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Certifying Safety for Nonlinear Time Delay Systems via Safety Functionals: A Discretization Based Approach

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***Abstract*— In this paper, we consider the safety of continuous time control systems with input delays. Safety functionals are constructed that define safety sets in the infinite-dimensional state space. Time-discretization is used in order to compute safety sets in finite dimensions and it is shown that these sets approach an infinite-dimensional safety set as the time step is decreased. A simple example of a nonlinear scalar system is used to demonstrate the convergence of the proposed methods.**

I. INTRODUCTION

In recent years there has been an increasing interest in the safety of control systems, with applications ranging from connected automated vehicles [1] to robotic systems [2]. Formally, safety means keeping the state of the system within a given set for all time by ensuring the forward invariance of this set under the closed loop dynamics. It is often achieved with the help of *safety functions* for which *barrier functions* provide a canonical example [3], [4].

Recently these concepts have been extended to time delay systems [5], [6], [7]. Discrete-time linear systems with input delay are discussed in [8], [9] and in [5], [10] using model reduction. Controllers with zero-order hold are analyzed in [11] without computing control invariant sets. Methods for computing invariant sets of autonomous systems in discrete and continuous-time are given in [12], [13]. In particular, one may utilize *safety functionals* (or *barrier functionals*) to ensure the forward invariance of safety sets in the infinite-dimensional state space [6]. However, to the best of our knowledge, there exists no systematic method to construct such functionals and compute the corresponding safety set.

In what follows, we compute safety sets for time delay systems. We approximate the system with a discrete-time map via discretization, and compute forward invariant sets in the corresponding finite-dimensional state space. We prove that by decreasing the discretization step the finite-dimensional safety sets approximate their infinite-dimensional counterpart. We demonstrate the proposed approach on a case study.

The rest of the paper is organized as follows. The problem is formally stated in Sec. II. Forward invariance theorems are listed in Sec. III for continuous-time control systems with delay and discrete-time systems. In Sec. IV safety functionals are constructed and computational tools are developed for the time-discretized system. The methods are applied to a simple example in Sec. V and the results are concluded in Sec. VI.

II. PROBLEM STATEMENT

Let us consider the control system

$$\dot{x}(t) = F\big(x(t), u(t)\big), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represent the state variables and control inputs, and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Formally, the safety of this system can be defined by keeping its state within a desired safe set $S_{\rm des} \subset \mathbb{R}^n$, assumed to be the 0-superlevel set of a function $h_{\rm des} \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$S_{\rm des} = \{x \in \mathbb{R}^n : h_{\rm des}(x) \geq 0\}, \tag{2}$$

where the choice of $h_{\rm des}$ and $S_{\rm des}$ depends on the application.

Using the results in [3], [4], one can design feedback laws of the form $u = k(x)$ with $k : \mathbb{R}^n \to \mathbb{R}^m$ that ensure safety under the closed loop dynamics

$$\dot{x}(t) = F\big(x(t), k(x(t))\big) = G\big(x(t)\big), \tag{3}$$

$G : \mathbb{R}^n \to \mathbb{R}^n$. Safety requires the forward invariance of $S_{\rm des}$ under (3), i.e., if $x(0) \in S_{\rm des}$ then $x(t) \in S_{\rm des}$ for all $t > 0$.

With input delay τ, the control system (1) becomes

$$\dot{x}(t) = F\big(x(t), u(t-\tau)\big). \tag{4}$$

Substituting the same feedback law $u = k(x)$ we obtain the delay differential equation (DDE)

$$\begin{aligned} &\dot{x}(t) = F\big(x(t), k(x(t-\tau))\big) = f\big(x(t), x(t-\tau)\big), \\ &x(\theta) = x_0(\theta), \quad \theta \in [-\tau, 0], \end{aligned} \tag{5}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed locally Lipschitz continuous in both arguments and the initial conditions are given by $x_0 \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ that lead to a unique solution $x(t)$.

Safety can be extended to the time delay system (5) by requiring that if $x(\theta) \in S_{\rm des}$ for all $\theta \in [-\tau, 0]$ then $x(t+\theta) \in S_{\rm des}$ for all $t > 0$ and $\theta \in [-\tau, 0]$. This condition can be reformulated by defining the state

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-\tau, 0], \tag{6}$$

that is an element of the Banach space $\mathcal{B} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ as illustrated in Fig. 1. Note that considering the state as a function over the delay interval is a well-established concept for time delay systems [14], [15].

Safety is then equivalent to the forward invariance of a set $\mathcal{S}_{\rm des} \subset \mathcal{B}$ defined over the Banach space, that is, if $x_0 \in \mathcal{S}_{\rm des}$

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then $x_t \in \mathcal{S}_{\rm des}$ for all $t > 0$. To formally define the desired safety set $\mathcal{S}_{\rm des}$ we use the functional $\mathcal{H}_{\rm des} : \mathcal{B} \to \mathbb{R}$ given by

$$\mathcal{H}_{\rm des}(x_t) = \inf_{\theta \in [-\tau, 0]} h_{\rm des}(x(t+\theta)), \tag{7}$$

and require

$$\mathcal{S}_{\rm des} = \{\phi \in \mathcal{B} : \mathcal{H}_{\rm des}(\phi) \geq 0\}. \tag{8}$$

However, safety for $\tau = 0$, i.e., the invariance of $S_{\rm des}$, does not imply safety for $\tau > 0$, i.e., the invariance of $\mathcal{S}_{\rm des}$, since k did not account for the delay. As of now, there exist no constructive method to design such "delay-resistant" controller. Thus, we intend to tackle the following problem.

Problem Statement 1: *Find the set $\mathcal{S} \subset \mathcal{S}_{\rm des} \subset \mathcal{B}$ that is invariant under the delayed dynamics (5) when the controller $u = k(x)$ is designed to ensure the invariance of $S_{\rm des} \subset \mathbb{R}^n$ in (2) under the non-delayed dynamics (1). Furthermore, identify the functional $\mathcal{H}(x_t)$ whose 0-superlevel set is $\mathcal{S}$.*

Calculating invariant sets over $\mathcal{B}$ is challenging due to the infinite-dimensional nature of this state space. Thus, we approximate (5) with a discrete-time map and we compute invariant sets in the finite-dimensional state space of this map. Then, we show the convergence of the sets as the discretization is refined, to finally obtain $\mathcal{S}$. Once $\mathcal{S}$ is found, one can potentially improve the controller to modify this set as desired; see this idea in [16]. This last step is beyond our scope and we simply study how delays affect set invariance.

III. FORWARD INVARIANCE THEOREMS

In this section, we discuss the safety of dynamical systems using the concept of forward invariant sets in state space. We state forward invariance theorems for DDEs via safety functionals and for discrete-time systems via safety functions.

A. Safety of continuous-time systems with delay

Consider the time delay system (5) and recall a theorem from [6] that ensures the forward invariance of a set $\mathcal{S}$ in the infinite-dimensional state space $\mathcal{B}$ as follows.

Definition 1: The set $\mathcal{S} \subset \mathcal{B}$ is the 0-superlevel set of the continuous functional $\mathcal{H} : \mathcal{B} \to \mathbb{R}$ if

$$\begin{aligned} \mathcal{S} &= \{\phi \in \mathcal{B} : \mathcal{H}(\phi) \geq 0\}, \\ \partial\mathcal{S} &= \{\phi \in \mathcal{B} : \mathcal{H}(\phi) = 0\}, \\ {\rm Int}(\mathcal{S}) &= \{\phi \in \mathcal{B} : \mathcal{H}(\phi) > 0\}. \end{aligned} \tag{9}$$

Theorem 1: *Given the set $\mathcal{S} \subset \mathcal{B}$ that is the 0-superlevel set of the continuously differentiable functional $\mathcal{H} : \mathcal{B} \to \mathbb{R}$, it is forward invariant if for all $x_t \in \mathcal{S}$*

$$\dot{\mathcal{H}}(x_t) \geq -\alpha(\mathcal{H}(x_t)), \tag{10}$$

*where $\dot{\mathcal{H}}(x_t)$ denotes the derivative along the solutions of (5) and α is an extended class $\mathcal{K}$ function. We refer to $\mathcal{S}$ as the **safety set** while $\mathcal{H}$ is called the **safety functional**.*

The proof can be found in [6]. The simplest extended class $\mathcal{K}$ function $\alpha(s) = \gamma s$, $\gamma > 0$ simplifies (10) to

$$\dot{\mathcal{H}}(x_t) + \gamma\mathcal{H}(x_t) \geq 0. \tag{11}$$

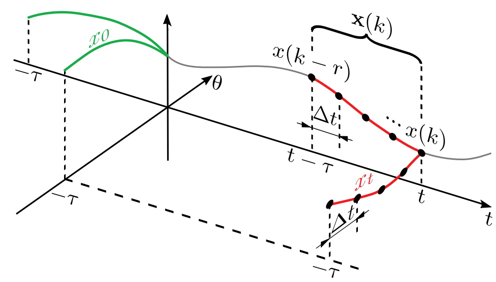

Fig. 1. Time evolution of the state (6) in the infinite-dimensional state space and schematic figure of time discretization (12). Green represents the initial condition x_0, while red highlights the state x_t.

B. Time discretization

Now we approximate DDE (5) by a discrete-time map via time discretization. Let us define the discrete time moments $t_k = k\Delta t$, $k \in \mathbb{N}$, $\Delta t = \tau/r$, where $r \in \mathbb{N}^+$ is the resolution of discretization. We approximate the state $x(t)$ of (5) at the discrete moments $t = t_k$ by a discretized state called $x(k) \in \mathbb{R}^n$ such that $x(k) \approx x(t_k)$. Furthermore, we define

$$\mathbf{x}(k) = \begin{bmatrix} x(k)^\top & x(k-1)^\top & \cdots & x(k-r)^\top \end{bmatrix}^\top, \tag{12}$$

that approximates $r+1$ sampled values from the state $x_t \in \mathcal{B}$ at $t = t_k$; see the illustration in Fig. 1.

To approximate (5), we construct the discrete-time map

$$\mathbf{x}(k+1) = \mathbf{f}\big(\mathbf{x}(k)\big), \tag{13}$$

where $\mathbf{f} \colon \mathbb{R}^{(r+1)n} \to \mathbb{R}^{(r+1)n}$ has the following structure:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ {\rm I} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & {\rm I} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} R(\mathbf{x}) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{14}$$

while I and 0 are n-dimensional identity and zero matrices, and $R \colon \mathbb{R}^{(r+1)n} \to \mathbb{R}^n$ depends on the right hand side of DDE (5) and the numerical scheme used for discretization.

A simple example of a discretization scheme is given below, although there exist several approaches to discretize DDEs [17], [18]. For such discretization schemes, it has been established [19] that when Δt is small enough the solution of (13,14) stays close to the solution of (5). More precisely,

$$\begin{aligned} &\lim_{r\to\infty} \|x(k) - x(t_k)\| = 0, \\ &\lim_{r\to\infty} \left\| \frac{R(\mathbf{x}(k)) - x(k)}{\Delta t} - f\big(x(t_k), x(t_{k-r})\big) \right\| = 0, \end{aligned} \tag{15}$$

$\forall k \in \mathbb{N}$. Throughout the paper $\|\cdot\|$ denotes arbitrary vector norm on $\mathbb{R}^n$ and $\mathbb{C}^m$, $m \in \mathbb{N}^+$ and the associated matrix norm on $\mathbb{C}^{m\times n}$. More details on discretion methods for DDEs, their convergence and examples for R are in [19].

For example, to discretize (5) one can use the first-order Euler-type discretization scheme

$$\dot{x}(t_k) \approx \big(x(k+1) - x(k)\big)/\Delta t. \tag{16}$$

This yields the discrete-time map (13,14) with

$$R(\mathbf{x}) = \mathbf{P}\mathbf{x} + \Delta t\, f(\mathbf{P}\mathbf{x}, \mathbf{Q}\mathbf{x}), \tag{17}$$

where matrices $\mathbf{P}$ and $\mathbf{Q}$ select the actual and delayed terms $x(k)$ and $x(k-r)$ from the discretized state $\mathbf{x}(k)$:

$$\mathbf{P} = \begin{bmatrix} \mathrm{I} & 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 0 & \dots & 0 & \mathrm{I} \end{bmatrix}. \tag{18}$$

Below we formulate safety conditions for map (13) and investigate how they change as the discretization is refined.

C. Safety of discrete-time systems

Safety functions for discrete-time maps were introduced in [20] and further details are given in [21]. Now we state and prove the forward invariance theorem. We use subscript r to highlight that the state space dimension depends on r.

Definition 2: The set $S_r \subset \mathbb{R}^{(r+1)n}$ is the 0-superlevel set of the continuous function $h_r : \mathbb{R}^{(r+1)n} \to \mathbb{R}$ if

$$\begin{aligned} S_r &= \{\mathbf{x} \in \mathbb{R}^{(r+1)n} : h_r(\mathbf{x}) \geq 0\}, \\ \partial S_r &= \{\mathbf{x} \in \mathbb{R}^{(r+1)n} : h_r(\mathbf{x}) = 0\}, \\ \mathrm{Int}(S_r) &= \{\mathbf{x} \in \mathbb{R}^{(r+1)n} : h_r(\mathbf{x}) > 0\}. \end{aligned} \tag{19}$$

Theorem 2: *Given the set $S_r \subset \mathbb{R}^{(r+1)n}$ that is the 0-superlevel set of the continuous function $h_r : \mathbb{R}^{(r+1)n} \to \mathbb{R}$, it is forward invariant if for all $\mathbf{x} \in S_r$*

$$\Delta h_r(\mathbf{x}) \geq -\hat{\alpha}_r\big(h_r(\mathbf{x})\big), \tag{20}$$

where Δh_r is the difference of h_r along the solutions of (13):

$$\Delta h_r(\mathbf{x}) = h_r(\mathbf{f}(\mathbf{x})) - h_r(\mathbf{x}), \tag{21}$$

*$\hat{\alpha}_r$ is a function of the form $\hat{\alpha}_r(s) = s - \alpha_r(s)$ where α_r is an extended class $\mathcal{K}$ function that satisfies $|\alpha_r(s)| < |s|$ for $s \neq 0$. Here S_r is the finite-dimensional representation of the **safety set**, while h_r is called the **safety function**.*

The proof is provided below. We remark that $\hat{\alpha}_r$ is also an extended class $\mathcal{K}$ function if it is strictly monotonically increasing, but this property is not required by Theorem 2. Again, the simplest function one can choose is $\hat{\alpha}_r(s) = \gamma_r s$ with the restriction $0 < \gamma_r < 1$, which simplifies (20) to

$$\Delta h_r(\mathbf{x}) + \gamma_r h_r(\mathbf{x}) \geq 0. \tag{22}$$

Proof. To prove forward invariance, we show $h_r(\mathbf{x}(k)) \geq 0$, $\forall k > 0$ if $h_r(\mathbf{x}(0)) \geq 0$. First, note that (20) is equivalent to

$$h_r\big(\mathbf{f}(\mathbf{x})\big) \geq \alpha_r\big(h_r(\mathbf{x})\big). \tag{23}$$

For the solutions of (13), we thus have

$$h_r(\mathbf{x}(k+1)) \geq \alpha_r\big(h_r(\mathbf{x}(k))\big). \tag{24}$$

Then, consider the system

$$y(k+1) = \alpha_r\big(y(k)\big), \quad y(0) = h_r(x(0)), \tag{25}$$

$y \in \mathbb{R}$. The solution $y(k) = \beta_r\big(h_r(\mathbf{x}(0)), k\big)$ of this system is given by an extended class $\mathcal{KL}$ function β_r, which implies $y(k) \geq 0$, $\forall k > 0$. Since α_r is strictly monotonically increasing, we can apply the discrete-time comparison lemma [21]:

$$h_r(\mathbf{x}(k)) \geq y(k) \geq 0, \tag{26}$$

$\forall k > 0$, which can also be verified by induction. ∎

Note that $|\alpha_r(s)| < |s|$ for $s \neq 0$ is required for (25) to be a contraction. If this does not hold, the trivial fixed point of (25) is not globally stable. Thus $y(k)$ is not given by an extended class $\mathcal{KL}$ function, and for certain initial conditions $y(k)$ converges to a positive fixed point or undergoes unstable increase. If the domain of α_r is bounded, such increase leads out of the domain in finite time.

IV. SAFETY ANALYSIS FOR DELAYED SYSTEMS

Theorems 1 and 2 address the safety of DDE (5) and discrete-time map (13,14) without considering that one is the discretization of the other. Now we link the two theorems.

We restrict ourselves to a class of nonlinear functionals:

$$\mathcal{H}(x_t) = g(\mathcal{L}(x_t)), \tag{27}$$

that is a continuously differentiable nonlinear function $g : \mathbb{C}^m \to \mathbb{R}$, $m \in \mathbb{N}^+$ of a linear functional $\mathcal{L} : \mathcal{B} \to \mathbb{C}^m$. The advantage of this formalism is all linear functionals can be written into the following form using a Stieltjes integral:

$$\mathcal{L}(x_t) = \eta(0)x_t(0) + \int_{-\tau}^{0} \mathrm{d}\eta(\theta) x_t(\theta), \tag{28}$$

where $\eta : [-\tau, 0] \to \mathbb{C}^{m \times n}$ is of bounded variation. The derivative of $\mathcal{H}$ in (27) along the solution of (5) is

$$\dot{\mathcal{H}}(x_t) = \nabla g(\mathcal{L}(x_t))^\top \dot{\mathcal{L}}(x_t), \tag{29}$$

where ∇ denotes gradient and

$$\dot{\mathcal{L}}(x_t) = \eta(0) f\big(x_t(0), x_t(-\tau)\big) + \int_{-\tau}^{0} \mathrm{d}\eta(\theta) x_t'(\theta), \tag{30}$$

with prime being the derivative with respect to θ. Furthermore, for linear functionals one has $\dot{\mathcal{L}}(x_t) = \mathcal{L}(\dot{x}_t)$.

As counterpart to the functional, we consider the function

$$h_r(\mathbf{x}) = g_r(L_r(\mathbf{x})) = g_r(\mathbf{W}_r \mathbf{x}), \tag{31}$$

that is a continuously differentiable nonlinear function $g_r : \mathbb{C}^m \to \mathbb{R}$ of a linear map L_r given by $\mathbf{W}_r \in \mathbb{C}^{m \times (r+1)n}$.

A. Safety under discretization

First, we discuss convergence for the linear map L_r and its difference ΔL_r along the solution of (13,14) to a linear functional $\mathcal{L}$ and its derivative $\dot{\mathcal{L}}$ along the solution of (5).

Lemma 1: *Consider the continuous time delay system (5) and its discretization (13,14) which converges according to (15). Furthermore, consider a sequence of matrices $\mathbf{W}_r \in \mathbb{C}^{m \times (r+1)n}$ whose column blocks $W_{r\ell} \in \mathbb{C}^{m \times n}$ defined by $\mathbf{W}_r = \begin{bmatrix} W_{r0} & W_{r1} & \dots & W_{rr} \end{bmatrix}$ have bounded norm, i.e., there exists $C > 0$ such that $\|W_{r\ell}\| \leq C$, $\forall \ell \in \{0, \dots, r\}$, $\forall r \in \mathbb{N}^+$. Assume that there exists a linear functional $\mathcal{L} : \mathcal{B} \to \mathbb{C}^m$ such that*

$$\lim_{r \to \infty} \left\| \sum_{\ell=0}^{r} W_{r\ell} \phi(-\ell \Delta t) - \mathcal{L}(\phi) \right\| = 0, \tag{32}$$

$\forall \phi \in \mathcal{B}$. Then, one has the following properties for all $k \in \mathbb{N}$ considering the solutions x_t and $\mathbf{x}(k)$ of (5) and (13,14):

$$\lim_{r \to \infty} \big\| L_r(\mathbf{x}(k)) - \mathcal{L}(x_{t_k}) \big\| = 0 \tag{33}$$

and

$$\lim_{r\to\infty} \left\| \Delta L_r(\mathbf{x}(k))/\Delta t - \dot{\mathcal{L}}(x_{t_k}) \right\| = 0, \tag{34}$$

where $L_r(\mathbf{x}) = \mathbf{W}_r \mathbf{x}$, $\Delta L_r(\mathbf{x}) = L_r(\mathbf{f}(\mathbf{x})) - L_r(\mathbf{x})$, *and the derivative is along the solution of (5).*

Proof. Using (12) and the column blocks $W_{r\ell}$, we have

$$\left\|\mathbf{W}_r \mathbf{x}(k) - \mathcal{L}(x_{t_k})\right\| \le \sum_{\ell=0}^{r} \left\|W_{r\ell}\right\| \left\|x(k-\ell) - x(t_{k-\ell})\right\| + \left\| \sum_{\ell=0}^{r} W_{r\ell} x_{t_k}(-\ell\Delta t) - \mathcal{L}(x_{t_k}) \right\|, \tag{35}$$

which leads to (33) considering (15,32) and that $\|W_{r\ell}\|$ is bounded. Using (5,12,14) and $\dot{\mathcal{L}}(x_{t_k}) = \mathcal{L}(\dot{x}_{t_k})$, we get

$$\begin{aligned}
&\left\|\mathbf{W}_r\big(\mathbf{f}(\mathbf{x}(k)) - \mathbf{x}(k)\big)/\Delta t - \dot{\mathcal{L}}(x_{t_k})\right\| \\
&\le \left\|W_{r0}\right\| \left\|\big(R(\mathbf{x}(k)) - x(k)\big)/\Delta t - f\big(x(t_k), x(t_{k-r})\big)\right\| \\
&+ \sum_{\ell=1}^{r} \left\|W_{r\ell}\right\| \left\|\big(x(k-\ell+1) - x(k-\ell)\big)/\Delta t - \dot{x}(t_{k-\ell})\right\| \\
&+ \left\| \sum_{\ell=0}^{r} W_{r\ell} \dot{x}_{t_k}(-\ell\Delta t) - \mathcal{L}(\dot{x}_{t_k}) \right\|,
\end{aligned} \tag{36}$$

which yields (34) via (15,32) and that $\|W_{r\ell}\|$ is bounded. ■

That is, if the weighted sum in (32) converges to an integral given by $\mathcal{L}$ in (28), then Lemma 1 guarantees convergence to linear functionals and their derivatives. Now we consider the nonlinear functional $\mathcal{H}$, and we state sufficient conditions under which the safety of the discretized system (ensured by Theorem 2) guarantees the safety of the time delay system (given by Theorem 1) at the limit $r \to \infty$.

Theorem 3: *Consider the continuous time delay system (5) and its discretization (13,14) which converges according to (15). Consider the continuously differentiable functional* $\mathcal{H} : \mathcal{B} \to \mathbb{R}$ *given by (27) and a sequence of continuously differentiable functions* $h_r : \mathbb{R}^{(r+1)n} \to \mathbb{R}$ *given by (31), which are chosen such that* $\mathbf{W}_r \in \mathbb{C}^{m\times(r+1)n}$ *has the properties listed in Lemma 1, while* g_r *and* g *satisfy*

$$\lim_{r\to\infty} g_r(\boldsymbol{\xi}) = g(\boldsymbol{\xi}), \quad \lim_{r\to\infty} \|\nabla g_r(\boldsymbol{\xi}) - \nabla g(\boldsymbol{\xi})\| = 0, \tag{37}$$

$\forall \boldsymbol{\xi} \in \mathbb{C}^m$. *Assume that the 0-superlevel set* S_r *of* h_r *is forward invariant under the discrete dynamics (13,14) for all* $r \in \mathbb{N}^+$*, i.e., (20) holds for all* $\mathbf{x} \in S_r$ *with* $\hat{\alpha}_r$ *having the properties listed in Theorem 2, and assume that the limit*

$$\alpha(s) = \lim_{r\to\infty} \hat{\alpha}_r(s)/\Delta t \tag{38}$$

exists and $\hat{\alpha}_r$ *is an extended class* $\mathcal{K}$ *function. Then, the 0-superlevel set* $\mathcal{S}$ *of* $\mathcal{H}$ *is forward invariant under the continuous delayed dynamics (5).*

Proof. Since g and g_r are continuous, (27,31,33,37) lead to

$$\mathcal{H}(x_{t_k}) = \lim_{r\to\infty} h_r(\mathbf{x}(k)). \tag{39}$$

Furthermore, one can prove

$$\dot{\mathcal{H}}(x_{t_k}) = \lim_{r\to\infty} \Delta h_r(\mathbf{x}(k))/\Delta t \tag{40}$$

as follows. By the definition (21,31) of Δh_r and the continuous differentiability of g_r, we apply the mean value theorem:

$$\begin{aligned}
\Delta h_r(\mathbf{x}(k)) &= g_r\big(\mathbf{W}_r \mathbf{f}(\mathbf{x}(k))\big) - g_r\big(\mathbf{W}_r \mathbf{x}(k)\big) \\
&= \nabla g_r\big(\mathbf{W}_r \mathbf{x}(k) + \boldsymbol{\varepsilon}_r\big)^\top \mathbf{W}_r (\mathbf{f}(\mathbf{x}(k)) - \mathbf{x}(k)),
\end{aligned} \tag{41}$$

where $\boldsymbol{\varepsilon}_r = \delta \mathbf{W}_r(\mathbf{f}(\mathbf{x}(k) - \mathbf{x}(k))$ with $\delta \in [0,1]$. Notice that $\lim_{r\to\infty} \|\boldsymbol{\varepsilon}_r\| = 0$ due to (34). Since g and g_r are continuously differentiable, we get (40) from (29,33,34,37,41). Then, using (38) and that $\hat{\alpha}_r$ is continuous, (39,40) give

$$\dot{\mathcal{H}}(x_{t_k}) + \alpha(\mathcal{H}(x_{t_k})) = \lim_{r\to\infty} (\Delta h_r(\mathbf{x}(k)) + \hat{\alpha}_r(h_r(\mathbf{x}(k))))/\Delta t. \tag{42}$$

Since (20) holds for all r, taking $r \to \infty$ leads to (10), and the rest of the proof follows from Theorem 1. ■

The simplest choice of α, $\hat{\alpha}_r$ are $\alpha(s) = \gamma s$, $\hat{\alpha}_r(s) = \gamma_r s$, $\gamma_r = \gamma \Delta t$ with $0 < \gamma_r < 1$, $0 < \gamma < 1/\Delta t$, where the upper bound on γ vanishes as $\Delta t \to 0$. Furthermore, we remark that the theorem can be extended to a combination of terms as $h_r(\mathbf{x}) = \sum_{j=0}^{r} g_r^j(\mathbf{W}_r^j \mathbf{x})$ and $\mathcal{H}(x_t) = \sum_{j=0}^{\infty} g^j(\mathcal{L}^j(x_t))$ provided that g_r^j, $\mathbf{W}_r^j$, g^j, $\mathcal{L}^j$ have the properties of g_r, $\mathbf{W}_r$, g, $\mathcal{L}$ for all $j \in \{0, \ldots, r\}$ and the sum is convergent. We will use this form in the case study of Sec.V.

B. Computation of safety sets

Now we discuss the computation of the safety set S_r corresponding to h_r in the finite-dimensional state space $\mathbb{R}^{(r+1)n}$. We approach this by constructing h_r with a free parameter $c_r \in \mathbb{R}$, denoted by $h_r(\mathbf{x}; c_r)$. The parameterization gives rise to a family of 0-superlevel sets $S_{r,c}$:

$$S_{r,c} = \{\mathbf{x} \in \mathbb{R}^{(r+1)n} : h_r(\mathbf{x}; c_r) \ge 0\}, \tag{43}$$

cf. (19), which facilitates finding the forward invariant ones amongst them by tuning parameter c_r. Moreover, we can search for S_r as the largest invariant set amongst $S_{r,c}$ to reduce conservativeness. Based on (22), we use the following condition for invariance:

$$\Delta h_r(\mathbf{x}; c_r) + \gamma_r h_r(\mathbf{x}; c_r) \ge 0. \tag{44}$$

To ensure safety, the set given by (43) must be inside the one defined by (44) as illustrated in Fig. 2(a), where the green domain is inside the red one. Varying c_r may enlarge the safety set, which becomes maximal for $c_r = c_{\rm cr}$ where the surfaces $h_r(\mathbf{x}; c_r) = 0$ and $\Delta h_r(\mathbf{x}; c_r) + \gamma_r h_r(\mathbf{x}; c_r) = 0$ are tangent to each other at a point $\mathbf{x} = \mathbf{x}_{\rm cr}$; see Fig. 2(b). Once the surfaces intersect, (44) is violated in a subset of (43), see Fig. 2(c), and the set in (43) is no longer forward invariant (not even in the subset where (44) holds).

We study the tangency of the surfaces via their gradients

$$\mathbf{w}(\mathbf{x}; c_r) = \nabla h_r, \quad \mathbf{v}(\mathbf{x}; c_r) = \nabla(\Delta h_r + \gamma_r h_r), \tag{45}$$

which are parallel when the surfaces are tangent at $\mathbf{x} = \mathbf{x}_{\rm cr}$ for parameter $c_r = c_{\rm cr}$. The largest invariant set is found by solving the following system of nonlinear algebraic equations

$$\begin{aligned}
&h_r(\mathbf{x}_{\rm cr}; c_{\rm cr}) = 0, \\
&\Delta h_r(\mathbf{x}_{\rm cr}; c_{\rm cr}) + \gamma_r h_r(\mathbf{x}_{\rm cr}; c_{\rm cr}) = 0, \\
&\mathbf{v}(\mathbf{x}_{\rm cr}; c_{\rm cr}) = \lambda \mathbf{w}(\mathbf{x}_{\rm cr}; c_{\rm cr}),
\end{aligned} \tag{46}$$

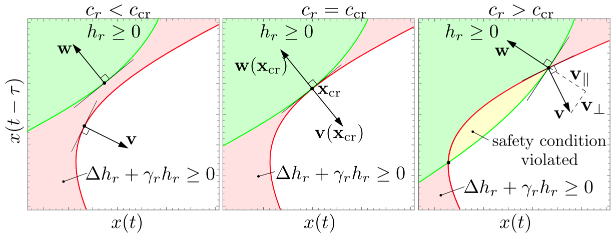

Fig. 2. Graphical representation of the 0-superlevel set (43) (green shading) and the safety condition (44) (red shading).

where $\lambda \in \mathbb{R}$. This contains $(r+1)n+2$ equations and $(r+1)n+2$ unknowns ($\mathbf{x}_{\mathrm{cr}} \in \mathbb{R}^{(r+1)n}$, c_{cr} and λ), thus, the method scales well with the dimension of the problem. Note that the existence of the roots requires appropriate selection of h_r since invariant sets are not of arbitrary shape.

V. CASE STUDY

In order to illustrate the above mathematical construction, we consider the following scalar nonlinear control system

$$\dot{x}(t) = x^3(t) + u(t-\tau), \tag{47}$$

with input delay τ where $x \in \mathbb{R}$ and $u \in \mathbb{R}$; cf. (4). Then, we consider the proportional control $u = -x$, which guarantees for the delay-free case ($\tau = 0$) that the set S_{des} given by $h_{\mathrm{des}}(x) = 1 - x^2$ is safe [6], i.e., $-1 \leq x(t) \leq 1$ for all $t > 0$ if $-1 \leq x(0) \leq 1$. As we will see, this desired set is not invariant when delays arise ($\tau > 0$), and we intend to find invariant sets and compare them to the desired one.

The controller $u = -x$ yields the closed loop dynamics

$$\dot{x}(t) = x^3(t) - x(t-\tau), \tag{48}$$

cf. (5). This system has three equilibria: $x(t) \equiv 0$, which is linearly stable for $0 \leq \tau < \pi/2$ and unstable otherwise, and $x(t) \equiv -1$, $x(t) \equiv 1$, which are unstable for any $\tau \geq 0$.

For $0 \leq \tau < \pi/2$ we expect an invariant domain around the origin, which we compute via discretization. System (48) can be discretized into the form (13,14) with

$$R(\mathbf{x}(k)) = x(k) + \Delta t\big(x(k)^3 - x(k-r)\big), \tag{49}$$

cf. (17). The disretized system has three fixed points: $\mathbf{x}(k) \equiv \mathbf{0}$, $\mathbf{x}(k) \equiv [-1 \cdots -1]^\top$ and $\mathbf{x}(k) \equiv [1 \cdots 1]^\top$. Again, the latter two are unstable for any $\tau \geq 0$ (that appears via $\Delta t = \tau/r$), while the former one is stable below a critical τ and unstable above. In order to make sure that this critical value is $\pi/2$ (as it was in the continuous time case), we rescale time with parameter $a_r = \frac{4r}{\pi} \sin \frac{\pi}{2(2r+1)}$. Such rescaling allows us to compare the results for small values of r, but it is not necessary for observing the convergence for larger values of r (since $a_r \to 1$ as $r \to \infty$).

Finally, we propose the quadratic safety function

$$h_r(\mathbf{x}; c_r) = c_r^2 - \mathbf{x}^\top \mathbf{P}_r \mathbf{x}, \tag{50}$$

where $\mathbf{P}_r = \overline{\mathbf{T}}_r^{-\top} \mathbf{T}_r^{-1} \in \mathbb{R}^{(r+1)\times(r+1)}$, overbar denotes conjugate, and $\mathbf{T}_r \in \mathbb{C}^{(r+1)\times(r+1)}$ is constructed such that its columns are the eigenvectors of the coefficient matrix $\partial \mathbf{f}/\partial \mathbf{x}(\mathbf{0})$ of the linear terms in (13,14,49). Note that (50) is of form $h_r(\mathbf{x}) = \sum_{j=0}^{r} g_r^j(\mathbf{W}_r^j \mathbf{x})$. Here $\mathbf{W}_r^j$ is the j-th row of $\mathbf{T}_r^{-1}$, and $\mathbf{W}_r^j \mathbf{x}$ represents a projection to the j-th eigendirection. This spectral projection has a continuous limit and $\mathbf{W}_r^j$ has the properties listed in Theorem 3. Furthermore, $g_r^j(\xi) = c_r^2/(r+1) - \overline{\xi}\xi$, $j \in \{1, \ldots, r\}$ that also satisfy the conditions in Theorem 3 as long as $c_r^2/(r+1)$ converges for $r \to \infty$, which will be shown below. Also note that, c_r can be considered as the size of the invariant set and one can potentially find the largest one.

The quadratic safety function (50) renders the safety set (43) to be an $(r+1)$-dimensional ellipsoid around the origin in the state space $\mathbb{R}^{(r+1)}$. The left-hand side of (44) becomes

$$\begin{aligned} &\Delta h_r(\mathbf{x}; c_r) + \gamma_r h_r(\mathbf{x}; c_r) \\ &= -\mathbf{f}^\top(\mathbf{x}) \mathbf{P}_r \mathbf{f}(\mathbf{x}) + (1-\gamma_r)\mathbf{x}^\top \mathbf{P}_r \mathbf{x} + \gamma_r c_r^2, \end{aligned} \tag{51}$$

while the gradient vectors in (45) read

$$\begin{aligned} \mathbf{w}(\mathbf{x}; c_r) &= -2\mathbf{P}_r \mathbf{x}, \\ \mathbf{v}(\mathbf{x}; c_r) &= -2\frac{\partial \mathbf{f}^\top(\mathbf{x})}{\partial \mathbf{x}} \mathbf{P}_r \mathbf{f}(\mathbf{x}) + 2(1-\gamma_r)\mathbf{P}_r \mathbf{x}. \end{aligned} \tag{52}$$

Using (50,51,52) in (46) allows us to find the largest invariant set S_r for different r values that are visualized in Fig. 3.

Figure 3a shows the case $r = 1$, where the state space and the safety set are two-dimensional. The state $\mathbf{x}(k)$ consists of the current state $x(k)$ (associated with $x(t)$ in the continuous-time system) and the delayed state $x(k-1)$ (associated with $x(t-\tau)$). The safety set is shaded as green while the safety condition is shown by red for different values of γ_r as indicated. The black curves bound the domain that is forward invariant according to direct numerical iteration of the nonlinear map (13,14,49) for various initial conditions.

Figure 3b illustrates the case $r = 2$ associated with the three-dimensional state space given by $x(k)$, $x(k-1)$ and $x(k-2)$ which are related to $x(t)$, $x(t-\tau/2)$ and $x(t-\tau)$ in the continuous-time system. The safety set is now three-dimensional as shown by the green ellipsoid. The safety condition is shown by the red surface, whereas the boundary of the invariant domain obtained by direct iteration is gray.

While safety sets can be depicted in two- or three-dimensional state spaces ($r = 1, 2$), visualizing them in higher dimensions and comparing them for different r values is challenging. For visualization and comparison only, we restrict to a subset of states, which can be illustrated in two dimensions. We consider states $x_t(\theta)$ that are linear in θ, i.e., in discrete time: $x(k-\ell) = x(k)(1-\ell/r) + x(k-r)\ell/r$, $\ell \in \{0, 1, \ldots, r\}$. Here $x(k)$ and $x(k-r)$ are associated with $x(t)$ and $x(t-\tau)$, thus invariant sets can be illustrated and compared in the plane $(x(t), x(t-\tau))$.

Figure 3c shows the two-dimensional slice of the invariant sets for different mesh numbers r. Green illustrates the safety sets based on forward invariance theorem, while grayscale curves visualize those obtained from direct iteration. As r increases, the green curves converge to an ellipse, which can be considered as the two-dimensional slice of the safety set of the continuous time delay system. The difference between the green and grey curves is due to the selected quadratic

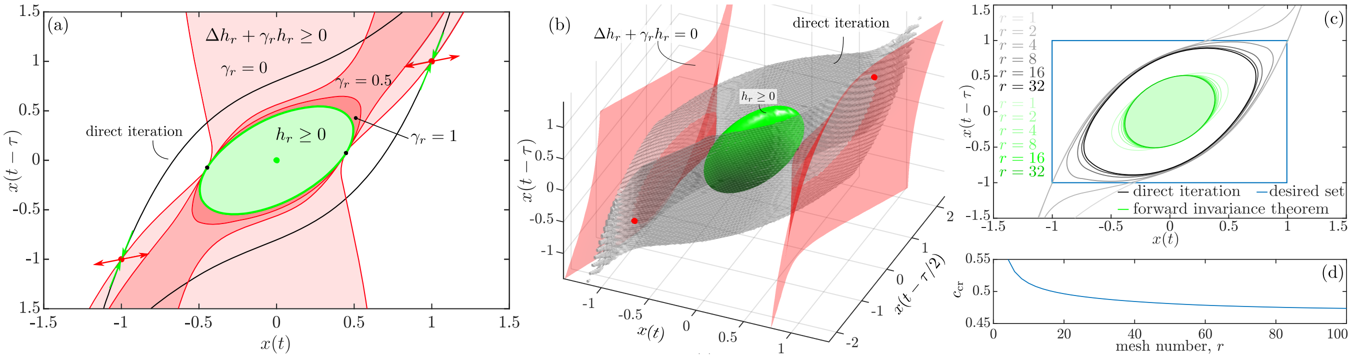

Fig. 3. Safety sets of (48) for $\tau = 1.3$. (a) Safety set in two dimensions ($r = 1$) for $\gamma_r = 0, 0.5$ and 1. Green and red show the safety sets and safety conditions, respectively. Black curves are the stable manifolds of the nontrivial equilibria (red dots) which bound the forward invariant domain around the trivial equilibrium (green dot) obtained by direct iteration. (b) Safety set in three dimensions ($r = 2$) for $\gamma_r = 0$. (c) Two-dimensional slice of the safety sets (green) and invariant domains computed by direct iteration (black) for different values of r. (d) Convergence of the parameter c_{cr} when increasing r.

form of the safety function in (50). We remark that the delay decreases the size of the invariant set; cf. the blue desired set S_{des} that is invariant without the delay.

In order to illustrate the convergence of the safety sets, we use the c_{cr} value in (50) as a linear measure to compare the size of different-dimensional ellipsoids for different mesh numbers. Figure 3d shows that this measure converges to a value as the mesh number r is increased.

Finally, note that direct iterations with various initial conditions can be computed within $\mathcal{O}(p^r)$ seconds for p different initial values in each state, that becomes computationally infeasible for large r. Meanwhile, the approximation of the invariant domain with the forward invariance theorem is computationally effective, we found the computation time for this example to be 1.036^r seconds on a normal laptop.

VI. Conclusions

Safety of time delay systems was analyzed by using safety functionals, time-discretization and safety functions. By discretizing the time delay systems, the infinite-dimensional state space was reduced to a finite-dimensional one, while the safety functionals were approximated by safety functions. This allowed computing forward invariant sets of time delay systems in finite dimensions. The convergence of this method for calculating invariant sets was demonstrated on a scalar example. Our future research includes the use of projections to finite-dimensional subspaces like the one corresponding to the output dynamics, comparing barrier function based methods to those obtained by formal methods [10], and analyzing the effect of the time delay on invariant domains.

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