# Forbidden subposet problems in the grid 

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#### Abstract

For posets $P$ and $Q$, extremal and saturation problems about weak and strong $P$-free subposets of $Q$ have been studied mostly in the case $Q$ is the Boolean poset $Q_{n}$, the poset of all subsets of an $n$-element set ordered by inclusion. In this paper, we study some instances of the problem with $Q$ being the grid, and its connections to the Boolean case and to the forbidden submatrix problem.


## 1 Introduction

In extremal combinatorics, Turán type problems ask for the largest size that a combinatorial object can have if it is does not contain a prescribed forbidden substructure. Graphs with the most number of edges not containing a fixed subgraph, set systems with the most number of sets not containing two with prescribed intersection size, etc. In this flavor, the forbidden subposet problem for posets $P$ and $Q$ asks for the size of the largest subset of $Q$ that does not contain $P$ as a subposet. There exist two notions of a subposet: we say that $P$ is a weak subposet of $R$ if there exists an injection $i: P \rightarrow R$ such that for every $p, p^{\prime} \in P, p \leqslant_{P} p^{\prime}$ implies $i(p) \leqslant_{R} i\left(p^{\prime}\right)$. If in addition the injection $i$ satisfies $p \leqslant_{P} p^{\prime}$ if and only if $i(p) \leqslant_{R} i\left(p^{\prime}\right)$, then we say that $P$ is a strong subposet of $R$. Otherwise we say that $R$ is weak / strong $P$-free. Strong subposets are also called induced subposets in the literature, while weak subposets are often referred to as subposets, and sometimes as not necessarily induced subposets. The extremal numbers $L a(Q, P)$ and $L a^{*}(Q, P)$ are defined as the size of the largest weak / strong $P$-free subposet of $Q$. Most of the research, initiated by Katona and Tarján [14] in the early eighties, focused on the case $Q=Q_{n}$ the Boolean cube poset $\{0,1\}^{n}$, i.e., the poset of all subsets of an $n$-element set ordered by inclusion. As it is usual in the literature, we use the notation $L a(n, P)$ and $L a^{*}(n, P)$ instead of $L a\left(Q_{n}, P\right)$ and $L a^{*}\left(Q_{n}, P\right)$. For a survey on the topic see [12] and an even more recent summary is Chapter 7 of 11 .

One of the most used tools in addressing forbidden subposet problems in the case of the Boolean cube is to find a simpler poset structure $R$ in the cube and apply some averaging argument to the result obtained for $R$. Simpler structures include the chain, the double chain, complementary chain pairs, intervals of a cycle, etc., see [5, 6, 10]. In this note, we focus on the grid $[k]^{d}=\{1,2, \ldots, k\}^{d}$ ordered coordinate-wise. Elements will be denoted by lower case letters $a, b, x, y, \ldots$ etc., and the $i$ th $(1 \leq i \leq d)$ coordinate by $a_{i}, b_{i}, x_{i}, y_{i}, \ldots$ etc. The order $(\leqslant)$ on $[k]^{d}$ is defined as follows: for $x, y \in[k]^{d}$ we have that $x \leqslant y$ if and only if $x_{i} \leq y_{i}$ for all $i=1,2, \ldots, d$. Let the $\operatorname{rank} r(x)$ of an element $x \in[k]^{d}$ be defined as $\sum_{i=1}^{d} x_{i}$. The set of all elements of rank $r(d \leq r \leq k d)$ will be denoted by $S_{k, d, r}$ and we write $s_{k, d, r}=\left|S_{k, d, r}\right|$. It is well-known [1] that the size $w_{k, d}$ of the largest antichain in $[k]^{d}$ is $s_{k, d,\lfloor(k+1) d / 2\rfloor}=(1+o(1)) \sqrt{\frac{6}{\pi d}} k^{d-1}$ as $\min \{k, d\} \rightarrow \infty$. Therefore, there exist constants $C=C_{d}$ and $\varepsilon=\varepsilon_{d}$ such that if $\left|i-\frac{(k+1) d}{2}\right|<\varepsilon k$, then $s_{k, d, i} \geq C k^{d-1}$.

Connection between forbidden subposet problems for the Boolean poset and some extremal problems on the grid was first established by Methuku and Pálvölgyi [19]. The dimension of a poset $P$ is the smallest
number $d$ for which there exist $d$ permutations $\pi_{1}, \pi_{2}, \ldots, \pi_{d}$ of the elements of $P$ such that $p<_{P} q$ if and only if $\pi_{i}(p)<\pi_{i}(q)$ for all $i=1,2, \ldots, d$. This is clearly equivalent to the fact that $d$ is the smallest integer for which $[|P|]^{d}$ contains a strong copy of $P$. Methuku and Pálvölgyi showed - by embedding any poset $P$ of the Boolean cube to some grid - that the forbidden subposet problem is naturally connected to the forbidden sub(hyper)matrix problem for the permutation hypermatrix defined by the $\pi_{i} \mathrm{~s}$. Applying a Marcus-Tardos-type theorem for hypermatrices [16, 17], they proved that for any poset $P$ there exists a constant $c_{P}$ such that $L a^{*}(n, P) \leq c_{P}\binom{n}{\lfloor n / 2\rfloor}$ (the analogous statement for weak subposets follows trivially from a result of Erdős [7]). Their result was strengthened by Méroueh [18] and Tomon [21].

Forbidden subposet problems on the grid and their connection to the case of the Boolean cube was first studied by Tomon [22] and Sudakov, Tomon, Wagner [20]. In [20], the following general framework was introduced. We say that a formula is affine, if it is built from variables, the lattice operators $\wedge$ and $\vee$ (or to avoid confusion, one might prefer to use $\cup$ and $\cap$ ), and parentheses (,) (constants are not allowed, e.g. $x \cap\{1,2,3\}$ is not an affine formula). Also, an affine statement is a statement of the form $f<g$ or $f=g$, where $f$ and $g$ are affine formulas. Finally, an affine configuration is a Boolean expression, which uses symbols $\wedge, \vee, \neg$ and whose variables are replaced with affine statements. Given an affine configuration $C$ with $k$ variables, a lattice $L$ contains $C$, if there exists $k$ distinct elements of $L$ that satisfy $C$, otherwise, say that $L$ avoids $C$. Let ex $(L, C)$ denote the size of the largest subposet $L^{\prime}$ of $L$ such that $L^{\prime}$ avoids $C$.

Note that in the Boolean cube $\{0,1\}^{n}$, the lattice operators are $\cup$ and $\cap$ and $<$ is simply $\subsetneq$. Given a poset $P$ with relation $\prec$, the weak and strong $P$-free properties can be described with the following affine configurations:

$$
C_{P}:=\bigwedge_{p, q \in P, p \prec q}(p<q) \quad C_{P}^{*}:=\bigwedge_{p, q \in P, p \prec q}(p<q) \wedge \bigwedge_{p, q \in P, p \nprec q, q \nprec p}(\neg(p<q) \wedge \neg(q<p)) .
$$

Theorem 1.1 (Theorem 3.1 in [20]). Let d be a positive integer, $C$ an affine configuration and $c, \alpha>0$ such that ex $\left([k]^{d}, C\right) \leq c k^{d-\alpha}$ holds for every sufficiently large $k \in \mathbb{N}$. Then we have

$$
e x\left(2^{[n]}, C\right) \leq(1+o(1)) c\left(\frac{2 d}{\pi n}\right)^{\alpha / 2} 2^{n}
$$

Applying Theorem 1.1 with $\alpha=1$ one obtains the following (see p. 17 in [20]).
Corollary 1.2. Let $d$ be a fix natural number.
(i) If $\lim \sup _{k \rightarrow \infty} L a\left([k]^{d}, P\right) \frac{\sqrt{d}}{k^{d-1}} \leq c$, then $\lim \sup _{n \rightarrow \infty} \frac{\operatorname{La(n,P)}}{\binom{n}{n / 2\rfloor}} \leq c$ holds.
(ii) If $\lim \sup _{k \rightarrow \infty} L a^{*}\left([k]^{d}, P\right) \frac{\sqrt{d}}{k^{d-1}} \leq c$, then $\lim \sup _{n \rightarrow \infty} \frac{L a^{*}(n, P)}{\left(\begin{array}{l}n \\ n \\ n\end{array}\right)} \leq c$ holds.

We are going to prove a theorem that is similar in flavour to Theorem 1.1. However the proof of Theorem 1.3 is much simpler than that of Theorem 1.1, as the authors of 20] applied an involved chain partition theorem to obtain their result, while for us a relatively simple averaging argument would suffice.

Theorem 1.3. Let $d$ be a positive integer, $C$ an affine configuration, and $c, \alpha>0$ such that ex $\left([k]^{d}, C\right) \leq$ $c k^{1-\alpha} \cdot w_{k, d}$ holds for every sufficiently large $k \in \mathbb{N}$. Then we have

$$
e x\left(2^{[n]}, C\right) \leq(1+o(1)) c n^{1-\alpha}\binom{n}{\lfloor n / 2\rfloor}
$$

Observe that the following strengthening of Corollary 1.2 is an immediate consequence of Theorem 1.3 with $\alpha=1$.

Corollary 1.4. Let $w_{k, d}$ denote the size of the largest antichain in $[k]^{d}$.
(i) If $\lim \sup _{k \rightarrow \infty} \frac{L a\left([k]^{d}, P\right)}{w_{k, d}} \leq c$, then $\lim \sup _{n \rightarrow \infty} \frac{\operatorname{La(n,P)}}{(\lfloor n / 2\rfloor)} \leq c$ holds.
(ii) If $\lim \sup _{k \rightarrow \infty} \frac{L a^{*}\left([k]^{d}, P\right)}{w_{k, d}} \leq c$, then $\lim \sup _{n \rightarrow \infty} \frac{L a^{*}(n, P)}{\left(\begin{array}{l}n / 2\rfloor\end{array}\right)} \leq c$ holds.

Also, one can compare the two theorems: Theorem 1.1 is stronger as long as $\alpha<1$ (moreover, Theorem 1.3 is meaningless if $\alpha<1 / 2$ ), while Theorem 1.3 is stronger for $\alpha>1$ and they are exactly of the same strength (apart from a multiplicative constant factor $\sqrt{\frac{6}{\pi}}$ ) if $\alpha=1$, i.e., the case of Corollary 1.2 and 1.4 .

Next we start investigating the forbidden subposet problem on the grid for specific posets. This topic is naturally connected to the area of forbidden sub(hyper)matrices. We say that a $0-1$ matrix $A$ contains another 0-1 matrix $M$ if $A$ has a submatrix $M^{\prime}$ that can be turned into $M$ by replacing some (possibly zero) 1 entries with 0 . Otherwise we say that $A$ avoids $M$. For $k, l \geq 1$ and a $0-1$ matrix $M$ let us define $e x(k, l, M)$ as the largest number of 1-entries in a $k \times l 0-1$ matrix $A$ that avoids $M$. (In the literature, $P$ is used for $M$ to hint at the word pattern for the avoided matrix $M$, but since we use $P$ for posets, to avoid confusion $M$ stands for the forbidden matrix in this paper.) An overview of results on this extremal function can be found in the introduction of [8]. Now consider a poset $P$ of dimension 2 and all possible embeddings of $P$ into $[|P|]^{2}$. Every such embedding $f$ naturally corresponds to a $|P| \times|P| 0-1$ matrix $T$ : the entry $t_{i, j}$ is 1 if and only if $f(p)=(i, j)$ for some $p \in P$. Let $M_{f}$ denote the submatrix of $T$ of all rows and columns that contain at least one 1-entry. A subposet $Q$ of $[k]^{2}$ again naturally corresponds to the $k \times k 0-1$ matrix $M_{Q}$ : its $(i, j)$-entry is 1 if and only if $(i, j) \in Q$. Then clearly, $Q$ is strong $P$-free if and only if $M_{Q}$ avoids $M_{f}$ for all embeddings $f$. This means that every strong forbidden subposet problem on $[k]^{2}$ corresponds to forbidding several submatrices. The same holds for weak forbidden subposet problems, as containing a weak copy of $P$ is equivalent to containing a strong copy from the family of posets $P^{\prime}$ that have the same number of elements as $P$ and that contain a weak copy of $P$.

A much studied [3, 8] permutation pattern is the $s \times s$ matrix $J_{s}$ that is obtained from the identity matrix by moving its last column to the beginning. Let $\vee_{s}$ denote the poset on $s+1$ elements $a, b_{1}, b_{2}, \ldots, b_{s}$ with $a<b_{1}, b_{2}, \ldots, b_{s}$. A copy of $J_{s}$ in an $n \times m$ matrix corresponds to one possible embedding of $\vee_{s-1}$ into $[n] \times[m]$. However, we conjecture that the extremal and saturation numbers for $J_{s}$ correspond to strong extremal and saturation numbers of $\vee_{s}$. (See Conjecture 4.5.)

Set $k_{2}=c_{2}=1$ and for $s \geq 3$ let us define $k_{s}$ to be the maximum $k$ such that $\sum_{j=2}^{k} j<s$. Finally, let $c_{s}=s-1-\sum_{j=2}^{k_{s}} j$.

## Theorem 1.5.

(i) For any $s \geq 1$, we have $L a\left([k]^{2}, \vee_{s}\right)=\left(k_{s}+\frac{c_{s}}{k_{s}+1}+o(1)\right) k$.
(ii) For $k, l \geq 1$ we have $L a^{*}\left([k] \times[l], \vee_{2}\right)=k+l-1$.
(iii) For $k, l \geq 2$ we have $L a^{*}\left([k] \times[l], \vee_{3}\right)=2(k+l)-4$.
(iv) $L a^{*}\left([k]^{2}, \vee_{s}\right) \geq 2(s-1) k-O\left(s^{2}\right)$.
(v) $L a^{*}\left([k] \times[l],\left\{\vee_{2}, \wedge_{2}\right\}\right)=k+l-1$.

For a finite poset $P$, let $h(P)$ denote its height, i.e., the number of elements of its largest complete subposet. Let $D_{k}$ denote the $k$-diamond, a poset on $k+2$ elements $a<b_{1}, b_{2}, \ldots, b_{k}, c$ with $a<b_{i}<c$ for all $1 \leq i \leq k$.

Proposition 1.6. For any poset $P$, we have $L a\left([k]^{2}, P\right) \leq\left(\frac{|P|+h(P)}{2}-1\right) k+O(1)$.
In particular, $L a\left([k]^{2}, D_{2}\right)=\frac{5}{2} k+O(1)$ and $L a\left([k]^{2}, D_{3}\right)=3 k+O(1)$.
Every extremal problem has its saturation counterpart. Very recently, there has been an increased attention [2, 3, 8, 9] to the saturation version of the forbidden submatrix problem. Let $\operatorname{sat}(n, m, M)$ denote
the minimum number of 1-entries of an $n \times m$ binary matrix $A$ that avoids $M$, such that any matrix $A^{\prime}$ that is obtained from $A$ by changing a 0 to a 1 , contains $M$. Fulek and Keszegh 8 proved that for any matrix $M$ one has $\operatorname{sat}(n, n, M)=O(1)$ or $\operatorname{sat}(n, n, M)=\Theta(n)$ and asked for a characterization of matrices with linear saturation number. Partial answers were given in [2, 9 .

We start investigating the saturation version of the forbidden subposet problem on the grid. We say that a subposet $Q^{\prime}$ of $Q$ is weak/strong $P$-saturated if it is weak/strong $P$-free, but adding any element of $Q \backslash Q^{\prime}$ to $Q^{\prime}$ creates a weak/strong copy of $P$. Let $\operatorname{sat}(Q, P)$ and $s a t^{*}(Q, P)$ denote the minimum size of a weak / strong $P$-saturated subposet of $Q$, respectively. Just as in the extremal case, the poset saturation problems on the grid are equivalent to matrix saturation problems with a family of matrices to be avoided. First we observe that in any dimension, the weak poset saturation number is always bounded by a constant.

Proposition 1.7. For any positive integers $p, d \geq 2$ and for any $p$-element poset $P$ and integer $k$ we have $\operatorname{sat}\left([k]^{d}, P\right) \leq \sum_{r=d}^{d+p-2} s_{k, d, r}$.

Note that $\sum_{r=d}^{d+p-2} s_{k, d, r}$ does not change once $k \geq p$ and thus the upper bound is a constant independent of $k$.

Let us remark that the proof of Proposition 1.7 stays valid for $\left[k_{1}\right] \times\left[k_{2}\right] \times \ldots \times\left[k_{n}\right]$, if we replace $\sum_{r=d}^{d+p-2} s_{k, d, r}$ with the size of the $p-1$ lowest levels.

Based on the above mentioned result of Fulek and Keszegh, we show an analogous theorem for the strong saturation number of posets.

Theorem 1.8. For any poset $P$ with $\operatorname{dim}(P)=2$ we either have sat $\left([k]^{2}, P\right)=O(1)$ or sat $\left([k]^{2}, P\right)=$ $\Theta(k)$.

Finally, we address the saturation problem for some specific posets.

## Theorem 1.9.

(i) For any $k, l$ we have sat ${ }^{*}\left([k] \times[l],\left\{\vee_{2}, \wedge_{2}\right\}\right)=\max \{k, l\}$.
(ii) For any integers $k, l$ we have sat $\left([k] \times[l], \vee_{2}\right)=L a^{*}\left([k] \times[l], \vee_{2}\right)=k+l-1$.

Theorem 1.10. Let $P$ be a poset with $\operatorname{dim}(P)=2$ such that a strong copy of $P$ in a two dimensional grid cannot contain two neighboring points. Then sat* $([k] \times[l], P) \geq \max \{k, l\}$.

Note that a poset, that satisfies the assumption of Theorem 1.10 is any $\vee_{s}$ with $s \geq 3$.

## 2 Connection to the hypercube - the proof of Theorem 1.3

Let us start with stating the theorem again. Recall that $w_{k, d}=s_{k, d,\lfloor(k+1) d / 2\rfloor}$ is the size of the largest antichain in $[k]^{d}$.

Theorem 1.3. Let $d$ be a positive integer, $C$ an affine configuration, and $c, \alpha>0$ such that ex $\left([k]^{d}, C\right) \leq$ $c k^{1-\alpha} \cdot w_{k, d}$ holds for every sufficiently large $k \in \mathbb{N}$. Then we have

$$
e x\left(2^{[n]}, C\right) \leq(1+o(1)) c n^{1-\alpha}\binom{n}{\lfloor n / 2\rfloor}
$$

Let us mention that our proof is similar to that of Methuku and Pálvölgyi 19. The main difference is that we only consider partitions with equal (or almost equal) parts and that we optimize our calculations in order to obtain the best possible constants.

Proof of Theorem 1.3. Let $\mathcal{F} \subset 2^{[n]}$ be a family that avoids $C$. Let us write $n$ in the following form: $n=d(k-1)+r$ with $0 \leq r<d$ and set $n^{\prime}=n-r$. We partition $\mathcal{F}$ into $2^{r}$ subfamilies $\mathcal{F}_{A}$ indexed with subsets of $[n] \backslash\left[n^{\prime}\right]$ such that $\mathcal{F}_{A}=\left\{F \backslash A: F \in \mathcal{F}, F \cap\left([n] \backslash\left[n^{\prime}\right]\right)=A\right\}$. If we can prove that for every $\mathcal{F}_{A}$ we have $\left|\mathcal{F}_{A}\right| \leq(1+o(1)) c n^{1-\alpha}\binom{n^{\prime}}{\left\lfloor\frac{n^{\prime}}{2}\right\rfloor}$, then $|\mathcal{F}|=\sum_{A}\left|\mathcal{F}_{A}\right| \leq 2^{r}(1+o(1)) c n^{1-\alpha}\binom{n-r}{\left\lfloor\frac{n-r}{2}\right\rfloor}=(1+o(1)) c n^{1-\alpha}\binom{n}{\left\lfloor\frac{n}{2}\right\rfloor}$.

We say that a subset $F$ of $\left[n^{\prime}\right]$ is a $\pi$-block for a permutation $\pi$ of $\left[n^{\prime}\right]$ if for every $0 \leq j \leq d-1$ the intersection $F \cap\{\pi(j(k-1)+1), \pi(j(k-1)+2), \ldots, \pi((j+1)(k-1))\}$ is an initial segment, i.e., $\{\pi(j(k-1)+1), \pi(j(k-1)+2), \ldots, \pi(j(k-1)+h)\}$ for some $h=0,1,2, \ldots, k-1$ (here $h=0$ means that the intersection is empty). Observe that union and intersection of any pair of $\pi$-blocks is a $\pi$-block again. More importantly, the set of $\pi$-blocks is isomorphic to $[k]^{d}$. A $\pi$-block $F$ is identified with the $d$-tuple of the size of the intersection of $F$ and the initial segments - all sizes increased by one.

Let $\mathcal{G} \subseteq 2^{\left[n^{\prime}\right]}$ be a family that avoids an affine configuration $C$. We count the pairs $(F, \pi)$ such that $F \in \mathcal{G}$ is a $\pi$-block. For a fixed permutation $\pi$, the number of pairs is clearly at most $e x\left([k]^{d}, C\right)$, so the total number of pairs is at most $n^{\prime}!\cdot e x\left([k]^{d}, C\right)$, which is, by the assumption, at most $(c+o(1)) n^{\prime 1-\alpha} w_{k, d} \cdot n^{\prime}!$.

On the other hand, for any set $F$, the permutations $\pi$ for which $F$ is a $\pi$-block are in bijection with the set $\left\{\left(\alpha, \beta,\left(f_{1}, f_{2}, \ldots, f_{d}\right)\right)\right\}$, where $\alpha$ is an order of the elements of $F, \beta$ is an order of elements of $\left[n^{\prime}\right] \backslash F$ and $\left(f_{1}+1, f_{2}+1, \ldots, f_{d}+1\right)$ satisfies $\sum_{i=1}^{d} f_{i}=|F|, f_{i} \leq k-1$ for all $i=1,2, \ldots, d$. Indeed, the $f_{i}$ s tell us how large the $i$ th initial segment of $F$ is in $\pi$. Therefore, the number of such permutations for a fixed $F$ is $s_{k, d,|F|+d}|F|!\left(n^{\prime}-|F|\right)!$. We obtained

$$
\begin{equation*}
\sum_{F \in \mathcal{G}} s_{k, d,|F|+d}|F|!\left(n^{\prime}-|F|\right)!\leq(c+o(1)) n^{\prime 1-\alpha} w_{k, d} \cdot n^{\prime}!. \tag{1}
\end{equation*}
$$

Claim 2.1. We have $s_{k, d, j+d} \cdot j!\left(n^{\prime}-j\right)!\geq(1-o(1)) w_{k, d}\left\lfloor\frac{n^{\prime}}{2}\right\rfloor!\left\lceil\frac{n^{\prime}}{2}\right\rceil!$.
Proof of Claim. First of all, by symmetry, it is enough to prove the statement for $j<\frac{(k-1) d}{2}$. On the one hand, we have $\frac{\left(\left\lfloor n^{\prime} / 2\right\rfloor-i\right)!\left(\left[n^{\prime} / 2\right\rceil+i\right)!}{\left\lfloor n^{\prime} / 2\right\rfloor!\left\lceil n^{\prime} / 2\right\rceil!}=\prod_{j=1}^{i} \frac{\left\lceil n^{\prime} / 2\right\rceil+i-j+1}{\left\lfloor n^{\prime} / 2\right\rfloor-j+1} \geq\left(1+\frac{2 i}{n^{\prime}}\right)^{i}$.

On the other hand, we have $s_{k, d, i}=s_{k, d, i-1}+s_{k, d-1, i-1}-s_{k, d-1, i-k}$. Indeed, the number of elements of $[k]^{d}$ with rank $i$ and first coordinate 1 is $s_{k, d-1, i-1}$. The elements with rank $i$ and non-one first coordinate are in 1-to-1 relationship with those of rank $i-1$ and first coordinate not $k$, so their number is $s_{k, d, i-1}-s_{k, d-1, i-k}$ (the bijection is established by subtracting 1 from the first coordinate). Rearranging and omitting a negative factor from the right hand side, we obtain $s_{k, d, i}-s_{k, d-1, i-1} \leq s_{k, d, i-1}$.

If $\varepsilon>0$ is small enough, then for $i \geq n^{\prime} \frac{1-\varepsilon}{2}$ we have that $s_{k, d, i}=\Theta\left(k^{d-1}\right)$ and $s_{k, d-1, i-1}=\Theta\left(k^{d-2}\right)$ as pointed out in the introduction. Therefore, $\frac{s_{k, d, i-1}}{s_{k, d, i}} \geq 1-\frac{s_{k, d-1, i-1}}{s_{k, d, i}} \geq 1-\frac{c_{0}}{n^{\prime}}$ for some absolute constant $c_{0}$.

Putting this together: if $j \geq n^{\prime} \frac{1-\varepsilon}{2}$, then writing $j=\left\lfloor n^{\prime} / 2\right\rfloor-j^{\prime}$

$$
\frac{j!\left(n^{\prime}-j\right)!s_{k, d, j+d}}{\left\lfloor n^{\prime} / 2\right\rfloor!\left\lceil n^{\prime} / 2\right\rceil!s_{k, d,\left\lfloor n^{\prime} / 2\right\rfloor+d}} \geq\left[\left(1+\frac{2 j^{\prime}}{n^{\prime}}\right)\left(1-\frac{c_{0}}{n^{\prime}}\right)\right]^{j^{\prime}} .
$$

The right hand side is greater than 1 , if $j$ is at least than $n^{\prime} / 2-c_{1}$ for some absolute constant $c_{1}$ and always $1-o(1)$.

Finally, if $j<n^{\prime} \frac{1-\varepsilon}{2}$, then $j!\left(n^{\prime}-j\right)$ ! is already larger than $w_{k, d}\left\lfloor n^{\prime} / 2\right\rfloor!\left\lceil n^{\prime} / 2\right\rceil$ !.
By Claim 2.1 and (1) we have

$$
|\mathcal{G}|(1-o(1)) w_{k, d}\left\lfloor\frac{n^{\prime}}{2}\right\rfloor!\left\lceil\frac{n^{\prime}}{2}\right\rceil!\leq(c+o(1)) n^{\prime 1-\alpha} w_{k, d} n^{\prime}!.
$$

After rearranging and using the first paragraph of this proof, the statement of the theorem follows.

## 3 Results for the grid

Recall that $k_{2}=c_{2}=1$, for $s \geq 3$ we defined $k_{s}$ to be the maximum $k$ such that $\sum_{j=2}^{k} j<s$, and $c_{s}=s-1-\sum_{j=2}^{k_{s}} j$. Theorem 1.5 (i) states that for any $s \geq 1$, we have $L a\left([k]^{2}, \vee_{s}\right)=\left(k_{s}+\frac{c_{s}}{k_{s}+1}+o(1)\right) k$.
Proof of Theorem 1.5 (i). For the lower bound, we need a construction. For every $1 \leq i \leq k$, let the leftmost element of $F$ in the $i$ th row be the element in the diagonal. Furthermore, in each row, let the element of $F$ form an interval of size $k_{s}$ or $k_{s}+1$. If $i \equiv 1,2, \ldots, c_{s} \bmod k_{s}+1$, then let the length be $k_{s}+1$, otherwise $k_{s}$. (See Figure 1). The minimal elements of $F$ are exactly those in the diagonal, and the number of other elements greater than one such element is at most $\sum_{j=2}^{k_{s}} j+c_{s}=s-1$. The size of $F$ is $\left(k_{s}+\frac{c_{s}}{k_{s}+1}\right) k-O(s)$.


Figure 1: The construction given for Theorem 1.5 (i). In this example $k=12$ and $s=8$, therefore $k_{s}=3$ and $c_{s}=2$.

Let $F \subseteq[k]^{2}$ be a $\vee_{s}$-free set of elements. Let $M \subseteq F$ denote the subset of minimal elements. Observe that elements of $M$ are both leftmost in their row, and lowest in their column. Also, to check whether $F$ is $\vee_{s}$-free, it is enough to check whether any element of $M$ is smaller than less than $s$ other elements of $F$. Because of this, we can assume that $F$ is convex, i.e., for any $f, f^{\prime}, g \in[k]^{2}$ with $f, f^{\prime} \in F, f<g<f^{\prime}$, we have $g \in F$. Indeed, if $g \notin F$, then we can replace $f^{\prime}$ by $g$, and as minimal elements remain the same, we preserve the $\vee_{s}$-free property. This implies that we can assume that in any row and column, the elements of $F$ form an interval, and if $m_{i}$ denotes the second coordinate of the smallest element of $F$ in the $i$ th column, then the $m_{i}$ s form a non-increasing sequence.

Let $b_{i}$ denote the number of elements of $F$ in the $i$ th row, so $\sum_{i=1}^{k} b_{i}=|F|$. Let $M^{\prime}$ denote the set containing the lowest element of all columns. By definition, $M \subseteq M^{\prime}$. For any pair $(m, f)$ with $m \in M^{\prime}, f \in$ $F, m \leqslant f$, we can appoint the pair $\left(f^{\prime}, f\right)$, where $f^{\prime}$ is the element at the intersection of the column of $m$ and the row of $f$. By the convexity of $F$, we must have $f^{\prime} \in F$. Also, as $m$ is the lowest element of its column, this is a bijection. Therefore, we obtained that the number of such pairs is $\sum_{i=1}^{k}\left(\binom{b_{i}}{2}+b_{i}\right)=\sum_{i=1}^{k}\binom{b_{i}+1}{2}$. This function is convex, thus for fixed $|F|$, its minimum is attained when the $b_{i}$ s differ by at most 1 . Therefore, if $|F|>\left(k_{s}+\frac{c_{s}}{k_{s}+1}+o(1)\right) k$, then the number of pairs $(f, b)$ with $f \in M^{\prime}, b \in F, f \leqslant b$ is more than $(s-1) k$, so there must exist an $m \in M^{\prime}$ smaller than at least $s$ many other elements.

Let us continue with Theorem 1.5 (iii), which states that for $k, l \geq 2$ we have $L a^{*}\left([k] \times[l], \vee_{3}\right)=2(k+l)-4$.

Proof of Theorem 1.5 (iii). To see the lower bound let us consider the set $\{(a, b): a=1$ or $l$ or $b=1$ or $k\}$.
We prove the upper bound by contradiction. Let us denote the elements of copies of $\vee_{3}$ by $A, B, C$ and $D$ : let $A$ be the smallest element and the other elements $B, C$ and $D$ ordered by their second coordinate decreasingly (equivalently, by their first coordinate increasingly). If $k=2$ or $l=2$, then the statement is trivially true. Let us consider a counterexample $F \subseteq[k] \times[l]$ with $k+l$ minimal and among these the sum of the coordinates of the elements is maximal. If $F$ contains at most 2 elements in a row or in a column, then we can delete them, and either we get a counterexample with smaller $k+l$ or either $k$ or $l$ is 2 . Therefore all rows and columns contain at least 3 elements of $F$.

The next observation is that one can put $(k, l)$ into $F$ without violating the strong $\vee_{3}$-freeness condition. If $(1,1) \in F$, then $F$ contains at most two elements from each diagonal, so we are done.
If $(1,1) \notin F$, then let $\left(k_{1}, 1\right)$ be the first element in $F$ in the first row (so $k_{1} \geq 2$ ). As we have at least two elements above $\left(k_{1}, 1\right)$ and at least two elements right to $\left(k_{1}, 1\right)$, we have that $\left(k_{1}+1,2\right) \notin F$ by the strong $\vee_{3}$-free property. Note that by the maximality of the sum of the coordinates in $F$ we cannot replace $\left(k_{1}, 1\right)$ by $\left(k_{1}+1,2\right)$ in $F$. This means that $F \backslash\left\{\left(k_{1}, 1\right)\right\} \cup\left\{\left(k_{1}+1,2\right)\right\}$ contains a strong $\vee_{3}$. Note also that $\left(k_{1}+1,2\right)$ can only play the role of $D$ in this strong $\vee_{3}$. The role of $A$ can be played only by an element $\left(k^{\prime}, 2\right)$ with $k^{\prime}<k_{1}$. Let $\left(k_{2}, 2\right)$ be the first element in the second row. Similar way as above we have that $\left(k_{2}+1,3\right) \notin F$ and we cannot put $\left(k_{2}, 2\right)$ into $\left(k_{2}+1,3\right)$ by the maximality of the sum of the coordinates of the elements in $F$, so $\left(k_{2}+1,3\right)$ would create a strong $\vee_{3}$. In that strong $\vee_{3}$, the element $\left(k_{2}+1,3\right)$ can only play the role of $D$ :

- it can not be $A$, as otherwise $\left(k_{2}, 2\right)$ could also play the role of $A$ instead,
- it can not be a $B$ as there is at least 1 element above $\left(k_{2}+1,3\right)$ in the $\left(k_{2}+1\right)$ th column, and that could play the role of $B$ as well,
- it can not be $C$ as there is no element $(x, y) \in F \backslash\left\{\left(k_{2}, 2\right)\right\}$ with $x \leq k_{2}$ and $y \leq 2$.

This implies that there is $\left(k^{\prime \prime}, 3\right) \in F$ with $k^{\prime \prime}<k_{2}$, and let $\left(k_{3}, 3\right)$ be the first element in $F$ in the third row. We can continue the same way: we have elements $\left(k_{1}, 1\right),\left(k_{2}, 2\right),\left(k_{3}, 3\right), \ldots$ with $k_{1}>k_{2}>k_{3}>\ldots$. If the first element in the first column is $(1, t)$, then we have $k_{1} \geq t$. By changing the role of columns and rows in the above reasoning, we obtain $k_{1} \leq t$, and so $k_{1}=t$. The previous argument also implies that $k_{j}=k_{1}+1-j$ for $1 \leq j \leq k_{1} ;\left\{\left(k_{1}, 1\right),\left(k_{2}, 2\right),\left(k_{3}, 3\right), \ldots,\left(1, k_{1}\right)\right\}=: M \subset F$ and $\left\{\left(k_{1}+1,2\right),\left(k_{2}+1,3\right),\left(k_{3}+\right.\right.$ $\left.1,4), \ldots,\left(2, k_{1}+1\right)\right\}=: M^{+}$is disjoint with $F$. (Observe that $k_{1}<\min \{k, l\}$ as every row and column contains at least 2 elements. Therefore there exist $k_{1}+1$ st column and row.)

Let us consider $F^{\prime}:=F \backslash M \cup M^{+}$. Note that we described the possible strong copies of $\vee_{3}$ in $F$. By that it is easy to see that $F^{\prime}$ is also strong $\vee_{3}$-free and the sum of the coordinates is larger, a contradiction.

Recall that Theorem 1.5 (iv) states that $L a^{*}\left([k]^{2}, \vee_{s}\right) \geq 2(s-1) k-O\left(s^{2}\right)$.
Proof of Theorem 1.5 (iv). To prove this part we will show a family $G \subseteq[k]^{2}$ avoiding strong $\vee_{s}$. Let $G$ be the union of the $(s-1)$ highest rows and the $(s-1)$ rightmost columns. Then $|G|=2(s-1) k-(s-1)^{2}$, and $G$ is strong $\vee_{s}$-free, since none of its elements can be the minimal element of a strong $\vee_{s}$.

We will prove Theorem 1.5 (ii) and (v) later, together with the corresponding saturation statements from Theorem 1.9. Let us turn to Proposition 1.6, which states that for any poset $P$, we have $L a\left([k]^{2}, P\right) \leq$ $\left(\frac{|P|+h(P)}{2}-1\right) k+O(1)$. Moreover, matching lower bounds are given in case of the posets $D_{2}$ and $D_{3}$.

Proof of Proposition 1.6. In [5], a poset called the infinite double chain is introduced. Its elements are $L_{i}, M_{i}, i \in \mathbb{Z}$. The defining relations between the elements are $i<j \Rightarrow L_{i}<L_{j}, L_{i}<M_{j}, M_{i}<L_{j}$. Burcsi and Nagy proved that if a subset of the infinite double chain is $P$-free for some finite poset $P$, then its size is at most $|P|+h(P)-2$.

Now let $F$ be a $P$-free subset of $[k]^{2}$. We call the points $(x, y) \in[k]^{2}$ for which $x-y$ is constant an increasing diagonal. Consider the union of four consecutive increasing diagonals. This structure is isomorphic to a subset of the infinite double chain: the middle two diagonals correspond to elements $L_{i}$ while the outer elements correspond to the elements $M_{i}$. Therefore there can be at most $|P|+h(P)-2$ elements of $F$ there. There are a total of $2 k-1$ increasing diagonals, so we can partition them into $\left\lceil\frac{k}{2}\right\rceil-1$ groups, leaving out the last one or three diagonals, a constant number of elements. In conclusion, $|F| \leq\left(\left\lceil\frac{k}{2}\right\rceil-1\right)(|P|+h(P)-2)+O(1) \leq\left(\frac{|P|+h(P)}{2}-1\right) k+O(1)$.

The upper bounds for $D_{2}$ and $D_{3}$ follow from the general result proved above. For the lower bounds, consider the following $D_{2}$-free and $D_{3}$-free families in $[k]^{2}$ :

$$
\begin{gathered}
\left\{(x, y) \in[k]^{2}: x+y \in\{k, k+2\}\right\} \cup\{(x, y): x+y=k+1 \text { and } x \text { is odd }\}, \\
\left\{(x, y) \in[k]^{2}: k \leq x+y \leq k+2\right\} .
\end{gathered}
$$

Let us turn to saturation problems. Recall that Proposition 1.7 states that the weak saturation number is always upper bounded by a constant that does not depend on $k$. More precisely, for any positive integers $p, d \geq 2$ and for any $p$-element poset $P$ and integer $k$ we have $\operatorname{sat}\left([k]^{d}, P\right) \leq \sum_{r=d}^{d+p-2} s_{k, d, r}$.

Proof of Proposition 1.7. For any enumeration $\pi=x_{1}, x_{2}, \ldots x_{k^{d}}$ of $[k]^{d}$, one can create the $P$-saturating family $\mathcal{F}_{\pi} \subseteq[k]^{d}$ with respect to $\pi$ greedily as follows: we let $\mathcal{F}_{0}=\emptyset$ and whenever $\mathcal{F}_{i-1}$ is defined, we set $\mathcal{F}_{i}=\mathcal{F}_{i-1} \cup\left\{x_{i}\right\}$ if $\mathcal{F}_{i-1} \cup\left\{x_{i}\right\}$ is $P$-free, and let $\mathcal{F}_{i}=\mathcal{F}_{i-1}$ otherwise. By definition, $\mathcal{F}_{\pi}:=\mathcal{F}_{k^{d}}$ is $P$-saturating.

Let $\pi=x_{1}, x_{2}, \ldots, x_{k^{d}}$ be an enumeration of $[k]^{d}$ such $r\left(x_{i}\right) \leq r\left(x_{j}\right)$ for any $i \leq j$. We claim that $\mathcal{F}_{\pi}$ is downward closed, i.e., if $x_{i} \in \mathcal{F}_{\pi}$, then any $y \leq x_{i}$ belongs to $\mathcal{F}_{\pi}$. Indeed, the property of the enumeration $\pi$ ensures that at any moment when we decide about whether to include an $x_{j}$, then any element of $\mathcal{F}_{j-1}$ that is in relation with $x_{j}$ must be smaller than $x_{j}$. Also, any $y \leq x_{i}$ is enumerated before $x_{i}$, so $y=x_{j}$ for some $j<i$. If $y=x_{j} \notin \mathcal{F}_{\pi}$, then it is because $\mathcal{F}_{j-1} \cup\{y\}$ contains a copy of $P$ that contains $y$, thus $y$ is a maximal element in that copy of $P$. But by the above, if we replace $y$ by $x_{i}$, then we get another copy of $P$. (Here we use that we look for a weak copy of $P$.)

Clearly, $\mathcal{F}_{\pi}$ cannot contain a chain of length $p$ as that is a weak copy of $P$. Therefore, we must have $\mathcal{F}_{\pi} \subseteq \cup_{r=d}^{d+p-2} S_{d, k, r}$ and the result follows.

Observe that enumerations considering low-rank elements first are not necessarily the best even among greedily picked $P$-saturating families. Indeed, if $P$ is the chain of length $p$, then one is much better off considering low and high ranked elements alternatingly. More formally, we say that an enumeration $\pi=$ $x_{1}, x_{2}, \ldots, x_{k^{d}}$ is middle comes last (MCL) if for any $i \leq j$ we have $\left|r\left(x_{i}\right)-\frac{d(k+1)}{2}\right| \geq\left|r\left(x_{j}\right)-\frac{d(k+1)}{2}\right|$. If $p=2 m+1$, then the greedy $C_{l}$-saturating family with respect to an MCL enumeration is $\cup_{r=d}^{d+m-1} S_{k, d, r} \cup$ $\cup_{r=k d-m+1}^{k d} S_{k, d, r}$, while the enumerations used in Theorem 1.7 yield $\cup_{r=d}^{d+2 m-1} S_{k, d, r}$ which is significantly larger. We do not know whether MCL enumerations always give the best greedy approach..

Let us continue with strong saturation. Recall that Theorem 1.8 states that for any poset $P$ with $\operatorname{dim}(P)=2$ we either have $s a t^{*}\left([k]^{2}, P\right)=O(1)$ or $s a t^{*}\left([k]^{2}, P\right)=\Theta(k)$.

Proof of Theorem 1.8. Recall that the strong saturation problem for a poset $P$ is equivalent to the saturation problem for a finite set of 0-1 matrices, one of which is a permutation matrix. A result of Marcus and Tardos [17] states that $e x(n, n, M)=O(n)$ for any permutation matrix $M$, which implies that $s a t^{*}\left([k]^{2}, P\right)=O(k)$ holds.

Fulek and Keszegh [8] showed that for any matrix $M$, if $\operatorname{sat}(n, n, M)$ is not constant, then it is at least linear. Their argument stays valid for any finite set $M_{1}, M_{2}, \ldots, M_{r}$ of matrices, so we just sketch it here. If
$M_{i}$ is a $q_{i} \times p_{i}$ matrix, then let $q:=\max \left\{q_{i}, p_{i}: i=1,2, \ldots, r\right\}$. If $\operatorname{sat}\left(n, n,\left\{M_{1}, M_{2} \ldots, M_{r}\right\}\right) \geq \frac{n}{q}$ for every large enough $n$, then clearly the saturation number grows at least linearly. Otherwise, there exist a large enough $n_{0}$ and an $n_{0} \times n_{0}$ matrix $A$ that is $\left\{M_{1}, M_{2}, \ldots, M_{r}\right\}$-saturated and $A$ contains less than $\frac{n_{0}}{q} 1$-entries. Then there must exist $q$ consecutive all- 0 rows and $q$ consecutive all- 0 columns of $A$. It is easy to check that for any $n>n_{0}$ if we add $n-n_{0}$ all- 0 rows and columns to $A$ such that together with the $q$ consecutive all- 0 rows and columns of $A$ they stay consecutive, then the obtained matrix $A^{\prime}$ is $\left\{M_{1}, M_{2}, \ldots, M_{r}\right\}$-saturated and contains the same number of 1 -entries as $A$.

This shows that for any finite set of $0-1$ matrices, the saturation number is either constant or at least linear. As the strong saturation problem for poset $P$ is equivalent to the saturation problem for a finite set of 0-1 matrices, this concludes the proof of Theorem 1.8

It is more convenient for us to prove Theorem 1.9 together with the corresponding results from Theorem 1.5. Theorem 1.9 (i) states that for any $k, l$ we have $\operatorname{sat}^{*}\left([k] \times[l],\left\{\vee_{2}, \wedge_{2}\right\}\right)=\max \{k, l\}$ and Theorem 1.5 (v) states that $L a^{*}\left([k] \times[l],\left\{\vee_{2}, \wedge_{2}\right\}\right)=k+l-1$. We will use the following notions in the proof. The comparability graph of a poset $P$ has vertex set $P$ and $p \neq q$ are joined by an edge if $p<q$ or $q<p$. A poset is connected if its comparability graph is connected, and a component of $P$ is a connected component of its comparability graph.

Proof of Theorem 1.9 (i) and Theorem 1.5 (v). Observe that $P$ is strong $\left\{\vee_{2}, \wedge_{2}\right\}$-free if and only if the components of $P$ are chains (or equivalently the components of its comparability graph are cliques). Let $F \subseteq[k] \times[k]$ be a strong $\left\{\vee_{2}, \wedge_{2}\right\}$-saturated set of elements, and let $C_{1}, C_{2}, \ldots, C_{h}$ be the components of $F$ (thus we know that each $C_{i}$ is a chain). For $i=1,2, \ldots, h$ let $\left(a_{i}, b_{i}\right)$ be the minimal element of $C_{i}$ and ( $x_{i}, y_{i}$ ) be the maximal element of $C_{i}$.

- As the chains are incomparable (they are the connected components of $F$ ), after renumbering the chains we can assume that for any $i<j$ we have $x_{i}<a_{j}$ and $y_{j}<b_{i}$.
- As $F$ is saturated, for every $i$ the component $C_{i}$ is a maximal chain between $\left(a_{i}, b_{i}\right)$ and $\left(x_{i}, y_{i}\right)$, as otherwise we could extend $C_{i}$ to such a chain that is still incomparable with the other chain components of $F$, thus the resulting larger set of elements would be $\left\{\vee_{2}, \wedge_{2}\right\}$-free, contradicting our assumption on $F$. In particular, $\left|C_{i}\right|=\left(x_{i}-a_{i}+1\right)+\left(y_{i}-b_{i}+1\right)-1$.
- For every $1 \leq \alpha \leq k$, there exists $i$ with $a_{i} \leq \alpha \leq x_{i}$. Indeed, if not then there would exist a counterexample $\alpha=x_{i}+1$ for some $i$. But then adding $\left(\alpha, y_{i}\right)$ to $C_{i}$ and to $F$ would keep the $\left\{\vee_{2}, \wedge_{2}\right\}$ free property contradicting the maximality of $F$. Similarly, for every $1 \leq \beta \leq l$, there exists $j$ with $b_{j} \leq \beta \leq y_{j}$. In particular, $\sum_{i=1}^{h}\left(x_{i}-a_{i}+1\right)=k$ and $\sum_{i=1}^{h}\left(y_{i}-b_{i}+1\right)=l$.

The above bullet points yield that $|F|=\sum_{i=1}^{h}\left|C_{i}\right|=k+l-h$. Thus the size of $F$ is largest if $h=1$ and thus $L a^{*}\left([k] \times[l],\left\{\vee_{2}, \wedge_{2}\right\}\right)=k+l-1$, while the size of $F$ is smallest if $h$ is as large as possible. Clearly, $h$ cannot be more than the width of $[k] \times[l]$, which is $\min \{k, l\}$, so $s a t^{*}\left([k] \times[l],\left\{\vee_{2}, \wedge_{2}\right\}\right) \geq \max \{k, l\}$ and the construction $\{(k, 1),(k-1,2), \ldots,(1, k),(1, k+1), \ldots,(1, l)\}$ shows that equality holds.

We continue with Theorem 1.9 (ii) and Theorem 1.5 (ii), which state that for any integers $k, l$ we have $s a t^{*}\left([k] \times[l], \vee_{2}\right)=L a^{*}\left([k] \times[l], \vee_{2}\right)=k+l-1$.

Proof of Theorem 1.9 (ii) and Theorem 1.5 (ii). We proceed by induction on $k+l$ with the base cases $k=1$ or $l=1$ being trivial. Let $F$ be any strong $\vee_{2}$-saturated subset of $[k] \times[l]$ and observe that $(k, l) \in F$ as it is not contained in any strong copy of $\vee_{2}$. Let $\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right), \ldots,\left(a_{h}, b_{h}\right)$ be the maximal elements of $F \backslash\{(k, l)\}$. By reordering, we may assume $a_{1}<a_{2}<\cdots<a_{h}$ and $b_{1}>b_{2}>\cdots>b_{h}$. Observe that by $\vee_{2}$-free property, any $f \in F$ is below only one $\left(a_{j}, b_{j}\right)$ and thus

$$
\begin{equation*}
F \backslash\{(k, l)\} \subseteq \cup_{j=1}^{h}\left[\left(a_{j-1}+1, b_{j+1}+1\right),\left(a_{j}, b_{j}\right)\right], \tag{2}
\end{equation*}
$$

where $a_{0}=b_{h+1}=0$ and $\left[f_{1}, f_{2}\right]=\left\{g: f_{1} \leqslant g \leqslant f_{2}\right\}$. We claim that $h \leq 2$ and if $h=2$, then $b_{1}=l$, $a_{2}=k$. Indeed, if there exists $j$ with $a_{j} \neq k, b_{j} \neq l$ and there exists $1 \leq i \leq h, i \neq j$, then for ( $a, b$ ) with $a=\max \left\{a_{i}, a_{j}\right\}, b=\max \left\{b_{i}, b_{j}\right\}$ we have $(a, b) \neq(k, l)$ and thus $(a, b) \notin F$. But $(a, b)$ does not create any strong copy of $\vee_{2}$ as there is only one element of $F$, namely $(k, l)$, that is larger than $(a, b)$, and also, by (2), any $f \in F$ that is smaller than $(a, b)$ is comparable only to elements in $\left\{f^{\prime}: f^{\prime} \leqslant(a, b)\right\} \cup\{(k, l)\}$.

We distinguish two cases. If there is a unique maximal element $(a, b)$ of $F \backslash\{(k, l)\}$, then adding a maximal chain from $(a, b)$ to $(k, l)$ does not violate the strong $\vee_{2}$-free property, thus $(a, b)=(k, l-1)$ or $(a, b)=(k-1, l)$. But then $F \backslash\{(k, l)\}$ is strong $\vee_{2}$-free saturating in $[a] \times[b]$, so by induction we obtain $|F|=(a+b-1)+1=k+l-1$.

Finally, if there are two maximal elements of $F \backslash\{(k, l)\}$, then these must be $(k, j)$ and $(i, l)$ for some $1 \leq j \leq l$ and $1 \leq i \leq k$. As any $f \in[(i+1,1),(k, j)]$ and $f^{\prime} \in[(1, j+1),(i, l)]$ form an incomparable pair, by (2), we have that $F \cap[(i+1,1),(k, j)]$ is strong $\vee_{2}$-saturated in $[(i+1,1),(k, j)]$ and $F \cap[(1, j+1),(i, l)]$ is strong $\vee_{2}$-saturated in $[(1, j+1),(i, l)]$. Thus by induction, we obtain $|F|=(k-i+j-1)+(i+l-j-1)+1=$ $k+l-1$.

We finish this section with the proof of Theorem 1.10, which states the following. Let $P$ be a poset with $\operatorname{dim}(P)=2$ such that a strong copy of $P$ in a two dimensional grid cannot contain two neighboring points. Then $\operatorname{sat}^{*}([k] \times[l], P) \geq \max \{k, l\}$.

Proof of Theorem 1.10. Suppose that $F$ is a $P$-saturated subset of $[k] \times[l]$. We prove that it must contain an element in each column, and by an analogous argument for the rows we are done.

We prove by contradiction: suppose there is an empty column. If $F$ is not empty, then we can suppose that there is an empty column next to a non-empty one: $\{(h, j): j \in[l]\} \cap F=\emptyset$ and $\{(g, j): j \in[l]\} \cap F \neq \emptyset$ with either $g=h-1$ or $g=h+1$. Let us define $y$ as $\max \{j:(g, j) \in F\}$ if $g=h-1$ and $\min \{j:(g, j) \in F\}$ if $g=h+1$. Let $a:=(g, y) \in F$ and $b:=(h, y) \notin F$.

Since $F$ is $P$-saturated, $F \cup\{b\}$ contains a strong copy of $P$ that includes $b$. By the property of $P$, the neighboring point $a$ is not in this copy. Also note that all points of the grid, except for certain points in the columns $g$ and $h$, compare the same way to $a$ and $b$ (smaller than, greater than or incomparable to both). By the selection of $a$ and $b$ these exceptional points are not in $F$. Therefore as far as subposets in $F$ are concerned, $a$ and $b$ are interchangeable. This means that there is a strong copy of $P$ with $a$ in the place of $b$. That contradicts the assumption that $F$ is strong $P$-free.

## 4 Open problems

The widely believed conjecture of forbidden subposet problems for the Boolean case appeared first in 4, 13] and considers the limits $\pi_{P}=\lim _{n \rightarrow \infty} \frac{L a(n, P)}{\binom{n}{\lfloor n / 2\rfloor}}$ and $\pi_{P}^{*}=\lim _{n \rightarrow \infty} \frac{L a^{*}(n, P)}{\left(\begin{array}{l}n n 2\rfloor\end{array}\right)}$. These limits are yet to be proved to exist, however, the following natural conjecture gives their possible values.

Conjecture 4.1. For a poset $P$ let us denote by $e(P)$ the largest integer $m$ such that for any $n$, any family $\mathcal{F} \subseteq 2^{[n]}$ consisting of $m$ consecutive levels is weak $P$-free and the parameter $e^{*}(P)$ is defined analogously for strong $P$-free families. Then $\pi_{P}=e(P)$ and $\pi_{P}^{*}=e^{*}(P)$ hold.

We conjecture that the values $\pi_{P}$ and $\pi_{P}^{*}$ can be obtained via forbidden subposet problems in the grid for any poset $P$ using Corollary 1.2 .

Conjecture 4.2. For any at most d-dimensional poset $P$ there exist $\pi_{d, P}=\lim _{k \rightarrow \infty} \frac{\operatorname{La([k]^{d},P)}}{w_{k, d}}$ and $\pi_{d, P}^{*}=$ $\lim _{k \rightarrow \infty} \frac{L a^{*}\left([k]^{d}, P\right)}{w_{k, d}}$. Moreover, $\lim _{d \rightarrow \infty} \pi_{d, P}=\pi_{P}$ and $\lim _{d \rightarrow \infty} \pi_{d, P}^{*}=\pi_{P}^{*}$ hold.

Corollary 1.2 implies $\pi_{P} \leq \pi_{d, P}$ and $\pi_{P}^{*} \leq \pi_{d, P}^{*}$ for any $P$ and $d$. The following conjecture states that $\pi_{d, P}$ is monotone decreasing in $d$.

Conjecture 4.3. For any at most d-dimensional poset $P$, we have $\pi_{d, P} \geq \pi_{d+1, P}$ and $\pi_{d, P}^{*} \geq \pi_{d+1, P}^{*}$
The smallest poset for which Conjecture 4.1 has not been verified is the diamond poset $D_{2}$. Theorem 1.6 determines $\pi_{2, D_{2}}$, and we have the following conjecture for larger values of $d$.

Conjecture 4.4. For any $d \geq 2$, we have $\pi_{d, D_{2}}=\pi_{d, D_{2}}^{*}=2+\frac{1}{d}$.
Concerning the connection of the permutation pattern $J_{s}$ and $\vee_{s}$, we state the following conjecture, the first part of which can already be found in [3].

Conjecture 4.5. For any $s \geq 3$, we have $\operatorname{sat}\left(n, n, J_{s}\right)=\operatorname{ex}\left(n, n, J_{s}\right)=\operatorname{sat}^{*}\left([n]^{2}, \vee_{s}\right)=L a^{*}\left([n]^{2}, \vee_{s}\right)$.
We finish with two problems in larger dimensional grids.
Conjecture 4.6. For any $d \geq 3$ and $k_{1}, k_{2}, \ldots, k_{d}$ we have sat $\left(\left[k_{1}\right] \times\left[k_{2}\right] \times \cdots \times\left[k_{d}\right], \vee_{2}\right)=\sum_{i=1}^{d} k_{i}-1$ and $L a^{*}\left([k]^{d}, \vee_{2}\right)=\left(1+\frac{1}{d}+o(1)\right) w_{k, d}$.

Problem 4.7. Determine the possible orders of magnitude of $s a t^{*}\left([k]^{d}, P\right)$.
Observe that the dichotomy part of the proof of Theorem 1.8 stays valid, showing that for any poset $P$, we have that $s a t^{*}\left([k]^{d}, P\right)$ is either constant or at least linear.

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