstable, we apply such an inequality for particular (x_0, z_0) provided by Property 4.4.

Step 2: Convergence and estimation of the order n^*.
According to Property 4.4, consider the non trivial solution (x_0, z_0) in $\mathcal{S}$ such that $|x_0| = 1$, $\|z_0\|_\infty = 1$ and that inequality $V(x_0, z_0) \leq -\frac{1}{2r}$ holds. Inequality (4.81) boils down to

$$V_n(x_0,z_0) \leq -\frac{1}{2r} + 2h\,|B|\,(1 + h\,|B|)\,\|U\|_\infty\,\|\tilde{z}_{0,n}\|_\infty + h^2\,|B|^2\,\|U\|_\infty\,\|\tilde{z}_{0,n}\|_\infty^2. \tag{4.84}$$

Since we have selected (x_0, z_0) in $\mathcal{S}$ such that $\left\|z_0^{(d)}\right\|_\infty \leq (hr)^d$ and thanks to the convergence properties of Legendre approximation provided in Theorem 2.2 in Chapter 2, its truncated error $\|\tilde{z}_{0,n}\|_\infty$ can be made sufficiently small, in a way that $V_n(x_0, z_0) \leq 0$, for sufficiently large orders n. Indeed, according to Theorem 2.2, there exists an order n^* such that the following inequality holds

$$h\,|B|\,\|\tilde{z}_{0,n}\|_\infty \leq -(1 + h\,|B|) + \sqrt{(1 + h\,|B|)^2 + \frac{1}{2r\,\|U\|_\infty}}, \quad \forall n \geq n^*. \tag{4.85}$$

Then, the negativity of V_n is satisfied from a certain order n^*,

$$V_n(x_0,z_0) \leq -\frac{1}{2r} + 2h\,|B|\,(1+h\,|B|)\,\|U\|_\infty\,\|\tilde{z}_{0,n}\|_\infty + h^2\,|B|^2\,\|U\|_\infty\,\|\tilde{z}_{0,n}\|_\infty^2 \leq 0, \quad \forall n \geq n^*. \tag{4.86}$$

We have found a non null vector $\xi_n = \begin{bmatrix} x_0 \\ \mathcal{I}_n \int_0^1 \ell_n(\theta) z_0(\theta) \mathrm{d}\theta \end{bmatrix}$ in $\mathbb{R}^{n_x + nn_z}$ such that $V_n(x_0, z_0) = \xi_n^\top \mathbf{P}_n \xi_n \leq 0$ and, consequently, $\mathbf{P}_n$ is not positive definite, for orders $n \geq n^*$. More precisely, an estimation of the order n^* from which the required precision is reached is made possible. According to Theorem 2.2, an estimation of the order n^* from which condition (4.85) is true is given by

$$n^* = \mathcal{N}_0\left(\frac{1}{h\,|B|}\left(-(1 + h\,|B|) + \sqrt{(1 + h\,|B|)^2 + \frac{1}{2r\,\|U\|_\infty}}\right)\right), \tag{4.87}$$

and concludes the proof. □

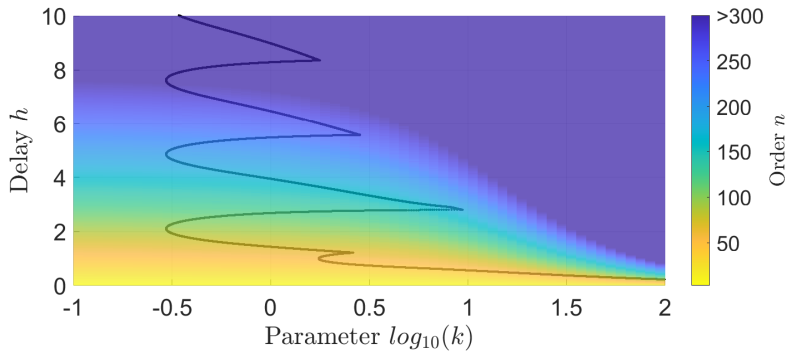
Remark 4.9. Note that two options can be considered to calculate n^* given by (4.78). From Property 4.43, an over estimation of n^* can be proposed using the upper bound $\rho_0 = \sqrt{n_x}\,\mathrm{e}^{h|M|}\left|N^{-1}\right||W|$ for $\|U\|_\infty$. A better estimation can also be provided by searching directly $\|U\|_\infty$ via Nelder-Mead algorithm [169]. The disadvantage is that it can be time consuming.

Numerical calculus. The function $\mathcal{E}$ undergoes numerical inaccuracies for small values of β_2 compared to β_1 (i.e. large values of r or $\|U\|_\infty$). Then, the error function is rather computed by

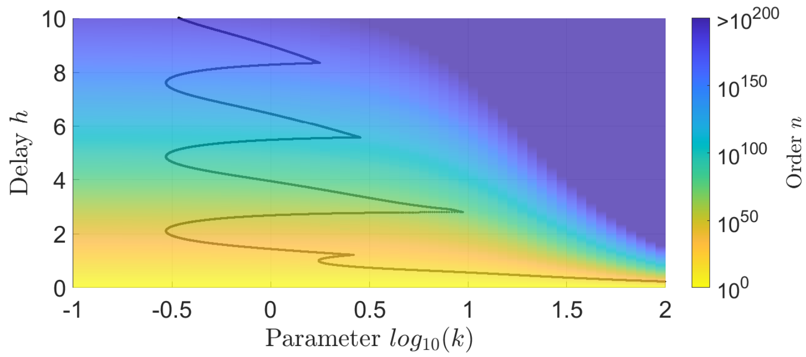
$$\mathcal{E}(\beta_1, \beta_2) = -\beta_1 + \sqrt{\beta_1^2 + \beta_2} = \frac{\beta_2}{\beta_1 + \sqrt{\beta_1^2 + \beta_2}}, \tag{4.88}$$

to improve the numerical accuracy, especially when $\beta_1 \gg \beta_2$.

Theorem 4.7 ensures the convergence of the necessity side of Theorem 4.6 and gives an estimation of the order from which the converse proposition is true. By combining Theorems 4.6 and 4.7, a new necessary and sufficient stability condition for ODE-transport systems can be formulated. For a given delay h and matrices A, B, C, the order n^* given by (4.78) is calculated. Then, matrices $\mathcal{Q}_{n^*}$ and $\mathcal{T}_{n^*}$ which are composed of matrices Γ_k and $\bar{\Gamma}_{jk}$ are calculated recursively via Propositions B.1 and B.2. The positivity condition $\underline{\sigma}(\mathbf{P}_n) > 0$ can finally be checked. If the test is true, then system $(\mathcal{S}_1)$ is stable. If the test is false, then system $(\mathcal{S}_1)$ is unstable.



(a) Application of Theorem 4.7.



(b) Comparison with [98, Theorem 4].

Figure 4.13: Estimation of the order n for the necessity of positivity conditions.

Figure 4.13a shows the over estimations of n^* given by (4.46) for Example 1.2 with $(k, h) \in [0.1, 100] \times [0, 10]$. From these orders, the sufficient condition of instability becomes necessary. As in the previous section, one can see that the order n^* increases as parameters k and h increase. It is also worth noticing that, along the critical lines in black, the same phenomena occur and n^* would tend to infinity because $\|U\|_\infty$ would blow up. Contrary to the previous section, the positivity test is faster than the LMI test. The establishment of the necessary and sufficient condition via the positivity of matrix $\mathbf{P}_n$ is therefore totally implementable, especially for small values of k and h.

In an equivalent way to our result, [98, Theorem 4] provides an estimation of the order n^*, from which matrix $\mathbf{K}_n$ given by (4.69) is not positive definite. This minimal order is given by

$$n^* = 1 + \left\lceil \mathrm{e}^{h|A|}\, h(2\,|A| + |A_d|) \left(\alpha^* + \sqrt{\alpha^*(\alpha^*+1)}\right) - h\,|A| \right\rceil, \tag{4.89}$$

where scalar

$$\alpha^* = 2r(1 + h + h^2)\rho_0.$$

Figure 4.13b show such order n^* for Example 1.2 with $(k, h) \in [0.1, 100] \times [0, 10]$. These orders seem extremely large and pessimistic. They are exponentially greater than the ones obtained by the Legendre approximation (i.e. for order $n^*_{\mathbf{P}}$ with Legendre polynomials we obtain $n^*_{\mathbf{K}} \propto 10^{n^*_{\mathbf{P}}}$ with discretization procedure [98]). The mitigated conclusions on the positivity test based on Legendre polynomials compared to [72, 98] have then to be balanced with the fast convergence rates of Legendre approximation and with its faculty to obtain tractable necessary and sufficient stability conditions.

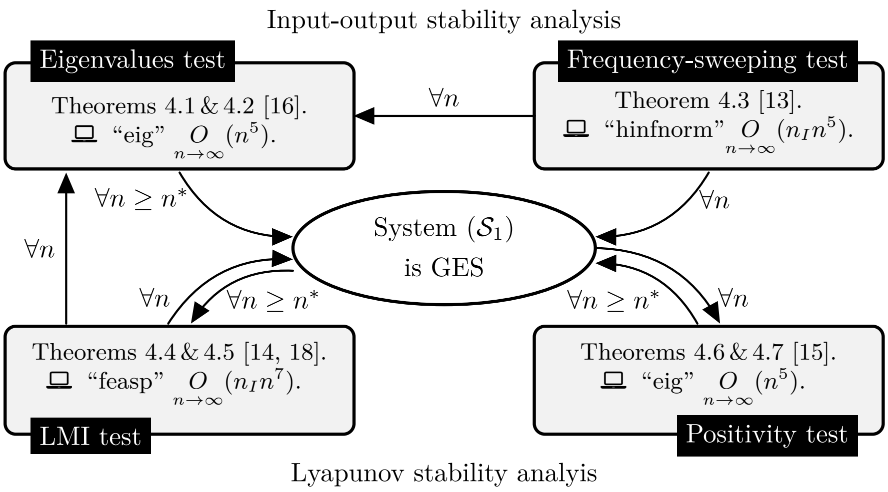


Figure 4.14: Summary of Chapter 4.

Conclusion

In this chapter, we have addressed the stability of an ODE interconnected to a transport PDE, which can be interpreted as a time-delay system. Four methods have been presented, all of them based on an intensive use of Legendre polynomials. A summary of the four results obtained is shown in Figure 4.14. The main benefits of these techniques rely on the fast convergence of Legendre approximation. Here, we have focused on the ODE-PDE modelling, knowing that the same calculus would lead to the same conclusion for time-delay systems.

In a natural way, we wonder if the developments introduced in this part can be applied to other systems. In that sense, the following part is dedicated to a system interconnected with the reaction-diffusion equation.

Part III

System interconnected with the reaction-diffusion equation

Chapter 5

Modelling of ODE-reaction-diffusion systems through approximation

"Two equivalent polygons can be converted one into the other after the manner of Chinese puzzles."
Reviews of Hibert's fundations of geometry, Poincaré.

Contents

This chapter is an opportunity to answer a number of questions all related to the approximation of ODE-reaction-diffusion interconnected systems. In several contexts, an approximation in frequencies or in time is made to compute the solution or analyze such systems. Most of the time, rational [53] or pseudo-spectral [101] approximations are considered.

- *What are the existing approximated models and is there links between them?*
- *Can they be used to pursue stability analysis via an expression of the model truncated error?*

In the first section, Padé, spectral and Legendre-*tau* modelling of the reaction-diffusion are recalled. In the second section, we present two new approximated models based on Legendre polynomial approximation. Comparisons with existing methods is addressed and an expression of the residual infinite-dimensional part is provided. In the last section, for our purposes, these models are looped with a finite-dimensional system to be used for stability analysis of ODE-reaction-diffusion interconnected systems.

5.1 Existing models for the reaction-diffusion equation

The reaction-diffusion equation is recalled below

$$\begin{cases} \partial_t z(t,\theta) = (\lambda + \nu \partial_{\theta\theta}) z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \\ z(t,\tfrac{1}{2}) = 0, & \forall t \in \mathbb{R}_+, \\ \partial_\theta z(t,1) = u(t), \quad \partial_\theta z(t,0) = u(t), & \forall t \in \mathbb{R}_+, \\ y(t) = z(t,1), & \forall t \in \mathbb{R}_+, \\ z(0,\theta) = z_0(\theta), & \forall \theta \in (0,1), \end{cases} \tag{$\mathcal{S}_{2\infty}$}$$

where the initial condition z_0 belongs to $H^2(0,1;\mathbb{R}^{n_z})$ and satisfies $z_0(\theta) = -z_0(1-\theta)$, for all θ in $[\frac{1}{2}, 1]$. Indeed, the reaction-diffusion on $[\frac{1}{2}, 1]$ has been duplicated by antisymmtery on the interval $[0, \frac{1}{2}]$ for technical reasons. It can be modeled in multiple ways and a non-exhaustive list of methods is drawn up in this section.

5.1.1 Padé methods

In the Laplace domain, recall Padé rational approximation technique [21, 175].

Definition 5.1. The rational approximation with numerator $N_p(s) = \sum_{i=0}^{p} a_i s^i$ at order p and denominator $D_q(s) = \sum_{i=0}^{q} b_i s^i$ at order q is called $(p|q)$ Padé approximant of function $H(s)$ around $s = \lambda$ if

$$H(s) - \frac{N_p(s)}{D_q(s)} = \underset{s\to\lambda}{O}((s-\lambda)^{p+q+1}). \tag{5.1}$$

In the light of Definition 5.1, consider Padé rational approximation $\mathcal{P}_{(p_n|q_n)}(s) = \frac{N_{p_n}(s)}{D_{q_n}(s)}$ around $s = \lambda$ of the reaction-diffusion transfer function $H(s) = \frac{\tanh\left(\frac{1}{2}\sqrt{\frac{s-\lambda}{\nu}}\right)}{\sqrt{\frac{s-\lambda}{\nu}}} I_{n_z}$ provided in (1.13) of system $(\mathcal{S}_{2\infty})$. Indices p_n and q_n are positive integers, which are given as functions of n in $\mathbb{N}$.

Proposition 5.1. *For any order n in $\mathbb{N}$, the transfer function $H(s) = \frac{\tanh\left(\frac{1}{2}\sqrt{\frac{s-\lambda}{\nu}}\right)}{\sqrt{\frac{s-\lambda}{\nu}}} I_{n_z}$ of system $(\mathcal{S}_{2\infty})$ can be split into two parts*

$$H(s) = \mathcal{P}_{(p_n|q_n)}(s) + \left(H(s) - \mathcal{P}_{(p_n|q_n)}(s)\right), \tag{5.2}$$

where $\tilde{\mathcal{P}}_{(p_n|q_n)}(s) = H(s) - \mathcal{P}_{(p_n|q_n)}(s)$ is the $(p_n|q_n)$ Padé remainder.

The finite-dimensional part $\mathcal{P}_{(p_n|q_n)}$ is used to approximate the behavior of the reaction-diffusion equation. However, the convergence radius of $\tanh(z)$ being limited to $\frac{\pi}{2}$, the approximation $\mathcal{P}_{(p_n|q_n)}$ converges to $H(s)$ on any closed subset of $\mathcal{B}(\lambda, \nu\pi^2)$ [175]. That is why Padé method is rarely used in practice. Alternative methods based on spectral or pseudo-spectral approximations in time are often privileged and are described in the sequel.

5.1.2 Spectral methods

In the time domain, the reaction-diffusion equation can be solved analytically via Fourier series [82] for periodic boundary conditions or Fokas method [59] for non-periodic boundary conditions. However, to solve the equations numerically, a finite-dimensional part and an infinite-dimensional remainder need to be separated. The Galerkin technique [102, 115] and the decomposition on the eigenbasis is an appropriate way to perform this separation.

Taking support from Fourier trigonometric functions, a first modelling can be derived. Focusing on $(\mathcal{S}_{2\infty})$, let us define the eigenbasis of the reaction-diffusion equation itself.

Definition 5.2. For any order n in $\mathbb{N}$, the state z can be split on the eigenbasis into an approximated function and a truncated error $\tilde{z}_n^\natural$ as follows

$$z(t,\theta) = \phi_n^{\natural\top}(\theta)\mathcal{Z}_n^\natural(t) + \tilde{z}_n^\natural(t,\theta), \quad \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \tag{5.3}$$

where the matrix $\mathcal{Z}_n^\natural$ in $\mathbb{R}^{nn_z}$, which collocates the n first spectral coefficients of the state z, is given by

$$\mathcal{Z}_n^\natural(t) = \underbrace{\left(\int_0^1 \phi_n^\natural(\theta)\phi_n^{\natural\top}(\theta)\mathrm{d}\theta\right)^{-1}}_{=2I_{nn_z}} \left(\int_0^1 \phi_n^\natural(\theta) z(t,\theta)\mathrm{d}\theta\right), \quad \forall t \in \mathbb{R}_+, \tag{5.4}$$

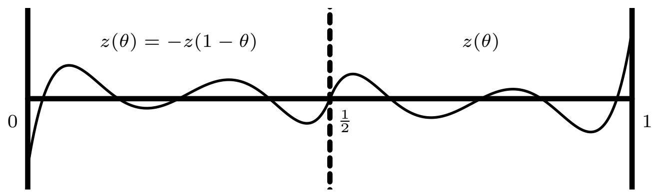


Figure 5.1: Function z extended on the interval $[0, 1]$.

and where the matrix $\phi_n^\natural$ in $\mathbb{R}^{nn_z \times n_z}$, which collocates the n first eigenfunctions $\{\varphi_k^\natural\}_{k\in\{0,\dots,n-1\}}$ of the system $(\mathcal{S}_{2\infty})$, is given by

$$\phi_n^\natural(\theta) = \left[\sin\left(\pi(\theta - \tfrac{1}{2})\right) I_{n_z} \quad \dots \quad \sin\left((2n-1)\pi(\theta - \tfrac{1}{2})\right) I_{n_z}\right]^\top, \quad \forall \theta \in [0, 1]. \tag{5.5}$$

Before going any further, it is worth noticing that system $(\mathcal{S}_{2\infty})$ has mixed boundary conditions on both bounds of the interval $[\frac{1}{2}, 1]$. Using the boundary condition $z(t, \frac{1}{2}) = 0$ and imposing $\partial_\theta z(t, 0) = u(t)$, system $(\mathcal{S}_{2\infty})$ has been extended on the interval $[0, 1]$ by the anti-symmetric relation

$$z(t, \theta) = -z(t, 1-\theta), \quad \forall (t, \theta) \in \mathbb{R}_+ \times [0, \tfrac{1}{2}]. \tag{5.6}$$

This extension has been depicted in Figure 5.1.
The spectral modelling is constructed in the following proposition.

Proposition 5.2. *For any order n in $\mathbb{N}$, system $(\mathcal{S}_{2\infty})$ is modeled by the spectral method as*

$$\begin{cases} \dot{\mathcal{Z}}_n^\natural(t) = \Lambda_n \mathcal{Z}_n^\natural(t) + 2\nu\big(\phi_n^\natural(1) - \phi_n^\natural(0)\big)u(t), & \forall t \in \mathbb{R}_+, \\ \partial_t \tilde{z}_n^\natural(t, \theta) = (\lambda + \nu\partial_{\theta\theta})\tilde{z}_n^\natural(t, \theta) + 2\nu\phi_n^{\natural\top}(\theta)\big(\phi_n^\natural(1) - \phi_n^\natural(0)\big)u(t), & \forall (t, \theta) \in \mathbb{R}_+ \times [0, 1], \\ \tilde{z}_n^\natural(t, \frac{1}{2}) = 0, & \forall t \in \mathbb{R}_+, \\ \partial_\theta \tilde{z}_n^\natural(t, 1) = u(t), \quad \partial_\theta \tilde{z}_n^\natural(t, 0) = u(t), & \forall t \in \mathbb{R}_+, \\ y(t) = \phi_n^{\natural\top}(1)\mathcal{Z}_n^\natural(t) + \tilde{z}_n^\natural(t, 1), & \forall t \in \mathbb{R}_+, \\ \mathcal{Z}_n^\natural(0) = 2\left(\int_0^1 \phi_n^\natural(\theta) z_0(\theta) \mathrm{d}\theta\right), \ \tilde{z}_n(0, \theta) = z_0(\theta) - \phi_n^{\natural\top}(\theta)\mathcal{Z}_n^\natural(0), & \forall \theta \in (0, 1), \end{cases} \tag{5.7}$$

where the diagonal matrix Λ_n collocates the n first eigenvalues of the system $(\mathcal{S}_{2\infty})$ and is given by

$$\Lambda_n = \lambda I_{nn_z} - \nu\pi^2 \mathrm{diag}\left(I_{n_z}, \dots, (2n-1)^2 I_{n_z}\right). \tag{5.8}$$

Proof. Firstly, applying two integrations by parts, the finite dimension dynamics are given by

$$\begin{aligned} \dot{\mathcal{Z}}_n^\natural(t) &= 2\int_0^1 \phi_n^\natural(\theta)\partial_t z(t, \theta)\mathrm{d}\theta = 2\nu\int_0^1 \phi_n^\natural(\theta)\partial_{\theta\theta} z(t, \theta)\mathrm{d}\theta + 2\lambda\int_0^1 \phi_n^\natural(\theta) z(t, \theta)\mathrm{d}\theta, \\ &= 2\nu\int_0^1 \phi_n^{\natural\prime\prime}(\theta) z(t, \theta)\mathrm{d}\theta + 2\nu\left[\phi_n^\natural(\theta)\partial_\theta z(t, \theta)\right]_0^1 - 2\nu\left[\phi_n^{\natural\prime}(\theta) z(t, \theta)\right]_0^1 + \lambda\mathcal{Z}_n^\natural(t). \end{aligned} \tag{5.9}$$

Deriving twice the trigonometric eigenfunctions (5.5), noticing that $\phi_n^{\natural\prime}(0) = \phi_n^{\natural\prime}(1) = 0$ and using the symmetry property (5.6) satisfied by z, one obtains

$$\dot{\mathcal{Z}}_n^\natural(t) = \Lambda_n \mathcal{Z}_n^\natural(t) + 2\nu\big(\phi_n^\natural(1) - \phi_n^\natural(0)\big)\partial_\theta z(t, 1). \tag{5.10}$$

Secondly, concerning the truncated error dynamics, it is worth noticing that

$$\begin{aligned} \partial_t \tilde{z}_n^\natural(t, \theta) &= \partial_t z(t, \theta) - \phi_n^{\natural\top}(\theta)\dot{\mathcal{Z}}_n^\natural(t), \\ &= (\lambda + \nu\partial_{\theta\theta})z(t, \theta) - \phi_n^{\natural\top}(\theta)\left(\Lambda_n \mathcal{Z}_n^\natural(t) + 2\nu\big(\phi_n^\natural(1) - \phi_n^\natural(0)\big)\partial_\theta z(t, 1)\right), \\ &= (\lambda + \nu\partial_{\theta\theta})(z(t, \theta) - \phi_n^{\natural\top}(\theta)\mathcal{Z}_n^\natural(t)) + 2\nu\phi_n^{\natural\top}(\theta)\big(\phi_n^\natural(1) - \phi_n^\natural(0)\big)\partial_\theta z(t, 1), \end{aligned} \tag{5.11}$$

which yields the result. Lastly, the boundary conditions are easily obtained. □

Here, notice that there is no *tau* modelling error in the dynamical parts. The truncated error $\tilde{z}_n^\natural$ satisfies a reaction-diffusion like equation as the original state z. The boundary conditions keep unchanged and the dynamics are perturbed only by the input $u(t)$.

5.1.3 Legendre-*tau* method

In the time domain, Galerkin-like technique [102] based on other Riesz basis can also be used. Such a method is commonly known as the *tau* method [173] and a focus on Legendre-*tau* method is made.

By the use of the anti-symmetric relation (5.6) satisfied by the state z on the interval $[0,1]$, all the even polynomials lead to null projected coefficients. Redefine notations (2.10) and (2.11) in accordance to take into account odd Legendre polynomials $\varphi_k := l_{2k+1}$ and to fulfill dimension $m = n_z$. These notations will be used in the two following chapters.

Definition 5.3. For any order n in $\mathbb{N}$, the state z can be split on Legendre polynomials basis into an approximated function and a truncated error $\tilde{z}_{2n}$ as follows

$$z(t,\theta) = \check{\ell}_n^\top(\theta)\check{\mathcal{Z}}_n(t) + \underbrace{\hat{\ell}_n^\top(\theta)\hat{\mathcal{Z}}_n(t)}_{=0} + \tilde{z}_{2n}(t,\theta), \quad \forall (t,\theta) \in \mathbb{R}_+ \times [0,1]. \tag{5.12}$$

where matrices $\check{\mathcal{Z}}_n$ and $\hat{\mathcal{Z}}_n$ in $\mathbb{R}^{nn_z \times n_z}$, which collocate respectively the n first odd and even Legendre coefficients of the state z, are given by

$$\begin{aligned} \check{\mathcal{Z}}_n(t) &= \left(\int_0^1 \check{\ell}_n(\theta)\check{\ell}_n^\top(\theta)\mathrm{d}\theta\right)^{-1}\left(\int_0^1 \check{\ell}_n(\theta)z(t,\theta)\mathrm{d}\theta\right), \\ \hat{\mathcal{Z}}_n(t) &= \left(\int_0^1 \hat{\ell}_n(\theta)\hat{\ell}_n^\top(\theta)\mathrm{d}\theta\right)^{-1}\left(\int_0^1 \hat{\ell}_n(\theta)z(t,\theta)\mathrm{d}\theta\right) = 0, \end{aligned} \quad \forall t \in \mathbb{R}_+, \tag{5.13}$$

and where matrices $\check{\ell}_n$ and $\hat{\ell}_n$ in $\mathbb{R}^{nn_z \times n_z}$, which collocate respectively the n first odd and even Legendre polynomials, are given by

$$\begin{aligned} \check{\ell}_n(\theta) &= \begin{bmatrix} l_1(\theta)I_{n_z} & l_3(\theta)I_{n_z} & \dots & l_{2n-1}(\theta)I_{n_z} \end{bmatrix}^\top, \\ \hat{\ell}_n(\theta) &= \begin{bmatrix} l_0(\theta)I_{n_z} & l_2(\theta)I_{n_z} & \dots & l_{2(n-1)}(\theta)I_{n_z} \end{bmatrix}^\top, \end{aligned} \quad \forall \theta \in [0,1]. \tag{5.14}$$

Remark 5.1. The even Legendre polynomial coefficients vanish on the interval $[0,1]$ thanks to the extended system on both sides of $\theta = \frac{1}{2}$ through the anti-symmetric relation (5.6).

Recall also the properties of Legendre polynomials and associated notations which are used all along.

Property 5.1. *In light of Property 2.1 of Legendre polynomials, the following properties are derived.*

- ***Orthogonality*** (2.5)***:*** *Legendre polynomials are orthogonal to each other and, for any $S \in \mathbb{S}_+^n$, they satisfy the orthogonality relation*

$$\begin{aligned} \check{\mathcal{I}}_n\left(\int_0^1 \check{\ell}_n(\theta)S\check{\ell}_n^\top(\theta)\mathrm{d}\theta\right)\check{\mathcal{I}}_n &= \check{\mathcal{I}}_n^S \text{ with } \check{\mathcal{I}}_n^S = \mathrm{diag}(3S, 7S, \dots, (4n-1)S) \in \mathbb{R}^{nn_z \times nn_z}, \\ \hat{\mathcal{I}}_n\left(\int_0^1 \hat{\ell}_n(\theta)S\hat{\ell}_n^\top(\theta)\mathrm{d}\theta\right)\hat{\mathcal{I}}_n &= \hat{\mathcal{I}}_n^S \text{ with } \hat{\mathcal{I}}_n^S = \mathrm{diag}(S, 5S, \dots, (4n-3)S) \in \mathbb{R}^{nn_z \times nn_z}. \end{aligned} \tag{5.15}$$

With a light abuse of notations, matrices $\check{\mathcal{I}}_n$ and $\hat{\mathcal{I}}_n$ stand for $\check{\mathcal{I}}_n^{I_{n_z}}$ and $\hat{\mathcal{I}}_n^{I_{n_z}}$.

- ***Point-wise values*** (2.15)***:*** *Legendre polynomials are evaluated point wisely by*

$$
\begin{aligned}
\check{\ell}_{n,0} &= \check{\ell}_n(0) = -\begin{bmatrix} I_{n_z} \\ \vdots \\ I_{n_z} \end{bmatrix} = -\ell_{n,1}, & \hat{\ell}_{n,0} &= \hat{\ell}_n(0) = \begin{bmatrix} I_{n_z} \\ \vdots \\ I_{n_z} \end{bmatrix} = \ell_{n,1} \quad \in \mathbb{R}^{nn_z \times n_z}, \\
\check{\ell}_{n,1} &= \check{\ell}_n(1) = \begin{bmatrix} I_{n_z} \\ \vdots \\ I_{n_z} \end{bmatrix} = \ell_{n,1}, & \hat{\ell}_{n,1} &= \hat{\ell}_n(1) = \begin{bmatrix} I_{n_z} \\ \vdots \\ I_{n_z} \end{bmatrix} = \ell_{n,1} \quad \in \mathbb{R}^{nn_z \times n_z}, \\
\check{\ell}_{n,2} &= \check{\ell}_n(1) - \check{\ell}_n(0) = 2\ell_{n,1}, & \hat{\ell}_{n,2} &= \hat{\ell}_n(1) + \hat{\ell}_n(0) = 2\ell_{n,1} \quad \in \mathbb{R}^{nn_z \times n_z},
\end{aligned}
\tag{5.16}
$$

- ***Derivation*** (2.19)-(2.20)***:*** *Legendre polynomials verify the following differentiation rule*

$$
\check{\ell}_n''(\theta) = \check{\mathcal{L}}_n \hat{\mathcal{I}}_n \hat{\mathcal{L}}_n \check{\mathcal{I}}_n \check{\ell}_n(\theta), \ \forall \theta \in [0,1] \quad \text{with} \ \begin{cases} \check{\mathcal{L}}_n = 2\mathrm{tril}(\ell_{n,1}\ell_{n,1}^\top), \\ \hat{\mathcal{L}}_n = \check{\mathcal{L}}_n - 2I_{nn_z}, \end{cases}
\tag{5.17}
$$

and, for any $z \in H_0^2(0,1;\mathbb{R}^{n_z})$, the projections satisfy

$$
\begin{aligned}
&\int_0^1 \check{\ell}_n(\theta) \partial_{\theta\theta} z(t,\theta) \mathrm{d}\theta = \hat{\mathcal{L}}_n^\top \hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \check{\mathcal{I}}_n \int_0^1 \check{\ell}_n(\theta) z(t,\theta) \mathrm{d}\theta \\
&+ \begin{bmatrix} \sum_{i=2}^{\infty} (2i+1)(\check{\ell}_{n,1} - (-1)^i \check{\ell}_{n,0}) \sum_{j=\max(2n,i+1)}^{\infty} (2j+1)(\hat{\ell}_{n,1} - (-1)^j \hat{\ell}_{n,0}) \langle l_j | z(t) \rangle \\ \vdots \\ \sum_{i=2n}^{\infty} (2i+1)(\check{\ell}_{n,1} - (-1)^i \check{\ell}_{n,0}) \sum_{j=\max(2n,i+1)}^{\infty} (2j+1)(\hat{\ell}_{n,1} - (-1)^j \hat{\ell}_{n,0}) \langle l_j | z(t) \rangle \end{bmatrix}.
\end{aligned}
\tag{5.18}
$$

Notice that we also have the following relations

$$
\check{\mathcal{L}}_n + \hat{\mathcal{L}}_n^\top = 2\ell_{n,1}\ell_{n,1}^\top, \quad \hat{\mathcal{L}}_n + \check{\mathcal{L}}_n^\top = 2\ell_{n,1}\ell_{n,1}^\top.
\tag{5.19}
$$

The Legendre-*tau* model writes as follows.

Proposition 5.3. *For any order n in $\mathbb{N}$, system ($\mathcal{S}_{2\infty}$) with an initial condition z_0 in $H_0^2(0,1;\mathbb{R}^{n_z})$ satisfying* (5.6) *is modeled by the Legendre-tau method as*

$$
\begin{cases}
\dot{\check{\mathcal{Z}}}_n(t) = \left(\lambda I_{nn_z} + \nu \check{\mathcal{I}}_n(-\check{\mathcal{L}}_n + \check{\mathcal{L}}_n \hat{\mathcal{I}}_n 2\ell_{n,1} \mathfrak{d}_n \ell_{n,1}^\top) \hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \right) \check{\mathcal{Z}}_n(t) + 2\nu \check{\mathcal{I}}_n \left(\ell_{n,1} - \check{\mathcal{L}}_n \hat{\mathcal{I}}_n \ell_{n,1} \mathfrak{d}_n \right) u(t) + \tau_n^z(t), \\
y(t) = \ell_{n,1}^\top \left(1 + \mathfrak{d}_n \hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \right) \check{\mathcal{Z}}_n(t) + \mathfrak{d}_n u(t) + \tau_n^y(t), \qquad \forall t \in \mathbb{R}_+, \\
\check{\mathcal{Z}}_n(0) = \check{\mathcal{I}}_n \int_0^1 \check{\ell}_n(\theta) z_0(\theta) \mathrm{d}\theta,
\end{cases}
\tag{5.20}
$$

where $\mathfrak{d}_n = (\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}$ and where both errors τ_n^z and τ_n^y depend on Legendre polynomial coefficients of z of degree higher or equal to $2(n+1)$.

Proof. If z is solution to system ($\mathcal{S}_{2\infty}$), then the n first odd Legendre coefficients satisfy the following dynamics,

$$
\check{\mathcal{I}}_n^{-1} \dot{\check{\mathcal{Z}}}_n(t) = \int_0^1 \check{\ell}_n(\theta) \partial_t z(t,\theta) \mathrm{d}\theta = \nu \int_0^1 \check{\ell}_n(\theta) \partial_{\theta\theta} z(t,\theta) \mathrm{d}\theta + \lambda \int_0^1 \check{\ell}_n(\theta) z(t,\theta) \mathrm{d}\theta,
\tag{5.21}
$$

Based on *tau* approximation applied to Legendre polynomials (5.18), we obtain

$$\int_0^1 \check{\ell}_n(\theta)\partial_{\theta\theta}z(t,\theta)\mathrm{d}\theta = \begin{bmatrix} I_{nn_z} & 0 \end{bmatrix} \hat{\mathcal{L}}_{n+1}^\top \hat{\mathcal{I}}_{n+1} \check{\mathcal{L}}_{n+1}^\top \check{\mathcal{Z}}_{n+1}(t) + \begin{bmatrix} \sum_{i=2}^{\infty}(2i+1)(\check{\ell}_{n,1}-(-1)^i\check{\ell}_{n,0}) \sum_{j=\max(2(n+1),i+1)}^{\infty} (2j+1)(\hat{\ell}_{n,1}-(-1)^j\hat{\ell}_{n,0}) \langle l_j | z(t)\rangle \\ \vdots \\ \sum_{i=2n}^{\infty}(2i+1)(\check{\ell}_{n,1}-(-1)^i\check{\ell}_{n,0}) \sum_{j=\max(2(n+1),i+1)}^{\infty} (2j+1)(\hat{\ell}_{n,1}-(-1)^j\hat{\ell}_{n,0}) \langle l_j | z(t)\rangle \end{bmatrix}. \tag{5.22}$$

Moreover, since z belongs to $H_0^2(0,1;\mathbb{R}^{n_z})$, point-wise convergence of Legendre approximation and its derivatives is ensured on the closed interval $[0,1]$. Then, from Definition 5.3 evaluated at the extremities $\theta \in \{0,1\}$, we are able to write

$$u(t) = \partial_\theta z(t,1) = \ell_{n,1}^\top \hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \check{\mathcal{Z}}_n(t) + \ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1}(4n+3)\langle l_{2n+1}|z(t)\rangle + \tau_n^u(t), \tag{5.23}$$

$$y(t) = z(t,1) = \ell_{n,1}^\top \check{\mathcal{Z}}_n(t) + (4n+3)\langle l_{2n+1}|z(t)\rangle + \sum_{i=n+1}^{\infty}(4i+3)\langle l_{2i+1}|z(t)\rangle, \tag{5.24}$$

where $\tau_n^u \in \mathbb{R}^{nn_z}$ is not detailed but can be seen as a series which depends on $\langle l_i|z\rangle$ for $i \geq 2(n+1)$. From (5.23), it is worth noticing that

$$(4n+3)\langle l_{2n+1}|z(t)\rangle = (\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}\left(u(t) - \ell_{n,1}^\top \hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \check{\mathcal{Z}}_n(t) - \tau_n^y(t)\right).$$

Then, the integral term (5.22) are given by

$$\begin{aligned}\int_0^1 \check{\ell}_n(\theta)\partial_{\theta\theta}z(t,\theta)\mathrm{d}\theta &= \hat{\mathcal{L}}_n^\top \hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \check{\mathcal{Z}}_n(t) + (\tfrac{1}{\nu}\check{\mathcal{I}}_n^{-1})\tau_n^z(t) \\ &\quad + [\,I_{nn_z}\ 0\,]\hat{\mathcal{L}}_{n+1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1}(\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}\left(u(t) - \ell_{n,1}^\top \hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \check{\mathcal{Z}}_n(t)\right),\end{aligned}$$

where $\tau_n^z \in \mathbb{R}^{nn_z}$ is not described but can be related to τ_n^x and depends on $\langle l_i|z\rangle$ for $i \geq 2(n+1)$. Noticing that $\hat{\mathcal{L}}_n^\top + \check{\mathcal{L}}_n = 2\ell_{n,1}\ell_{n,1}^\top$, the above expression simplifies to

$$\begin{aligned}\int_0^1 \check{\ell}_n(\theta)\partial_{\theta\theta}z(t,\theta)\mathrm{d}\theta &= \left(-\check{\mathcal{L}}_n + 2\ell_{n,1}\ell_{n,1}^\top\right)\hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \check{\mathcal{Z}}_n(t) \\ &\quad -[\,I_{nn_z}\ 0\,]2\ell_{n+1,1}\underbrace{\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1}(\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}}_{=1}\ell_{n,1}^\top \hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \check{\mathcal{Z}}_n(t) \\ &\quad +[\,I_{nn_z}\ 0\,]\check{\mathcal{L}}_{n+1}\hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1}(\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}\ell_{n,1}^\top \hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \check{\mathcal{Z}}_n(t) \\ &\quad +[\,I_{nn_z}\ 0\,]2\ell_{n+1,1}\underbrace{\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1}(\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}}_{=1}u(t) \\ &\quad -[\,I_{nn_z}\ 0\,]\check{\mathcal{L}}_{n+1}\hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1}(\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}u(t) \\ &\quad +(\tfrac{1}{\nu}\check{\mathcal{I}}_n^{-1})\tau_n^z(t), \\ &= \left(-\check{\mathcal{L}}_n + \check{\mathcal{L}}_n\hat{\mathcal{I}}_n 2\ell_{n,1}(\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}\ell_{n,1}^\top\right)\hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \check{\mathcal{Z}}_n(t) \\ &\quad + \left(2\ell_{n,1} - \check{\mathcal{L}}_n\hat{\mathcal{I}}_n 2\ell_{n,1}(\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}\right)u(t) \\ &\quad +(\tfrac{1}{\nu}\check{\mathcal{I}}_n^{-1})\tau_n^z(t).\end{aligned}$$

Therefore, the dynamics (5.21) gives

$$\begin{aligned}\check{\mathcal{I}}_n^{-1}\dot{\check{\mathcal{Z}}}_n(t) &= \left(\lambda\check{\mathcal{I}}_n^{-1} + \nu\left(-\check{\mathcal{L}}_n + \check{\mathcal{L}}_n\hat{\mathcal{I}}_n 2\ell_{n,1}(\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}\ell_{n,1}^\top\right)\hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top\right)\check{\mathcal{Z}}_n(t) \\ &\quad +\nu\left(2\ell_{n,1} - \check{\mathcal{L}}_n\hat{\mathcal{I}}_n 2\ell_{n,1}(\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1}\right)u(t) + \check{\mathcal{I}}_n^{-1}\tau_n^z(t).\end{aligned} \tag{5.25}$$

Finally, the output defined by (5.24) is given by

$$y(t) = \ell_{n,1}^\top \check{\mathcal{Z}}_n(t) + (\ell_{n+1,1}^\top \hat{\mathcal{I}}_{n+1} 2\ell_{n+1,1})^{-1} \left(u(t) - \ell_{n,1}^\top \hat{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \check{\mathcal{Z}}_n(t)\right) + \tau_n^y(t).$$

where $\tau_n^y \in \mathbb{R}^{n_z}$ is not described but resumes in a linear combination of the truncated error on $\partial_\theta z(1)$ and $z(1)$. To conclude, these two equations lead to system (5.20). □

For simulation goals, model (5.20) is often used [162, 163] by considering the the finite-dimensional model and neglecting errors τ_n^z and τ_n^y. From the initial condition $\hat{\mathcal{Z}}_n(0)$, an estimation of the state $\hat{\mathcal{Z}}_n(t)$ is drawn from the dynamics and a representation of the solution $z(t,\theta) = \hat{\ell}_n^\top(\theta)\hat{\mathcal{Z}}_n(t)$ is possible.

For stability analysis goals, the dynamical error τ_n^z and an output error τ_n^y need to be considered. A better description of these errors remains to be done thanks to a dual interpretation of this modelling in the Laplace and time domain. That is why, in the next section, two new models will be proposed to describe completely the residual infinite-dimensional part.

5.2 Legendre modelling for the reaction-diffusion equation

A novel Legendre modelling technique is set up in this section. As a supplement to Legendre-*tau* modelling, the error put aside is permanently retained and interpreted either in the Laplace domain or in time domain. Using Definition 2.6, the Legendre polynomial decomposition of the transfer function $G(s,\theta) = \frac{\sinh\left(\sqrt{\frac{s-\lambda}{\nu}}(\theta-\frac{1}{2})\right)}{\sqrt{\frac{s-\lambda}{\nu}}\cosh\left(\frac{1}{2}\sqrt{\frac{s-\lambda}{\nu}}\right)} I_{n_z}$ from the input $U(s)$ to the state $Z(s,\theta)$ of the reaction-diffusion system given by (1.12) is taken into account. The counterpart of Definition 5.3 in the Laplace domain leads to the following definition.

Definition 5.4. For any order n in $\mathbb{N}$, transfer function G can be split on Legendre polynomials basis into an approximated function and a truncated error $\tilde{G}_{2n}$ as follows

$$G(s,\theta) = \check{\ell}_n^\top(\theta)\check{\mathcal{G}}_n(s) + \underbrace{\hat{\ell}_n^\top(\theta)\hat{\mathcal{G}}_n(s)}_{=0} + \tilde{G}_{2n}(s,\theta), \quad \forall (s,\theta) \in \mathbb{C} \times [0,1], \tag{5.26}$$

where matrices $\check{\mathcal{G}}_n$ and $\hat{\mathcal{G}}_n$ in $\mathbb{C}^{nn_z \times n_z}$, which collocate respectively the n first odd and even Legendre coefficients of the transfer function G, are given by

$$\begin{aligned} \check{\mathcal{G}}_n(s) &= \left(\int_0^1 \check{\ell}_n(\theta)\check{\ell}_n^\top(\theta)\mathrm{d}\theta\right)^{-1} \left(\int_0^1 \check{\ell}_n(\theta) G(s,\theta)\mathrm{d}\theta\right), \\ \hat{\mathcal{G}}_n(s) &= \left(\int_0^1 \hat{\ell}_n(\theta)\hat{\ell}_n^\top(\theta)\mathrm{d}\theta\right)^{-1} \left(\int_0^1 \hat{\ell}_n(\theta) G(s,\theta)\mathrm{d}\theta\right) = 0, \end{aligned} \quad \forall s \in \mathbb{C}, \tag{5.27}$$

and where matrices $\check{\ell}_n$ and $\hat{\ell}_n$ in $\mathbb{R}^{nn_z \times n_z}$, which collocate respectively the n first odd and even Legendre polynomials, are given by (5.14).

Remark 5.2. The even Legendre polynomial coefficients of G vanish on the interval $[0,1]$ thanks to the extended system on both sides of $\theta = \frac{1}{2}$ through the anti-symmetric relation

$$G(s,\theta) = -G(s,1-\theta), \quad \forall (s,\theta) \in \mathbb{C} \times [0,\tfrac{1}{2}]. \tag{5.28}$$

Indeed, the anti-symmetric relation (5.6) satisfied by the state z is also true for the transfer function G and is depicted on Figure 5.2.

In the sequel, the procedure will be similar to the one used for *tau* modelling. The differences rely on the fact that the state space $H^2(0,1;\mathbb{R}^{n_z})$ instead of $H_0^2(0,1;\mathbb{R}^{n_z})$ is considered and that a cross-check between frequency and time domains is carried out. The following models turn out to be realizations of $(n-1|n)$ and $(n|n)$ Padé approximants but also to give dynamics, formula and sense to the Padé infinite-dimensional error.

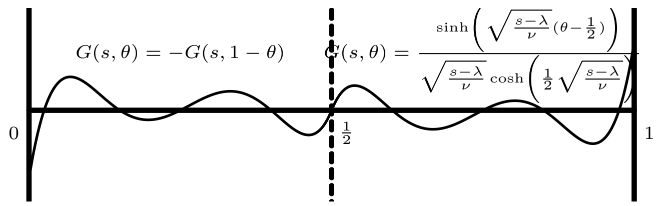


Figure 5.2: Transfer function G extended on the interval $[0,1]$.

5.2.1 Model I: a complete realization of $(n-1|n)$ Padé model

The first modelling given below is strictly causal and is set up to put aside the truncated error $\tilde{G}_{2n}$.

Proposition 5.4. *For any order n in $\mathbb{N}$, system $(\mathcal{S}_{2\infty})$ with an initial condition z_0 in $H^2(0,1;\mathbb{R}^{n_z})$ satisfying (5.6) can be modeled as follows*

$$\begin{cases} \dot{\check{\mathcal{Z}}}_n(t) = \left(\lambda I_{nn_z} - \nu\check{\mathcal{I}}_n\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\right)\check{\mathcal{Z}}_n(t) + 2\nu\check{\mathcal{I}}_n\ell_{n,1}u(t) - 2\nu\check{\mathcal{I}}_n\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}e_n(t), & (5.29a)\\ \partial_t\tilde{z}_{2n}(t,\theta) = (\lambda+\nu\partial_{\theta\theta})\tilde{z}_{2n}(t,\theta) & \\ \qquad + 2\nu\check{\ell}_n^\top(\theta)\check{\mathcal{I}}_n\big(\ell_{n,1}\ell_{n,1}^\top\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t) - \ell_{n,1}u(t) + \check{\mathcal{L}}_n\check{\mathcal{I}}_n\ell_{n,1}e_n(t)\big), & (5.29b)\\ \tilde{z}_{2n}(t,\tfrac{1}{2}) = 0, & (5.29c)\\ \partial_\theta\tilde{z}_{2n}(t,1) = u(t) - \ell_{n,1}^\top\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t), \quad \partial_\theta\tilde{z}_{2n}(t,1) = u(t) - \ell_{n,1}^\top\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t), & (5.29d)\\ y(t) = \ell_{n,1}^\top\check{\mathcal{Z}}_n(t) + e_n(t), & (5.29e)\\ e_n(t) = \tilde{z}_{2n}(t,1), & (5.29f) \end{cases}$$

for all (t,θ) in $\mathbb{R}_+\times[0,1]$, with the initial condition

$$\begin{cases} \check{\mathcal{Z}}_n(0) = \check{\mathcal{I}}_n\displaystyle\int_0^1 \check{\ell}_n(\theta)z_0(\theta)\mathrm{d}\theta, \\ \tilde{z}_n(0,\theta) = z_0(\theta) - \check{\mathcal{I}}_n\displaystyle\int_0^1 \check{\ell}_n(\theta)z_0(\theta)\mathrm{d}\theta, \quad \forall\theta\in[0,1]. \end{cases} \tag{5.30}$$

The error $e_n(t)$ is the boundary output of (5.29) *and can be seen in the Laplace domain as*

$$E_n(s) = \tilde{G}_{2n}(s,1)U(s), \tag{5.31}$$

where $\tilde{G}_{2n}(s,1)$ is the Legendre truncated error of function $G(s,\theta)$ at order n given by (5.26) *evaluated at $\theta = 1$.*

Proof. Assume that z is solution to system $(\mathcal{S}_{2\infty})$. Firstly, applying two integrations by parts to (5.21), the following equation holds

$$\check{\mathcal{I}}_n^{-1}\dot{\check{\mathcal{Z}}}_n(t) = \lambda\int_0^1 \check{\ell}_n(\theta)z(t,\theta)\mathrm{d}\theta + \nu\int_0^1 \check{\ell}_n''z(t,\theta)\mathrm{d}\theta + \nu[\check{\ell}_n(\theta)\partial_\theta z(t,\theta)]_0^1 - \nu[\check{\ell}_n'(\theta)z(t,\theta)]_0^1,$$

which can be rewritten using the derivation rule in (5.17) satisfied by Legendre polynomials as

$$\check{\mathcal{I}}_n^{-1}\dot{\check{\mathcal{Z}}}_n(t) = \left(\lambda\check{\mathcal{I}}_n^{-1} + \nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\hat{\mathcal{L}}_n\right)\check{\mathcal{Z}}_n(t) + 2\nu\ell_{n,1}u(t) - 2\nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}z(t,1), \tag{5.32}$$

Furthermore, from Legendre approximation applied to state z, Definition 5.3 in the time domain and Definition 5.4 in the Laplace domain allow decomposing the output into two parts

$$\begin{cases} y(t) = z(t,1) = \ell_{n,1}^\top\check{\mathcal{Z}}_n(t) + \tilde{z}_{2n}(t,1), \\ Y(s) = Z(s,1) = \left(\ell_{n,1}^\top\check{\mathcal{G}}_n(s) + \tilde{G}_{2n}(s,1)\right)U(s). \end{cases} \tag{5.33}$$

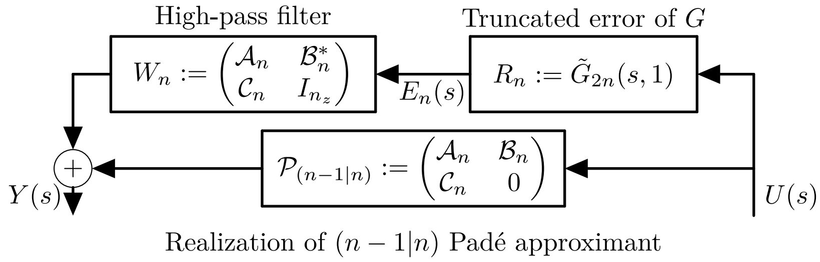


Figure 5.3: Legendre-modelling I of the reaction-diffusion equation ($\mathcal{S}_{2\infty}$).

The truncated error $e_n(t) = \tilde{z}_{2n}(t,1)$ is given in Laplace domain by $E_n(s) = \tilde{G}_{2n}(s,1)U(s)$. Then, replacing $z(t,1)$ by the expression given by (5.33) yields

$$\check{\mathcal{I}}_n^{-1}\dot{\check{\mathcal{Z}}}_n(t) = \left(\lambda\check{\mathcal{I}}_n^{-1} - \nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\right)\check{\mathcal{Z}}_n(t) + 2\nu\ell_{n,1}u(t) - 2\nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\tilde{z}_{2n}(t,1), \tag{5.34}$$

which corresponds to (5.29a). Next, the Legendre truncated error satisfies along the trajectories of system ($\mathcal{S}_{2\infty}$) the following dynamics

$$\begin{aligned} \partial_t\tilde{z}_{2n}(t,\theta) &= \partial_t\left(z(t,\theta) - \ell_n^\top(\theta)\check{\mathcal{Z}}_n(t)\right), \\ &= (\lambda + \nu\partial_{\theta\theta})z_{2n}(t,\theta) - \check{\ell}_n^\top(\theta)\dot{\check{\mathcal{Z}}}_n(t), \\ &= (\lambda + \nu\partial_{\theta\theta})\tilde{z}_{2n}(t,\theta) + \ell_n^\top(\theta)\left((\lambda + \nu\check{\mathcal{I}}_n\hat{\mathcal{L}}_n^\top\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top)\check{\mathcal{Z}}_n(t) - \dot{\check{\mathcal{Z}}}_n(t)\right) \end{aligned} \tag{5.35}$$

by the use of the derivation property (5.17). Using (5.34), the last term rewrites as

$$(\lambda + \nu\check{\mathcal{I}}_n\hat{\mathcal{L}}_n^\top\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top)\check{\mathcal{Z}}_n(t) - \dot{\check{\mathcal{Z}}}_n(t) = \nu\check{\mathcal{I}}_n\left((\hat{\mathcal{L}}_n^\top + \check{\mathcal{L}}_n)\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t) - 2\ell_{n,1}u(t) + 2\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\tilde{z}_{2n}(t,1)\right),$$

and the relation $\hat{\mathcal{L}}_n^\top + \check{\mathcal{L}}_n = 2\ell_{n,1}\ell_{n,1}^\top$ in (5.19) provides (5.29b). Lastly, the boundary input and output come from an application of Legendre expansion (5.12). □

The finite-dimensional part of this model is strictly causal and will be used to analyze the stability of system (1.13) with respect to the approximated order n. It is worth noticing that the error $e_n(t)$ appears in the output equation but also in the dynamical part (5.29a) since we proceed to a quasi-spectral decomposition. However, as for spectral decomposition, the dynamics of the remainder (5.29b) is a modified reaction-diffusion equation subject to a distributed input orthogonal to the first Legendre polynomials.

In addition, thanks to the frequency point of view, such a modelling turns out to be a Padé modelling as illustrated in Figure 5.3.

Property 5.2. *System* $\begin{pmatrix} \mathcal{A}_n & \mathcal{B}_n \\ \mathcal{C}_n & 0 \end{pmatrix}$ *is a realization of the* $(n-1|n)$ *Padé approximant of* $H(s) = \frac{\tanh\left(\frac{1}{2}\sqrt{\frac{s-\lambda}{\nu}}\right)}{\sqrt{\frac{s-\lambda}{\nu}}}I_{n_z}$ *and the corresponding Padé remainder is given by the following expression*

$$\tilde{\mathcal{P}}_{(n-1|n)}(s) = \underbrace{\left(I_{n_z} + \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n^*\right)}_{W_n(s)}\tilde{G}_{2n}(s,1). \tag{5.36}$$

where matrices are given by

$$\begin{aligned} \mathcal{A}_n &= \lambda I_{nn_z} - \nu\check{\mathcal{I}}_n\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top, & \mathcal{B}_n &= 2\nu\check{\mathcal{I}}_n\ell_{n,1}, \\ \mathcal{C}_n &= \ell_{n,1}^\top, & \mathcal{B}_n^* &= -2\nu\check{\mathcal{I}}_n\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}, \end{aligned} \tag{5.37}$$

and where $\tilde{G}_{2n}(s,1)$ *is the Legendre truncated error of function* $G(s,\theta)$ *at order* n *evaluated at* $\theta = 1$.

Proof. In the Laplace domain, the output $Y(s)$ of system (5.29) is decomposed as

$$Y(s) = \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n U(s) + \left(I_{n_z} + \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n^*\right) E_n(s)$$

Moreover, according to (5.31), we have $E_n(s) = \tilde{G}_{2n}(s,1)U(s)$. Therefore, the transfer function from U to Y is given by

$$H(s) = \underbrace{\mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n}_{H_n(s)} + \underbrace{\left(I_{n_z} + \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n^*\right) \tilde{G}_{2n}(s,1)}_{\tilde{H}_n(s)}.$$

The objective is now to prove that $\tilde{H}_n(s)$ is "small" when s tends to λ. To begin with, we denote $\mathfrak{A}_n = -\nu\check{\mathcal{I}}_n\check{\mathcal{L}}_n\hat{\mathcal{I}}_n$, which is a lower triangular matrix with non-null diagonal coefficients. Using the relation $\check{\mathcal{L}}_n^\top + \hat{\mathcal{L}}_n = 2\ell_{n,1}\ell_{n,1}^\top$, we have

$$\begin{aligned} I_{n_z} + \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n^* &= I_{n_z} + \ell_{n,1}^\top\left((s-\lambda)I_{nn_z} - \mathfrak{A}_n\check{\mathcal{L}}_n^\top\right)^{-1} 2\mathfrak{A}_n\ell_{n,1}, \\ &= I_{n_z} + \ell_{n,1}^\top\left((s-\lambda)I_{nn_z} + \mathfrak{A}_n\hat{\mathcal{L}}_n - 2\mathfrak{A}_n\ell_{n,1}\ell_{n,1}^\top\right)^{-1} 2\mathfrak{A}_n\ell_{n,1}. \end{aligned}$$

The technical Lemma B.4, postponed in Appendix B.2, is applied with $L = \mathfrak{A}_n\hat{\mathcal{L}}_n$, $u = -2\mathfrak{A}_n\ell_{n,1}$, and $v = \ell_{n,1}$ and gives

$$I_{n_z} + \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n^* = \underset{s\to\lambda}{O}((s-\lambda)^n).$$

Moreover, noticing that

$$\left\|\partial_\theta^{(2n+1)}G(s)\right\|_\infty = \sup_{[0,1]}\left|\left(\frac{s-\lambda}{\nu}\right)^n \frac{\sinh(\sqrt{\frac{s-\lambda}{\nu}}(\theta-1))}{\cosh(\sqrt{\frac{s-\lambda}{\nu}})}\right| \le \left|\frac{s-\lambda}{\nu}\right|^n,$$

holds, the application of Lemma 2.2 at order $d = 2n$ leads to

$$\left|\tilde{G}_{2n+1}(s,1)\right| \le \frac{\left|\frac{s-\lambda}{\nu}\right|^n}{2^{2n+1}(2n)(2n-\frac{1}{2})\dots(\frac{3}{2})}.$$

Then, noticing that $\left|\partial_\theta^{(2n+1)}\tilde{G}_{2n+1}(\lambda,1)\right| \neq 0$, we have $\tilde{G}_{2n}(s,1) = \tilde{G}_{2n+1}(s,1) = \underset{s\to\lambda}{O}((s-\lambda)^n)$. Thus, $\tilde{H}_n(s) = \underset{s\to\lambda}{O}((s-\lambda)^{2n})$ holds and, from Definition 5.1, we recognize $H_n(s) := \mathcal{P}_{(n-1|n)}(s)$ and $\tilde{H}_n(s) := \tilde{\mathcal{P}}_{(n-1|n)}(s)$ and conclude the proof.

□

Remark 5.3. As a complement, another proof has also been provided in [17] relying on the Maclaurin series of $\tilde{G}_{2n}$ near $s = \lambda$.

Pushing the Legendre development up to the next order, another modelling is proposed below. It turns out to be a Padé model to higher order.

5.2.2 Model II: a complete realization of $(n|n)$ Padé model

The second modelling given below consists to push the approximation up to order $n+1$ by keeping aside the truncated error $\tilde{G}_{2(n+1)}(s)$. As in the previous section, the following modelling presents two dynamics. From one side, the dynamics of the n first odd Legendre polynomial coefficients with respect to an error at order $n+1$. On the other side, the dynamics of this residual infinite-dimensional error as well as its analytic expression in Laplace and time domains.

Proposition 5.5. *For any order n in $\mathbb{N}$, system $(\mathcal{S}_{2\infty})$ with an initial condition z_0 in $H^2(0,1;\mathbb{R}^{n_z})$ satisfying* (5.6) *can be modeled as follows*

$$\begin{cases}
\dot{\check{\mathcal{Z}}}_n(t) = \left(\lambda I_{nn_z} + \nu\check{\mathcal{I}}_n(-\check{\mathcal{L}}_n + 2\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\ell_{n,1}^\top\mathfrak{d}_n)\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\right)\check{\mathcal{Z}}_n(t) \\
\qquad + 2\nu\check{\mathcal{I}}_n\left(\ell_{n,1} - \check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\mathfrak{d}_n\right)u(t) - 2\nu\check{\mathcal{I}}_n\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}e_n^\flat(t), & (5.38a)\\
\partial_t\tilde{z}_{2n}(t,\theta) = (\lambda + \nu\partial_{\theta\theta})\tilde{z}_{2n}(t,\theta) + 2\nu\check{\ell}_n^\top(\theta)\check{\mathcal{I}}_n\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}e_n^\flat(t) \\
\quad +2\nu\check{\ell}_n^\top(\theta)\check{\mathcal{I}}_n\Big((\ell_{n,1}\ell_{n,1}^\top - \check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\ell_{n,1}^\top\mathfrak{d}_n)\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t) - (\ell_{n,1} - \check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\mathfrak{d}_n)u(t)\Big), & (5.38b)\\
\tilde{z}_{2n}(t,\tfrac{1}{2}) = 0, & (5.38c)\\
\partial_\theta\tilde{z}_{2n}(t,1) = u(t) - \ell_{n,1}^\top\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t), \quad \partial_\theta\tilde{z}_{2n}(t,0) = u(t) - \ell_{n,1}^\top\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t), & (5.38d)\\
y(t) = \ell_{n,1}^\top(I_{n_z} - \mathfrak{d}_n\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top)\check{\mathcal{Z}}_n(t) + \mathfrak{d}_n u(t) + e_n^\flat(t), & (5.38e)\\
e_n^\flat(t) = \tilde{z}_{2(n+1)}(t,1) - \mathfrak{d}_n\partial_\theta\tilde{z}_{2(n+1)}(t,1), & (5.38f)
\end{cases}$$

for all (t,θ) in $\mathbb{R}_+ \times [0,1]$, with $\mathfrak{d}_n = (\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}$ and with initial condition (5.30)*. The error $e_n^\flat(t)$ is given in the Laplace domain by*

$$E_n^\flat(s) = \left(\tilde{G}_{2(n+1)}(s,1) - \mathfrak{d}_n\partial_\theta\tilde{G}_{2(n+1)}(s,1)\right)U(s), \tag{5.39}$$

and corresponds to a combination of the Legendre truncated error of function $G(s,\theta)$ at order $n+1$ given by (5.26) *and its derivatives both evaluated at $\theta = 1$.*

Proof. Similarly to *tau* method, the approximation of $z(t,1)$ and $\partial_\theta z(t,1)$ pushed to order $n+1$ gives

$$u(t) = \partial_\theta z(t,1) = \ell_{n,1}^\top\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t) + \ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1}(4n+3)\langle l_{2n+1}|z(t)\rangle + \partial_\theta\tilde{z}_{2(n+1)}(t,1), \tag{5.40}$$
$$y(t) = z(t,1) = \ell_{n,1}^\top\check{\mathcal{Z}}_n(t) + (4n+3)\langle l_{2n+1}|z(t)\rangle + \tilde{z}_{2(n+1)}(t,1), \tag{5.41}$$

Combining these two equations, an error of order $n+1$ is put aside as a linear combination of $\tilde{z}_{2(n+1)}(t,1)$ and $\partial_\theta\tilde{z}_{2(n+1)}(t,1)$ as follows

$$\begin{aligned}
z(t,1) &= \ell_{n,1}^\top\check{\mathcal{Z}}_n(t) + (\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}\Big(u(t) - \ell_{n,1}^\top\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t) - \partial_\theta\tilde{z}_{2(n+1)}(t,1)\Big) + \tilde{z}_{2(n+1)}(t,1), \\
&= \ell_{n,1}^\top\Big(I_{n_z} - (\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\Big)\check{\mathcal{Z}}_n(t) + (\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}u(t) + e_n^\flat(t).
\end{aligned} \tag{5.42}$$

The truncated error in the time domain or in the Laplace domain is identified

$$\begin{cases}
e_n^\flat(t) = \tilde{z}_{2(n+1)}(t,1) - (\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}\partial_\theta\tilde{z}_{2(n+1)}(t,1), \\
E_n^\flat(s) = \Big(\tilde{G}_{2(n+1)}(s,1) - (\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}\partial_\theta\tilde{G}_{2(n+1)}(s,1)\Big)U(s).
\end{cases} \tag{5.43}$$

The dynamics of the first odd Legendre coefficients $\check{\mathcal{Z}}_n(t)$ along the trajectories of system $(\mathcal{S}_{2\infty})$ are given by (5.32) recalled hereafter

$$\check{\mathcal{I}}_n^{-1}\dot{\check{\mathcal{Z}}}_n(t) = \left(\lambda\check{\mathcal{I}}_n^{-1} + \nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\hat{\mathcal{L}}_n\right)\check{\mathcal{Z}}_n(t) + 2\nu\ell_{n,1}u(t) - 2\nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}z(t,1),$$

Then, replacing $z(t,1)$ by the expression (5.42), one obtains

$$\begin{aligned}
\check{\mathcal{I}}_n^{-1}\dot{\check{\mathcal{Z}}}_n(t) \;=\; &\left(\lambda\check{\mathcal{I}}_n^{-1} + \nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\hat{\mathcal{L}}_n - 2\nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\ell_{n,1}^\top\right)\check{\mathcal{Z}}_n(t) \\
&+2\nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\ell_{n,1}^\top(\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t) \\
&+2\nu\left(\ell_{n,1} - \check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}(\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}\right)u(t) - 2\nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}e_n^\flat(t),
\end{aligned} \tag{5.44}$$

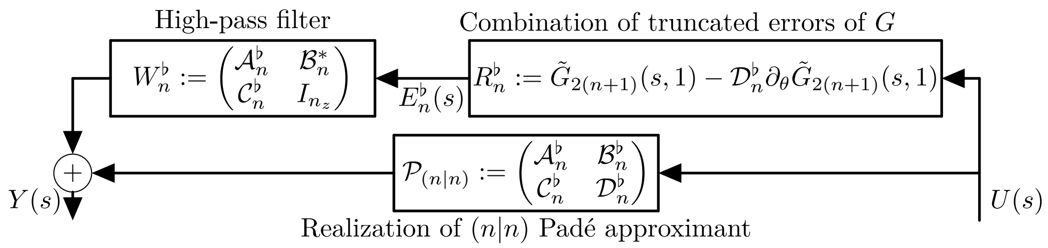


Figure 5.4: Legendre-modelling II of the reaction-diffusion equation $(\mathcal{S}_{2\infty})$.

where we simplify $\nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n(\hat{\mathcal{L}}_n - 2\ell_{n,1}\ell_{n,1}^\top) = -\nu\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top$ according to (5.19) and recognize (5.38a). The PDE part is finally issued from the previous calculations (5.35) with the slight difference that the last term is expressed in terms of $\check{\mathcal{Z}}_n(t)$, $u(t)$ and $e_n^\flat(t)$

$$\partial_t\tilde{z}_{2n}(t,\theta) = (\lambda + \nu\partial_{\theta\theta})\tilde{z}_{2n}(t,\theta) + \ell_n^\top(\theta)\left((\lambda + \nu\check{\mathcal{I}}_n\hat{\mathcal{L}}_n^\top\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top)\check{\mathcal{Z}}_n(t) - \dot{\check{\mathcal{Z}}}_n(t)\right),$$

$$= (\lambda + \nu\partial_{\theta\theta})\tilde{z}_{2n}(t,\theta) + 2\nu\ell_n^\top(\theta)\check{\mathcal{I}}_n \begin{pmatrix} \left(\ell_{n,1}\ell_{n,1}^\top - \check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\ell_{n,1}^\top(\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}\right)\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\check{\mathcal{Z}}_n(t) \\ -\left(\ell_{n,1} - \check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}(\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}\right)u(t) \\ +\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}e_n^\flat(t) \end{pmatrix}.$$

One recognizes PDE dynamics (5.38b) and concludes the proof. □

As previously, we able to describe the infinite-dimensional residual part in the time domain by the PDE dynamics and in the Laplace domain by an input-output transfer function.

Such a model is depicted in Figure 5.4. It is similar to the model I but has the advantage to put aside an error of order $n+1$ at the price of not beeing strictly causal anymore.

This effort to move to order $n+1$ is rewarded by the following properties.

Property 5.3. *Input-output models* (5.20) *and* (5.38) *are identical.*

Proof. The proof is trivial since the state, input, output, and transfer direct matrices are identical. □

In fact, we found a method to obtain *tau* models allowing us to keep into account the error term.

Furthermore, thanks to the frequency viewpoint, such a modelling is once again related to the Padé model. This time, the numerator and denominator of the rational approximation are both of order n.

Property 5.4. *System* $\begin{pmatrix} \mathcal{A}_n^\flat & \mathcal{B}_n^\flat \\ \mathcal{C}_n^\flat & \mathcal{D}_n^\flat \end{pmatrix}$ *is a realization of the* $(n|n)$ *Padé approximant of* $H(s) = \frac{\tanh\left(\frac{1}{2}\sqrt{\frac{s-\lambda}{\nu}}\right)}{\sqrt{\frac{s-\lambda}{\nu}}}I_{n_z}$ *and the corresponding Padé remainder is given by the following expression*

$$\tilde{\mathcal{P}}_{(n|n)}(s) = \underbrace{\left(I_{n_z} + \mathcal{C}_n^\flat(sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^*\right)}_{W_n^\flat(s)}\underbrace{\left(\tilde{G}_{2(n+1)}(s,1) - \mathcal{D}_n^\flat\partial_\theta\tilde{G}_{2(n+1)}(s,1)\right)}_{R_n^\flat(s)}. \tag{5.45}$$

where matrix $\mathcal{B}_n^*$ *is given in* (5.37) *and the other matrices are given by*

$$\begin{aligned} \mathcal{A}_n^\flat &= \lambda I_{nn_z} + \nu\check{\mathcal{I}}_n(-\check{\mathcal{L}}_n + 2\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\ell_{n,1}^\top\mathcal{D}_n^\flat)\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top, & \mathcal{B}_n^\flat &= 2\nu\left(\ell_{n,1} - \check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}\mathcal{D}_n^\flat\right), \\ \mathcal{C}_n^\flat &= \ell_{n,1}^\top(I_{n_z} - \mathcal{D}_n^\flat\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top), & \mathcal{D}_n^\flat &= \mathfrak{d}_n = (\ell_{n+1,1}^\top\hat{\mathcal{I}}_{n+1}2\ell_{n+1,1})^{-1}, \end{aligned} \tag{5.46}$$

and where $\tilde{G}_{2(n+1)}(s)$ *is the Legendre truncated error of function* $G(s)$ *at order* $n+1$.

Proof. In the Laplace domain, the output $Y(s)$ of system (5.29) is decomposed as

$$Y(s) = \left(\mathcal{C}_n^\flat(sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^\flat + \mathcal{D}_n^\flat\right)U(s) + \left(I_{n_z} + \mathcal{C}_n^\flat(sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^*\right)E_n^\flat(s).$$

Moreover, according to (5.31), we have $E_n(s) = \tilde{G}_{2n}(s,1)U(s)$. Therefore, the transfer function from U to Y is given by

$$\begin{aligned}H(s) = &\underbrace{\left(\mathcal{C}_n^\flat(sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^\flat + \mathcal{D}_n^\flat\right)}_{H_n^\flat(s)} \\ &+ \underbrace{\left(I_{n_z} + \mathcal{C}_n^\flat(sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^*\right)\left(\tilde{G}_{2(n+1)}(s,1) - \mathcal{D}_n^\flat\partial_\theta\tilde{G}_{2(n+1)}(s,1)\right)}_{\tilde{H}_n^\flat(s)}.\end{aligned}$$

The objective is now to prove that $\tilde{H}_n^\flat(s)$ is small when s tends to λ. As in the previous section, we denote $\mathfrak{A}_n = -\nu\check{\mathcal{I}}_n\check{\mathcal{L}}_n\check{\mathcal{I}}_n$, which is a lower triangular matrix with non-null diagonal coefficients. Using the relation $\check{\mathcal{L}}_n^\top + \hat{\mathcal{L}}_n = 2\ell_{n,1}\ell_{n,1}^\top$ in (5.19), we have

$$\begin{aligned}I_{n_z} + \mathcal{C}_n^\flat(sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^* &= I_{n_z} + \mathcal{C}_n^\flat\left((s-\lambda)I_{nn_z} - \mathfrak{A}_n\check{\mathcal{L}}_n^\top + 2\mathfrak{A}_n\ell_{n,1}\ell_{n,1}^\top\mathcal{D}_n^\flat\hat{\mathcal{I}}_n\check{\mathcal{L}}_n^\top\right)^{-1}2\mathfrak{A}_n\ell_{n,1}, \\ &= I_{n_z} + \mathcal{C}_n^\flat\left((s-\lambda)I_{nn_z} + \mathfrak{A}_n\hat{\mathcal{L}}_n - 2\mathfrak{A}_n\ell_{n,1}\mathcal{C}_n^\flat\right)^{-1}2\mathfrak{A}_n\ell_{n,1}.\end{aligned}$$

The technical Lemma B.4, postponed in Appendix B.2, is applied with $L = \mathfrak{A}_n\hat{\mathcal{L}}_n$, $u = -2\mathfrak{A}_n\ell_{n,1}$, and $v = \mathcal{C}_n^{\flat\top}$ and gives

$$I_{n_z} + \mathcal{C}_n^\flat(sI_{nn_z} - \mathcal{A}_n^\flat)^{-1}\mathcal{B}_n^* = \underset{s\to\lambda}{O}((s-\lambda)^n).$$

Moreover, noticing that

$$\left\|\partial_\theta^{(2n+3)}G(s)\right\|_\infty = \sup_{[0,1]}\left|\left(\frac{s-\lambda}{\nu}\right)^{(n+1)}\frac{\sinh(\sqrt{\frac{s-\lambda}{\nu}}(\theta-1))}{\cosh(\sqrt{\frac{s-\lambda}{\nu}})}\right| \le \left|\frac{s-\lambda}{\nu}\right|^{(n+1)},$$

holds, the application of Lemma 2.2 and 2.3 at order $d = 2n+3$ leads respectively to $\tilde{G}_{2(n+1)}(s,1) = \underset{s\to\lambda}{O}((s-\lambda)^{n+1})$ and $\partial_\theta\tilde{G}_{2(n+1)}(s,1) = \underset{s\to\lambda}{O}((s-\lambda)^{n+1})$. Thus, $\tilde{H}_n^\flat(s) = \underset{s\to\lambda}{O}((s-\lambda)^{2n+1})$ holds and, from Definition 5.1, the identification of $H_n^\flat(s) := \mathcal{P}_{(n|n)}(s)$ and $\tilde{H}_n^\flat(s) := \tilde{\mathcal{P}}_{(n|n)}(s)$ is made possible. □

Model II can be viewed as an extension of model I as the finite-dimensional part is more accurate in terms of eigenvalues as it will be seen in the next chapter. Nevertheless, in terms of stability analysis, for simplicity reasons, only model I will be taken into account.

5.3 Proposed models for ODE-reaction-diffusion systems

Consider the ODE-reaction-diffusion interconnected system $(\mathcal{S}_2)$ recalled below

$$\begin{cases}\dot{x}(t) = Ax(t) + Bz(t,1), & \forall t \in \mathbb{R}_+, \\ \partial_t z(t,\theta) = (\nu\partial_{\theta\theta} + \lambda)z(t,\theta), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \\ z(t,\tfrac{1}{2}) = 0, & \forall t \in \mathbb{R}_+, \\ \partial_\theta z(t,1) = Cx(t), \quad \partial_\theta z(t,0) = Cx(t), & \forall t \in \mathbb{R}_+, \\ \left[\begin{smallmatrix} x(0) \\ z(0,\theta)\end{smallmatrix}\right] = \left[\begin{smallmatrix} x_0 \\ z_0(\theta)\end{smallmatrix}\right] \in \mathcal{D}_2 := \left\{\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right] \in \mathbb{R}^{n_x} \times H^2(0,1;\mathbb{R}^{n_z}) \text{ s.t. } \left[\begin{smallmatrix} z(\frac{1}{2}) \\ \partial_\theta z(1) \\ z_0(\theta)\end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ Cx \\ -z_0(1-\theta)\end{smallmatrix}\right]\right\}. \end{cases} \tag{$\mathcal{S}_2$}$$

In this section, the previous developments are followed to design models that can be used to pursue stability analysis of system $(\mathcal{S}_2)$. A focus is made on the trigonometric model (5.7) and Legendre polynomial model given by (5.29) once interconnected with the finite-dimensional system $\left(\begin{smallmatrix} A & B \\ C & 0\end{smallmatrix}\right)$.

5.3.1 Extension of trigonometric models to ODE-reaction-diffusion systems

Supplying on the eigenfunctions of the reaction-diffusion system ($\mathcal{S}_{2\infty}$) expands the finite-dimensional part. As the order n increases, the first trigonometric coefficient $\mathcal{Z}_n^\natural$ defined by (5.4) are incorporated to the finite-dimensional state x. The corresponding augmented model is given below.

Proposition 5.6. *For any order n in $\mathbb{N}$, system ($\mathcal{S}_2$) can be modeled as follows*

$$\begin{cases} \dot{\xi}_n^\natural(t) = \mathbf{A}_n^\natural \xi_n^\natural(t) + \mathbf{B}_n^\natural e_n^\natural(t), & \forall t \in \mathbb{R}_+, \\ \partial_t \tilde{z}_n^\natural(t,\theta) = (\lambda + \nu\partial_{\theta\theta})\tilde{z}_n^\natural(t,\theta) + 2\nu\phi_n^{\natural\top}(\theta)\big(\phi_n^\natural(1) - \phi_n^\natural(0)\big)u(t), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \\ \tilde{z}_n^\natural(t,\tfrac{1}{2}) = 0, & \forall t \in \mathbb{R}_+, \\ \partial_\theta \tilde{z}_n^\natural(t,1) = u(t), \quad \partial_\theta \tilde{z}_n^\natural(t,0) = u(t), & \forall t \in \mathbb{R}_+, \\ u(t) = \mathbf{C}_n^\natural \xi_n^\natural(t), & \forall t \in \mathbb{R}_+, \\ e_n^\natural(t) = \tilde{z}_n^\natural(t,1), & \forall t \in \mathbb{R}_+, \end{cases} \tag{$\mathcal{S}_{2n}^\natural$}$$

where the matrices are given by

$$\mathbf{A}_n^\natural = \begin{bmatrix} A & B\phi_n^{\natural\top}(1) \\ 2\nu\big(\phi_n^\natural(1) - \phi_n^\natural(0)\big)C & \Lambda_n \end{bmatrix}, \quad \mathbf{B}_n^\natural = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \mathbf{C}_n^\natural = \begin{bmatrix} C & 0 \end{bmatrix}, \tag{5.47}$$

The initial condition is given by

$$\begin{cases} \xi_n^\natural(0) = \begin{bmatrix} x_0 \\ 2\int_0^1 \phi_n^\natural(\theta) z_0(\theta)\mathrm{d}\theta \end{bmatrix}, \\ \tilde{z}_n^\natural(0,\theta) = z_0(\theta) - 2\ell_n^\top(\theta)\displaystyle\int_0^1 \phi_n^\natural(\theta) z_0(\theta)\mathrm{d}\theta, \quad \forall \theta \in [0,1], \end{cases} \tag{5.48}$$

where $\left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right]$ belongs to the set $\mathcal{D}_2 = \left\{ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in \mathbb{R}^{n_x} \times H^2(0,1;\mathbb{R}^{n_z}) \text{ s.t. } \left[\begin{smallmatrix} z(\frac{1}{2}) \\ \partial_\theta z(1) \\ z_0(\theta) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ Cx \\ -z_0(1-\theta) \end{smallmatrix}\right] \right\}$.

Proof. The proof is simply a Redheffer product of the finite-dimensional part ($\mathcal{S}_2$a) with the model of the reaction-diffusion equation ($\mathcal{S}_{2\infty}$) proposed in Proposition 5.2. □

It is the most intuitive way to model an ODE-reaction-diffusion system and has been used in [144, 145]. The finite-dimensional part is governed by the dynamics of $\xi_n^\natural = \left[\begin{smallmatrix} x \\ \mathcal{Z}_n^\natural \end{smallmatrix}\right]$, where x is the ODE state and $\mathcal{Z}_n^\natural$ are the first PDE spectral coefficients.

Note that, by interconnection with the finite-dimensional system $\left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$, it is no longer possible to talk about spectral modelling. Indeed, the trigonometric basis which has been used to extend the state is no more the eigenbasis of the whole interconnected system.

Insofar as the spectral approach does not longer exhibit spectral properties of the interconnected system, one can wonder if pseudo-spectral approaches would regain interest.

5.3.2 Extension of Legendre polynomial models to ODE-reaction-diffusion systems

A less intuitive approach consists in using a polynomial basis. Considering our strictly-causal model I given by Proposition 5.4 for the reaction-diffusion system ($\mathcal{S}_{2\infty}$), we expand the finite-dimensional state x with the first Legendre polynomial coefficients $\check{\mathcal{Z}}_n$. Such an extension leads to the following Legendre polynomial model.

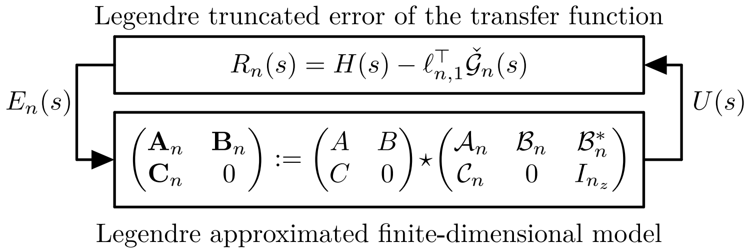


Figure 5.5: Legendre-modelling of the ODE-reaction-diffusion interconnected system $(\mathcal{S}_2)$.

Proposition 5.7. *For any order n in $\mathbb{N}$, system $(\mathcal{S}_2)$ can be modeled as follows*

$$\begin{cases} \dot{\check{\xi}}_n(t) = \mathbf{A}_n \check{\xi}_n(t) + \mathbf{B}_n e_n(t), & \forall t \in \mathbb{R}_+, \\ \partial_t \tilde{z}_{2n}(t,\theta) = (\lambda + \nu\partial_{\theta\theta})\tilde{z}_{2n}(t,\theta) - 2\nu \check{\ell}_n^\top(\theta)\check{\mathcal{I}}_n \Big(\ell_{n,1}\tilde{\mathbf{C}}_n\check{\xi}_n(t) + \check{\mathcal{L}}_n\check{\mathcal{I}}_n\ell_{n,1}e_n(t)\Big), & \forall (t,\theta) \in \mathbb{R}_+ \times [0,1], \\ \tilde{z}_{2n}(t,\tfrac{1}{2}) = 0, & \forall t \in \mathbb{R}_+, \\ \partial_\theta \tilde{z}_{2n}(t,1) = \tilde{\mathbf{C}}_n\check{\xi}_n(t), \quad \partial_\theta \tilde{z}_{2n}(t,0) = \tilde{\mathbf{C}}_n\check{\xi}_n(t), & \forall t \in \mathbb{R}_+, \\ u(t) = \mathbf{C}_n\check{\xi}_n(t), & \forall t \in \mathbb{R}_+, \\ e_n(t) = \tilde{z}_{2n}(t,1), & \forall t \in \mathbb{R}_+, \end{cases} \tag{$\mathcal{S}_{2n}$}$$

where the matrices are given by

$$\mathbf{A}_n = \begin{bmatrix} A & B\mathcal{C}_n \\ \mathcal{B}_n C & \mathcal{A}_n \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} B \\ \mathcal{B}_n^* \end{bmatrix}, \quad \mathbf{C}_n = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \tilde{\mathbf{C}}_n = \begin{bmatrix} C & -\ell_{n,1}^\top \check{\mathcal{I}}_n \check{\mathcal{L}}_n^\top \end{bmatrix}, \tag{5.49}$$

and where matrices $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{C}_n$ and $\mathcal{B}_n^*$ are given by (5.37). For any $\left[\begin{smallmatrix} x_0 \\ z_0 \end{smallmatrix}\right]$ in $\mathcal{D}_2$, the initial condition is given by

$$\begin{cases} \check{\xi}_n(0) = \begin{bmatrix} x_0 \\ \check{\mathcal{I}}_n \int_0^1 \check{\ell}_n(\theta) z_0(\theta)\mathrm{d}\theta \end{bmatrix}, \\ \tilde{z}_{2n}(0,\theta) = z_0(\theta) - \check{\ell}_n^\top(\theta)\check{\mathcal{I}}_n \displaystyle\int_0^1 \check{\ell}_n(\theta) z_0(\theta)\mathrm{d}\theta, \quad \forall \theta \in [0,1]. \end{cases} \tag{5.50}$$

This modelling is depicted in Figure 5.5. In the lower part, the finite-dimensional part embeds the dynamics of the first Legendre polynomial coefficients. Thanks to Property 5.2, it can be seen as a Padé approximation and have nice convergence properties when s tends to λ. In the upper part, the infinite-dimensional part has been preserved and can be described analytically.

Conclusion

In this chapter, a particular attention has been paid to the definition of two augmented models for ODE-reaction-diffusion system $(\mathcal{S}_2)$. The first one $(\mathcal{S}_{2n}^\natural)$ has been issued from the spectral method on a trigonometric basis. The second one $(\mathcal{S}_{2n})$ has been issued from the Legendre-*tau* method. Interestingly, the use of Legendre polynomials provides an interpretation of the projection operation, strongly linked with Padé approximations, and enables describing the infinite-dimensional residual part.

In the next chapter, we look at classic finite-dimensional tools to analyze the stability of system $(\mathcal{S}_2)$ and compare the relevance and effectiveness of our two extended models $(\mathcal{S}_{2n}^\natural)$ and $(\mathcal{S}_{2n})$.

Chapter 6

Stability analysis of ODE-reaction-diffusion systems

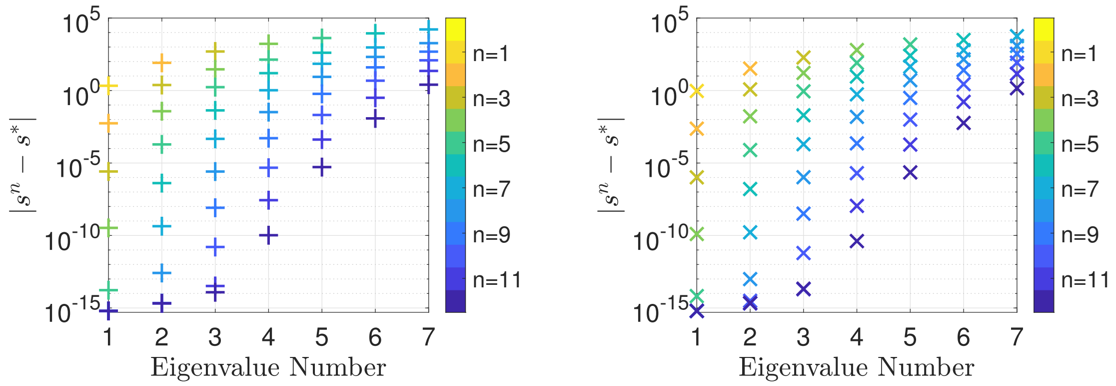
"Quantitative laws express a sameness beneath sensed difference, a constancy beneath the change."
Reality & Stability: From Parmenides to Einstein, R. Tallis.

Contents

ENCOURAGED by the stability results for ODE-transport interconnected systems obtained in Chapter 4, an extension to other ODE-PDE interconnected systems is envisioned. The ways of approaching stability follow the same tracks. The main idea is to apply finite-dimensional criteria to the augmented systems proposed in Chapter 5. From one side, input-output stability analysis [57] is investigated to take maximum advantage of the input-output relations of the approximated model. From the other side, Lyapunov stability analysis [144, 145, 187] is pursued on trigonometric and polynomial augmented models to derive several sufficient conditions of stability depending on the approximated order n, and we aim to answer, at least partially, the following questions.

- *How do evaluate the stability or instability of ODE-PDE interconnected systems?*
- *What are the reasons to choose a model rather than another to address stability properties?*
- *What are the challenges and difficulties to overcome to achieve the same results as for ODE-transport interconnected systems?*

This chapter is dedicated to the stability analysis of an ODE interconnected with a reaction-diffusion PDE either in the frequency or time domains. Firstly, spectral properties and input-output stability properties are discussed. Secondly, Lyapunov arguments are followed to obtain under and over estimates of the stability regions via linear matrix inequality test and positivity test, respectively. Questions about the hierarchy and convergence are still pending and avenues for future research are then discussed.



(a) Accuracy of Legendre model I.

(b) Accuracy of Legendre model II.

Figure 6.1: Accuracy of the eigenvalues with respect to the order n.

6.1 Input-output stability analysis

In the Laplace domain, similar conclusions as the one drawn in Chapter 4 can be obtained. From one side, the eigenvalues of the approximated matrix of systems $(\mathcal{S}_{2n})$ converge to the expected characteristic roots of system $(\mathcal{S}_2)$. On the other side, considering the whole modeling $(\mathcal{S}_{2n})$, the application of the small gain theorem leads to a sufficient condition of stability.

6.1.1 Characteristic roots approximation

Focusing on the reaction-diffusion system $(\mathcal{S}_{2\infty})$, the characteristic roots can be given analytically. They are denoted $s_k^* = \lambda - \nu\pi^2(2k+1)^2$, for any $k \in \mathbb{N}$, and correspond to the diagonal coefficients of matrix Λ_n introduced through the trigonometric spectral model (5.7).
However, the finite-dimensional part $\mathcal{A}_n$ and $\mathcal{A}_n^\flat$ of Legendre models I and II issued from revisited pseudo-spectral methods, have also been presented as approximated models. Indeed, with the benefit of realizing Padé approximated transfer functions, the two Legendre models are in capacity to approximate the expected characteristic roots $\{s_k^*\}_{k\in\mathbb{N}}$.

The n first eigenvalues of matrices $\mathcal{A}_n$ or $\mathcal{A}_n^\flat$ (denoted $\{s_k^n\}_{k\in\{1,\ldots,n\}}$) and the expected characteristic roots of the reaction-diffusion (denoted $\{s_k^*\}_{k\in\{1,\ldots,n\}}$) are compared. First, it is worth noticing that $\{s_k^n\}_{k\in\{1,\ldots,n\}}$ and $\{s_k^*\}_{k\in\{1,\ldots,n\}}$ are dispatched on the real axis and deviated more and more to the left as the eigenvalue number k increases. Then, the norm of the error $\{|s_k^n - s_k^*|\}_{k\in\{1,\ldots,n\}}$ is computed. On Figure 6.1, such an error is reported in function of the eigenvalue number k with colored plus and cross signs for $\mathcal{A}_n$ and $\mathcal{A}_n^\flat$, respectively. For any eigenvalue number k, both errors are converging to zero as n increases. Moreover, the convergence is faster for small numbers k. This means that the eigenvalues close to $s = \lambda$ converge better than those more distant. Moving forward, the convergence of $(n-1|n)$ and $(n|n)$ Padé approximants on $\mathcal{B}(\lambda, \nu\pi^2)$ can be interpreted as the convergence of the error to zero, at least for $k = 1$.
Lastly, the error made by the eigenvalues of matrices $\mathcal{A}_n$ and $\mathcal{A}_n^\flat$ are confronted in Figure 6.1a and 6.1b. The comparison is fine but we can see that all $\times$ signs on the right are lower than $+$ signs on the left. For a given order n, the root locus of matrix $\mathcal{A}_n^\flat$ is slightly more accurate than for the one of $\mathcal{A}_n$. This is totally logical since Legendre model II comes from the $(n|n)$ Padé rational approximant (see Property 5.4) whereas the strictly-causal Legendre model I is limited to $(n-1|n)$ Padé rational approximant (see Property 5.2).

Returning to the interconnection with the finite-dimensional system, augmented systems $(\mathcal{S}_{2n}^\natural)$ and $(\mathcal{S}_{2n})$ allow us to deduce information on the spectrum. The main difference is that the finite-dimensional part of the trigonometric model $\mathbf{A}_n^\natural$ and polynomial model $\mathbf{A}_n$ are next to the original system and do not provide the exact location of the characteristic roots. However, focusing on matrix $\mathbf{A}_n$, convergence properties of its eigenvalues towards the expected characteristic roots could be proven as the approximated order n grows.

Theorem 6.1. *If system $(\mathcal{S}_2)$ contains K characteristic roots s_k^* with multiplicities μ_k^*, for k in $\{1,\ldots,K\}$, inside the open ball $\mathcal{B}(\lambda, r)$ for any radius $0 \le r \le \nu\pi^2$, then $\sum_{k=1}^{K} \mu_k^*$ eigenvalues $\{s_{k,i}^n\}_{\substack{k\in\{1,\ldots,K\}\\ i\in\{\mu_1^*,\ldots,\mu_K^*\}}}$ of matrix $\mathbf{A}_n$ converges towards them.*
More precisely, for any $\varepsilon > 0$, there exists an order n^* such that

$$\max_{\substack{k\in\{1,\ldots,K\}\\ i\in\{\mu_1^*,\ldots,\mu_k^*\}}} \left|s_{k,i}^n - s_k^*\right| \le \varepsilon, \quad \forall n \ge n^*. \tag{6.1}$$

Proof. The proof provided in Chapter 4 Section 4.1 can be conducted on the ball $\mathcal{B}(\lambda, \nu\pi^2)$. □

Note that, due to the radius of convergence of Padé approximation, the proof of convergence for polynomial model $(\mathcal{S}_{2n})$ can be conducted only on open balls $\mathcal{B}(\lambda, \nu\pi^2)$. This result is already interesting since it means that the most unstable characteristic roots can be approximated accurately. Like for ODE-transport interconnected systems, from a certain order, it could lead to a test on the eigenvalues of matrices $\mathbf{A}_n^\natural$ or $\mathbf{A}_n$ to ensure the stability of system $(\mathcal{S}_2)$. Here, such an order n^* has not been given analytically.
Numerical simulation has not been proposed to the extent that the root locus is unknown but the comments given previously could be maintained.

In the sequel, a sufficient stability condition is derived by applying the small-gain theorem.

6.1.2 Frequency-sweeping test

From robust analysis, when a finite-dimensional system is interconnected with an uncertainty, the application of the small gain theorem leads to an initial stability result [140]. In the context of a system $\left(\begin{smallmatrix} A & B \\ C & 0\end{smallmatrix}\right)$ interconnected with a reaction-diffusion transfer function $H(s) = \frac{\tanh\left(\frac{1}{2}\sqrt{\frac{s-\lambda}{\nu}}\right)}{\sqrt{\frac{s-\lambda}{\nu}}} I_{n_z}$, the small gain criterion $|\left(\begin{smallmatrix} A & B \\ C & 0\end{smallmatrix}\right)|_{\mathcal{H}_\infty} |H|_{\mathcal{H}_\infty} \le 1$ is a sufficient stability condition for system $(\mathcal{S}_2)$. Nevertheless, this condition imposes that the two interconnected system are GES, which is extremely restrictive.

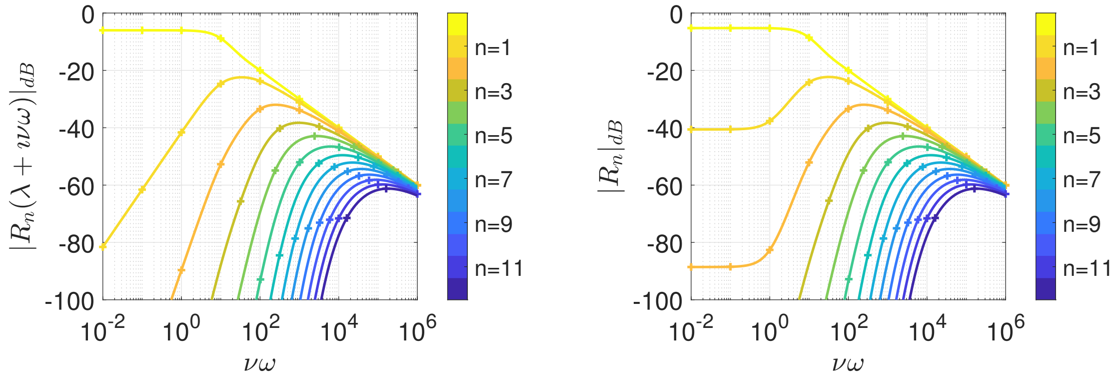
To go beyond these limitations, one uses augmented models such as system $(\mathcal{S}_{2n})$ which takes support on a Legendre polynomial basis. As developed and explained in the previous chapter, the modelling $(\mathcal{S}_{2n})$ enriches the finite-dimensional part by the first Legendre polynomials dynamics while keeping into consideration the infinite-dimensional error. Referring to Property 5.2, the finite-dimensional part turns out to be a Padé approximation around $s = \lambda$ but the constructional method also split the Padé remainder into two parts

$$\tilde{\mathcal{P}}_{(n-1|n)}(s) = \underbrace{\left(I_{n_z} + \mathcal{C}_n(sI_{nn_z} - \mathcal{A}_n)^{-1}\mathcal{B}_n^*\right)}_{W_n(s) = \underset{s\to\lambda}{O}((s-\lambda)^n)} \quad \underbrace{\tilde{G}_{2n}(s,1)}_{R_n(s) = \underset{s\to\lambda}{O}((s-\lambda)^n)} \quad . \tag{6.2}$$

On the left, it provides the design of a substantive finite-dimensional system W_n. On the right, the remaining part is kept into account and can be given analytically, for instance by

$$R_n(s) = \tilde{G}_{2n}(s,1) = H(s) - \ell_{n,1}^\top \left((s-\lambda)I_{nnz} - \nu\check{\mathcal{I}}_n\check{\mathcal{L}}_n\hat{\mathcal{I}}_n\hat{\mathcal{L}}_n\right)^{-1} 2\nu\check{\mathcal{I}}_n \left(\ell_{n,1} - \check{\mathcal{L}}_n\hat{\mathcal{I}}_n\ell_{n,1}H(s)\right). \tag{6.3}$$

Figure 6.2a shows the $\mathcal{H}_\infty$ norm of $R_n(\lambda + \imath\nu\omega)$ with respect to the frequencies ω independently of parameters λ and ν, for several approximated orders n. For $n = 0$, without taking into account any Legendre coefficients, the yellow plot corresponds to $|H(\lambda + \imath\nu\omega)|_{\mathcal{H}_\infty}$ which is inherently a strictly proper transfer function. Increasing the order n, the low-frequency behavior of H is shifted to the finite-dimensional part. Therefore, the infinite-dimensional left-over of R_n is very small for low frequencies and depicts H for high frequencies. More precisely, for low frequencies, we have proven that $R_n(s) = \underset{s\to\lambda}{O}((s-\lambda)^n)$ holds, which is reflected by a slope of $20n$dB by decade in low frequencies in Figure 6.2a. For high frequencies, we have $\lim_{s\to\infty} |R_n(s)| = \lim_{s\to\infty} |H(s)| = 0$. Transfer function $R_n(\lambda + \imath\nu\omega)$ is a band-pass filter, for any order $n \ge 1$.



(a) Magnitude of $\tilde{R}_n(\lambda + \nu\imath\omega)$. (b) Bode diagram of $\tilde{R}_n$ for $\lambda = 1$.

Figure 6.2: Representation of the error transfer function R_n of modelling $(\mathcal{S}_{2n})$.

Figure 6.2b depicts the $\mathcal{H}_\infty$ norm of $R_n(\imath\nu\omega)$ with respect to the frequencies ω, for $\lambda = 1$, any $\nu > 0$ and several orders n. The difference with Figure 6.2a is that a constant depending on λ is always present for very low frequencies. Such a constant is equal to $|R_n(0)|_{\mathcal{H}_\infty}$ and decreases as the order n increases.

This layout promotes an investigation track by applying the small gain theorem. As for ODE-transport interconnected systems, the splitting $\tilde{\mathcal{P}}_{(n-1|n)} = W_n R_n$ is essential to provide extensive stability conditions [210, 211]. The strategy is to encapsulate the transfer function $R_n(\imath\omega)$ into a ball $\mathcal{B}(0, \frac{1}{\gamma_n})$ and to consider it as an uncertainty Δ_n. By application of the small gain theorem, we obtain the following theorem.

Theorem 6.2. *For any order n in $\mathbb{N}$, if the $\mathcal{H}_\infty$ norm of system $\left(\begin{smallmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & 0 \end{smallmatrix}\right)$ is lower than γ_n then system $(\mathcal{S}_2)$ is GES.*

Proof. The proof is the same as the one provided in Chapter 4 Section 4.2. Assuming that the inequality $|\left(\begin{smallmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & 0 \end{smallmatrix}\right)|_{\mathcal{H}_\infty} < \gamma_n$ holds, we have the small-gain condition $|\left(\begin{smallmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & 0 \end{smallmatrix}\right)|_{\mathcal{H}_\infty} |R_n|_{\mathcal{H}_\infty} < 1$. Applying the small-gain theorem on the augmented system $(\mathcal{S}_{2n})$, the interconnected system $(\mathcal{S}_{2n})$ is input-output stable which means that the system $(\mathcal{S}_{2n})$ and so on system $(\mathcal{S}_2)$ is GES. $\square$

Compared to ODE-transport interconnected systems, the bound γ_n is ν-independent but λ-dependent and cannot be stored upstream in a table. The computation load is then impacted since γ_n has to be determined before doing each frequency-sweeping test and no numerical tests have been proposed here.

However, the frequency analysis is not to be forgotten and could lead to interesting and promising results. For instance, as the reaction-diffusion equation is strictly proper, we have hope that γ_n tends to 0 when n tends to infinity. The frequency test proposed in Theorem 6.2 could then be summarized as an eigenvalues test ensuring that $\mathbf{A}_n$ is Hurwitz for sufficiently larger orders. These avenues of research have not been pushed to their limits but could be very promising. Last but not least, μ-analysis and the use of integral quadratic constraints [26, 77] could reduce the conservatism of the stability test. These developments have not been pushed in this manuscript.

In the margin of the stability analysis, it should not be forgotten that *tau* models have been used for a long time in optimal control [166, 167]. In the case of strictly proper systems such that parabolic PDEs, the solution to the finite-dimensional Ricatti equation at order n converges towards the infinite-dimensional one when n tends to infinity [165]. Our frequency mapping mechanic could be put in the Ricatti format to synthesize controllers. Future work could be devoted to the estimation of the order from which the finite-dimensional controller stabilizes the infinite-dimensional system and to the comparison of the convergence rates with existing techniques.

Let us now focus on another way to assess the stability of system $(\mathcal{S}_2)$ using Lyapunov energy arguments.

6.2 Lyapunov stability analysis

In the time domain, the stability analysis of ODE-reaction-diffusion interconnected system ($\mathcal{S}_2$) can also be assessed by application of the Lyapunov theorem. As emphasized in Appendix A, one chooses a positive Lyapunov candidate functional $\mathcal{V}$ and the negativity of its derivatives along the trajectories of system ($\mathcal{S}_2$) is a sufficient condition of stability. In the opposite way, one builds a converse Lyapunov functional V such that its derivatives are equal to $\dot{V} = -|x|^2$ along the trajectories of system ($\mathcal{S}_2$) and the positivity of V is a sufficient condition of instability.

6.2.1 Linear matrix inequality test

Let us start with the sufficient condition of stability. For a quadratic Lyapunov candidate functional $\mathcal{V}$, the main objective is to guarantee the negativity of $\dot{\mathcal{V}}$ along the trajectories of system ($\mathcal{S}_2$). Due to the reaction term λ, which can be positive, the negativity can be obtained but requires one more step: the application of Wirtinger-like inequalities [127, 129].

Application of basic Wirtinger inequality

To begin with, for any P in $\mathbb{S}_+^{n_x}$, one takes support on a basic quadratic Lyapunov candidate functional

$$\mathcal{V}(x,z) = x^\top P x + \frac{1}{2\nu}\|z\|^2, \tag{6.4}$$

which is positive definite and continuous, with continuous derivatives.

By derivation along the trajectories of system ($\mathcal{S}_2$), calculations done for proving the existence of solutions (1.41) are recovered to obtain

$$\dot{\mathcal{V}}(x,z) = \begin{bmatrix} x \\ z(1) \end{bmatrix}^\top \begin{bmatrix} \mathcal{H}(PA) & PB + C^\top \\ * & 0 \end{bmatrix} \begin{bmatrix} x \\ z(1) \end{bmatrix} + \frac{\lambda}{\nu}\|z\|^2 - \|\partial_\theta z\|^2. \tag{6.5}$$

A first idea is to use a combination of the Jensen's inequality (i.e. Bessel's inequality at order $n = 0$) and Wirtinger's inequalities of the second kind [66] summed up below.

Lemma 6.1. *Assume $\lambda < \nu\pi^2$. For any z in $H^1(0,1;\mathbb{R}^{n_z})$, there exists a scalar $\varepsilon > 0$ such that the following inequality holds*

$$\frac{\lambda}{\nu}\|z\|^2 - \|\partial_\theta z\|^2 \leq -4\left(1 - \frac{\max(0, \frac{\lambda}{\nu} + \varepsilon)}{\pi^2}\right)|z(1)|^2 - \varepsilon\|z\|^2. \tag{6.6}$$

Proof. Using the boundary condition $z(\frac{1}{2}) = 0$ and applying the Wirtinger inequality of the second kind as in [161], we obtain a first Wirtinger-like inequality

$$\begin{aligned} \int_{\frac{1}{2}}^{1} |\partial_\theta z(\theta)|^2 \,\mathrm{d}\theta &\geq \pi^2 \int_{\frac{1}{2}}^{1} |z(\theta)|^2 \,\mathrm{d}\theta, \\ \frac{1}{2}\|\partial_\theta z\|^2 &\geq \pi^2 \frac{1}{2}\|z\|^2. \end{aligned} \tag{6.7}$$

For any $\frac{\lambda}{\nu} \in \mathbb{R}$, this implies that

$$\frac{\lambda}{\nu}\|z\|^2 - \|\partial_\theta z\|^2 \leq \left(\frac{\max(0, \frac{\lambda}{\nu} + \varepsilon)}{\pi^2} - 1\right)\|\partial_\theta z\|^2 - \varepsilon\|z\|^2, \tag{6.8}$$

holds. Then, using the fact that anti-symmetric relation $z(0) = -z(1)$, the application of modified Bessel inequality provided by Theorem 2.4 at order $n = 0$ gives

$$\|\partial_\theta z\|^2 \geq 4|z(1)|^2. \tag{6.9}$$

By selection of a sufficiently small scalar $\varepsilon > 0$ such that coefficient $\left(\frac{\max(0,\frac{\lambda}{\nu}+\varepsilon)}{\pi^2} - 1\right)$ is negative, inequality (6.8) is combined to (6.9) and yields the result. □

Theorem 6.3. *Assume $\lambda < \nu\pi^2$. If there exists matrix P in $\mathbb{S}_+^{n_x}$ such that the following linear matrix inequality is satisfied*

$$\begin{bmatrix} \mathcal{H}(PA) & PB + C^\top \\ * & -4\left(1 - \frac{\max(0,\frac{\lambda}{\nu})}{\pi^2}\right) \end{bmatrix} \prec 0, \tag{6.10}$$

then system $(\mathcal{S}_2)$ is GES.

Proof. By application of Lemma (6.1), the derivatives of the Lyapunov candidate functional (6.4) along the trajectories of system $(\mathcal{S}_2)$ given by (6.5) are upper bounded by

$$\dot{\mathcal{V}}(x,z) \leq \begin{bmatrix} x \\ z(1) \end{bmatrix}^\top \begin{bmatrix} \mathcal{H}(PA) & PB + C^\top \\ * & -4\left(1 - \frac{\max(0,\frac{\lambda}{\nu}+\varepsilon)}{\pi^2}\right) \end{bmatrix} \begin{bmatrix} x \\ z(1) \end{bmatrix} - \varepsilon \left\| z \right\|^2,$$

for a sufficiently small scalar $\varepsilon > 0$. The proof ends by application of the Lyapunov theorem. □

Theorem 6.4 proposes a first sufficient stability condition that is tractable by a semi-definite programming toolbox. Such a condition is really basic and can be seen as the counterpart of the input-output stability analysis where the reaction-diffusion equation is considered as a perturbation. Note that two necessary conditions appear. Firstly, as $\lambda < \nu\pi^2$, the PDE part must be stable. Secondly, as $\mathcal{H}(PA) \prec 0$ with $P \in \mathbb{S}_+^{n_x}$, the ODE part must be also stable. In order to relax these hypotheses and reduce the conservatism of the LMI (6.10), the proposed methodology consists in using the equivalent systems $(\mathcal{S}_{2n}^\natural)$ and $(\mathcal{S}_{2n})$ to $(\mathcal{S}_2)$ proposed in Chapter 5 and in enriching the Lyapunov candidate functional (6.4) in accordance.

Application of modified Wirtinger-Fourier inequality

The Lyapunov stability analysis on Fourier trigonometric basis resumes choosing a Lyapunov candidate functional quadratic with respect to the state $\xi_n^\natural$ in accordance with system $(\mathcal{S}_{2n}^\natural)$. For any order n in $\mathbb{N}$ and matrix $\mathbf{P}_n$ in $\mathbf{S}_+^{n_x+nn_z}$, consider then

$$\mathcal{V}_n^\natural(x,z) = \begin{bmatrix} x \\ 2\int_0^1 \phi_n^\natural(\theta) z(\theta)\mathrm{d}\theta \end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} x \\ 2\int_0^1 \phi_n^\natural(\theta) z(\theta)\mathrm{d}\theta \end{bmatrix} + \frac{1}{2\nu}\left\| \tilde{z}_n^\natural \right\|^2. \tag{6.11}$$

The advantage of working with system $(\mathcal{S}_{2n}^\natural)$ instead of $(\mathcal{S}_2)$ is the fact that the ODE-PDE interconnection is taken into account. Indeed, the state z is decomposed by its first trigonometric coefficients and its truncated error. With the help of the first coefficients, the interconnection between the state x and z is represented via extra-diagonal coefficients in matrix $\mathbf{P}_n$. For $n = 0$, the Lyapunov function (6.11) corresponds to the basic functional (6.4) but it adds degree of freedom as the order n increases.

With the splitting at order n, the basic Wirtinger inequality (6.7) can be rewritten and improved.

Lemma 6.2. *For any n in $\mathbb{N}$ and z in $H^1(0,1;\mathbb{R}^{n_z})$, the following inequality holds*

$$\left\| \partial_\theta \tilde{z}_n^\natural \right\|^2 \geq \left((2n+1)\pi\right)^2 \left\| \tilde{z}_n^\natural \right\|^2, \tag{6.12}$$

where $\tilde{z}_n^\natural$ is the Fourier truncated error of z.

Proof. On the left-hand part of the inequality, the L^2 norm is given by

$$\left\| \tilde{z}_n^\natural \right\|^2 = \sum_{k=n}^{\infty} 2\left(\int_0^1 \varphi_k^\natural(\theta) z(\theta)\mathrm{d}\theta\right)^2. \tag{6.13}$$

On the right-hand part of the inequality, the L^2 norm of the derivatives is given by

$$\left\|\partial_\theta \tilde{z}_n^\natural\right\|^2 = \sum_{k=n}^{\infty} 2\big((2k+1)\pi\big)^2 \left(\int_0^1 \varphi_k^\natural(\theta) z(\theta)\mathrm{d}\theta\right)^2. \tag{6.14}$$

Together, it yields the result. □

In addition, Jensen's inequality (6.9) needs to be revisited for the truncated error $\tilde{z}_n^\natural$. An extension via Bessel-Fourier inequality gives nothing but $\left\|\partial_\theta \tilde{z}_n^\natural\right\|^2 \geq 0$ because of the periodicity of trigonometric functions $\varphi_k^\natural$. The key comes from Cauchy-Schwarz inequality used in [128, 144].

Lemma 6.3. *For any n in $\mathbb{N}^*$ and z in $H^1(0,1;\mathbb{R}^{n_z})$, the following inequality holds*

$$\left\|\partial_\theta \tilde{z}_n^\natural\right\|^2 \geq (2n-1)\pi^2 \left|\tilde{z}_n^\natural(1)\right|^2, \tag{6.15}$$

where $\tilde{z}_n^\natural$ is the Fourier truncated error of z.

Proof. From the Fourier development of function $z \in H^1(0,1;\mathbb{R}^{n_z})$ provided by Definition 5.2 and evaluated at $\theta = 1$, we have

$$\left|\tilde{z}_n^\natural(1)\right|^2 = \left(\sum_{k=n}^{\infty} 2(-1)^k \int_0^1 \varphi_k^\natural(\theta) z(\theta)\mathrm{d}\theta\right)^2, \tag{6.16}$$

From Cauchy-Schwarz inequality, with $a_k = \frac{\sqrt{2}}{(2k+1)\pi}$ and $b_k = \sqrt{2}(-1)^k(2k+1)\pi \int_0^1 \varphi_k^\natural(\theta) z(\theta)\mathrm{d}\theta$, we obtain

$$\begin{aligned} \left|\tilde{z}_n^\natural(1)\right|^2 = \left(\sum_{k=n}^{\infty} a_k b_k\right)^2 &\leq \left(\sum_{k=n}^{\infty} a_k^2\right)\left(\sum_{k=n}^{\infty} b_k^2\right), \\ &\leq \left(\sum_{k=n}^{\infty} \frac{2}{\big((2k+1)\pi\big)^2}\right) \left\|\partial_\theta \tilde{z}_n^\natural\right\|^2. \end{aligned} \tag{6.17}$$

The series exists by integral test for convergence and since the state z belongs to $H^1(0,1;\mathbb{R}^{n_z})$. In addition, for any $n \geq 1$, the integral test for convergence gives

$$\left|\tilde{z}_n^\natural(1)\right|^2 \leq \left(\int_{x=n-1}^{\infty} \frac{2}{\big((2x+1)\pi\big)^2}\mathrm{d}x\right) \left\|\partial_\theta \tilde{z}_n^\natural\right\|^2 = \frac{1}{(2n-1)\pi^2} \left\|\partial_\theta \tilde{z}_n^\natural\right\|^2, \tag{6.18}$$

which ends the proof. □

With the help of these two lemmas, we obtain the following sufficient conditions of stability in the linear matrix inequality framework.

Theorem 6.4. *For a given order n in $\mathbb{N}^*$, assume $\lambda \leq \nu\big((2n+1)\pi\big)^2$. If there exists matrix $\mathbf{P}_n$ in $\mathbb{S}_+^{n_x+nn_z}$ such that the following linear matrix inequality is satisfied*

$$\Xi_n^\natural := \begin{bmatrix} \mathcal{H}(\mathbf{P}_n \mathbf{A}_n^\natural) & \mathbf{P}_n \mathbf{B}_n^\natural + \mathbf{C}_n^{\natural\top} \\ * & -(2n-1)\pi^2 \left(1 - \frac{\max(0,\frac{\lambda}{\nu})}{\left((2n+1)\pi\right)^2}\right) I_{n_z} \end{bmatrix} \prec 0, \tag{6.19}$$

then system $(\mathcal{S}_2)$ is GES. In addition, this condition is hierarchic with respect to n.

Proof. For the sake of simplicity, the time argument will be omitted in the following proof. Consider the Lyapunov functional $\mathcal{V}_n^\natural$ given by (6.11) constructed with the help of the n first Fourier coefficients.

Step 1: Positivity of the Lyapunov functional.

For any order n in $\mathbb{N}$, the Lyapunov functional $\mathcal{V}_n^\natural$ given by (6.11) satisfies

$$\alpha_1 \left|(x,z)\right|^2 \le \mathcal{V}_n^\natural(x,z) \le \alpha_2 \left|(x,z)\right|^2, \tag{6.20}$$

with positive scalars $\alpha_1 = \min(\underline{\sigma}(\mathbf{P}_n), \frac{1}{2\nu})$ and $\alpha_2 = \bar{\sigma}(\mathbf{P}_n) + \frac{1}{2\nu}$.

Step 2: Negativity of the derivative of the Lyapunov functional along $(\mathcal{S}_2)$.
Along the trajectories of system $(\mathcal{S}_2)$, the use of the augmented system $(\mathcal{S}_{2n}^\natural)$ leads to

$$\begin{aligned}\dot{\mathcal{V}}_n^\natural(x,z) = &\begin{bmatrix}\xi_n^\natural\\ z(1)\end{bmatrix}^\top \begin{bmatrix}\mathcal{H}(\mathbf{P}_n\mathbf{A}_n^\natural) & \mathbf{P}_n\mathbf{B}_n^\natural\\ * & 0\end{bmatrix}\begin{bmatrix}\xi_n^\natural\\ z(1)\end{bmatrix} + \int_0^1 (\frac{\lambda}{\nu} + \partial_{\theta\theta})\tilde{z}_n^{\natural\top}(\theta)\tilde{z}_n^\natural(\theta)\mathrm{d}\theta\\ &+ 2\left(\cancel{\int_0^1 \tilde{z}_n^{\natural\top}(\theta)\phi_n^{\natural\top}(\theta)\mathrm{d}\theta}\right)(\phi_n^\natural(1) - \phi_n^\natural(0))\mathbf{C}_n^\natural\xi_n^\natural,\end{aligned} \tag{6.21}$$

where the last term vanishes because of the orthogonality of Fourier trigonometric basis. By integration by parts, we obtain

$$\dot{\mathcal{V}}_n^\natural(x,z) = \begin{bmatrix}\xi_n^\natural\\ z(1)\end{bmatrix}^\top \begin{bmatrix}\mathcal{H}(\mathbf{P}_n\mathbf{A}_n^\natural) & \mathbf{P}_n\mathbf{B}_n^\natural + \mathbf{C}_n^{\natural\top}\\ * & 0\end{bmatrix}\begin{bmatrix}\xi_n^\natural\\ z(1)\end{bmatrix} + \frac{\lambda}{\nu}\left\|\tilde{z}_n^\natural\right\|^2 - \left\|\partial_\theta \tilde{z}_n^\natural\right\|^2. \tag{6.22}$$

where the augmented state is denoted $\xi_n^\natural = \begin{bmatrix} x \\ 2\int_0^1 \phi_n^\natural(\theta)z(\theta)\mathrm{d}\theta\end{bmatrix}$. By successive applications of Lemmas 6.2 and 6.3, we obtain

$$\dot{\mathcal{V}}_n^\natural(x,z) \le \begin{bmatrix}\xi_n^\natural\\ z(1)\end{bmatrix}^\top \begin{bmatrix}\mathbf{P}_n\mathbf{A}_n^\natural & \mathbf{P}_n\mathbf{B}_n^\natural + \mathbf{C}_n^{\natural\top}\\ * & -(2n-1)\pi^2\left(1 - \frac{\max(0,\frac{\lambda}{\nu}+\varepsilon)}{\left((2n+1)\pi\right)^2}\right)I_{n_z}\end{bmatrix}\begin{bmatrix}\xi_n^\natural\\ z(1)\end{bmatrix} - \varepsilon\left\|\tilde{z}_n^\natural\right\|^2, \tag{6.23}$$

for a sufficiently small scalar $\varepsilon > 0$. If LMI (6.19) holds, there exists $\alpha_3 = \min\left(\underline{\sigma}(\Xi_n^\natural), \varepsilon\right) > 0$ such that $\dot{\mathcal{V}}_n^\natural(x,z) \le -\alpha_3\left|(x,z)\right|^2$. Then, applying Lyapunov theorem, system $(\mathcal{S}_2)$ is GES.

Step 3: Hierarchy.
Lastly, since $\mathcal{V}_{n+1}^\natural$ is an enhancement of $\mathcal{V}_n^\natural$, it is possible to show the hierarchy of the LMIs. In other terms, if there exists $\mathbf{P}_n^\natural \succ 0$ such that $\Xi_n^\natural \prec 0$ is true, then there exists $\mathbf{P}_{n+1}^\natural = \begin{bmatrix}\mathbf{P}_n & 0\\ * & \frac{1}{4\nu}I_{n_z}\end{bmatrix} \succ 0$ such that $\Xi_{n+1}^\natural \prec 0$ is true. □

Theorem 6.4 provides sufficient conditions of stability for several orders n. The condition $\Xi_n^\natural \prec 0$ is a numerical stability criterion, which can be implemented via Matlab LMI toolbox. Moreover, we are able to reduce the conservatism as n increases.

Remark 6.1. Note that $\mathbf{A}_n$ Hurwitz is a necessary condition for the LMI condition $\Xi_n^\natural \prec 0$.

It is also worth noticing that the term of in second diagonal bloc is

$$\Gamma_n^\natural = (2n-1)\pi^2\left(1 - \frac{\max(0,\frac{\lambda}{\nu})}{\left((2n+1)\pi\right)^2}\right)I_{n_z} \succ 0, \tag{6.24}$$

and has its eigenvalues which grow in $\underset{n\to\infty}{O}(n)$. As in [127], the following alternative result is obtained.

Corollary 6.1. *If $\mathbf{A}_n^\natural$ is Hurwitz for n sufficiently large, then system $(\mathcal{S}_2)$ is GES.*

Proof. Firstly, if $\mathbf{A}_n^\natural$ is Hurwitz, it is proven in [127, Theorem 3.1] that there exists $\mathbf{P}_n \succ 0$ which does not depend on n such that $\mathcal{H}(\mathbf{P}_n\mathbf{A}_n^\natural) = -I_{n_x+nn_z}$. From Schur's complement, the LMI condition (6.19) is then rewritten as

$$\mathcal{H}(\mathbf{P}_n\mathbf{A}_n^\natural) - \mathbf{G}_n(\Gamma_n^\natural)^{-1}\mathbf{G}_n^\top \prec 0, \tag{6.25}$$

where $\Gamma_n^\natural$ is given by (6.24) and where

$$\mathbf{G}_n = \mathbf{P}_n\mathbf{B}_n + \mathbf{C}_n^{\natural\top}.$$

Since $\mathbf{G}_n = \underset{n\to\infty}{O}(1)$ and $\Gamma_n^\natural = \underset{n\to\infty}{O}(n)$, inequalities (6.25) holds for sufficiently large order n. Thus, there exists n^* in $\mathbb{N}^*$ such that Theorem 6.4 holds, for all $n > n^*$, and then system $(\mathcal{S}_2)$ is GES. □

This result presented in [127] is very interesting but cannot be exploited numerically since the sufficiently large order n cannot be quantified. In the numerical section, we will then only use our LMI condition (6.19) for low orders.

Concerning the convergence of the sufficient condition proposed in Theorem 6.4, we are more skeptical with regard to Chapter 2. Indeed, only simple convergence results are obtained with Fourier approximation due to non-periodic boundary conditions, which makes it difficult to adapt the procedure proposed in Section 4.3. That is why the Legendre polynomial approximation has also been regarded in the sequel.

Application of modified Wirtinger-Legendre inequality

Another way to find quadratic Lyapunov candidate functional is to use pseudo-spectral models. Focusing on Legendre augmented system $(\mathcal{S}_{2n})$, an enhanced Lyapunov functional is proposed. It takes support on the augmented state $\check{\xi}_n \in \mathbb{R}^{n_x+nn_z}$, which is composed of the finite-dimensional state x enriched by the n first odd Legendre polynomials coefficients. For any order n in $\mathbb{N}$ and matrix $\mathbf{P}_n$ in $\mathbf{S}_+^{n_x+nn_z}$, consider

$$\mathcal{V}_n(x,z) = \begin{bmatrix} x \\ \check{\mathcal{I}}_n \int_0^1 \check{\ell}_n(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} x \\ \check{\mathcal{I}}_n \int_0^1 \check{\ell}_n(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix} + \frac{1}{2\nu}\|\tilde{z}_{2n}\|^2. \tag{6.26}$$

As previously, this Lyapunov functional allows us to take into account more and more interconnected elements between the ODE and the PDE as the order n increases.

Thanks to Legendre polynomials properties, we can lean on the modified Wirtinger and Bessel inequalities introduced in Chapter 2. Both inequalities are recalled hereafter.

Lemma 6.4. *For any n in $\mathbb{N}^*$ and z in $H^1(0,1;\mathbb{R}^{n_z})$, the following inequality holds*

$$\|\partial_\theta \tilde{z}_{2n}\|^2 - (2\pi)^2\|\tilde{z}_{2n}\|^2 \geq \upsilon_n^W |\tilde{z}_{2n}(1)|^2, \tag{6.27}$$

where $\tilde{z}_{2n}$ is the Legendre truncated error of z and where

$$\upsilon_n^W = \left(4n(2n-1) + \frac{(2\pi)^2}{4n-1}\right) I_{n_z} \succ 0. \tag{6.28}$$

Lemma 6.5. *For any n in $\mathbb{N}$ and z in $H^1(0,1;\mathbb{R}^{n_z})$, the following inequality holds*

$$\|\partial_\theta \tilde{z}_{2n}\|^2 \geq \upsilon_n^B |\tilde{z}_{2n}(1)|^2, \tag{6.29}$$

where $\tilde{z}_{2n}$ is the Legendre truncated error of z and where

$$\upsilon_n^B = 4(n+1)(2n+1)I_{n_z} \succ 0. \tag{6.30}$$

Remark 6.2. Note that Wirtinger-Legendre inequality exposed in Lemma 6.4 is restrictive compared to Fourier-Wirtinger inequality in Lemma 6.2. Indeed, the Wirtinger constant is limited to $(2\pi)^2$. With a different approach developed in [77] based on the consideration of the Legendre approximation of function z', one could go beyond this limitation as n increases.

Subsequently, the following theorem can be established [17]. It can be seen as the counterpart of Theorem 6.4 when Fourier trigonometric are replaced by Legendre polynomials.

Theorem 6.5. *Take an order n in $\mathbb{N}^*$ and assume $\lambda < \nu(2\pi)^2$. If there exists matrix $\mathbf{P}_n$ in $\mathbb{S}_+^{n_x+nn_z}$ such that the following linear matrix inequality is satisfied*

$$\Xi_n := \begin{bmatrix} \mathcal{H}(\mathbf{P}_n\mathbf{A}_n) & \mathbf{P}_n\mathbf{B}_n + \mathbf{C}_n^\top \\ * & -\left(1 - \frac{\max(0,\frac{\lambda}{\nu})}{(2\pi)^2}\right)\upsilon_n^B - \frac{\max(0,\frac{\lambda}{\nu})}{(2\pi)^2}\upsilon_n^W \end{bmatrix} \prec 0, \tag{6.31}$$

with matrices $\upsilon_n^B, \upsilon_n^W$ given by (6.30) *and* (6.28)*, respectively, then system* ($\mathcal{S}_2$) *is GES. In addition, this condition is hierarchic with respect to n.*

Proof. For the sake of simplicity, the time argument will be omitted in the following proof. Consider the Lyapunov functional $\mathcal{V}_n$ given by (6.26) constructed with the help of the n first Legendre polynomials coefficients.

Step 1: Positivity of the Lyapunov functional.
For any order n in $\mathbb{N}$, the Lyapunov functional $\mathcal{V}_n$ given by (6.26) satisfies

$$\alpha_1 \left|(x,z)\right|^2 \leq \mathcal{V}_n(x,z) \leq \alpha_2 \left|(x,z)\right|^2, \tag{6.32}$$

with positive scalars $\alpha_1 = \min(\underline{\sigma}(\mathbf{P}_n), \frac{1}{2\nu})$ and $\alpha_2 = \bar{\sigma}(\mathbf{P}_n) + \frac{1}{2\nu}$.

Step 2: Negativity of the derivative of the Lyapunov functional along ($\mathcal{S}_2$).
Along the trajectories of system ($\mathcal{S}_2$), the use of the augmented system ($\mathcal{S}_{2n}$) leads to

$$\begin{aligned} \dot{\mathcal{V}}_n(x,z) &= \begin{bmatrix} \check{\xi}_n \\ z(1) \end{bmatrix}^\top \begin{bmatrix} \mathcal{H}(\mathbf{P}_n\mathbf{A}_n) & \mathbf{P}_n\mathbf{B}_n \\ * & 0 \end{bmatrix} \begin{bmatrix} \check{\xi}_n \\ z(1) \end{bmatrix} + \frac{\lambda}{\nu}\left\|\tilde{z}_{2n}\right\|^2 + \int_0^1 \tilde{z}_{2n}^\top(\theta)\partial_{\theta\theta}\tilde{z}_{2n}(\theta)\mathrm{d}\theta \\ &\quad -2\cancel{\left(\int_0^1 \tilde{z}_{2n}^\top(\theta)\check{\ell}_n^\top(\theta)\mathrm{d}\theta\right)}\check{\mathcal{I}}_n\left(\ell_{n,1}\partial_\theta\tilde{z}_{2n}(t,1) + \check{\mathcal{L}}_n\check{\mathcal{I}}_n\ell_{n,1}\tilde{z}_{2n}(t,1)\right), \end{aligned} \tag{6.33}$$

where the last term vanishes because of the orthogonality of Legendre polynomial basis. By integration by parts, we obtain

$$\dot{\mathcal{V}}_n(x,z) = \begin{bmatrix} \check{\xi}_n \\ z(1) \end{bmatrix}^\top \begin{bmatrix} \mathcal{H}(\mathbf{P}_n\mathbf{A}_n) & \mathbf{P}_n\mathbf{B}_n + \mathbf{C}_n^\top \\ * & 0 \end{bmatrix} \begin{bmatrix} \check{\xi}_n \\ z(1) \end{bmatrix} + \frac{\lambda}{\nu}\left\|\tilde{z}_{2n}\right\|^2 - \left\|\partial_\theta\tilde{z}_{2n}\right\|^2, \tag{6.34}$$

where the augmented state is denoted $\check{\xi}_n = \begin{bmatrix} x \\ \check{\mathcal{I}}_n\int_0^1 \phi_n^\natural(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix}$. By successive applications of Lemmas 6.2 and 6.3, we obtain

$$\dot{\mathcal{V}}_n(x,z) \leq \begin{bmatrix} \check{\xi}_n \\ z(1) \end{bmatrix}^\top \begin{bmatrix} \mathcal{H}(\mathbf{P}_n\mathbf{A}_n) & \mathbf{P}_n\mathbf{B}_n + \mathbf{C}_n^\top \\ * & -\left(1 - \frac{\max(0,\frac{\lambda}{\nu}+\varepsilon)}{(2\pi)^2}\right)\upsilon_n^B - \frac{\max(0,\frac{\lambda}{\nu}+\varepsilon)}{(2\pi)^2}\upsilon_n^W \end{bmatrix} \begin{bmatrix} \check{\xi}_n \\ z(1) \end{bmatrix} - \varepsilon\left\|\tilde{z}_{2n}\right\|^2, \tag{6.35}$$

for a sufficiently small scalar $\varepsilon > 0$. If LMI (6.19) holds, there exists $\alpha_3 = \min\left(-\bar{\sigma}(\Xi_n^\natural), \varepsilon\right) > 0$ such that $\dot{\mathcal{V}}_n(x,z) \leq -\alpha_3\left|(x,z)\right|^2$. Then, applying Lyapunov theorem, system ($\mathcal{S}_2$) is GES.

Step 3: Hierarchy. Lastly, since $\mathcal{V}_{n+1}$ is an enhancement of $\mathcal{V}_n$, it is possible to show the hierarchy of the LMIs. In other terms, if there exists $\mathbf{P}_n \succ 0$ such that $\Xi_n \prec 0$ is true, then there exists $\mathbf{P}_{n+1} = \begin{bmatrix} \mathbf{P}_n & 0 \\ * & \frac{1}{2(2n+1)\nu}I_{n_z} \end{bmatrix} \succ 0$ such that $\Xi_{n+1} \prec 0$ is true. A detailed proof would also require careful attention to the reduction of conservatism of modified Bessel and Wirtinger inequalities as the order increases. □

Theorem 6.5 proposes a tractable sufficient stability condition for system ($\mathcal{S}_2$) in the LMI framework. The LMI test depends on the ODE parameters A, B, and C and PDE parameters λ and ν. Performing the test for several orders n and various parameters, the feasibility of $\Xi_n \prec 0$ provides pockets of stability, which give an under estimates of the exact stability regions.

Compared to the basic LMI condition given by Theorem 6.3, we are here able to consider the reaction-diffusion part which is unstable, or at least with one of the PDE characteristic roots on the right-half plane. It is also important to recall that no reaction terms have been considered in previous works [30]. However, compared to the Fourier LMI condition given by Theorem 6.4, we are still not able to deal with large reaction coefficients and with PDE with more than one characteristic root on the right-half plane. This is not a drastic problem because, to the best of our knowledge, there are no examples with $\lambda \geq \nu(2\pi)^2$ (i.e. with two unstable PDE characteristic roots) such that the interconnected system is stable in practice. In the literature, finite-dimensional observers have been designed for a reaction-diffusion equation for any $\lambda \in \mathbb{R}$ in [128], but rely on a perfect match between unstable PDE characteristic roots and ODE eigenvalues and can be really sensitive to model perturbations. To deal with such cases with Legendre LMI conditions, Wirtinger-Legendre inequality should be rewritten as in [77] taking support on the Legendre approximation of $\partial_\theta z$ instead of z. These technical details are not presented here, but are kept for future direction of research.

Remark 6.3. Note that $\mathbf{A}_n$ Hurwitz is a necessary condition for the LMI condition $\Xi_n^\natural \prec 0$.

Remark 6.4. Note that the eigenvalue of matrix $\mathbf{A}_n$ could also be a sufficient condition of stability for sufficiently large orders. Indeed, following the approach outlined in [127], the term in the second diagonal bloc is

$$\Gamma_n = \left(1 - \frac{\max(0, \frac{\lambda}{\nu})}{(2\pi)^2}\right)\upsilon_n^B + \frac{\max(0, \frac{\lambda}{\nu})}{(2\pi)^2}\upsilon_n^W \succ 0, \tag{6.36}$$

and has its eigenvalues which grow in $\underset{n\to\infty}{O}(n^2)$. However, we were not able to prove the following statement for sufficiently large orders: "if $\mathbf{A}_n$ Hurwitz, then system $(\mathcal{S}_2)$ is GES". The main blocking element is the fact that coefficients in matrices $\mathbf{A}_n$, $\mathbf{B}$ and $\mathbf{C}_n$ grow as n increases.

Remark 6.5. It is also worth noticing that the proposed methodology for developing LMI conditions can be generalized to other ODE-reaction-diffusion interconnected systems as shown in [19], where Robin's boundary conditions have been considered.

For future works, a legitimate question is now to wonder if inner estimates of the stability areas converge towards the exact ones. Following the ODE-transport example, this result could be based on the existence of a complete Lyapunov functional $\mathcal{V}$ for system $(\mathcal{S}_2)$ (see Appendix A.2) and the convergence of $\mathcal{V}_n$ towards $\mathcal{V}$ when n tends to infinity. As well as for ODE-transport interconnected systems, the supergeometric convergence properties satisfied by Legendre polynomials could be exploited. This could also be a promising research direction.

Comparison of the linear matrix inequalities

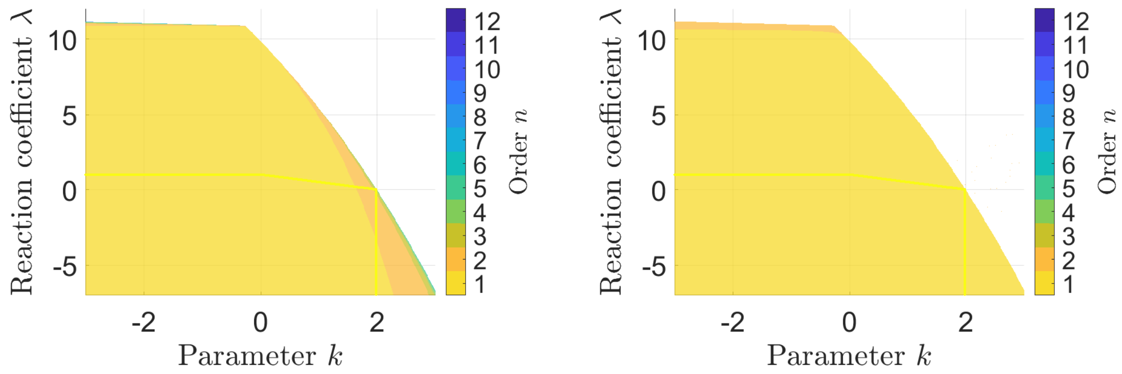
In this paragraph, comparative elements between Fourier and Legendre methods are provided regarding the scalar Example 1.3 with $A = -1$, $B = k \in \mathbb{R}$, $C = 1$, $\nu = 1$ and $\lambda \in \mathbb{R}$.

To begin with, the basic LMI condition (6.10) is solved for parameters $(k, \lambda) \in [-3, 3] \times [-7, 12]$. The test succeeds for $k \leq 2$ and $\lambda \leq 0$ and the boundaries of such a stability region are colored in yellow in Figure 6.3. Elsewhere, we cannot conclude on the stability or instability of system $(\mathcal{S}_2)$. To have a better precision on the stability areas, Fourier and Legendre LMI conditions are solved for several approximated orders.

In Figure 6.3a and Figure 6.3b, Theorem 6.4 and 6.5 are respectively applied for low orders n in $\{1, \dots, 12\}$ and stability assessed areas are represented in color gradation with respect to n.

Computational load. Both tests have the same numerical complexity with respect to order n. Indeed, the number of LMI variables is $N_{var} = \frac{(n_x+nn_z)(n_x+nn_z+1)}{2} = \underset{n\to\infty}{O}(n^2)$. To solve such LMIs, the interior point method is used and requires at least $\underset{n\to\infty}{O}(n_I n^7)$, where n_I is the number of iterations as explained in Appendix A.3.

In terms of similarities, both theorems provide an inner estimation of the stability regions. It is also worth noticing that the proposed tests confirm to be hierarchic and converge towards the expected bound of stability in a fast manner (from order $n = 3$). Notice that, thanks to the hierarchy, for a given precision on the reaction coefficient or other system parameters, incremental tests can even be set up. In terms of differences, the required order seems to be larger when parameters k or λ increase. All the elements of comparison are gathered together and stored in Table 6.1.



(a) Fourier LMI test.

(b) Legendre LMI test.

Figure 6.3: Allowable sets of stability given by Theorems 6.4 and 6.5 with respect to the orders $n \in \{1, \dots, 12\}$ and for various pairs of parameters (λ, k).

Remark 6.6. Compared to the generalized Sum-Of-Square formulation [181], the Lyapunov analysis and inequalities used have been dedicated to the system under study. For understanding the counterweights in play, the LMI framework is more appropriate. Moreover, the result is condensed and the numerical burden is improved.

Aspects	Fourier	Legendre
Basics functions	Trigonometric functions	Orthogonal polynomials
Convergence properties for $z \in C_\infty(0, 1; \mathbb{R}^{n_z})$	Simple convergence	Uniform convergence with decay rate $\underset{n\to\infty}{O}\left(\frac{1}{n!}\right)$
Inequalities to rely $\|z\|$, $\|\partial_\theta z\|$ and $\tilde{z}_n(1)$	Wirtinger inequality Cauchy-Schwarz inequality	Wirtinger inequality Bessel inequality
Structural constraint	$\lambda \in \mathbb{R}$	$\lambda < \nu(2\pi)^2$
Number of LMI variables	$N_{var} = \underset{n\to\infty}{O}(n^2)$	$N_{var} = \underset{n\to\infty}{O}(n^2)$
Necessary requirement	$\mathbf{A}_n$ Hurwitz	$\mathbf{A}_n$ Hurwitz
Hierarchy of the LMI	Yes	Yes
Convergence of the LMI	-	Expected for $n \geq n^*$ (see [18] for ODE-transport systems)
Convergence rate	-	Expected to be supergeometric
Asymptotic behavior	$\Gamma_n^\natural = \underset{n\to\infty}{O}(n)$	$\Gamma_n = \underset{n\to\infty}{O}(n^2)$
	$\mathbf{A}_{n\geq n^*}$ Hurwitz $\Rightarrow$ $(\mathcal{S}_2)$ GES	Expected

Table 6.1: Advantages and drawbacks of Fourier and Legendre LMI conditions.

6.2.2 Positivity test

In this last section, we reverse the stability question.

- *How to assess instability of system $(\mathcal{S}_2)$ using tractable numerical tools?*
- *Can we propose finite-dimensional sufficient conditions of instability, which turn out to be necessary as the order increases?*
- *Is it possible to estimate the necessary order?*

An answer can be provided using the converse Lyapunov theorem and Theorem 1.1 emphasized in Chapter 1 but relies on the following assumption.

Assumption 6.1. Assume that there exists a quadratic, continuous and differentiable converse Lyapunov functional V satisfying $\dot{V}(x,z) = -|x|^2$ and given by

$$V(x,z) = \underbrace{x^\top F_0 x}_{V_a(x,z)} + \underbrace{2x^\top \int_0^1 F_1(\theta)Bz(\theta)\mathrm{d}\theta}_{V_b(x,z)} + \underbrace{\int_0^1\int_0^1 z^\top(\theta_1)B^\top F_2(\theta_1,\theta_2)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2}_{V_c(x,z)}, \tag{6.37}$$

where functions F_0, F_1 and F_2 is the unique continuous solution of system (A.24)-(A.25).

As discussed in Appendix A.1, we are optimistic that such a hypothesis consists in discarding systems with eigenvalues on the imaginary axis. Moreover, the ordinary-partial differential matrix coupled equations (A.24)-(A.25) which needs to be verified by F_0, F_1 and F_2 have been detailed in Appendix A.2.2.

Under Assumption 6.1, it is possible to propose an outer estimate of the stability regions which converges towards the exact stability regions.

Sufficient condition of instability

Consider the approximated Lyapunov functional V_n of the form (6.26) and written as

$$V_n(x,z) = \begin{bmatrix} x \\ \check{\mathcal{I}}_n \int_0^1 \check{\ell}_n(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix}^\top \underbrace{\begin{bmatrix} P & \mathcal{Q}_n \\ * & \mathcal{T}_n \end{bmatrix}}_{\mathbf{P}_n} \begin{bmatrix} x \\ \check{\mathcal{I}}_n \int_0^1 \check{\ell}_n(\theta)z(\theta)\mathrm{d}\theta \end{bmatrix}, \tag{6.38}$$

where matrices P, $\mathcal{Q}_n$ and $\mathcal{T}_n$ are given by

$$\begin{aligned} P &= F_0, \\ \mathcal{Q}_n &= \int_0^1 F_1(\theta)B\check{\ell}_n^\top(\theta)\mathrm{d}\theta, \\ \mathcal{T}_n &= \int_0^1\int_0^1 \check{\ell}_n(\theta_1)B^\top F_2(\theta_1,\theta_2)B\check{\ell}_n^\top(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2, \end{aligned} \tag{6.39}$$

where functions F_0, F_1 and F_2 is the unique continuous solution of system (A.24)-(A.25) provided in Appendix A.2.2.

The contrapositive proposition of the converse Lyapunov theorem yields the following theorem.

Theorem 6.6. *Take an order n in $\mathbb{N}^*$ and assume $\lambda < \nu\pi^2$. If matrix $\mathbf{P}_n$ is not positive definite, then system $(\mathcal{S}_2)$ is not GES. In addition, this condition is hierarchic with respect to order n, this means that if $\mathbf{P}_{n_0}$ is not positive definite then $\mathbf{P}_n$ is not positive definite for all $n \geq n_0$.*

Proof. The proof of the proposition and hierarchy are separated below.

Step 1: Negativity of the converse Lyapunov functional.
Assume that matrix $\mathbf{P}_n$ is not positive definite. Then, there exist a positive scalar $\alpha > 0$ and a non

null vector $\left[\begin{smallmatrix} x_0 \\ \zeta_n \end{smallmatrix}\right] \in \mathbb{R}^{n_x+n_z(n+1)}\backslash\{0\}$ such that

$$\begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix}^\top \mathbf{P}_n \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix} \leq -\alpha \left| \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix} \right|^2 . \tag{6.40}$$

Furthermore, for any vector $\left[\begin{smallmatrix} x_0 \\ \zeta_n \end{smallmatrix}\right] \in \mathbb{R}^{n_x+n_z(n+1)}$, consider (x, z) expressed as follows

$$x = x_0, \quad z(\tfrac{1}{2}) = 0, \quad \partial_\theta z(\theta) = \begin{cases} Cx_0 & \text{if } \theta = 1, \\ \hat{\ell}_n^\top(\theta)\hat{\mathcal{I}}_n\check{\mathcal{L}}_n\zeta_n & \text{if } \theta \in (0,1), \\ Cx_0 & \text{if } \theta = 0, \end{cases} \tag{6.41}$$

in order to have $z(\theta) = \check{\ell}_n^\top(\theta)\zeta_n$ and $(x, z) \in \mathcal{D}_2$. Then, we obtain

$$\begin{cases} V_n(x,z) = \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix}^\top \begin{bmatrix} P & \mathcal{Q}_n \\ * & \mathcal{T}_n \end{bmatrix} \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix} = V(x,z), \\ \left| \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix} \right|^2 \geq \left|(x,z)\right|^2, \end{cases} \quad \forall \begin{bmatrix} x_0 \\ \zeta_n \end{bmatrix} \in \mathbb{R}^{n_x+n_x(n+1)}\backslash\{0\}. \tag{6.42}$$

Therefore, there exist a positive scalar $\alpha > 0$ and a non null state $(x, z) \in \mathcal{D}_1\backslash\{(0, 0)\}$ given by (4.59) such that

$$V(x,z) \leq -\alpha \left|(x,z)\right|^2 . \tag{6.43}$$

Step 2: Conclusion by contradiction.
Assuming that system $(\mathcal{S}_2)$ is GES, we will reach a contradiction and conclude that system $(\mathcal{S}_2)$ cannot be GES. Indeed, for any $\alpha_1 > 0$, introduce the functional W as

$$W(x,z) = V(x,z) + \alpha_1 \left|(x,z)\right|^2 .$$

According to (4.23), differentiating W along the trajectories of system $(\mathcal{S}_2)$ and using (1.42) based on Wirtinger and Cauchy-Schwarz inequalities, we obtain

$$\dot{W}(x,z) = \begin{bmatrix} x \\ z(1) \\ \|z\| \end{bmatrix}^\top \begin{bmatrix} -I_{n_x} + \alpha_1\mathcal{H}(A) & \alpha_1(B + 2\nu C^\top) & 0 \\ * & -2\alpha_1\nu\pi^2 I_{n_z} & 2\alpha_1\pi^2 \mathrm{sign}(z(1))I_{n_z} \\ * & * & 2\alpha_1(\lambda - \nu\pi^2)I_{n_z} \end{bmatrix} \begin{bmatrix} x \\ z(1) \\ \|z\| \end{bmatrix}. \tag{6.44}$$

Then, under the assumption $\lambda < \nu\pi^2$, there exists a sufficiently small positive number $\alpha_1 < \alpha$, such that $\dot{W}(x,z) \leq 0$ holds. Pursuing an integration in time from 0 to ∞, we obtain

$$\lim_{t\to\infty} W\big(x(t), z(t)\big) - W\big(x(0), z(0)\big) \leq 0, \quad \forall\big(x(0), z(0)\big) \in \mathcal{D}_1.$$

From the assumption that system $(\mathcal{S}_2)$ is GES, we have $\big(x(t), z(t)\big) \underset{t\to\infty}{\longrightarrow} (0, 0)$. Then, the inequality $W\big(x(0), z(0)\big) \geq 0$ holds for all $\big(x(0), z(0)\big) \in \mathcal{D}_1$ and leads to

$$V_{I_{n_x}}(x,z) \geq -\alpha_1 \left|(x,z)\right|^2 > -\alpha \left|(x,z)\right|^2, \quad \forall (x,z) \in \mathcal{D}_1. \tag{6.45}$$

We end up with a contradiction: assuming that system $(\mathcal{S}_2)$ is GES, the inequality (6.43) cannot be verified. Thus, matrix $\mathbf{P}_n$ not positive definite implies that system $(\mathcal{S}_2)$ is not GES.

Step 3: Hierarchy.
Concerning the hierarchy, consider matrix $\mathbf{P}_n$ at order $n \geq n_0$ given by

$$\mathbf{P}_n = \begin{bmatrix} \mathbf{P}_{n_0} & \bar{\mathcal{Q}}_{n_0:n} \\ * & \bar{\mathcal{T}}_{n_0:n} \end{bmatrix}, \tag{6.46}$$

where

$$\begin{aligned}
\bar{\mathcal{Q}}_{n_0:n} &= h\int_0^1 F_1(\theta)B\bar{\ell}_{n_0:n}(\theta)\mathrm{d}\theta,\\
\bar{\mathcal{T}}_{n_0:n} &= h^2\int_0^1\int_0^1 \bar{\ell}_{n_0:n}(\theta_1)B^\top F_2(\theta_1,\theta_2)B\bar{\ell}^\top_{n_0:n}(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2,\\
\bar{\ell}_{n_0:n} &= \begin{bmatrix} l_{n_0}I_{n_z} & \dots & l_{n-1}I_{n_z}\end{bmatrix}^\top .
\end{aligned}$$

If $\mathbf{P}_{n_0}$ is not positive definite, then $\mathbf{P}_n$ cannot be positive definite for all $n \geq n_0$. □

Theorem 6.6 is a sufficient condition of instability holding for orders n in $\mathbb{N}$. It allows the detection of unstable systems. The hierarchy of such a positivity condition with respect to order n has also been proven and leads to incremental outer estimates of the stability regions.

For the moment these conditions cannot be solved numerically as the matrices functions F_0, F_1, and F_2 are unknown analytically as far as we know. They correspond to the solution to an ODE-PDE interconnected system given in Appendix A.2 for which we were not able to find a solution.

In the next paragraph, a proof for the convergence of this sufficient condition of instability towards its necessity is sketched. Future works will also allow us to validate and reinforce these last results.

Convergence of the positivity condition of instability

The convergence of these positivity conditions can be proven by the use of Theorem 1.1 introduced in Chapter 1 and recalled below.

Property 6.1. *If system $(\mathcal{S}_2)$ is not GES, then there exists*

$$(x_0, z_0) \in \mathcal{S} := \mathcal{D}_2 \cap \left\{(x,z) \in \mathbb{R}^{n_x} \times C_\infty(0,1;\mathbb{R}^{n_z}) \text{ s.t. } |x| = 1, \left\|z^{(d)}\right\|_\infty \leq \left(\frac{r}{\nu}\right)^{\frac{d-1}{2}}, \forall d \in \mathbb{N}\right\}, \tag{6.47}$$

such that the following inequality

$$V(x_0,z_0) \leq -\frac{1}{2r} < 0 \qquad \text{with } r = \max\left(\lambda, |A| + |B|\right). \tag{6.48}$$

Proof. The proof corresponds to the one given in Theorem 1.1 in Chapter 1 for the case of ODE-reaction-diffusion interconnected system $(\mathcal{S}_2)$. □

Using the negative bound $-\frac{1}{2r}$, the approximated converse Lyapunov functional V_n given by (6.38) will also be negative for sufficiently large orders n. The reasoning is the same as for ODE-transport interconnected systems [15] and follows the arguments provided in [98, 156]. Roughly speaking, we take the benefits of the supergeometric convergence of the Legendre polynomial approximation provided in Theorems 2.2 and 2.3 in Chapter 2.

Assuming that system $(\mathcal{S}_2)$ is not GES, two main results are derived:

- the non positive definiteness of matrix $\mathbf{P}_n$ as n tends to infinity,
- the estimation of the order n^* from which the positivity condition $\underline{\sigma}(\mathbf{P}_n) \geq 0$ is false.

Theorem 6.7. *If system $(\mathcal{S}_2)$ is not GES, then there exists an order n^* such that for all $n \geq n^*$ matrix $\mathbf{P}_n$ is not positive definite. Moreover, this order can be computed by the following formula*

$$n^* = \left\lceil \frac{1}{2}\mathcal{N}_0\left(\frac{\mathcal{E}(\beta_1,\beta_2)}{|B|}\right)\right\rceil, \tag{6.49}$$

where scalars β_1, β_2 are given by

$$\beta_1 = \frac{\|F_1\|_\infty + |B| \, \|F_2\|_\infty}{\|F_2\|_\infty}, \quad \beta_2 = \frac{1}{2r \, \|F_2\|_\infty}, \quad r = \max(\lambda, |A| + |B|), \tag{6.50}$$

and where functions $\mathcal{N}_0$ and $\mathcal{E}$ are defined by

$$\begin{aligned} \mathcal{N}_0(\varepsilon) &= 2 + \left\lceil \frac{1}{2}\sqrt{\frac{r}{\nu}}\, \mathrm{e}^{1+\mathcal{W}\left((\frac{1}{2}\sqrt{\frac{r}{\nu}})^{-1} \log\left(\frac{\sqrt{\frac{r}{\nu}}}{2\,\mathrm{e}^{\lfloor \frac{1}{2}\sqrt{\frac{r}{\nu}} \rfloor}\, \varepsilon} \right) \right)} \right\rceil, \\ \mathcal{E}(\beta_1, \beta_2) &= -\beta_1 + \sqrt{\beta_1^2 + \beta_2}. \end{aligned} \tag{6.51}$$

Proof. In the proof, we first make the link between the approximated functional V_n and the converse Lyapunov functional $V_{I_{n_x}}$ and then we used convergence arguments to conclude the proof.

Step 1: Error between the approximated Lyapunov functional V_n and the original one V.
The passage from V (6.37) to V_n (6.38) is issued from Legendre approximation of the state z. Then, putting aside the truncated error $\tilde{z}_{2n}$, we have

$$\begin{aligned} V_n(x,z) = V(x,z) &- \underbrace{2x^\top \int_0^1 F_1(\theta) B \tilde{z}_{2n}(\theta) \mathrm{d}\theta}_{\tilde{V}_{nb}(x,z)} - \underbrace{\int_0^1\int_0^1 \tilde{z}_{2n}^\top(\theta_1) B^\top F_2(\theta_1,\theta_2) B \tilde{z}_{2n}(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2}_{\tilde{V}_{nc,1}(x,z)}, \\ &- \underbrace{\int_0^1\int_0^1 z^\top(\theta_1) B^\top F_2(\theta_1,\theta_2) B \tilde{z}_{2n}(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2}_{\tilde{V}_{nc,2}(x,z)} - \underbrace{\int_0^1\int_0^1 \tilde{z}_{2n}^\top(\theta_1) B^\top F_2(\theta_1,\theta_2) B z(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2}_{\tilde{V}_{nc,3}(x,z)}, \end{aligned} \tag{6.52}$$

where $\tilde{z}_{2n}$ is the Legendre truncated error of z at order n defined in Definition 5.3 by

$$\tilde{z}_{2n}(\theta) = z(t,\theta) - \check{\ell}_n^\top(\theta) \check{\mathcal{Z}}_n(t), \quad \forall (t,\theta) \in \mathbb{R}_+ \times [0,1]. \tag{6.53}$$

where $\check{\mathcal{Z}}_n$ are the odd Legendre polynomials coefficients since the even ones are zero due to the anti-symmetry of the system. Roughly bounding V_n, we obtain

$$V_n(x,z) \leq V(x,z) + 2\,|B| \left(\|F_1\|_\infty |x| + |B| \, \|F_2\|_\infty |z| \right) \|\tilde{z}_{2n}\|_\infty + |B|^2 \, \|F_2\|_\infty \, \|\tilde{z}_{2n}\|_\infty^2. \tag{6.54}$$

In the next step, assuming that the system is unstable, we apply such an inequality for particular (x_0, z_0) provided by Property 6.1.

Step 2: Convergence and estimation of the order n^*.
According to Property 6.1, there exists a non trivial solution (x_0, z_0) in $\mathcal{S}$ such that inequality $V(x_0, z_0) \leq -\frac{1}{2r}$ which means that

$$V_n(x_0, z_0) \leq -\frac{1}{2r} + 2\,|B| \left(\|F_1\|_\infty + |B| \, \|F_2\|_\infty \right) \|\tilde{z}_{0,2n}\|_\infty + |B|^2 \, \|F_2\|_\infty \, \|\tilde{z}_{0,2n}\|_\infty^2. \tag{6.55}$$

Since we have selected (x_0, z_0) in $\mathcal{S} \subset \mathbb{R}^{n_x} \times C_\infty(0,1;\mathbb{R}^{n_z})$ and thanks to the convergence properties of Legendre approximation provided in Theorems 2.2 in Chapter 2, the truncated error $\|\tilde{z}_{0,2n}\|_\infty$ can be made as small as desired and $V_n(x_0, z_0) \leq 0$ for sufficiently large orders n. Indeed, according to Theorem 2.2, there exists an order n^* such that the following inequality holds

$$|B| \, \|\tilde{z}_{0,2n}\|_\infty \leq -\frac{\|F_1\|_\infty + |B| \, \|F_2\|_\infty}{\|F_2\|_\infty} + \sqrt{\left(\frac{\|F_1\|_\infty + |B| \, \|F_2\|_\infty}{\|F_2\|_\infty} \right)^2 + \frac{1}{2r \, \|F_2\|_\infty}}, \quad \forall n \geq n^*. \tag{6.56}$$

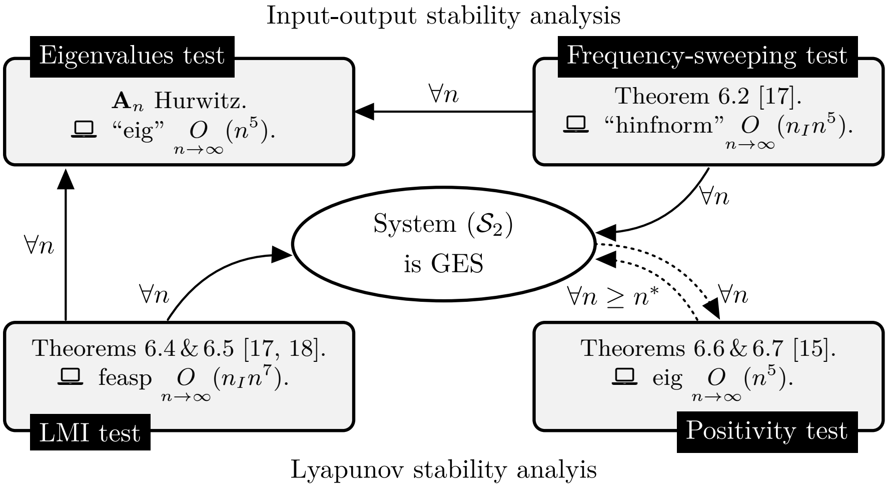


Figure 6.4: Summary of Chapter 6.

Then, the negativity of V_n is satisfied from a certain order n^*,

$$V_n(x_0, z_0) \leq -\frac{1}{2r} + 2\,|B|\,(\|F_1\|_\infty + |B|\,\|F_2\|_\infty)\,\|\tilde{z}_{0,2n}\|_\infty + |B|^2\,\|F_2\|_\infty\,\|\tilde{z}_{0,2n}\|_\infty^2 \leq 0, \quad \forall n \geq n^*. \tag{6.57}$$

We have found a non trivial vector $\xi_n = \begin{bmatrix} x_0 \\ \mathcal{I}_n \int_0^1 \ell_n(\theta) z_0(\theta) \mathrm{d}\theta \end{bmatrix}$ in $\mathbb{R}^{n_x + n n_z}$ such that $V_n(x_0, z_0) = \xi_n^\top \mathbf{P}_n \xi_n \leq 0$ and, consequently, matrix $\mathbf{P}_n$ is not positive definite, for orders $n \geq n^*$. More precisely, an estimation of the order n^* from which the required precision is reached is made possible. According to Theorem 2.2, an estimation of the order n^* from which condition (6.56) is true is given by

$$2n^* = \mathcal{N}_0\left(\frac{1}{|B|}\left(-\frac{\|F_1\|_\infty + |B|\,\|F_2\|_\infty}{\|F_2\|_\infty} + \sqrt{\left(\frac{\|F_1\|_\infty + |B|\,\|F_2\|_\infty}{\|F_2\|_\infty}\right)^2 + \frac{1}{2r\,\|F_2\|_\infty}}\right)\right), \tag{6.58}$$

and concludes the proof. □

Once again, the result cannot be illustrated numerically since matrix functions F_1 and F_2 have not been exhibited and remain unknown.

Notice also that the results have been presented using Legendre approximation basis but also hold for other approximation basis. Forthcoming works will concern this direction.

Conclusion

In this chapter, we have addressed the stability of an ODE interconnected to a reaction-diffusion PDE. Avenues of research have been evoked and discussed. Through augmented models and the application of the small-gain theorem, input-output stability conditions have been presented in Section 6.1. Through the Lyapunov approach, a sufficient stability condition has been proposed in terms of linear matrix inequalities in Section 6.2.1. It relies on the augmented models based on trigonometric or polynomial coefficients of the reaction-diffusion part. Through the converse Lyapunov approach, a necessary and sufficient instability condition has been sketched in Section 6.2.2. It relies on the expression of some functions F_0, F_1, and F_2, which still be unknown and not elucidated. All these results have been reported in Figure 6.4.

Applying our methodology to ODE-PDE interconnected systems remains at its beginning. This chapter shows that our results could be generalized to a large panel of infinite-dimensional systems and lead to certificates on the stability or on the reliability of controllers or observers once implemented.

Part IV

Conclusion

Chapter 7

Conclusions and perspectives

"The history of science is rich in example of the fruitfulness of bringing two sets of techniques, two sets of ideas, developed in separate contexts for the pursuit of new truth"
Science and the common understanding, J.R. Oppenheimer.

Contents

7.1 Conclusions

In this manuscript, a generic methodology proposing numerical stability criteria for linear ODE-PDE interconnected systems has been developed. In a first step, augmented systems have been constructed in order to separate a finite-dimensional approximate model from an infinite-dimensional structured error. In a second step, frequency and time conditions issued from the finite-dimensional field have been traced to these augmented systems. It has lead to conditions of stability or instability depending on the order of approximation. More precisely, inner and outer estimates of the stability regions have been proposed, and proven to converge to the expected stability regions as the order increases.

In Chapter 1, the problem statement and the class of systems under consideration has been exposed. Two particular linear ODE-PDE interconnections have been presented and served as case studies throughout the manuscript. In the first case, the PDE is hyperbolic. More particularly, we have considered a transport PDE. In the second case, the PDE is parabolic. More particularly, we have considered a reaction-diffusion PDE. The well-posedness, the uniqueness of the equilibrium, and the definition of exponential stability have been recalled. In both cases, the main objective was to find numerical criteria to ensure stability.

In Chapter 2, different approximation techniques that can be used during the stability analysis have been described. A focus has been made on Legendre polynomials approximation. The supergeometric convergence property of this approximation has been emphasized. Moreover, a new formulation of the Bessel and Wirtinger inequalities, often used for stability analysis, has been provided to fit with the context of the Legendre approximation. With all these preliminary results, it is possible to deal with our two case studies.

Parts II and III have been devoted to the stability analysis of ODE-transport and ODE-reaction-diffusion interconnected systems, respectively. To do so, the construction process of Legendre modelling was explained in Chapters 3 and 5. From one part, a finite-dimensional system in the state space representation $\left(\begin{smallmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & 0 \end{smallmatrix}\right)$ has been exhibited. From the other part, a residual infinite-dimensional transfer function $R_n(s)$ has been preserved. Links with existing techniques such as Padé or *tau* models have been highlighted. Then, numerical tests that assess stability were expressed in Chapters 4 and 6

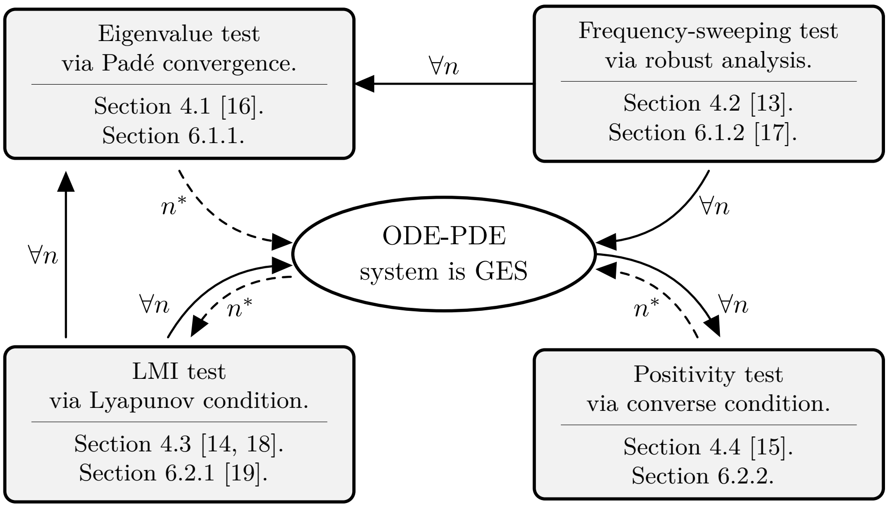


Figure 7.1: Summary of the main results.

leaning on the augmented systems designed in Chapters 3 and 5. The results have been divided into four categories illustrated in Figure 7.1 and detailed below

- Using Padé approximation, if the state matrix $\mathbf{A}_n$ is Hurwitz for sufficiently large orders, then the trivial equilibrium of the system is stable.
- Using the small gain theorem, if the inequality $\left|\left(\begin{smallmatrix}\mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & 0\end{smallmatrix}\right)\right|_{\mathcal{H}_\infty} \leq \gamma_n$ holds where γ_n is a bound of a transfer function representing the neglected infinite-dimensional part of the augmented system, then the trivial equilibrium of the system is stable.
- By application of the Lyapunov theorem, if a linear matrix inequality $\Xi_n \prec 0$ is satisfied, then the trivial equilibrium of the system is stable. This condition gives an inner estimate of the stability regions.
- By application of the converse Lyapunov theorem, if a well-chosen matrix $\mathbf{P}_n$ is not positive definite, then the trivial equilibrium of the system is unstable. This condition gives an outer estimate of the stability regions.

The main result especially concerns the case of time-delay systems. Indeed, based on the existence of a well-known complete Lyapunov-Krasovskii functional, it has been shown that our modelling on Legendre polynomials leads to under and over estimates of the stability regions which converge exponentially fast towards the expected ones.

7.2 Perspectives

The first axis of perspectives would be to push to the same stage all the results obtained for time-delay systems to the ODE-reaction-diffusion system. For instance, the convergence of the eigenvalues of the approximating matrix has not been proven, the hierarchy and convergence of the conditions resulting from the application of the Lyapunov theorem with respect to the approximation order are still to be verified. The first lock, concerning the spectral analysis, lies in the finite radius of Padé approximation for the reaction-diffusion equation. The second and more important lock refers to the fact that the complete Lyapunov functional is not given in the literature and has an unknown analytical expression. Without this, the convergence analysis cannot be carried out and the orders from which the converse conditions are satisfied cannot be estimated. For the moment, the finite-dimensional necessary and sufficient conditions for the ODE-reaction diffusion system are at an early stage, and its numerical complexity cannot be evaluated numerically. Once these locks are removed, it will be easier to compare different approximation approaches in terms of conservatism.

Continuing with this idea of generalization, future theoretical works could rewrite the method for any system in the Callier-Desoer class. Extensions could also include time-delay systems with multiple and distributed delays or integral delay systems. All in all, the establishment of our results depends on our ability to find a finite-dimensional object which converges to the solution of the infinite-dimensional Lyapunov equation. By conducting a very formal approach, one could even generalize to many types of approximations that satisfy uniform convergence properties.

For extensions on the numerical side, a natural question is to ask what is the stability analysis tool and the approximation strategy leading to the lowest computational load. This work would consist in balancing the velocity of the numerical test with the accuracy of the approximation with respect to order n. It is possible that hybrid methods are the most advantageous. For instance, as for the finite element method, the state could be discretized and then approximated on each sub-interval. One could also use the first Legendre coefficients to enhance the feature of the state and then make a Fourier series decomposition which is less time-consuming.

Future works could finally be dedicated to the synthesis of finite-dimensional controllers or observers for linear ODE-PDE interconnected systems by following two lines of investigation.
On the one hand, late-lumping approaches can be followed. Infinite-dimensional control laws such as Volterra integral equations resulting from the backstepping method can be approximated and stability certificates can be obtained for sufficiently large approximation orders using our convergence results.
On the other hand, early-lumping approaches can also be investigated. Finite-dimensional dynamical controllers can be designed and aligned with the augmented systems presented in this manuscript. Thanks to robust synthesis techniques, it derives from a linear matrix inequality and could be proven to stabilize the original system for sufficient large approximation orders. This could be put in perspective with the convergence of finite-dimensional control laws towards the solution of infinite-dimensional Ricatti equations.

Part V

Appendices

Appendix **A**

Lyapunov analysis and convex optimization

"Nothing is lost, nothing is created, everything is transformed."
Elementary Treatise on Chemistry, A. Lavoisier.

Contents

A.1 Lyapunov analysis for linear systems

A.1.1 Lyapunov candidate functionals

The energy is preserved along the time and is transferred between several physical systems. It can be decomposed in several parts such as the kinetic and the potential energy in mechanics. More generally, one distinguishes the internal and external energy of a system. Lyapunov functionals are mathematical tools based on the internal energy of dynamical systems in order to rule on its stability properties [10].

Finite-dimensional case

Consider a linear finite-dimensional systems given by

$$\dot{x}(t) = Ax(t), \quad \forall t \geq 0, \tag{A.1}$$

with matrix A in $\mathbb{R}^{n\times n}$ constant and known.

For any linear finite-dimensional systems, the following definition can be provided.

Definition A.1. A Lyapunov candidate function is a scalar function defined on $\mathbb{R}^n$ that is globally continuous, differentiable, positive definite and radially unbounded. More precisely, there exists positive scalars α_1 and α_2 such that

$$\alpha_1 |x|^2 \leq \mathcal{V}(x) \leq \alpha_2 |x|^2, \quad \forall x \in \mathbb{R}^n. \tag{A.2}$$

For instance, the Lyapunov candidate functional is quadratic and given by

$$\mathcal{V}(x) = x^\top P x, \quad \forall x \in \mathbb{R}^n, \tag{A.3}$$

where P belongs to $\mathbb{S}^n_+$.
The method to find such functionals is well-known and is recalled in the next section.

Infinite-dimensional case

Focusing now on linear infinite-dimensional systems

$$\dot{z}(t) = \mathcal{A}z(t), \quad \forall t \geq 0, \tag{A.4}$$

with a linear operator $\mathcal{A}$ on the Hilbert space $\mathcal{D}$.
For any linear infinite-dimensional systems, the following definition can be provided.

Definition A.2. A Lyapunov candidate functional is a scalar functional defined on $\mathcal{D}$ that is globally continuous, differentiable, positive definite and radially unbounded. More precisely, there exist positive scalars α_1 and α_2 such that

$$\alpha_1 \|z\|^2_{\mathcal{D}} \leq \mathcal{V}(z) \leq \alpha_2 \|z\|^2_{\mathcal{D}}, \quad \forall z \in \mathcal{D}. \tag{A.5}$$

Forinstance, the Lyapunov candidate functional is quadratic and given by

$$\mathcal{V}(z) = \langle z | \mathcal{P} z \rangle, \quad \forall z \in \mathcal{D}, \tag{A.6}$$

where $\mathcal{P}$ is a positive Hermitian endomorphism on $\mathcal{D}$.

A physical interpretation of the problem sometimes allows to build this kind of functions. For example, for the heat equation, the most natural function is the L^2 norm of the signal governed by the dynamics. Physically, at the macroscopic scale, this corresponds to the energy $E_m = \int_0^1 c\mathrm{d}T$, where c denotes the volumetric heat capacity and T the temperature.

When two systems are put in cascade, the choice of Lyapunov candidate functional resumes to the sum of the different energies. However, as soon as the two systems are interconnected, the superposition theorem is no more relevant and the sum is often too restrictive. The determination of Lyapunov candidate functionals which take into consideration products between the two states is then mandatory. Such interconnected Lyapunov candidate functionals are constructed in the manuscript.

A.1.2 Lyapunov theorem

Lyapunov theorem has been introduced in [150] and has been proved to be a very efficient tool to assess stability properties of a system in the Lyapunov sense [130, Chapter 4]. The strategy is to take a Lyapunov candidate functional and verify if it decreases along the trajectories of the system. Most of the time, it serves as a sufficient condition of stability. The emergence of linear semi-definite programming tools has further increased its use.

Remark A.1. Note that, for linear systems, assymptotic and exponential stability are equivalent. Moreover, when the Lyapunov candidate functional $\mathcal{V}$ is defined on the whole state space and is radially unbounded, the stability is automatically global.

Remark A.2. Hereafter, strong Lyapunov functional are used. Notice that stability can also be assessed applying using weak Lyapunov functional for which the definiteness property is relaxed [10].

Finite-dimensional case

For any linear finite-dimensional systems, considering Lyapunov candidate functions within the meaning of Definition A.1, the following lemma holds.

Lemma A.1. *Let $\mathcal{V}$ a Lyapunov candidate function on $\mathbb{R}^n$, denote $\dot{\mathcal{V}}$ its time derivatives along the trajectories of system* (A.1) *and assume $\dot{\mathcal{V}}$ is negative definite on $\mathbb{R}^n$.*
More precisely, there exist positive scalars $\alpha_1, \alpha_2, \alpha_3 > 0$ such that

$$\begin{aligned} \alpha_1 |x|^2 \leq \mathcal{V}(x) \leq \alpha_2 |x|^2, \qquad \forall x \in \mathbb{R}^n \backslash \{0\}, \\ \dot{\mathcal{V}}(x) \leq -\alpha_3 |x|^2, \quad \forall x \in \mathbb{R}^n \backslash \{0\}. \end{aligned} \tag{A.7}$$

Then, the origin of system (A.1) *is globally exponentially stable (GES) in the sense of the norm $|\cdot|$.*

Infinite-dimensional case

As an extension to the case of linear infinite-dimensional systems, considering Lyapunov candidate functionals within the meaning of Definition A.2, the following lemma holds.

Lemma A.2. *Let $\mathcal{V}$ a Lyapunov candidate functional on $\mathcal{D}$, denote $\dot{\mathcal{V}}$ its time derivatives along the trajectories of system* (A.4) *and assume $\dot{\mathcal{V}}$ is negative definite on $\mathcal{D}$.*
More precisely, there exists positive scalars $\alpha_1, \alpha_2, \alpha_3 > 0$ such that

$$\begin{aligned} \alpha_1 \|z\|_{\mathcal{D}}^2 \leq \mathcal{V}(z) \leq \alpha_2 \|z\|_{\mathcal{D}}^2, \qquad \forall z \in \mathcal{D} \backslash \{(0,0)\}, \\ \dot{\mathcal{V}}(z) \leq -\alpha_3 \|z\|_{\mathcal{D}}^2, \quad \forall z \in \mathcal{D} \backslash \{(0,0)\}. \end{aligned} \tag{A.8}$$

Then, the origin of system (A.4) *is globally exponentially stable (GES) in the sense of the norm $\|\cdot\|_{\mathcal{D}}$.*

This lemma is used in this manuscript to derive sufficient stability conditions for ODE-PDE interconnected systems by selecting appropriate Lyapunov candidate functionals.

A.1.3 Converse Lyapunov theorem

A converse Lyapunov theorem can also be used [130, Section 4.7]. It is less common for stability analysis purpose but is equally important. Indeed, assuming that the system is stable, it ensures the existence of a Lyapunov candidate functional. It allows to prove the necessity side of the above lemmas. In addition, it also allows to propose instability criteria [72, 155].

Remark A.3. In the literature, such a converse Lyapunov method is also used for nonlinear systems. The exact direct generation of the converse Lyapunov functional provides necessary and sufficient conditions of stability and an estimation of the stability domains [104, Chapter 5].

Finite-dimensional case

For linear finite-dimensional system, a converse Lyapunov function can be introduced. It correspond to look at the problem in reverse and to build the Lyapunov function which satisfies $V(x) = -x^\top Q x$. Assuming $Q \in \mathbb{S}_+^n$, there exists a Lyapunov matrix $P \in \mathbb{S}^n$ such that the Lyapunov equation given below holds

$$PA + A^\top P = -Q. \tag{A.9}$$

The corresponding Lyapunov function $V(x) = x^\top P x$ is called converse Lyapunov functional. Note that (A.9) is a Sylvester's equation and that the solution P is unique when matrices A and $-A$ have non common eigenvalues. Such a matrix P can be computed thanks to the Matlab routine “lyap(A,Q)”, which solves the Sylvester equation (A.9) with a cost $\underset{n\to\infty}{O}(n^6)$.

The necessary side of the Lyapunov theorem can then be formulated.

Lemma A.3. *If system* (A.1) *is exponentially stable then the symmetric Lyapunov matrix P satisfying* (A.9) *is positive definite and is given by*

$$P = \int_0^\infty e^{A^\top t} Q e^{At} \mathrm{d}t. \tag{A.10}$$

Subsequently, a sufficient condition of instability can be derived: if the unique solution P of (A.9) is not positive definite, then system (A.1) is unstable.

Infinite-dimensional case

For linear infinite-dimensional system (A.4) where linear operator $\mathcal{A}$ generates a strongly semigroup $\mathcal{T}$ defined on space $\mathcal{D}$, this lemma is extended and remains valid [58]. Let $\mathcal{A}^*$ the adjoint operator of $\mathcal{A}$. For any positive definite Hermitian endomorphism $\mathcal{Q}$, the Lyapunov equation is written as

$$\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P} = -\mathcal{Q}. \tag{A.11}$$

which is still an infinite-dimensional Sylvester's equation and has a unique Hermitian endomorphism solution $\mathcal{P}$ when operators $\mathcal{A}$ and $-\mathcal{A}^*$ have non common spectra. The converse Lyapunov functional is then given by $V(z) = \langle z|\mathcal{P}z\rangle_{\mathcal{D}}$.
Extending Lemma A.3 to infinite dimension, the following lemma is derived.

Lemma A.4. *If system* (A.4) *is exponentially stable then the operator $\mathcal{P}$ satisfying* (A.11) *is a Hermitian positive definite endomorphism on $\mathcal{D}$ and is given by*

$$\mathcal{P} = \int_0^\infty \mathcal{T}^*(t)\mathcal{Q}\mathcal{T}(t)\mathrm{d}t, \tag{A.12}$$

where $\mathcal{T}$ and $\mathcal{T}^*$ are the strongly semigroup generated by $\mathcal{A}$ and $-\mathcal{A}^*$, respectively.

Here, a theoretical sufficient condition of instability can be derived too. If the unique solution $\mathcal{P}$ of (A.11) is not positive definite, then system (A.1) is unstable.
However, it is hard to find numerically tractable criterion from this extension. In order to obtain necessary conditions of stability, the solution $\mathcal{P}$ needs to be known. For linear ODE-PDE interconnected systems a trivial methodology is proposed in the next section. For this class of systems, one can settle for operator $\mathcal{Q}$ acting only on the ODE part. In that case, $\mathcal{Q}$ is positive but non-definite but $\mathcal{P}$ still to be positive definite. Indeed, solutions living in the kernel of the semi-definite operator $\mathcal{Q}$ is then reduced to the trivial solution. It is in this way that non positivity tests to assess instability have been proposed in this manuscript.

A.2 Converse Lyapunov functionals for ODE-PDE interconnected systems

A.2.1 System interconnected with the transport equation

In the case of linear functional differential equations, converse Lyapunov functional have been constructed in the literature [131]. More precisely, a closed-form solution can be given for time-delay systems with multiple commensurate delays or distributed delays where the distributed function can be seen as an impulse response of a linear finite dimensional system [131]. For two incommensurate delays, some tracks are available in [64].
The construction of such complete functionals is retrieves via ODE-PDE modelization $(\mathcal{S}_1)$. Guess a quadratic Lyapunov functional, for all $(x, z) \in \mathcal{D}_1$,

$$V(x,z) = \underbrace{x^\top F_0 x}_{V_a(x,z)} + \underbrace{2hx^\top \int_0^1 F_1(\theta)Bz(\theta)\mathrm{d}\theta}_{V_b(x,z)} + \underbrace{h^2 \int_0^1\int_0^1 z^\top(\theta_1)B^\top F_2(\theta_1,\theta_2)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2}_{V_c(x,z)}, \tag{A.13}$$

where matrices F_0 and $F_1(\theta)$ are respectively in $\mathbb{S}^{n_x}$ and $\mathbb{R}^{n_x \times n_x}$ and where matrix $F_2(\theta_1, \theta_2)$ in $\mathbb{R}^{n_x \times n_x}$ satisfies $F_2(\theta_1, \theta_2) = F_2^\top(\theta_2, \theta_1)$ for all $(\theta_1, \theta_2) \in [0, 1]^2$.

For any $W \in \mathbb{S}^{n_x}$, the objective is to find functions F_0, F_1 and F_2 such that

$$\dot{V}(x,z) = -x^\top W x. \tag{A.14}$$

The derivatives of (A.13) with respect to the trajectories of system $(\mathcal{S}_1)$ is decomposed as

$$\begin{aligned}
\dot{V}_0(x,z) &= x^\top \mathcal{H}(F_0 A)x + \boxed{2x^\top F_0 B z(0)},\\
\dot{V}_1(x,z) &= x^\top \mathcal{H}(F_1(1)BC)x - \boxed{2x^\top F_1(0)Bz(0)} + \boxed{2hz^\top(0)\int_0^1 B^\top F_1(\theta)Bz(\theta)\mathrm{d}\theta}\\
&\quad + 2x^\top \int_0^1 \left(-F_1'(\theta) + hA^\top F_1(\theta)\right) Bz(\theta)\mathrm{d}\theta,\\
\dot{V}_2(x,z) &= -h\int_0^1\int_0^1 z^\top(\theta_1)B^\top(\partial_{\theta_1} + \partial_{\theta_2})F_2(\theta_1,\theta_2)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2\\
&\quad + 2hx^\top \int_0^1 (BC)^\top F_2(1,\theta)Bz(\theta)\mathrm{d}\theta - \boxed{2hz^\top(0)\int_0^1 B^\top F_2(0,\theta)Bz(\theta)\mathrm{d}\theta}.
\end{aligned}$$

The boxed terms directly give $F_0 = F_1(0)$, $F_1(\theta) = F_2(0,\theta)$ and, in order to remove the double integral term under the symmetric condition, one easily find

$$F_2(\theta_1,\theta_2) = \begin{cases} F_1^\top(\theta_1-\theta_2) & \text{if } \theta_1 \geq \theta_2,\\ F_1(\theta_2-\theta_1) & \text{if } \theta_1 < \theta_2. \end{cases} \tag{A.15}$$

Denoting $A_d = BC$, the other terms lead to the following set of equations

$$\begin{cases} F_1'(\theta) - hA^\top F_1(\theta) - hA_d^\top F_1^\top(1-\theta) = 0, \quad \forall\theta \in (0,1),\\ \mathcal{H}(F_0A) + \mathcal{H}(F_1(1)A_d) = -W. \end{cases} \tag{A.16}$$

Taking $U(\theta) := F_1(\theta)$ and $V(\theta) := F_1^\top(1-\theta)$, one obtains

$$\begin{cases} \begin{bmatrix} U'(\theta)\\ V'(\theta)\end{bmatrix} - h\begin{bmatrix} A^\top U(\theta) + A_d^\top V(\theta)\\ -U(\theta)A_d - V(\theta)A\end{bmatrix} = 0, \quad \forall\theta \in (0,1),\\ U(0)A + A^\top U(0) + U(1)A_d + A_d^\top V(0) = -W,\\ U(0) - V(1) = 0. \end{cases} \tag{A.17}$$

This last equation can be solved by vectorization technique and lead to the following closed-form expression

$$\begin{bmatrix} \mathrm{vec}(U(\theta))\\ \mathrm{vec}(V(\theta))\end{bmatrix} = \mathrm{e}^{\theta M} N^{-1}\begin{bmatrix} -\mathrm{vec}(W)\\ 0\end{bmatrix}, \quad \forall\theta \in [0,1], \tag{A.18}$$

where matrices

$$\begin{aligned} M &= h\begin{bmatrix} I_{n_x}\otimes A^\top & I_{n_x}\otimes A_d^\top\\ -A_d^\top \otimes I_{n_x} & -A^\top \otimes I_{n_x}\end{bmatrix},\\ N &= \begin{bmatrix} A^\top\otimes I_{n_x} + I_{n_x}\otimes A^\top & I_{n_x}\otimes A_d^\top\\ I_{n_x^2} & 0\end{bmatrix} + \begin{bmatrix} A_d^\top \otimes I_{n_x} & 0\\ 0 & -I_{n_x^2}\end{bmatrix} e^M. \end{aligned} \tag{A.19}$$

using $\mathrm{vec}(AQB) = (B^\top \otimes A)\mathrm{vec}(Q)$ where $\otimes$ denotes the Kronecker product.

Thanks to the existence of an analytical solution for the converse Lyapunov functional, we are able to proposed necessary conditions of stability for ODE-transport systems in Part II of this manuscript.

A.2.2 System interconnected with the reaction-diffusion equation

Consider now system $(\mathcal{S}_2)$ and we expect a quadratic Lyapunov functional given by

$$V(x,z) = \underbrace{x^\top F_0 x}_{V_a(x,z)} + \underbrace{2x^\top \int_0^1 F_1(\theta)Bz(\theta)\mathrm{d}\theta}_{V_b(x,z)} + \underbrace{\int_0^1\int_0^1 z^\top(\theta_1)B^\top F_2(\theta_1,\theta_2)Bz(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2}_{V_c(x,z)}, \tag{A.20}$$

where matrices F_0, $F_1(\theta)$ and $F_2(\theta_1,\theta_2)$ are respectively in $\mathbb{S}^{n_x}$, $\mathbb{R}^{n_x\times n_x}$ and $\mathbb{R}^{n_x\times n_x}$ with the additional symmetric constraint

$$F_2(\theta_1,\theta_2) = F_2^\top(\theta_2,\theta_1), \quad \forall(\theta_1,\theta_2) \in [0,1]^2. \tag{A.21}$$

For any $W \in \mathbb{S}^{n_x}$, the objective is to find functions F_0, F_1 and F_2 such that (A.14) holds. The derivatives of (A.20) along the trajectories of system $\mathcal{S}_2$ give the following terms

$$
\begin{aligned}
\dot{V}_0(x,z) &= x^\top \mathcal{H}(F_0 A)x + \boxed{2x^\top F_0 B z(1)},\\
\dot{V}_1(x,z) &= \cancel{2\nu x^\top F_1(1) B \partial_\theta z(1)} - \cancel{2\nu x^\top F_1(0) B \partial_\theta z(1)} - \boxed{4\nu x^\top F_1'(1) B z(1)}\\
&\quad + \boxed{2z^\top(1)\int_0^1 B^\top F_1(\theta) B z(\theta)\mathrm{d}\theta} + 2x^\top \int_0^1 (\nu F_1''(\theta) + \lambda F_1(\theta) + A^\top F_1(\theta)) B z(\theta)\mathrm{d}\theta,
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{V}_2(x,z) = &\int_0^1\int_0^1 z^\top(\theta_1) B^\top (\nu\partial_{\theta_1\theta_1} + \nu\partial_{\theta_2\theta_2} + \lambda) F_2(\theta_1,\theta_2) B z(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2\\
&+ \cancel{2\nu\partial_\theta z(1)^\top \int_0^1 (BC)^\top F_2(1,\theta) B z(\theta)\mathrm{d}\theta} - \cancel{2\nu\partial_\theta z(1)^\top \int_0^1 (BC)^\top F_2(0,\theta) B z(\theta)\mathrm{d}\theta}\\
&- \boxed{4\nu z^\top(1)\int_0^1 B^\top \partial_{\theta_1} F_2(1,\theta) B z(\theta)\mathrm{d}\theta}.
\end{aligned}
$$

Firstly, the boxed terms rely functions F_0, F_1, F_2 and the canceled terms give some boundary conditions. It boils down to

$$
F_0 = 2\nu F_1'(1)B, \qquad F_1(1) = F_1(0), \tag{A.22}
$$
$$
F_1(\theta) = 2\nu\partial_{\theta_1} F_2(1,\theta), \qquad F_2(1,\theta) = F_2(0,\theta), \quad \forall\theta\in(0,1), \tag{A.23}
$$

Then, one need to solve the ordinary differential equation satisfied by function F_1 with the boundary condition issued from (A.22) and from the term in x^2

$$
\begin{cases}
\left((\nu\dfrac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \lambda)I_{n_x} + A^\top\right) F_1(\theta) = 0, \quad \forall\theta\in(0,1),\\
F_1'(1)A + A^\top F_1'(1) = -W,\\
F_1(1) - F_1(0) = 0.
\end{cases} \tag{A.24}
$$

Finally, on the triangle $\mathbf{T} = \{(\theta_1,\theta_2)\in[0,1]^2 \text{ s.t. } \theta_1 \geq \theta_2\}$, one need to solve the partial differential equation satisfied by function F_2 with the boundary conditions (A.23)

$$
\begin{cases}
(\nu\partial_{\theta_1\theta_1} + \nu\partial_{\theta_2\theta_2} + \lambda)F_2(\theta_1,\theta_2) = 0, & \forall(\theta_1,\theta_2)\in\mathbf{T},\\
2\nu\partial_{\theta_1} F_2(1,\theta) = F_1(\theta), & \forall\theta\in(0,1),\\
F_2(\theta,0) = F_2^\top(1,\theta), & \forall\theta\in(0,1).
\end{cases} \tag{A.25}
$$

On the hypotenuse of the triangle $\mathbf{T}$, the boundary condition is given by the symmetric condition (A.21), i.e. $F_2(\theta,\theta) = F_2^\top(\theta,\theta)$ for all $\theta\in[0,1]$.

Remark A.4. Note that these equations are similar to the one used for the synthesis of backstepping controllers for ODE-PDE interconnected systems [137, Chapter 15]. There could be connections with the search of backstepping kernels and Volterra integral equations.

For the moment, we did not found analytical solution for such a set of equation and did not even know if the problem is well-posed. That is why the necessity of stability conditions for a system interconnected with the reaction-diffusion equation cannot be addressed in Part III. To find these functionals, it may be useful to follow the Lyapunov-Krasovskii idea [131] with the analytical solutions of the reaction-diffusion equation given by the Fokas method [59]. These extension are kept for future works.

A.3 Resolution of linear matrix inequalities

A.3.1 Convex optimization

Define the notion of convexity for sets and functions.

Definition A.3. A set $C \subset \mathbb{R}^m$ is said to be convex if

$$\forall(\mathbf{x}_1, \mathbf{x}_2) \in C^2,\ \forall\lambda \in [0,1],\ \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in C. \tag{A.26}$$

A function $f : C \to \mathbb{R}$ is said to be convex if C is convex and if

$$\forall(\mathbf{x}_1, \mathbf{x}_2) \in C^2,\ \forall\lambda \in [0,1],\ f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2). \tag{A.27}$$

A convex optimization problem consists in minimizing a convex function on a convex set. The main advantage is that there is no local minimum but only a global optimum. Solvers are then dedicated to find the solution.

A.3.2 Linear matrix inequalities

One or more linear matrix inequalities can always take the form $\mathbf{f}(\mathbf{x}) := \mathfrak{f}_0 + \mathfrak{x}_1\mathfrak{f}_1 + \cdots + \mathfrak{x}_m\mathfrak{f}_m \succ 0$ where $\mathbf{x} = \begin{bmatrix} \mathfrak{x}_1 & \cdots & \mathfrak{x}_m \end{bmatrix}^\top$ are variables in $\mathbb{R}^m$ and $\mathfrak{f}_0, \ldots, \mathfrak{f}_m$ are symmetric matrices in $\mathbb{S}^n$. Linear matrix inequalities constraints can then be represented by the intersection of the cone of positive semi-definite matrices with an affine space. The corresponding formed shape is called spectrahedron and is convex. In the wide class of convex optimization problems, semi-definite programming concerns the minimization of linear functions over a spectrahedron. Therefore, the problem of feasibility of linear matrix inequalities can be reformulated as a semi-definite programming problem. Asking if there exists $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{f}(\mathbf{x}) \succ 0$? amounts to find

$$(\lambda^*, \mathbf{x}^*) = \min_{\lambda\in\mathbb{R}, \mathbf{x}\in\mathbb{R}^m} \lambda \text{ subject to } \lambda I_m + \mathbf{f}(\mathbf{x}) \succ 0 \ \Leftrightarrow\ \bar{\mathbf{x}}^* = \min_{\bar{\mathbf{x}}\in\mathbb{R}^{m+1}} \mathbf{c}^\top\bar{\mathbf{x}} \text{ subject to } \bar{\mathbf{f}}(\bar{\mathbf{x}}) \succ 0, \tag{A.28}$$

with $\mathbf{c} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^\top$. The solution indicates if the problem is feasible or not. Indeed, if the optimal value is strictly negative ie. $\lambda^* = \mathbf{c}^\top\bar{\mathbf{x}}^* < 0$ holds, the problem is feasible and the solution is $\mathbf{x}^*$ since $\mathbf{f}(\mathbf{x}^*) \succ -\lambda^* I_m \succ 0$. Otherwise, if the optimal value is positive ie. $\lambda^* = \mathbf{c}^\top\bar{\mathbf{x}}^* \geq 0$ holds, the problem is unfeasible.

A.3.3 The interior point method

Dedicated algorithms are deployed to obtain a solution to semidefinte programming up to an additive error ε. The computation time is polynomial with respect to the size of the problem and to $\log(\frac{1}{\varepsilon})$. Hereafter, the algorithm based on the interior method is detailed. It is the technique used by *Lmilab*, *Mosek*, *Sedumi* and *SDPT3* solvers.

The interior point method algorithm

- Initialization:
  - Set $\bar{\mathbf{x}}^{(0,0)} = \begin{bmatrix} \lambda^{(0,0)} \\ \mathbf{x}^{(0,0)} \end{bmatrix}$ with a feasible point, ie. $\lambda^{(0,0)} > \bar{\sigma}(-\mathbf{f}(0))$ and $\mathbf{x}^{(0,0)} = 0$.
  - Take $y^{(0)} > c^\top\bar{\mathbf{x}}^{(0,0)} = \lambda^{(0,0)}$.
- Main loop: for $k = 0, \ldots, n_k$,
  - Let $J_k(\bar{\mathbf{x}}) = -\log(\det\bar{\mathbf{f}}(\bar{\mathbf{x}})) - \log(y^{(k)} - \mathbf{c}^\top\bar{\mathbf{x}})$ and denote by H_k, g_k the hessian matrix and gradient vector of the cost function J_k.
  - Inside loop (Newton's algorithm): for $l = 1, \ldots n_l$, and some step size $\alpha^{(l)} \in (0,1)$, compute

    $$\bar{\mathbf{x}}^{(k,l+1)} = \bar{\mathbf{x}}^{(k,l)} - \alpha^{(l)} \left(H_k(\bar{\mathbf{x}}^{(k,l)})\right)^{-1} g_k(\bar{\mathbf{x}}^{(k,l)}),$$

    to approximate the minimum $\bar{\mathbf{x}}^{(k)}$ of J_k.
  - Update $\bar{\mathbf{x}}^{(k+1,0)} = \bar{\mathbf{x}}^{(k,n_l)}$ and, by relaxation, $y^{(k+1)} = (1-\theta)\mathbf{c}^\top\bar{\mathbf{x}}^{(k,n_l)} + \theta y^{(k)}$, for some $\theta \in (0,1)$.

By cumulating costs of the inversion, the Hessian and gradient operations, the cost of such algorithm is in the range of $\underset{n\to\infty}{O}(n_k n_l n^7)$.
To reduce the storage and computation of a large Hessian matrix, first order techniques can also be used like alternating direction method of multipliers [206].

A.3.4 Application to standard stability problem

Consider finite dimensional system (A.1). From the Lyapunov theorem A.1 and A.3, the equilibrium point $x_e = 0$ of system (A.1) is globally exponentially stable if and only if there exits $P \in \mathbb{S}_+^n$ such that $PA + A^\top P \prec 0$.
This stability condition boils down to a feasibility problem: find out $\mathbf{x} = \begin{bmatrix} \mathfrak{x}_{11} & \mathfrak{x}_{21} & \mathfrak{x}_{22} & \mathfrak{x}_{31} & \cdots & \mathfrak{x}_{nn} \end{bmatrix}^\top$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{i} \mathfrak{x}_{ij} \begin{bmatrix} \mathfrak{f}_{ij} & 0 \\ 0 & -\mathfrak{f}_{ij} A - A^\top \mathfrak{f}_{ij} \end{bmatrix} \succ 0, \tag{A.29}$$

with matrix $\mathfrak{f}_{ij} = \begin{bmatrix} \delta_{(1,1),(i,j)} & \cdots & \delta_{(1,n),(i,j)} \\ \vdots & \ddots & \vdots \\ \delta_{(n,1),(i,j)} & \cdots & \delta_{(n,n),(i,j)} \end{bmatrix}$, where $\delta_{(i_1,j_1),(i_2,j_2)}$ denotes the Kronecker delta. The sequence $\{\mathfrak{f}_{ij}\}_{i=\{1,\dots,n\}, j\le i}$ spans the set of symmetric matrices of size n.
Thanks to the expression (A.29), which is a linear matrix inequality, this stability problem can be solved by the algorithms introduced previously. In practice, on Matlab or Yalmip, we use "feasp" or "optimize" routines. The corresponding codes are provided below.

Algorithms to assess stability with linear matrix inequalities

With Matlab toolbox

- Set state matrix A
  $A = \ldots$; $n = \mathrm{size}(A, 1)$;
  $\varepsilon = 1e - 15$; % Precision Matlab
- Set LMIs symmetric variables
  setlmis([]); $P = \mathrm{lmivar}(1, [n\,1])$;
- Configure LMIs
  lmiterm([1 1 1 P], $-1, 1$); % lmi1
  lmiterm([2 1 1 P], $1, A,' s'$); % lmi2
  lmis = getlmis;
- Solve the feasibility problem
  options = [0 0 0 0 0];
  $[t_{feas}, x_{feas}]$ = feasp(lmis,options);
- Solution of the problem
  if $t_{feas} < -\varepsilon$, then (A.1) is stable
  otherwise (A.1) is unstable

With Yalmip toolbox

- Set state matrix A
  $A = \ldots$; $n = \mathrm{size}(A, 1)$;
  $\varepsilon = 1e - 15$; % Precision Matlab
- Set LMIs symmetric variables
  $P = sdpvar(n, n)$;
- Configure LMIs
  lmi1 = $[P >= \varepsilon]$;
  lmi2 = $[A'P + PA <= -\varepsilon]$;
  lmis = [lmi1 lmi2];
- Solve the feasibility problem
  options = sdpsettings('solver',...);
  sol = optimize(lmis,[],options);
- Solution of the problem
  If sol.problem == 0, then (A.1) is stable,
  otherwise (A.1) is unstable

Every linear matrix inequality encountered in this manuscript have been solved likewise.

Appendix **B**

Matrix manipulations

"Mathematics is a dangerous science: it reveals deceptions and miscalculations." G. Galilei.

Contents

B.1 A lemma for complex singular matrix

The next lemma which has also been formulated in [95, Lemma 3] is useful for proving the main results.

Lemma B.1. *Assume $M \in \mathbb{C}^{m\times m}$ such that $\det(M) = 0$. Then, there exist two vectors u_1 and u_2 in $\mathbb{R}^m$ such that the following expressions hold*

$$\begin{cases} M(u_1 + \imath u_2) = 0, & \text{(B.1a)} \\ |u_1| = 1, & \text{(B.1b)} \\ |u_2| \leq 1, & \text{(B.1c)} \\ u_1^\top u_2 = 0. & \text{(B.1d)} \end{cases}$$

Proof. Since $\det(M) = 0$ holds, there exists a non trivial complex vector $v = v_1 + \imath v_2$ with v_1 and v_2 in $\mathbb{R}^m$ such that $Mv = 0$. For any $b \in \mathbb{R}$, one can consider the following vectors as linear combination of v_1, v_2 above

$$w_1 = v_1 + bv_2, \qquad w_2 = -bv_1 + v_2.$$

This selection leads to

$$w_1^\top w_2 = (1 - b^2)v_1^\top v_2 - b(|v_1|^2 - |v_2|^2).$$

Then, we choose number b such that $w_1^\top w_2 = 0$. In the case that $v_1^\top v_2 = 0$, one can take $b = 0$. Otherwise, one can take one of the two real solutions of the following equation

$$b^2 + b\frac{|v_1|^2 - |v_2|^2}{v_1^\top v_2} - 1 = 0.$$

Lastly, since at least one of the two vectors is non null, vectors u_1, u_2 are constructed by homothety

$$u_1 = \begin{cases} \dfrac{w_1}{|w_1|}, \text{ if } |w_1| \geq |w_2|, \\ \dfrac{w_2}{|w_2|}, \text{ if } |w_1| < |w_2|, \end{cases} \qquad u_2 = \begin{cases} \dfrac{w_2}{|w_1|}, \text{ if } |w_1| \geq |w_2|, \\ \dfrac{w_1}{|w_2|}, \text{ if } |w_1| < |w_2|. \end{cases}$$

□

B.2 A technical matrix lemma

Recall the matrix inversion lemma, also called Woodbury matrix identity.

Lemma B.2. *For any vectors u, v in $\mathbb{R}^m$ and non singular matrix M in $\mathbb{R}^{m\times m}$, the following identity holds*

$$1 - v^T(M + uv^\top)^{-1}u = (1 + v^\top M^{-1}u)^{-1}. \tag{B.2}$$

Proof. Some matrix multiplications show that

$$\begin{aligned}(1+v^\top M^{-1}u)(1-v^\top(M + uv^\top)^{-1}u) &= 1+v^\top M^{-1}u - (1 + v^\top M^{-1}u)v^\top(M + uv^\top)^{-1}u, \\ &= 1+v^\top M^{-1}u - (v^\top + v^\top M^{-1}uv^\top)(M + uv^\top)^{-1}u, \\ &= 1+v^\top M^{-1}u - v^\top M^{-1}(M + uv^\top)(M + uv^\top)^{-1}u, \\ &= 1,\end{aligned}$$

holds and concludes the proof. □

Recall also the determinant lemma.

Lemma B.3. *For any vectors u, v in $\mathbb{R}^m$ and non singular matrix M in $\mathbb{R}^{m\times m}$, the following identity holds*

$$\det(M + uv^T) = \det(M)(1 + v^T M^{-1}u). \tag{B.3}$$

Proof. Since det is a homomorphism, we have

$$\det(M + uv^\top) = \det(M)\det(I_m + M^{-1}uv^\top).$$

Then, the result follows from the following equality

$$\begin{bmatrix} I_m & 0 \\ v^\top & 1 \end{bmatrix}\begin{bmatrix} I_m+(M^{-1}u)v^\top & (M^{-1}u) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I_m & 0 \\ -v^\top & 1 \end{bmatrix} = \begin{bmatrix} I_m & (M^{-1}u) \\ 0 & 1+v^\top(M^{-1}u) \end{bmatrix},$$

which means that $\det(I_m + M^{-1}uv^\top) = 1 + v^\top M^{-1}u$ holds. □

Derived from (B.2),(B.3), a usefull lemma can then be formulated.

Lemma B.4. *For any $u \in \mathbb{R}^m$ with a non-zero first component, any non trivial vector $v \in \mathbb{R}^m$ and any strictly lower triangular matrix $L \in \mathbb{R}^{m\times m}$ such that* $\text{rank}(L) = m - 1$*, one obtains*

$$1 - v^T(sI_n + L + uv^\top)^{-1}u = \underset{s\to 0}{O}(s^n). \tag{B.4}$$

Proof. The matrix inversion lemma (B.2) applied to vectors u, v and matrix $M = sI_n + L$ gives

$$1 - v^\top(sI_n + L + uv^\top)^{-1}u = (1 + v^\top(sI_n + L)^{-1}u)^{-1},$$

and the matrix determinant lemma (B.3) applied to the same components leads to

$$\det(sI_n + L + uv^\top) = \det(sI_n + L)(1 + v^\top(sI_n + L)^{-1}u).$$

Then, bringing it together, one obtains

$$1 - v^\top (sI_n + L + uv^\top)^{-1} u = \frac{\det(sI_n + L)}{\det(sI_n + L + uv^T)}.$$

Since L is strictly lower triangular, we have

$$\det(sI_n + L) = \det(sI_n) = s^n.$$

and, because L has non-zero under diagonal coefficients and under the hypothesis done on vectors u, v, matrix $L + uv^\top$ has full rank which means $\det(sI_n + L + uv^\top) \neq 0$ in a neighborhood of $s = 0$. That yields the result for s tending to 0. □

B.3 Legendre polynomial coefficients of exponential matrices

B.3.1 Integration on an interval

In this section, for any matrix M in $\mathbb{R}^{m\times m}$, some techniques are exposed to compute

$$\Gamma_k = \int_0^1 l_k(\theta)\, \mathrm{e}^{M\theta}\, \mathrm{d}\theta, \quad k \in \mathbb{N}, \tag{B.5}$$

where $\{l_k\}_{k\in\mathbb{N}}$ are the Legendre polynomials.

Software of formal calculation can be used but are too slow to be used in practice. If M is a singular matrix, the singular part can be seen as a nilpotent matrix of index ι and expressed on the ι first Legendre polynomials. If M is a non singular matrix, we propose a way to compute quickly such coefficients by induction.

Proposition B.1. *Consider a non singular matrix M in $\mathbb{R}^{m\times m}$. Matrices Γ_k defined by* (B.5) *can be computed by the recursive relation*

$$\Gamma_{k+1} = \Gamma_{k-1} - 2(2k+1)M^{-1}\Gamma_k, \quad \forall k \in \mathbb{N}^*, \tag{B.6}$$

where the initialization is done with

$$\begin{bmatrix}\Gamma_0 \\ \Gamma_1\end{bmatrix} = \begin{bmatrix} M^{-1}(\mathrm{e}^M - I_m) \\ M^{-1}(\mathrm{e}^M + I_m - 2\Gamma_0)\end{bmatrix}. \tag{B.7}$$

Proof. An integration by parts on (B.5) yields

$$\begin{aligned}
\Gamma_{k+1} - \Gamma_{k-1} &= \int_0^1 \big(l_{k+1}(\theta) - l_{k-1}(\theta)\big)\, \mathrm{e}^{\theta M}\, \mathrm{d}\theta, \\
&= -2(2k+1)M^{-1}\int_0^1 l_k(\theta)\, \mathrm{e}^{\theta M}\, \mathrm{d}\theta + M^{-1}\left[\big(l_{k+1}(\theta) - l_{k-1}(\theta)\big)\, \mathrm{e}^{\theta M}\right]_0^1, \\
&= -2(2k+1)M^{-1}\Gamma_k, \qquad \forall k \in \mathbb{N}^*,
\end{aligned}$$

using the differential property $l'_{k+1}(\theta) - l'_{k-1}(\theta) = 2(2k+1)l_k(\theta)$ for all $\theta \in [0,1]$ (2.16) and the point-wise property $l_{k+1}(\theta) = l_{k-1}(\theta)$ for $\theta \in \{0,1\}$ (2.15) satisfied by Legendre polynomials [89]. Moreover, we know that the initial matrices are given by

$$\begin{aligned}
\Gamma_0 &= \int_0^1 \mathrm{e}^{\theta M}\, \mathrm{d}\theta = M^{-1}(\mathrm{e}^M - I_m), \\
\Gamma_1 &= \int_0^1 (2\theta - 1)\, \mathrm{e}^{\theta M}\, \mathrm{d}\theta = M^{-1}(\mathrm{e}^M + I_m) - 2M^{-1}\Gamma_0,
\end{aligned}$$

which concludes the proof. □

The numerical cost to compute the set of matrices $\{\Gamma_k\}_{k\in\{0,\dots,n\}}$ is reduced to a number of operations

in $\underset{n\to\infty}{O}(n)$. The most expensive operation is finally the inversion of matrix M in $\underset{m\to\infty}{O}(m^3)$ by Gauss-Jordan elimination, which is done only once at the beginning.
However, note that the robustness of system (B.6) should be investigated in order to avoid numerical issues for large sizes m or low conditional number for matrix M.

B.3.2 Integration on a triangle

In this section, for any matrix M in $\mathbb{R}^{m\times m}$, an inductive method is exposed to compute

$$\bar{\Gamma}_{jk} = \iint_{\mathbf{T}} l_j(\theta_1) l_k(\theta_2)\, \mathrm{e}^{(\theta_1-\theta_2)M}\, \mathrm{d}\theta_1 \mathrm{d}\theta_2. \tag{B.8}$$

on the triangle $\mathbf{T} = \{(\theta_1, \theta_2) \in [0,1]^2 \text{ s.t. } \theta_1 \geq \theta_2\}$.

Proposition B.2. *Consider a non singular matrix M in $\mathbb{R}^{m\times m}$. Matrices $\bar{\Gamma}_{jk}$ defined by* (B.8) *can be computed by the recursive relation*

$$\bar{\Gamma}_{jk} = \begin{cases} (-1)^{j+k}\bar{\Gamma}_{kj}, & \forall k < j, \\ \bar{\Gamma}_{jk-2} + 2(2k-1)M^{-1}\bar{\Gamma}_{jk-1} - \frac{1}{2j+1}(\delta_{jk} - \delta_{jk-2})M^{-1}, & \forall k \geq \max(2, j), \end{cases} \tag{B.9}$$

where the initialization is done with

$$\begin{bmatrix} \bar{\Gamma}_{00} \\ \bar{\Gamma}_{01} \\ \bar{\Gamma}_{11} \end{bmatrix} = \begin{bmatrix} M^{-1}(\Gamma_0 - I_m) \\ -M^{-1}\Gamma_1 \\ -M^{-1}\left(2\bar{\Gamma}_{01} + \Gamma_1 + \frac{1}{3}I_m\right) \end{bmatrix}. \tag{B.10}$$

Proof. As in the proof of Proposition B.1, an integration by parts on (B.9) and the use of Legendre polynomials properties ensures that $\bar{\Gamma}_{jk}$ satisfies the recursive relation

$$\begin{aligned} \bar{\Gamma}_{jk+1} - \bar{\Gamma}_{jk-1} &= \int_0^1 l_j(\theta_1) \left(\int_0^{\theta_1} \big(l_{k+1}(\theta_2) - l_{k-1}(\theta_2)\big)\, \mathrm{e}^{(\theta_1-\theta_2)M}\, \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1, \\ &= 2(2k+1)M^{-1}\int_0^1 l_j(\theta_1) \left(\int_0^{\theta_1} l_k(\theta_2)\, \mathrm{e}^{(\theta_1-\theta_2)M}\, \mathrm{d}\theta_1 \right) \mathrm{d}\theta_2 \\ &\quad - M^{-1} \int_0^1 l_j(\theta_1)\big(l_{k+1}(\theta_1) - l_{k-1}(\theta_1)\big)\mathrm{d}\theta_1, \\ &= 2(2k+1)M^{-1}\bar{\Gamma}_{jk} - \frac{1}{2j+1}\big(\delta_{jk+1} - \delta_{jk-1}\big)M^{-1}, \quad \forall (j,k) \in \mathbb{N}\times\mathbb{N}^*. \end{aligned}$$

Moreover, using the symmetry of Legendre polynomials $l_k(1-\theta) = (-1)^k l_k(\theta)$ [89], the changes of variables $\theta_1' = 1-\theta_1$ and $\theta_2' = 1-\theta_2$ lead to

$$\begin{aligned} \bar{\Gamma}_{jk} &= \int_0^1 l_j(\theta_1) \left(\int_0^{\theta_1} l_k(\theta_2)\, \mathrm{e}^{(\theta_1-\theta_2)M}\, \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1, \\ &= (-1)^{j+k} \int_0^1 l_j(\theta_1') \left(\int_{\theta_1'}^1 l_k(\theta_2')\, \mathrm{e}^{(\theta_2'-\theta_1')M}\, \mathrm{d}\theta_2' \right) \mathrm{d}\theta_1', \\ &= (-1)^{j+k} \int_0^1 l_k(\theta_2') \left(\int_0^{\theta_1'} l_j(\theta_1')\, \mathrm{e}^{(\theta_2'-\theta_1')M}\, \mathrm{d}\theta_1' \right) \mathrm{d}\theta_2', \\ &= (-1)^{j+k}\bar{\Gamma}_{kj}, \qquad \forall (j,k) \in \mathbb{N}^2, \end{aligned}$$

The initial values are finally given by

$$\bar{\Gamma}_{00} = \int_0^1 \left(\int_0^{\theta_1} \mathrm{e}^{(\theta_1-\theta_2)M} \, \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 = -M^{-1} \int_0^1 \big(I_m - e^{\theta_1 M} \big) \, \mathrm{d}\theta_1 = M^{-1}(\Gamma_0 - I_m),$$

$$\bar{\Gamma}_{10} = \int_0^1 (2\theta_1 - 1) \left(\int_0^{\theta_1} \mathrm{e}^{(\theta_1-\theta_2)M} \, \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 = -M^{-1} \int_0^1 (2\theta_1 - 1) \big(I_m - \mathrm{e}^{\theta_1 M} \big) \mathrm{d}\theta_1 = M^{-1}\Gamma_1,$$

$$\bar{\Gamma}_{11} = \int_0^1 (2\theta_1 - 1) \left(\int_0^{\theta_1} (2\theta_2 - 1) \mathrm{e}^{(\theta_1-\theta_2)M} \mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 = 2M^{-1}\bar{\Gamma}_{10} - M^{-1} \left(\frac{1}{3} I_m + \Gamma_1 \right),$$

which concludes the proof. □

The numerical cost of such operations is given by $\underset{n\to\infty}{O}(n^2)$ for spanning $(j,k) \in \{0, \ldots, n\}^2$ and by $\underset{m\to\infty}{O}(m^3)$ for inverting matrix M in $\mathbb{R}^{m\times m}$.

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