POLYNOMIAL VALUES OF SURFACE POINT COUNTING POLYNOMIALS

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Dedicated to Attila Pethő on the occasion of his 70th birthday.

Abstract. There are many results in the literature concerning power values, equal values or more generally, polynomial values of lattice point counting polynomials. In the present paper we prove various finiteness results for polynomial values of polynomials counting the lattice points on the surface of an n-dimensional cube, pyramid and simplex.

1. Introduction

There are many papers in the literature about equal values and polynomial values of lattice point counting polynomials. Their survey would be an enormous task; here we only mention the most important results from our viewpoint. These results have an interesting common feature, namely that they all hold a nice geometric meaning: they describe those polynomials whose values taken in integers, infinitely often give back the number of lattice points in certain regular bodies.

In the present paper, among regular bodies we focus on the n-dimensional cube, pyramid and simplex. As it is well known, the number of integral points in the interior of these bodies in $\mathbb{R}^n$ (in case of their usual placement) is given by the polynomials

$$(1) \qquad (x+1)^n, \quad 1^{n-1}+2^{n-1}+\cdots+(x+1)^{n-1}, \quad \binom{x+n}{n},$$

respectively.

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The polynomial values of the first polynomial in (1), namely the so called superelliptic equation

$$(x+1)^n = g(y)$$

has been studied by many mathematicians. Here g is a polynomial with rational coefficients and x, y are integral unknowns. Results of Tijdeman [22] and Schinzel and Tijdeman [20] imply that under certain necessary assumptions here n can be effectively bounded. Baker (see [1, 2]) and Brindza [7] showed that given n, under some assumptions one can also bound the absolute values of x, y, as well. For further related results see the book of Shorey and Tijdeman [21].

The second polynomial in (1) is denoted by $S_{n-1}(x+1)$. The polynomial values of this polynomial, i.e. the equation

$$S_{n-1}(x+1) = g(y)$$

where g is a polynomial with rational coefficients and x, y are integral unknowns, has also been intensively studied. In the special case where g is of the form $g(y) = y^\ell$, a classical result of Schäffer [19] shows that (apart from certain completely described exceptions) the above equation has only finitely many solutions for n fixed. When $g(y)$ is a shifted power, or more generally it is of the shape $g(y) = Ay^\ell + B$ with $A, B \in \mathbb{Q}$, $A \neq 0$, Győry, Tijdeman and Voorhoeve [14] obtained deep finiteness results - again, with n fixed. Later, the same authors derived even more general finiteness results concerning shifts of $S_{n-1}(x+1)$ with polynomials (see [23]). The general case has been taken up by Rakaczki [18]. He proved that the previous equation for any fixed n, apart from certain well-described exceptions, has only finitely many solutions in integers x, y. For more related results see e.g. the papers Bennett, Győry and Pintér [5], Győry and Pintér [13], Bazsó [3] and Hajdu [15] and the references given there.

Finally, the investigation of the third polynomial in (1) reduces to the equation

$$\binom{x+n}{n} = g(y)$$

in integers x, y, where g is a polynomial with rational coefficients again. This is also a famous equation, studied by several authors. In the case where $g(y) = y^\ell$, the equation has been completely solved by Erdős [10] (for $n \geq 4$) and Győry [11] (for $n = 2, 3$). When g is of the shape $g(y) = Ay^\ell + B$ with $A, B \in \mathbb{Q}$, $A \neq 0$, Yuan [24] gave effective upper bounds for the absolute values of x, y. In the general case, Kulkarni and Sury [17] gave an ineffective finiteness theorem for the solutions of the previous equation.

Besides the above mentioned results, there are many more related papers in the literature. The interested reader may consult e.g. the paper of Bilu, Brindza, Kirschenhofer, Pintér and Tichy [8] or the survey paper of Győry, Kovács, Péter and Pintér [12] and the references therein.

In the present paper we study the polynomials describing the number of lattice points on the *surfaces* of the above mentioned regular bodies. These polynomials can be obtained by certain differences of the polynomials in (1). Namely, one can easily check that the number of integral points on the surfaces of the n-dimesional cube, pyramid and simplex (for arbitrary $n \geq 1$) can be given by the polynomials

$$(2)\quad F_n(x) = (x+1)^n - (x-1)^n, \quad G_n(x) = (x+1)^{n-1} + x^{n-1},$$
$$H_n(x) = \binom{x+n}{n} - \binom{x-1}{n},$$

respectively. We provide various finiteness results for the polynomial values of $F(x), G(x), H(x)$, that is for the integer solutions of the equations

$$F_n(x) = g(y), \quad G_n(x) = g(y), \quad H_n(x) = g(y)$$

where g is a polynomial with rational coefficients. In the general case our theorems are ineffective. However, in the case where g is of the form $g(y) = Ay^\ell + B$ with $A, B \in \mathbb{Q}$, $A \neq 0$ then we can provide effective finiteness results. In our proofs (among others) we combine Baker's method and the Bilu-Tichy theorem [9]. To apply these methods (as we shall see) we need to get precise information on the root structures of the polynomials, their derivatives and their shifts from (2). We shall also have to understand the decomposability properties of these polynomials. It is worth to mention that to prove the related properties of the difference polynomials (2) in many cases is significantly more difficult than in case of the original polynomials (1). Finally, we note that related investigations (i.e. papers concerned with differences of combinatorial polynomials) are known in the literature: see e.g. the paper of Liptai, Luca, Pintér and Szalay [16] (and the references there), where the equation $S_k(x-1) = S_\ell(y-1) - S_\ell(x)$ has been studied.

The structure of the paper is the following. In the next section we give our main results. In Section 3 we describe the root structures of the polynomials (2) and of their derivatives and shifts, together with their decomposability properties. Then we provide the proofs of our effective statements. Finally, we give the proofs of our ineffective results.

2. Main results

In our paper we examine the equation

$$f(x) = g(y) \tag{3}$$

where $f(x)$ is one of the polynomials $F_n(x)$, $G_n(x)$, $H_n(x)$ ($n \geq 1$) from (2), and $g(y) \in \mathbb{Q}[y]$. Our purpose is to prove finiteness results for the integer solutions x, y of (3). First we provide a general theorem for the problem considered. This result is ineffective, so it only shows the finiteness of the number of solutions, it does not give bounds for the solutions themselves.

Theorem 2.1. *Let $n \geq 6$ and $\deg(g) \geq 2$. If equation (3) has infinitely many solutions in integers x, y then either*

$$g(y) = f(P(y))$$

where $P(y) \in \mathbb{Q}[y]$, or

$$g(y) = \hat{f}(Q(y))$$

where $Q(y) \in \mathbb{Q}[y]$ with at most two roots of odd multiplicity, n is odd, and in case of $f(x) = F_n(x)$, $G_n(x)$, $H_n(x)$ the polynomial $\hat{f}$ is φ_1, φ_2, φ_3, respectively, with

$$\varphi_1(x) = 2\binom{n}{1}x^{\frac{n-1}{2}} + 2\binom{n}{3}x^{\frac{n-3}{2}} + \cdots + 2\binom{n}{n-2}x + 2,$$

$$\varphi_2(x) = 2x^{\frac{n-1}{2}} + \frac{1}{2}\binom{n-1}{2}x^{\frac{n-3}{2}} + \cdots + \frac{1}{2^{n-4}}\binom{n-1}{n-3}x + \frac{1}{2^{n-2}},$$

$$\varphi_3(x) = \frac{2}{n!}(s_1 x^{\frac{n-1}{2}} + \cdots + s_n) \quad \text{where } s_j = \sum_{\substack{A \subseteq \{1,\ldots,n\} \\ |A| = j}} \prod_{a \in A} a \ \ (j = 1, \ldots, n).$$

Remark 2.1. *Clearly, we have to exclude polynomials g with $\deg(g) = 1$, so the assumption $\deg(g) \geq 2$ is necessary. The condition $n \geq 6$ is necessary, too. For $n \leq 5$ one can easily find counterexamples (which is not surprising in view of the many free parameters involved; see the proof of the theorem).*

In the case where $g(y) = Ay^\ell + B$ with $A, B \in \mathbb{Q}$ with $A \neq 0$, we can give an effective upper bound for the absolute values of the integer solutions x, y of the equation (3).

Theorem 2.2. *Let $n \geq 1$ and consider the equation*

$$f(x) = Ay^\ell + B \tag{4}$$

where $f(x)$ *is one of the polynomials* $F_n(x), G_n(x), H_n(x)$ *from* (2), A, B *are given rationals with* $A \neq 0$, *and* x, y *and* $\ell \geq 2$ *are integer unknowns.*
i) Let $n \geq 4$. *Then there exists an effectively computable constant* $C_1(A, B, n)$, *depending only on* A, B, n *such that*

$$\ell < C_1(A, B, n)$$

for every solutions of (4) *with* $|y| > 1$.
ii) Let $\ell \geq 2$ *be arbitrary but fixed and* $n \geq 8$. *Then there exists an effectively computable constant* $C_2(A, B, n)$, *depending only on* A, B, n *such that*

$$\max(|x|, |y|) \leq C_2(A, B, n)$$

for every integer solution x, y *of* (4).

Remark 2.2. *Also in this case, the assumptions made for* n *are all necessary; one could easily find counterexamples in the excluded cases.*

For the formulation of our last theorem we need some more notions and notation.

By the *decomposition* of a polynomial $T(x)$ over a field K we mean a composition of the form $T(x) = U_1(U_2(x))$, where $U_1(x), U_2(x) \in K[x]$. We say that the decomposition is *nontrivial* if $\deg(U_1) > 1$ and $\deg(U_2) > 1$. Two decompositions $T(x) = U_1(U_2(x))$ and $T(x) = V_1(V_2(x))$ are *equivalent* if there exists a linear polynomial $t(x) \in K[x]$ such that $U_1(x) = V_1(t(x))$ and $V_2(x) = t(U_2(x))$. If $T(x)$ has at least one nontrivial decomposition over K then we say that $T(x)$ is *decomposable*; otherwise $T(x)$ is *indecomposable.*

In the proof of our ineffective results the following theorem plays an important role. It completely describes the decompositions of the polynomial families $F_n(x)$, $G_n(x)$, $H_n(x)$.

Theorem 2.3. *Let* $n \geq 2$. *If* n *is even then the polynomials* $F_n(x)$, $G_n(x)$, $H_n(x)$ *are indecomposable. If* n *is odd, then all the decompositions of these polynomials are equivalent with*

$$F_n(x) = \varphi_1(x^2), \quad G_n(x) = \varphi_2\left(\left(x + \frac{1}{2}\right)^2\right), \quad H_n(x) = \varphi_3(x^2),$$

respectively. Here φ_1, φ_2, φ_3 *are the same polynomials as in Theorem 2.1.*

3. Root structures and decompositions

In this section we describe the root structures of the studied polynomial families, of their derivatives and of their shifts. We also prove

Theorem 2.3 in this section, characterizing the decompositions of the polynomials $F_n(x)$, $G_n(x)$, $H_n(x)$.

We note that obviously

$$\deg(F_n) = \deg(G_n) = \deg(H_n) = n - 1 \quad (n \geq 1).$$

3.1. **The polynomial family $F_n(x)$.** In this subsection we describe the root structure of $F_n(x)$, of its derivative and of its shifts. We start with $F_n(x)$ itself.

Lemma 3.1. *Let $n \geq 2$. Then all the roots of $F_n(x) = (x+1)^n - (x-1)^n$ are simple.*

Proof. Taking derivative we get

$$F_n'(x) = n(x+1)^{n-1} - n(x-1)^{n-1}$$

whence

$$nF_n(x) - (x+1)F_n'(x) = 2n(x-1)^{n-1}.$$

Thus the common roots of $F_n(x)$ and $F_n'(x)$ are also roots of $(x-1)^{n-1}$. As clearly there are no such roots, our claim follows. □

As a simple consequence we obtain the following statement concerning $F_n'(x)$.

Corollary 3.1. *Let $n \geq 3$. Then all the roots of $F_n'(x)$ are simple.*

Proof. Since $F_n'(x) = nF_{n-1}(x)$, the statement immediately follows from Lemma 3.1. □

In the next lemma we examine the root structure of the shifted polynomials of $F_n(x)$.

Lemma 3.2. *Let $n \geq 2$. Then for any $r \in \mathbb{C}$ the polynomial $F_n(x) + r$ has at most two multiple roots, which are at most double.*

Proof. By Lemma 3.1 we may assume that $r \neq 0$. Differentiating $F_n(x) + r$ we obtain

$$(F_n(x) + r)' = n(x+1)^{n-1} - n(x-1)^{n-1}.$$

Thus

$$n(F_n(x) + r) - (x-1)(F_n(x) + r)' = 2n(x+1)^{n-1} + nr.$$

So any common root α of $F_n(x) + r$ and $(F_n(x) + r)'$ is a root of $2(x+1)^{n-1} + r$, which implies that

$$|\alpha + 1| = \sqrt[n-1]{\left|\frac{r}{2}\right|}. \tag{5}$$

On the other hand, we also have

$$n(F_n(x)+r)-(x+1)(F_n(x)+r)'=2n(x-1)^{n-1}+nr.$$

So α is also a root of $2(x-1)^{n-1}+r$ whence

$$|\alpha-1| = \sqrt[n-1]{\left|\frac{r}{2}\right|}. \tag{6}$$

Combining (5) and (6) we see that the multiple roots of $F_n(x)+r$ are on the intersection of two circles on the complex plane. This shows that there are at most two such roots. The multiplicity of these roots (if they exist) cannot be greater than two, since by Corollary 3.1 all the roots of $(F_n(x)+r)'=F_n'(x)$ are simple. □

Finally, we need the following assertion to prove the corresponding part of Theorem 2.3.

Lemma 3.3. *Let $n \geq 2$. Then $\max_{\lambda\in\mathbb{C}} \deg(\gcd(F_n(x)-\lambda, F_n'(x))) \leq 2$.*

Proof. We know from Lemma 3.2 that for any $\lambda \in \mathbb{C}$ the polynomial $F_n(x)-\lambda$ has at most two multiple roots, which are at most double. Hence $\deg(\gcd(F_n(x)-\lambda, F_n'(x))) \leq 2$, and our claim follows. □

3.2. **The polynomial family $G_n(x)$.** Now we examine the polynomials $G_n(x)$. Since this family is similar to $F_n(x)$, the proofs here are very similar to those in the previous subsection, and we shall omit the details.

Lemma 3.4. *For any $n \geq 2$ the roots of the polynomial $G_n(x) = (x+1)^{n-1}+x^{n-1}$ are all simple.*

Proof. By taking derivative, the proof is similar to that of Lemma 3.1. □

By the previous statement we can easily describe the root structure of $G_n'(x)$, as well.

Corollary 3.2. *For any $n \geq 3$ the roots of the polynomial $G_n'(x)$ are all simple.*

Proof. As $G_n'(x)=(n-1)G_{n-1}(x)$, the statement immediately follows from Lemma 3.4. □

Now we examine the root structure of the shifts of $G_n(x)$.

Lemma 3.5. *Let $n \geq 2$. Then in case of any $r \in \mathbb{C}$ the polynomial $G_n(x)+r$ has at most two multiple roots, which are at most double.*

Proof. The proof follows the same line as that of Lemma 3.2. We omit the details. □

To prove the corresponding part of Theorem 2.3 we also need the following lemma.

Lemma 3.6. *Let* $n \geq 2$. *Then* $\max_{\lambda \in \mathbb{C}} \deg(\gcd(G_n(x) - \lambda, G_n'(x))) \leq 2$.

Proof. The statement is an easy consequence of Lemma 3.5 (like Lemma 3.3 of Lemma 3.2). □

3.3. **The polynomial family** $H_n(x)$**.** Finally, we examine the family $H_n(x)$.

Lemma 3.7. *For all* $n \geq 2$*, all the roots of*

$$H_n(x) = \frac{1}{n!}\left((x+1)\dots(x+n) - (x-1)\dots(x-n)\right)$$

are simple. Further, the real part of any root of $H_n(x)$ *is zero.*

Proof. We distinguish two cases, according to the parity of n.

Assume first that n is odd. Observe that then the coefficients of the odd powers of $h(x)$ are zero. In particular, we easily see that $\deg(h) = n - 1$ and 0 is not a root of $h(x)$. It is also obvious that if α is a root of $h(x)$, so is $-\alpha$. Thus it is sufficient to show that $h(x)$ has roots of the form $a_k i$ with distinct positive real numbers a_k ($k = 1, \dots, (n-1)/2$). For arbitrary positive real a, we have

$$h(ai) = |ai+1|\dots|ai+n|(\cos(\alpha_1 + \dots + \alpha_n) + i\sin(\alpha_1 + \dots + \alpha_n))$$

$$-|ai-1|\dots|ai-n|(\cos(n\pi - \alpha_1 - \dots - \alpha_n) + i\sin(n\pi - \alpha_1 + \dots - \alpha_n))$$

$$= 2|ai+1|\dots|ai+n|\cos\left(\sum_{j=1}^{n}\arctan\left(\frac{a}{j}\right)\right).$$

Here we wrote α_j for the argument of $ai + j$, and used that $\alpha_j = \arctan(a/j)$ ($j = 1, \dots, n$). Put

$$s(a) = \sum_{j=1}^{n}\arctan\left(\frac{a}{j}\right).$$

Observe that $s(a)$ is strictly monotone increasing in a, and

$$0 < s(a) < n\arctan(a) < \frac{\pi}{2}$$

if $\arctan(a) < \pi/2n$, and

$$(n-1)\frac{\pi}{2} < n\arctan\left(\frac{a}{n}\right) < s(a)$$

if $(n-1)\pi/2n < \arctan(a/n)$. So by the continuity of $s(a)$, we get that there exist distinct positive real numbers a_k such that

$$s(a_k) = (2k-1)\pi/2 \quad (k = 1, \ldots, (n-1)/2). \tag{7}$$

However, then we have $h(a_k i) = 0$ $(k = 1, \ldots, (n-1)/2)$, and the statement follows in this case.

Assume now that n is even. Observe that then the coefficients of the even powers of $h(x)$ are zero. In particular, we easily see that $\deg(h) = n-1$ and 0 is a simple root of $h(x)$. Again, if α is a root of $h(x)$, so is $-\alpha$. Thus it is sufficient to show that $h(x)$ has roots of the form $a_k i$ with distinct positive real numbers a_k $(k = 1, \ldots, (n-2)/2)$. Similarly as for n odd, for arbitrary positive real a, we have

$$h(ai) = 2i|ai+1| \ldots |ai+n| \sin\left(\sum_{j=1}^{n} \arctan\left(\frac{a}{j}\right)\right).$$

Now we get that there exist distinct positive real numbers a_k such that $s(a_k) = k\pi$ $(k = 1, \ldots, (n-2)/2)$. However, then we have $h(a_k i) = 0$ $(k = 1, \ldots, (n-2)/2)$, and the statement follows also in this case. $\square$

The characterization of the root structure of $H_n'(x)$ is much more complicated then for $F_n'(x)$ and $G_n'(x)$.

Lemma 3.8. *For all $n \geq 2$, all the roots of $H_n'(x)$ are simple.*

Proof. According to the parity of n we distinguish two cases again.

Assume first that n is odd. Then $H_n(x)$ can be written in the form

$$H_n(x) = u_{n-1}x^{n-1} + u_{n-3}x^{n-3} + \ldots + u_2x^2 + u_0.$$

We know, that the roots of $H_n(x)$ are on the imaginary axis: $(n-1)/2$ roots are on its positive part and $(n-1)/2$ roots are on its negative part. We introduce the polynomials $H_n^*(x) := H_n(ix)$. Then we have

$$H_n^*(x) = (-1)^{\frac{n-1}{2}} u_{n-1}x^{n-1} + (-1)^{\frac{n-3}{2}} u_{n-3}x^{n-3} + \ldots + u_2x^2 + u_0.$$

It is easy to check that $H^*(x) \in \mathbb{R}[x]$ and $H_n^*(a) = 0$ if and only if $H_n(ia) = 0$ $(a \in \mathbb{R})$. Since the roots of $H_n(x)$ have the shape $\pm a_k i$ $(k = 1, \ldots, (n-1)/2)$ with $0 < a_1 < \ldots < a_{(n-1)/2}$, the roots of $H_n^*(x)$ are $\pm a_k$ $(k = 1, \ldots, (n-1)/2)$. Applying Rolle's theorem, we get that $(H_n^*)'(x)$ has a root in every interval $[-a_{j+1}, -a_j]$ and $[a_j, a_{j+1}]$ $(j = 1, \ldots, (n-3)/2)$, so every root of $(H_n^*)'(x)$ is simple and real. Observe that

$$(H_n^*)'(x) = x\left((-1)^{\frac{n-1}{2}}(n-1)u_{n-1}x^{n-3} + \ldots + 4u_4x^2 - 2u_2\right), \tag{8}$$

hence 0 is a root of $(H_n^*)'(x)$, and if b is a root of $(H_n^*)'(x)$, then so is $-b$. Thus the roots of $(H_n^*)'(x)$ are given by

$$b_0, \pm b_1, \ldots, \pm b_{\frac{n-3}{2}}$$

with

$$0 = b_0 < b_1 < \ldots < b_{\frac{n-3}{2}}$$

and

$$-a_{\frac{n-1}{2}} < -b_{\frac{n-3}{2}} < -a_{\frac{n-3}{2}} < \ldots < -b_1 < -a_1 < b_0 < a_1 < \tag{9}$$

$$< b_1 < a_2 < \ldots < a_{\frac{n-3}{2}} < b_{\frac{n-3}{2}} < a_{\frac{n-1}{2}}. \tag{10}$$

Also, $(H_n')^*(b) = 0$ if and only if $H_n'(ib) = 0$. (Here * stands for the earlier transformation: in case of $g(x) \in \mathbb{C}[x]$, $g^*(x) = g(ix)$.) We show that $(H_n')^*(x) = (-i)(H_n^*)'(x)$. It easily follows, since

$$H_n'(x) = x\left((n-1)u_{n-1}x^{n-3} + (n-3)u_{n-3}x^{n-5} + \ldots + 4u_4x^2 + 2u_2\right)$$

and thus

$$(H_n')^*(x) = ix\left((-1)^{\frac{n-3}{2}}(n-1)u_{n-1}x^{n-3} + \ldots - 4u_4x^2 + 2u_2\right)$$

which by (8) gives our claim. Hence the roots of $H_n'(x)$ are given by

$$0 = b_0, \ \pm b_j i \quad \left(j = 1, \ldots, \frac{n-3}{2}\right). \tag{11}$$

So the statement is true for n odd.

Assume next that n is even. Using the transformation $H_n^{\times}(x) := iH_n(ix)$ in place of $H_n^*(x)$, a very similar argument works as in case of n odd. We omit the details. □

Now we examine the root structures of the shifts of $H_n(x)$.

Corollary 3.3. *For any $n \geq 2$ and for all $r \in \mathbb{C}$, the multiplicities of the roots of $H_n(x) + r$ are at most two.*

Proof. The statement is an immediate consequence of Lemma 3.8. □

To prove the corresponding part of Theorem 2.3 we need one more lemma, similar to Lemmas 3.3 and 3.6.

Lemma 3.9. *Let $n \geq 2$. Then* $\max_{\lambda\in\mathbb{C}} \deg(\gcd(H_n(x) - \lambda, H_n'(x))) \leq 2$.

Proof. We split the proof into two parts, according to the parity of n again.

If n is odd then recalling (11) from the proof of Lemma 3.8, we know that the roots of $H_n'(x)$ can be written in the form

$$b_0, \ \pm b_k i \quad (k = 1, \ldots, (n-3)/2)$$

where $0 = b_0 < b_1 < \ldots < b_{(n-3)/2}$ are real numbers. Also, by (9) here

$$0 = b_0 < a_1 < b_1 < a_2 < b_2 < \ldots < a_{\frac{n-3}{2}} < b_{\frac{n-3}{2}} < a_{\frac{n-1}{2}}$$

holds, where $\pm a_k i$ ($k = 1, \ldots, (n-1)/2$) are the roots of $H_n(x)$. Furthermore, for all

$$a_{k-1} \le t \le a_k \quad \left(k = 2, \ldots, \frac{n-1}{2}\right)$$

we have

$$|H_n(ti)| \le |H_n(b_{k-1}i)|.$$

Recall that by (7)

$$\sum_{j=1}^{n} \arctan\left(\frac{a_k}{j}\right) = (2k-1)\frac{\pi}{2} \quad \left(k = 1, \ldots, \frac{n-1}{2}\right)$$

also holds. Now, let $\hat{b}_s \in [a_k, a_{k+1}]$ ($k = 1, \ldots, (n-3)/2$) be that unique real number for which

$$\sum_{j=1}^{n} \arctan\left(\frac{\hat{b}_k}{j}\right) = k\pi.$$

With this notation, since

$$\prod_{j=1}^{n} |\hat{b}_k i + j| > \prod_{j=1}^{n} |b_{k-1} i + j| \quad \left(k = 1, \ldots, \frac{n-3}{2}\right)$$

is obviously true, we obtain

$$|H_n(b_{k-1}i)| < |H_n(\hat{b}_k i)| \le |H_n(b_k i)|.$$

It can be similarly proved (in fact, it also follows by symmetry) that

$$|H_n(-b_{k-1}i)| < |H_n(-b_k i)| \quad \left(k = 1, \ldots, \frac{n-3}{2}\right).$$

This implies that H_n' cannot have three different roots $\beta_1, \beta_2, \beta_3$ with

$$H_n'(\beta_1) = H_n'(\beta_2) = H_n'(\beta_3).$$

As the roots of H_n' are simple, we get that for any $\lambda \in \mathbb{C}$

$$\deg(\gcd(H_n'(x), H_n(x) - \lambda)) \le 2,$$

and the statement is proved for n odd.

In case of n even our claim follows by a rather similar argument, so we omit the details. $\square$

3.4. **The decomposition properties of $F_n(x)$, $G_n(x)$, $H_n(x)$.** In this subsection we prove Theorem 2.3.

Proof of Theorem 2.3. Let the polynomial $f(x)$ be one of $F_n(x)$, $G_n(x)$, $H_n(x)$ ($n \geq 2$) and suppose that it is decomposable. Then f is of the shape $f(x) = T_1(T_2(x))$ with some $T_1, T_2 \in \mathbb{Q}[x]$, $\deg(T_1), \deg(T_2) > 1$. It is well-known (see e.g. the proof of Theorem 4.3 in [8]) that we have

$$\deg(T_2) \leq \max_{\lambda\in\mathbb{C}} \deg(\gcd(f(x)-\lambda, f'(x))).$$

Thus based on Lemmas 3.3, 3.6 and 3.9 we obtain $\deg(T_2) \leq 2$. This immediately shows that $n-1$ (the common degree of F_n, G_n, H_n) must be even, or in other words, that our polynomials are indecomposable for n even.

So assume that n is odd. Then by direct checking we get

$$\begin{aligned} F_n(x) &= (x+1)^n - (x-1)^n = \\ &= 2\binom{n}{1}x^{n-1} + 2\binom{n}{3}x^{n-3} + \cdots + 2\binom{n}{n-2}x^2 + 2 = \varphi_1(x^2) \end{aligned}$$

with

$$\varphi_1(t) = 2\binom{n}{1}t^{\frac{n-1}{2}} + 2\binom{n}{3}t^{\frac{n-3}{2}} + \cdots + 2\binom{n}{n-2}t + 2,$$

$$\begin{aligned} G_n(x) &= (x+1)^{n-1} + x^{n-1} = \\ &= \left(\left(x+\frac{1}{2}\right)+\frac{1}{2}\right)^{n-1} + \left(\left(x+\frac{1}{2}\right)-\frac{1}{2}\right)^{n-1} = \\ &= 2\left(x+\frac{1}{2}\right)^{n-1} + \frac{1}{2}\binom{n-1}{2}\left(x+\frac{1}{2}\right)^{n-3} + \cdots \\ &\cdots + \frac{1}{2^{n-4}}\binom{n-1}{n-3}\left(x+\frac{1}{2}\right)^2 + \frac{1}{2^{n-2}} = \varphi_2\left(\left(x+\frac{1}{2}\right)^2\right), \end{aligned}$$

with

$$\varphi_2(t) = 2t^{\frac{n-1}{2}} + \frac{1}{2}\binom{n-1}{2}t^{\frac{n-3}{2}} + \cdots + \frac{1}{2^{n-4}}\binom{n-1}{n-3}t + \frac{1}{2^{n-2}}$$

and

$$\begin{aligned} H_n(x) =& \frac{1}{n!}((x+1)\cdots(x+n) - (x-1)\cdots(x-n)) \\ =& \frac{2}{n!}(s_1x^{n-1} + s_3x^{n-3} + \cdots + s_n), \end{aligned}$$

where

$$s_j = \sum_{\substack{A\subseteq\{1,\dots,n\}\\ |A|=j}} \prod_{a\in A} a \quad (j = 1, \dots, n),$$

so $H_n(x) = \varphi_3(x^2)$ with

$$\varphi_3(t) = \frac{2}{n!}(s_1 t^{\frac{n-1}{2}} + s_3 t^{\frac{n-3}{2}} + \dots + s_{n-2}t + s_n).$$

We only need to show that if n is odd then all decompositions of $F_n(x)$, $G_n(x)$ and $H_n(x)$ are equivalent to the above ones, respectively. We only check it for $F_n(x)$, the two other cases can be handled similarly. Since in any decomposition we must have $\deg(T_2) = 2$, we can write $T_2(x) = \alpha(x-\beta)^2 + \gamma$ with some $\alpha, \beta, \gamma \in \mathbb{C}$. Then the decomposition

$$F_n(x) = T_1(\alpha(x-\beta)^2 + \gamma)$$

is equivalent to a decomposition of the form $F_n(x) = P((x-\beta)^2)$ with some $P(x) \in \mathbb{C}[x]$. Thus the roots of $F_n(x)$ are symmetric about β. So necessarily $\beta = 0$, which proves our statement. □

4. The proofs of our effective results

In this section we prove our effective results. For this, we give two lemmas (which will be also used in the proofs of the ineffective theorems).

Let $f(x) \in \mathbb{Z}[x]$ and A be an integer with $A \neq 0$, and consider the equation

$$f(x) = Ay^\ell, \tag{12}$$

in unknown integers x, y, ℓ with $\ell \geq 2$. In the proofs of our theorems we will need two lemmas concerning the solutions of equation (12). The first lemma is a special case of a theorem of Bérczes, Brindza and Hajdu [6]. We mention that the first result of this kind is due to Tijdeman [22] and Schinzel and Tijdeman [20].

Lemma 4.1. *If $f(x)$ has at least two different roots, then for all solutions of (12) with $|y| > 1$*

$$\ell < C_3(A, d, H)$$

holds. Here $C_3(A, d, H)$ is an effectively computable constant which depends only on A, the degree d of $f(x)$ and the height H (the maximum of the absolute values of the coefficients) of $f(x)$.

The second lemma is a special case of the main result of Brindza [7]. To formulate the statement, we need some new notation.

Let S be a finite set of primes, and let $\mathbb{Z}_S$ be the set of those rationals whose denominators are composed exclusively of primes from S.

By the height $h(q)$ of a rational number q we mean the maximum of the absolute value of its denominator and numerator.

Lemma 4.2. *Let $f(x) \in \mathbb{Z}[x]$, and write*

$$f(x) = a\prod_{i=1}^{k}(x-\gamma_i)^{r_i},$$

where a is the leading coefficient of f, and $\gamma_1, \dots, \gamma_k$ are the distinct complex roots of $f(x)$, with multiplicities $r_1, \dots, r_k$, respectively. Further, fix ℓ with $\ell \geq 2$, and put

$$t_i = \frac{\ell}{(\ell, r_i)} \qquad (i = 1, \dots, k).$$

Suppose that $(t_1, \dots, t_k)$ is not a permutation of any of the k-tuples

$$(t, 1, \dots, 1)\ (t \geq 1), \quad (2, 2, 1, \dots, 1).$$

Then for any finite set S of primes, the solutions $x, y \in \mathbb{Z}_S$ of (12) *satisfy*

$$\max(h(x), h(y)) < C_4(A, \ell, d, H, S),$$

where $C_4(A, \ell, d, H, S)$ is an effectively computable constant depending only on A, ℓ, d, H, S, where d is the degree and H is the height of $f(x)$.

Remark 4.1. *Note that if $\ell \geq 3$ and $f(x)$ has at least two simple roots, or if $\ell = 2$ and $f(x)$ has at least three simple roots, then the conditions of Lemma 4.2 are satisfied.*

Proof of Theorem 2.2. Recall that for any $B \in \mathbb{Q}$, we have $\deg(f(x) - B) = n - 1$. If $n \geq 4$ then by Lemmas 3.2, 3.5 and Corollary 3.3 the polynomial $f(x) - B$ has at least two distinct roots. Hence by Lemma 4.1 part i) of the theorem follows.

To prove part ii) of the statement, we examine the cases $f(x) = F_n(x), G_n(x), H_n(x)$ separately. We shall always assume that $n \geq 8$. Further, note that by part i), ℓ is bounded in terms of A, d, H, unless $|y| \leq 1$ - but in that case the statement is obvious.

Lemmas 3.2 and 3.5 yield that the polynomials $F_n(x) - B$ and $G_n(x) - B$ have at least three simple roots. Thus Lemma 4.2 immediately shows that the statement is true for $f(x) = F_n(x), G_n(x)$.

So finally, let $f(x) = H_n(x)$. If $\ell \geq 3$ then Corollary 3.3 gives that $H_n(x) - B$ has at least three distinct roots, and the multiplicity of these roots cannot be larger then two. Thus our statement follows from

Lemma 4.2 also in this case. Let now $\ell = 2$. Clearly, our statement follows from Lemma 4.2 also in this case, unless we have

$$H_n(x) - B = u(x)(v(x))^2$$

with some $u(x), v(x) \in \mathbb{Q}[x]$ with $\deg(u) \leq 2$. However, if $H_n(x) - B$ is of the above shape, then we have

$$(H_n(x) - B)' = v(x)\left(u'(x)v(x) + 2u(x)v'(x)\right).$$

Hence the roots of $v(x)$ are also roots of $H_n'(x)$. However, these are roots of the polynomial $H_n(x) - B$ as well. According to Lemma 3.9, we know that the degree of the greatest common divisor of $H_n'(x)$ and $H_n(x) - B$ is at most two. Hence $\deg(v) \leq 2$, so $n \leq 7$. This contradicts our assumption $n \geq 8$, and our statement follows. $\square$

5. The proofs of our ineffective results

In this section we prove Theorem 2.1. For this we need some notation and a deep result of Bilu and Tichy [9].

Let α, β, $\delta \in \mathbb{Q}\setminus\{0\}$, μ, ν, q be positive integers, r be a non-negative integer, and $v(x) \in \mathbb{Q}[x]$ a polynomial, which is not identically zero. Write $D_\mu(x, \delta)$ for the μ-th Dickson polynomial, that is

$$D_\mu(x, \delta) = \sum_{i=0}^{\lfloor \mu/2 \rfloor} d_{\mu,i} x^{\mu - 2i},$$

where

$$d_{\mu,i} = \frac{\mu}{\mu - i}\binom{\mu - i}{i}(-\delta)^i.$$

We say that the polynomials $F(x)$ and $G(x)$ form a standard pair over $\mathbb{Q}$, if $(F(x), G(x))$ or $(G(x), F(x))$ appears in Table 1.

Kind	Standard pair	Parameter restrictions
First	$(x^q, \alpha x^r v(x)^q)$	$0 \leq r < q, (r, q) = 1$, $r + \deg(v(x)) > 0$
Second	$(x^2, (\alpha x^2 + \beta)v(x)^2)$	-
Third	$(D_\mu(x, \alpha^\nu), D_\nu(x, \alpha^\mu))$	$(\mu, \nu) = 1$
Fourth	$(\alpha^{\frac{-\mu}{2}} D_\mu(x, \alpha), -\beta^{\frac{-\nu}{2}} D_\nu(x, \beta))$	$(\mu, \nu) = 2$
Fifth	$((\alpha x^2 - 1)^3, 3x^4 - 4x^3)$	-

Table 1. Standard pairs

The following lemma is the main result of Bilu and Tichy [9].

Lemma 5.1. *Let $f(x), g(x) \in \mathbb{Q}[x]$ be non-constant polynomials. Then the following two assertions are equivalent.*
i) The equation

$$f(x) = g(y)$$

has infinitely many solutions with a bounded denominator.
ii) We have $f = \varphi \circ F \circ \lambda$ and $g = \varphi \circ G \circ \kappa$, where $\lambda(x), \kappa(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$ and $(F(x), G(x))$ is a standard pair over $\mathbb{Q}$ such that the equation $F(x) = G(x)$ has infinitely many solutions with a bounded denominator.

Now we give the proof of Theorem 2.1. We mention that the polynomials F_n and G_n ($n \geq 1$) form so-called Appell families. Thus in the proof we could use some results of Bazsó and Pink [4]. However, as we should handle certain cases separately anyhow, to keep the presentation coherent, we proceed differently.

Proof of Theorem 2.1. Suppose that (3) has infinitely many solutions in integers x, y. Then according to Lemma 5.1 we have $f = \varphi \circ F \circ \lambda$ and $g = \varphi \circ G \circ \kappa$, where $\varphi, \lambda, \kappa \in \mathbb{Q}[x], \deg(\lambda) = \deg(\kappa) = 1$ and F, G form a standard pair. Thus using Theorem 2.3 we obtain that only one of the following cases is possible:

- $\deg(\varphi) = n - 1$ and $\deg(F) = 1$,
- $n - 1$ is even, $\deg(\varphi) = (n-1)/2$, $\deg(F) = 2$, and φ is one of the polynomials φ_1, φ_2, φ_3,
- $\deg(\varphi) = 1$ and $\deg(F) = n - 1$.

In the first case we get that $\varphi(x) = f(\tau(x))$, where $\tau(x) \in \mathbb{Q}[x]$ is a linear polynomial. Thus we have $g(y) = f(P(y))$, where $P(y) \in \mathbb{Q}[y]$, and our statement follows.

In the second case we have one of

$$f(x) = \varphi_1(x^2),\ \varphi_2\left(\left(x + \frac{1}{2}\right)^2\right),\ \varphi_3(x^2).$$

Hence now $g(y) = \hat{f}(Q(y))$ holds, where $\hat{f}$ is one of φ_1, φ_2, φ_3. Furthermore, Lemma 5.1 implies that the equation

$$x^2 = Q(y), \quad \left(x + \frac{1}{2}\right)^2 = Q(y), \quad x^2 = Q(y),$$

respectively, must have infinitely many solutions in rational numbers with bounded denominators. So by Lemma 4.2 we deduce that $Q(y)$ can have at most two roots with odd multiplicity. This proves our claim also in this case.

Finally, assume that we are in the third case, that is $\deg(\varphi) = 1$ and $\deg(F) = n - 1$. Then we can write

$$f(x) = AF(ax + b) + B$$

with some $A, B, a, b \in \mathbb{Q}$, $Aa \neq 0$, were F is a member of one of the five standard pairs. We shall check all the five standard pairs in turn. Recall that by our assumption we have $n \geq 6$. Further, one can easily see that the theorem for $\deg(g) = 2$ follows from the case $\ell = 2$ of Theorem 2.2. Hence we may also assume that $\deg(g) \geq 3$.

We start with the case where $F(x)$ is from a standard pair of the fifth kind. It can be easily seen that then $f'(x)$ has multiple roots. However, this is not possible because of Corollaries 3.1, 3.2 and Lemma 3.8. So this possibility cannot hold.

Assume next that $F(x)$ is from a standard pair of the first kind. If

$$f(x) = A(ax + b)^q + B,$$

then

$$f'(x) = Aaq(ax + b)^{q-1}.$$

However, by Corollaries 3.1, 3.2 and Lemma 3.8 we know that the roots of $f'(x)$ are simple. Thus we get $q \leq 2$, contradicting $n \geq 6$. If f is of the form

$$f(x) = A\alpha(ax + b)^r v(ax + b)^q + B$$

where $0 \leq r < q$, $(r, q) = 1$, $r + \deg(v) > 0$, then

$$f'(x) = A\alpha a(ax + b)^{r-1} v(ax + b)^{q-1}(rv(ax + b) + q(ax + b)v'(ax + b)).$$

This similarly as above yields that $r \leq 2$, and either $q \leq 2$ or $\deg(v) = 0$. If $\deg(v) = 0$ then we get back to the previous case, which has already been excluded. Thus we may assume that $\deg(v) > 0$, and by $r < q$ and $\gcd(r, q) = 1$ also that $(r, q) = (0, 1), (1, 2)$. Now as

$$g = \varphi \circ G \circ \kappa$$

where $\deg(\varphi) = \deg(\kappa) = 1$ and $\deg(G) = q$, we get $\deg(g) \leq 2$, which is excluded. Thus our theorem follows also in this case.

Consider now the case where $F(x)$ is a member of a standard pair of the second kind. Now we easily get that either $\deg(f) = 2$, contradicting our assumption $n \geq 6$, or $\deg(g) = 2$, contradicting the condition $\deg(g) \geq 3$.

Finally, assume that $F(x)$ is from a standard pair of the third or fourth kind. We give detailed arguments only for the case $f(x) = F_n(x)$, since $f(x) = G_n(x), H_n(x)$ can be handled similarly. So let $f(x) = F_n(x)$. Then

$$F_n(x) = AD_{n-1}(ax + b) + B,$$

where $D_{n-1}(x)$ is a Dickson-polynomial, with some parameter δ. So

$$2\binom{n}{1}x^{n-1}+2\binom{n}{3}x^{n-3}+2\binom{n}{5}x^{n-5}+2\binom{n}{7}x^{n-7}+\dots$$
$$=Ad_0(ax+b)^{n-1}+Ad_1(ax+b)^{n-3}+Ad_2(ax+b)^{n-5}+\dots$$

with

$$d_i=\frac{n-1}{n-1-i}\binom{n-1-i}{i}(-\delta)^i \quad (i\geq 0).$$

Comparing the leading coefficients, we get that $Aa^{n-1}=2n$, and from the coefficients of x^{n-2} we immediately see that $b=0$. Now checking the coefficients of x^{n-3}, we obtain

$$-\frac{\delta}{a^2}=\frac{n-2}{6}.$$

Then the comparison of the coefficients of x^{n-5} yields a contradiction, and our theorem follows in this case. As we mentioned, in the case of the other two polynomial families a similar argument applies, and thus the proof of our theorem is complete. $\square$

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