

Monetary and Fiscal Regimes Policy Rules in a Discrete Time Model

(Peraturan Dasar Rejim Monetari dan Fiskal dalam Model Masa Diskret)

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ABSTRACT

This paper determines how fiscal policy rule interacts with monetary policy rule affect the conditions of equilibrium determinacy when moving from continuous to discrete time. The monetary authority follows an interest rate-targeting rule while the fiscal authority follows a debt-targeting rule. It is shown that the local determinacy of an equilibrium path is determinate under the active monetary/passive fiscal regime, while the examination of other regimes is shown to be indeterminate. These findings are in stark contrast with the case of the continuous time model, suggesting that the timing assumptions play an important role in determining local equilibrium.

Keywords: Monetary policy; fiscal policy; equilibrium indeterminacy

ABSTRAK

Artikel ini menentukan bagaimana peraturan dasar fiskal berinteraksi dengan peraturan dasar kewangan mempengaruhi ketetapan keseimbangan apabila bertukar dari model penetapan masa berterusan kepada model masa diskret. Pembuat dasar monetari mengikuti peraturan kadar faedah manakala pembuat dasar fiskal mengikuti peraturan sasaran hutang. Hasil kajian menunjukkan bahawa penetapan keseimbangan setempat dapat ditentukan di bawah rejim aktif monetari / pasif fiskal, sementara rejim-rejim lain menunjukkan tiada penetapan keseimbangan. Penemuan ini sangat berbeza dengan kes model masa yang berterusan, menunjukkan bahawa model penentuan masa memainkan peranan penting dalam menentukan keseimbangan setempat.

Kata kunci: Dasar monetari; dasar fiskal; penentuan keseimbangan

INTRODUCTION

The issue of local determinacy or local uniqueness of equilibrium has a considerable influence on policy makers as they can then respond to the impact of policy changes. According to Woodford (2001), the presence of indeterminacy is undesirable not only because it permits the existence of non-fundamental shocks but also because it amplifies the persistence and volatility of the equilibrium paths of inflation, interest rates and output in response to fundamental shocks. This also implies that a policy that could produce determinacy of equilibrium is desired as this enables the policy makers to respond immediately by changing their policy with respect to target variables.

Although there has been a considerable amount of research studying the possibility of equilibrium determinacy, most of the existing literatures have made some notable simplification such as the assumption of a fixed regime policy, the lack of monetary-fiscal interaction and the absence of capital accumulation. The characterisation of regime policy rules, i.e. 'active' and 'passive', are however crucial, as empirical studies have found evidence of regime shifts in which the policy rules vary substantially over different periods. This has been proved by the empirical findings of Woodford (1999),

Clarida et al. (2000) and Favero and Monacelli (2003) for the United States of America's (US) and Khalid and Marwan (2013) for Malaysia, Singapore and Thailand as well as Khalid et al. (2014) and Khalid et al. (2018) for Malaysia's economy. Overall, they find that monetary and fiscal policy reaction functions always fluctuate between active and passive depending on the economic cycles and shocks. Furthermore, most of the existing studies on equilibrium determinacy ignore the interaction between monetary and fiscal policies.

Studies that examined implication on the equilibrium determinacy for monetary policy includes Meng and Yip (2004), Carlstrom and Fuerst (2005), Fujisaki (2008) and Paul Kitney (2018) for Taylor-type monetary policy rules, Airaudo & Zanna (2012) and Airaudo et. al (2015) and Svensson and Woodford (2014) for inflation targeting, Carlstrom and Fuerst (2003) for money growth targeting and Kareken and Wallace (1981) and Hagedorn (2018) for analysis of monetary policy in open economies and used exchange rate targeting. Most of these studies concluded that macroeconomic fluctuations driven by self-fulfilling expectations may results in high risk of indeterminacy. In addition, following a rule in which the central bank responds to endogenous variables may introduce real indeterminacy. Policy rules may renders equilibrium determinate if the central bank's



commitment to it policy can be made credible to private sector.

Although there was no fiscal policy in the model, they have explicitly assumed a passive fiscal policy. However, the choice of a monetary policy in ensuring equilibrium uniqueness is also related to the decision or the choice of a fiscal policy. Sargent and Wallace (1981) in their *Unpleasant Monetarist Arithmetic* are the first example, which studied the intertemporal relationship between policy instruments. They emphasize the role of a fiscal policy for the determination of the inflationary consequences of monetary policies. This is also consistent with the Fiscal Theory of the Price Level highlighted by Leeper (1991), Sims (1994) and Woodford (2001). The theory says that for the price level to be stable or to control inflation, the fiscal authority must make sure there is a balanced budget over the business cycle and that it is sustainable.

The past literatures using a New Keynesian model tend to relax the assumption of capital accumulation or investment in their model. Adding capital however, has important implications on the determinacy of equilibrium since it adds another jump variable (in the language of Blanchard & Kahn 1980) and this will affect the conditions of the determinacy equilibrium. This has been proved by Dupor (2001) and Carlstrom and Fuerst (2005) in a standard New Keynesian model which shows that their findings change dramatically when the investment spending is included in the model. Carlstrom and Fuerst (2005) analyse the condition of equilibrium determinacy under an interest rate policy with endogenous capital accumulation and partial nominal price adjustment in a discrete time. Basically, they compare the outcome of equilibrium determinacy by Dupor (2001) with continuous time and found that by changing to a discrete time model, this timing assumption has an important implication on equilibrium determinacy. Dupor (2001) finds that a passive rule is a necessary and sufficient condition for local equilibrium determinacy. In contrast, Carlstrom and Fuerst (2005) find that the monetary policy should react aggressively to current movements in inflation for a sufficient condition of local determinacy. The key reason why they produce different results is attributable to the Euler equations for investment. In continuous time, the marginal productivity of capital today must equal the interest rate. In discrete time however, the marginal productivity of capital for the next period must equal the interest rate. Since capital is predetermined, there is an extra-predetermined variable in the continuous time model that does not appear in the discrete time model. Both papers give emphasis to endogenous capital in their modelling. This implies no possibility of arbitrage between the expected real return on bonds and expected real return on capital. However, clearly both papers do not include a fiscal policy in their analysis while in this paper we investigate both monetary and fiscal policies. Most recently, Tsuzuki (2014;

2015) and Tsuzuki (2016) examine the effect of policy lag on determinacy of equilibrium under continuous time framework for monetary and fiscal respectively. The results show that the length of policy lag play an important role in determining equilibrium determinacy. Fiscal authorities seem to take a longer time than monetary authorities to implement policy and therefore suggesting that one way to ensure the effectiveness of policy aimed at stabilization is to adjust the timing of its implementation.

Motivated by these shortcomings, the current paper addresses the gap in the literatures by establishing the necessary condition of local equilibrium determinacy under a discrete time model for both monetary and fiscal regime policy rules. It is assumed that the policy makers commit to simple feedback rules. The monetary authority follows an interest rate-targeting rule, i.e. Taylor rule, while the fiscal authority follows a debt-targeting rule. Specifically, this study attempts to determine the conditions of policy rules that need to be satisfied for determinacy of equilibrium when the price rigidity and accumulation of capital are present. Intuitively, in discrete time, we allow a distinction between the marginal productivity of capital for today and the future.

The determination of local equilibrium determinacy here is crucial due to a study by Dupor (2001) in continuous time that found a different result compared with those of Carlstrom and Fuerst (2005) in discrete time. The structure of the model is very close to the idea of Carlstrom and Fuerst (2005) by taking into account capital accumulation and analysing in discrete time. However, Carlstrom and Fuerst (2005) analyse local equilibrium determinacy for the monetary policy only associated with an interest rate rule that follows both forward-looking and current-looking rules. On the other hand, this paper extends the work of Carlstrom and Fuerst (2005) by adding a fiscal policy rule.

The study by Leeper (1991) was the first to characterise different regimes of policy rules and analyse their equilibria and their properties. However, he analyses equilibrium determinacy without having capital accumulation in the model. Following that, these regimes have been discussed from different perspectives and assumptions such as in the Ricardian or non-Ricardian framework (Benassy 2003 ; Leith & Von Thadden 2008), a rational expectation model, a general equilibrium model (Branch et al. 2008), and an overlapping generation model (OGM) (Annicchiarico et al. 2007; Benassy 2003) and whether money and capital should be included or not. Most recently, many researchers have focused on the New Keynesian framework in accessing 'active' vs. 'passive' monetary and fiscal policies with different assumptions and methodology. For instance, Annicchiarico et al. (2007) evaluate the effects of different fiscal policy regimes on the performance of Taylor's interest rate rules in a modified Dynamic New Keynesian macroeconomic model in the presence of wealth effects. Aloui and

Guillard (2008) contribute to Leeper's model by taking into account the wealth effect in a non-Ricardian economy with capital and a zero lower bound interest rate. Aloui and Guillard find that the four types of equilibria share the same properties as the equilibria described by Leeper (1991) for a unique set of policy parameter space. Nevertheless, Leith and Von Thadden (2008) provide a New Keynesian model of Blanchard (1985) by assuming all taxation is a lump sum, a departure from Ricardian equivalence through a change in the probability of the death of consumers and with capital accumulation.

In this paper, we also employ a New Keynesian framework in assessing the necessary conditions for equilibrium determinacy for policy rules. We assume the firms in the intermediate goods sector are producing in a monopolistically competitive market. They produce differentiated intermediate goods, which allow them to have some monopoly power over the price of the goods. These firms are subject to some constraints on the frequency of adjusting their prices of the goods and services. Hence, the idea of sticky price setting is to assert that firms are indeed price-setters, rather than price-takers. Since our main objective is to see how the departures from a continuous to a discrete time model affect the conditions on equilibrium determinacy, we try to be as close as possible to the assumptions used by Carlstrom and Fuerst (2005). Accordingly, we employ Yun (1996)'s sticky price model by log linearizing around zero steady state inflation with capital accumulation and analysing in a discrete time model. We only extend Carlstrom and Fuerst (2005) by introducing a fiscal policy in the model and then examine to what extent the properties of the model economy change. Overall, we can see the contribution of this paper as an extension of that of Leeper (1991) by employing New Keynesian framework with capital accumulation in discrete time.

The structure of this paper is as follows. Section 2 develops the basic model. In section 3, the existence of steady states as well as the equation of local dynamics around steady states will be established. Section 4 concludes and discusses the findings.

THE MODEL

Consider a New Keynesian model where the basic structure is similar to that of Carlstrom and Fuerst (2005). In contrast to Carlstrom and Fuerst (2005), this paper analyses both monetary and fiscal policy regimes using a discrete time model, indexed by $t \equiv 0, 1, 2$. The economy is populated by a large number of households, firms and the government. The government consists of both fiscal and monetary authorities.

Households Consider an economy that is populated by a large number of identical and infinite-life households. All households have the same preferences and choose the level of consumption, C_t , and the level of real

money balances, $\frac{M_{t+1}}{P_t}$. Assume that the labor market is in equilibrium for every period. Each household is endowed with one unit of labour, which is supplied exogenously and earns a wage rate, w_t . In this model, we use the cash-when-I'm-done (CWID) timing in which the end-of-period money balances enter into the utility function. The preferences are given by:

$$E_0 \sum_{i=0}^{\infty} \beta^i U \left[C_t, \frac{M_{t+1}}{P_t} \right] \quad (1)$$

where $\beta \in (0, 1)$ is a discount factor, C_t is the household's consumption in period t , P_t is the price level, and M_{t+1} is the household's nominal money balances at the end of period. The sub-utility function $U(\cdot)$ is:

$$U \left[C_t, \frac{M_{t+1}}{P_t} \right] \equiv \left[\gamma \ln C_t + \phi \ln \frac{M_{t+1}}{P_t} \right] \text{ for } \phi > 0 \quad (2)$$

At the beginning of period t , the consumer owns predetermined levels of the aggregate stock of capital (K_t), real government bonds $\left(\frac{B_t}{P_t} \right)$ and real money balances $\left(\frac{M_{t+1}}{P_{t+1}} \right)$ resulting from the decisions undertaken in period $t-1$. The flow of the budget constraint that incorporates the holding of money and bond explicitly takes the form:

$$\begin{aligned} \frac{M_{t+1}}{P_t} + \frac{B_{t-1}R_{t-1}}{P_{t-1}} + M_t^s(\tilde{g}_t^m - 1) + p_t^k K_t + w_t L_t - \dot{Y}_t \\ + \Pi_t - T_t + \frac{M_{t+1}}{P_t} + \frac{B_t}{P_t} \end{aligned} \quad (3)$$

where R_{t-1} is the gross nominal rate of return on bond holding from $t-1$ to $M_t^s(\tilde{g}_t^m - 1)$ is a lump sum transfer of money from the monetary authority in period t where M_t^s denotes the money supply per capita and \tilde{g}_t^m is the gross money growth rate. L_t is the household's labour supply in period t . Denote \dot{Y}_t as the household's demand for goods in period t , while w_t, p_t^k, Π_t and T_t are real wage, real capital rate, profit flow from firms and taxes, respectively.

Consider the case of endogenous capital accumulation, where both the supply and demand for capital are determined endogenously. Denote that I_t is the household's investment during period and capital is depreciated at rate δ , the capital accumulation is therefore:

$$K_{t+1} = I_t + (1 - \delta)K_t \quad (4)$$

The market clearing condition is given by $Y_t = C_t + I_t + G_t$. By combining this with (4) and substituting into (3), we now have the flow of budget constraint with endogenous capital accumulation as the following:

$$\begin{aligned} \frac{M_{t+1}}{P_t} + \frac{B_{t-1}R_{t-1}}{P_{t-1}} + M_t^s(\tilde{g}_t^m - 1) + p_t^k K_t + w_t L_t - C_t \\ - K_{t+1} + (1 - \delta)K_t - G_t + \Pi_t - T_t + \frac{M_{t+1}}{P_t} + \frac{B_t}{P_t} \end{aligned} \quad (5)$$

At each date, the objective of the household is to maximise (2) subject to (5) with respect to $C_t, \frac{M_{t+1}}{P_t} + \frac{B_t}{P_t}$ and K_{t+1} . The first order conditions (FOCs) for this problem are shown in Appendix A.1. Generally, using the FOCs, we obtain three equations. First is a non-arbitrage condition:

$$R_t = p_{t+1}^k + (1 - \delta) \tag{6}$$

This non-arbitrage condition implies that there are no profit gains between investing in bonds or capital, since the expected real return on bonds is equal to the expected real return on capital. Second, we obtain a Fisher equation:

$$R_t = \frac{C_{t+1}}{\beta C_t} \tag{7}$$

Finally, the asset accumulation margin is

$$\frac{1}{C_t} = \beta \frac{1}{C_{t+1}} [p_{t+1}^k + (1 - \delta)] \tag{8}$$

Firms This study follows Calvo (1983) and Yun’s (1996) sticky price model in order to analyse a model of imperfect competition. Consider two types of firms, an intermediate good firm and a final good firm. The firms are perfectly competitive in the final good sector and a monopolistic competitor (price-setter) in the intermediate good sector. Denote Y_t as the final good and $Y_t(i)$ as the continuum of intermediate firms, each indexed by $i \in [0,1]$. The final good sector produces the final good Y_t according to the constant elasticity of substitution (CES) final good aggregator:

$$Y_t = \int_0^1 [Y_t(i)^{\frac{\varepsilon}{\varepsilon-1}} di]^{\frac{\varepsilon-1}{\varepsilon}} \tag{9}$$

where ε is the elasticity of substitution. A final good firm’s problem is to maximise:

$$Max \{Y_t(i)P_t Y_t - \int_0^1 P_t(i)P_t(i)di \} \tag{10}$$

subject to production technology above where P_t is the nominal price of the final good and $P_t(i)$ is the nominal price of the differentiated good i . By substituting the CES final good aggregator into (10), the problem of the final good firm is to maximise:

$$Max \{Y_t(i)\} P_t \left[\int_0^1 [Y_t(i)^{\frac{\varepsilon}{\varepsilon-1}} di]^{\frac{\varepsilon-1}{\varepsilon}} \right] - \int_0^1 P_t(i)Y_t(i)di \tag{11}$$

Taking a derivative of this function with respect to results in the first order condition which can be simplified to the demand for good:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{\frac{1}{\varepsilon-1}} \tag{12}$$

Substituting this demand for firm i ’s output into the final good aggregator gives the final good price:

$$P_t = \int_0^1 [P_t(i)^{1-\varepsilon} di]^{\frac{1}{1-\varepsilon}} \tag{13}$$

The intermediate goods market is monopolistically competitive where each firm produces differentiated intermediate goods. A monopolist producer of intermediate good for instance, produces intermediate good using a constant returns to scale (CRS) Cobb-Douglas production function denoted by:

$$Y(i)_t = K(i)_t^\alpha L(i)_t^{1-\alpha} \tag{14}$$

where $L(i)_t$ and $K(i)_t$ are labour and capital in the competitive market respectively. Denote w_t and p_t^k as the wage and the rental rate of capital and due to symmetry, the cost minimization in input market implies the optimal combination of labour and capital when the labour supply is exogenous:

$$\frac{K_t(i)}{L_t(i)} = K_t = \frac{\alpha}{1-\alpha} \frac{w_t}{p_t^k} \tag{15}$$

Assume that this p_t^k is identical to the risk-free rate, r_t and satisfies the zero-profit condition, $p_t^k = r_t + \delta$. Due to market power in the intermediate good market, the factor payments are distorted according to:

$$w_t = (1 - \alpha) \left(\frac{K_t}{L_t} \right)^\alpha mc_t \tag{16}$$

and

$$p_t^k = \alpha \left(\frac{K_t}{L_t} \right)^{1-\alpha} mc_t \tag{17}$$

Given that the cost function is linear in output, the is given by:

$$mc_t = (p_t^k)^\alpha w_t^{1-\alpha} \alpha^{-\alpha} (1 - \alpha)^{\alpha-1} \tag{18}$$

Since the market for input factors is competitive, the intermediate good firms take as given and this mc_t is independent of the level of output given the assumption of CRS.

Following Yun (1996), it is assumed that the intermediate good firms choose how much to produce in every period but do not choose the price of their good every period. Therefore, in each period, only a fraction $1 - \theta$ of randomly selected firms are permitted to set their price for period t, P_t^* while the remaining fraction of firms θ must update prices by a stationary state gross inflation rate denoted by π . This type of staggering implies that the price index in each period evolves over time according to the recursive form given by:

$$p_t^{1-\varepsilon} = (1 - \theta)P_t^{*1-\varepsilon} + \theta\pi P_{t-1}^{1-\varepsilon} \tag{19}$$

Denote θ^k as the probability for new price commitment and $\beta \in (0,1)$ as the discount factor, where the optimisation problem is given by:

$$\text{Max} \{P_t(i)\} \left\{ \sum_{k=0}^{\infty} (\theta\beta)^k \left[\left(\frac{\pi_t^k P_t(i)}{P_t} \right)^{1-\varepsilon} Y_t - \left(\frac{\pi_t^k P_t(i)}{P_t} \right)^{-\varepsilon} Y_t m c_t \right] \right\} \quad (20)$$

The differentiated firm i take as given the demand of their output by the final good firm given by (12). By maximising this function as derived in Appendix A.2, one obtains the optimal pricing condition as the following:

$$P_t(i) = \frac{\varepsilon \sum_{k=0}^{\infty} (\alpha\beta)^k P_{t+k}^{\varepsilon} Y_{t+k} m c_{t+k}}{\varepsilon - 1 \sum_{k=0}^{\infty} (\alpha\beta\pi)^k P_{t+k}^{\varepsilon-1} Y_{t+k}} \quad (21)$$

Combining equations (4), (6), (7), (8), (16), and (17) yields:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \left[\alpha \left(\frac{L_{t+1}}{K_{t+1}} \right)^{1-\alpha} m c_{t+1} + (1-\delta) \right] \quad (22)$$

and

$$K_{t+1} = K_t L_t^{1-\alpha} + (1-\delta)K_t - C_t - G \quad (23)$$

Notice that equations (22 and 23) are central to real business cycle conditions distorted by marginal cost and the effect of real money balances on the marginal utility of consumption.

The Government Following Leeper (1991), the monetary and fiscal policy rules will be characterised according to constraints faced by the policy authorities. It is assumed that the monetary and fiscal policies follow two policy rules with a simple feedback structure. Considering the interest rate as a policy rule for the monetary policy, the monetary authority sets the Taylor rule in which the nominal interest rate responds to the inflation rate and the output gap. Given the assumption of nominal stickiness based on Calvo's pricing contract and then generalising the Taylor rule, it is assumed that the monetary agent has, in the short run, leverage over the real interest rate that responds to the deviation of inflation from its long run target (π^*):

$$r_t = r + f^M(\pi_t - \pi^*), \quad \pi^* = 0 \quad (24)$$

where r is the steady state level of the real interest rate and f^M is the Taylor coefficient. Consider an inflation target of zero, $\pi^* = 0$ under an optimal monetary policy rule. Consequently, the monetary policy is called 'active' ('passive') if this Taylor coefficient is larger (smaller) than zero, i.e. if the real interest rate rises (falls) in the inflation rate. Notice that here, (24) has a critical value of the Taylor coefficient equal to zero and not unity as this feedback rule is expressed in terms of real and not nominal values.

Meanwhile, the fiscal policy follows a debt targeting rule which specifies how the fiscal rule, T_t (lump sum tax), reacts to deviations of the actual level of real government debt from a target level of real debt (b^*):

$$T_t = T + f^F(b_t - b^*), \quad G_t = G(\text{fixed}) \quad (25)$$

where f^F is the feedback coefficient of taxes on debt and T is the steady state level of taxes. Since the real government debt is growing at the steady state of the interest rate, in line with Leeper (1991), the fiscal policy is called 'passive' ('active') if this coefficient is larger (smaller) than the steady state of real interest rate. The dynamics of real government debt outstanding is given by:

$$G_t + r_t b_t = T_t + b_{t+1} - b_t \quad (26)$$

where we assume that $G_t = G > 0$ denotes an exogenous and constant stream of government expenditures in terms of the aggregate final output.

STEADY STATE

In this section, we first evaluate the existence of steady states for the model, then the local dynamics around steady states are characterised. Using Uhlig's (1995) method, we log-linearized the summary of equations in terms of deviation from the steady state and then reduced in a system of equations.¹ Finally, the condition on equilibrium determinacy is investigated for monetary dynamics as well as for monetary and fiscal dynamics.

In the steady state, the output when the labour supply is exogenous is given as:

$$Y_{ss} = K_{ss}^{\alpha} \quad (27)$$

where subscripts ss denote the variables in the steady state. Saving, denoted as s , is a constant fraction of output, and since in the steady state, $K_{t+1} = K_t = K_{ss}$, the capital in the steady state is given by:

$$K_{ss} = \left(\frac{s}{\delta} \right)^{\frac{1}{1-\alpha}} \quad (28)$$

Given that the capital accumulation in the steady state is just simply $I_{ss} - \delta K_{ss}$, the steady state of consumption can be found from the resource constraint:

$$C_{ss} = K_{ss}^{\alpha} - \delta K_{ss} - G \quad (29)$$

In the steady state, the rule for determining the final good price becomes:

$$p_{ss}^{1-e} = (1-\theta)p_{ss}^{*1-e} + \theta\pi p_{ss}^{1-e} \quad (30)$$

or

$$P_{ss} = P_{ss}^*(i) = P_{ss}(i) \quad (31)$$

This implies that in the steady state, whatever the rate growth of money, all firms charge the same price as the updating rule coincides with the steady state optimal pricing rule. Thus, in the steady state, there is no issue about relative price since all firms set the same price. Substituting (31) into the demand function for the intermediate good i gives:

$$Y_{ss}(i) = Y_{ss} \left(\frac{P_{ss}}{P_{ss}(i)} \right)^\varepsilon = Y_{ss} \quad (32)$$

The price-setting function for an intermediate good firm i in the steady state becomes:

$$\frac{P_{ss}^*(i)}{P_{ss}} = \frac{\varepsilon}{\varepsilon - 1} mc_{ss} \quad (33)$$

This implies the markup is equal to one over the real marginal cost, as follows:

$$\frac{\varepsilon}{\varepsilon - 1} = \frac{1}{mc_{ss}} \quad (34)$$

Furthermore, the factor payments in terms of the markup under exogenous labour supply:

$$w_{ss} = (1 - \alpha) K_{ss}^\alpha \frac{\varepsilon - 1}{\varepsilon} \quad (35)$$

$$r_{ss} = \alpha K_{ss}^{\alpha-1} \frac{\varepsilon - 1}{\varepsilon} - \delta \quad (36)$$

In the steady state equilibrium with $\pi = 0$ and $G_t = G$, the real government bond is given by:

$$b = \frac{T - G}{r} \quad (37)$$

This implies that the real government bond must be completely backed by the present value of future primary fiscal surpluses.

LOCAL EQUILIBRIUM DYNAMICS

The local dynamics around the steady states can be characterised by a dynamic system in b_t, C_t, K_t and π_t . The first order conditions and resource constraints together with the policy rules characterise a system of non-linear difference equations. In order to analyse the equilibrium dynamics of the model, a first order approximation is taken around a steady state to replace the non-linear equilibrium system with an approximation that is linear. This paper uses a method of log-linearization as proposed by Uhlig (1995), which is an easier and almost automatic way to do linearization without taking the derivatives.³

Using Uhlig's rules for linearization, a system of equations is created, all of which are linear in deviation. The process of log-linearization for dynamic variables b_t, C_t, K_t and π_t are explained in Appendix A.3. The system of equations in the summary of equilibrium conditions will be reduced to a system in b_t, C_t, K_t and π_t . The equilibrium law of motion for each variable is:

$$\hat{b}_{t+1} = (r - f^F + 1)\hat{b}_t + f^F \hat{\pi}_t \quad (38)$$

$$\hat{K}_{t+1} = \left[\alpha \frac{Y_{ss}}{K_{ss}} - \delta + 1 \right] \hat{K}_t - \frac{C_{ss}}{K_{ss}} \hat{C}_t \quad (39)$$

$$\hat{C}_{t+1} = [1 - \beta(1 - \delta) + 1] \frac{f^F}{r + \delta} \pi_t + \hat{C}_t \quad (40)$$

$$\hat{\pi}_{t+1} = \left[\frac{1}{B} - \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} \frac{f^M}{r + \delta} \right] \pi_t - \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} (1 - \alpha) \hat{K}_t \quad (41)$$

The system characterises (b_t, K_t) as the two predetermined variables or state variables, while the other variables (C_t, p_t) are characterised as two forward-looking jump variables. The local equilibrium dynamics will be examined by using the Blanchard and Kahn's conditions (1980). Equations (38) - (41) constitute a system of four linear difference equations and can be represented by the following matrix:

$$\Gamma_0 Z_t = \Gamma_1 Z_{t-1}$$

where $Z_t = [b_t, C_t, K_t, n_t]$. Γ_0 and Γ_1 are 4x4 matrix. If Γ_0 is non-singular, we can rewrite this matrix as:

$$Z_t = \Gamma_0^{-1} \Gamma_1 Z_{t-1}$$

According to the Blanchard and Kahn's conditions (1980), the dynamic behaviour of the system is governed by the eigenvalues of the reduced form coefficient matrix $\Gamma_0^{-1} \Gamma_1$. Indeed, the existence and uniqueness of perfect foresight equilibrium depends on the number of explosive eigenvalues of matrix $\Gamma_0^{-1} \Gamma_1$, that is eigenvalues that are bigger than unity in absolute terms. The general condition for determinacy is that the number of stable eigenvalues must be equal to the number of predetermined variables. In other words, if the number of roots outside the unit circle is equal to the number of forward looking variables, then there exists a unique stable solution or equilibrium path and the system has saddle path stability.

Monetary Dynamics First, consider the determinacy properties for the monetary dynamics. So, monetary dynamics can be readily inferred from the 3x3 sub-system in C, K and π . This sub-system is characterised by one state variable (K) and two forward-looking jump variables (C, π). In this case, since there are two forward-looking equations, the determinacy requires that two roots lie outside the unit circle. Denoting J as the Jacobian matrix of the system, this system of a linear difference equation can be written in matrix form as:

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{C}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix} = J \begin{bmatrix} \hat{K}_t \\ \hat{C}_t \\ \hat{\pi}_t \end{bmatrix}$$

where

$$J = \begin{bmatrix} \alpha \frac{Y_{ss}}{K_{ss}} - \delta + 1 & -\frac{C_{ss}}{K_{ss}} & 0 \\ 0 & 1 & [1 - \beta(1 - \delta)] \frac{f^M}{r + \delta} \\ -\frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} (1 - \alpha) & 0 & \frac{1}{\beta} - \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} \frac{f^M}{r + \delta} \end{bmatrix}$$

Proposition 4.1. Consider the dynamics sub-system in C , K and π implied by matrix in (42). Since this sub-system is characterised by a one state variable (K) and two forward-looking variables, the determinacy requires that the two roots lie outside the unit circle, then a unique stable solution exists.

1. Assume that the monetary policy is passive (PM) ($f^M < 0$). Then the dynamics are indeterminate.
2. Assume that the monetary policy is active (AM) ($f^M > 0$). Then the dynamics are determinate if and only if it satisfies two necessary conditions which are that $f^M > 0$ and that f^M has to be in one of the following two regions:

$$\begin{aligned} & \frac{\left(2 + A + \frac{1}{\beta}\right)(r + \delta)}{N} < f^M \\ & < \min \left\{ \frac{A}{\beta} \frac{r + \delta}{N[A - Z(1 - \alpha)]}, \frac{F}{N} \frac{r + \delta}{2(1 + A) - Z(1 - \alpha)} \right\} \\ & 0 < f^M < \frac{\left(A + \frac{1}{\beta}\right)(r + \delta)}{N} \end{aligned} \quad (43)$$

$$A = \alpha \frac{Y_{ss}}{K_{ss}} - \delta + 1$$

$$N = \frac{(1 - \theta)(a - \theta\beta)}{\theta\beta}$$

$$F = \left[2 + 2A + \left(1 + \frac{1}{\beta}\right) \right]$$

$$Z = \frac{C_{ss}}{K_{ss}} [1 - \beta(1 - \delta)]$$

(See Appendix A.4 for the proof of proposition)

Monetary And Fiscal Dynamics Now, we consider the determinacy properties for both monetary and fiscal dynamics. By adding the fiscal policy, we now add another state variable, b , and an additional eigenvalue $\lambda = r - f^F + 1$. As discussed before, this fiscal feedback coefficient is ‘active’ if $f^F < r$ and ‘passive’ if $f^F > r$. This implies that the eigenvalue associated with $\lambda = r - f^F + 1$ is negative if the fiscal policy is ‘passive’ and it is positive if the fiscal policy is ‘active’. The 4x4 sub-system is now characterised by two state variables (b, K) and two forward-looking variables (C, π). In this case, since there are two forward-looking equations, the determinacy requires that two roots lie outside the unit circle.

$$\begin{bmatrix} \hat{b}_{t+1} \\ \hat{C}_{t+1} \\ \hat{K}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix} = J \begin{bmatrix} \hat{b}_t \\ \hat{C}_t \\ \hat{K}_t \\ \hat{\pi}_t \end{bmatrix}$$

where

$$J = \begin{bmatrix} r - f^F + 1 & 0 & 0 \\ \alpha \frac{Y_{ss}}{K_{ss}} - \delta + 1 & -\frac{C_{ss}}{K_{ss}} \\ 0 & 0 & 1 \\ 0 & -\frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta}(1 - \alpha) & 0 \\ & & f^M \\ & & 0 \\ & & [1 - \beta(1 - \delta)] \frac{f^M}{r + \delta} \\ & & \frac{1}{\beta} - \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} \frac{f^M}{r + \delta} \end{bmatrix} \quad (44)$$

Proposition 4.2. Consider the dynamic sub-system in C , K and π implied by matrix in equation (44) (see Appendix A.4).

1. Assume that both monetary and fiscal policies are active (AM/AF). Then, the dynamics have too many unstable roots and thus equilibrium paths are explosive and there is no solution.
2. Assume that the monetary policy is active and the fiscal policy is passive (AM/PF). Then, the dynamics are determinate and thus the equilibrium path has a unique solution.
3. Assume that the monetary policy is passive and the fiscal policy is active (PM/AF). Then, the dynamics are indeterminate and thus equilibrium paths have multiple solutions.
4. Assume that both monetary and fiscal policies are passive (PM/PF). Then, the dynamics are indeterminate and thus equilibrium paths have multiple solutions.

CONCLUSIONS AND POLICY IMPLICATIONS

The issue of time setting in determining the existence of equilibrium determinacy was first explored by Dupor (2001) and Carlstrom and Fuerst (2005) in continuous and discrete time respectively. Both models consider the Taylor rule with capital accumulation for the determination of equilibrium determinacy. In continuous time, Dupor (2001) finds that an active monetary policy leads to indeterminacy while a passive monetary policy is needed to ensure determinacy. These findings are however, contradictory to the findings by Carlstrom and Fuerst (2005) in discrete time. According to Carlstrom and Fuerst (2005), the monetary authority must be active by raising the interest rate instrument more than one-for-one with increases in inflation. On the other hand,

a passive monetary policy only leads to indeterminacy. The main reason for the difference in their results is due to the existence of capital accumulation as highlighted in Carlstrom and Fuerst (2005).

Given these results, this paper takes one further step by adding fiscal policy rules and analyses the effect on the equilibrium determinacy. Basically, this study takes most of the assumptions used by Carlstrom and Fuerst (2005) such as the New Keynesian framework, capital accumulation and discrete time model. Thus, simply by adding the fiscal policy rule, this paper attempts to see what the differences are in terms of model prediction on equilibrium determinacy. To characterise policy regimes, we assume that the monetary authority focuses on achieving the specific interest rate-targeting rule where the nominal interest rate responds to the inflation rate. Also, the fiscal authority is assumed to follow a debt-targeting rule in which it specifies how the lump sum tax reacts to the deviations of the actual level of real government debt from a target level of debt. In this paper, we have determined the conditions of policy rules that must be held in order to ensure equilibrium determinacy using the Blanchard and Kahn's conditions.

This paper showed that first, under the monetary dynamics, an active monetary policy rule is a necessary condition for determinacy. These findings imply that in an economy that assumes capital matters for determinacy through production cost and firms' pricing behaviour and thus the New Keynesian Phillips curve, a Taylor principle must be active to produce determinacy. However, this result is contradict to the results obtained in continuous time model, such as done by Dupor (2001) where a passive monetary policy is a necessary condition for determinacy. The key difference between a discrete and a continuous time with endogenous capital accumulation is the fact that there is no arbitrage relationship between bonds and capital. In continuous time, the marginal productivity of capital today must be equal to the interest rate while in discrete time, the marginal productivity of capital next period must equal to the interest rate. Since capital is predetermined, there is an extra-predetermined variable in the continuous time model that does not appear in the discrete time model and this in turn affects the local determinacy properties.

Second, for the case of monetary and fiscal dynamics, the local determinacy of the equilibrium path is determinate and has a unique solution if the monetary policy is active and the fiscal policy is passive (AM/PF), while the examination of other regimes show no solution (AM/AF) and indeterminacy (PM/AF and PM/PF) results. This implies an active monetary policy that is designed to stabilise aggregate demands and inflation requires the fiscal policy to be passive and adjusts taxes in response to debt. Our findings suggest that the timing assumptions i.e., discrete or continuous time play an important role in determining the local equilibrium under different policy regimes.

However, the results for monetary dynamics here is limited for policy rule that follows interest rate targeting or so called Taylor rule. Certainly, different types of policy rules may lead to different in(determinacy) solutions as discussed in the Introduction Section. Policy rules may renders equilibrium determinate if the central bank's commitment to it policy can be made credible to private sector.

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NOTES

- ¹ In the steady state condition, all the variables are constant over time, or by the definition $\bar{X} = \dot{X} = X = 0$.
- ² Basically, the idea is to replace each variable by the product of its steady state level and the deviation from it. For instance, one can write the original variable X_t as $X_t = X_{ss}e^{\hat{x}_t} \simeq X_{ss}(1 + \hat{x}_t)$ where \hat{x}_t is the deviation from steady state, X_{ss} .

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APPENDIX

A.1: HOUSEHOLD MAXIMISATION PROBLEM

At each date, the objective of the household is to maximise its utility (2) subject to (5) with respect to C_t ,

$\frac{M_{t+1}}{P_t}$, $\frac{B_t}{P_t}$ and K_{t+1} . The Lagrangian is given by:

$$\mathcal{L} = \sum_{i=0}^{\infty} \beta_i \left[\gamma \ln C_t + \phi \ln \frac{M_{t+1}}{P_t} \right] + \lambda_t$$

$$\frac{M_t}{P_{t-1}} + \frac{B_{t-1}R_{t-1}}{P_{t-1}} + M_t^s(g_t^m - 1) + p_t^k K_t + w_t L_t - C_t - K_{t+1}$$

$$+ (1 - \delta)K_t - g + \pi_t - T_t = \frac{M_{t+1}}{P_t}, \frac{B_t}{P_t} \quad (45)$$

The first order necessary conditions for this Lagrangian

with respect to C_t , $\frac{M_{t+1}}{P_t}$, $\frac{B_t}{P_t}$ and K_{t+1} are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 = \frac{\gamma}{C_t} - \gamma_t \Rightarrow \frac{\gamma}{C_t} = \gamma_t \quad (46)$$

$$\frac{\partial \mathcal{L}}{\partial \frac{M_{t+1}}{P_t}} = 0 = \phi \left(\frac{P_t}{M_{t+1}} \right)^{-1} + \beta \lambda_t - \lambda_t \Rightarrow \lambda_t = \phi \left(\frac{P_t}{M_{t+1}} \right)^{-1} + \beta \lambda_{t+1} \quad (47)$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = \beta \lambda_{t+1} R_t - \lambda_t = 0 \Rightarrow R_t = \frac{\lambda_t}{\beta \lambda_{t+1}} \quad (48)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K_{t+1}} &= \beta \lambda_{t+1} [p_{t+1}^k + (1 - \delta)] - \lambda_t = 0 \Rightarrow p_{t+1}^k + (1 - \delta) \\ &= \frac{\lambda_t}{\beta \lambda_{t+1}} \end{aligned} \quad (49)$$

From (48) and (49):

$$R_t = p_{t+1}^k + (1 - \delta)$$

Transform (46) and using (49):

$$\begin{aligned} \frac{C_{t+1}}{\beta C_t} &= \beta \lambda_{t+1} + (1 - \delta) \\ C_{t+1} &= C_t [\beta (p_{t+1}^k + (1 - \delta))] \end{aligned}$$

A.2 INTERMEDIATE GOOD FIRM MAXIMISATION PROBLEM

An intermediate good firm i chooses the price in period t , $P_t(i)$ to maximise its profits:

$$\text{Max } \{P_t(i)\} \left\{ \sum_{i=0}^{\infty} (\theta\beta)^k \frac{A_{t+k}}{A_t} \left[\left(\frac{\pi_t^k P_t(i)}{P_t} \right)^{1-\varepsilon} Y_t - \left(\frac{\pi_t^k P_t(i)}{P_t} \right)^{-\varepsilon} Y_t mc_t \right] \right\} \quad (50)$$

The first order condition for this problem is:

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} (\theta\beta)^k \frac{A_{t+k}}{A_t} \\ &\left[Y_t \left[-\varepsilon \left(\frac{\pi_t^k P_t(i)}{P_t} \right)^{-\varepsilon-1} \frac{\pi_t^k}{P_t} \left(\frac{\pi_t^k P_t(i)}{P_t} - mc_t \right) \right. \right. \\ &\quad \left. \left. + \frac{\pi_t^k}{P_t} \left(\frac{\pi_t^k P_t(i)}{P_t} \right)^{-\varepsilon} \right] \right] \quad (51) \end{aligned}$$

Rearrange this and we get:

$$\begin{aligned} \pi_t^k (\varepsilon - 1) \sum_{k=0}^{\infty} (\theta\beta)^k A_{t+k} Y_t P_t^{\varepsilon-1} &= \\ P_t(i)^{-1} \varepsilon \sum_{k=0}^{\infty} (\theta\beta)^k A_{t+k} Y_t P_t^{\varepsilon-1} mc_t & \quad (52) \end{aligned}$$

The optimal price for an intermediate firm is given as:

$$P_t(i) = \frac{\sum_{k=0}^{\infty} (\alpha\beta)^k A_{t+k} P_{t+k}^{\varepsilon} Y_{t+k} mc_{t+k}}{-1 \sum_{k=0}^{\infty} (\alpha\beta\pi)^k A_{t+k} P_{t+k}^{\varepsilon-1} Y_{t+k}} \quad (53)$$

A.3: DERIVATION OF LINEARIZED EQUATIONS

Output

The production function is denoted by:

$$Y_t = K_t^\alpha L_t^{1-\alpha} \quad (54)$$

Linearizing equation gives:

$$Y_{ss} + Y_{ss} \hat{Y}_t = K_{ss}^\alpha L_{ss}^{1-\alpha} + K_{ss}^\alpha L_{ss}^{1-\alpha} (a \hat{K}_t + (1 - \alpha) \hat{L}_t)$$

Use the steady state relationship $Y_{ss} = K_{ss}^\alpha L_{ss}^{1-\alpha}$ (to get:

$$Y_{ss} \hat{Y}_t = K_{ss}^\alpha L_{ss}^{1-\alpha} (a \hat{K}_t + (1 - \alpha) \hat{L}_t)$$

Or under an exogenous labour supply:

$$\hat{Y}_t = a \hat{K}_t$$

Capital Accumulation Equation

The initial equation for capital accumulation is given as:

$$K_{t+1} - K_t = Y_t - \delta K_t - C_t \quad (55)$$

Linearizing the equation gives:

$$\begin{aligned} K_{ss}(1 + \hat{K}_{t+1}) - K_{ss}(1 + \hat{K}_t) &= Y_{ss}(1 + \hat{Y}_{t+1}) - C_{ss}(1 + \hat{C}_t) \\ &\quad - \delta K_{ss}(1 + \hat{K}_t) \end{aligned} \quad (56)$$

Use the steady state relationship $0 = Y_{ss} - C_{ss} - \delta K_{ss}$ and substituting the approximation of output equation, into the above equation gives:

$$K_{t+1} - K_t = \left[\alpha \frac{Y_{ss}}{K_{ss}} - \delta \right] K_t - \frac{Y_{ss}}{K_{ss}} - \hat{C}_t \quad (57)$$

Real Government Debt Equation

Using equations 24, 25 and 26, we obtain a real government debt dynamics equation which is as follows:

$$b_{t+1} - b_t = rb_t + f^M \pi_t b_t + G - T - f^F b_t + f^F b^* \quad (58)$$

Linearizing the equation gives:

$$b_{ss} e^{\hat{b}_{t+1}} - b_{ss} e^{\hat{b}_t} = rb_{ss} e^{\hat{b}_t} + f^M \pi_{ss} e^{\hat{b}_t} e^{\hat{\pi}_t} + G - T - f^F b_{ss} e^{\hat{b}_t} + f^F b^* \quad (59)$$

Use the steady-state relationship $0 = rb_{ss} + f^M \pi_{ss} + G - T - f^F b_{ss} + f^F b^*$ and since the New Keynesian Phillips curve (NKPC) is obtained by log-linearizing equilibrium conditions around a steady-state characterised by zero trend inflation, the above equation is simplified to get:

$$\hat{b}_{t+1} - \hat{b}_t = (r - f^F) \hat{b}_t + f^M \hat{\pi}_t \quad (60)$$

Marginal Cost Equation

From equation (18), the marginal cost equation is given as:

$$mc_t = (p_t^k)^\alpha w_t^{1-\alpha} (\alpha^{-\alpha})^{\alpha-1} \quad (61)$$

Linearizing the equation gives:

$$mc_{ss}(1 + \hat{m}\hat{c}_t) \approx p_{ss}^{k\alpha} w_{ss}^{1-\alpha} (1 + \alpha \hat{p}_t^k + (1 - \alpha) \hat{w}_t) \alpha^{-\alpha} (1 - \alpha)^{\alpha-1} \quad (62)$$

Use the steady-state relationship $mc_{ss} = p_{ss}^{k\alpha} w_{ss}^{1-\alpha} \alpha^{-\alpha} (1 - \alpha)^{\alpha-1}$ to get:

$$\hat{m}\hat{c}_t = \alpha \hat{p}_t^k + (1 - \alpha) \hat{w}_t \quad (63)$$

Using the monetary policy rule as in (24) and given that $p_t^k = r_t + \delta$, one obtains:

$$p_t^k = r + f^M \pi_t + \delta \quad (64)$$

Linearizing the equation gives:

$$\hat{p}_t^k (1 + \hat{p}_t^k) \approx r + f^M \pi_t + \delta$$

Use the steady-state relationship $p_{ss}^k = r + f^M \pi + \delta$ to get:

$$\hat{p}_t^k = \frac{f^M \pi_t}{r + \delta} \quad (65)$$

since $p_t^k = r + \delta$. From equation (15), under an exogenous labour supply the optimal wages is given by:

$$w_t = K_t \frac{1 - \alpha}{\alpha} p_t^k \quad (66)$$

Linearizing this equation and using the steady-state gives:

$$\hat{w}_t = \hat{K}_t + \hat{p}_t^k \quad (67)$$

By substituting (65) and (67) into (63) yields:

$$\hat{m}\hat{c}_t = \frac{f^M \pi_t}{r + \delta} + (1 - \alpha) \hat{K}_t \quad (68)$$

Consumption Equation

The initial equation for consumption dynamics is given from equations (8) and (17) which is:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \left[(1 - \alpha) \left(\frac{L_{t+1}}{K_{t+1}} \right)^\alpha mc_{t+1} + (1 - \delta) \right]$$

or rewrite this equation as:

$$E_t \frac{C_{t+1}}{C_t} = \beta E_t \left[(1 - \alpha) \left(\frac{L_{t+1}}{K_{t+1}} \right)^\alpha mc_{t+1} + (1 - \delta) \right] \quad (69)$$

Log-linearizing this equation gives:

$$\frac{f^M \pi_t}{r + \delta} \approx \beta \left[(1 - \alpha) \left(\frac{L_{ss}}{K_{ss}} \right)^\alpha mc_{ss} + (1 - \delta) + \left(1 - \alpha \left(\frac{L_{ss}}{K_{ss}} \right)^\alpha mc_{ss} (\alpha \hat{L}_{t+1} - \alpha \hat{K}_{t+1} + m \hat{c}_{t+1}) \right) \right] \quad (70)$$

Use the steady-state relationship $1 - \beta(1 - \delta) = \beta(1 - \alpha) \left(\frac{L_{ss}}{K_{ss}} \right)^\alpha mc_{ss}$ and substituting for mc from (68),

we obtain the log-linearized version of consumption dynamics under an exogenous labour supply, which is:

$$\hat{C}_{t+1} - \hat{C}_t = [1 - \beta(1 - \delta)] \frac{f^M}{r + \delta} \pi_t \quad (71)$$

Inflation Equation

The behaviour of final good and intermediate good firms in a New Keynesian framework will be used to find the log-linearized version of inflation dynamics. First, an expression for a final good firm will be obtained. The final good pricing rule is given as:

$$P_t = \int_0^1 [P_t(i)^{1-\varepsilon} di]^{1/\varepsilon} \quad (72)$$

Following Yun (1996), it is assumed that all the adjusting firms set the new price in period t that is indexed to trend inflation while non-adjusting firms keep their price as it was in the previous period. Hence the price level updated version is:

$$P_t^{1-\varepsilon} = (1 - \theta) P_t^{*1-\varepsilon} + \theta \pi P_{t-1}^{1-\varepsilon} \quad (73)$$

In the stationary state of an economy without money growth, this equation implies that both the final good price and the intermediate good prices are the same. Thus, $P_{ss} = P_{ss}^*(i)$ and therefore

$$P_{ss}^{1-\varepsilon} = (1 - \theta) P_t^{*1-\varepsilon} + \theta \pi P_{t-1}^{1-\varepsilon} \quad (74)$$

Next, the final good pricing rule will be log-linearized. Using Uhlig's method gives:

$$P_{ss}^{1-\varepsilon} (1 + (1 - \varepsilon) \hat{P}_t) \approx \theta \pi P_{t-1}^{1-\varepsilon} (1 + (1 - \varepsilon) \hat{P}_{t-1}) + (1 - \theta) P_{ss}^{1-\varepsilon} (1 + (1 - \varepsilon) \hat{P}_t^*) \quad (75)$$

Using the steady-state relationship as in 74, to get:

$$\hat{P}_t^* = \theta \pi P_{t-1} (1 - \theta) \hat{P}_t^* \quad (76)$$

The next step is to log-linearize the intermediate good firm's price as in equation 53. Yun (1996) shows the log-linear pricing rule for the intermediate goods firm as:

$$\hat{P}_t^* = (1 - \theta\beta) \sum_{k=0}^{\infty} (\theta\beta)^k [\hat{P}_{t+k} + \widehat{m}\widehat{c}_{t+k}] \quad (77)$$

Inserting this into (76), we obtain:

$$\hat{P}_t^* = \theta\pi P_{t-1} + (1 - \theta)(1 - \theta\beta) \sum_{k=0}^{\infty} (\theta\beta)^k [\hat{P}_{t+k} + \widehat{m}\widehat{c}_{t+k}] \quad (78)$$

Removing from (68), the log-linearized version of inflation dynamics is:

$$\hat{\pi}_{t+1} = \left[\frac{1}{\beta} \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} \frac{f^M}{r + \delta} \right] \pi_t - (1 - \alpha) \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} \hat{K}_t \quad (79)$$

Thus, Yun's log-linearized optimal price setting rule coincides with the log-linearization of a typical Calvo framework obtained as an approximation around zero trend inflation.

A.4: PROOF OF PROPOSITIONS

Monetary Dynamics

Proposition 5.1. Consider the dynamics sub-system in C , K and π . implied by matrix in 80. According to Blanchard and Kahn (1980), if the number of eigenvalues outside the unit circle (that is eigenvalues that are bigger than unity in absolute terms) equal to two (the number of forward-looking variables), then there exists a unique stable solution.

1. Assume that the monetary policy is passive (PM) ($f^M < 0$). Then the dynamics are indeterminate.
2. Assume that the monetary policy is active (AM) ($f^M > 0$). Then the dynamics are determinate if and only if it satisfies two necessary conditions which are that $f^M > 0$ and that has to be in one of the following two regions:

$$\frac{\left(2 + A + \frac{1}{\beta}\right)(r + \delta)}{N} < f^M < \min \left\{ \frac{A}{\beta} \frac{r + \delta}{N[A - Z(1 - \alpha)]}, \right.$$

$$\left. \frac{F}{N} \frac{r + \delta}{2(1 + A) - Z(1 - \alpha)} \right\} 0 < f^M < \frac{\left(A + \frac{1}{\beta}\right)(r + \delta)}{N}$$

$$A = \alpha \frac{Y_{ss}}{K_{ss}} - \delta + 1$$

$$N = \frac{(1 - \theta)(a - \theta\beta)}{\theta\beta}$$

$$F = (2 + 2A) + \left(1 + \frac{1}{\beta}\right)$$

$$Z = \frac{C_{ss}}{K_{ss}} [1 - \beta](1 - \delta)]$$

Proof: Equations (39) - (41) constitute a system of three linear difference equations and can be represented by the following matrix:

$$\Gamma_0 Z_t = \Gamma_1 Z_{t-1}$$

where $Z_t = [C_t, K_t, \pi_t]$. Γ_0 and Γ_1 are 3x3 matrix. If Γ_0 is non-singular, we can rewrite this matrix as:

$$Z_t = \Gamma_0^{-1} \Gamma_1 Z_{t-1}$$

The dynamic behaviour of the system is governed by the eigenvalues of the reduced form coefficient matrix $\Gamma_0^{-1} \Gamma_1$. Denote $J = \Gamma_0^{-1} \Gamma_1$ as the Jacobian matrix of the system, where this system of a linear difference equation can be written in matrix form as:

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{C}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix} = J \begin{bmatrix} \hat{K}_t \\ \hat{C}_t \\ \hat{\pi}_t \end{bmatrix}$$

where

$$J = \begin{bmatrix} \alpha \frac{Y_{ss}}{K_{ss}} - \delta + 1 & \frac{C_{ss}}{K_{ss}} & 0 \\ -(1 - \alpha) \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} & 0 & \frac{1}{\beta} \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} \frac{f^M}{r + \delta} \end{bmatrix} \quad (80)$$

and can be simplified as:

$$J = \begin{bmatrix} A - \lambda & B & 0 \\ 0 & 1 - \lambda & C \\ D & 0 & E - \lambda \end{bmatrix} \quad (81)$$

where

$$A = \alpha \frac{Y_{ss}}{K_{ss}} - \delta + 1$$

$$B = -\frac{Y_{ss}}{K_{ss}}$$

$$C = [1 - \beta(1 - \delta)] \frac{f^M}{r + \delta}$$

$$D = -(1 - \alpha) \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta}$$

$$E = \frac{1}{\beta} \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} \frac{f^M}{r + \delta}$$

Since this sub-system is characterised by a one state variable (K) and two forward-looking variables (C, π), the determinacy requires that the two roots lie outside the unit circle. The eigenvalues associated with matrix in equation 80 are given by the relevant cubic function as below:

$$J_3 \lambda^3 + J_2 \lambda^2 + J_1 \lambda + J_0 \quad (82)$$

where:

$$\begin{aligned} J_3 &= -1 \\ J_2 &= A + 1 + E \\ J_1 &= -A - AE - E \\ J_0 &= AE + BCD \end{aligned}$$

The above implies:

$$\begin{aligned} J(0) &= \frac{A}{\beta} + N \frac{f^M}{r + \delta} [Z(1 - \alpha) - A] \\ J(1) &= Z \frac{f^M}{r + \delta} N(1 - \alpha) \\ J(-1) &= F + N \frac{f^M}{r + \delta} [Z(1 - \alpha) - 2(1 + A)] \end{aligned} \quad (83)$$

where:

$$\begin{aligned} N &= \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} \\ Z &= \frac{C_{ss}}{K_{ss}} [(1 - \beta(1 - \delta))] \\ F &= -(2 - 2A) \left(1 + \frac{1}{\beta} \right) \end{aligned}$$

Hence, if $f^M < 0$, then $J(1) < 0$ and $J(1) > 0$. By means of the Intermediate Value Theorem, this implies one root is in $(0, 1)$. This implies, under a passive monetary rule ($f^M < 0$), the dynamics are never determinate since there is only one eigenvalue outside the unit circle. Now, suppose that monetary policy is given by active monetary rule ($f^M > 0$). Under this restriction $J(1) > 0$, so in order to have two roots outside the unit circle, we need additional necessary conditions which are $J(1) > 0$ and $J(-1) > 0$. These put an upper bound on f^M .

$$f^M < \min \left\{ \frac{A}{\beta} \frac{r + \delta}{N[A - Z(1 - \alpha)]}, \frac{F}{N} \frac{r + \delta}{2(1 + A) - Z(1 - \alpha)} \right\} \quad (84)$$

Assuming the roots are real and $f^M > 0$, we have two potential regions of determinacy:

$$\frac{-J_2}{3J_3} < -1 \quad \text{and} \quad \frac{-J_2}{3J_3} > 1 \quad (85)$$

These put a lower bound on f^M :

$$\frac{\left(2 + A + \frac{1}{\beta} \right) (r + \delta)}{N} < f^M \quad (86)$$

and

$$0 < f^M < \frac{\left(2 + A + \frac{1}{\beta} \right) (r + \delta)}{N} \quad (87)$$

Combining 84 and 86, as well as 87, we obtain two necessary conditions for determinacy which are and must be in one of the following two regions:

Monetary and Fiscal Dynamics

Proposition 5.2. Consider the dynamic sub-system in b , C , K and π implied by matrix in equation 89.

1. Assume that both monetary and fiscal policies are active (AM/AF). Then, the dynamics have too many unstable roots and thus equilibrium paths are explosive and there is no solution.
2. Assume that the monetary policy is active and the fiscal policy is passive (AM/PF). Then, the dynamics are determinate and thus the equilibrium path has a unique solution.
3. Assume that the monetary policy is passive and the fiscal policy is active (PM/AF). Then, the dynamics are indeterminate and thus equilibrium paths have multiple solutions.
4. Assume that both monetary and fiscal policies are passive (PM/PF). Then, the dynamics are indeterminate and thus equilibrium paths have multiple solutions.

Proof: Equations (38) - (41) constitute a system of four linear difference equations and can be represented by the following matrix:

$$\Gamma_0 Z_t = \Gamma_1 Z_{t-1}$$

where $Z_t = [b_t, C_t, K_t, \pi_t]$. Γ_0 and Γ_1 are 4×4 matrix. If Γ_0 is non-singular, we can rewrite this matrix as:

$$Z_t = \Gamma_0^{-1} \Gamma_1 Z_{t-1}$$

The dynamic behaviour of the system is governed by the eigenvalues of the reduced form coefficient matrix $\Gamma_0^{-1} \Gamma_1$. Denoting $J = \Gamma_0^{-1} \Gamma_1$ as the Jacobian matrix of the system, this system of linear difference equation can be written in matrix form as:

$$\begin{bmatrix} \hat{b}_{t+1} \\ \hat{C}_{t+1} \\ \hat{K}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix} = J \begin{bmatrix} \hat{b}_t \\ \hat{C}_t \\ \hat{K}_t \\ \hat{\pi}_t \end{bmatrix}$$

where

$$J = \begin{bmatrix} r - f^F + 1 & 0 & 0 \\ \alpha \frac{Y_{ss}}{K_{ss}} - \delta + 1 & -\frac{C_{ss}}{K_{ss}} \\ 0 & 0 & 1 \\ 0 & -\frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} (1 - \alpha) & 0 \end{bmatrix}$$

$$\begin{bmatrix} f^M \\ 0 \\ [1 - \beta(1 - \delta)] \frac{f^M}{r + \delta} \\ \frac{1}{\beta} - \frac{(1 - \theta)(1 - \theta\beta)}{\theta\beta} \frac{f^M}{r + \delta} \end{bmatrix} \quad (89)$$

and can be simplified as:

$$J = \begin{bmatrix} A - \lambda & B & 0 & G \\ 0 & A - \lambda & B & 0 \\ 0 & 0 & 1 - \lambda & C \\ 0 & D & 0 & E - \lambda \end{bmatrix} \quad (90)$$

where

$$F = r - f^F + 1 \\ G = f^M$$

One root is given by:

$$F - \lambda = 0 \quad \text{or} \quad \lambda = F = r - f^F + 1 = \lambda \quad (91)$$

Hence, this root is positive if the fiscal policy is ‘active’ ($f^F < r$) and negative if the fiscal policy is ‘passive’ ($f^F > r$). This eigenvalue is positive and greater than one ($r - f^F + 1 > 1$) if and only if the fiscal policy is ‘active’ ($f^F < r$), while the remaining two eigenvalues are identical to the eigenvalues given by the monetary dynamics. Combining all these eigenvalues gives the determinacy conditions for various policy regimes as follows:

1. Assume that both monetary and fiscal policies are active (AM/AF). Then, the dynamics have too many unstable roots and thus equilibrium paths are explosive and there is no solution.
2. Assume that the monetary policy is active and the fiscal policy is passive (AM/PF). Then, the dynamics are determinate and thus the equilibrium path has a unique solution.
3. Assume that the monetary policy is passive and the fiscal policy is active (PM/AF). Then, the dynamics are indeterminate and thus equilibrium paths have multiple solutions.
4. Assume that both monetary and fiscal policies are passive (PM/PF). Then, the dynamics are indeterminate and thus equilibrium paths have multiple solutions.