EXTREMA OF POLYNOMIALS WITH REAL ROOTS AND DIOPHANTINE EQUATIONS

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Abstract. There are many results in the literature concerning polynomial values and (shifted) power values of polynomials with consecutive integer roots, or more generally, with roots forming an arithmetic progression. It is an interesting question that how far one can 'disturb' the structure of the roots such that the finiteness results still remain valid. Also there are many results into this direction, with adding or removing one or more terms (roots).

In this paper we study a case where (part of) the symmetric root structure is preserved, however, we allow (possibly large) increasing gaps between the roots. We prove that the finiteness of the solutions can also be guaranteed under these generalized circumstances. In our proofs we combine Baker's method and the Bilu-Tichy theorem with a new result providing an increasing property of the extremal values of polynomials with distinct real roots satisfying certain symmetry and increasing gap properties.

1. Introduction

There are many results in the literature concerning polynomial values and (shifted) power values of polynomials with consecutive integer roots, or more generally, with roots forming an arithmetic progression. We only mention a classical result of Erdős and Selfridge [5] saying that the product of consecutive integers can never be a perfect power, a theorem of Győry, Hajdu and Pintér [6] giving an alike result concerning arithmetic progressions up to 35 terms, and a paper by Kulkarni and Sury [10] providing finiteness results for the polynomial values of

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Date: May 17, 2022.

2010 *Mathematics Subject Classification.* 11D41, 11C08.

Key words and phrases. Extrema of polynomials with real roots, polynomial values.

Research supported in part by the by the Eötvös Loránd Research Network (ELKH), the NKFIH grants K115479, K128088 and 130909, the project EFOP-3.6.1-16-2016-00022 co-financed by the European Union and the European Social Fund, and by the ÚNKP-21-3 New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund.

products of consecutive integers. It is an interesting question that how far one can 'disturb' the structure of the roots such that the finiteness results still remain valid. Also there are many results into this direction, with adding or removing one or more terms (roots). Here we only recall results of Saradha and Shorey [11, 12] concerning power values of products of consecutive integers with one term missing, Hajdu and Papp [7] and Hajdu, Papp and Tijdeman [8] about polynomial values and shifted power values of products of consecutive terms of arithmetic progressions with one and with several missing terms, respectively, and Hajdu and Varga [9] with one term added. We suggest the interested reader to consult the references of the mentioned papers, as well.

In this paper we study a case where (part of) the symmetric root structure is preserved, however, having increasing (possibly large) gaps between the roots. We prove that the finiteness of the solutions can also be guaranteed under these generalized circumstances. Our results can be considered to be generalizations of the corresponding finiteness results, e.g. from [10]. (This will be explained in Remark 2.1.) In our proofs we combine Baker's method and the Bilu-Tichy theorem with a new result guaranteeing an increasing property for the extremal values of polynomials, with distinct real roots satisfying certain symmetry and increaing gap properties. The structure of the paper is the following. In the next section we provide our main results. Then we give their proofs (together with the corresponding lemmas and auxiliary results) in separate sections. The reason of this is that we need different tools in the proofs of our results, and we would like to present the necessary tools close to their actual use (as much as possible). We note that we give the proofs of our results not in the order of stating the theorems, but in the 'logical' order (which in fact is just the opposite order).

2. Main results

We say that a finite sequence $b_1, \ldots, b_k$ in $\mathbb{R}$ with $b_1 < \cdots < b_k$ is symmetric, if there exists a $c \in \mathbb{R}$ such that $b_i + b_{k+1-i} = 2c$ for $i = 1, 2, \ldots, k$. We say that c is the center of symmetry for the sequence. A symmetric sequence is called centrally convex, if $b_\ell, b_{\ell+1}, \ldots, b_k$ form a convex sequence, that is

$$b_i - b_{i-1} \leq b_{i+1} - b_i \quad (\ell < i < k) \tag{1}$$

holds. For example, $-2, 0, 1, 2, 4$ is a centrally convex symmetric sequence: the center of symmetry is $c = 1$, and we have

$$2 - 1 \leq 4 - 2.$$

To see an example of a centrally convex symmetric sequence with an even number of elements, consider $-10, -2, 1, 4, 6, 9, 12, 20$: now the center of symmetry is $c = 5$ and we have

$$6 - 4 \leq 9 - 6 \leq 12 - 9 \leq 20 - 12.$$

Remark 2.1. *Our results concern polynomials with simple real roots forming a centrally convex symmetric sequence. We find it important to emphasize a few points here. In the first place, any arithmetic progression $h_1, \dots, h_N$ forms a centrally convex symmetric sequence. Indeed, the sequence is symmetric to the point $c := (h_1 + h_N)/2$ (i.e. $h_i + h_{N+1-i} = 2c$ for all $i = 1, \dots, N$), and since the gaps between the terms after the middle point are non-decreasing (certainly, the gap is constant), the centrally convex property* (1) *is also satisfied. So our Theorem 2.1 below provides an extension of the main result of* [10]. *However, in fact our results are much more general than that. For example, as one can easily check, the numbers*

$$-k^2, -(k-1)^2, \dots, -4, -1, 0, 1, 4, \dots, (k-1)^2, k^2$$

also form a centrally convex symmetric sequence, so our results provide finiteness conditions for the equations appearing in Theorems 2.1 and 2.2 involving the corresponding polynomial

$$f(x) = x \prod_{j=1}^{k} (x - j^2)(x + j^2).$$

First we provide a general, ineffective theorem for the common integer values of a polynomial $f(x) \in \mathbb{Q}[x]$ having distinct roots forming a centrally convex symmetric sequence with any polynomial $g(x) \in \mathbb{Q}[x]$.

Theorem 2.1. *Let $f(x) \in \mathbb{Q}[x]$ have distinct real roots forming a centrally convex symmetric sequence,* $\deg(f) > 6$ *and let $g(x) \in \mathbb{Q}[x]$ with* $\deg(g) \geq 2$. *If the equation*

$$f(x) = g(y) \tag{2}$$

has infinitely many solutions in integers x, y then either

$$g(y) = f(P(y))$$

with some $P(y) \in \mathbb{Q}[y]$ of degree ≥ 1, or $\deg(f) = 2k$ *is even and*

$$g(y) = \hat{f}(Q(y))$$

with some $Q(y) \in \mathbb{Q}[y]$ having at most two roots of odd multiplicity, where

$$\hat{f}(x) = b_0(x - (b_1 - c)^2) \cdots (x - (b_k - c)^2).$$

Here b_0 is the leading coefficient of f, b_i $(i = 1, \ldots, 2k)$ are the roots of f in increasing order, and c is the center of symmetry for them.

Remark 2.2. *It is easy to see that $\hat{f}(x) \in \mathbb{Q}[x]$. We shall show this in the proof of the theorem.*

If we have $g(x) = Ax^m + B$ with some fixed $A, B \in \mathbb{Q}$ $(A \neq 0)$ and m is an integer variable with $m \geq 2$, we are able to provide an effective upper bound for the absolute values of the integer solutions x, y and also of m in equation (2). By the height of a polynomial in $\mathbb{Q}[x]$ we mean the maximum of the absolute values of the numerators and denominators of its coefficients.

Theorem 2.2. *Let $f(x) \in \mathbb{Q}[x]$ have distinct real roots forming a centrally convex symmetric sequence and suppose that* $\deg(f) > 6$. *Let A, B be given rationals with $A \neq 0$, and consider the equation*

$$f(x) = Ay^m + B \tag{3}$$

in integers x, y, m with $m \geq 2$, with the convention that $m \leq 3$ if $|y| \leq 1$. Then there exists an effectively computable constant $C_1(A, B, d, H)$, depending only on A, B and the degree d and height H of f such that

$$\max(|x|, |y|, m) \leq C_1(A, B, d, H)$$

for every integer solution x, y, m of (3).

Remark 2.3. *The assumptions of Theorems 2.1 and 2.2 are necessary. Clearly, we need to exclude the case* $\deg(g) = 1$ *in Theorem 2.1. To see an example with* $\deg(f) = 6$ *such that both* (2) *and* (3) *have infinitely many solutions, put*

$$f(x) = (x + 8)(x + 4)(x + 1)(x - 1)(x - 4)(x - 8)$$

and

$$g(y) = Ay^m + B = 29y^2 + 3136.$$

(Observe that the roots $-8, -4, -1, 1, 4, 8$ form a centrally convex symmetric sequence.) Then both (2) *and* (3) *can be written as*

$$(x^2 - 65)(x^2 - 8)^2 = 29y^2.$$

Since the generalized Pell equation

$$u^2 - 29v^2 = 65$$

has infinitely many integer solutions u, v (the 'smallest' one is given by $(u, v) = (23, 4)$*),* (2) *and* (3) *admit infinitely many solutions $x, y \in \mathbb{Z}$. However, we mention that the condition* $\deg(f) > 6$ *is necessary only for $m = 2$. When $m \geq 3$, in fact the assumption* $\deg(f) > 2$ *is sufficient. This can be easily seen from the proof of Theorem 2.2.*

We also note that requiring only distinct real roots for f is certainly not necessary: see e.g. the identities involving Dickson polynomials in [3]. *That is, some further requirement for the roots is necessary.*

To formulate the next theorem we need some more notation. Let K be a field and $T(x) \in K[x]$. By the decomposition of $T(x)$ over K we mean a composition of the form $T(x) = P_1(P_2(x))$ with $P_1(x), P_2(x) \in K[x]$. A decomposition is called nontrivial if $\deg P_1 > 1$ and $\deg P_2 > 1$. Two decompositions $T(x) = P_1(P_2(x))$ and $T(x) = Q_1(Q_2(x))$ are called equivalent if there exists a linear polynomial $r(x) \in K[x]$ such that $P_1(x) = Q_1(r(x))$ and $Q_2(x) = r(P_2(x))$. If $T(x)$ has a nontrivial decomposition then it is decomposable; otherwise it is indecomposable over K.

In the proof of Theorem 2.1 the following result plays an important role. It gives a complete description of the decompositions of polynomials over $\mathbb{Q}$ with simple real roots forming a centrally convex symmetric sequence.

Theorem 2.3. *Let $f(x) \in \mathbb{Q}[x]$ have distinct real roots forming a centrally convex symmetric sequence. If $\deg(f)$ is odd or $\deg(f) = 2$ then f is indecomposable over $\mathbb{Q}$. If $\deg(f) \geq 4$ is even then f is decomposable over $\mathbb{Q}$, and all the decompositions of f over $\mathbb{Q}$ are equivalent to*

$$f(x) = b_0((x-c)^2 - (b_1-c)^2)\dots((x-c)^2 - (b_k-c)^2),$$

where b_0 is the leading coefficient of f, $b_1, \dots, b_{2k}$ are the roots of f in increasing order, and c is their center of symmetry.

Remark 2.4. *Using the notation introduced in Theorem 2.1, the above decomposition can also be written as*

$$f(x) = \hat{f}((x-c)^2).$$

So (for $\deg(f) \geq 4$ even) this decomposition is over $\mathbb{Q}$, indeed.

Finally, we give a theorem providing information about the extrema of polynomials having simple real roots forming a centrally convex symmetric sequence. As we shall see, this result will play a key role in the proofs of our theorems given above - however, we find it of possible independent interest.

Theorem 2.4. *Let $f(x) \in \mathbb{R}[x]$ have distinct real roots, which form a centrally convex symmetric sequence. Then the extremal values of f are strictly increasing in absolute value moving away from the center of symmetry of the roots.*

Remark 2.5. *In the statement neither the centrally convex nor the symmetric properties can be dropped. We illustrate it with two examples.*

Take first

$$f(x) = (x+3)(x+2)(x+1)(x-1)(x-2)(x-3).$$

We see that the roots form a symmetric sequence (with center of symmetry being 0*), and for the gaps only one 'centrally convex inequality' is violated (namely,* $1-(-1) \leq 2-1$ *does not hold). However, a simple calculation with Maple shows that the roots of* $f'(x)$ *are given by*

$$-\sqrt{7}, \quad -\sqrt{\frac{7}{3}}, \quad 0, \quad \sqrt{\frac{7}{3}}, \quad \sqrt{7},$$

and the extremal values of $f(x)$ *are*

$$-36, \quad \frac{400}{27}, \quad -36, \quad \frac{400}{27}, \quad -36$$

at these values, respectively. So we see that the strictly monotone increasing property of the absolute values of the extremal values (moving away from the center of symmetry) does not hold in this case.

Let now

$$f(x) = (x+9)(x+6)(x+3)x(x-1)(x-2)(x-3).$$

We see that the roots satisfy an 'increasing gap property' starting from the middle root (which is 0*), into both the positive and the negative direction. (Since we dropped symmetry here, certainly we cannot use a 'center of symmetry'.) However, a simple calculation with Maple shows that the extremal value of* f *between the roots* 0 *and* 1 *is larger in absolute value than that between the roots* 1 *and* 2*. (Since the data are non-rational and cannot be expressed easily, we suppress the details.) So the strictly increasing extremal value property does not hold in this case, too.*

3. Proof of Theorem 2.4

As we shall see, Theorem 2.4 is a simple consequence of the following two propositions. They are rather similar, but because of technical reasons it is worth to formulate them separately.

Proposition 3.1. *Let* $0 = a_0 < a_1 < \cdots < a_n$ *be real numbers with*

$$a_i - a_{i-1} \leq a_{i+1} - a_i \quad (1 \leq i \leq n-1), \tag{4}$$

and let

$$f_1(x) = x\prod_{i=1}^{n}(x-a_i)(x+a_i).$$

Let α_i be the extremum of f_1 between a_i and a_{i+1} for $i = 0, \dots, n-1$. Then we have

$$|f_1(\alpha_0)| < |f_1(\alpha_1)| < \cdots < |f_1(\alpha_{n-1})|.$$

Proposition 3.2. *Let $0 < a_1 < \cdots < a_n$ be real numbers with*

$$3a_1 \le a_2 \quad \text{and} \quad a_i - a_{i-1} \le a_{i+1} - a_i \quad (2 \le i \le n-1), \tag{5}$$

and let

$$f_2(x) = \prod_{i=1}^{n}(x-a_i)(x+a_i).$$

Let α_i be the extremum of f_2 between a_i and a_{i+1} for $i = 1, \dots, n-1$. Then we have

$$|f_2(0)| < |f_2(\alpha_1)| < \cdots < |f_2(\alpha_{n-1})|.$$

Remark 3.1. *Note that by Rolle's theorem the extrema of f_1 and f_2 are situated in the way indicated in Propositons 3.1 and 3.2, respectively - that is, they are between the roots.*

To prove Propositions 3.1 and 3.2 we shall need some lemmas. The first one concerns certain properties of the Γ function, defined by

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}} \quad (z \in \mathbb{C}\setminus\mathbb{Z}_{\le 0}),$$

where $\mathbb{Z}_{\le 0}$ is the set of non-positive integers. Note that there are many other possibilities to define $\Gamma(z)$, the above form is called Euler's formula.

Lemma 3.1. *The following assertions hold.*

i) *For any $z \in \mathbb{C}\setminus\mathbb{Z}_{\le 0}$ we have*

$$z\Gamma(z) = \Gamma(z+1).$$

ii) *For any $z \in \mathbb{C}\setminus\mathbb{Z}$ we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

iii) *For any $u_1, u_2, v_1, v_2 \in \mathbb{C}\setminus\mathbb{Z}_{\le 0}$ with $u_1 + u_2 = v_1 + v_2$ we have*

$$\prod_{k=0}^{\infty}\frac{(k+u_1)(k+u_2)}{(k+v_1)(k+v_2)} = \frac{\Gamma(v_1)\Gamma(v_2)}{\Gamma(u_1)\Gamma(u_2)}.$$

Proof. The assertions i), ii) and iii) can be found in Sections 12.12, 12.14 and 12.13 of [15], respectively. □

In the proof of Proposition 3.1 we shall need the following assertion.

Lemma 3.2. *Let $0 < a_1 < \cdots < a_n$ be real numbers with*
$$a_i - a_{i-1} \le a_{i+1} - a_i \quad (2 \le i \le n-1).$$
Then for every i_1, i_2 with $1 \le i_1 \le i_2 \le n$ we have
$$\frac{a_{i_2}}{a_{i_1}} \ge \frac{i_2}{i_1}.$$

Proof. We prove the statement by induction on i_2. For $i_2 = 1$ the assertion is obvious. Assume that the statement holds for some i_2 with $1 \le i_2 < n$. Observe that using the assertion concerning the gaps between the a_i, we have
$$\frac{a_{i_2+1}}{a_{i_2}} = \frac{a_{i_2+1} - a_{i_2}}{a_{i_2}} + 1 \ge \frac{a_{i_2} - a_{i_2-1}}{a_{i_2}} + 1 =$$
$$= 2 - \frac{a_{i_2-1}}{a_{i_2}} \ge 2 - \frac{i_2 - 1}{i_2} = \frac{i_2 + 1}{i_2}.$$
Here, we also used the induction hypothesis. Now take any i_1 with $1 \le i_1 < i_2$. Then using the above assertion and the induction hypothesis we have
$$\frac{a_{i_2+1}}{a_{i_1}} = \frac{a_{i_2+1}}{a_{i_2}} \cdot \frac{a_{i_2}}{a_{i_1}} \ge \frac{i_2+1}{i_2} \cdot \frac{i_2}{i_1} = \frac{i_2+1}{i_1}.$$
Hence, the lemma follows. □

In the proof of Proposition 3.2 we shall need the following variant of Lemma 3.2.

Lemma 3.3. *Let $0 < a_1 < \cdots < a_n$ be real numbers with*
$$3a_1 \le a_2$$
and
$$a_i - a_{i-1} \le a_{i+1} - a_i \quad (2 \le i \le n-1).$$
Then for every i_1, i_2 with $1 \le i_1 \le i_2 \le n$ we have
$$\frac{a_{i_2}}{a_{i_1}} \ge \frac{2i_2 - 1}{2i_1 - 1}.$$

Proof. The proof is similar to that of Lemma 3.2. However, for the convenience of the reader we summarize the main steps, but we give less details.

We apply induction on i_2. For $i_2 = 1$ the statement is clear. Assume that the statement holds for some i_2 with $1 \le i_2 < n$. Using the assumption on the gaps between the a_i and the induction hypothesis, we have

$$\frac{a_{i_2+1}}{a_{i_2}} \geq 2 - \frac{a_{i_2-1}}{a_{i_2}} \geq \frac{2i_2+1}{2i_2-1}.$$

Now for any i_1 with $1 \leq i_1 < i_2$ we obtain

$$\frac{a_{i_2+1}}{a_{i_1}} = \frac{a_{i_2+1}}{a_{i_2}} \cdot \frac{a_{i_2}}{a_{i_1}} \geq \frac{2i_2+1}{2i_1-1},$$

and the lemma follows. □

Now we give the proof of Proposition 3.1.

Proof of Proposition 3.1. Let i be fixed with $1 \leq i \leq n-1$. First we show that for any t with $0 < t < 1$ we have

$$f^*(t) := \left| \frac{f_1(a_i - t(a_i - a_{i-1}))}{f_1(a_i + t(a_i - a_{i-1}))} \right| < 1, \tag{6}$$

from this the assertion will easily follow. Note that by our assumption on the gaps between the a_j, we have

$$a_{i-1} < a_i - t(a_i - a_{i-1}) < a_i < a_i + t(a_i - a_{i-1}) < a_{i+1} \quad (0 < t < 1).$$

Putting

$$d = \frac{a_i - a_{i-1}}{a_i} \quad \text{and} \quad s_j = \frac{a_j}{a_i} \quad (1 \leq j \leq n)$$

we can write (6) as

$$f^*(t) = \left(\frac{1-td}{1+td} \cdot \frac{2-td}{2+td} \right) \times \prod_{j=1}^{i-1} \left(\frac{1-s_j-td}{1-s_j+td} \cdot \frac{1+s_j-td}{1+s_j+td} \right) \times \\ \times \prod_{j=i+1}^{n} \left(\frac{s_j-1+td}{s_j-1-td} \cdot \frac{s_j+1-td}{s_j+1+td} \right). \tag{7}$$

Here, the first block corresponds to the roots $a_0 = 0$ and $\pm a_i$ of f_1. Further, note that we have $1 - s_j > td > 0$ for $j = 1, \dots, i-1$ and $0 < td < s_j - 1$ for $j = i+1, \dots, n$. (That is why it is worth to split the product for $j \geq 0$, $j \neq i$ according as $j < i$ or $j > i$.) Now we deal with the second and third terms on the right hand side of (7) in turn. We start with the second term. First observe that

$$\frac{X-Y}{X+Y} \text{ is strictly monotone increasing in } X > 0, \text{ for any } Y > 0. \tag{8}$$

Further, in view of the gap property of the a_j we have

$$1 - s_j = \frac{a_i - a_j}{a_i} = \frac{(a_i - a_{i-1}) + \cdots + (a_{j+1} - a_j)}{a_i} \leq (i-j)d,$$

and by Lemma 3.2 we see that

$$s_j \leq \frac{j}{i}$$

for all j with $1 \leq j < i$. Combining the above assertions, we obtain

$$\prod_{j=1}^{i-1} \left(\frac{1 - s_j - td}{1 - s_j + td} \cdot \frac{1 + s_j - td}{1 + s_j + td} \right) \leq \prod_{j=1}^{i-1} \left(\frac{i - j - t}{i - j + t} \cdot \frac{j + i - itd}{j + i + itd} \right). \tag{9}$$

Now we estimate the third product in (7). For this, observe that

$$\frac{s_j - 1 + td}{s_j - 1 - td} \cdot \frac{s_j + 1 - td}{s_j + 1 + td} = \frac{s_j^2 - (1 - td)^2}{s_j^2 - (1 + td)^2}$$

and that the function

$$\frac{X^2 - (1 - td)^2}{X^2 - (1 + td)^2}$$

is strictly decreasing in X for $X > 1 + td$. Hence in view of the inequality

$$s_j = 1 + \frac{a_j - a_i}{a_i} = 1 + \frac{(a_j - a_{j-1}) + \cdots + (a_{i+1} - a_i)}{a_i} \geq 1 + (j - i)d$$

valid for any $j > i$ obtained by the gap property of the a_j, we get

$$\prod_{j=i+1}^{n} \left(\frac{s_j - 1 + td}{s_j - 1 - td} \cdot \frac{s_j + 1 - td}{s_j + 1 + td} \right) \leq \tag{10}$$

$$\leq \prod_{j=i+1}^{n} \left(\frac{j - i + t}{j - i - t} \cdot \frac{j - i + \frac{2}{d} - t}{j - i + \frac{2}{d} + t} \right).$$

On combining (7), (9) and (10), we obtain

$$f^*(t) \leq \left(\frac{1 - td}{1 + td} \cdot \frac{2 - td}{2 + td} \right) \times \prod_{j=1}^{i-1} \left(\frac{i - j - t}{i - j + t} \cdot \frac{j + i - itd}{j + i + itd} \right) \times \tag{11}$$

$$\times \prod_{j=i+1}^{n} \left(\frac{j - i + t}{j - i - t} \cdot \frac{j - i + \frac{2}{d} - t}{j - i + \frac{2}{d} + t} \right).$$

In view of (8),

$$\frac{j - i + \frac{2}{d} - t}{j - i + \frac{2}{d} + t}$$

is monotone increasing in $2/d$ - so it is monotone decreasing in d. On the other hand, using (the negative of) (8) again, we see that all the terms in the first and second terms on the right hand side of (11) (which depend on d) are strictly monotone decreasing in d. Altogether, we obtain that the right hand side of (11) is monotone decreasing in d (for any fixed t). In view of

$$d = \frac{a_i - a_{i-1}}{a_i} = 1 - \frac{a_{i-1}}{a_i} \geq 1 - \frac{i-1}{i} = \frac{1}{i}$$

obtained by Lemma 3.2, substituting $d = 1/i$ in (11) gives

$$f^*(t) \leq -\frac{i-t}{i+t} \cdot \prod_{j=1}^{n} \left(\frac{j-i+t}{j-i-t} \cdot \frac{j+i-t}{j+i+t} \right).$$

(The negative sign comes from the factor $t/(-t)$ in case $j = i$.) Now using parts iii), i) and ii) of Lemma 3.1 (in this order), in view of that

$$\frac{j-i+t}{j-i-t} \cdot \frac{j+i-t}{j+i+t} > 1 \quad \text{for } j > n$$

we obtain

$$f^*(t) < -\frac{i-t}{i+t} \cdot \prod_{j=1}^{\infty} \left(\frac{j-i+t}{j-i-t} \cdot \frac{j+i-t}{j+i+t} \right) =$$
$$= -\frac{i-t}{i+t} \cdot \frac{\Gamma(1-i-t)\Gamma(1+i+t)}{\Gamma(1-i+t)\Gamma(1+i-t)} = -\frac{\Gamma(1-i-t)\Gamma(i+t)}{\Gamma(1-i+t)\Gamma(i-t)} =$$
$$= -\frac{\sin \pi(i-t)}{\sin \pi(i+t)} = \frac{\sin \pi(t-i)}{\sin \pi(t+i)} = 1.$$

Thus

$$f^*(t) < 1 \quad \text{for all } t \in (0,1),$$

and our claim (6) follows.

Put now

$$t_0 := \frac{a_i - \alpha_{i-1}}{a_i - a_{i-1}}.$$

Observe that $0 < t_0 < 1$ and that

$$\alpha_{i-1} = a_i - t_0(a_i - a_{i-1}) < a_i < a_i + t_0(a_i - a_{i-1}) < a_{i+1}.$$

Thus (6) implies

$$|f_1(\alpha_i)| = |f_1(a_i - t_0(a_i - a_{i-1}))| < |f_1(a_i + t_0(a_i - a_{i-1}))| \leq |f_1(\alpha_{i+1})|,$$

and the proposition follows. □

Now we give the proof of Proposition 3.2. It is rather similar to that of Proposition 3.1, however, with considerable technical differences. So we indicate all the important steps, but we suppress some details.

Proof of Proposition 3.2. Fix i with $2 \le i \le n-1$. First we prove that for any t with $0 < t < 1$ we have

$$f^*(t) := \left| \frac{f_2(a_i - t(a_i - a_{i-1}))}{f_2(a_i + t(a_i - a_{i-1}))} \right| < 1. \tag{12}$$

Put

$$d = \frac{a_i - a_{i-1}}{a_i} \quad \text{and} \quad s_j = \frac{a_j}{a_i} \quad (1 \le j \le n)$$

and rewrite (12) as

$$f^*(t) = \frac{2 - td}{2 + td} \times \prod_{j=1}^{i-1} \left(\frac{1 - s_j - td}{1 - s_j + td} \cdot \frac{1 + s_j - td}{1 + s_j + td} \right) \times \\ \times \prod_{j=i+1}^{n} \left(\frac{s_j - 1 + td}{s_j - 1 - td} \cdot \frac{s_j + 1 - td}{s_j + 1 + td} \right). \tag{13}$$

The first term corresponds to the root a_i of f_2. Note that $1 - s_j > td > 0$ for $j = 1, \ldots, i-1$ and $0 < td < s_j - 1$ for $j = i+1, \ldots, n$. To estimate the second term we follow the arguments in the proof of Proposition 3.1. Applying

$$1 - s_j = \frac{a_i - a_j}{a_i} = \frac{(a_i - a_{i-1}) + \cdots + (a_{j+1} - a_j)}{a_i} \le (i - j)d$$

again, but now combining it with

$$s_j \le \frac{2j - 1}{2i - 1} \quad (i < j \le n)$$

obtained by Lemma 3.2, we get

$$\prod_{j=1}^{i-1} \left(\frac{1 - s_j - td}{1 - s_j + td} \cdot \frac{1 + s_j - td}{1 + s_j + td} \right) \le \\ \le \prod_{j=1}^{i-1} \left(\frac{i - j - t}{i - j + t} \cdot \frac{2j - 1 + (2i-1)(1 - td)}{2j - 1 + (2i-1)(1 + td)} \right). \tag{14}$$

On the other hand, in the same way as in the proof of Proposition 3.1, for the third term of (13) we obtain

$$\prod_{j=i+1}^{n} \left(\frac{s_j - 1 + td}{s_j - 1 - td} \cdot \frac{s_j + 1 - td}{s_j + 1 + td} \right) \le \\ \le \prod_{j=i+1}^{n} \left(\frac{j - i + t}{j - i - t} \cdot \frac{j - i + \frac{2}{d} - t}{j - i + \frac{2}{d} + t} \right). \tag{15}$$

Combining (13), (14) and (15), we conclude

$$(16)\quad f^*(t) \le \frac{2-td}{2+td} \times \prod_{j=1}^{i-1}\left(\frac{i-j-t}{i-j+t}\cdot\frac{2j-1+(2i-1)(1-td)}{2j-1+(2i-1)(1+td)}\right) \times$$
$$\times \prod_{j=i+1}^{n}\left(\frac{j-i+t}{j-i-t}\cdot\frac{j-i+\frac{2}{d}-t}{j-i+\frac{2}{d}+t}\right).$$

Similarly as in the proof of Proposition 3.1, we can check that the right hand side of (16) is monotone decreasing in d (for any fixed t). Since

$$d = \frac{a_i - a_{i-1}}{a_i} = 1 - \frac{a_{i-1}}{a_i} \ge 1 - \frac{2i-3}{2i-1} = \frac{2}{2i-1}$$

by Lemma 3.3, substituting $d = 2/(2i-1)$ in (16) we obtain

$$f^*(t) \le -\prod_{j=1}^{n}\left(\frac{j-i+t}{j-i-t}\cdot\frac{j+i-1-t}{j+i-1+t}\right).$$

Now using parts iii), i) and ii) of Lemma 3.1 (in this order) we obtain

$$f^*(t) < -\prod_{j=1}^{\infty}\left(\frac{j-i+t}{j-i-t}\cdot\frac{j+i-1-t}{j+i-1+t}\right) = -\frac{\Gamma(1-i-t)\Gamma(i+t)}{\Gamma(1-i+t)\Gamma(i-t)} =$$
$$= -\frac{\sin\pi(i-t)}{\sin\pi(i+t)} = \frac{\sin\pi(t-i)}{\sin\pi(t+i)} = 1.$$

Thus,

$$f^*(t) < 1 \quad \text{for all } t \in (0,1),$$

and our claim (12) follows. From this, just as in the proof of Proposition 3.1 we get

$$|f_2(\alpha_i)| < |f_2(\alpha_{i+1})|,$$

implying

$$|f_2(\alpha_1)| < \cdots < |f_2(\alpha_{n-1})|.$$

Thus, to prove the statement, it remains to show that

$$|f_2(0)| < |f_2(\alpha_1)|.$$

For this, first we show that

$$|f_2(0)| < |f_2(2a_1)|.$$

Plainly, we have

$$f_2(0) = \prod_{j=1}^{n} a_j^2.$$

On the other hand,

$$|f_2(2a_1)| = \left|\prod_{j=1}^{n}(2a_1 - a_j)(2a_1 + a_j)\right| = 3a_1^2\prod_{j=2}^{n}(a_j^2 - 4a_1^2).$$

Thus,

$$\left|\frac{f_2(0)}{f_2(2a_1)}\right| = \frac{1}{3}\prod_{j=2}^{n}\frac{\delta_j^2}{\delta_j^2 - 4},$$

where

$$\delta_j = \frac{a_j}{a_1} \quad (2 \le j \le n).$$

Since the function

$$\frac{X^2}{X^2 - 4}$$

is strictly decreasing in $X \ge 3$ and

$$\delta_j \ge 2j - 1 \quad (2 \le j \le n),$$

we obtain

$$\left|\frac{f_2(0)}{f_2(2a_1)}\right| < \frac{1}{3}\cdot\prod_{j=2}^{\infty}\frac{(2j-1)^2}{(2j-1)^2 - 4} =$$

$$= \frac{1}{3}\cdot\prod_{j=2}^{\infty}\frac{\left(j-\frac{1}{2}\right)^2}{\left(j-\frac{3}{2}\right)\left(j+\frac{1}{2}\right)} = \frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)} = 1.$$

Here we used part iii) of Lemma 3.1. From this, by $a_1 < 2a_1 < a_2$, we get

$$|f_2(0)| < |f_2(2a_1)| \le f_2(\alpha_1)|$$

and hence the proposition follows. □

Now we are ready to give the proof of Theorem 2.4.

Proof of Theorem 2.4. Let b_0 be the leading coefficient of $f(x)$, write $b_1, \ldots, b_k$ for the roots of f in increasing order, and let c be the center of symmetry of them. Observe that then $b_1 - c, \ldots, b_k - c$ can be written as

$$-a_n, \ldots, -a_1, (a_0 = 0), a_1, \ldots, a_n$$

with $k = 2n+1$ or $k = 2n$, according as k is odd or k is even. Thus, we have

$$f(x+c) = \begin{cases} b_0 f_1(x) & \text{if } n \text{ is odd,} \\ b_0 f_2(x) & \text{if } n \text{ is even,} \end{cases}$$

with $f_1(x)$ and $f_2(x)$ defined in Propositions 3.1 and 3.2, respectively. Further, by the centrally convex property of $b_1, \ldots, b_k$, we see that (4) in Proposition 3.1 or (5) in Proposition 3.2 is also satisfied, respectively. (Note that $3a_1 \leq a_2$ in (5) can be written as $a_1 - (-a_1) \leq a_2 - a_1$.) Since $f_1(x)$ and $f_2(x)$ are symmetric with respect to 0, and $f(x+c)$ is just a shift of $f(x)$ along the x axis, the statement immediately follows from Propositions 3.1 and 3.2. □

4. Proof of Theorem 2.3

As we shall see, Theorem 2.3 follows from Theorem 2.4 after some simple considerations.

Proof of Theorem 2.3. Suppose that f is decomposable over $\mathbb{Q}$. Then we can write $f(x) = T_1(T_2(x))$ with some polynomials $T_1, T_2 \in \mathbb{Q}[x]$ where $\deg(T_1) > 1$ and $\deg(T_2) > 1$. As one can easily check (or see e.g. the proof of Theorem 4.3 in [2]) we have

$$\deg(T_2) \leq \max_{\lambda \in \mathbb{C}} \deg(\gcd(f(x) - \lambda, f'(x))).$$

Observe that since $f(x) \in \mathbb{Q}[x]$ and the roots of $f'(x)$ are simple and real, if $\deg(\gcd(f(x) - \lambda, f'(x))) \geq 1$, then λ is an extremal value of f (in particular, $\lambda \in \mathbb{R}$). However, Theorem 2.4 shows that there are no three (or more) extremal values of f which are equal. Hence, $\deg(T_2) = 2$. So, if $\deg(f)$ is odd, then f is indecomposable. On the other hand, if $\deg(f)$ is even then we have

$$f(x) = b_0((x-c)^2 - (b_1 - c)^2) \cdots ((x-c)^2 - (b_k - c)^2). \tag{17}$$

Indeed, the degree and the leading coefficient of the right hand side in (17) are the same as those of f. Further, $b_1, \ldots, b_k$ are obviously roots of the right hand side - and by

$$(b_i - c)^2 = (b_{2k+1-i} - c)^2 \quad (i = 1, \ldots, k)$$

the same is true for $b_{k+1}, \ldots, b_{2k}$.

Write

$$\hat{f}(x) = b_0(x - (b_1 - c)^2) \cdots (x - (b_k - c)^2).$$

(Note that this is the same polynomial that appears in Theorem 2.2.) We show that this polynomial has rational coefficients. First observe that we have

$$2kc = b_1 + \cdots + b_{2k}.$$

Since the right hand side above is just the negative of the coefficient of x^{2k-1} in $f(x)$, this implies that $c \in \mathbb{Q}$. Thus $f(x+c) = \hat{f}(x^2) \in \mathbb{Q}[x]$. But then we also have $\hat{f}(x) \in \mathbb{Q}[x]$.

Finally, it is easy to check that any other decomposition of $f(x)$ over $\mathbb{Q}$ is equivalent to (17), and the theorem follows.

□

5. Proof of Theorem 2.2

In the proof of Theorem 2.2 we shall need the following lemmas. Let $T(x) \in \mathbb{Z}[x]$ and A be an integer with $A \neq 0$, and consider the equation

$$T(x) = Ay^m, \tag{18}$$

in unknown integers x, y, m with $m \geq 2$, under the convention that $m \leq 3$ if $|y| \leq 1$. The next result is due to Schinzel and Tijdeman [13] (see also Tijdeman [14]).

Lemma 5.1. *If $T(x)$ has at least two different roots, then for all solutions of* (18)

$$m < C_2(A, d, H)$$

holds. Here $C_2(A, d, H)$ is an effectively computable constant depending only on A, the degree d and the height H of $T(x)$.

The following lemma is a special case of the main result of Brindza [4]. In order to formulate it we need some new notation. Let S be a finite set of primes, and let $\mathbb{Z}_S$ be the set of those rationals whose denominators are composed exclusively of primes from S. By the height $h(q)$ of a rational number q we mean the maximum of the absolute value of its denominator and numerator.

Lemma 5.2. *Let $T(x) \in \mathbb{Z}[x]$, and write*

$$T(x) = a \prod_{i=1}^{k} (x - \gamma_i)^{r_i},$$

where a is the leading coefficient of T, and $\gamma_1, \ldots, \gamma_k$ are the distinct complex roots of $T(x)$, with multiplicities $r_1, \ldots, r_k$, respectively. Further, fix m with $m \geq 2$, and put

$$t_i = \frac{m}{(m, r_i)} \qquad (i = 1, \ldots, k).$$

Suppose that $(t_1, \ldots, t_k)$ is not a permutation of any of the k-tuples

$$(t, 1, \ldots, 1)\ (t \geq 1), \quad (2, 2, 1, \ldots, 1).$$

Then for any finite set S of primes, the solutions $x, y \in \mathbb{Z}_S$ of (18) *satisfy*

$$\max\left(h(x), h(y)\right) < C_3(A, m, d, H, S),$$

where $C_3(A, m, d, H, S)$ is an effectively computable constant depending only on A, m, d, H, S, where d is the degree and H is the height of $T(x)$.

Now we can give the proof of Theorem 2.2.

Proof of Theorem 2.2. Since $\deg(f) > 6$ and $f'(x) = (f(x) - B)'$ has simple roots, it is clear that $f(x) - B$ has at least two distinct roots. Hence, we can apply Lemma 5.1 to get an effective upper bound for m as claimed.

In particular, from this point on we may assume that $m \geq 2$ is fixed. Using again that $f'(x) = (f(x) - B)'$ has simple roots, we see that $f(x) - B$ has at most double roots. Thus the second part of the statement immediately follows from Lemma 5.2 for $m \geq 3$.

So we may assume that $m = 2$. Then, the second part of the theorem also follows from Lemma 5.2, unless we have

$$f(x) - B = p(x)(q(x))^2 \tag{19}$$

with some $p, q \in \mathbb{Q}[x]$, $\deg(p) \leq 2$. Differentiating both sides of (19) we get

$$f'(x) = q(x)(p'(x)q(x) + 2p(x)q'(x)).$$

So writing $\alpha_1, \ldots, \alpha_{d-1}$ for the (real, simple) roots of $f'(x)$, we see that the roots of $q(x)$ are among them. However, if α_i is a root of $q(x)$, then (19) yields $f(\alpha_i) = B$. However, this may hold at most for two α_i-s. That is, $\deg(q) \leq 2$. Hence, $d \leq 6$, which is excluded, and the theorem follows. $\square$

6. Proof of Theorem 2.1

To prove Theorem 2.1, we need some more notation and a deep result of Bilu and Tichy [3].

Let α, β, $\delta \in \mathbb{Q}\setminus\{0\}$, μ, ν, q be positive integers, r be a non-negative integer, and $v(x) \in \mathbb{Q}[x]$ a polynomial, which is not identically zero. Write $D_\mu(x, \delta)$ for the μ-th Dickson polynomial, that is

$$D_\mu(x, \delta) = \sum_{i=0}^{\lfloor \mu/2 \rfloor} d_{\mu,i} x^{\mu-2i},$$

where

$$d_{\mu,i} = \frac{\mu}{\mu - i}\binom{\mu - i}{i}(-\delta)^i.$$

We say that the polynomials $F(x)$ and $G(x)$ form a standard pair over $\mathbb{Q}$, if $(F(x), G(x))$ or $(G(x), F(x))$ appears in Table 1.

The following lemma is the main result of Bilu and Tichy [3].

Lemma 6.1. *Let $f(x), g(x) \in \mathbb{Q}[x]$ be non-constant polynomials. Then the following two assertions are equivalent.*
i) The equation

$$f(x) = g(y)$$

Kind	Standard pair	Parameter restrictions
First	$(x^q, \alpha x^r v(x)^q)$	$0 \leq r < q, (r,q) = 1,$ $r + \deg v(x) > 0$
Second	$(x^2, (\alpha x^2 + \beta) v(x)^2)$	-
Third	$(D_\mu(x, \alpha^\nu), D_\nu(x, \alpha^\mu))$	$(\mu, \nu) = 1$
Fourth	$(\alpha^{\frac{-\mu}{2}} D_\mu(x, \alpha), -\beta^{\frac{-\nu}{2}} D_\nu(x, \beta))$	$(\mu, \nu) = 2$
Fifth	$((\alpha x^2 - 1)^3, 3x^4 - 4x^3)$	-

TABLE 1. Standard pairs

has infinitely many solutions with a bounded denominator.
ii) We have $f(x) = \varphi(F(\lambda(x)))$ *and* $g(x) = \varphi(G(\kappa(x)))$*, where* $\lambda(x)$ *and* $\kappa(x)$ *are linear polynomials in* $\mathbb{Q}[x]$*,* $\varphi(x) \in \mathbb{Q}[x]$ *and* $(F(x), G(x))$ *is a standard pair over* $\mathbb{Q}$ *such that the equation* $F(x) = G(y)$ *has infinitely many solutions with a bounded denominator.*

Now we give the proof of Theorem 2.1.

Poof of Theorem 2.1. First observe that if $\deg(g) = 2$, then by a linear substitution we may get rid of the coefficient of the linear term in g, and hence the statement follows from Theorem 2.2. So, from this point on we shall assume that $\deg(g) \geq 3$.

Suppose that (2) has infinitely many solutions in integers x, y. Then according to Lemma 6.1 we have $f(x) = \varphi(F(\lambda(x)))$ and $g(x) = \varphi(G(\kappa(x)))$, where $\lambda(x)$ and $\kappa(x)$ are linear polynomials in $\mathbb{Q}[x]$, $\varphi(x) \in \mathbb{Q}[x]$ and $(F(x), G(x))$ is a standard pair over $\mathbb{Q}$. Based on Theorem 2.3 we have three possible cases:

(1) $\deg(\varphi) = \deg(f)$ and $\deg(F) = 1$,
(2) $\deg(f) = 2k$ even, $\deg(\varphi) = k$ and $\deg(F) = 2$,
(3) $\deg(\varphi) = 1$ and $\deg(F) = \deg(f)$.

In the first case $\varphi(x) = f(\tau(x))$, where τ is a rational linear polynomial. Hence, we have $g(y) = f(P(y))$, where $P(y) \in \mathbb{Q}[y]$ is arbitrary with degree ≥ 1, and the theorem follows in this case.

In the second case we have $\varphi = \hat{f}$ and $f(x) = \hat{f}((x-c)^2)$. (Note that from the proof of Theorem 2.2 we already know that $\hat{f}(x) \in \mathbb{Q}[x]$.) Thus, we have $g(y) = \hat{f}(Q(y))$ with some $Q(y) \in \mathbb{Q}[y]$. Lemma 6.1 implies that the equation

$$(x-c)^2 = Q(y)$$

must have infinitely many solutions in $x, y \in \mathbb{Q}$ with a bounded denominator. Thus, according to Lemma 5.2, $Q(y)$ can have at most two roots with odd multiplicity. So the statement is proved also in this case.

In the third case, we have

$$f(x) = AF(ax + b) + B \tag{20}$$

with $A, B, a, b \in \mathbb{Q}$ with $Aa \neq 0$, and F is a member of one of the five standard pairs from Table 1. We check the cases of the five standard pairs in turn.

Assume first that F comes from a standard pair of the fifth kind. Then differentiating both sides of (20) we see that $f'(x)$ has a double root. However, this contradicts the fact that the roots of $f'(x)$ are simple (and real). So, this case cannot occur.

Suppose next that F belongs to a standard pair of the first kind. By our conditions $\deg(f) > 6$ and $\deg(g) \geq 3$ we see that $q \geq 3$. Further, as $f'(x)$ has simple (real) roots, we obtain that $F(x) = \alpha x^r v(x)^q$ must be valid, but with $v(x)$ being constant and $r \leq 2$. However, this contradicts $\deg(f) > 6$, so this case also cannot occur.

The case that F belongs to a standard pair of the second kind also cannot hold, since then we would get $\deg(f) = 2$ or $\deg(g) = 2$, which are excluded.

So we are left with the possibilities where $F(x)$ comes from a standard pair of the third or fourth kind. In both cases, (20) yields an equality of the form

$$f(x) = AD_n(ax + b, \delta) + B$$

where $n \geq 3$ and δ is a non-zero rational. By Proposition 3.3 of Bilu [1] we see that $D_n(x, \delta)$ has precisely two different extremal values, and this property is certainly inherited to $AD_n(ax + b, \delta) + B$. However, Theorem 2.4 shows that f has at least three extremal values already for $\deg(f) > 4$. Hence, this case also cannot occur, and the theorem is proved. □

Acknowledgements

The authors are grateful to the Referee for the useful and helpful remarks.

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