The complexity of recognizing minimally tough graphs

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Abstract

A graph is called t-tough if the removal of any vertex set S that disconnects the graph leaves at most |S|/t components. The toughness of a graph is the largest t for which the graph is t-tough. A graph is minimally t-tough if the toughness of the graph is t and the deletion of any edge from the graph decreases the toughness. The complexity class DP is the set of all languages that can be expressed as the intersection of a language in NP and a language in coNP. In this paper, we prove that recognizing minimally t-tough graphs is DP-complete for any positive rational number t. We introduce a new notion called weighted toughness, which has a key role in our proof.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let $\omega(G)$ denote the number of components and $\alpha(G)$ denote the independence number of a graph G. (Using $\omega(G)$ to denote the number of components may be confusing, however, most of the literature on toughness uses this notation.) For a graph G and a vertex set $V' \subseteq V(G)$, let G[V'] denote the subgraph of G induced by V'. For a connected graph G, a vertex set $S \subseteq V(G)$ is called a cutset if its removal disconnects the graph.

The notion of toughness was introduced by Chvátal [3].

Definition 1.1. Let t be a real number. A graph G is called t-tough if $|S| \ge t\omega(G-S)$ holds for any vertex set $S \subseteq V(G)$ that disconnects the graph (i.e. for any $S \subseteq V(G)$ with $\omega(G-S) > 1$). The toughness of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \ge 1$. We say that a cutset $S \subseteq V(G)$ is a tough set if $\omega(G-S) = |S|/\tau(G)$.

It follows directly from the definition of toughness that every t-tough noncomplete graph is 2t-connected; therefore, the minimum degree of any t-tough noncomplete graph is at least $\lceil 2t \rceil$.

Clearly, the more edges a graph has, the larger its connectivity can be, so the graphs whose toughness decreases whenever one of their edges are removed might have some interesting properties.

Definition 1.2. A graph G is minimally t-tough if $\tau(G) = t$ and $\tau(G - e) < t$ for all $e \in E(G)$.

The motivation for our research is the following conjecture.

Conjecture 1.3 (Kriesell [4]). Every minimally 1-tough graph has a vertex of degree 2.

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The above conjecture can be naturally generalized to any positive rational number t as follows: every minimally t-tough graph has a vertex of degree $\lceil 2t \rceil$. Note that this conjecture is an analogue of a theorem of Mader stating that for any positive integer k, every minimally k-connected graph has a vertex of degree k, see [11].

Kriesell's conjecture is still open, but in [5] we presented some related results, in particular that in the class of claw-free graphs the conjecture is true in a very strong sense, namely, the only minimally 1-tough, claw-free graphs are cycles of length at least four. On the other hand, we also proved that the class of minimally t-tough graphs is large for any positive rational number t: any graph can be embedded as an induced subgraph into a minimally t-tough graph.

Therefore it is natural to ask, how "large" the set of minimally t-tough graphs is for different t values and for various graph classes. In the present paper we investigate the first question from a complexity theoretical viewpoint. Similar results for the second question are presented in [6].

Let t be an arbitrary positive rational number and consider the following problem.

t-Tough

Instance: a graph G.

Question: is it true that $\tau(G) \geq t$?

Note that in this problem, t is not part of the input.

It is easy to see that for any positive rational number t, the problem t-Tough is in coNP: a witness is a vertex set S whose removal disconnects the graph and leaves more than |S|/t components. By reducing a variant of the independent set problem to the complement of t-Tough, Bauer et al. proved the following.

Theorem 1.4 ([1]). For any positive rational number t, the problem t-Tough is coNP-complete.

However, in some graph classes the toughness can be computed in polynomial time, for instance, in the class of split graphs.

Theorem 1.5 ([15]). For any rational number t > 0, the class of t-tough split graphs can be recognized in polynomial time.

The focus of our investigation is on the critical version of the problem t-Tough. Let t be an arbitrary positive rational number and consider the following problem.

MIN-t-Tough

Instance: a graph G.

Question: is it true that G is minimally t-tough?

Since extremal problems usually seem not to belong to NP \cup coNP, the complexity class called DP was introduced by Papadimitriou and Yannakakis in [13].

Definition 1.6. A language L is in the class DP if there exist two languages $L_1 \in NP$ and $L_2 \in coNP$ such that $L = L_1 \cap L_2$.

A language is called DP-hard if all problems in DP can be reduced to it in polynomial time. A language is DP-complete if it is in DP and it is DP-hard.

It should be emphasized that $DP \neq NP \cap coNP$ if $NP \neq coNP$. Moreover, $NP \cup coNP \subseteq DP$.

To prove that MIN-t-TOUGH is DP-complete for any positive rational number t, we use the following problem for reduction.

α -Critical

Instance: a graph G and a positive integer k.

Question: is it true that $\alpha(G) < k$, but $\alpha(G - e) \ge k$ for any edge $e \in E(G)$?

Note that, unlike in t-TOUGH or MIN-t-TOUGH, in this problem k is part of the input. The DP-completeness of the problem α -Critical is a trivial consequence of the following theorem.

Theorem 1.7 ([14]). The following problem is DP-complete.

CRITICALCLIQUE

Instance: a graph G and a positive integer k.

Question: is it true that G has no clique of size k, but adding any missing edge e to G, the resulting graph G + e has a clique of size k?

Corollary 1.8. The problem α -Critical is DP-complete.

Our main result is the following.

Theorem 1.9. The problem Min-t-Tough is DP-complete for any positive rational number t.

Note that since the toughness of any noncomplete graph is a rational number, there exist no minimally tough graphs with irrational toughness.

To prove the case $t \ge 1$, we introduce a new notion called weighted toughness. However, we believe that this might be an interesting idea on its own.

Definition 1.10. Let t be a positive real number. Given a graph G and a positive weight function w on its vertices, we say that the graph G is weighted t-tough with respect to the weight function w if

$$\omega(G-S) \le \frac{w(S)}{t}$$

holds for any vertex set $S \subseteq V(G)$ whose removal disconnects the graph, where

$$w(S) = \sum_{v \in S} w(v).$$

The weighted toughness of a noncomplete graph (with respect to the weight function w) is the largest t for which the graph is weighted t-tough, and we define the weighted toughness of complete graphs (with respect to w) to be infinity.

Note that the weighted toughness of a graph with respect to the weight function that assigns 1 to every vertex is the toughness of the graph.

The paper is organized as follows. Section 2 gathers the properties of minimally tough graphs and α -critical graphs that are needed to prove Theorem 1.9. Since the proof of this theorem is fairly complicated, Section 3 discusses some of its special cases in the hope of fostering a better understanding. The proof of Theorem 1.9 considers three cases: when 1/2 < t < 1, when $t \ge 1$, and when $t \le 1/2$; they are proved in Sections 4, 5 and 6, respectively.

2 Preliminaries

In this section we collect some useful properties of minimally tough graphs and α -critical graphs.

2.1 Minimally tough graphs

Proposition 2.1. Let $t \leq 1$ be a positive rational number and G a graph with $\tau(G) = t$. Then

$$\omega(G-S) \leq \frac{|S|}{t}$$

for any nonempty proper subset S of V(G).

Proof. If S is a cutset in G, then by the definition of toughness $\omega(G-S) \leq |S|/t$ holds.

If S is not a cutset in G, then $\omega(G-S)=1$ (since $S\neq V(G)$). On the other hand, $|S|/t\geq 1$ since $S\neq\emptyset$ and $t\leq 1$. Therefore $\omega(G-S)\leq |S|/t$ holds in this case as well.

As is clear from its proof, the above proposition holds even if S is not a cutset. However, it does not hold if t > 1 and S is not a cutset: if t > 1, then the graph cannot contain a cut-vertex; therefore $\omega(G - S) = 1$ for any subset S with |S| = 1, while |S|/t = 1/t < 1.

Proposition 2.2. Let G be a connected noncomplete graph on n vertices. Then $\tau(G)$ is a positive rational number, and if $\tau(G) = a/b$, where a, b are relatively prime positive integers, then $1 \le a, b \le n-1$.

Proof. By definition,

$$\tau(G) = \min_{\substack{S \subseteq V(G) \\ \omega(G-S) \ge 2}} \frac{|S|}{\omega(G-S)}$$

for a noncomplete graph G. Since G is connected and noncomplete, $1 \le |S| \le n-2$ for every $S \subseteq V(G)$ with $\omega(G-S) \ge 2$. Obviously, $\omega(G-S) \ge 2$, and since G is connected, $\omega(G-S) \le n-1$.

Corollary 2.3. Let G and H be two connected noncomplete graphs on n vertices. If $\tau(G) \neq \tau(H)$, then

$$\left|\tau(G) - \tau(H)\right| > \frac{1}{n^2}.$$

Proof. Let a, b and a', b' be two pairs of relative prime positive integers such that $\tau(G) = a/b$ and $\tau(H) = a'/b'$. Proposition 2.2 implies that $1 \le a, b, a', b' \le n - 1$. Since $\tau(G) \ne \tau(H)$,

$$\left|\tau(G) - \tau(H)\right| = \left|\frac{a}{b} - \frac{a'}{b'}\right| = \left|\frac{ab' - a'b}{bb'}\right| > \frac{1}{n^2}.$$

Proposition 2.4. For every positive rational number t, the problem Min-t-Tough belongs to DP.

Proof. For any positive rational number t,

$$\begin{aligned} \text{Min-t-Tough} &= \left\{ G \text{ graph } \middle| \ \tau(G) = t \text{ and } \tau(G - e) < t \text{ for all } e \in E(G) \right\} \\ &= \left\{ G \text{ graph } \middle| \ \tau(G) \geq t \right\} \cap \left\{ G \text{ graph } \middle| \ \tau(G) \leq t \right\} \\ &\cap \left\{ G \text{ graph } \middle| \ \tau(G - e) < t \text{ for all } e \in E(G) \right\}. \end{aligned}$$

Let

$$L_{1,1} = \{G \text{ graph } \mid \tau(G - e) < t \text{ for all } e \in E(G)\},$$
$$L_{1,2} = \{G \text{ graph } \mid \tau(G) \le t\}$$

and

$$L_2 = \{ G \text{ graph } | \tau(G) \ge t \}.$$

Notice that $L_2 = t$ -Tough and it is known to be in coNP: if a graph G is not t-tough, then a witness is a vertex set $S \subseteq V(G)$ whose removal disconnects G and leaves more than |S|/t components. Similarly, $L_{1,1} \in \text{NP}$, since a witness is a set of vertex sets $\{S_e \subseteq V(G) \mid e \in E(G)\}$, where for any $e \in E(G)$ the removal of S_e disconnects G - e and leaves more than $|S_e|/t$ components.

Now we show that $L_{1,2} \in NP$, i.e. we can express $L_{1,2}$ in the form

$$L_{1,2} = \{ G \text{ graph } | \tau(G) < t + \varepsilon \},$$

which is the complement of a language belonging to coNP. Let G be an arbitrary graph on n vertices. If G is disconnected, then $\tau(G) = 0$, and if G is complete, then $\tau(G) = \infty$, so in both cases $\tau(G) \leq t$ if and only if $\tau(G) < t + \varepsilon$ for any positive number ε . If G is connected and noncomplete, then from Corollary 2.3 it follows that $\tau(G) \leq t$ if and only if $\tau(G) < t + 1/n^2$. Therefore

$$L_{1,2} = \left\{ G \text{ graph } \middle| \tau(G) \le t \right\} = \left\{ G \text{ graph } \middle| \tau(G) < t + \frac{1}{|V(G)|^2} \right\},$$

so $L_{1,2} \in NP$.

Since $L_{1,1} \cap L_{1,2} \in \text{NP}$ and $L_2 \in \text{coNP}$ and MIN-t-Tough = $(L_{1,1} \cap L_{1,2}) \cap L_2$, we can conclude that MIN-t-Tough $\in \text{DP}$.

Proposition 2.5. Let t be a positive rational number and G a minimally t-tough graph. For every edge e of G,

1. the edge e is a bridge in G, or

2. there exists a vertex set $S = S(e) \subseteq V(G)$ with

$$\omega(G-S) \le \frac{|S|}{t}$$
 and $\omega((G-e)-S) > \frac{|S|}{t}$,

and the edge e is a bridge in G - S.

In the first case, we define $S = S(e) = \emptyset$.

Proof. Let e be an arbitrary edge of G which is not a bridge. Since G is minimally t-tough, $\tau(G-e) < t$. Since e is not a bridge, G-e is still connected, so there exists a cutset $S = S(e) \subseteq V(G-e) = V(G)$ in G-e satisfying $\omega((G-e)-S) > |S|/t$.

By Proposition 2.1, if $t \le 1$, then $\omega(G - S) \le |S|/t$. So assume that t > 1. Now there are two cases.

Case 1: (t > 1 and) S is a cutset in G.

Since $\tau(G) = t$ and S is a cutset, $\omega(G - S) \leq |S|/t$. This is only possible if e connects two components of (G - e) - S.

Case 2: (t > 1 and) S is not a cutset in G.

Then $\omega(G-S)=1$. Since S is a cutset in G-e, the edge e must connect two components of (G-e)-S, so

$$\omega((G-e)-S)=2.$$

Now we show that $\omega(G-S) \leq |S|/t$. Suppose to the contrary that $\omega(G-S) > |S|/t$. Since $\omega(G-S) = 1$, this implies that |S| < t. Moreover, since $\tau(G) = t$, the graph G is $\lceil 2t \rceil$ -connected, and thus it has at least 2t+1 vertices. From this it follows that S and one of the endpoints of e form a cutset in G, otherwise G would only have

$$|S| + 2 < t + 2 < 2t + 1$$

vertices (where the latter inequality is valid since t > 1). Let S' denote this cutset. Since G is t-tough and S' is a cutset in G,

$$2 \le \omega(G - S') \le \frac{|S'|}{t} = \frac{|S| + 1}{t},$$

so $|S| \ge 2t - 1$. Therefore

$$2t - 1 \le |S| < t,$$

which implies that t < 1 and that is a contradiction.

2.2 Almost minimally 1-tough graphs

The graphs K_2 and K_3 behave similarly as minimally 1-tough graphs: they are 1-tough, and the removal of any of their edges decreases their toughness below 1. However, they are not minimally 1-tough since their toughness is infinity. To handle these kinds of graphs, we introduce the following definition.

Definition 2.6. A graph G is almost minimally 1-tough if $\tau(G) \ge 1$ and $\tau(G - e) < 1$ for all $e \in E(G)$.

In fact, the only almost minimally 1-tough graphs are minimally 1-tough graphs and the graphs K_2 and K_3 .

Claim 2.7. For a graph G the following are equivalent.

- (1) The graph G is almost minimally 1-tough.
- (2) The graph G is 1-tough and for every $e \in E(G)$, the edge e is a bridge or there exists a vertex set $S = S(e) \subseteq V(G)$ with

$$\omega(G-S) = |S|$$
 and $\omega((G-e)-S) = |S|+1$.

(If e is a bridge, we define $S = S(e) = \emptyset$.)

(3) The graph G is either minimally 1-tough or $G \simeq K_2$ or $G \simeq K_3$.

Proof.

 $(1)\Longrightarrow (2)$: Let e be an arbitrary edge of G, and let us assume that it is not a bridge. Since $\tau(G-e)<1$ and G-e is still connected, there exists a cutset $S=S(e)\subseteq V(G-e)=V(G)$ in G-e satisfying $\omega((G-e)-S)>|S|$.

Now there are two cases.

Case 1: S is a cutset in G.

Since $\tau(G) \ge 1$ and S is a cutset, $\omega(G-S) \le |S|$. This is only possible if e connects two components of (G-e)-S, which means that

$$\omega((G-e)-S) = |S|+1$$
 and $\omega(G-S) = |S|$.

Case 2: S is not a cutset in G.

Then $\omega(G-S)=1$. On the other hand,

$$\omega((G-e)-S) \ge 2$$

since S is a cutset in G - e. This is only possible if e connects two components of (G - e) - S, which means that

$$\omega((G-e)-S)=2.$$

Since

$$\omega((G-e)-S) > |S|,$$

this implies that $|S| \leq 1$. Moreover, |S| = 1 since e is not a bridge in G. Hence,

$$\omega((G-e)-S) = |S|+1$$
 and $\omega(G-S) = |S|$.

 $(2)\Longrightarrow (3)$: Then $\tau(G)\geq 1$ and $\tau(G-e)<1$ for every $e\in E(G)$. Let us assume that G is not minimally 1-tough, i.e. $\tau(G)>1$. We need to show that $G\simeq K_2$ or $G\simeq K_3$.

Suppose to the contrary that G has at least 4 vertices. Let $e \in E(G)$ be an arbitrary edge, and let $S = S(e) \subseteq V(G)$ be a vertex set for which

$$\omega(G-S) = |S|$$
 and $\omega((G-e)-S) = |S|+1$.

Since $\tau(G) > 1$ and $\omega(G - S) = |S|$, the vertex set S cannot be a cutset in G, so $|S| \le 1$ must hold. Since G has at least 4 vertices, S and one of the endpoints of e form a cutset of size at most 2, so $\tau(G) \le 1$, which is a contradiction. This means that $G \simeq K_2$ or $G \simeq K_3$, since there are no other 1-tough graphs on at most 3 vertices with at least one edge.

$$(3) \Longrightarrow (1)$$
: Trivial.

Proposition 2.8. Let G be an almost minimally 1-tough graph. Then $\omega(G-S) \leq |S|$ for any nonempty proper subset S of V(G).

Proof. By Claim 2.7, the graph G is either minimally 1-tough or $G \simeq K_2$ or $G \simeq K_3$. If G is minimally 1-tough, then $\tau(G) = 1$, and we already covered this case in Proposition 2.1. If $G \simeq K_2$ or $G \simeq K_3$, then $\omega(G - S) = 1$ and $1 \le |S| \le 2$ hold for any nonempty proper subset S of V(G).

2.3 α -critical graphs

First, we cite some results on α -critical graphs.

Proposition 2.9 (Problem 12 of §8 in [9]). If G is an α -critical graph without isolated vertices, then every point is contained in at least one maximum independent vertex set.

Lemma 2.10 (Problem 14 of §8 in [9]). If we replace a vertex of an α -critical graph with a clique, and connect every neighbor of the original vertex with every vertex in the clique, then the resulting graph is still α -critical.

Lemma 2.11 ([10]). Let G be an α -critical graph and w an arbitrary vertex of degree at least 2. Split w into two vertices y and z, each of degree at least 1, add a new vertex x and connect it to both y and z. Then the resulting graph G' is α -critical, and $\alpha(G') = \alpha(G) + 1$.

For one of our proofs we also need the following observation, which is a straightforward consequence of Corollary 1.8 and Lemmas 2.10 and 2.11.

Proposition 2.12. For any positive integers l and m, the following variant of the problem α -Critical is DP-complete.

Instance: an l-connected graph G and a positive integer k that is divisible by m. Question: is it true that $\alpha(G) < k$, but $\alpha(G - e) \ge k$ for any edge $e \in E(G)$?

3 On some special cases of Theorem 1.9

This section aims to highlight the key steps of the proof of Theorem 1.9 by considering some simpler cases of it. In the view of this intention, technical details are omitted here.

Let $n \geq 2$ be an integer and let G be a complete graph of size n on the vertex set $\{v_1, \ldots, v_n\}$. Add the vertices u_1, \ldots, u_n and w to G, and for all $i \in [n]$ connect v_i and u_i , and also u_i and w, and let G' denote the obtained graph. (For an example see Figure 10 in the Appendix.) It is easy to see that G' is a minimally 1-tough graph, and it is due to the fact that complete graphs are α -critical. This plain construction inspires all the others proposed in this paper. This construction can be generalized for α -critical graphs in general to obtain a minimally 1-tough graph. (See Figure 11.) The construction for minimally integer-tough graphs can be seen as a "blow-up" of the minimally 1-tough construction. (See Figure 12.) These constructions are described in details in the following subsection.

3.1 On the case of minimally t-tough graphs, where t is a positive integer

Let t, k and $n \ge t + 1$ be positive integers, let G be an arbitrary $\lceil (t+1)/2 \rceil$ -connected graph on the vertices v_1, \ldots, v_n , and let $G'_{t,k}$ be defined as follows. For all $i \in [n]$ and $j \in [k]$ let

$$V_{i,j} = \{v_{i,j,l} \mid l \in [t]\}.$$

For all $i \in [n]$ let

$$V_i = \bigcup_{j \in [k]} V_{i,j}$$

and place a complete graph on its vertices. For all $i_1, i_2 \in [n]$ if $v_{i_1}v_{i_2} \in E(G)$, then place a complete bipartite graph on $(V_{i_1}; V_{i_2})$. For all $i \in [n]$ and $j \in [k]$ add the vertex set

$$U_{i,j} = \left\{ u_{i,j,l} \mid l \in [t] \right\}$$

to the graph and place a complete graph on the vertices of $U_{i,j}$. For all $i \in [n], j \in [k], l \in [t]$ connect $v_{i,j,l}$ to $u_{i,j,l}$. For all $j \in [k]$ add the vertex set

$$W_j = \{w_{j,1}, \dots, w_{j,t}\}$$

to the graph and for all $i \in [n]$ place a complete bipartite graph on $(U_{i,j}; W_j)$. Let

$$V = \bigcup_{i=1}^{n} V_i, \qquad U = \bigcup_{i=1}^{n} \bigcup_{j=1}^{k} U_{i,j}, \qquad W = \bigcup_{j=1}^{k} W_j.$$

See Figure 1. (For examples see Figures 11 and 12.)

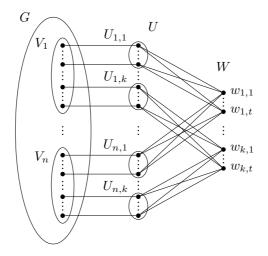


Figure 1: The graph $G'_{t,k}$, when t is a positive integer.

Claim 3.1. Let G be an arbitrary $\lceil (t+1)/2 \rceil$ -connected graph. Then G is α -critical with $\alpha(G) = k$ if and only if $G'_{t,k}$ is minimally 1-tough.

Proof. The cases t = 1 and $t \ge 2$ should be handled separately, but since the main steps of the proofs are similar, only the (easier) case t = 1 is presented here.

The proof of the following lemma is omitted now. (In Section 5 a similar lemma is proved, but for a more complex graph, see Lemma 5.6.)

Lemma 3.2. If $\alpha(G) \leq k$, then $\tau(G'_{1,k}) = 1$.

Accepting this lemma, all we have left to show is that

- if G is α -critical with $\alpha(G)=k$, then $\tau(G'_{1,k}-e)<1$ holds for any $e\in E(G)$,
- if $\alpha(G) > k$, then $\tau(G'_{1,k}) < 1$, and
- if either $\alpha(G) = k$ but the graph G is not α-critical or $\alpha(G) < k$, then there exists an edge $e \in E(G)$ for which $\tau(G'_{1,k} e) = 1$.

Assume first that G is α -critical with $\alpha(G)=k$. Let $e\in E(G'_{1,k})$ be an arbitrary edge. If e is incident to one of the vertices of U, i.e., to a vertex of degree 2, then clearly $\tau(G'_{1,k}-e)<1$. If e is not incident to any of the vertices of U, then it connects two vertices of V. By Lemma 2.10, the subgraph $G'_{1,k}[V]$ is α -critical, so in $G'_{1,k}[V]-e$ there exists an independent vertex set I of size $\alpha(G)+1$. Let

$$S = (V \setminus I) \cup W.$$

Then it is easy to see that

$$|S| = |V| - 1$$
 and $\omega((G'_{1,k} - e) - S) = |V|$

hold, so $\tau(G'_{1,k} - e) < 1$.

Now assume $\alpha(G) > k$. Then let I be an independent vertex set of size $\alpha(G)$ in $G'_{1,k}[V]$, and let

$$S = (V \setminus I) \cup W.$$

Then

$$|S| < |V|$$
 and $\omega(G'_{1,k} - S) = |V|$

hold, so $\tau(G'_{1,k}) < 1$.

Finally, assume that either $\alpha(G) = k$ but the graph G is not α -critical or $\alpha(G) < k$. Then there exists an edge $e \in E(G)$ such that $\alpha(G - e) \le k$. By Lemma 3.2, the graph $(G - e)'_{1,k}$ is 1-tough, but we can obtain $(G - e)'_{1,k}$ from $G'_{1,k}$ by edge-deletion, which means that $G'_{1,k}$ is not minimally 1-tough.

Corollary 3.3. For any positive integer t, the problem Min-t-Tough is DP-complete.

Proof. In Proposition 2.4 we already proved that the problem Min-t-Tough is in DP, and it follows from Claim 3.1 that we can reduce α -Critical to it, but for this it should be also noted that $G'_{t,k}$ can be constructed from G in polynomial time.

The above construction works only in the case when t is a positive integer for the simple reason that the sets $V_{i,j}$, $U_{i,j}$ and W_j consist of t vertices.

3.2 On the case of minimally 1/b-tough graphs, where $b \ge 2$ is an integer

Up to this point, we only handled the case when t is a positive integer. To prove Theorem 1.9 for the noninteger cases, we modify the previous constructions and here we illustrate these modifications with the following simple example.

Let $b \ge 2$ be an integer, let t = 1/b, let G be an arbitrary connected graph, and let G_t be defined as follows. Add b-1 independent vertices for each original vertex $v \in V(G)$ to the graph G, and connect them to v (see Figure 2). (For an example see Figure 13.)

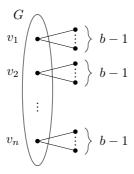


Figure 2: The graph G_t when t = 1/b, where $b \ge 2$ is an integer.

Claim 3.4. Let G be an arbitrary connected graph. Then G_t is minimally t-tough if and only if G is almost minimally 1-tough.

Similarly as before, we can conclude the following.

Corollary 3.5. For every integer $b \ge 2$, MIN-1/b-TOUGH is DP-complete.

In Section 6, this latter idea is extended to the case when t <= 1/2 by "gluing" some other graph to the vertices of the original graph G. (See Figure 14.) It is worth noting that in the case when t = 1/b for some integer $b \ge 2$, the obtained graph in Section 6 is exactly the same as the graph G_t constructed here. After this "gluing", the vertices of G become cut-vertices in the obtained graph G_t , thus the toughness of G_t can be at most 1/2. The plan for the cases when t > 1/2 is to perform this so called "gluing" by identifying not only one, but 2t vertices of a smaller and a larger graph, where the larger graph resembles a minimally $\lceil t \rceil$ -tough graph and the "gluing" procedure aims to decrease its toughness to the desired value t. In fact, in Sections 4 and 5 this larger graph is chosen to be a slight modification of $G'_{\lceil t \rceil,k}$.

4 Minimally t-tough graphs, where 1/2 < t < 1

Before proving Theorem 1.9 for any positive rational number 1/2 < t < 1, we need some preparation. First, we construct some auxiliary graphs.

4.1 The auxiliary graph $H_{t,k}^{**}$ when 1/2 < t < 1

Let t be a rational number such that 1/2 < t < 1. Let a, b be relatively prime positive integers such that t = a/b. Let k be a positive integer, and let

$$W = \{w_1, \dots, w_{ak}\}$$
 and $W' = \{w'_1, \dots, w'_{(b-1)k}\}.$

Place a clique on the vertices of W and a complete bipartite graph on (W;W'). Obviously, the toughness of this complete split graph is a/(b-1) > t. Deleting an edge may decrease the toughness, and now we delete edges incident to W' until the toughness remains at least t but the deletion of any other such edge would result in a graph with toughness less than t. Let $H_{t,k}^*$ denote the obtained split graph. Then $\tau(H_{t,k}^*) \geq t$, and $\tau(H_{t,k}^* - e) < t$ for any edge $e \in E(H_{t,k}^*)$ incident to W', i.e. there exists a vertex set $S = S(e) \subseteq W$ whose removal disconnects $H_{t,k}^* - e$ and

$$\omega((H_{t,k}^* - e) - S) > \frac{|S|}{t}.$$

Now delete all the edges induced by W, and let $H_{t,k}^{**}$ denote the obtained bipartite graph.

4.2 The auxiliary graph H''_t when 1/2 < t < 1

Let t be a rational number such that 1/2 < t < 1. Let a, b be relatively prime positive integers such that t = a/b and let H_t be constructed as follows. Let

$$A = \{v_1, v_2, \dots, v_a\}, \quad B = \{u_1, u_2, \dots, u_b\}.$$

For any $i \in [a]$ and $j \in [b-1]$ connect v_i to u_j , and connect u_b to v_1 and v_a . (In other words, H_t can be obtained from the complete bipartite graph $K_{a,b}$ by deleting a-2 edges incident to one vertex of the color class of size b. See Figure 3.)

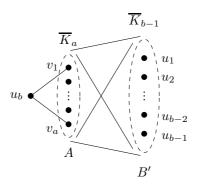


Figure 3: The graph H_t , when 1/2 < t < 1.

Claim 4.1. Let t be a rational number such that 1/2 < t < 1. Then $\tau(H_t) = t$.

Proof. Let $B' = B \setminus \{u_b\}$ and let S be an arbitrary cutset in H_t . Now we show that $\omega(H_t - S) \leq |S|/t$.

Case 1: $A \subseteq S$.

Then $|S| \ge a$ and $\omega(H_t - S) \le b$. Since t = a/b < 1, it follows that

$$\omega(H_t - S) \le b = \frac{a}{t} \le \frac{|S|}{t}.$$

Case 2: $B' \subseteq S$.

If $u_b \in S$ as well, then $|S| \geq b$ and $\omega(H_t - S) \leq a$. Since t = a/b < 1, it follows that

$$\omega(H_t - S) \le a = bt < \frac{b}{t} \le \frac{|S|}{t}.$$

If $u_b \notin S$, then $|S| \ge b-1$ and $\omega(H_t-S) \le a-1$. Since t=a/b < 1, it follows that

$$\omega(H_t - S) \le a - 1 \le \frac{b - 1}{t} \le \frac{|S|}{t}.$$

Case 3: $A \nsubseteq S$ and $B' \nsubseteq S$.

Then $\omega(H_t - S) \leq 2$, but since S is a cutset, $\omega(H_t - S) = 2$. Obviously, there is no cut-vertex in H_t , thus $|S| \geq 2$. Since t < 1, it follows that

$$\omega(H_t - S) = 2 < \frac{2}{t} \le \frac{|S|}{t}.$$

Hence $\tau(H_t) \geq t$. On the other hand, the vertex set S = A is a cutset in H_t with |S| = a and $\omega(H_t - S) = b$, so $\tau(H_t) \leq t$.

Therefore,
$$\tau(H_t) = t$$
.

By repeatedly deleting some edges of H_t , eventually we obtain a minimally t-tough graph, let us denote it with H'_t (i.e. if there exists an edge whose deletion does not decrease the toughness, then we delete it). Obviously, we could not delete the edges incident to u_b , so the vertex u_b still has degree 2. Let e denote the edge connecting v_1 and u_b and let $H''_t = H'_t - e$. Note that H''_t is a bipartite graph with color classes A and B.

4.3 The proof of Theorem 1.9 when 1/2 < t < 1

Theorem 4.2. For any rational number t with 1/2 < t < 1, the problem Min-t-Tough is DP-complete.

Proof. Let t be a rational number such that 1/2 < t < 1. In Proposition 2.4 we already proved that the problem MIN-t-TOUGH is in DP. To show that it is DP-hard, we reduce α -CRITICAL to it.

Let a, b be relatively prime positive integers such that t = a/b, let G be an arbitrary 2-connected graph on the vertices v_1, \ldots, v_n and let $G_{t,k}$ be defined as follows. For all $i \in [n]$ let

$$V_i = \{v_{i,j} \mid i \in [n], j \in [ak]\}$$

and place a clique on the vertices of V_i . For all $i_1, i_2 \in [n]$ if $v_{i_1}v_{i_2} \in E(G)$, then place a complete bipartite graph on $(V_{i_1}; V_{i_2})$. (This subgraph is denoted by \tilde{G} in Figure 4.) For all $i \in [n], j \in [ak]$ "glue" the graph H''_t to the vertex $v_{i,j}$ by identifying $v_{i,j}$ with the vertex v_1 of H''_t and let $H^{i,j}$ denote the (i,j)-th copy of H''_t and let $A^{i,j}$ denote the (i,j)-th copies of the vertices v_a and v_b , respectively. Let

$$V = \bigcup_{i=1}^{n} V_i,$$

$$V' = \{ v'_{i,j} \mid i \in [n], j \in [ak] \}$$

and

$$U = \big\{u_{i,j} \mid i \in [n], j \in [ak]\big\}.$$

Add the vertex sets

$$W = \left\{ w_j \mid j \in [ak] \right\}$$

and

$$W' = \{w'_1, \dots, w'_{(b-1)k}\}$$

to the graph and place the bipartite graph $H_{t,k}^{**}$ on (W;W'). For all $i \in [n]$ and $j \in [ak]$ connect w_j to $u_{i,j}$. See Figure 4. (For an example see Figure 15.) Now k is part of the input of the problem α -Critical, therefore the graph $H_{t,k}^{**}$ must be constructed in polynomial time and by Theorem 1.5, this can be done. On the other hand, t is not part of the input of the problem MIN-t-Tough, therefore the graph H_t'' can be constructed in advance. Hence, $G_{t,k}$ can be constructed from G in polynomial time.

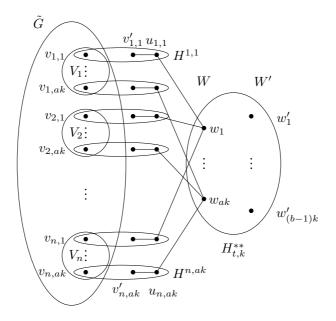


Figure 4: The graph $G_{t,k}$, when 1/2 < t < 1.

To show that G is α -critical with $\alpha(G) = k$ if and only if $G_{t,k}$ is minimally t-tough, first we prove the following lemma.

Lemma 4.3. Let G be a 2-connected graph with $\alpha(G) \leq k$. Then $G_{t,k}$ is t-tough.

Proof. Let $S \subseteq V(G_{t,k})$ be a cutset in $G_{t,k}$. We need to show that $\omega(G_{t,k} - S) \leq |S|/t$. First, we show that the following assumption can be made for S.

(1) $U \cap S = \emptyset$.

Suppose that $u_{i,j} \in S$ for some $i \in [n], j \in [ak]$. If $v'_{i,j} \in S$, then after the removal of $v'_{i,j}$, the vertex $u_{i,j}$ has degree 1, so there is no need to remove it. Similarly, if $w_j \in S$, then we can also assume that $u_{i,j} \notin S$. If $v'_{i,j}, w_j \notin S$, then considering $S' = S \setminus \{u_{i,j}\}$ instead of S decreases the number of components only by one, meaning that if S' is a cutset in $G_{t,k}$, then it is enough to show that $\omega(G_{t,k} - S') \leq |S'|/t$ since it implies

$$\omega(G_{t,k} - S) = \omega(G_{t,k} - S') + 1 \le \frac{|S'|}{t} + 1 = \frac{|S| - 1}{t} + 1 \le \frac{|S|}{t},$$

where the last inequality is valid since t < 1. If S' is not a cutset in $G_{t,k}$, then $\omega(G_{t,k} - S) = 2$ and $|S| \ge 2$ since $u_{i,j}$ has degree 2 and is not a cut-vertex in $G_{t,k}$, i.e.

$$\omega(G_{t,k} - S) = 2 \le |S| \le \frac{|S|}{t},$$

where again the last inequality is valid since t < 1. This completes the validation of assumption (1).

Now there are two cases.

Case 1: $W \subseteq S$.

After the removal of W, the vertices of W' are isolated; therefore we can assume that $W' \cap S = \emptyset$. To write up a formula for |S| and $\omega(G_{t,k} - S)$, we need to introduce some notations. Let

$$C = \{(i,j) \in [n] \times [ak] \mid v_{i,j} \in V \cap S\},$$
$$c_{i,j} = |V(H^{i,j}) \cap S| - 1$$

for all $(i, j) \in C$, and

$$d_{i,j} = |V(H^{i,j}) \cap S|$$

for all $(i, j) \in ([n] \times [ak]) \setminus C$. Finally, let

$$D = \{(i,j) \in ([n] \times [ak]) \setminus C \mid d_{i,j} > 0\}.$$

Using these notations it is clear that

$$|S| = \sum_{(i,j)\in[n]\times[ak]} |V(H^{i,j})\cap S| + |W| = |C| + \sum_{(i,j)\in C} c_{i,j} + \sum_{(i,j)\in D} d_{i,j} + ak.$$

By the assumption that $W \subseteq S$, in $G_{t,k} - S$ the (b-1)k vertices of W' are isolated. Since $\alpha(G_{t,k}[V]) = \alpha(G)$, the removal of $V \cap S$ from $G_{t,k}[V]$ leaves at most $\alpha(G)$ components. By Claim 4.1 and Proposition 2.1, for any $(i,j) \in C$ the removal of $V(H^{i,j}) \cap S$ from $H^{i,j}$ leaves at most $(c_{i,j}+1)/t$ components. By Proposition 2.1, for any $(i,j) \in D$ the removal of $V(H^{i,j}) \cap S$ from $H^{i,j}$ leaves at most $d_{i,j}/t+1$ components, but the component of $v_{i,j}$ has been already counted. Hence

$$\omega(G_{t,k} - S) \le (b - 1)k + \alpha(G) + \sum_{(i,j) \in C} \frac{c_{i,j} + 1}{t} + \sum_{(i,j) \in D} \frac{d_{i,j}}{t}$$
$$\le bk + \frac{|C| + \sum_{(i,j) \in C} c_{i,j} + \sum_{(i,j) \in D} d_{i,j}}{t} = \frac{|S|}{t},$$

using that $\alpha(G) \leq k$.

Case 2: $W \nsubseteq S$.

Assume that $w_{j_0} \notin S$ for some $j_0 \in [ak]$. In this case, using assumption (1), we can also assume the following.

(2) There exists at most one $i \in [n]$ for which $v_{i,j_0} \in S$.

Suppose that $v_{i_1,j_0}, v_{i_2,j_0} \in S$ for some $i_1, i_2 \in [n]$. By assumption (1), the component of w_{j_0} contains all of the vertices $u_{1,j_0}, u_{2,j_0}, \ldots, u_{n,j_0}$. Now considering the cutset $S' = S \cup \{w_{j_0}\}$ instead of S increases the number of components by at least two: it disconnects both u_{i_1,j_0} and u_{i_2,j_0} from the vertices $\{u_{i,j_0} \mid i \in [n] \setminus \{i_1,i_2\}\}$, and it also disconnects u_{i_1,j_0} from u_{i_2,j_0} (and of course it can also disconnect other vertices of $\{u_{i,j_0} \mid i \in [n]\}$ from each other). Then it is enough to show that $\omega(G_{t,k} - S') \leq |S'|/t$ since it implies

$$\omega(G_{t,k} - S) \le \omega(G_{t,k} - S') - 2 \le \frac{|S'|}{t} - 2 = \frac{|S| + 1}{t} - 2 < \frac{|S|}{t},$$

where the last inequality is valid since t > 1/2. Proceeding further, we can obtain a cutset S^* for which $W \subseteq S^*$ holds; and such sets were already handled in Case 1.

(3) $(G_{t,k} - S)[V]$ is connected.

By assumption (2), there exists at most one $i \in [n]$ for which $V_i \subseteq S$. Since G is 2-connected, this implies that $(G_{t,k} - S)[V]$ is connected.

- (4) There exists at most one $i \in [n]$ for which v_{i,j_0} and u_{i,j_0} belong to different components in $G_{t,k} S$. Suppose that v_{i_1,j_0}, u_{i_1,j_0} belong to different components in $G_{t,k} - S$, and so do v_{i_2,j_0}, u_{i_2,j_0} for some $i_1, i_2 \in [n]$. Similarly as in the proof of assumption (2), considering the cutset $S' = S \cup \{w_{j_0}\}$ instead of S increases the number of components by at least two, so it is enough to show that $\omega(G_{t,k} - S') \leq |S'|/t$.
- (5) In $G_{t,k} S$ all the remaining vertices of $\{v_{i,j_0}, u_{i,j_0} \mid i \in [n]\}$ belong to the component of w_{j_0} . It follows directly from assumptions (1), (2) and (3).
- (6) In $G_{t,k} S$ all the remaining vertices of V belong to the component of w_{j_0} . It follows directly from assumptions (3) and (5).

(7) In $G_{t,k} - S$ all the remaining vertices of $V \cup W$ belong to the same component.

It follows directly from assumptions (5) and (6).

By assumption (7), in $G_{t,k} - S$ there is a component containing all the remaining vertices of $V \cup W$, and every other component is either an isolated vertex of W' (since $G_{t,k}[W \cup W']$ is a bipartite graph) or a component of $H^{i,j} - (V(H^{i,j}) \cap S)$ for some $i \in [n], j \in [ak]$. Hence we can also assume the following.

(8) $W' \cap S = \emptyset$.

By assumption (5) and Proposition 2.1 and the properties of $H_{t,k}^{**}$, the removal of $W \cap S$ from $H_{t,k}^{**}$ leaves at most $|W \cap S|/t$ components, but the component of w_{j_0} has been already counted.

Using the previous notations,

$$|S| = |C| + \sum_{(i,j) \in C} c_{i,j} + \sum_{(i,j) \in D} d_{i,j} + |W \cap S|$$

and

$$\omega(G_{t,k} - S) \le 1 + \sum_{(i,j) \in C} \frac{c_{i,j} + 1}{t} + \sum_{(i,j) \in D} \frac{d_{i,j}}{t} + \left(\frac{|W \cap S|}{t} - 1\right)$$
$$= \frac{|C| + \sum_{(i,j) \in C} c_{i,j} + \sum_{(i,j) \in D} d_{i,j}}{t} + \frac{|W \cap S|}{t} = \frac{|S|}{t}.$$

This means that $\tau(G_{t,k}) \geq t$.

Now we return to the proof of Theorem 4.2 and we show that G is α -critical with $\alpha(G) = k$ if and only if $G_{t,k}$ is minimally t-tough.

Let us assume that G is α -critical with $\alpha(G) = k$. By Lemma 4.3, the graph $G_{t,k}$ is t-tough, i.e. $\tau(G_{t,k}) \geq t$.

Let I be an independent vertex set of size $\alpha(G)$ in $G_{t,k}[V]$.

Recall the definition of $A^{i,j}$ from the beginning of the proof: it is the color class A in the corresponding copy of H''_t . Let

$$J = \{(i,j) \in [n] \times [ak] \mid v_{i,j} \in I\}$$

and

$$S = \left(\bigcup_{(i,j) \notin J} A^{i,j}\right) \cup W.$$

Then S is a cutset in $G_{t,k}$ with

$$|S| = a(|V| - \alpha(G)) + ak = a|V|$$

and

$$\omega(G_{t,k} - S) = \alpha(G) + b(|V| - \alpha(G)) + (b-1)k = b|V| = \frac{|S|}{t},$$

so $\tau(G_{t,k}) \leq t$.

Therefore, $\tau(G_{t,k}) = t$.

Let $e \in E(G_{t,k})$ be an arbitrary edge. We need to show that $\tau(G_{t,k} - e) < t$. Now we have four cases.

Case 1: e has an endpoint in U.

Then this endpoint has degree 2, so $\tau(G_{t,k} - e) \leq 1/2 < t$.

Case 2: e has an endpoint in W'.

By the properties of $H_{t,k}^*$, there exists a cutset $S \subseteq W$ in $H_{t,k}^* - e$ for which

$$\omega((H_{t,k}^* - e) - S) > \frac{|S|}{t}.$$

Note that S is also a cutset in $G_{t,k} - e$ and

$$\omega((G_{t,k}-e)-S)>\frac{|S|}{t},$$

so $\tau(G_{t,k} - e) < t$.

Case 3: e is induced by H^{i_0,j_0} for some $i_0 \in [n], j_0 \in [ak]$. By Proposition 2.5, there exists a vertex set $S \subseteq V(H'_t)$ for which

$$\omega((H'_t - e) - S) > \frac{|S|}{t}.$$

Consider the (i_0, j_0) -th copy of the vertex set S in $G_{t,k} - e$; let us denote it with S_{i_0,j_0} . If $v_{i_0,j_0} \in S_{i_0,j_0}$, then S_{i_0,j_0} is a cutset in $G_{t,k} - e$ and

$$\omega((G_{t,k}-e)-S_{i_0,j_0})=\omega((H'_t-e)-S)>\frac{|S|}{t},$$

so $\tau(G_{t,k}-e) < t$. Now assume that $v_{i_0,j_0} \notin S_{i_0,j_0}$. Let I be an independent vertex set of size $\alpha(G)$ in $G_{t,k}[V]$ that contains v_{i_0,j_0} (by Proposition 2.9, such an independent vertex set exists). Let

$$J = \{(i,j) \in [n] \times [ak] \mid v_{i,j} \in I\}$$

and

$$S' = S_{i_0, j_0} \cup \left(\bigcup_{(i, j) \notin J} A^{i, j}\right) \cup W.$$

Then S' is a cutset in $G_{t,k} - e$ with

$$|S'| = |S| + a(|V| - \alpha(G)) + ak = |S| + a|V|$$

and

$$\omega((G_{t,k} - e) - S') > \frac{|S|}{t} + \alpha(G) + b(|V| - \alpha(G)) + (b - 1)k = \frac{|S|}{t} + b|V| = \frac{|S'|}{t},$$

so $\tau(G_{t,k} - e) < t$.

Case 4: e connects two vertices of V.

By Lemma 2.10, the graph $G_{t,k}[V]$ is α -critical, so in $(G_{t,k}-e)[V]$ there exists an independent vertex set I of size $\alpha(G)+1$. Let

$$J = \{(i, j) \in [n] \times [ak] \mid v_{i, j} \in I\}$$

and

$$S = \left(\bigcup_{(i,j) \notin J} A^{i,j}\right) \cup W.$$

Then S is a cutset in $G_{t,k} - e$ with

$$|S| = a(|V| - \alpha(G) - 1) + ak = a|V| - a$$

and

$$\omega((G_{t,k}-e)-S) = \alpha(G)+1+b(|V|-\alpha(G)-1)+(b-1)k=b|V|-b+1>\frac{|S|}{t},$$

so $\tau(G_{t,k} - e) < t$.

Therefore, if G is α -critical with $\alpha(G) = k$, then $G_{t,k}$ is minimally t-tough.

Now let us assume that G is not α -critical with $\alpha(G) = k$, i.e. either $\alpha(G) \neq k$ or even though $\alpha(G) = k$, the graph G is not α -critical.

Case I: $\alpha(G) > k$.

Let I be an independent vertex set of size $\alpha(G)$ in $G_{t,k}[V]$ and let

$$J = \{(i,j) \in [n] \times [ak] \mid v_{i,j} \in I\}$$

and

$$S = \left(\bigcup_{(i,j) \notin J} A^{i,j}\right) \cup W.$$

Then S is a cutset in $G_{t,k} - e$ with

$$|S| = a(|V| - \alpha(G)) + ak = a|V| - a(\alpha(G) - k)$$

and

$$\omega(G_{t,k} - S) = \alpha(G) + b(|V| - \alpha(G)) + (b - 1)k = b|V| - (b - 1)(\alpha(G) - k)$$
$$> b|V| - b(\alpha(G) - k) = \frac{|S|}{t},$$

so $\tau(G_{t,k}) < t$, which means that $G_{t,k}$ is not minimally t-tough.

Case II: $\alpha(G) \leq k$.

Since G is not α -critical with $\alpha(G) = k$, there exists an edge $e \in E(G)$ such that $\alpha(G - e) \leq k$. By Lemma 4.3, the graph $(G - e)_{t,k}$ is t-tough, but it can be obtained from $G_{t,k}$ by edge-deletion, which means that $G_{t,k}$ is not minimally t-tough.

5 Minimally t-tough graphs, where $t \geq 1$

This whole section resembles the previous one in structure. However, it requires some additional ideas that make the proofs more complicated. First, again, we construct some auxiliary graphs.

Let $t \ge 1$ be a rational number. It is easy to see that either $\lceil 2t \rceil = 2\lceil t \rceil$ or $\lceil 2t \rceil = 2\lceil t \rceil - 1$. Let $T = \lceil t \rceil$,

$$T' = \lceil 2t \rceil - \lceil t \rceil = \begin{cases} T & \text{if } \lceil 2t \rceil = 2\lceil t \rceil, \\ T - 1 & \text{if } \lceil 2t \rceil = 2\lceil t \rceil - 1, \end{cases}$$

and

$$M = \left\lceil \frac{2\lceil t \rceil}{\lceil 2t \rceil} \right\rceil = \begin{cases} 1 & \text{if } \lceil 2t \rceil = 2\lceil t \rceil, \\ 2 & \text{if } \lceil 2t \rceil = 2\lceil t \rceil - 1. \end{cases}$$

Let a, b be the smallest positive integers such that $b \geq 3$ and t = a/b.

5.1 The auxiliary graph $H_{t,k}^{**}$ when $t \ge 1$

Let k be a positive integer that is divisible by a. Note that in this case

$$\left(\frac{MT'}{t}-1\right)k = \begin{cases} \frac{Tbk}{a}-k & \text{if } \lceil 2t \rceil = 2\lceil t \rceil, \\ \frac{2(T-1)bk}{a}-k & \text{if } \lceil 2t \rceil = 2\lceil t \rceil - 1 \end{cases}$$

is a positive integer. Let

$$W = \{ w_{j,l,m} \mid j \in [k], l \in [T'], m \in M \}$$

and

$$W' = \{w'_1, \dots, w'_{(MT'/t-1)k}\}.$$

Place a clique on the vertices of W and a complete bipartite graph on (W; W'). Obviously, the toughness of this complete split graph is

$$\frac{kMT'}{(MT'/t-1)k} = \frac{1}{\frac{1}{t} - \frac{1}{MT'}} > t.$$

Deleting an edge may decrease the toughness, and now we delete edges incident to W' until the toughness remains at least t but the deletion of any other such edge would result in a graph with toughness less than t. Let $H_{t,k}^*$ denote the obtained split graph. Then $\tau(H_{t,k}^*) \geq t$, and $\tau(H_{t,k}^* - e) < t$ for any edge $e \in E(H_{t,k}^*)$ incident to W', i.e. there exists a vertex set $S = S(e) \subseteq W$ whose removal disconnects $H_{t,k}^* - e$ and

$$\omega((H_{t,k}^* - e) - S) > \frac{|S|}{t}$$

Now delete all the edges induced by W, and let $H_{t,k}^{**}$ denote the obtained bipartite graph.

5.2 The auxiliary graph $H''_{t,k}$ when $t \ge 1$

Let H_t be constructed as follows. Let

$$\begin{split} V_1' &= \{v_1', \dots, v_T'\}, \qquad V_2' &= \{v_{T+1}', \dots, v_{2T}'\}, \qquad V_3' &= \{v_{2T+1}', \dots, v_{aT}'\}, \\ V'' &= \{v_1'', \dots, v_T''\}, \\ U_1' &= \{u_1', \dots, u_T'\}, \qquad U_2' &= \{u_{T+1}', \dots, u_{2T}'\}, \qquad U_3' &= \{u_{2T+1}', \dots, u_{bT-1}'\}, \\ U'' &= \{u_1'', \dots, u_{T'}'\}, \end{split}$$

and

$$U_1'' = \{u_1'', \dots, u_T''\}.$$

Place a clique on the vertices of V_1' , V_2' , V_3' , and U''. For all $l \in [T]$ connect v_l'' to v_l' and to u_l' , and connect v_{T+l}' to u_{T+l}' . Connect all the vertices of V_3' to all the vertices of $V_1' \cup U_1' \cup U_2'$, and connect all the vertices of V_2' to all the vertices of U''. Finally, add a new vertex x to the graph and connect it to all the vertices of $V_1' \cup U''$. See Figure 5.

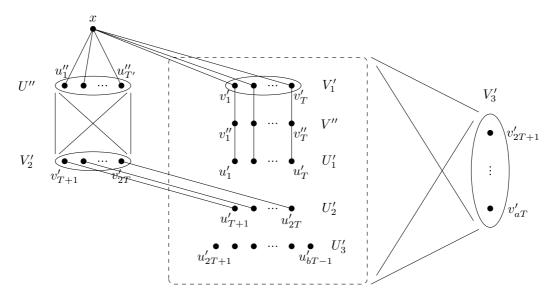


Figure 5: The graph H_t , when $t \ge 1$.

Claim 5.1. For any rational number $t \geq 1$ the graph H_t has weighted toughness t with respect to the weight function w that assigns weight 1 to all the vertices of H_t except for the vertex x, to which it assigns weight t.

Proof. Let S be an arbitrary cutset of H_t . We need to show that $\omega(H_t - S) \leq w(S)/t$.

We can assume that either $V_3' \cap S = \emptyset$ or $V_3' \subseteq S$ since removing only a proper subset of V_3' does not disconnect anything from the graph. Similarly, we can also assume that either $U'' \cap S = \emptyset$ or $U'' \subseteq S$.

Case 1: $V_3' \cap S = \emptyset$ and $U'' \cap S = \emptyset$.

Then $H_t - S$ has at most 2 components, and to obtain 2 components, the following must hold:

 $-u'_{T+l} \in S$ or $v'_{T+l} \in S$ for all $l \in [T]$, and

$$-x \in S \text{ or } V_1' \subseteq S.$$

Hence $w(S) \geq T + t$ and

$$\omega(H_t - S) = 2 \le \frac{T + t}{t} \le \frac{w(S)}{t}.$$

Case 2: $V_3' \cap S = \emptyset$ and $U'' \subseteq S$.

Now we can assume that $x \notin S$ since after the removal of U'' removing x does not disconnect anything from the graph. Similarly, we can also assume that $V_2' \nsubseteq S$. Then $H_t - S$ has at most 3 components. To obtain three components, the following must hold:

- (i) $u'_{T+l} \in S$ or $v'_{T+l} \in S$ for all $l \in [T]$ (but $V'_2 \nsubseteq S$), and
- (ii) $V_1' \subseteq S$.

Hence $w(S) \geq T' + 2T = \lceil 2t \rceil + T$ and

$$\omega(H_t - S) = 3 \le \frac{\lceil 2t \rceil + T}{t} = \frac{w(S)}{t}.$$

To obtain two components, either (i) or (ii) must hold; in both cases $w(S) \geq \lceil 2t \rceil$ and

$$\omega(H_t - S) = 2 \le \frac{\lceil 2t \rceil}{t} \le \frac{w(S)}{t}.$$

Case 3: $V_3' \subseteq S$.

First we show that the following assumptions can be made for S.

(1) $(U_1' \cup U_2' \cup U_3') \cap S = \emptyset$.

After the removal of V_3' , removing any of the vertices of $U_1' \cup U_2' \cup U_3'$ does not disconnect anything from the graph.

(2) There exists at most one $l \in [T]$ for which $v'_{T+l} \notin S$, i.e. $|V'_2 \setminus S| \leq 1$.

Suppose that there exist $l_1, l_2 \in [T]$ for which $l_1 \neq l_2$ and $v'_{T+l_1}, v'_{T+l_2} \notin S$. By assumption (1), considering the cutset $S' = S \cup \{v'_{T+l_2}\}$ instead of S increases both the number of components and the weight of the removed vertex set by 1. Hence it is enough to show that

$$\omega(H_t - S') \le \frac{w(S')}{t}$$

since it implies

$$\omega(H_t - S) = \omega(H_t - S') - 1 \le \frac{w(S')}{t} - 1 = \frac{w(S) + 1}{t} - 1 \le \frac{w(S)}{t},$$

where the last inequality is valid since $t \geq 1$.

(3) For all $l \in [T]$ if $v'_l \in S$, then $v''_l \notin S$.

After the removal of V_3' and v_1' removing v_1'' does not disconnect anything from the graph.

(4) For all $l \in [T]$ if $v'_l \notin S$, then $v''_l \in S$.

Suppose that there exists $l \in [T]$ for which $v'_l, v''_l \notin S$. By assumption (1), considering the cutset $S' = S \cup \{v''_l\}$ instead of S increases both the number of components and the weight of the removed vertex set by 1. Hence, similarly as in assumption (2), it is enough to show that

$$\omega(H_t - S') \le \frac{w(S')}{t}.$$

(5) $|(V_1' \cup V'') \cap S| = T$.

It follows directly from assumptions (3) and (4).

Case 3.1: $(V_3' \subseteq S \text{ and}) U'' \subseteq S$.

Now we can assume that $x \notin S$ since after the removal of U'' removing x does not disconnect anything from the graph. Similarly, by assumption (2), we can also assume that $V_2' \nsubseteq S$, i.e. $|V_2' \cap S| = T - 1$. Hence

$$w(S) = |V_3'| + |U''| + |(V_1' \cup V'') \cap S| + |V_2' \cap S| = (aT - 2T) + T' + T + (T - 1) = aT + T' - 1$$

and every component of $H_t - S$ contains exactly one of the vertices $u'_1, \ldots, u'_{bT-1}, x$, i.e.

$$\omega(H_t - S) = bT = \frac{aT}{t} \le \frac{aT + T' - 1}{t} = \frac{w(S)}{t}.$$

Case 3.2: $(V_3' \subseteq S \text{ and}) U'' \cap S = \emptyset.$

In this case we can make some further assumptions for S.

(6) If $V_1' \subseteq S$, then $x \notin S$.

After the removal of V'_1 removing x does not disconnect anything from the graph.

(7) If $V_1' \nsubseteq S$, then $x \in S$.

Suppose that $x \notin S$. Then considering the cutset $S' = S \cup \{x\}$ instead of S increases the number of components by 1 and the weight of the removed vertex set by t. Hence it is enough to show that $\omega(H_t - S') \leq w(S')/t$ since it implies

$$\omega(H_t - S) = \omega(H_t - S') - 1 \le \frac{w(S')}{t} - 1 = \frac{w(S) + t}{t} - 1 = \frac{w(S)}{t}.$$

(8) $V_2' \subseteq S$.

Suppose that $V_2' \nsubseteq S$. Then by assumption (2), there exists $l \in [T]$ for which $V_2' \setminus S = \{v_{T+l}'\}$. But by assumption (1), considering the cutset $S' = S \cup \{v_{T+l}'\}$ instead of S increases both the number of components and the weight of the removed vertex set by 1. Then, similarly as in assumption (2), it is enough to show that $\omega(H_t - S') \leq w(S')/t$.

Case 3.2.1: $(V_3' \subseteq S, U'' \cap S = \emptyset \text{ and}) V_1' \subseteq S$.

Hence

$$w(S) = |V_3'| + |V_2'| + |V_1'| = aT$$

and

$$\omega(H_t - S) = bT = \frac{w(S)}{t}.$$

Case 3.2.2: $(V_3' \subseteq S, U'' \cap S = \emptyset \text{ and}) V_1' \nsubseteq S$.

Hence

$$w(S) = |V_3'| + |V_2'| + |(V_1' \cup V'') \cap S| + w(x) = aT + t$$

and

$$\omega(H_t - S) = bT + 1 = \frac{w(S)}{t}.$$

Therefore H_t is weighted t-tough with respect to w (meaning that the weighted toughness of H_t is at least t).

Consider the cutset

$$S = V_1' \cup V_2' \cup V_3'$$
.

Since w(S) = aT and

$$\omega(H_t - S) = bT = \frac{w(S)}{t},$$

the weighted toughness of H_t with respect to w is at most t.

Thus the weighted toughness of H_t with respect to w is exactly t.

Deleting an edge may decrease the weighted toughness, and now we delete edges not induced by U'' until the weighted toughness with respect to the weight function w remains at least t but the deletion of any other edge not induced by U'' would result in a graph with weighted toughness less than t. Let H'_t denote the obtained graph.

According to the following claim we could not delete the edges induced by V_1' or incident to any of the vertices of $\{x\} \cup V_2' \cup U''$.

Claim 5.2. Let $t \ge 1$ be a rational number. For any edge $e \in E(H_t)$ induced by V_1' or incident to any of the vertices of $\{x\} \cup V_2' \cup U''$, there exists a cutset $S = S(e) \subseteq V(H_t)$ in $H_t - e$ for which

$$\omega((H_t - e) - S) > \frac{w(S)}{t}.$$

Proof. Let $e \in E(H_t)$ be an arbitrary edge induced by V_1' or incident to any of the vertices of $\{x\} \cup V_2' \cup U''$.

Case 1: e is incident to a vertex of $\{x\} \cup V_2'$.

Let $y \in \{x\} \cup V_2'$ denote one of the endpoints of e, and let z denote the other one. Let S be the neighborhood of the vertex y except for z. Since y has degree $\lceil 2t \rceil$ and all of its neighbors have weight 1,

$$w(S) = \lceil 2t \rceil - 1.$$

Since the removal of S from $H_t - e$ leaves the vertex y isolated.

$$\omega((H_t - e) - S) \ge 2 = \frac{2t}{t} > \frac{\lceil 2t \rceil - 1}{t} = \frac{w(S)}{t}.$$

Case 2: e is incident to a vertex of U''.

If e is incident to a vertex of U'', then either it is incident to a vertex of $\{x\} \cup V'_2$ and this case was already settled in Case 1, or it is induced by U'' and therefore it was not allowed to be deleted.

Case 3: e is induced by V_1' , i.e. $e = v_{l_1}'v_{l_2}'$ for some $l_1, l_2 \in [T], l_1 \neq l_2$.

Then

$$S = \left(V_1' \setminus \{v_{l_1}', v_{l_2}'\}\right) \cup \{v_{l_1}'', v_{l_2}''\} \cup V_2' \cup V_3' \cup \{x\}$$

is a cutset in $H_t - e$ such that

$$w(S) = (T-2) + 2 + T + (aT - 2T) + t = aT + t$$

and

$$\omega((H_t - e) - S) = bT + 2 = \frac{aT + t}{t} + 1 = \frac{w(S)}{t} + 1 > \frac{w(S)}{t}.$$

Claim 5.3. Let $t \ge 1$ be a rational number and $H''_t = H'_t - \{x\}$. Then the following hold.

- (i) The graph H''_t is connected.
- (ii) For any cutset S of H''_t ,

$$\omega(H_t''-S) \le \frac{|S|}{t} + 1.$$

(iii) If $V_1' \subseteq S$ holds for a cutset S of H_t'' , then

$$\omega(H_t''-S) \le \frac{|S|}{t}.$$

(iv) For any edge $e \in E(H''_t)$ not induced by U'' there exists a vertex set S = S(e) whose removal from $H''_t - e$ disconnects the graph and

$$\omega((H_t''-e)-S) > \frac{|S|}{t}.$$

Proof.

(i) Suppose to the contrary that H''_t is not connected. Then x is a cut-vertex in H'_t . Since the weighted toughness of H'_t with respect to w is t,

$$2 \le \omega (H'_t - \{x\}) \le \frac{w(x)}{t} = \frac{t}{t} = 1,$$

which is a contradiction.

(ii) Let S be an arbitrary cutset of H''_t . Since S is a cutset in H''_t , the vertex set $S \cup \{x\}$ is a cutset in H'_t , and

$$\omega(H_t'' - S) = \omega \left(H_t' - (S \cup \{x\}) \right) \le \frac{w(S \cup \{x\})}{t} = \frac{|S| + t}{t} = \frac{|S|}{t} + 1.$$

(iii) Let S be a cutset of H''_t for which $V'_1 \subseteq S$. We can assume that $U'' \cap S = \emptyset$ since removing any of the vertices of U'' from H''_t does not disconnect anything from the graph. Then all the neighbors of the vertex x belong to the same component in $H''_t - S$, so S is a cutset in H'_t as well and

$$\omega(H_t'' - S) = \omega(H_t' - S) \le \frac{w(S)}{t} = \frac{|S|}{t},$$

where the last equality is valid since $x \notin S$.

(iv) Let $e \in E(H''_t)$ be an arbitrary edge not induced by U''. Then by the properties of H'_t , there exists a vertex set $S \subseteq V(H'_t)$ whose removal from $H'_t - e$ disconnects the graph and

$$\omega((H'_t - e) - S) > \frac{w(S)}{t} \ge \frac{|S|}{t},$$

where the last inequality is valid since $t \geq 1$. Let $S' = S \setminus \{x\}$. Then

$$\omega((H_t''-e)-S') \ge \omega((H_t'-e)-S) > \frac{|S|}{t} \ge \frac{|S'|}{t}.$$

5.3 The cutsets X and Y_1, \ldots, Y_T in H''_t when $t \ge 1$

Let

$$X = V_1' \cup V_2' \cup V_3'$$

and for all $l \in [T]$ let

$$Y_l = (V_1' \setminus \{v_l'\}) \cup \{v_l''\} \cup V_2' \cup V_3'.$$

Proposition 5.4. The sets X and Y_1, \ldots, Y_T are all cutsets in H''_t and

$$\omega(H_t'' - X) = \frac{|X|}{t},$$

and

$$\omega(H_t'' - Y_l) = \frac{|Y_l|}{t} + 1$$

for all $l \in [T]$.

Proof. It is easy to see that

$$\omega(H_t'' - X) = bT = \frac{aT}{t} = \frac{|X|}{t}$$

and

$$\omega(H_t'' - Y_l) = bT + 1 = \frac{aT}{t} + 1 = \frac{|Y_l|}{t} + 1.$$

5.4 The proof of Theorem 1.9 when $t \ge 1$

Theorem 5.5. For any rational number $t \geq 1$, the problem Min-t-Tough is DP-complete.

Proof. Let $t \geq 1$ be a rational number. In Proposition 2.4 we already proved that the problem MIN-t-Tough is in DP. To show that it is DP-hard, we reduce the variant of α -Critical defined in Proposition 2.12 to it.

Let $T = \lceil t \rceil$, and $T' = \lceil 2t \rceil - \lceil t \rceil$, and $M = \lceil 2\lceil t \rceil / \lceil 2t \rceil \rceil$ as before. Let a, b be the smallest positive integers such that $b \geq 3$ and t = a/b, let G be an arbitrary 3-connected graph on the vertices v_1, \ldots, v_n with $n \geq t+1$, let k be a positive integer that is divisible by a and let $G_{t,k}$ be defined as follows. For all $i \in [n], j \in [k], m \in [M]$ let

$$V_{i,j,m} = \{v_{i,j,l,m} \mid l \in [T]\}.$$

For all $i \in [n]$ let

$$V_i = \bigcup_{\substack{j \in [k], \\ m \in [M]}} V_{i,j,m}$$

and place a clique on the vertices of V_i . For all $i_1, i_2 \in [n]$ if $v_{i_1}v_{i_2} \in E(G)$, then place a complete bipartite graph on $(V_{i_1}; V_{i_2})$. (This subgraph is denoted by \tilde{G} in Figure 6.) For all $i \in [n], j \in [k], m \in [M]$ "glue" the graph H''_t to the vertex set $V_{i,j,m}$ by identifying $v_{i,j,l,m}$ with the vertex v'_l of H''_t for all $l \in [T]$. For all $i \in [n], j \in [k], m \in [M]$ let $H^{i,j,m}, U''_{i,j,m}$ and $X_{i,j,m}$ denote the (i,j,m)-th copies of H''_t, U'' and X_i respectively. For all $i \in [n], j \in [k], l \in [T'], m \in [M]$ let $U''_{i,j,l,m}$ denote the (i,j,m)-th copy of V_l . For all $i \in [m], m \in [M]$ add the vertex set

$$W_{j,m} = \{ w_{j,l,m} \mid l \in [T'] \}$$

to the graph and for all $i \in [n], j \in [k], l \in [T'], m \in [M]$ connect $w_{j,l,m}$ to $u''_{i,j,l,m}$. Let

$$V = \bigcup_{i \in [n]} V_i, \qquad U'' = \bigcup_{\substack{i \in [n], \\ j \in [k], \\ m \in [M]}} U''_{i,j,m}, \qquad W = \bigcup_{\substack{j \in [k], \\ m \in [M]}} W_{j,m},$$

and let

$$U = \left(\bigcup_{\substack{i \in [n], \\ j \in [k], \\ m \in [M]}} V(H^{i,j,m})\right) \setminus V.$$

Add the vertex set

$$W' = \{w'_1, \dots, w'_{(MT'/t-1)k}\}\$$

to the graph and place the bipartite graph $H_{t,k}^{**}$ on (W;W'). See Figure 6. Now k is part of the input of the problem α -Critical, therefore the graph $H_{t,k}^{**}$ must be constructed in polynomial time and by Theorem 1.5, this can be done. On the other hand, t is not part of the input of the problem Mint-Tough, therefore the graph H_t'' can be constructed in advance. Hence, $G_{t,k}$ can be constructed from G in polynomial time.

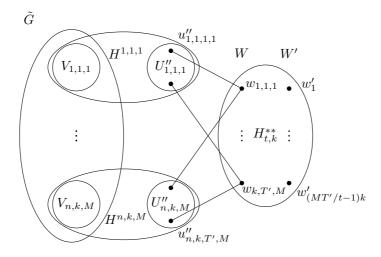


Figure 6: The graph $G_{t,k}$, when $t \geq 1$.

To show that G is α -critical with $\alpha(G) = k$ if and only if $G_{t,k}$ is minimally t-tough, first we prove the following lemma.

Lemma 5.6. Let G be an arbitrary 3-connected graph on $n \ge t + 1$ vertices with $\alpha(G) \le k$. Then $G_{t,k}$ is t-tough.

Proof. Let $S \subseteq V(G_{t,k})$ be a cutset in $G_{t,k}$. We need to show that $\omega(G_{t,k}-S) \leq |S|/t$.

Case 1: $W \subseteq S$.

After the removal of W, the vertices of W' are isolated, therefore we can assume that $W' \cap S = \emptyset$. Let

$$C = C(S) = \{(i, j, m) \in [n] \times [k] \times [M] \mid V_{i,j,m} \subseteq S\},$$
$$c_{i,j,m} = |V(H^{i,j,m}) \cap S| - T$$

for all $(i, j, m) \in C$, and

$$d_{i,j,m} = \left| V(H^{i,j,m}) \cap S \right|$$

for all $(i, j, m) \in ([n] \times [k] \times [M]) \setminus C$. Let

$$D = D(S) = \{(i, j, m) \in ([n] \times [k] \times [M]) \setminus C \mid d_{i,j,m} > 0\}.$$

Using these notations it is clear that

$$|S| = |C| \cdot T + \sum_{(i,j,m) \in C} c_{i,j,m} + \sum_{(i,j,m) \in D} d_{i,j,m} + kMT'.$$

By the assumption that $W \subseteq S$, in $G_{t,k} - S$ the (MT'/t - 1)k vertices of W' are isolated. Since $\alpha(G_{t,k}[V]) = \alpha(G)$, the removal of $V \cap S$ from $G_{t,k}[V]$ leaves at most $\alpha(G)$ components. By Claim 5.3, for any $(i,j,m) \in C$ the removal of $V(H^{i,j,m}) \cap S$ from $H^{i,j,m}$ leaves at most

$$\max\left(\frac{|V(H^{i,j,m})\cap S|}{t},1\right) = \max\left(\frac{c_{i,j,m}+T}{t},1\right) = \frac{c_{i,j,m}+T}{t}$$

components. By Claim 5.3, for any $(i, j, m) \in D$ the removal of $V(H^{i,j,m}) \cap S$ from $H^{i,j,m}$ leaves at most

$$\max\left(\frac{|V(H^{i,j,m}) \cap S|}{t} + 1, 1\right) = \max\left(\frac{d_{i,j,m}}{t} + 1, 1\right) = \frac{d_{i,j,m}}{t} + 1$$

components, but the component of the remaining vertices of $V_{i,j,m}$ has been already counted. Hence

$$\omega(G_{t,k} - S) \le \left(\frac{MT'}{t} - 1\right)k + \alpha(G) + \sum_{(i,j,m)\in C} \frac{c_{i,j,m} + T}{t} + \sum_{(i,j,m)\in D} \frac{d_{i,j,m}}{t} \le \frac{kMT' + |C| \cdot T + \sum_{(i,j,m)\in C} c_{i,j,m} + \sum_{(i,j,m)\in D} d_{i,j,m}}{t} = \frac{|S|}{t}.$$

Case 2: $W \nsubseteq S$.

There are four types of components in $G_{t,k} - S$:

- (a) components containing at least one vertex of V,
- (b) components containing at least one vertex of U but no vertices of V,
- (c) components containing at least one vertex of W but no vertices of $U \cup V$,
- (d) isolated vertices of W'.

Let $w_{j_0,l_0,m_0} \in W \setminus S$ be fixed. First we show that the following assumptions can be made for S.

(1) $S \cap U'' = \emptyset$.

Obviously, the number of vertices of W that belong to a component of type (c) is at most $|S \cap U''|/n$. Since the neighborhood of any vertex of U'' spans a clique in $G_{t,k}$, considering the cutset

$$S' = (S \setminus U'') \cup \{w \in W \mid w \text{ belongs to a component of type (c)}\}\$$

instead of S can only increase the number of components of types (a), (b) and (d), while it decreases the number of components of type (c) to 0, i.e., by at most $|S \cap U''|/n$. Hence,

$$|S'| \le |S| - |S \cap U''| + \frac{|S \cap U''|}{n}$$

and

$$\omega(G_{t,k} - S') \ge \omega(G_{t,k} - S) - \frac{|S \cap U''|}{n}.$$

Then it is enough to prove that $\omega(G_{t,k} - S') \leq |S'|/t$ since it implies

$$\omega(G_{t,k} - S) \le \omega(G_{t,k} - S') + \frac{|S \cap U''|}{n} \le \frac{|S'|}{t} + \frac{|S \cap U''|}{n}$$

$$\le \frac{|S| - |S \cap U''| + |S \cap U''|/n}{t} + \frac{|S \cap U''|}{n} = \frac{|S|}{t} - |S \cap U''| \cdot \frac{n - t - 1}{nt} \le \frac{|S|}{t},$$

where the last inequality is valid since $n \geq t + 1$.

- (2) There are no components of type (c) in $G_{t,k} S$. It follows directly from assumption (1).
- $(3) \left| \left\{ i \in [n] \mid V_{i,j_0,m_0} \subseteq S \right\} \right| \leq \lceil T'/t \rceil.$

$$I = I(w_{j_0, l_0, m_0}, S) = \{i \in [n] \mid V_{i, j_0, m_0} \subseteq S\}$$

and suppose that $|I| \ge \lceil T'/t \rceil + 1$. By assumption (1), the component of w_{j_0,l_0,m_0} contains every vertex of $\bigcup_{i=1}^n U_{i,j_0,m_0}$ and therefore all the remaining vertices of W_{j_0,m_0} . Now considering the cutset

$$S' = S \cup \{W_{j_0,m_0}\}$$

instead of S increases the number of removed vertices by at most T', and it increases the number of components by at least $\lceil T'/t \rceil$ since it disconnects the vertex sets U_{i,j_0,m_0} , $i \in I$ from each other. Then it is enough to show that $\omega(G_{t,k} - S') \leq |S'|/t$ since it implies

$$\omega(G_{t,k} - S) \le \omega(G_{t,k} - S') - \left\lceil \frac{T'}{t} \right\rceil \le \frac{|S'|}{t} - \left\lceil \frac{T'}{t} \right\rceil \le \frac{|S| + T'}{t} - \left\lceil \frac{T'}{t} \right\rceil \le \frac{|S|}{t}.$$

Proceeding further, we can obtain a cutset S^* for which $W \subseteq S^*$ holds; and such sets were already handled in Case 1.

(4) $(G_{t,k} - S)[V]$ is connected, i.e. there is only one component of type (a). Since $t \ge 1$,

$$\left\lceil \frac{T'}{t} \right\rceil \le \left\lceil \frac{t+1}{t} \right\rceil = 1 + \left\lceil \frac{1}{t} \right\rceil \le 2.$$

Since G is 3-connected, assumption (2) implies that $(G_{t,k} - S)[V]$ is connected.

Using the previous notations,

$$|S| = |C| \cdot T + \sum_{(i,j,m) \in C} c_{i,j,m} + \sum_{(i,j,m) \in D} d_{i,j,m} + |S \cap (W \cup W')|.$$

By assumption (2), there are no components of type (c), and by assumption (4), there is only one component of type (a). By the properties of $H_{t,k}^{**}$, the removal of $S \cap (W \cup W')$ from $H_{t,k}^{*}$ leaves at most

$$\max\left(\frac{|S\cap (W\cup W')|}{t},1\right)$$

components, one of them is the component of w_{j_0,l_0,m_0} , hence there are at most

$$\max\left(\frac{|S\cap(W\cup W')|}{t},1\right)-1\leq\frac{|S\cap(W\cup W')|}{t}$$

components of type (d). Similarly as before, for any $(i, j, m) \in C$ the removal of $V(H^{i,j,m}) \cap S$ from $H^{i,j,m}$ leaves at most

$$\frac{c_{i,j,m} + T}{t}$$

components, all of which can be of type (b). For any $(i, j, m) \in D$ the removal of $V(H^{i,j,m}) \cap S$ from $H^{i,j,m}$ leaves at most

$$\frac{d_{i,j,m}}{t}+1$$

components; the component of the remaining vertices of $V_{i,j,m}$ is of type (a), all the others can be of type (b). By assumption (1), all the vertices of $\bigcup_{i=1}^n U_{i,j_0,m_0}$ belong to the component of w_{j_0,l_0,m_0} , hence the component of w_{j_0,l_0,m_0} has been counted multiple times (more than once). Therefore,

$$\omega(G_{t,k} - S) \le \left(1 + \sum_{(i,j,m) \in C} \frac{c_{i,j,m} + T}{t} + \sum_{(i,j,m) \in D} \frac{d_{i,j,m}}{t} + \frac{|S \cap (W \cup W')|}{t}\right) - 1 = \frac{|S|}{t}.$$

Thus,
$$\tau(G_{t,k}) \geq t$$
.

Now we return to the proof of Theorem 5.5 and we show that G is α -critical with $\alpha(G) = k$ if and only if $G_{t,k}$ is minimally t-tough.

Let us assume that G is α -critical with $\alpha(G)=k$. By Lemma 5.6, the graph $G_{t,k}$ is t-tough, i.e. $\tau(G_{t,k}) \geq t$.

Let I be an independent vertex set of size $\alpha(G)$ in G, and recall the definition of the sets X and Y_1, \ldots, Y_T constructed in Subsection 5.3. Let

$$J = \{ i \in [n] \mid v_i \in I \}$$

and

$$S = \left(\bigcup_{i \in J} Y_{i,1,1,1}\right) \cup \left(\bigcup_{\substack{i \in J, \\ j \in [k] \setminus \{1\}, \\ m \in [M] \setminus \{1\}}} X_{i,j,m}\right) \cup \left(\bigcup_{\substack{i \notin J, \\ j \in [k], \\ m \in [M]}} X_{i,j,m}\right) \cup W.$$

Then S is a cutset in $G_{t,k}$ with

$$|S| = nkMaT + kMT'$$

and after the removal of S from $G_{t,k}$, the vertices of W' are isolated and the other components of $G_{t,k} - S$ are exactly the components of $H^{i,j,m} - (S \cap V(H^{i,j,m}))$ for all $(i,j,m) \in [n] \times [k] \times [M]$. By Proposition 5.4,

$$\omega(H^{i,j,m} - X_{i,j,m}) = bT$$

and

$$\omega(H^{i,j,m} - Y_{i,1,1,1}) = bT + 1$$

for all $(i, j, m) \in [n] \times [k] \times [M]$. Since $|J| = \alpha(G)$ and

$$|W'| = \left(\frac{MT'}{t} - 1\right)k,$$

it follows that

$$\omega(G_{t,k}-S) = nkMbT + \alpha(G) + \left(\frac{MT'}{t} - 1\right)k = nkMbT + \frac{kMT'}{t} = \frac{nkMaT + kMT'}{t} = \frac{|S|}{t},$$

so $\tau(G_{t,k}) \leq t$.

Therefore, $\tau(G_{t,k}) = t$.

Let $e \in E(G_{t,k})$ be an arbitrary edge. We need to show that $\tau(G_{t,k}-e) < t$. Now we have four cases.

Case 1: e has an endpoint in U''.

Then this endpoint has degree $\lceil 2t \rceil - 1$ in $G_{t,k} - e$, so

$$\tau(G_{t,k} - e) \le \frac{\lceil 2t \rceil - 1}{2} < \frac{2t}{2} = t.$$

Case 2: e has an endpoint in W'.

By the properties of $H_{t,k}^*$, there exists a cutset $S \subseteq W$ in $H_{t,k}^* - e$ for which

$$\omega((H_{t,k}^* - e) - S) > \frac{|S|}{t}.$$

Note that S is also a cutset in $G_{t,k} - e$ and

$$\omega((G_{t,k}-e)-S) = \omega((H_{t,k}^*-e)-S) > \frac{|S|}{t},$$

so $\tau(G_{t,k} - e) < t$.

Case 3: e is induced by H^{i_0,j_0,m_0} for some $i_0 \in [n], j_0 \in [k], m_0 \in [M]$.

The case when e is induced by U''_{i_0,j_0,m_0} was already covered in Case 1. So assume that e is not induced by U''_{i_0,j_0,m_0} . Then by Claim 5.3, there exists a vertex set $S \subseteq V(H''_t)$ for which

$$\omega((H_t''-e)-S) > \frac{|S|}{t}.$$

Consider the (i_0, j_0, m_0) -th copy of the vertex set S in $G_{t,k} - e$; let us denote it with S_{i_0,j_0,m_0} . If $V_{i_0,j_0,m_0} \subseteq S_{i_0,j_0,m_0}$, then S_{i_0,j_0,m_0} is a cutset in $G_{t,k} - e$ and

$$\omega((G_{t,k}-e)-S_{i_0,j_0,m_0})=\omega((H_t''-e)-S)>\frac{|S|}{t},$$

so $\tau(G_{t,k}-e) < t$. Assume that $V_{i_0,j_0,m_0} \nsubseteq S_{i_0,j_0,m_0}$. Let I be an independent vertex set of size $\alpha(G)$ in G that contains v_{i_0} (by Proposition 2.9, such an independent vertex set exists). Let

$$J = \{ i \in [n] \mid v_i \in I \}$$

and

$$S' = S_{i_0, j_0, m_0} \cup \left(\bigcup_{\substack{j \in [k], \\ m \in [M], \\ (j, m) \neq (j_0, m_0)}} X_{i_0, j, m}\right) \cup \left(\bigcup_{\substack{i \in J \setminus \{i_0\} \\ j \in [k] \setminus \{1\}, \\ m \in [M]}} X_{i, j, m}\right) \cup \left(\bigcup_{\substack{i \notin J, \\ j \in [k], \\ m \in [M]}} X_{i, j, m}\right) \cup W.$$

Then S' is a cutset in $G_{t,k} - e$ with

$$|S'| = |S| + (nkM - 1)aT + kMT'$$

and similarly as before,

$$\omega((G_{t,k} - e) - S') > \frac{|S|}{t} + (nkM - 1)bT + \alpha(G) + \left(\frac{MT'}{t} - 1\right)k = \frac{|S|}{t} + (nkM - 1)bT + \frac{kMT'}{t} = \frac{|S'|}{t},$$
so $\tau(G_{t,k} - e) < t$.

Case 4: e connects two vertices of V.

Since the case when e is induced by H^{i_0,j_0,m_0} for some $i_0 \in [n], j_0 \in [k], m_0 \in [M]$ was settled in Case 3, we can assume that there do not exist $i \in [n], j \in [k], m \in [M]$ for which e is induced by $V_{i,j,m}$. By Lemma 2.10, the graph $G_{t,k}[V]$ is α -critical, so in $G_{t,k}[V] - e$ there exists an independent vertex set I of size $\alpha(G) + 1$. Let

$$J = \{(i, j, l, m) \in [n] \times [k] \times [T] \times [M] \mid v_{i, j, l, m} \in I\},$$
$$J'_{1} = \{(i, j, m) \in [n] \times [k] \times [M] \mid \exists ! l \in [T] : v_{i, j, l, m} \in I\}.$$

and

$$J_2' = \big\{(i,j,m) \in [n] \times [k] \times [M] \bigm| \nexists l \in [T]: \ v_{i,j,l,m} \in I\big\}.$$

By the assumption that there do not exist $i \in [n], j \in [k], m \in [M]$ for which e is induced by $V_{i,j,m}$,

$$J_1' \cup J_2' = [n] \times [k] \times [M],$$

so

$$S = \left(\bigcup_{(i,j,l,m)\in J} Y_{i,j,l,m}\right) \cup \left(\bigcup_{(i,j,m)\in J'_2} X_{i,j,m}\right) \cup W$$

is a (well-defined) cutset in $G_{t,k} - e$. Then

$$|S| = nkMaT + kMT'$$

and similarly as before,

$$\omega((G_{t,k}-e)-S) = nkMbT + \alpha(G) + 1 + \left(\frac{MT'}{t} - 1\right)k = \frac{nkMaT}{t} + \frac{kMT'}{t} + 1 > \frac{|S|}{t},$$

so
$$\tau(G_{t,k} - e) < t$$
.

Now let us assume that G is not α -critical with $\alpha(G) = k$, i.e. either $\alpha(G) \neq k$ or even though $\alpha(G) = k$, the graph G is not α -critical.

Case I: $\alpha(G) > k$.

Let I be an independent vertex set of size $\alpha(G)$ in G. Let

$$J = \{ i \in [n] \mid v_i \in I \}$$

and

$$S = \left(\bigcup_{i \in J} Y_{i,1,1,1}\right) \cup \left(\bigcup_{\substack{i \in J, \\ j \in [k] \setminus \{1\}, \\ m \in [M] \setminus \{1\}}} X_{i,j,m}\right) \cup \left(\bigcup_{\substack{i \notin J, \\ j \in [k], \\ m \in [M]}} X_{i,j,m}\right) \cup W.$$

Then S is a cutset in $G_{t,k} - e$ with

$$|S| = nkMaT + kMT'$$

and similarly as before,

$$\omega\left((G_{t,k}-e)-S\right)=nkMbT+\alpha(G)+\left(\frac{MT'}{t}-1\right)k>nkMbT+\frac{kMT'}{t}=\frac{nkMaT+kMT'}{t}=\frac{|S|}{t},$$

so $\tau(G_{t,k}) < t$, which means that $G_{t,k}$ is not minimally t-tough.

Case II: $\alpha(G) \leq k$.

Since G is not α -critical with $\alpha(G) = k$ there exists an edge $e \in E(G)$ such that $\alpha(G - e) \leq k$. By Lemma 5.6, the graph $(G - e)_{t,k}$ is t-tough, but it can be obtained from $G_{t,k}$ by edge-deletion, which means that $G_{t,k}$ is not minimally t-tough.

Therefore the problem MIN-1-TOUGH is DP-complete, so by Claim 2.7, we can conclude the following.

Corollary 5.7. Recognizing almost minimally 1-tough graphs is DP-complete.

Let Almost-Min-1-Tough denote the problem of determining whether a given graph is almost minimally 1-tough.

6 Minimally t-tough graphs with $t \le 1/2$

The case when $t \le 1/2$ is special in some sense: graphs with toughness at most 1/2 can have cut-vertices. Unlike in the previous cases, we reduce Almost-Min-1-Tough to this problem. But first, again, we construct an auxiliary graph.

6.1 The auxiliary graph H'_t when $t \leq 1/2$

Let $t \le 1/2$ be a positive rational number. Let a, b be relatively prime positive integers such that t = a/b and let H_t be constructed as follows. Let

$$V = \{v_1, v_2, \dots, v_a\}, \qquad U = \{u_1, u_2, \dots, u_{b-a}\}, \qquad W = \{w_1, w_2, \dots, w_a\}.$$

Place a clique on the vertices of V, connect every vertex of V to every vertex of U, and connect v_i to w_i for all $i \in [n]$. See Figure 7.

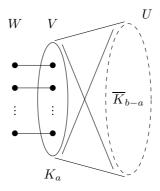


Figure 7: The graph H_t when $t \leq 1/2$.

Proposition 6.1. Let $t \leq 1/2$ be a positive rational number. Then $\tau(H_t) = t$.

Proof. Let S be an arbitrary cutset of H_t . We can assume that $S \cap (U \cup W) = \emptyset$ since removing any of the vertices of $U \cup W$ does not disconnect anything in the graph. Then $S \subseteq V$, so

$$\omega(H_t - S) = \begin{cases} a + (b - a) = b & \text{if } S = V, \\ |S| + 1 & \text{if } S \neq V, \end{cases}$$

which implies that

$$\tau(H_t) = \min \left\{ \frac{|S|}{\omega(H_t - S)} \mid S \subseteq V, S \neq \emptyset \right\} = \frac{a}{b} = t.$$

By repeatedly deleting some edges of H_t , eventually we obtain a minimally t-tough graph; let us denote it with H'_t (i.e. if there exists an edge whose deletion does not decrease the toughness, then we delete it). Obviously, we could not delete the edges between V and W, so the vertices of W still have degree 1 in H'_t .

Note that V is a tough set of H'_t . For further reference (to avoid confusion with other sets denoted by V), we introduce a new name for it.

Notation 6.2. Let S_t denote the tough set V in H'_t .

6.2 "Gluing"

Definition 6.3. Let H be a graph with a vertex u of degree 1, and let v be the neighbor of u. Let G be an arbitrary graph, and "glue" $H - \{u\}$ separately to all vertices of G by identifying each vertex of G with v. Let $G \oplus_v H$ denote the obtained graph. (See Figure 8.)

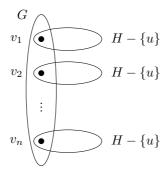


Figure 8: The graph $G \oplus_v H$.

6.3 The proof of Theorem 1.9 when $t \le 1/2$

Theorem 6.4. For any positive rational number $t \leq 1/2$, the problem Min-t-Tough is DP-complete.

Proof. Let $t \le 1/2$ be a positive rational number. In Proposition 2.4 we already proved that the problem MIN-t-Tough is in DP. To show that it is DP-hard, we reduce Almost-MIN-1-Tough to it.

Let G be an arbitrary graph and n = |V(G)|. Consider the graph H'_t and let $u \in U$ be an arbitrary vertex of H'_t having degree 1, and let v be its neighbor. Let

$$H_t'' = H_t' - \{u\}$$

and let H^i denote the *i*-th copy of H''_t "glued" to the vertex $v_i \in V(G)$ for all $i \in [n]$. (For examples see Figures 13 and 14.)

Now we show that G is almost minimally 1-tough if and only if $G_t = G \oplus_v H'_t$ is minimally t-tough. First, let G be almost minimally 1-tough. We need to show that G_t is minimally t-tough.

Let $S \subseteq V(G_t)$ be an arbitrary cutset of G_t . Let

$$C = C(S) = \{i \in [n] \mid v_i \in V(G) \cap S\},$$
$$c_i = |V(H^i) \cap S| - 1$$

for all $i \in C$, and

$$d_i = |V(H^i) \cap S|$$

for all $i \in [n] \setminus C$ (see Figure 9). Finally, let

$$D = D(S) = \{i \in [n] \setminus C \mid d_i > 0\}.$$

Using these notations it is clear that

$$|S| = |C| + \sum_{i \in C} c_i + \sum_{i \in D} d_i.$$

By Proposition 2.8, the removal of $V(G) \cap S$ from G leaves at most $|V(G) \cap S| = |C|$ components. By Proposition 2.1, the removal of $V(H'_t) \cap S$ from H'_t leaves at most $|V(H'_t) \cap S|/t$ components. But for all $i \in [n] \setminus C$ we have already counted the component of $G'_t - S$ which contains v_i , and for all $i \in C$ we do not need to count the component $\{u\}$ of H'_t . Hence

$$\omega(G_t - S) \le |C| + \sum_{i \in C} \left(\frac{c_i + 1}{t} - 1\right) + \sum_{i \in D} \left(\frac{d_i}{t} - 1\right)$$
$$= \frac{|C| + \sum_{i \in C} c_i + \sum_{i \in D} d_i}{t} - |D| \le \frac{|S|}{t},$$

which means that $\tau(G_t) \geq t$.

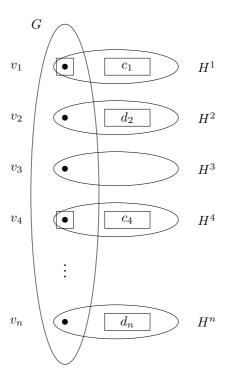


Figure 9: The graph G_t and the cutset S, when $t \leq 1/2$.

Now let S_0 be a tough set of H'_t . Since u has degree 1, we can assume that $u \notin S_0$. Let $S_0^1 \subseteq V(H^1)$ be the first copy of S_0 . Obviously, S_0^1 is a cutset in G_t , and

$$\omega(G_t - S_0^1) = \omega(H_t' - S_0) = \frac{|S_0|}{t} = \frac{|S_0^1|}{t},$$

which means that $\tau(G_t) \leq t$.

Therefore, $\tau(G_t) = t$.

Let $e \in E(G_t)$ be an arbitrary edge. We need to show that $\tau(G_t - e) < t$ for all $e \in E(G_t)$. Now we have two cases.

Case 1: $e \in E(G)$.

If e is a bridge in G, then $\tau(G_t - e) = 0 < t$. So assume that e is not a bridge in G. Let $S = S(e) \neq \emptyset$ be a vertex set in G guaranteed by Claim 2.7, and for all $i \in [n]$ let $S_t^i \subseteq V(H^i)$ be the i-th copy of the tough set S_t defined in Notation 6.2. (Note that $v \in S_t$ and $u \notin S_t$.) Let

$$J = J(S) = \{i \in [n] \mid v_i \in S\}$$

and consider the vertex set

$$S' = S \cup \left(\bigcup_{i \in I} S_t^i\right) = \bigcup_{i \in I} S_t^i.$$

Then S' is a cutset in $G_t - e$ with

$$|S'| = \sum_{i \in J} |S_t^i| = |S| \cdot |S_t|$$

and

$$\omega((G_t - e) - S') > |S| + |S| \left(\frac{|S_t|}{t} - 1\right) = \frac{|S| \cdot |S_t|}{t} = \frac{|S'|}{t},$$

which means that $\tau(G_t - e) < t$.

Case 2: $e \in E(H^{i_0})$ for some $i_0 \in [n]$.

If e is a bridge in H'_t , then $\tau(G_t - e) = 0 < t$. So assume that e is not a bridge in H'_t and let $S = S(e) \neq \emptyset$ be a vertex set in H'_t guaranteed by Proposition 2.5. Again, since u has degree 1, we can assume that $u \notin S$. Let $S^{i_0} \subseteq V(H^{i_0})$ be the i_0 -th copy of S. Obviously, S^{i_0} is a cutset in $G_t - e$ and

$$\omega((G_t - e) - S^{i_0}) = \omega((H'_t - e) - S) > \frac{|S|}{t} = \frac{|S^{i_0}|}{t},$$

which means that $\tau(G_t - e) < t$.

Therefore, the graph G_t is minimally t-tough.

Now we show that if G_t is minimally t-tough, then G is almost minimally 1-tough.

First, we prove that $\tau(G) \geq 1$. Suppose to the contrary that $\tau(G) < 1$. Obviously, G must be connected (otherwise $\tau(G_t) = 0 \neq t$), so there exists a cutset $S \subseteq V(G)$ in G satisfying

$$\omega(G-S) > |S|.$$

For all $i \in [n]$ let $S_t^i \subseteq V(H^i)$ be the *i*-th copy of the tough set S_t defined in Notation 6.2. (Note that $v \in S_t$ and $u \notin S_t$.) Let

$$J = J(S) = \{ i \in [n] \mid v_i \in S \}$$

and consider the vertex set

$$S' = S \cup \left(\bigcup_{i \in J} S_t^i\right) = \bigcup_{i \in J} S_t^i.$$

Then S' is a cutset in G_t with

$$|S'| = \sum_{i \in J} |S_t^i| = |S| \cdot |S_t|$$

and

$$\omega(G_t - S') > |S| + |S| \left(\frac{|S_t|}{t} - 1\right) = \frac{|S| \cdot |S_t|}{t} = \frac{|S'|}{t},$$

which means that $\tau(G_t) < t$ and that is a contradiction. So $\tau(G) > 1$.

Now we prove that $\tau(G-e) < 1$ for all $e \in E(G)$. Let $e \in E(G)$ be an arbitrary edge. If e is a bridge in G, then $\tau(G-e) = 0 < 1$. Let us assume that e is not a bridge in G. Then e is not a bridge in G either. Let $S = S(e) \neq \emptyset$ be a vertex set guaranteed by Proposition 2.5. Consider the vertex set $S_0 = S \cap V(G)$. Since e is a bridge in $G - S_0$ as well, S_0 is a cutset in G - e. Let

$$C = C(S) = \{i \in [n] \mid v_i \in S_0\},$$
$$c_i = |V(H^i) \cap S| - 1$$

for all $i \in C$ and

$$d_i = \left| V(H^i) \cap S \right|$$

for all $i \in [n] \setminus C$. Let

$$D = D(S) = \{ i \in [n] \setminus C \mid d_i > 0 \}.$$

Then

$$\omega((G-e)-S_0) > |S_0| = |C|$$

must hold, otherwise, similarly as before,

$$\omega((G'-e)-S) \le |C| + \sum_{i \in C} \left(\frac{c_i+1}{t}-1\right) + \sum_{i \in D} \left(\frac{d_i}{t}-1\right)$$

$$= \frac{|C| + \sum_{i \in C} c_i + \sum_{i \in D} d_i}{t} - |D| \le \frac{|S|}{t},$$

which is a contradiction. So $\tau(G-e) < 1$.

Therefore, G is almost minimally 1-tough.

7 Conclusion

In this paper we proved that recognizing minimally t-tough graphs is DP-complete for any positive rational number t. On the other hand, in [6] we proved that in some special graph classes, this problem belongs to P:

- in the class of split graphs,
- in the class of claw-free graphs when $t \leq 1$, and
- in the class of $2K_2$ -free graphs.

These results are not really surprising since the toughness of split, or claw-free, or $2K_2$ -free graphs can be computed in polynomial time, see [15], [12], and [2], respectively.

It is also known that recognizing t-tough bipartite graphs is coNP-complete for any positive rational number $t \leq 1$, see [8] and [7], but determining the complexity of recognizing minimally t-tough, bipartite graphs is still open.

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Appendix

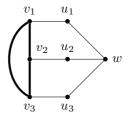


Figure 10: The minimally 1-tough graph G' constructed in the beginning of Section 3, when $G \simeq K_3$. The edges of K_3 are drawn with thick lines.

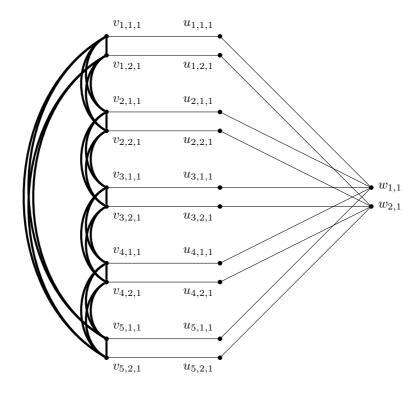


Figure 11: The graph $G'_{1,2}$ constructed in Subsection 3.1, when $G \simeq C_5$. Since the graph C_5 is connected and α -critical with $\alpha(C_5) = 2$, the choice k = 2 results in a minimally 1-tough graph. The edges of the "blown-up" C_5 are drawn with thick lines.

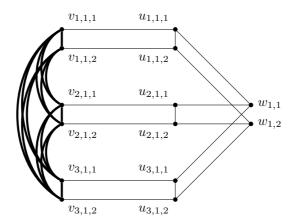


Figure 12: The graph $G'_{2,1}$ constructed in Subsection 3.1, when $G \simeq K_3$. Since the graph K_3 is 2-connected and α -critical with $\alpha(K_3) = 1$, the choice k = 1 results in a minimally 2-tough graph. The edges of the "blown-up" K_3 are drawn with thick lines.

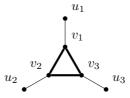


Figure 13: The graph $G_{1/2}$ constructed in Subsection 3.2, when $G \simeq K_3$. Since the graph K_3 is almost minimally 1-tough, this graph is minimally 1/2-tough. The edges of K_3 are drawn with thick lines.



Figure 14: The graph $G_{2/5}$ constructed in Subsection 6.3, when $G \simeq K_3$. Since the graph K_3 is almost minimally 1-tough, this graph is minimally 2/5-tough. The edges of K_3 are drawn with thick lines.

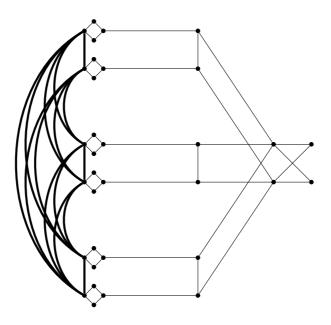


Figure 15: The graph $G_{2/3,1}$ constructed in Subsection 4.3, when $G \simeq K_3$. Since the graph K_3 is 2-connected and α -critical with $\alpha(K_3) = 1$, the choice k = 1 results in a minimally 2/3-tough graph. The edges of the "blown-up" K_3 are drawn with thick lines.

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