# An Almost Optimal Bound on the Number of Intersections of Two Simple Polygons 

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#### Abstract

What is the maximum number of intersections of the boundaries of a simple $m$-gon and a simple $n$-gon, assuming general position? This is a basic question in combinatorial geometry, and the answer is easy if at least one of $m$ and $n$ is even: If both $m$ and $n$ are even, then every pair of sides may cross and so the answer is $m n$. If exactly one polygon, say the $n$-gon, has an odd number of sides, it can intersect each side of the $m$-gon at most $n-1$ times; hence there are at most $m n-m$ intersections. It is not hard to construct examples that meet these bounds. If both $m$ and $n$ are odd, the best known construction has $m n-(m+n)+3$ intersections, and it is conjectured that this is the maximum. However, the best known upper bound is only $m n-\left(m+\left\lceil\frac{n}{6}\right\rceil\right)$, for $m \geq n$. We prove a new upper bound of $m n-(m+n)+C$ for some constant $C$, which is optimal apart from the value of $C$.


2012 ACM Subject Classification Theory of computation $\rightarrow$ Computational geometry; Mathematics of computing $\rightarrow$ Combinatorial problems

Keywords and phrases Simple polygon, Ramsey theory, combinatorial geometry
Funding Eyal Ackerman: The main part of this work was performed during a visit to Freie Universität Berlin which was supported by the Freie Universität Alumni Program.
Balázs Keszegh: Research supported by the Lendület program of the Hungarian Academy of Sciences (MTA), under the grant LP2017-19/2017 and by the National Research, Development and Innovation Office - NKFIH under the grant K 116769.

## 1 Introduction

To determine the union of two or more geometric objects in the plane is one of the basic computational geometric problems. In strong relation to that, determining the maximum complexity of the union of two or more geometric objects is a basic extremal geometric problem. We study this problem when the two objects are simple polygons.

Let $P$ and $Q$ be two simple polygons with $m$ and $n$ sides, respectively, where $m, n \geq 3$. For simplicity we always assume general position in the sense that no three vertices (of $P$ and $Q$ combined) lie on a line and no two sides (of $P$ and $Q$ combined) are parallel. We are interested in the maximum number of intersections of the boundaries of $P$ and $Q$.

This naturally gives an upper bound for the complexity of the union of the polygon areas as well. (In the worst case all the $m+n$ vertices of the two polygons contribute to the complexity of the boundary in addition to the intersection points.)

This paper is to appear in the proceedings of the 36th International Symposium on Computational Geometry (SoCG 2020) in June 2016; Editors: Sergio Cabello and Danny Chen, Leibniz International Proceedings in Informatics.

(a)

(b)

(c)

Figure 1 (a) Optimal construction for $m=n=8$, with $8 \times 8=64$ intersections. (b) Optimal construction for $m=8, n=7$, with $8 \times 6=48$ intersections. (c) Lower-bound construction for $m=9, n=7$. There are $8 \times 6+2=50$ intersections.

This problem was first studied in 1993 by Dillencourt, Mount, and Saalfeld [2]. The cases when $m$ or $n$ is even are solved there. If $m$ and $n$ are both even, then every pair of sides may cross and so the answer is $m n$. Figure 1 a shows one of many ways to achieve this number. If one polygon, say $Q$, has an odd number $n$ of sides, no line segment $s$ can be intersected $n$ times by $Q$, because otherwise each side of $Q$ would have to flip from one side of $s$ to the other side. Thus, each side of the $m$-gon $P$ is intersected at most $n-1$ times, for a total of at most $m n-m$ intersections. It is easy to see that this bound is tight when $P$ has an even number of sides, see Figure 1p.

When both $m$ and $n$ are odd, the situation is more difficult; the bound that is obtained by the above argument remains at $m n-\max \{m, n\}$, because the set of $m$ intersections that are necessarily "missing" due to the odd parity of $n$ might conceivably overlap with the $n$ intersections that are "missing" due to the odd parity of $m$. However, the best known family of examples gives only $m n-(m+n)+3=(m-1)(n-1)+2$ intersection points, see Figure 1. Note that in Figure 11 all vertices of the polygons contribute to the boundary of the union of the polygon areas.

- Conjecture 1. Let $P$ and $Q$ be simple polygons with $m$ and $n$ sides, respectively, such that $m, n \geq 3$ are odd numbers. Then there are at most $m n-(m+n)+3$ intersection points between sides of $P$ and sides of $Q$.

In [2] an unrecoverable error appears in a claimed proof of Conjecture 1. Another attempted proof [5] also turned out to have a fault. The only correct improvement over the trivial upper bound is an upper bound of $m n-\left(m+\left\lceil\frac{n}{6}\right\rceil\right)$ for $m \geq n$, due to Černý, Kára, Král', Podbrdský, Sotáková, and Šámal [1]. We will briefly discuss their proof in Section 2

We improve the upper bound to $m n-(m+n)+O(1)$, which is optimal apart from an additional constant:

- Theorem 1. There is an absolute constant $C$ such that the following holds. Suppose that $P$ and $Q$ are simple polygons with $m$ and $n$ sides, respectively, such that $m$ and $n$ are odd numbers. Then there are at least $m+n-C$ pairs of a side of $P$ and a side of $Q$ that do not intersect. Hence, there are at most $m n-(m+n)+C$ intersections.

The value of the constant $C$ that we obtain in our proof is around $2^{2^{67}}$. We did not make a large effort to optimize this value, and obviously, there is ample space for improvement.


Figure 2 The edge-labeled multigraph $G_{0}$ in Proposition 2 .


Figure 3 The unfolded graph $G_{0}^{\prime}$

## 2 Overview of the Proof

First we establish the crucial statement that the odd parity of $m$ and $n$ allows us to associate to any two consecutive sides of one polygon a pair of consecutive sides of the other polygon with a restricted intersection pattern among the four involved sides (Lemma 5 and Figure 5). This is the only place where we use the odd parity of the polygons.

A simple observation (Observation 3) relates the bound on $C$ in Theorem 1 to the number of connected components of the bipartite "disjointness graph" between the polygon sides of $P$ and $Q$. Our goal is therefore to show that there are few connected components.

We proceed to consider two pairs of associated pairs of sides ( 4 consecutive pairs with 8 sides in total). Unless they form a special structure, they cannot belong to four different connected components (Lemma 7). (Four is the maximum number of components that they could conceivably have.) The proof involves a case distinction with a moderate amount of cases. This structural statement allows us to reduce the bound on the number of components by a constant factor, and thereby, we can already improve the best previous result on the number of intersections (Proposition 9 in Section 6).

Finally, to get a constant bound on the number of components, our strategy is to use Ramsey-theoretic arguments like the Erdős-Szekeres Theorem on caps and cups or the pigeonhole principle (see Section 7) in order to impose additional structure on the configurations that we have to analyze. This is the place in the argument where we give up control over the constant $C$ in exchange for useful properties that allow us to derive a contradiction. This eventually boils down again to a moderate number of cases (Section 8.2. .

By contrast, the proof of the bound $m n-\left(m+\left\lceil\frac{n}{6}\right\rceil\right)$ for $m \geq n$ by Černý et al. proceeds in a more local manner. The core of their argument [1, Lemma 3], which is proved by case distinction, is that it is impossible to have 6 consecutive sides of one polygon together with 6 distinct sides of the other polygon forming a perfect matching in the disjointness graph. This statement is used to bound the number of components of the disjointness graph. (Lemma 8 below uses a similar argument.)

## 3 An Auxiliary Lemma on Closed Odd Walks

We begin with the following seemingly unrelated claim concerning a specific small edge-labeled multigraph. Let $G_{0}=\left(V_{0}, E_{0}\right)$ be the undirected multigraph shown in Figure 2, It has four nodes $V_{0}=\{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ and five edges $E_{0}=\left\{e_{1}=\{\mathrm{II}, \mathrm{IV}\}, e_{2}=\{\mathrm{I}, \mathrm{IV}\}, e_{3}=\{\mathrm{I}, \mathrm{II}\}, e_{4}=\right.$ $\left.\{\mathrm{I}, \mathrm{III}\}, e_{5}=\{\mathrm{I}, \mathrm{III}\}\right\}$. Every edge $e_{i} \in E_{0}$ has a label $L\left(e_{i}\right) \in\{a, b, *\}$ as follows: $L\left(e_{1}\right)=*$, $L\left(e_{2}\right)=L\left(e_{4}\right)=a, L\left(e_{3}\right)=L\left(e_{5}\right)=b$.

- Proposition 2. If $W$ is a closed walk in $G_{0}$ of odd length, then $W$ contains two cyclically consecutive edges of labels $a$ and $b$.

Proof. Suppose for contradiction that $W$ does not contain two consecutive edges of labels $a$ and $b$. Since $W$ cannot switch between the $a$-edges and the $b$-edges in I or III, we can split I (resp., III) into two nodes $\mathrm{I}_{a}$ and $\mathrm{III}_{b}$ (resp., $\mathrm{III}_{a}$ and $\mathrm{III}_{b}$ ) such that every $a$-labeled edge that is incident to I (resp., III) in $G_{0}$ becomes incident to $\mathrm{I}_{a}$ (resp., $\mathrm{III}_{a}$ ) and every $b$-labeled edge that is incident to I (resp., III) in $G_{0}$ becomes incident to $\mathrm{I}_{b}$ (resp., $\mathrm{III}_{b}$ ). In the resulting graph $G_{0}^{\prime}$, which is shown in Figure 3, we can find a closed walk $W^{\prime}$ that corresponds to $W$ and that uses the edges with the same name as $W$. Since $G_{0}^{\prime}$ is a path, every closed walk has even length. Thus, $W$ cannot have odd length.

## 4 General Assumptions and Notations

Let $P$ and $Q$ be two simple polygons with sides $p_{0}, p_{1}, \ldots, p_{m-1}$ and $q_{0}, q_{1}, \ldots, q_{n-1}$. We assume that $m \geq 3$ and $n \geq 3$ are odd numbers. Addition and subtraction of indices is modulo $m$ or $n$, respectively. We consider the sides $p_{i}$ and $q_{j}$ as closed line segments. The condition that the polygon $P$ is simple means that its edges are pairwise disjoint except for the unavoidable common endpoints between consecutive sides $p_{i}$ and $p_{i+1}$. Throughout this paper, unless stated otherwise, we regard a polygon as a piecewise linear closed curve, and we disregard the region that it encloses. Thus, by intersections between $P$ and $Q$, we mean intersection points between the polygon boundaries.

As mentioned, we assume that the vertices of $P$ and $Q$ are in general position (no three of them on a line), and so every intersection point between $P$ and $Q$ is an interior point of two polygon sides.

The Disjointness Graph. As in [1] our basic tool of analysis is the disjointness graph of $P$ and $Q$, which we denote by $G^{\mathrm{D}}=\left(V^{\mathrm{D}}, E^{\mathrm{D}}\right)$. (Its original name in [1] is non-intersection graph.) It is a bipartite graph with node set $V^{\mathrm{D}}=\left\{p_{0}, p_{1}, \ldots, p_{m-1}\right\} \cup\left\{q_{0}, q_{1}, \ldots, q_{n-1}\right\}$ and edge set $E^{\mathrm{D}}=\left\{\left(p_{i}, q_{j}\right) \mid p_{i} \cap q_{j}=\emptyset\right\}$. (Since we are interested in the situation where almost all pairs of edges intersect, the disjointness graph is more useful than its more commonly used complement, the intersection graph.)

Our goal is to bound from above the number of connected components of $G^{\mathrm{D}}$.

- Observation 3. If $G^{\mathrm{D}}$ has at most $C$ connected components, then $G^{\mathrm{D}}$ has at least $m+n-C$ edges. Thus, there are at least $m+n-C$ pairs of a side of $P$ and a side of $Q$ that do not intersect, and there are at most $m n-(m+n)+C$ crossings between $P$ and $Q$.

Geometric Notions. Let $s$ and $s^{\prime}$ be two line segments. We denote by $\ell(s)$ the line through $s$ and by $I\left(s, s^{\prime}\right)$ the intersection of $\ell(s)$ and $\ell\left(s^{\prime}\right)$ see Figure 4 . We say that $s$ and $s^{\prime}$ are avoiding if neither of them contains $I\left(s, s^{\prime}\right)$. (This requirement is stronger than just disjointness.) If $s$ and $s^{\prime}$ are avoiding or share an endpoint, we denote by $\vec{r}_{s^{\prime}}(s)$ the ray from $I\left(s, s^{\prime}\right)$ to infinity that contains $s$, and by $\vec{r}_{s}\left(s^{\prime}\right)$ the ray from $I\left(s, s^{\prime}\right)$ to infinity that contains $s^{\prime}$. Moreover, we denote by $\operatorname{Cone}\left(s, s^{\prime}\right)$ the convex cone with apex $I\left(s, s^{\prime}\right)$ between these two rays.

- Observation 4. If a segment $s^{\prime \prime}$ that does not go through $I\left(s, s^{\prime}\right)$ has one of its endpoints in the interior of Cone $\left(s, s^{\prime}\right)$, then $s^{\prime \prime}$ cannot intersect both $\vec{r}_{s^{\prime}}(s)$ and $\vec{r}_{s}\left(s^{\prime}\right)$. In particular, it cannot intersect both $s$ and $s^{\prime}$.

For a polygon side $s$ of $P$ or $Q, \mathrm{CC}(s)$ denotes the connected component of the disjointness graph $G^{\mathrm{D}}$ to which $s$ belongs.

### 4.1 Associated Pairs of Consecutive Sides

- Lemma 5. Let $p_{a}$ and $p_{b}$ be two sides of $P$ that are either consecutive or avoiding such that $\mathrm{CC}\left(p_{a}\right) \neq \mathrm{CC}\left(p_{b}\right)$. Then there are two consecutive sides $q_{i}, q_{i \pm 1}$ of $Q$ such that $\left(p_{a}, q_{i}\right),\left(p_{b}, q_{i \pm 1}\right) \in E^{\mathrm{D}}$ and $\left(p_{a}, q_{i \pm 1}\right),\left(p_{b}, q_{i}\right) \notin E^{\mathrm{D}}$. Furthermore, $I\left(p_{a}, p_{b}\right) \in \operatorname{Cone}\left(q_{i}, q_{i \pm 1}\right)$ or $I\left(q_{i}, q_{i \pm 1}\right) \in \operatorname{Cone}\left(p_{a}, p_{b}\right)$.

The sign ' $\pm$ ' is needed since we do not know which of the consecutive sides intersects $p_{i}$ and is disjoint from $p_{i+1}$.

Proof. We may assume without loss of generality that $I\left(p_{a}, p_{b}\right)$ is the origin, $p_{a}$ lies on the positive $x$-axis and the interior of $p_{b}$ is above the $x$-axis. The lines $\ell\left(p_{a}\right)$ and $\ell\left(p_{b}\right)$ partition the plane into four convex cones ("quadrants"). Denote them in counterclockwise order by I, II, III, IV, starting with $\mathrm{I}=\operatorname{Cone}\left(p_{a}, p_{b}\right)$, see Figure 4. Every side of $Q$ must intersect $p_{a}$



Figure 4 How an odd polygon $Q$ can intersect two segments. The segments $p_{a}$ and $p_{b}$ are avoiding, whereas $s$ and $s^{\prime}$ are disjoint but non-avoiding.
or $p_{b}$ (maybe both), since $\mathrm{CC}\left(p_{a}\right) \neq \mathrm{CC}\left(p_{b}\right)$. One can now check that traversing the sides of $Q$ in order generates a closed walk $W$ in the graph $G_{0}$ of Figure 2 For example, a side of $Q$ that we traverse from its endpoint in I to its endpoint in III and that intersects $p_{a}$ corresponds to traversing the edge $e_{4}=\{\mathrm{I}, \mathrm{III}\}$ from I to III, whose label is $L\left(e_{4}\right)=a$. We do not care which of $p_{a}$ and $p_{b}$ are crossed when we move between II and IV.

It follows from Proposition 2 that $Q$ has two consecutive sides $q_{i}, q_{i \pm 1}$ such that $q_{i}$ intersects $p_{b}$ and does not intersect $p_{a}$, while $q_{i \pm 1}$ intersects $p_{a}$ and does not intersect $p_{b}$. Hence, $\left(p_{a}, q_{i}\right),\left(p_{b}, q_{i \pm 1}\right) \in E^{\mathrm{D}}$ and $\left(p_{a}, q_{i \pm 1}\right),\left(p_{b}, q_{i}\right) \notin E^{\mathrm{D}}$. Furthermore, $I\left(q_{i}, q_{i \pm 1}\right)$ must be either in I or III as these are the only nodes in $G_{0}$ that are incident both to an edge labeled $a$ and an edge labeled $b$. In the latter case $I\left(p_{a}, p_{b}\right) \in \operatorname{Cone}\left(q_{i}, q_{i \pm 1}\right)$, and in the former case $I\left(q_{i}, q_{i \pm 1}\right) \in \operatorname{Cone}\left(p_{a}, p_{b}\right)$.

Let $p_{i}, p_{i+1}$ be two sides of $P$ such that $\operatorname{CC}\left(p_{i}\right) \neq \operatorname{CC}\left(p_{i+1}\right)$. Then by Lemma 5 there are sides $q_{j}, q_{j \pm 1}$ of $Q$ such that $\left(p_{i}, q_{j}\right),\left(p_{i+1}, q_{j \pm 1}\right) \in E^{\mathrm{D}}$. We say that the pair $q_{j}, q_{j \pm 1}$ is associated to $p_{i}, p_{i+1}$. By Lemma 5 we have $I\left(q_{j}, q_{j \pm 1}\right) \in \operatorname{Cone}\left(p_{i}, p_{i+1}\right)$ or $I\left(p_{i}, p_{i+1}\right) \in$ $\operatorname{Cone}\left(q_{j}, q_{j \pm 1}\right)$. If the first condition holds we say that $p_{i}, p_{i+1}$ is hooking and $q_{j}, q_{j \pm 1}$ is hooked, see Figure 5. In the second case we say that $p_{i}, p_{i+1}$ is hooked and $q_{j}, q_{j \pm 1}$ is hooking. Note that it is possible that a pair of consecutive sides is both hooking and hooked (with respect to two different pairs from the other polygon or even with respect to a single pair, as in Figure 5f).

- Observation 6 (The Axis Property). If the pair $p_{i}, p_{i+1}$ and the pair $q_{j}, q_{j \pm 1}$ are associated such that $\left(p_{i}, q_{j}\right),\left(p_{i+1}, q_{j \pm 1}\right) \in E^{\mathrm{D}}$, then the line through $I\left(p_{i}, p_{i+1}\right)$ and $I\left(q_{j}, q_{j \pm 1}\right)$ separates $p_{i}$ and $q_{j \pm 1}$ on the one side from $p_{i+1}$ and $q_{j}$ on the other side.

(a)

(b)

(c)

Figure 5 Hooking and hooked pairs of consecutive sides. (a) The pair $p_{i}, p_{i+1}$ is hooking and the associated pair $q_{j}, q_{j \pm 1}$ is hooked. (b) vice versa. (c) Both pairs are both hooking and hooked.

We call this line the axis of the associated pairs. In our figures it appears as a dotted line when it is shown.

## 5 The Principal Structure Lemma about Pairs of Associated Pairs

- Lemma 7. Let $p_{i}, p_{i+1}, p_{j}, p_{j+1}$ be two pairs of consecutive sides of $P$ that belong to four different connected components of $G^{\mathrm{D}}$. Then it is impossible that both $p_{i}, p_{i+1}$ and $p_{j}, p_{j+1}$ are hooked or that both pairs are hooking.

Proof. Suppose first that both pairs $p_{i}, p_{i+1}$ and $p_{j}, p_{j+1}$, are hooking and let $q_{i^{\prime}}, q_{i^{\prime} \pm 1}$ and $q_{j^{\prime}}, q_{j^{\prime} \pm 1}$ be their associated (hooked) pairs such that: $\left(p_{i}, q_{i^{\prime}}\right),\left(p_{i+1}, q_{i^{\prime} \pm 1}\right) \in E^{\mathrm{D}}$, $\left(p_{j}, q_{j^{\prime}}\right),\left(p_{j+1}, q_{j^{\prime} \pm 1}\right) \in E^{\mathrm{D}}, I\left(q_{i^{\prime}}, q_{i^{\prime} \pm 1}\right) \in \operatorname{Cone}\left(p_{i}, p_{i+1}\right)$ and $I\left(q_{j^{\prime}}, q_{j^{\prime} \pm 1}\right) \in \operatorname{Cone}\left(p_{j}, p_{j+1}\right)$.

For better readability, we rename $p_{i}, p_{i+1}$ and $q_{i^{\prime}}, q_{i^{\prime} \pm 1}$ as $a, b$ and $A, B$, and we rename $p_{j}, p_{j+1}$ and $q_{j^{\prime}}, q_{j^{\prime} \pm 1}$ as $a^{\prime}, b^{\prime}$ and $A^{\prime}, B^{\prime}$. The small letters denote sides of $P$ and the capital letters denote sides of $Q$. In the new notation, $a, b$ are consecutive sides of $P$ with an associated pair $A, B$ of consecutive sides of $Q$, and $a^{\prime}, b^{\prime}$ are two other consecutive sides of $P$ with an associated pair $A^{\prime}, B^{\prime}$ of consecutive sides of $Q$. The disjointness graph $G^{\mathrm{D}}$ contains the edges $(a, A),(b, B),\left(a^{\prime}, A^{\prime}\right),\left(b^{\prime}, B^{\prime}\right)$. Since $a, b, a^{\prime}, b^{\prime}$ belong to different connected components of $G^{\mathrm{D}}$, it follows that the nodes $A, B, A^{\prime}, B^{\prime}$, to which they are connected, belong to the same four different connected components. There can be no more edges among these eight nodes, and they induce a matching in $G^{\mathrm{D}}$. One can remember as a rule that every side of $P$ intersects every side of $Q$ among the eight involved sides, except when their names differ only in their capitalization. In particular, each of $A^{\prime}$ and $B^{\prime}$ intersects each of $a$ and $b$. and hence they must lie as in Figure $\sqrt{6}$ a. To facilitate the future discussion, we will now normalize the positions of these four sides.

We first ensure that the intersection $I\left(A^{\prime}, b\right)$ is directly adjacent to the two polygon vertices $I(a, b)$ and $I\left(A^{\prime}, B^{\prime}\right)$ in the arrangement of the four sides, as shown in Figure 6 . This can be achieved by swapping the labels $a, A$ with the labels $b, B$ if necessary, and by independently swapping the labels $a^{\prime}, A^{\prime}$ with $b^{\prime}, B^{\prime}$ if necessary. Our assumptions are invariant under these swaps.

By an affine transformation we may finally assume that $I\left(A^{\prime}, b\right)$ is the origin; $b$ lies on the $x$-axis and is directed to the right; and $A^{\prime}$ lies on the $y$-axis and is directed upwards. Then $a$ has a positive slope and its interior is in the upper half-plane, and $B^{\prime}$ has a positive slope and its interior is to the right of the $y$-axis, see Figure 6;

The arrangement of the lines through $a, b, A^{\prime}, B^{\prime}$ has 11 faces, some of which are marked as $F_{1}, \ldots, F_{6}$ in Figure 6 Our current assumption is that both $a, b$ and $a^{\prime}, b^{\prime}$ are hooking:


Figure 6 Normalizing the position of $a, b, A^{\prime}, B^{\prime}$


Figure 7 Case $1 . I(A, B) \in F_{1}, I\left(a^{\prime}, b^{\prime}\right) \in F_{2}$


Figure 8 Case $2 I(A, B) \in F_{1}, I\left(a^{\prime}, b^{\prime}\right) \in F_{4}$

The hooking of $a, b$ means that $I(A, B) \in \operatorname{Cone}(a, b)=F_{1} \cup F_{2} \cup F_{3}$. By the Axis Property (Observation 6), the line through $I\left(A^{\prime}, B^{\prime}\right)$ and $I\left(a^{\prime}, b^{\prime}\right)$ must separate $A^{\prime}$ from $B^{\prime}$. Therefore, the vertex $I\left(a^{\prime}, b^{\prime}\right)$ can lie only in $F_{2} \cup F_{4} \cup F_{5} \cup F_{6}$. Thus, based on the faces that contain $I(A, B)$ and $I\left(a^{\prime}, b^{\prime}\right)$, there are 12 cases to consider. Some of these cases are symmetric, and all can be easily dismissed, as follows.

In the figures, the four sides $a^{\prime}, b^{\prime}, A^{\prime}, B^{\prime}$, which are associated to the second associated pair are dashed. All dashed sides of one polygon must intersect all solid sides of the other polygon.

1. $I(A, B) \in F_{1}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{2}$, see Figure 7 (symmetric to $I(A, B) \in F_{2}$ and $I\left(a^{\prime}, b^{\prime}\right) \in$ $F_{4}$ ). Let $r_{a}$ (resp., $r_{b}$ ) be the ray on $\ell(a)$ (resp., $\ell(b)$ ) that goes from the right endpoint of $a$ (resp., $b$ ) to the right. Since $a^{\prime}$ is not allowed to cross $b$, the only way for $a^{\prime}$ to intersect $A$ is by crossing $r_{b}$. Similarly, in order to intersect $B, a^{\prime}$ has to cross $r_{a}$. However, it cannot intersect both $r_{a}$ and $r_{b}$, by Observation 4
Since we did not use the assumption that $A, B$ are hooked, the analysis holds for the symmetric Case $6 . I(A, B) \in F_{2}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{4}$, as well.
2. $I(A, B) \in F_{1}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{4}$, see Figure 8 Since $a^{\prime}$ is not allowed to cross $b$, the only way for $a^{\prime}$ to intersect $B$ is by crossing $r_{b}$. However, in this case $a^{\prime}$ cannot intersect $A$.
3. $I(A, B) \in F_{1}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{5}$, see Figure 9 (symmetric to $I(A, B) \in F_{3}$ and $I\left(a^{\prime}, b^{\prime}\right) \in$ $F_{4}$ ). Both $a^{\prime}$ and $b^{\prime}$ must intersect $A$, and they have to go below the line $\ell(b)$ to do so. However, $a^{\prime}$ can only cross $\ell(b)$ to the right of $b$, and $b^{\prime}$ can only cross $\ell(b)$ to the left of $b$,


Figure 9 Case $3(A, B) \in F_{1}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{5}$

(a) At least one of the sides $a^{\prime}$ and $b^{\prime}$ has an endpoint in $F_{4}$.

(b) None of the sides $a^{\prime}$ and $b^{\prime}$ has an endpoint in $F_{4}$.

Figure 10 Case $4(A, B) \in F_{1}\left(\right.$ or $I(A, B) \in F_{2}$, which is similar $)$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{6}$.
and therefore they cross $A$ from different sides. This is impossible, because $a^{\prime}$ and $b^{\prime}$ start from the same point.
4. $I(A, B) \in F_{1}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{6}$. If one of the polygon sides $a^{\prime}$ and $b^{\prime}$ has an endpoint in $F_{4}$ (see Figure 10a), then this side cannot intersect $B$. So assume otherwise, see Figure 10b The side $a^{\prime}$ intersects $B^{\prime}$ and is disjoint from $A^{\prime}$, while $b^{\prime}$ is disjoint from $B^{\prime}$ and intersects $A^{\prime}$. (Due to space limitation some line segments are drawn schematically as curves.) Thus, each of $a^{\prime}$ and $b^{\prime}$ has an endpoint in $F_{2} \cup F_{5}$. But then $I(A, B) \in \operatorname{Cone}\left(a^{\prime}, b^{\prime}\right)$ and it follows from Observation 4 that neither $A$ nor $B$ can intersect both $a^{\prime}$ and $b^{\prime}$.
5. $I(A, B) \in F_{2}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{2}$, see Figure 11 Since $a^{\prime}, b^{\prime}$ is hooking, $I\left(A^{\prime}, B^{\prime}\right) \in$ Cone $\left(a^{\prime}, b^{\prime}\right)$, and the line segments $a^{\prime}, b^{\prime}, A^{\prime}, b, B^{\prime}$ enclose a convex pentagon. The polygon side $A$ must intersect $b, a^{\prime}$ and $b^{\prime}$, but it is restricted to $F_{2} \cup F_{4}$. It follows that $A$ must intersect three sides of the pentagon, which is impossible. (This is in fact the only place where we need the assumption that $a^{\prime}, b^{\prime}$ is hooking.)
6. $I(A, B) \in F_{2}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{4}$. This is symmetric to Case 1
7. $I(A, B) \in F_{2}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{5}$, see Figure 12 (symmetric to $I(A, B) \in F_{3}$ and $I\left(a^{\prime}, b^{\prime}\right) \in$ $F_{2}$ ). Then $A$ is restricted to $F_{2} \cup F_{4}$, while $a^{\prime}$ and $b^{\prime}$ do not intersect $F_{2}$ and $F_{4}$. Therefore


Figure 11 Case $5 I(A, B) \in F_{2}, I\left(a^{\prime}, b^{\prime}\right) \in F_{2}$


Figure 12 Case $7(A, B) \in F_{2}, I\left(a^{\prime}, b^{\prime}\right) \in F_{5}$
$A$ can intersect neither $a^{\prime}$ nor $b^{\prime}$.
8. $I(A, B) \in F_{2}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{6}$. This case is very similar to Case 4 , where $I(A, B) \in F_{1}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{6}$, see Figure 10 If one of the polygon sides $a^{\prime}$ and $b^{\prime}$ has an endpoint in $F_{4}$, then it cannot intersect $B$. Otherwise, $I(A, B) \in \operatorname{Cone}\left(a^{\prime}, b^{\prime}\right)$ and therefore, neither $A$ nor $B$ can intersect both $a^{\prime}$ and $b^{\prime}$.
9. $I(A, B) \in F_{3}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{2}$. This is symmetric to Case 7
10. $I(A, B) \in F_{3}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{4}$. This is symmetric to Case 3


Figure 13 Case $11 . I(A, B) \in F_{3}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{5}$
11. $I(A, B) \in F_{3}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{5}$, see Figure 13. Then the intersection of $b^{\prime}$ and $A$ can lie only in the lower left quadrant. It follows that the triangle whose vertices are $I\left(a^{\prime}, b^{\prime}\right)$, $I\left(a^{\prime}, A\right)$ and $I\left(A, b^{\prime}\right)$ contains $a$ and does not contain $I(A, B)$. This in turn implies that $B$ cannot intersect both $b^{\prime}$ and $a$, without intersecting $B^{\prime}$.
12. $I(A, B) \in F_{3}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{6}$, see Figure 14 As in Case 4, we may assume that neither $a^{\prime}$ nor $b^{\prime}$ has an endpoint in $F_{4}$, since then this side could not intersect $B$. We may also assume that $I(A, B) \notin \operatorname{Cone}\left(a^{\prime}, b^{\prime}\right)$ for otherwise neither $A$ nor $B$ intersects both of $a^{\prime}$
and $b^{\prime}$, according to Observation 4. If $a^{\prime}$ has an endpoint in $F_{2}$, then it cannot intersect $B$ (see Figure 14a). Otherwise, if $a^{\prime}$ has an endpoint in $F_{5}$, then $B$ cannot intersect $b^{\prime}$ (see Figure 14b).

(a) If $a^{\prime}$ has an endpoint in $F_{2}$, then it cannot intersect $B$.

(b) If $a^{\prime}$ has an endpoint in $F_{5}$, then $B$ cannot intersect $b^{\prime}$.

Figure 14 Case $12 I(A, B) \in F_{3}$ and $I\left(a^{\prime}, b^{\prime}\right) \in F_{6}$.
We have finished the case that $a, b$ and $a^{\prime}, b^{\prime}$ are hooking. Suppose now that $a, b$ and $a^{\prime}, b^{\prime}$ are hooked, with respect to some pairs $A, B$ and $A^{\prime}, B^{\prime}$. Then $A, B$ is hooking with respect to $a, b$ and $A^{\prime}, B^{\prime}$ is hooking with respect to $a^{\prime}, b^{\prime}$. Recall that $A, B, A^{\prime}$ and $B^{\prime}$ belong to four different connected components. Hence, this case can be handled as above, after exchanging the capital letters with the small letters (i.e., exchanging $P$ and $Q$ ).

## 6 A Weaker Bound

The principal structure lemma is already powerful enough to get an improvement over the previous best bound:

- Lemma 8. $G^{\mathrm{D}}$ has at most $(n+5) / 2$ connected components.

Proof. Partition the sides $q_{0}, q_{1}, \ldots, q_{n-1}$ of $Q$ into $(n-1) / 2$ disjoint pairs $q_{2 i}, q_{2 i+1}$, discarding the last side $q_{n-1}$. Let $H_{+}$denote the subset of these pairs that are hooked. Suppose first that this set contains some pair $q_{2 i_{0}}, q_{2 i_{0}+1}$ of sides that are in two different connected components. Combining $q_{2 i_{0}}, q_{2 i_{0}+1}$ with any of the remaining pairs $q_{2 i}, q_{2 i+1}$ of $H_{+}$, Lemma 7 tells us that the sides $q_{2 i}$ and $q_{2 i+1}$ must either belong to the same connected component, or one of them must belong to $\mathrm{CC}\left(q_{2 i_{0}}\right)$ or $\mathrm{CC}\left(q_{2 i_{0}+1}\right)$. In other words, each remaining pair contributes at most one "new" connected component, and it follows that the sides in $H_{+}$ belong to at most $\left|H_{+}\right|+1$ connected components. This conclusion holds also in the case that $H_{+}$contains no pair $q_{2 i_{0}}, q_{2 i_{0}+1}$ of sides that are in different connected components.

The same argument works for the complementary subset $H_{-}$of pairs that are not hooked, but hooking. Along with $\mathrm{CC}\left(q_{n-1}\right)$ there are at most $\left(\left|H_{+}\right|+1\right)+\left(\left|H_{-}\right|+1\right)+1=$ $(n-1) / 2+3=(n+5) / 2$ components.

Together with Observation 3 this already improves the previous bound $m n-\left(m+\left\lceil\frac{n}{6}\right\rceil\right)$ for a large range of parameters, namely when $m \geq n \geq 11$ :

- Proposition 9. Let $P$ and $Q$ be simple polygons with $m$ and $n$ sides, respectively, such that $m$ and $n$ are odd and $m \geq n \geq 3$. Then there are at most $m n-\left(m+\frac{n-5}{2}\right)$ intersection points between $P$ and $Q$.


## 7 Ramsey-Theoretic Tools

We recall some classic results.
A tournament is a directed graph that contains between every pair of nodes $x, y$ either the arc $(x, y)$ or the arc $(y, x)$ but not both. A tournament is transitive if for every three nodes $x, y, z$ the existence of the $\operatorname{arcs}(x, y)$ and $(y, z)$ implies the existence of the arc $(x, z)$. Equivalently, the nodes can be ordered on a line such that all arcs are in the same direction. The following is easy to prove by induction.

- Lemma 10 (Erdős and Moser [3]). Every tournament on a node set $V$ contains a transitive sub-tournament on $1+\left\lfloor\log _{2}|V|\right\rfloor$ nodes.

Proof. Choose $v \in V$ arbitrarily, and let $N \subseteq V-\{v\}$ with $|N| \geq(|V|-1) / 2$ be the set of in-neighbors of $v$ or the set of out-neighbors of $v$, whichever is larger. Then $v$ together with a transitive sub-tournament of $N$ gives a transitive sub-tournament of size one larger.

A set of points $p_{1}, p_{2}, \ldots, p_{r}$ in the plane sorted by $x$-coordinates (and with distinct $x$-coordinates) forms an $r$-cup (resp., $r$-cap) if $p_{i}$ is below (resp., above) the line through $p_{i-1}$ and $p_{i+1}$ for every $1<i<r$.

- Theorem 11 (Erdős-Szekeres Theorem for caps and cups in point sets [4]). For any two integers $r \geq 2$ and $s \geq 2$, the value $E S(r, s):=\binom{r+s-4}{r-2}$ fulfills the following statement:

Suppose that $P$ is a set of $E S(r, s)+1$ points in the plane with distinct $x$-coordinates such that no three points of $P$ lie on a line. Then $P$ contains an $r$-cup or an $s$-cap.

Moreover, $\operatorname{ES}(r, s)$ is the smallest value that fulfills the statement.
A similar statement holds for lines by the standard point-line duality. A set of lines $\ell_{1}, \ell_{2}, \ldots, \ell_{r}$ sorted by slope forms an $r$-cup (resp., r-cap) if $\ell_{i-1}$ and $\ell_{i+1}$ intersect below (resp., above) $\ell_{i}$ for every $1<i<r$.

- Theorem 12 (Erdős-Szekeres Theorem for lines). For the numbers $E S(r, s)$ from Theorem 11 , the following statement holds for any two integers $r \geq 2$ and $s \geq 2$ :

Suppose that $L$ is a set of $E S(r, s)+1$ non-vertical lines in the plane no two of which are parallel and no three of which intersect at a common point. Then $L$ contains an r-cup or an s-cap.

- Theorem 13 (Erdős-Szekeres Theorem for monotone subsequences [4]). For any integer $r \geq 0$, a sequence of $r^{2}+1$ distinct numbers contains either an increasing subsequence of length $r+1$ or a decreasing subsequence of length $r+1$.


## 8 Proof of Theorem 1

### 8.1 Imposing More Structure on the Examples

Going back to the proof of Theorem 1 recall that in light of Observation 3it is enough to prove that $G^{\mathrm{D}}$, the disjointness graph of $P$ and $Q$, has at most constantly many connected components.

We will use the following constants: $C_{6}:=6 ; C_{5}:=\left(C_{6}\right)^{2}+1=37 ; C_{4}:=E S\left(C_{5}, C_{5}\right)+1=$ $\binom{70}{35}+1=112,186,277,816,662,845,433<2^{70} ; C_{3}:=2^{C_{4}-1} ; C_{2}:=C_{3}+5 ; C_{1}:=8 C_{2} ;$ $C:=C_{1}-1<2^{2^{70}}$.

We claim that $G^{\mathrm{D}}$ has at most $C$ connected components. Suppose that $G^{\mathrm{D}}$ has at least $C_{1}=C+1$ connected components, numbered as $1,2, \ldots, C_{1}$. For each connected component $j$, we find two consecutive sides $q_{i_{j}}, q_{i_{j}+1}$ of $Q$ such that $\operatorname{CC}\left(q_{i_{j}}\right)=j$ and $\mathrm{CC}\left(q_{i_{j}+1}\right) \neq j$. We call $q_{i_{j}}$ the primary side and $q_{i_{j}+1}$ the companion side of the pair. We take these $C_{1}$ consecutive pairs in their cyclic order along $Q$ and remove every second pair. This ensures that the remaining $C_{1} / 2$ pairs are disjoint, in the sense that no side of $Q$ belongs to two different pairs.

We apply Lemma 5 to each of the remaining $C_{1} / 2$ pairs $q_{i_{j}}, q_{i_{j}+1}$ and find an associated pair $p_{k_{j}}, p_{k_{j} \pm 1}$ such that $\left(q_{i_{j}}, p_{k_{j}}\right),\left(q_{i_{j}+1}, p_{k_{j} \pm 1}\right) \in E^{\mathrm{D}}$. Therefore, $\mathrm{CC}\left(q_{i_{j}}\right)=\mathrm{CC}\left(p_{k_{j}}\right)$ and $\mathrm{CC}\left(q_{i_{j}+1}\right)=\mathrm{CC}\left(p_{k_{j} \pm 1}\right) \neq \mathrm{CC}\left(q_{i_{j}}\right)$. Again, we call $p_{k_{j}}$ the primary side and $p_{k_{j} \pm 1}$ the companion side. As before, we delete half of the pairs $p_{k_{j}}, p_{k_{j} \pm 1}$ in cyclic order along $P$, along with their associated pairs from $Q$, and thus we ensure that the remaining $C_{1} / 4$ pairs are disjoint also on $P$.

At least $C_{1} / 8$ of the remaining pairs $q_{i_{j}}, q_{i_{j}+1}$ are hooking or at least $C_{1} / 8$ of them are hooked. We may assume that at least $C_{2}=C_{1} / 8$ of the pairs $q_{i_{j}}, q_{i_{j}+1}$ are hooking with respect to their associated pair, $p_{k_{j}}, p_{k_{j} \pm 1}$, for otherwise, $p_{k_{j}}, p_{k_{j} \pm 1}$ is hooking with respect to $q_{i_{j}}, q_{i_{j}+1}$ and we may switch the roles of $P$ and $Q$. Let us denote by $Q_{2}$ the set of $C_{2}$ hooking consecutive pairs $\left(q_{i_{j}}, q_{i_{j} \pm 1}\right)$ at which we have arrived. (Because of the potential switch, we have to denote the companion side by $q_{i_{j} \pm 1}$ instead of $q_{i_{j}+1}$ from now on.)

By construction, all $C_{2}$ primary sides $q_{i_{j}}$ of these pairs belong to distinct components. We now argue that all $C_{2}$ adjacent companion sides $q_{i_{j} \pm 1}$ with at most one exception lie in the same connected component, provided that $C_{2} \geq 4$.

We model the problem by a graph whose nodes are the connected components of $G^{\mathrm{D}}$. For each of the $C_{2}$ pairs $q_{i_{j}}, q_{i_{j} \pm 1}$, we insert an edge between $\operatorname{CC}\left(q_{i_{j}}\right)$ and $\operatorname{CC}\left(q_{i_{j} \pm 1}\right)$. The result is a multigraph with $C_{2}$ edges and without loops. Two disjoint edges would represent two consecutive pairs of the form $\left(q_{i_{j}}, q_{i_{j} \pm 1}\right)$ whose four sides are in four distinct connected components, but this is a contradiction to Lemma 7 Thus, the graph has no two disjoint edges, and such graphs are easily classified: they are the triangle (cycle on three vertices) and the star graphs $K_{1 t}$, possibly with multiple edges. Overall, the graph involves at least $C_{2} \geq 4$ distinct connected components $\operatorname{CC}\left(q_{i_{j}}\right)$, and therefore the triangle graph is excluded. Let $v$ be the central vertex of the star. There can be at most one $j$ with $\operatorname{CC}\left(q_{i_{j}}\right)=v$, and we discard it. All other sides $q_{i_{j}}$ have $\operatorname{CC}\left(q_{i_{j}}\right) \neq v$, and therefore $\mathrm{CC}\left(q_{i_{j} \pm 1}\right)$ must be the other endpoint of the edge, that is, $v$.

In summary, we have found $C_{2}-1$ adjacent pairs $q_{i_{j}}, q_{i_{j} \pm 1}$ with the following properties.

- The primary sides $q_{i_{j}}$ belong to $C_{2}-1$ distinct components.
- All companion sides $q_{i_{j} \pm 1}$ belong to the same component, distinct from the other $C_{2}-1$ components.
- All $2 C_{2}-2$ sides of the pairs $q_{i_{j}}, q_{i_{j} \pm 1}$ are distinct.
- Each $q_{i_{j}}, q_{i_{j} \pm 1}$ is hooking with respect to an associated pair $p_{k_{j}}, p_{k_{j} \pm 1}$.
- All $2 C_{2}-2$ sides of the pairs $p_{k_{j}}, p_{k_{j} \pm 1}$ are distinct.

Let us denote by $Q_{2}^{\prime}$ the set of $C_{2}-1$ sides $q_{i_{j}}$.

- Proposition 14. There are no six distinct sides $q_{a}, q_{b}, q_{c}, q_{d}, q_{e}, q_{f}$ among the $C_{2}-1$ sides $q_{i_{j}} \in Q_{2}^{\prime}$ such that $q_{a}, q_{b}$ are avoiding or consecutive, $q_{c}, q_{d}$ are avoiding or consecutive, and $q_{e}, q_{f}$ are avoiding or consecutive.

Proof. Suppose for contradiction that there are six such sides. It follows from Lemma 5 that there are two consecutive sides $p_{a^{\prime}}$ and $p_{b^{\prime}}$ of $P$ such that $\mathrm{CC}\left(p_{a^{\prime}}\right)=\mathrm{CC}\left(q_{a}\right)$ and $\mathrm{CC}\left(p_{b^{\prime}}\right)=\mathrm{CC}\left(q_{b}\right)$.

Similarly, we find a pair of consecutive sides $p_{c^{\prime}}$ and $p_{d^{\prime}}$ of $P$ such that $\mathrm{CC}\left(p_{c^{\prime}}\right)=\mathrm{CC}\left(q_{c}\right)$ and $\operatorname{CC}\left(p_{d^{\prime}}\right)=\mathrm{CC}\left(q_{d}\right)$, and the same story for $e$ and $f$. By the pigeonhole principle, two of the three consecutive pairs $\left(p_{a^{\prime}}, p_{b^{\prime}}\right),\left(p_{c^{\prime}}, p_{d^{\prime}}\right),\left(p_{e^{\prime}}, p_{f^{\prime}}\right)$ are hooking or two of them are hooked. This contradicts Lemma 7

Define a complete graph whose nodes are the $C_{2}-1$ sides $q_{i_{j}} \in Q_{2}^{\prime}$, and color an edge $\left(q_{i_{j}}, q_{i_{k}}\right)$ red if $q_{i_{j}}$ and $q_{i_{k}}$ are avoiding or consecutive and blue otherwise. Proposition 14 says that this graph contains no red matching of size three. This means that we can get rid of all red edges by removing at most 4 nodes. To see this, pick any red edge and remove its two nodes from the graph. If any red edge remains, remove its two nodes. Then all red edges are gone, because otherwise we would find a matching with three red edges.

We conclude that there is a blue clique of size $C_{3}=C_{2}-5$, i.e., there is a set $Q_{3} \subset Q_{2}^{\prime}$ of $C_{3}$ polygon sides among the $C_{2}-1$ sides $q_{i_{j}} \in Q_{2}^{\prime}$ that are pairwise non-avoiding and disjoint, i.e., they do not share a common endpoint.

Our next goal is to find a subset of 7 segments in $Q_{3}$ that are arranged as in Figure 15 To define this precisely, we say for two segments $q$ and $q^{\prime}$ that $q$ stabs $q^{\prime}$ if $I\left(q, q^{\prime}\right) \in q^{\prime}$. Among any two non-avoiding and non-consecutive sides $q$ and $q^{\prime}$, either $q$ stabs $q^{\prime}$ or $q^{\prime}$ stabs $q$, but not both. Define a tournament $T$ whose nodes are the $C_{3}$ sides $q_{i_{j}} \in Q_{3}$, and the arc between each pair of nodes is oriented towards the stabbed side. It follows from Lemma 10 that $T$ has a transitive sub-tournament of size $1+\left\lfloor\log _{2} C_{3}\right\rfloor=C_{4}$.

Furthermore, since $C_{4}=\operatorname{ES}\left(C_{5}, C_{5}\right)+1$, it follows from Theorem 12 that there is a subset of $C_{5}$ sides such that the lines through them form a $C_{5}$-cup or a $C_{5}$-cap. By a vertical reflection if needed, we may assume that they form a $C_{5}$-cup.

We now reorder these $C_{5}$ sides $q_{i_{j}}$ of $Q$ in stabbing order, according to the transitive subtournament mentioned above. By the Erdős-Szekeres Theorem on monotone subsequences (Theorem 13), there is a subsequence of size $C_{6}+1=\sqrt{C_{5}-1}+1=7$ such that their slopes form a monotone sequence. By a horizontal reflection if needed, we may assume that they have decreasing slopes.

We rename these 7 segments to $a_{0}, a_{1}, \ldots, a_{6}$, and we denote the line $\ell\left(a_{i}\right)$ by $\ell_{i}$, see Figure 15. We have achieved the following properties:

- The lines $\ell_{0}, \ldots, \ell_{6}$ form a 7 -cup, with decreasing slopes in this order.
- The segments $a_{i}$ are pairwise disjoint and non-avoiding.
- $a_{i}$ stabs $a_{j}$ for every $i<j$.

These properties allow $a_{0}$ to lie between any two consecutive intersections on $\ell_{0}$. There is no such flexibility for the other sides: Every side $a_{j}$ is stabbed by every preceding side $a_{i}$. For $1 \leq i<j, a_{i}$ cannot stab $a_{j}$ from the right, because then $a_{0}$ would not be able to stab $a_{i}$. Hence, the arrangement of the sides $a_{1}, \ldots, a_{6}$ must be exactly as shown in Figure 15, in the sense that the order of endpoints and intersection points along each line $\ell_{i}$ is fixed. We will ignore $a_{0}$ from now on.

### 8.2 Finalizing the Analysis

Recall that every $a_{i}$ is the primary side of two consecutive sides $a_{i}, b_{i}$ of $Q$ that are hooking with respect to an associated pair $A_{i}, B_{i}$ of consecutive sides of $P$. The sides $a_{i}$ and $A_{i}$ are the primary sides and $b_{i}$ and $B_{i}$ are the companion sides. All these $4 \times 6$ sides are distinct,


Figure 15 The seven sides $a_{0}, a_{1}, \ldots, a_{6}$. The lines $\ell_{0}, \ldots, \ell_{6}$ form a 7 -cup.
and they intersect as follows: $a_{i}$ intersects $B_{i}$ and is disjoint from $A_{i} ; b_{i}$ intersects $A_{i}$ and is disjoint from $B_{i}$; and $I\left(A_{i}, B_{i}\right) \in \operatorname{Cone}\left(a_{i}, b_{i}\right)$.

Figure 16 summarizes the intersection pattern among these sides. A side $A_{i}$ must intersect every side $a_{j}$ with $j \neq i$ and every side $b_{j}$ since $\operatorname{CC}\left(A_{i}\right)=\operatorname{CC}\left(a_{i}\right) \neq \mathrm{CC}\left(a_{j}\right)$ and $\mathrm{CC}\left(A_{i}\right)=\mathrm{CC}\left(a_{i}\right) \neq \mathrm{CC}\left(b_{i}\right)=\mathrm{CC}\left(b_{j}\right)$. (Recall that all companion sides $b_{i}$ belong to the same component.) Similarly, every side $B_{i}$ must intersect every side $a_{j}$. We have no information about the intersection between $B_{i}$ and $b_{j}$, as these sides belong to the same connected component.


Figure 16 The subgraph of $G^{\mathrm{D}}$ induced on two pairs of consecutive sides $a_{i}, b_{i}$ and $a_{j}, b_{j}$ of $P$ and their associated partner pairs $A_{i}, B_{i}$ and $A_{j}, B_{j}$ of $Q$. Parts of $P$ and $Q$ are shown to indicate consecutive sides. The dashed edges may or may not be present.

We will now derive a contradiction through a series of case distinctions.

Case 1: There are three segments $A_{i}$ with the property that $A_{i}$ crosses $\ell_{i}$ to the left of $a_{i}$. Without loss of generality, assume that these segments are $A_{1}, A_{2}, A_{3}$, see Figure 17 . The segments $A_{1}, A_{2}, A_{3}$ must not cross because $P$ is a simple polygon. Therefore $A_{1}$ intersects $a_{2}$ to the right of $I\left(a_{1}, a_{2}\right)$ because otherwise $A_{1}$ would cross $A_{2}$ on the way between its intersections with $\ell_{2}$ and with $a_{1} . A_{3}$ must cross $\ell_{3}, a_{2}, a_{1}$ in this order, as shown. But then $A_{1}$ and $A_{3}$ (and $a_{2}$ ) block $A_{2}$ from intersecting $a_{3}$.


Figure 17 The assumed intersection points between $A_{i}$ and $\ell_{i}$ are marked.

Case 2: There at most two segments $A_{i}$ with the property that $A_{i}$ crosses $\ell_{i}$ to the left of $a_{i}$. In this case, we simply discard these segments. We select four of the remaining segments and renumber them from 1 to 4 .

From now on, we can make the following assumption:
General Assumption: For every $1 \leq i \leq 4$, the segment $A_{i}$ does not cross $\ell_{i}$ at all, or it crosses $\ell_{i}$ to the right of $a_{i}$.

This implies that $A_{3}$ must intersect the sides $a_{2}, a_{1}, a_{4}$ in this order, and it is determined in which cell of the arrangement of the lines $\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4}$ the left endpoint of $A_{3}$ lies (see Figures 15 and 18 . For the right endpoint, we have a choice of two cells, depending on whether $A_{3}$ intersects $\ell_{3}$ or not.

We denote by left(s) and right $(s)$ the left and right endpoints of a segment $s$. We distinguish four cases, based on whether the common endpoint of $A_{3}$ and $B_{3}$ lies at $\operatorname{left}\left(A_{3}\right)$ or $\operatorname{right}\left(A_{3}\right)$, and whether the common endpoint of $a_{3}$ and $b_{3}$ lies at left $\left(a_{3}\right)$ or right $\left(a_{3}\right)$.

Case 2.1: $I\left(A_{3}, B_{3}\right)=\operatorname{left}\left(A_{3}\right)$ and $I\left(a_{3}, b_{3}\right)=\operatorname{right}\left(a_{3}\right)$, see Figure 18
As indicated in the figure, we leave it open whether and where $A_{3}$ intersects $\ell_{3}$. We know that $b_{3}$ must lie below $\ell_{3}$ because $I\left(A_{3}, B_{3}\right) \in \operatorname{Cone}\left(a_{3}, b_{3}\right)$.

We claim that $A_{2}$ cannot have the required intersections with $a_{1}, a_{3}$, and $b_{3}$. Let us first consider $a_{1}$ : It is cut into three pieces by $A_{3}$ and $B_{3}$.

If $A_{2}$ intersects the middle piece of $a_{1}$ in the wedge between $A_{3}$ and $B_{3}$, then $A_{2}$ intersects exactly one of $a_{3}$ and $b_{3}$ inside the wedge, because these parts together with $a_{1}$ are three sides of a convex pentagon. If $A_{2}$ intersects $a_{3}$, then it has crossed $\ell_{3}$ and it cannot cross $b_{3}$ thereafter. If $A_{2}$ intersects $b_{3}$, it must cross $\ell_{4}$ before leaving the wedge, and then it cannot $\operatorname{cross} a_{3}$ thereafter.

Suppose now that $A_{2}$ crosses the bottom piece of $a_{1}$. Then it cannot go around $A_{3}, B_{3}$ to the right in order to reach $a_{3}$ because it would have to intersect $\ell_{4}$ twice. $A_{2}$ also cannot pass to the left of $A_{3}, B_{3}$ because it cannot cross $\ell_{2}$ through $a_{2}$ or, by the general assumption, to the left of $a_{2}$.

Suppose finally that $A_{2}$ crosses the top piece of $a_{1}$. Then it would have to cross $\ell_{3}$ twice before reaching $b_{3}$.


Figure 18 Case 2.1, $I\left(A_{3}, B_{3}\right)=\operatorname{left}\left(A_{3}\right)$ and $I\left(a_{3}, b_{3}\right)=\operatorname{right}\left(a_{3}\right)$. A hypothetical segment $A_{2}$ is shown as a dashed curve. The side $a_{2}$ and the part of $\ell_{2}$ to the left of $a_{2}$ is blocked for $A_{2}$.

Case 2.2: $\quad I\left(A_{3}, B_{3}\right)=\operatorname{left}\left(A_{3}\right)$ and $I\left(a_{3}, b_{3}\right)=\operatorname{left}\left(a_{3}\right)$.
If $\ell\left(A_{3}\right)$ does not intersect $a_{3}$, we derive a contradiction as follows, see Figure 19 . We know that the sides $a_{2}, a_{3}, a_{4}$ must be arranged as shown. The segment $A_{3}$ crosses $a_{2}$ but not $a_{3}$. Now, the parts of $a_{3}$ and $A_{3}$ to the left of $\ell_{2}$ form two opposite sides of a quadrilateral, as shown in the figure. If this quadrilateral were not convex, then either $\ell\left(A_{3}\right)$ would intersect $a_{3}$, which we have excluded by assumption, or $\ell_{3}$ would intersect $A_{3}$ left of $a_{3}$, contradicting the General Assumption. Thus, the sides $a_{3}$ and $A_{3}$ violate the Axis Property (Observation 6), which requires $a_{3}$ and $A_{3}$ to lie on different sides of the line through $I\left(A_{3}, B_{3}\right)$ and $I\left(a_{3}, b_{3}\right)$.

Looking back at this proof, we have seen that the configuration of the segments $a_{1}, a_{2}, a_{3}, a_{4}$ according to Figure 15 in connection with the particular case assumptions make the situation sufficiently constrained that the case can be dismissed by looking at the drawing. The treatment of the other cases will be proofs by picture in a similar way, but we will not always spell out the arguments in such detail.


Figure 19 Case 2.2. $I\left(A_{3}, B_{3}\right)=\operatorname{left}\left(A_{3}\right)$, $I\left(a_{3}, b_{3}\right)=\operatorname{left}\left(a_{3}\right), \ell\left(A_{3}\right)$ does not intersect $a_{3}$.


Figure 20 Case 2.3. $I\left(A_{3}, B_{3}\right)=\operatorname{right}\left(A_{3}\right)$, and $I\left(a_{3}, b_{3}\right)=\operatorname{right}\left(a_{3}\right), A_{3}$ lies below $\ell_{3}$.

If $\ell\left(A_{3}\right)$ intersects $a_{3}$, the situation must be as shown in Figure 21 the pair $A_{3}, B_{3}$ is hooked by $a_{3}$ and $b_{3}$. The analysis of Case 2.1 (Figure 18) applies verbatim, except that the word "pentagon" must be replaced by "hexagon".


Figure 21 Case $2.2, I\left(A_{3}, B_{3}\right)=\operatorname{left}\left(A_{3}\right), I\left(a_{3}, b_{3}\right)=\operatorname{left}\left(a_{3}\right)$, and $\ell\left(A_{3}\right)$ intersects $A_{3}$. A hypothetical segment $A_{2}$ is shown as a dashed curve.

Case 2.3: $I\left(A_{3}, B_{3}\right)=\operatorname{right}\left(A_{3}\right)$, and $I\left(a_{3}, b_{3}\right)=\operatorname{right}\left(a_{3}\right)$.
If $A_{3}$ lies entirely below $\ell_{3}$, then $A_{3}$ together with $a_{3}$ violates the Axis Property (Observation 6), see Figure 20


Figure 22 Case 2.3. $A_{3}$ intersects $\ell_{3}$.

Let us therefore assume that $A_{3}$ intersects $\ell_{3}$ (to the right of $a_{3}$ ), and thus right $\left(A_{3}\right)=$ $I\left(A_{3}, B_{3}\right)$ lies above $\ell_{3}$, see Figure 22 a . Then $b_{3}$ must also lie above $\ell_{3}$, because $a_{3}, b_{3}$ is supposed to be hooking, that is, $I\left(A_{3}, B_{3}\right) \in \operatorname{Cone}\left(a_{3}, b_{3}\right)$.

It follows that $A_{3}$ cannot intersect $\ell_{3}$ to the right of $I\left(a_{3}, a_{4}\right)$ (the option shown as a dashed curve), because otherwise it would miss $b_{3}: b_{3}$ is blocked by $a_{4}$.

Therefore, the situation looks as shown in Figure 22 a . Figure 22 shows the position of the relevant pieces. The segments $a_{4}, B_{3}, a_{3}, b_{3}, A_{3}$ enclose a convex pentagon. Now, the segment $A_{2}$ should intersect $a_{3}, b_{3}$, and $a_{4}$ without crossing $A_{3}$ and $B_{3}$, like the dashed curve in the figure. This is impossible.

Case 2.4: $I\left(A_{3}, B_{3}\right)=\operatorname{right}\left(A_{3}\right)$ and $I\left(a_{3}, b_{3}\right)=\operatorname{left}\left(a_{3}\right)$.
If $A_{3}$ intersects $\ell_{3}$ (to the right of $a_{3}$ ), then $A_{3}$ together with $a_{3}$ violates the Axis Property (Observation 6), see Figure 23 We thus assume that $A_{3}$ lies entirely below $\ell_{3}$.


Figure 23 Case 2.4. $A_{3}$ intersects $\ell_{3}$.


Figure 24 Case 2.4. $A_{3}$ lies below $\ell_{3}$.
If $\ell\left(A_{3}\right)$ passes above $I\left(a_{3}, b_{3}\right)=\operatorname{left}\left(a_{3}\right)$, the sides $a_{3}$ and $A_{3}$ violate the Axis Property see Figure 24 a . On the other hand, if $\ell\left(A_{3}\right)$ passes below $I\left(a_{3}, b_{3}\right)=\operatorname{left}\left(a_{3}\right)$, as shown in Figure 24 b , then $b_{3}$ must cross $\ell_{1}$ to the right of $a_{1}$ in order to reach $A_{2}$. Again by the Axis Property, $B_{3}$ must remain above the dotted axis line through $I\left(A_{3}, B_{3}\right)=\operatorname{right}\left(A_{3}\right)$ and $I\left(a_{3}, b_{3}\right)=\operatorname{left}\left(a_{3}\right)$. On $\ell_{1}, b_{3}$ separates $a_{1}$ from the axis line, and hence $a_{1}$ lies below the axis line. Therefore $B_{3}$ and $a_{1}$ cannot intersect.

This concludes the proof of Theorem 1.

Acknowledgement. We thank the reviewers for helpful suggestions.

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