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Impedance boundary conditions for acoustic time-harmonic wave propagation in viscous gases in two dimensions

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We present impedance boundary conditions for the viscoacoustic equations for approximative models that are in terms of the acoustic pressure or in terms of the macroscropic acoustic velocity. The approximative models are derived by the method of multiple scales up to order 2 in the boundary layer thickness. The boundary conditions are stable and asymptotically exact, which is justified by a complete mathematical analysis. The models can be discretized by finite element methods without resolving boundary layers. In difference to an approximation by asymptotic expansion for which for each order 1 PDE system has to be solved, the proposed approximative are solutions to one PDE system only. The impedance boundary conditions for the pressure of first and second orders are of Wentzell type and include a second tangential derivative of the pressure proportional to the square root of the viscosity and take thereby absorption inside the viscosity boundary layer of the underlying velocity into account. The conditions of second order incorporate with curvature the geometrical properties of the wall. The velocity approximations are described by Helmholtz-like equations for the velocity, where the Laplace operator is replaced by $\nabla \mathrm{div}$, and the local boundary conditions relate the normal velocity component to its divergence. The velocity approximations are for the so-called far field and do not exhibit a boundary layer. Including a boundary corrector, the so-called near field, the velocity approximation is accurate even up to the domain boundary. The results of numerical experiments illustrate the theoretical foundations.

KEYWORDS

acoustics wave propagation, asymptotic expansions, impedance boundary conditions, singularly perturbed PDE

MSC CLASSIFICATION

35C20; 35J25; 35B40; 41A60; 76Q05

1 | INTRODUCTION

In this study, we are investigating the viscoacoustic equations in the framework of Landau and Lifschitz[1] as a perturbation of the Navier–Stokes equations around a stagnant uniform fluid, with mean density ρ_0 and without heat flux. For gases and many liquids, the (dynamic) viscosity η is very small and leads to *viscosity boundary layers* close to walls[2, Section 10.4] that have been studied in previous studies[3–5] and goes back to the boundary layer theory of Prandtl.[6,7] To resolve the boundary layers with (quasi-)uniform meshes, the mesh size has to be of the same order, which leads to very large linear systems to be solved. This is especially the case for the very small boundary layers of acoustic waves. Finite difference schemes or finite element meshes specially adapted close to walls have been proposed for various model problems with boundary layers[8–11] which regain the optimal convergence rate of the numerical schemes; see also the review papers.[12,13] With impedance boundary conditions, the boundary layers need not to be resolved at all as they are posed for the macroscopic part of the solution. Impedance boundary conditions have been originally proposed for solid conductors by Shchukin[14] and Leontovich[15] and developed for several equations and geometrical setting; see, for example, Senior and Volakis[16] and Yuferev and Ida.[17] Based on asymptotic expansion techniques,[18] especially the method of multiscale expansions or the method of matched asymptotic expansion, for many models, rigorous error estimates were shown; see, for example, for conducting bodies[19] or thin sheets.[20,21]

In acoustics, they are known as wall boundary conditions and derived at first order for viscothermal boundary layers[2, Section 10.4] and for acoustic plane waves in presence of a shear flow of first order for planar[3] and of higher order for the guided modes in a cylindrical wave guide[22,23] using the method of matched asymptotic expansions. Moreover, first-order wall boundary conditions in a stagnant curvilinear coordinates were derived[24] and for viscothermal boundary layers for a flat wall,[25] which can be written in terms of the acoustic pressure only. It is argued in Berggren et al[25] that first-order impedance boundary conditions for flat walls may be used for curved walls as the minimal radius of curvature is typically much larger than boundary layer thicknesses. The first-order impedance conditions in terms of the pressure have been shown to be well-posed[25] for Lipschitz domain if on a part of the boundary radiation conditions are posed. Well-posedness of the models with higher order impedance conditions and also an analysis of the error based on a stability analysis of the singularly perturbed system is outstanding. In an earlier work,[26] we derived a complete asymptotic expansion for the viscoacoustic equations based on the technique of multiscale expansion in powers of $\sqrt{\eta}$ which takes into account curvature effects. The limit acoustic pressure or velocity and the correcting terms of different order can be defined and solved iteratively after each other. This asymptotic expansion was rigorously justified with optimal error estimates.

In this article, we propose and rigorously justify, based on the multiscale expansion in Schmidt et al.,[26] (effective) *impedance boundary conditions* for the velocity and the pressure for possibly curved boundaries, where no boundary layers need to be resolved and pressure or velocity approximations of orders 0, 1, and 2 are defined separately and can be computed in one step. The boundary layers can be locally computed from the far-field approximation and added to the far-field solution an uniform approximation of a particular order is obtained. The analysis of the modeling error requires some smoothness of the domain and to exclude eigensolutions. The error analysis justifies that the effect due to viscosity is dominant in relation to curvature for smooth curved walls. The approximative model of order 1 was stated without derivation and justification already in Semin and Schmidt[27] where semi-infinite wave guides were considered for which Dirichlet-to-Neumann absorbing boundary conditions were introduced that take into account the absorption due to viscosity on the infinite walls of the guide.

The article is subdivided as follows. In Section 2, we define the model problem of the viscous acoustic equations for acoustic velocity and pressure and derive the approximative models with impedance boundary conditions for the velocity and for the pressure on the basis of the asymptotic expansion presented in Schmidt et al[26] and state the stability and modeling error estimates. The well-posedness and estimates of the modeling error of the approximative models with the impedance boundary conditions will be shown in Sections 3 and 4. Results of some numerical experiments in Section 5 shall emphasize the validity of the theoretical findings.

2 | MODEL PROBLEM DEFINITION AND MAIN RESULTS

2.1 | Geometry and model problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with boundary $\partial\Omega$, where $\mathbf{n}$ denotes the outer normalized normal vector. If $\partial\Omega$ is piecewise C^2, then κ denotes the (signed) curvature a.e. on $\partial\Omega$ which is positive on convex parts of $\partial\Omega$.

We consider the time-harmonic acoustic velocity $\mathbf{v}$ and acoustic pressure p (the time regime is $e^{-i\omega t}$, $\omega \in \mathbb{R}^+$) which are described in the framework of Landau and Lifshitz[1] by the coupled system:

$$-i\omega\rho_0\mathbf{v} + \nabla p - \eta\Delta\mathbf{v} - \eta'\nabla\mathrm{div}\,\mathbf{v} = \mathbf{f}, \text{in } \Omega, \tag{2.1a}$$

$$-i\omega p + \rho_0 c^2\, div\mathbf{v} = 0, \text{in } \Omega, \tag{2.1b}$$

$$\mathbf{v} = \boldsymbol{0}, \text{on } \partial\Omega. \tag{2.1c}$$

In the *momentum equation* (2.1a) with some known source term $\mathbf{f}$, the viscous dissipation in the momentum is not neglected as we consider near wall regions. Since in this study we are mainly interested in the viscous effects, we neglect nonlinear convection. Here, ρ_0 is the density of the media, c is the speed of sound, $\eta > 0$ is the dynamic viscosity, and $\eta' = \frac{1}{3}\eta + \zeta$ with the second (volume) viscosity $\zeta > 0$. Both η and η' shall take small values, and we call $\gamma' = \eta'/\eta$ their quotient. The system is completed by *no-slip* boundary conditions and mathematically analyzed in Schmidt et al.[26] Similar acoustic equations have been derived and studied in previous studies[1,28,29] for a stagnant flow and in previous studies[3,5,28,30,31] for the case that a mean flow is present.

Note that with $\Delta = \nabla\mathrm{div} - \mathbf{curl}_{2D}\,\mathrm{curl}_{2D}$, the momentum equation (2.1a) can be written with the two-dimensional rotation operators $\mathrm{curl}_{2D} = div(\cdot^{\perp})$, $\mathbf{curl}_{2D} = (\nabla\cdot)^{\perp}$ where $\mathbf{u}^{\perp} := (u_2, -u_1)^{\top}$ denotes a vector rotated clockwise by 90°.

It is well known that the acoustic velocity field exhibits a boundary layer of thickness $O\left(\sqrt{\eta/\omega}\right)$, starting at the rigid wall; see, for example, previous studies[2–4,26] and the references therein. In the following, we propose definitions of far-field velocities, which approximate the acoustic velocity outside the boundary layer, the correcting near-field velocities, and approximative acoustic pressure in the whole domain.

In Schmidt et al,[26] the stability was proven for the nonresonant case, which we consider here as well, that is, for vanishing viscosity and absorption the kernel of the system is empty—there is no eigensolution. The eigenvalues of the limit problem coincide with the Neumann eigenvalue of $-\Delta$.

Lemma 1 (Stability for the nonresonant case). *For any* $\mathbf{f} \in (H_0(div,\Omega)) \cap H(curl_{2D},\Omega))'$, *the system (2.1) has a unique solution* $(\mathbf{v}^\epsilon, p^\epsilon) \in H_0(div,\Omega) \cap H(curl_{2D},\Omega) \times L^2(\Omega)$. *If* $\frac{\omega^2}{c^2}$ *is not a Neumann eigenvalue of* $-\Delta$*, then there exists a constant* $C > 0$ *independent of* ϵ*, such that*

$$\|\mathbf{v}\|_{H(div,\Omega)} + \sqrt{\eta}\,\|curl_{2D}\mathbf{v}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \le C\,\|\mathbf{f}\|_{(H_0(div,\Omega)\cap H(curl_{2D},\Omega))'}, \tag{2.2a}$$

$$\|\nabla p\|_{L^2(\Omega)} \le C\,\|\mathbf{f}\|_{L^2(\Omega)}. \tag{2.2b}$$

A proof can be found in Schmidt et al.[26, Lemma 2.2] Even so, in this work, C^∞ was assumed, the proof of (2.2a) and (2.2b) does not rely on a higher regularity assumption; see Marus%ić-Paloka.[32]

2.2 | Asymptotic expansion for small viscosities

To investigate the solution of (2.1) for small viscosities, we introduce a small dimensionless parameter $\varepsilon \ll 1$, $\varepsilon \in \mathbb{R}^+$ and replace η, η' by $\varepsilon^2\omega\rho_0/2$, $\varepsilon^2\gamma'\omega\rho_0/2$ (corresponding to $\eta_0 = \omega\rho_0/2$, $\eta_0' = \gamma'\omega\rho_0/2$ in Schmidt et al.[26]), respectively. In this way, the boundary layer thickness will become proportional to ε.

Close to the boundary $\partial\Omega$, we introduce a local coordinate system (t,s) that is uniquely defined by

$$\mathbf{x}(t,s) = \mathbf{x}_{\partial\Omega}(t) - s\mathbf{n}(t), \tag{2.3}$$

where the boundary is described by the mapping $\mathbf{x}_{\partial\Omega}(t)$ from an interval $T \subset \mathbb{R}$ and s is the distance from the boundary (see Figure 1A). Without loss of generality, we assume an arc length parametrization, that is, $|\mathbf{x}'_{\partial\Omega}(t)| = 1$ for all $t \in T$, and the tangential derivative is given by $\partial_\Gamma v(\mathbf{x}) = \partial_t v(\mathbf{x}(t,s))$.

Then, inspired by the framework of Vishik and Lyusternik,[18] the solution of (2.1) can be written as

$$\mathbf{v} = \sum_{j=0}^{\infty}\varepsilon^j\left(\mathbf{v}^j + \varepsilon\mathbf{curl}_{2D}(\phi^j\chi)\right), \qquad p = \sum_{j=0}^{\infty}\varepsilon^j p^j, \tag{2.4}$$

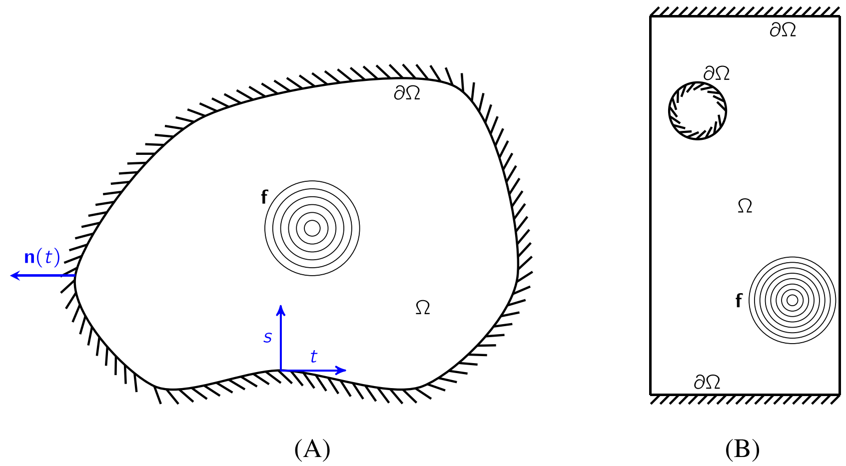

FIGURE 1 (A) Definition of a general domain with a local coordinate system (t, s) close to the wall and (B) definition of a torus domain for numerical simulations [Colour figure can be viewed at wileyonlinelibrary.com]

where $\mathbf{v}^j(x, y)$ and $p^j(x, y)$ are terms of the *far-field* expansion, the *near-field* terms $\phi^j\left(t, \frac{s}{\varepsilon}\right)$ represent the boundary layer close to the wall, and χ is an admissible cut-off function. We denote a monotone function $\chi \in C^\infty(\Omega)$ as an admissible cut-off function, if there exist constants $0 < s_1 < s_0 < \frac{1}{2}\|\kappa\|^{-1}_{L^\infty(\Gamma)}$ such that $\chi \equiv 0$ outside an s_0-neighborhood of $\partial\Omega$ and otherwise $\chi(\mathbf{x}) = \hat{\chi}(s)$, where $\hat{\chi}(s) = 1$ for $s < s_1$. For an admissible cut-off function χ, we denote as $\overline{\mathrm{supp}(\chi)}$ the χ-neighborhood of $\partial\Omega$.

The method of multiscale expansion separates the far- and near-field terms. We restrict ourselves to $j = 0, 1, 2$, as these will be used for the derivation of the impedance boundary conditions where the equations for general $j \in \mathbb{N}$ can be found in Schmidt et al.[26] The limit far-field velocity term $\mathbf{v}^0$ satisfies the PDE system

$$\nabla\mathrm{div}\ \mathbf{v}^0 + \frac{\omega^2}{c^2}\mathbf{v}^0 = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f},\ \text{in}\ \Omega, \tag{2.5a}$$

$$\mathbf{v}^0 \cdot \mathbf{n} = 0,\ \text{on}\ \partial\Omega, \tag{2.5b}$$

the first-order corrector $\mathbf{v}^1$

$$\nabla\mathrm{div}\ \mathbf{v}^1 + \frac{\omega^2}{c^2}\mathbf{v}^1 = 0,\ \text{in}\ \Omega, \tag{2.6a}$$

$$\mathbf{v}^1 \cdot \mathbf{n} = (1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_\Gamma^2\mathrm{div}\ \mathbf{v}^0 - (1-\mathrm{i})\frac{1}{2\omega\rho_0}\partial_\Gamma(\mathbf{f}\cdot\mathbf{n}^\perp),\ \text{on}\ \partial\Omega, \tag{2.6b}$$

and the second-order corrector $\mathbf{v}^2$

$$\nabla\mathrm{div}\ \mathbf{v}^2 + \frac{\omega^2}{c^2}\mathbf{v}^2 = \frac{\mathrm{i}\omega^2}{2c^2}\Delta\mathbf{v}^0 + \frac{\mathrm{i}\gamma'\omega^2}{2c^2}\nabla\mathrm{div}\ \mathbf{v}^0,\ \text{in}\ \Omega, \tag{2.7a}$$

$$\mathbf{v}^2 \cdot \mathbf{n} = (1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_\Gamma^2\mathrm{div}\ \mathbf{v}^1 + \frac{c^2}{\omega^2}\left(\frac{\mathrm{i}}{4}\partial_\Gamma\left(\kappa\partial_\Gamma\mathrm{div}\ \mathbf{v}^0\right)\right) - \frac{1}{4\omega\rho_0}\partial_\Gamma\left(\kappa\,\mathbf{f}\cdot\mathbf{n}^\perp\right),\ \text{on}\ \partial\Omega. \tag{2.7b}$$

The far-field pressure terms p^j are then given for any $j \in \mathbb{N}_0$ by

$$p^j = -\frac{\mathrm{i}\rho_0 c^2}{\omega}\mathrm{div}\ \mathbf{v}^j,\ \text{in}\ \Omega. \tag{2.8}$$

2.3 | Approximative models with impedance boundary conditions

In this section, we derive and state approximative models for the far-field velocity $\mathbf{v}_{\mathrm{appr},N}$ of orders 0, 1, and 2 (Section 2.3.1) and for the far-field pressure $p_{\mathrm{appr},N}$ of orders 0, 1, and 2 (Section 2.3.2), which include in particular impedance boundary conditions. The approximative models will be derived from the asymptotic expansion of the solution of (2.1). While the terms of the asymptotic expansion are defined order by order, we introduce approximative models that can be computed in one step and take into account all terms up to order $N = 0, 1$, or 2. Even so the governing equations couple velocity and pressure and can not be written in terms of the pressure only, we present approximative models in terms of the acoustic pressure or the acoustic velocity only. For both kinds of approximative models, the approximations to the respective other quantity, acoustic velocity $\mathbf{v}_{\text{appr},N}$ or pressure $p_{\text{appr},N}$, result in a postprocessing step by algebraic equations from $p_{\text{appr},N}$ or $\mathbf{v}_{\text{appr},N}$, respectively. Moreover, velocity boundary layer correctors $\mathbf{v}^{BL}{}_{\text{appr},N}$ can be computed from the far-field velocity. They will be derived in Section 2.6 for smooth boundaries, but their (weak) formulations can be defined if the domain Ω is Lipschitz, and piecewise C^2 boundary is required for the models of order 2 that include the curvature.

Note that the approximative models incorporate only boundary conditions for the scalar acoustic pressure or normal component of acoustic velocity in difference to the original viscoacoustic model (2.1) where with the no-slip boundary conditions all acoustic velocity components are prescribed. With this, the approximative models are of different nature, even so the no-slip boundary conditions is present in an effective way.

2.3.1 | Approximative models for the acoustic velocity

We start with the far-field asymptotic expansions $\mathbf{v}^{\varepsilon,N} := \sum_{j=0}^{N} \varepsilon^j \mathbf{v}^j$ and $p^{\varepsilon,N} := \sum_{j=0}^{N} \varepsilon^j p^j$ of order N. For their computation, $N+1$ PDE systems have to be solved.

The far-field limit $\mathbf{v}^0$ is that solution of one system, namely, (2.5), which is the natural approximative model of order 0:

$$\nabla \text{div}\, \mathbf{v}_{\text{appr},0} + \frac{\omega^2}{c^2}\mathbf{v}_{\text{appr},0} = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}, \text{ in } \Omega, \tag{2.9a}$$

$$\mathbf{v}_{\text{appr},0} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega. \tag{2.9b}$$

To obtain the approximative model of order $N = 1$

$$\nabla \text{div}\, \mathbf{v}_{\text{appr},1} + \frac{\omega^2}{c^2}\mathbf{v}_{\text{appr},1} = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}, \text{ in } \Omega, \tag{2.10a}$$

$$\mathbf{v}_{\text{appr},1} \cdot \mathbf{n} - (1+\mathrm{i})\frac{c^2}{\omega^2}\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma^2\, div\, \mathbf{v}_{\text{appr},1} = \frac{(\mathrm{i}-1)}{\omega\rho_0}\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma(\mathbf{f}\cdot\mathbf{n}^\perp), \text{ on } \partial\Omega, \tag{2.10b}$$

which is only one system providing the approximation $\mathbf{v}_{\text{appr},1}$, we consider the boundary conditions and PDEs that are solved by $\mathbf{v}^{\varepsilon,1}$ and neglect the terms of order ε^2.

We find that $\mathbf{v}^{\varepsilon,1}$ satisfies the boundary condition

$$\begin{aligned}\mathbf{v}^{\varepsilon,1}\cdot\mathbf{n} &= \varepsilon(1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_\Gamma^2\,\text{div}\,\mathbf{v}^0 - \varepsilon(1-\mathrm{i})\frac{1}{2\omega\rho_0}\partial_\Gamma(\mathbf{f}\cdot\mathbf{n}^\perp),\\ &= \varepsilon(1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_\Gamma^2\,\text{div}\,\mathbf{v}^{\varepsilon,N} - \varepsilon(1-\mathrm{i})\frac{1}{2\omega\rho_0}\partial_\Gamma(\mathbf{f}\cdot\mathbf{n}^\perp) - \varepsilon^2(1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_\Gamma^2\text{div}\,\mathbf{v}^1,\end{aligned}$$

where we were using (2.5b) and (2.6b). Now, neglecting the term of order ε^2 on the right-hand side and using the equality $\eta = \varepsilon^2\omega\rho_0/2$, we obtain the boundary condition (2.10b) for $\mathbf{v}_{\text{appr},1}$. The PDE (2.10a) for $\mathbf{v}_{\text{appr},1}$ follows in the same way where no term has to be neglected as the right-hand side of the PDE (2.6a) for $\mathbf{v}^1$ does not depend on $\mathbf{v}^0$—it is even zero.

The approximative model of order $N = 2$

$$\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\right)\nabla\text{div}\,\mathbf{v}_{\text{appr},2} + \frac{\omega^2}{c^2}\mathbf{v}_{\text{appr},2} = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f} + \frac{\eta}{\rho_0^2 c^2}\mathbf{curl}_{2D}\,\text{curl}_{2D}\,\mathbf{f}, \text{ in } \Omega, \tag{2.11a}$$

$$\begin{aligned}&\mathbf{v}_{\text{appr},2}\cdot\mathbf{n} - \frac{c^2}{\omega^2}\left((1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma^2 div \mathbf{v}_{\text{appr},2} + \frac{\mathrm{i}\eta}{2\omega\rho_0}\partial_\Gamma\left(\kappa\partial_\Gamma\text{div}\,\mathbf{v}_{\text{appr},2}\right)\right)\\ &\quad = \frac{(\mathrm{i}-1)}{\omega\rho_0}\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma\left(\mathbf{f}\cdot\mathbf{n}^\perp\right) - \frac{\eta}{2\omega^2\rho_0^2}\partial_\Gamma\left(\kappa\,\mathbf{f}\cdot\mathbf{n}^\perp\right), \text{ on } \partial\Omega,\end{aligned} \tag{2.11b}$$

we obtain similarly to the first order one. Using (2.5b), (2.6b), and (2.7b), we find that $\mathbf{v}^{\varepsilon,2}$ satisfies

$$
\begin{aligned}
\mathbf{v}^{\varepsilon,2}\cdot\mathbf{n} &= \varepsilon(1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_\Gamma^2\mathrm{div}\,\mathbf{v}^0-\varepsilon(1-\mathrm{i})\frac{1}{2\omega\rho_0}\partial_\Gamma\left(\mathbf{f}\cdot\mathbf{n}^\perp\right)\\
&\quad+\varepsilon^2(1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_\Gamma^2\mathrm{div}\,\mathbf{v}^1+\varepsilon^2\frac{c^2}{\omega^2}\left(\frac{\mathrm{i}}{4}\partial_\Gamma(\kappa\partial_\Gamma\mathrm{div}\,\mathbf{v}^0)\right)-\varepsilon^2\frac{1}{4\omega\rho_0}\partial_\Gamma\left(\kappa\mathbf{f}\cdot\mathbf{n}^\perp\right)\\
&=\varepsilon(1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_\Gamma^2\mathrm{div}\,\mathbf{v}^{\varepsilon,N}+\varepsilon^2\frac{c^2}{\omega^2}\left(\frac{\mathrm{i}}{4}\partial_\Gamma\left(\kappa\partial_\Gamma\mathrm{div}\,\mathbf{v}^{\varepsilon,N}\right)\right)-\varepsilon(1-\mathrm{i})\frac{1}{2\omega\rho_0}\partial_\Gamma\left(\mathbf{f}\cdot\mathbf{n}^\perp\right)-\varepsilon^2\frac{1}{4\omega\rho_0}\partial_\Gamma\left(\kappa\mathbf{f}\cdot\mathbf{n}^\perp\right)\\
&\quad-\varepsilon^3(1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_\Gamma^2\mathrm{div}\,\mathbf{v}^2-\varepsilon^3\frac{c^2}{\omega^2}\left(\frac{\mathrm{i}}{4}\partial_\Gamma\left(\kappa\partial_\Gamma\mathrm{div}\,\mathbf{v}^1\right)\right)-\varepsilon^4\frac{c^2}{\omega^2}\left(\frac{\mathrm{i}}{4}\partial_\Gamma\left(\kappa\partial_\Gamma\mathrm{div}\,\mathbf{v}^2\right)\right)
\end{aligned}
$$

Now, neglecting the terms of order ε^3 on the right-hand side and using the equality $\eta=\varepsilon^2\omega\rho_0/2$, we obtain the boundary condition (2.11b) for $\mathbf{v}_{appr,2}$. Using (2.5a), (2.6a), and (2.7a), we find that $\mathbf{v}^{\varepsilon,2}$ satisfies

$$
\begin{aligned}
\nabla\mathrm{div}\,\mathbf{v}^{\varepsilon,2}+\frac{\omega^2}{c^2}\mathbf{v}^{\varepsilon,2} &= \varepsilon^2\frac{\mathrm{i}\omega^2}{2c^2}\Delta\mathbf{v}^0+\varepsilon^2\frac{\mathrm{i}\gamma'\omega^2}{2c^2}\nabla\mathrm{div}\,\mathbf{v}^0+\frac{\mathrm{i}\omega}{\rho_0c^2}\mathbf{f}\\
&=\varepsilon^2\frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\nabla\mathrm{div}\,\mathbf{v}^0+\frac{\mathrm{i}\omega}{\rho_0c^2}\mathbf{f}+\varepsilon^2\frac{\omega}{2c^2\rho_0}\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{f}\\
&=\varepsilon^2\frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\nabla\mathrm{div}\,\mathbf{v}^{\varepsilon,2}+\frac{\mathrm{i}\omega}{\rho_0c^2}\mathbf{f}+\varepsilon^2\frac{\omega}{2c^2\rho_0}\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{f}-\varepsilon^3\frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\nabla\mathrm{div}\,(\mathbf{v}^1+\varepsilon\mathbf{v}^2).
\end{aligned}
$$

Now, neglecting the terms of order ε^3 on the right-hand side and with $\eta=\varepsilon^2\omega\rho_0/2$, we obtain the PDE (2.11a) for $\mathbf{v}_{\mathrm{appr},2}$. Using (2.8) for $j=0,1,2$, we find that $p^{\varepsilon,N}=-\frac{\mathrm{i}\rho_0c^2}{\omega}\mathrm{div}\,\mathbf{v}^{\varepsilon,N}$, and the pressure approximations

$$
p_{\mathrm{appr},N}=-\frac{\mathrm{i}\rho_0c^2}{\omega}\mathrm{div}\,\mathbf{v}_{\mathrm{appr},N} \tag{2.12}
$$

of order 0, 1, and 2 follow that can be computed from the far-field velocity in an a posteriori step. Also, a near-field velocity corrector $\mathbf{v}^{BL}_{\mathrm{appr},N}$ close to the wall can be computed afterwards which will be derived and given in Section 2.6.

The impedance boundary conditions (2.10b) and (2.11b) have similarities with Wentzell's boundary conditions,[33–36] where, however, the second tangential derivative applies to the Neumann trace div $\mathbf{v}_{\mathrm{appr},N}$, and not to the Dirichlet trace, which is here $\mathbf{v}_{\mathrm{appr},N}\cdot\mathbf{n}$. The limit velocity model and the approximative models of higher order are of different kind as the exact model (2.1) since the terms $\Delta\mathbf{v}_{\mathrm{appr},N}$ and so $\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\mathbf{v}_{\mathrm{appr},N}$ are not present and—that is consistent—there is no condition on the tangential component.

2.3.2 | Approximative models for the pressure

The far-field pressure limit p^0 is the solution of the Helmholtz equation with one boundary condition,[26] which is again the natural approximative model of order 0:

$$
\Delta p_{\mathrm{appr},0}+\frac{\omega^2}{c^2}p_{\mathrm{appr},0}=\mathrm{div}\,\mathbf{f},\text{ in }\Omega, \tag{2.13a}
$$

$$
\nabla p_{\mathrm{appr},0}\cdot\mathbf{n}=\mathbf{f}\cdot\mathbf{n},\text{ on }\partial\Omega. \tag{2.13b}
$$

The PDE results by taking the divergence of (2.9a) and using (2.12) for $N=0$, and the boundary conditions follow from (2.9) as

$$
\nabla p_{\mathrm{appr},0}\cdot\mathbf{n}=-\frac{\mathrm{i}\rho_0c^2}{\omega}\nabla\mathrm{div}\,\mathbf{v}_{\mathrm{appr},0}\cdot\mathbf{n}=\mathrm{i}\rho_0\omega\mathbf{v}_{\mathrm{appr},0}\cdot\mathbf{n}+\mathbf{f}\cdot\mathbf{n}=\mathbf{f}\cdot\mathbf{n}.
$$

If the source $\mathbf{f}$ is localized away from the boundary $\partial\Omega$, then the boundary conditions (2.13b) are homogeneous, likewise the following impedance conditions of higher order.

In the same way, the approximative model of order $N=1$ for the pressure

$$
\Delta p_{\mathrm{appr},1}+\frac{\omega^2}{c^2}p_{\mathrm{appr},1}=\mathrm{div}\,\mathbf{f},\text{ in }\Omega, \tag{2.14a}
$$

$$\nabla p_{\mathrm{appr},1} \cdot \mathbf{n} + (1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma^2 p_{\mathrm{appr},1} = \mathbf{f}\cdot\mathbf{n} - (1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma\left(\mathbf{f}\cdot\mathbf{n}^\perp\right), \text{on } \partial\Omega, \tag{2.14b}$$

follows where just the divergence of (2.10a) is taken to obtain the PDE. The boundary conditions results using (2.10) by

$$\begin{aligned}\nabla p_{\mathrm{appr},1}\cdot\mathbf{n} &= \mathrm{i}\rho_0\omega\mathbf{v}_{\mathrm{appr},1}\cdot\mathbf{n} + \mathbf{f}\cdot\mathbf{n} = \frac{\mathrm{i}\rho_0 c^2}{\omega}(1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma^2 div\mathbf{v}_{\mathrm{appr},1}\cdot\mathbf{n} + \mathbf{f}\cdot\mathbf{n} - (1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\partial\left(\mathbf{f}\cdot\mathbf{n}^\perp\right)\\ &= -(1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma^2 div p_{\mathrm{appr},1}\cdot\mathbf{n} + \mathbf{f}\cdot\mathbf{n} - (1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma\left(\mathbf{f}\cdot\mathbf{n}^\perp\right).\end{aligned}$$

Similarly, the approximative model of order $N = 2$ for the pressure

$$\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\right)\Delta p_{\mathrm{appr},2} + \frac{\omega^2}{c^2}p_{\mathrm{appr},2} = \mathrm{div}\,\mathbf{f} \qquad \text{in } \Omega, \tag{2.15a}$$

$$\begin{aligned}&\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\right)\nabla p_{\mathrm{appr},2}\cdot\mathbf{n} + (1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma^2 p_{\mathrm{appr},2} + \frac{\mathrm{i}\eta}{2\omega\rho_0}\partial_\Gamma(\kappa\partial_\Gamma p_{\mathrm{appr},2})\\ &=\mathbf{f}\cdot\mathbf{n} - (1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma\left(\mathbf{f}\cdot\mathbf{n}^\perp\right) - \frac{\mathrm{i}\eta}{2\omega\rho_0}\partial_\Gamma\left(\kappa\mathbf{f}\cdot\mathbf{n}^\perp\right) - \frac{\mathrm{i}\eta}{\omega\rho_0}\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{f}\cdot\mathbf{n}, \text{ on } \partial\Omega,\end{aligned} \tag{2.15b}$$

is obtained similarly, where for the PDE follows after applying the divergence to (2.11a) and using that div $\mathbf{curl}_{2D}$ vanishes for smooth enough functions. The boundary conditions follows using (2.11) and

$$\begin{aligned}\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\right)\nabla p_{\mathrm{appr},2}\cdot\mathbf{n} &= -\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\right)\frac{\mathrm{i}\rho_0 c^2}{\omega}\nabla\mathrm{div}\,\mathbf{v}_{\mathrm{appr},2}\cdot\mathbf{n}\\ &= \mathrm{i}\rho_0\omega\mathbf{v}_{\mathrm{appr},2}\cdot\mathbf{n} + \mathbf{f}\cdot\mathbf{n} - \frac{\mathrm{i}\eta}{\rho_0\omega}\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{f}\cdot\mathbf{n}\\ &= \frac{\mathrm{i}\rho_0 c^2}{\omega}\left((1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma^2 div\mathbf{v}_{\mathrm{appr},2} + \frac{\mathrm{i}\eta}{2\omega\rho_0}\partial_\Gamma\left(\kappa\partial_\Gamma\mathrm{div}\,\mathbf{v}_{\mathrm{appr},2}\right)\right)\\ &\quad - \mathrm{i}\rho_0\omega\left(\frac{(\mathrm{i}-1)}{\omega\rho_0}\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma\left(\mathbf{f}\cdot\mathbf{n}^\perp\right) - \frac{\eta}{2\omega^2\rho_0^2}\partial_\Gamma\left(\kappa\mathbf{f}\cdot\mathbf{n}^\perp\right)\right) + \mathbf{f}\cdot\mathbf{n} - \frac{\mathrm{i}\eta}{\rho_0\omega}\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{f}\cdot\mathbf{n},\end{aligned}$$

and using again (2.12).

When the approximative far-field pressure of orders 0, 1, or 2 is computed using (2.13), (2.14), or (2.15), we may obtain a posteriori the approximative far-field velocity of order 0, 1, or 2 by

$$\mathbf{v}_{\mathrm{appr},N} = \frac{\mathrm{i}}{\rho_0\omega}(\mathbf{f} - \nabla p_{\mathrm{appr},N}), \text{ for } N = 0, 1, \text{ in } \Omega, \tag{2.16a}$$

$$\mathbf{v}_{\mathrm{appr},2} = \frac{\mathrm{i}}{\rho_0\omega}\mathbf{f} - \frac{\mathrm{i}}{\rho_0\omega}\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\right)\nabla p_{\mathrm{appr},2} + \frac{\eta}{\rho_0^2\omega^2}\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{f}, \text{ in } \Omega, \tag{2.16b}$$

2.4 | Weak formulations

The weak formulations for the approximative models of order 0, that is, (2.13) for the pressure and (2.9) for the velocity, read: Seek $p_{\mathrm{appr},0} \in H^1(\Omega)$ such that

$$\int_\Omega \nabla p_{\mathrm{appr},0}\cdot\nabla q' - \frac{\omega^2}{c^2}p_{\mathrm{appr},0}q\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\nabla q'\mathrm{d}\mathbf{x} \qquad \text{for all } q' \in H^1(\Omega) \tag{2.17}$$

and seek $\mathbf{v}_{\text{appr},0} \in H_0(\text{div}\,,\Omega)$ such that

$$\int_\Omega \text{div}\,\mathbf{v}_{\text{appr},0}\text{div}\,\mathbf{v}' - \frac{\omega^2}{c^2}\mathbf{v}_{\text{appr},0}\cdot\mathbf{v}'\text{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}'\text{d}\mathbf{x} \qquad \text{for all } \mathbf{v}' \in H_0(\text{div}\,,\Omega). \tag{2.18}$$

The impedance boundary conditions (2.14b) of order $N = 1$ and (2.15b) of order $N = 2$ are of Wentzell type; see Bonnaillie-Noël et al[37] and Schmidt and Heier[38] for the functional framework. With the Sobolev space $H^1(\Omega)\cap H^1(\partial\Omega)$ with functions that are in $H^1(\Omega)$ and whose traces are in $H^1(\partial\Omega)$, the weak formulations for the systems (2.14) and (2.15) are given as

Seek $p_{\text{appr},N} := H^1(\Omega)\cap H^1(\partial\Omega)$ such that

$$\begin{aligned}&\int_\Omega \left(1 - \frac{\text{i}\omega(\eta+\eta')\delta_{N=2}}{\rho_0 c^2}\right)\nabla p_{\text{appr},N}\cdot\nabla q' - \frac{\omega^2}{c^2}p_{\text{appr},N}q'\,\text{d}\mathbf{x} - \int_{\partial\Omega}\left((1+\text{i})\sqrt{\frac{\eta}{2\omega\rho_0}} + \frac{\text{i}\eta\delta_{N=2}}{2\omega\rho_0}\kappa\right)\partial_\Gamma p_{\text{appr},N}\partial_\Gamma q'\,\text{d}\sigma(\mathbf{x})\\ &= \int_\Omega \mathbf{f}\cdot\nabla q'\text{d}\mathbf{x} - \int_{\partial\Omega}\left((1+\text{i})\sqrt{\frac{\eta}{2\omega\rho_0}} + \frac{\text{i}\eta\delta_{N=2}}{2\omega\rho_0}\kappa\right)\mathbf{f}\cdot\mathbf{n}^\perp\partial_\Gamma q' + \frac{\text{i}\eta\delta_{N=2}}{\omega\rho_0}\mathbf{curl}_{2D}\,\text{curl}_{2D}\,\mathbf{f}\cdot\mathbf{n}q'\,\text{d}\sigma(\mathbf{x})\\ &\text{for all } q' \in H^1(\Omega)\cap H^1(\partial\Omega).\end{aligned} \tag{2.19}$$

Introducing the Lagrange multipliers $\lambda_{\text{appr},N} = \left(1 - \frac{\text{i}\omega(\eta+\eta')\delta_{N=2}}{\rho_0 c^2}\right)\text{div}\,\mathbf{v}_{\text{appr},N}$, $N = 1, 2$ on $\partial\Omega$, we find the mixed variational formulations for the systems (2.10) and (2.11): Seek $(\mathbf{v}_{\text{appr},N}, \lambda_{\text{appr},N}) \in H(\text{div}\,,\Omega)\times H^1(\partial\Omega)$ such that for all $(\mathbf{v}', \lambda') \in H(div,\Omega)\times H^1(\partial\Omega)$,

$$\begin{aligned}&\int_\Omega \left(1 - \frac{\text{i}\omega(\eta+\eta')\delta_{N=2}}{\rho_0 c^2}\right)\text{div}\,\mathbf{v}_{\text{appr},N}\text{div}\,\mathbf{v}' - \frac{\omega^2}{c^2}\mathbf{v}_{\text{appr},N}\cdot\mathbf{v}'\,\text{d}\mathbf{x}\\ &-\int_{\partial\Omega}\lambda_{\text{appr},N}\mathbf{v}'\cdot\mathbf{n}\,\text{d}\sigma(\mathbf{x}) = \int_\Omega\left(\mathbf{f} + \frac{\eta\,\delta_{N=2}}{\rho_0^2c^2}\mathbf{curl}_{2D}\,\text{curl}_{2D}\,\mathbf{f}\right)\cdot\mathbf{v}'\,\text{d}\mathbf{x},\end{aligned} \tag{2.20a}$$

$$\int_{\partial\Omega}\mathbf{v}_{\text{appr},N}\cdot\mathbf{n}\,\lambda' + \frac{c^2}{\omega^2}\frac{(1+\text{i})\sqrt{\frac{\eta}{2\omega\rho_0}} + \frac{\text{i}\eta\,\delta_{N=2}}{2\omega\rho_0}\kappa}{1 - \frac{\text{i}\omega(\eta+\eta')\delta_{N=2}}{\rho_0c^2}}\partial_\Gamma\lambda_{\text{appr},N}\partial_\Gamma\lambda'\,\text{d}\sigma(\mathbf{x}) = \int_{\partial\Omega}\left(\frac{1-\text{i}}{\omega\rho_0}\sqrt{\frac{\eta}{2\omega\rho_0}} + \frac{\eta\,\delta_{N=2}}{2\omega^2\rho_0^2}\kappa\right)\mathbf{f}\cdot\mathbf{n}^\perp\partial_\Gamma\lambda'\,\text{d}\sigma(\mathbf{x}). \tag{2.20b}$$

2.5 | Well-posedness and modeling error

Obviously, the system for the limit pressure (2.13) has no unique solutions for frequencies $\omega > 0$ for that $\frac{\omega^2}{c^2}$ is an eigenvalue of—Δ with Neumann boundary conditions—the eigenfrequencies. In Schmidt et al,[26] we have shown that the limit velocity system (2.9) has eigensolutions for the same frequencies, for which it does not provide a unique solution. If ω takes such a value by the Fredholm alternative,[39] the systems provide solutions if the source is orthogonal to all eigenfunctions. This is, however, in practise rather unlikely.

In the pressure and velocity systems of order 1, there is an additional dissipative term, which is, however, not sufficient to guarantee uniqueness for all frequencies in general. There might be eigenfunctions of the pressure limit systems that do not vary on $\partial\Omega$ such that they satisfy the first-order pressure system (2.14) with $\mathbf{f} = \boldsymbol{0}$. Note that the eigenfunctions $\mathbf{v}$ of the velocity limit systems whose Neumann trace div $\mathbf{v}$ is constant along on $\partial\Omega$ are also eigenfunctions of the first-order velocity system (2.10).

Only the volumic dissipative term of the two systems of order 2 guarantee, as for the original model, for existence and uniqueness for all frequencies $\omega > 0$. These properties will be shown and discussed by numerical experiments in Section 5. However, in the analysis, we assume that ω is not an eigenfrequency of the limit system.

Theorem 1 (Stability, existence, and uniqueness of $(\mathbf{v}_{\text{appr},N}, p_{\text{appr},N})$). *Let Ω be an open Lipschitz domain whose boundary is piecewise C^2 for $N = 2$, and let $\frac{\omega^2}{c^2}$ be distinct from the Neumann eigenvalues of $-\Delta$ of Ω and let $\mathbf{f} \in H(curl_{2D},\Omega)$ and $\boldsymbol{curl}_{2D}\ curl_{2D}\ \mathbf{f} \in H(curl_{2D},\Omega)$ for $N = 2$. Then, there exists a constant $\eta_0 > 0$ such that for all $\eta \in (0,\eta_0)$, each of the systems (2.13)–(2.15) provides a unique solution $p_{appr,N} \in H^1(\Omega)$, $N = 0, 1, 2$ and each of the the systems (2.9)–(2.11)* *provide a unique solution $\mathbf{v}_{appr,N} \in H(div\ ,\Omega) \cap H(\boldsymbol{curl},\Omega)$, $N = 0, 1, 2$, respectively. Furthermore, there exists a constant C independent of η such that the stability estimates*

$$\|\mathbf{v}_{appr,N}\|_{H(div,\Omega)} + \|p_{appr,N}\|_{H^1(\Omega)} \leq C\left(\|\mathbf{f}\|_{L^2(\Omega)} + \eta\,\delta_{N=2}\|\boldsymbol{curl}_{2D}\ curl_{2D}\ \mathbf{f}\|_{L^2(\Omega)}\right), \tag{2.21a}$$

$$\|curl_{2D}\mathbf{v}_{appr,N}\|_{L^2(\Omega)} \leq C\left(\|curl_{2D}\ \mathbf{f}\|_{L^2(\Omega)} + \eta\,\delta_{N=2}\|curl_{2D}\boldsymbol{curl}_{2D}\ curl_{2D}\ \mathbf{f}\|_{L^2(\Omega)}\right) \tag{2.21b}$$

hold. Moreover, the approximative models are equivalent as the identities (2.12) and (2.16) hold.

The proof will be given in Section 3. Note that the equivalent systems (2.15), (2.16b) and (2.11), (2.12) provide a unique solution $(\mathbf{v}_{\text{appr},2}, p_{\text{appr},2}) \in H(div,\Omega) \cap H(\text{curl}_{2D},\Omega) \times H^1(\Omega)$ for any $\omega > 0$, however, with a constant $C = C(\eta)$ that may blow up for $\eta \to 0$. This is due to the factor $(1 - \mathrm{i}\omega(\eta + \eta')/(\rho_0 c^2))$ in front of $\Delta p_{\text{appr},2}$ or $\nabla\text{div}\ \mathbf{v}_{\text{appr},2}$, respectively, that has a nonvanishing imaginary part and implies an elliptic bilinear form with ellipticity constant of order $1/\eta$.

The approximative models were derived to be formally consistent with the asymptotic expansion to the respective order, and the following lemma conditions are given under which the approximative solutions are close to the respective asymptotic expansion.

Lemma 2 (Approximative solution is close to asymptotic expansion). *Let Ω be an open smooth domain, and let $\frac{\omega^2}{c^2}$ be distinct from the Neumann eigenvalues and let $\mathbf{f} \in (L^2(\Omega))^2$ where $curl_{2D}\ \mathbf{f} \in H^m(\Omega)$ for any $m \in \mathbb{N}$ and $\mathbf{f} \in H^m(\Omega_\Gamma))^2$ for any $m \in \mathbb{N}$ in some neighborhood $\Omega_\Gamma \subset \Omega$ of $\partial\Omega$, that is, $\partial\Omega \subset \partial\Omega_\Gamma$.*

Then, it holds for the solution $\mathbf{v}_{appr,N}$ of the approximative models (2.10) and (2.11) for $N = 1, 2$, respectively, that

$$curl_{2D}\mathbf{v}_{appr,N} - \sum_{j=0}^{N}\left(\frac{2\eta}{\omega\rho_0}\right)^{\frac{j}{2}} curl_{2D}\mathbf{v}^j = 0,$$

and there exist constants η_0 and C independent of η such that for $\mathbf{v}_{appr,N}$ and for $p_{appr,N}$ for $N = 1, 2$ given by (2.12) and any $\eta \in (0, \eta_0)$ it holds

$$\left\|\mathbf{v}_{appr,N} - \sum_{j=0}^{N}\left(\frac{2\eta}{\omega\rho_0}\right)^{\frac{j}{2}}\mathbf{v}^j\right\|_{H(div,\Omega)} + \left\|p_{appr,N} - \sum_{j=0}^{N}\left(\frac{2\eta}{\omega\rho_0}\right)^{\frac{j}{2}}p^j\right\|_{H^1(\Omega)} \leq C\eta^{\frac{N+1}{2}}. \tag{2.22}$$

The lemma means in other words that the asymptotic expansions of the exact solution and the approximative solution of the respective order coincide. The proof of the lemma that will be given in Section 4 is not straightforward due to the singular perturbed nature of the equations.

As the error of the asymptotic expansion was shown in Schmidt et al.,[26, Lemma 2.2] application of the triangle inequality and Lemma 2 implies a bound of the modeling error as stated in the following.

Theorem 2 (Modeling error). *Let the assumptions of Lemma 2 be fulfilled. Then, the approximative solution $(\mathbf{v}_{appr,N}, p_{appr,N})$ for $N = 0, 1, 2$ satisfies*

$$\|p - p_{appr,N}\|_{H^1(\Omega)} \leq C\eta^{\frac{N+1}{2}}, \tag{2.23a}$$

and for any $\delta > 0$

$$\|\mathbf{v} - \mathbf{v}_{appr,N}\|_{(H^1(\Omega\setminus\bar{\Omega}_\delta))^2} \leq C_{\delta,N}\,\eta^{\frac{N+1}{2}}, \tag{2.23b}$$

where Ω_δ is the original domain without a δ-neighborhood of $\partial\Omega$ and where the constants C, $C_{\delta,N} > 0$ do not depend on η.

2.6 | Approximative boundary layer correctors for the velocity

Close to the wall, the far-field velocities have to be corrected by an (approximative) boundary layer velocity field

$$\mathbf{v}^{BL}{}_{\text{appr},N} = \sqrt{\frac{2\eta}{\omega\rho_0}}\,\mathbf{curl}_{2D}(\phi_{\text{appr},N}\,\chi), \tag{2.24}$$

where χ is an admissible cut-off function. To define $\phi_{\text{appr},N}$, we take the near-field terms $\mathbf{v}_{BL}^j = \sqrt{\frac{2\eta}{\omega\rho_0}}\mathbf{curl}_{2D}(\phi^j\chi)$ of the asymptotic expansion[26] with

$$\phi^0(\mathbf{x}) = \frac{1}{2}(1+\mathrm{i})\mathrm{e}^{-(1-\mathrm{i})\frac{s}{\varepsilon}}\mathbf{v}^0\cdot\mathbf{n}^\perp, \tag{2.25a}$$

$$\phi^1(\mathbf{x}) = \frac{1}{2}(1+\mathrm{i})\mathrm{e}^{-(1-\mathrm{i})\frac{s}{\varepsilon}}\left(\mathbf{v}^2\cdot\mathbf{n}^\perp + \frac{1}{4}(3+\mathrm{i})\frac{\kappa s}{\varepsilon}\mathbf{v}^1\cdot\mathbf{n}^\perp\right), \tag{2.25b}$$

$$\begin{aligned}\phi^2(\mathbf{x}) = \frac{1}{2}(1+\mathrm{i})\mathrm{e}^{-(1-\mathrm{i})\frac{s}{\varepsilon}}\Bigg(&\mathbf{v}^2\cdot\mathbf{n}^\perp + \frac{1}{4}(3+\mathrm{i})\frac{\kappa s}{\varepsilon}\mathbf{v}^1\cdot\mathbf{n}^\perp \\ &+\left(\frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2} + \frac{3}{16}\kappa^2\left(\mathrm{i}+(1+\mathrm{i})\frac{s}{\varepsilon}+\frac{2s^2}{\varepsilon^2}\right)\right)\mathbf{v}^0\cdot\mathbf{n}^\perp + \frac{1}{4}\left(\mathrm{i}+(1+\mathrm{i})\frac{s}{\varepsilon}\right)\partial_\Gamma^2\mathbf{v}^0\cdot\mathbf{n}^\perp\Bigg).\end{aligned} \tag{2.25c}$$

Writing near-field terms for $\sum_{j=0}^N \varepsilon^j\mathbf{v}_{BL}^j$ and so $\sum_{j=0}^N \varepsilon^j\phi^j$ and neglecting the terms of order $N+1$, we find

$$\phi_{\text{appr},0}(\mathbf{x}) = \frac{1}{2}(1+\mathrm{i})\mathrm{e}^{-(1-\mathrm{i})s\sqrt{\frac{\omega\rho_0}{2\eta}}}(\mathbf{v}_{\text{appr},0}\cdot\mathbf{n}^\perp)(\mathbf{x}_{\partial\Omega}), \tag{2.26a}$$

$$\phi_{\text{appr},1}(\mathbf{x}) = \frac{1}{2}(1+\mathrm{i})\mathrm{e}^{-(1-\mathrm{i})s\sqrt{\frac{\omega\rho_0}{2\eta}}}\left(1+\frac{1}{4}(3+\mathrm{i})\kappa s\right)(\mathbf{v}_{\text{appr},1}\cdot\mathbf{n}^\perp)(\mathbf{x}_{\partial\Omega}),$$

$$\begin{aligned}\phi_{\text{appr},2}(\mathbf{x}) = \frac{1}{2}(1+\mathrm{i})\mathrm{e}^{-(1-\mathrm{i})s\sqrt{\frac{\omega\rho_0}{2\eta}}}\Bigg(\Bigg(&1+\frac{\mathrm{i}}{\omega\rho_0}\left(\frac{\omega^2}{c^2}(\eta+\eta')+\frac{3}{8}\kappa^2\eta\right)+\frac{\kappa s}{4}\left((3+\mathrm{i})+\frac{3}{4}\kappa\sqrt{\frac{2\eta}{\omega\rho_0}}\right) \\ &+\frac{3}{8}\kappa^2 s^2\Bigg)(\mathbf{v}_{\text{appr},2}\cdot\mathbf{n}^\perp)(\mathbf{x}_{\partial\Omega}) + \frac{1}{4}\left(\frac{2\eta\mathrm{i}}{\omega\rho_0}+(1+\mathrm{i})s\sqrt{\frac{2\eta}{\omega\rho_0}}\right)\partial_\Gamma^2(\mathbf{v}_{\text{appr},2}\cdot\mathbf{n}^\perp)(\mathbf{x}_{\partial\Omega})\Bigg).\end{aligned}$$

The tangential component of the approximative velocity has to be evaluated for each point $\mathbf{x}$ close to the boundary on its nearest point $\mathbf{x}_{\partial\Omega}$ on the boundary. To be able to define the near-field velocity correctors, enough smoothness of the boundary is required. Adding the near-field correction $\mathbf{v}^{BL}{}_{\text{appr},N}$ to the far-field velocity $\mathbf{v}_{\text{appr},N}$ for $N = 0, 1, 2$, the tangential and the normal components of the sum vanish on the boundary up to terms of order $N+1$ which follows from the same properties of the multiscale expansion.[26]

3 | STABILITY OF THE APPROXIMATIVE MODELS

In this section, we first define generalized approximative pressure and velocity systems that generalizes the derived approximative models and show their well-posedness. Even so we derived the approximative models for smooth domains, the analysis of the generalized approximative systems requires less regularity. Considering the generalized systems, we will not only benefit from a more compact notation, but more general source terms will allow us to prove the bounds on the modeling error in Section 4.

3.1 | Well-posedness for a generalized approximative pressure system

In this section, we analyze the well-posedness of a class of generalized approximative pressure problems

$$\operatorname{div}\ \left(\alpha_\eta\nabla p_\eta\right) + \frac{\omega^2}{c^2}p_\eta = -div\mathbf{g}_\eta,\ \text{in}\ \Omega, \tag{3.1a}$$

$$\alpha_\eta\nabla p_\eta\cdot\mathbf{n} + \partial_\Gamma\left(\beta_\eta\partial_\Gamma p_\eta\right) = \mathbf{g}_\eta\cdot\mathbf{n} + \partial_\Gamma h_\eta,\ \text{ on }\ \partial\Omega, \tag{3.1b}$$

with $\alpha_\eta, \beta_\eta \in L^\infty(\partial\Omega)$. For Ω smooth enough, the weak formulation of (3.1) is given as follows: Seek $p_\eta \in H^1_{\beta_\eta} := H^1(\Omega) \cap H^1(\partial\Omega)$ such that for all $q' \in H^1(\Omega) \cap H^1(\partial\Omega)$

$$\int_\Omega \alpha_\eta \nabla p_\eta \cdot \nabla q' - \frac{\omega^2}{c^2} p_\eta q' \,\mathrm{d}\mathbf{x} - \int_{\partial\Omega} \beta_\eta \partial_\Gamma p_\eta \partial_\Gamma q' \,\mathrm{d}\sigma(\mathbf{x}) = \int_\Omega \mathbf{g}_\eta \cdot \nabla q' \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} h_\eta \partial_\Gamma q' \,\mathrm{d}\sigma(\mathbf{x}). \tag{3.2}$$

The approximative pressure systems (2.14) and (2.15) of orders 1 or 2, respectively, belong to this generalized approximative pressure system. If we indicate the respective functions for the system of order N with a superscript N, we find that

$$\alpha^1_\eta = 1,\ \alpha^2_\eta = 1 - \frac{\mathrm{i}\omega(\eta + \eta')}{\rho_0 c^2}, \beta^1_\eta = (1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}, \beta^2_\eta = \beta^1_\eta + \frac{\mathrm{i}\eta}{2\omega\rho_0}\kappa,$$

$$\mathbf{g}^1_\eta = \mathbf{g}^2_\eta = \mathbf{f}, h^1_\eta = -\frac{1+\mathrm{i}}{\omega\rho_0}\sqrt{\frac{\eta}{2\omega\rho_0}}\mathbf{f}\cdot\mathbf{n}^\perp, h^2_\eta = h^1_\eta + \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\mathbf{f}\cdot\mathbf{n} - \frac{\mathrm{i}\eta}{2\omega\rho_0}\kappa\mathbf{f}\cdot\mathbf{n}^\perp.$$

Lemma 3 (Well-posedness of the generalized approximative pressure system). *Let Ω be a Lipschitz domain and $\frac{\omega^2}{c^2}$ be distinct from the Neumann eigenvalues of $-\Delta$ in Ω. Moreover, let $\mathbf{g}_\eta \in (L^2(\Omega))^2$, $h_\eta \in L^2(\partial\Omega)$ for all $\eta \in \eta > 0$, $\alpha_\eta \in L^\infty(\Omega)$ with $\alpha_\eta \to 1$ for $\eta \to 0$ and $Im\,\alpha_\eta \le 0$, $\beta_\eta \in L^\infty(\partial\Omega)$ with $\beta_\eta \to 0$ for $\eta \to 0$ and $Im\,\beta_\eta \ge c|\beta_\eta|$ for some $c > 0$ and invertible with $\beta_\eta^{-1} \in L^\infty(\partial\Omega)$. Then, there exists a constant $\eta_m > 0$ such that for any $\eta \in (0, \eta_m)$ the formulation (3.2) has a unique solution $p_\eta \in H^1(\Omega)$. Furthermore, there exists a constant $C = C(\eta_m) > 0$ not depending on η such that*

$$\|p_\eta\|_{H^1(\Omega)} + \sqrt{|\beta_\eta|}|p_\eta|_{H^1(\partial\Omega)} \le C\left(\|\mathbf{g}_\eta\|_{(L^2(\Omega))^2} + \|\beta_\eta^{-1/2} h_\eta\|_{L^2(\partial\Omega)}\right). \tag{3.3}$$

Proof. The proof is by contradiction and we suppose, contrary to our claim, that the estimate (3.3) is false. Then, there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ with $\eta_n \to 0$, a bounded sequence $\{p_n\}_{n\in\mathbb{N}}$ with $\|p_n\|_{H^1(\Omega)} + |p_n|_{H^1(\partial\Omega)} = 1$ and a sequence $\{(\mathbf{g}_n, h_n)\}_{n\in\mathbb{N}}$ with

$$\|\mathbf{g}_n\|_{(L^2(\Omega))^2} + \|\beta_{\eta_n}^{-1/2} h_n\|_{L^2(\partial\Omega)} \to 0, \tag{3.4}$$

such that p_n is a solution of (3.2) where $\mathbf{g}_\eta$, h_η, and η are replaced by $\mathbf{g}_n$, h_n, and η_n.

Then, there exists a weakly converging subsequence, again called $\{p_n\}_{n\in\mathbb{N}}$, whose limit p for $n \to \infty$ is with the assumptions $\alpha_{\eta_n} \to 1$ and $\beta_{\eta_n} \to 0$, the solution of the limit problem:

$$\int_\Omega \nabla p \cdot \nabla q' - \frac{\omega^2}{c^2} p q' \,\mathrm{d}\mathbf{x} = 0 \ \forall q' \in H^1(\Omega).$$

By the assumption on ω, it has a unique solution $p = 0$. Hence,

$$p_n \rightharpoonup 0 \ \text{ in } \ H^1(\Omega).$$

As $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, we have the strong convergence

$$p_n \to 0 \ \text{ in } \ L^2(\Omega).$$

Now, testing the variational formulation for p_n with $q' = -\overline{p_n}$ and taking the imaginary part, we obtain

$$E_n := -\int_\Omega Im\alpha_{\eta_n}|p_n|^2 \mathrm{d}\mathbf{x} + \int_{\partial\Omega} Im\beta_{\eta_n}|\partial_\Gamma p_n|^2 \mathrm{d}\sigma(\mathbf{x}) = -Im\int_\Omega \mathbf{g}_n \cdot \nabla\bar{p}_n \mathrm{d}\mathbf{x} - Im\int_{\partial\Omega} h_n \partial_\Gamma \bar{p}_n \mathrm{d}\sigma(\mathbf{x}).$$

With the assumption we have $c|\beta_n| \le Im\beta_n$ for some $c > 0$ and using $0 \le -Im\alpha_n$ we obtain a lower bound and using the Cauchy-Schwarz inequality a upper bound and conclude that

$$c|\beta_{\eta_n}|\,|p_n|^2_{H^1(\partial\Omega)} \le E_n \le \|\mathbf{g}_n\|_{(L^2(\Omega))^2}|p_n|_{H^1(\Omega)} + \|h_n\|_{L^2(\partial\Omega)}|p_n|_{H^1(\partial\Omega)}.$$

Using the inequality $2ab \leq \delta^{-1}a^2 + \delta b^2$ for all $a, b \in \mathbb{R}$ and $\delta > 0$, we find for $\delta = c|\beta_{\eta_n}|$

$$\frac{c}{2}|\beta_{\eta_n}|\,|p_n|^2_{H^1(\partial\Omega)} \leq \|\mathbf{g}_n\|_{(L^2(\Omega))^2}|p_n|_{H^1(\Omega)} + \frac{1}{2c|\beta_{\eta_n}|}\|h_n\|^2_{L^2(\partial\Omega)} = \|\mathbf{g}_n\|_{(L^2(\Omega))^2}|p_n|_{H^1(\Omega)} + \frac{1}{2c}\|\beta_{\eta_n}^{-1/2}h_n\|^2_{L^2(\partial\Omega)}.$$

With $|p_n|_{H^1(\Omega)} \leq 1$, $\|\mathbf{g}_n\|_{(L^2(\Omega))^2} \to 0$ and $\|\beta_{\eta_n}^{-1/2}h_n\|_{L^2(\partial\Omega)} \to 0$ by (3.4), we conclude that

$$\sqrt{|\beta_{\eta_n}|}|p_n|_{H^1(\partial\Omega)} \to 0,\ |p_n|_{H^1(\Omega)} \to 1. \tag{3.5}$$

Finally, testing the variational formulation for p_n with $q' = \overline{p_n}$, we find for n large enough

$$\frac{1}{2}|p_n|^2_{H^1(\Omega)} \leq C\left(\|\mathbf{g}_n\|_{L^2(\Omega)}\|p_n\|_{L^2(\Omega)} + \|\beta_{\eta_n}^{-1/2}h_n\|_{L^2(\partial\Omega)}|\sqrt{\beta_{\eta_n}}p_n|_{H^1(\partial\Omega)} + \frac{\omega^2}{c^2}\|p_n\|^2_{L^2(\Omega)} + |\sqrt{\beta_{\eta_n}}p_n|^2_{H^1(\partial\Omega)}\right) \to 0 \text{ for } n \to \infty.$$

This contradicts (3.5), and hence, the assumption and (3.3) follow for η small enough. The stability estimates the uniqueness of a solution, and with the Fredholm alternative, its existence follows. This completes the proof. ☐

3.2 | Well-posedness for a generalized approximative velocity system

In this section, we analyze the well-posedness of a class of approximative velocity problems

$$\nabla\left(\alpha_\eta \text{div } \mathbf{w}_\eta\right) + \frac{\omega^2}{c^2}\mathbf{w}_\eta = \mathbf{g}_\eta, \text{ in } \Omega, \tag{3.6a}$$

$$\mathbf{w}_\eta \cdot \mathbf{n} - \partial_\Gamma\left(\beta_\eta \partial_\Gamma div \mathbf{w}_\eta\right) = \partial_\Gamma h_\eta, \text{ on } \partial\Omega, \tag{3.6b}$$

with $\alpha_\eta, \beta_\eta \in L^\infty(\partial\Omega)$, to which the approximative velocity systems (2.10) and (2.11) of orders 1 or 2, respectively, belong to. If we indicate the respective functions for the system of order N with a superscript, we find that

$$\alpha_\eta^1 = 1, \alpha_\eta^2 = 1 - \frac{\mathrm{i}\omega(\eta + \eta')}{\rho_0 c^2}, \beta_\eta^1 = (1+\mathrm{i})\frac{c^2}{\omega^2}\sqrt{\frac{\eta}{2\omega\rho_0}}, \beta_\eta^2 = \beta_\eta^1 + \frac{c^2}{\omega^2}\frac{\mathrm{i}\eta}{2\omega\rho_0}\kappa,$$
$$\mathbf{g}_\eta^1 = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}, \mathbf{g}_\eta^2 = \mathbf{g}_\eta^1 + \frac{\eta}{\rho_0^2 c^2}\mathbf{curl}_{2D}\ \mathrm{curl}_{2D}\ \mathbf{f}, h_\eta^1 = \frac{\mathrm{i}-1}{\omega\rho_0}\sqrt{\frac{\eta}{2\omega\rho_0}}\mathbf{f}\cdot\mathbf{n}^\perp, h_\eta^2 = \alpha_\eta^2\left(h_\eta^1 - \frac{\eta}{2\omega^2\rho_0^2}\kappa\mathbf{f}\cdot\mathbf{n}^\perp\right).$$

Moreover, the system (3.6) will be useful for error estimates.

With $\lambda_\eta = \alpha_\eta div \mathbf{w}_\eta$ on $\partial\Omega$ the variational formulation for (3.6) is given by: Seek $(\mathbf{w}_\eta, \lambda_\eta) \in H(\text{div}\ ,\Omega) \times H^1(\partial\Omega)$ such that

$$\int_\Omega \alpha_\eta \text{div } \mathbf{w}_\eta \text{div } \mathbf{v}' - \frac{\omega^2}{c^2}\mathbf{w}_\eta \cdot \mathbf{v}'\mathrm{d}\mathbf{x} - \int_{\partial\Omega}\lambda_\eta \mathbf{v}' \cdot \mathbf{n}\mathrm{d}S = -\int_\Omega \mathbf{g}_\eta \cdot \mathbf{v}'\mathrm{d}\mathbf{x} \qquad \forall \mathbf{v}' \in H(\text{div}\ ,\Omega), \tag{3.7a}$$

$$\int_{\partial\Omega}\mathbf{w}_\eta \cdot \mathbf{n}\lambda' + \alpha_\eta^{-1}\beta_\eta\partial_\Gamma\lambda_\eta\partial_\Gamma\lambda'\mathrm{d}S = -\int_{\partial\Omega}h_\eta\partial_\Gamma\lambda'\mathrm{d}S\ \forall \lambda' \in H^1(\partial\Omega). \tag{3.7b}$$

The system (3.7) is a saddle point problem with penalty term.[40, Chapter III, §4] Note that due to integration by parts, we can consider (3.7) with sources $h_\eta \in L^2(\partial\Omega)$.

Lemma 4 (Well-posedness of the generalized approximative velocity system)**.** *Let the assumption of Lemma 3 be fulfilled. Then, there exists a constant $\eta_m > 0$ such that for any $\eta \in (0, \eta_m)$, the system (3.7) has a unique solution $\mathbf{w}_\eta \in H(div\ ,\Omega)$. Furthermore, there exists a constant $C = C(\eta_m) > 0$ not depending on η such that*

$$\|\mathbf{w}_\eta\|_{H(div,\Omega)} \leq C\left(\|\mathbf{g}_\eta\|_{(L^2(\Omega))^2} + \|\beta_\eta^{-1/2}h_\eta\|_{L^2(\partial\Omega)}\right). \tag{3.8a}$$

If, moreover, $curl_{2D}\mathbf{g}_\eta \in L^2(\Omega)$, then it holds

$$\|curl_{2D}\mathbf{w}_\eta\|_{L^2(\Omega)} = \frac{c^2}{\omega^2}\|curl_{2D}\mathbf{g}_\eta\|_{L^2(\Omega)}. \tag{3.8b}$$

Proof. We start with the Helmholtz decomposition $\mathbf{w}_\eta = \nabla\psi_\eta + \mathbf{w}_{\eta,0}$ with the scalar potential $\psi_\eta \in H^1_\star(\Omega) := \{\psi \in H^1(\Omega), \int_\Omega \psi \,\mathrm{d}\mathbf{x} = 0\}$ that is uniquely defined as the vanishing mean is prescribed and $\mathbf{w}_{\eta,0} \in (\nabla H^1_\star(\Omega))^\perp := \{\mathbf{w}_0 \in H(\mathrm{div}\,,\Omega) : \int_\Omega \mathbf{w}_0\cdot\nabla\psi\,\mathrm{d}\mathbf{x} = 0 \text{ for all } \psi \in H^1_\star(\Omega)\}$ in the orthogonal complement. As by integration by parts

$$\int_\Omega \mathbf{w}_0\cdot\nabla\psi\,\mathrm{d}\mathbf{x} = -\int_\Omega \psi\,\mathrm{div}\,\mathbf{w}_0\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\psi\mathbf{w}_0\cdot\mathbf{n}\,\mathrm{d}\sigma(\mathbf{x}) = 0,$$

it is indeed $(\nabla H^1_\star(\Omega))^\perp = H_0(\mathrm{div}\,0,\Omega) := \{\mathbf{w}_0 \in H(div,\Omega) : div\mathbf{w}_0 = 0, \mathbf{w}_0\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$, that is, the functions $\mathbf{w}_{\eta,0}$ are divergence free and vanish in its normal component on the boundary.

Now, testing (3.7a) with $\mathbf{v}' \in H_0(div0,\Omega)$, we find that $\mathbf{w}_{\eta,0}$ is uniquely defined as c^2/ω^2 times the $L^2(\Omega)$-projection of $\mathbf{g}_\eta$ onto $H_0(div0,\Omega)$. Hence, the estimates (3.8) hold for the component $\mathbf{w}_{\eta,0}$.

Furthermore, let $\phi_\eta := \mathrm{div}\,\mathbf{w}_\eta = \Delta\psi_\eta$, and hence, $\lambda_\eta = \alpha_\eta\mathrm{div}\,\mathbf{w}_\eta = \alpha_\eta\phi_\eta$ on $\partial\Omega$. Inserting ϕ_η in (3.6a), we obtain

$$\alpha_\eta\nabla\phi_\eta = \mathbf{g}_\eta - \frac{\omega^2}{c^2}\left(\mathbf{w}_{\eta,0} + \nabla\psi_\eta\right) \in L^2(\Omega). \tag{3.9a}$$

As we have a Poincaré inequality for functions in $H^1_\star(\Omega)$, we find $\|\phi_\eta\|_{L^2(\Omega)} \le C|\phi_\eta|_{H^1(\Omega)}$ for some $C > 0$ and so $\phi_\eta \in H^1(\Omega)$. Then, inserting ϕ_η in (3.6b), we find

$$\partial_\Gamma(\beta_\eta\partial_\Gamma\phi_\eta) = \nabla\psi_\eta\cdot\mathbf{n} - \partial_\Gamma h_\eta \in H^{-1}(\partial\Omega), \tag{3.9b}$$

and so $\phi_\eta \in H^1(\Omega)\cap H^1(\partial\Omega)$. Now, multiplying (3.9a) by $\nabla\phi'$ for $\phi' \in H^1(\Omega)\cap H^1(\partial\Omega)$ and integrating over Ω and using integration by parts for the term with $\nabla\phi_\eta$ and inserting $\phi_\eta = \Delta\psi_\eta$, we find the equality

$$\int_\Omega \alpha_\eta\nabla\phi_\eta\cdot\nabla\phi' - \frac{\omega^2}{c^2}\phi_\eta\phi'\,\mathrm{d}\mathbf{x} + \frac{\omega^2}{c^2}\int_{\partial\Omega}\nabla\psi_\eta\cdot\mathbf{n}\phi'\,\mathrm{d}\sigma(\mathbf{x}) = \int_\Omega \mathbf{g}_\eta\cdot\nabla\phi'\,\mathrm{d}\mathbf{x}. \tag{3.10}$$

Then, inserting (3.9b) and using integration by parts, we obtain a variational formulation for ϕ_η: Seek $\phi_\eta \in H^1(\Omega)\cap H^1(\partial\Omega)$ such that for all $\phi' \in H^1(\Omega)\cap H^1(\partial\Omega)$

$$\int_\Omega \alpha_\eta\nabla\phi_\eta\cdot\nabla\phi' - \frac{\omega^2}{c^2}\phi_\eta\phi'\,\mathrm{d}\mathbf{x} - \frac{\omega^2}{c^2}\int_{\partial\Omega}\beta_\eta\partial_\Gamma\phi_\eta\partial_\Gamma\phi'\,\mathrm{d}\sigma(\mathbf{x}) = \int_\Omega \mathbf{g}_\eta\cdot\nabla\phi'\,\mathrm{d}\mathbf{x} + \frac{\omega^2}{c^2}\int_{\partial\Omega}h_\eta\partial_\Gamma\phi'\,\mathrm{d}\sigma(\mathbf{x}). \tag{3.11}$$

Following the lines of the proof of Lemma 3, we see that this formulation provides a unique solution $\phi_\eta \in H^1(\Omega)\cap H^1(\partial\Omega)$ with

$$\|div\nabla\psi_\eta\|_{L^2(\Omega)} \le \|\phi_\eta\|_{H^1(\Omega)} \le C\left(\|\mathbf{g}_\eta\|_{(L^2(\Omega))^2} + \|\beta_\eta^{-1/2}h_\eta\|_{L^2(\partial\Omega)}\right).$$

Now, applying the triangle inequality to (3.9a) and using the estimate above and the estimate on $\mathbf{w}_{\eta,0}$, we find

$$\|\nabla\psi_\eta\|_{L^2(\Omega)} \le C\left(\|\mathbf{g}_\eta\|_{(L^2(\Omega))^2} + \|\beta^{-1/2}h_\eta\|_{L^2(\partial\Omega)}\right).$$

Hence, $\nabla\psi_\eta$ fulfills (3.8a) as well, and so $\mathbf{w}_\eta = \mathbf{w}_{\eta,0} + \nabla\psi_\eta$. Finally, applying curl_{2D} to (3.6a), we find that

$$\mathrm{curl}_{2D}\mathbf{w}_\eta = \frac{c^2}{\omega^2}\mathrm{curl}_{2D}\mathbf{g}_\eta$$

and so the second estimate. This completes the proof. □

3.3 | Well-posedness and equivalence of approximative models for pressure and velocity

With the well-posedness of the generalized approximative models for pressure and velocity, we are in the position to prove the well-posedness and the equivalence of the approximative models.

Proof of Theorem. The well-posedness of the approximative models (2.13) and (2.9) of order 0 was proven in Schmidt et al.[26]

The well-posedness of the approximative models (2.14), (2.15) for $p_{\mathrm{appr},N}$, $N = 1, 2$ and (2.10), (2.11) for $\mathbf{v}_{\mathrm{appr},N}$, $N = 1, 2$ follows from Lemmas 3 and 4, where the assumption on the smoothness of the boundary $\partial\Omega$ guarantees that the curvature $\kappa \in L^\infty(\partial\Omega)$. It remains to show the equivalence of the definitions (2.9)–(2.12) and (2.13)–(2.16) of $(\mathbf{v}_{\mathrm{appr},N}, p_{\mathrm{appr},N})$.

First, we take $(\mathbf{v}_{\mathrm{appr},N}, p_{\mathrm{appr},N})$ defined by (2.9)–(2.12). With the assumption of $\mathbf{f} \in L^2(\Omega)$, it follows that $\nabla \mathrm{div}\, \mathbf{v}_{\mathrm{appr},N} \in L^2(\Omega)$, and hence, $\mathrm{div}\, \mathbf{v}_{\mathrm{appr},N} \in H^1(\Omega)$. Then, $p_{\mathrm{appr},N} = -(\mathrm{i}\rho_0 c^2)/\omega \mathrm{div}\, \mathbf{v}_{\mathrm{appr},N}$, see (2.12), fulfills with the derivations in Section 2.3.2, the systems (2.13)–(2.15). Then, replacing $\nabla \mathrm{div}\, \mathbf{v}_{\mathrm{appr},N}$ in (2.9a), (2.10a), and (2.11a) by $(\mathrm{i}\omega)/(\rho_0 c^2)\nabla p_{\mathrm{appr},N}$ and multiplying by c^2/ω^2, we see that $\mathbf{v}_{\mathrm{appr},N}$ fulfills (2.16).

Now, we take $(\mathbf{v}_{\mathrm{appr},N}, p_{\mathrm{appr},N})$ defined by (2.13)–(2.16). Then, applying the divergence to $\mathbf{v}_{\mathrm{appr},N}$ and inserting (2.13)–(2.15), we find that

$$\mathrm{div}\, \mathbf{v}_{\mathrm{appr},N} = \frac{\mathrm{i}}{\rho_0\omega} div\, \mathbf{f} - \frac{\mathrm{i}}{\rho_0\omega}\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')\delta_{N=2}}{\rho_0 c^2}\right)\Delta p_{\mathrm{appr},2} = \frac{\mathrm{i}\omega}{\rho_0 c^2} p_{\mathrm{appr},N} \in H^1(\Omega). \tag{3.12}$$

Then, applying ∇, multiplying with $\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')\delta_{N=2}}{\rho_0 c^2}\right)$, and using (2.16), we obtain

$$\begin{aligned}\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')\delta_{N=2}}{\rho_0 c^2}\right)\nabla \mathrm{div}\, \mathbf{v}_{\mathrm{appr},N} &= \frac{\mathrm{i}\omega}{\rho_0 c^2}\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')\delta_{N=2}}{\rho_0 c^2}\right)\nabla p_{\mathrm{appr},N}\\ &= \frac{\omega^2}{c^2}\left(-\mathbf{v}_{\mathrm{appr},N} + \frac{\mathrm{i}}{\rho_0\omega}\mathbf{f} + \frac{\eta}{\rho_0^2\omega^2}\delta_{N=2}\mathbf{curl}_{2D}\, \mathrm{curl}_{2D}\, \mathbf{f}\right),\end{aligned}$$

which is (2.9a), (2.10a), or (2.11a), respectively.

Taking the normal trace of (2.16) on $\partial\Omega$ and inserting (2.13b), (2.14b), or (2.15b), respectively, we find

$$\begin{aligned}\mathbf{v}_{\mathrm{appr},N}\cdot\mathbf{n} &= \frac{\mathrm{i}}{\rho_0\omega}\left(\mathbf{f}\cdot\mathbf{n} - \left(1 - \frac{\mathrm{i}\omega(\eta+\eta')\delta_{N=2}}{\rho_0 c^2}\right)\nabla p_{\mathrm{appr},N}\cdot\mathbf{n}\right) + \frac{\eta}{\rho_0^2\omega^2}\delta_{N=2}\mathbf{curl}_{2D}\, \mathrm{curl}_{2D}\, \mathbf{f}\cdot\mathbf{n}\\ &= \frac{\mathrm{i}}{\rho_0\omega}\left((1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\delta_{N\ge 1}\left(\partial_\Gamma^2 p_{\mathrm{appr},N} + \partial_\Gamma\left(\mathbf{f}\cdot\mathbf{n}^\perp\right)\right) + \frac{\mathrm{i}\eta}{2\omega\rho_0}\delta_{N=2}\left(\partial_\Gamma(\kappa\partial_\Gamma p_{\mathrm{appr},N}) + \partial_\Gamma\left(\kappa\mathbf{f}\cdot\mathbf{n}^\perp\right)\right)\right) \in H^{-1/2}(\partial\Omega),\end{aligned}$$

since with (2.14b), we have $\partial_\Gamma^2 p_{\mathrm{appr},1} \in H^{-1/2}(\partial\Omega)$ and with (2.15b), it follows $\partial_\Gamma^2 p_{\mathrm{appr},2} + \frac{1+\mathrm{i}}{2}\sqrt{\frac{\eta}{2\omega\rho_0}}\partial_\Gamma(\kappa\partial_\Gamma p_{\mathrm{appr},N}) \in H^{-1/2}(\partial\Omega)$. Now, taking the trace of (2.16) on $\partial\Omega$ and inserting it in the previous identity, we find that

$$\begin{aligned}\mathbf{v}_{\mathrm{appr},N}\cdot\mathbf{n} = (1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\delta_{N\ge 1}&\left(\frac{\omega^2}{c^2}\partial_\Gamma^2 div\mathbf{v}_{\mathrm{appr},N} + \frac{\mathrm{i}}{\rho_0\omega}\partial_\Gamma\left(\mathbf{f}\cdot\mathbf{n}^\perp\right)\right)\\ &+ \frac{\mathrm{i}\eta}{2\omega\rho_0}\delta_{N=2}\left(\frac{\omega^2}{c^2}\partial_\Gamma(\kappa\partial_\Gamma div\mathbf{v}_{\mathrm{appr},N}) + \frac{\mathrm{i}}{\rho_0\omega}\partial_\Gamma\left(\kappa\mathbf{f}\cdot\mathbf{n}^\perp\right)\right),\end{aligned}$$

which is (2.9b), (2.10b), or (2.11b), respectively. Finally, in view of (3.12), $p_{\mathrm{appr},N}$ fulfills (2.12). This finishes the proof. □

4 | ASYMPTOTIC EXACTNESS OF THE APPROXIMATIVE MODELS

In this section, we give the proof of Lemma 2, and so the approximative solutions of order N are asymptotically close to the asymptotic far-field expansions of the exact solution. As the asymptotic expansions are justified, the estimates for the modeling error in Theorem 2 follow immediately. The proof relies on the stability estimates for the generalized approximative velocity system in Lemma 4. Due to the singular perturbed nature of the equations, half a order in $\sqrt{\eta}$ is lost. So just estimating the residual and a direct application of the stability estimates gives only nonoptimal estimates.

We follow therefore the idea in Schmidt and Tordeux[20] with the asymptotic expansion of the approximative pressure and velocity in terms of $\varepsilon = \sqrt{2\eta/(\omega\rho_0)}$, subtract the asymptotic expansion of the solution $(\mathbf{v}, p)$ of the exact model (2.1) to obtain optimal error estimates. To prove these results together for the approximative models up to order 2, we rewrite the PDE system for far-field velocity and pressure terms $(\mathbf{v}^j, p^j)$ as

$$\nabla \mathrm{div}\, \mathbf{v}^j + \frac{\omega^2}{c^2}\mathbf{v}^j = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}\cdot\delta_{j=0} + \frac{\mathrm{i}\omega^2}{2c^2}\Delta\mathbf{v}^{j-2} + \frac{\mathrm{i}\gamma'\omega^2}{2c^2}\nabla\mathrm{div}\,\mathbf{v}^{j-2}, \text{ in } \Omega, \tag{4.1a}$$

$$\mathbf{v}^j\cdot\mathbf{n} = \sum_{\ell=1}^{j} G_\ell(\partial_\Gamma \mathrm{div}\,\mathbf{v}^{j-\ell}) + H_j(\mathbf{f}), \text{ on } \partial\Omega, \tag{4.1b}$$

$$p^j = -\frac{\mathrm{i}\rho_0 c^2}{\omega}\mathrm{div}\,\mathbf{v}^j, \text{ in } \Omega, \tag{4.1c}$$

where $\mathbf{v}^{-1} = \mathbf{v}^{-2} = \boldsymbol{0}$, $G_\ell : C^\infty(\Gamma) \to C^\infty(\Gamma)$, and $H_\ell : C^\infty(\Gamma) \to C^\infty(\Gamma)$ are tangential differential operators acting on traces of terms of lower orders or the trace of $\mathbf{f}$ on $\partial\Omega$, respectively. Furthermore, $\delta_{j=0}$ stands for the Kronecker symbol which is 1 if $j = 0$ and 0 otherwise. The operators G_0 and H_0 vanish, and the G_ℓ and H_ℓ for $\ell = 1, 2$ are given by

$$G_1(v) = (1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_\Gamma v, H_1(\mathbf{f}) = -(1-\mathrm{i})\frac{1}{2\omega\rho_0}\partial_\Gamma\left(\mathbf{f}\cdot\mathbf{n}^\perp\right), \tag{4.2a}$$

$$G_2(v) = \frac{c^2}{\omega^2}\left(\frac{\mathrm{i}}{4}\partial_\Gamma(\kappa v)\right), H_2(\mathbf{f}) = -\frac{1}{4\omega\rho_0}\partial_\Gamma\left(\kappa\mathbf{f}\cdot\mathbf{n}^\perp\right). \tag{4.2b}$$

For the proof, we need some higher regularity of the terms of the asymptotic expansion.

Lemma 4.1. *Let the assumptions of Lemma 2 be fulfilled. Then, there exists a neighborhood Ω_Γ of $\partial\Omega$ such that for all $j \in \{0,1,2\}$ and any $m \in \mathbb{N}_0$, it holds div $\mathbf{v}^j \in H^1(\Omega) \cap H^m(\Omega_\Gamma)$.*

Proof. By Schmidt et al.,[26, Lemma 2.3] all terms $\mathbf{v}^j \in (H^1(\Omega))^2$, and by Schmidt et al.,[26, Lemma 4.6] the terms $\mathbf{v}^j$ have any Sobolev regularity in any subdomain of Ω_Γ. Applying curl_{2D} to (4.1a), we obtain

$$\mathrm{curl}_{2D}\mathbf{v}^j = \frac{\mathrm{i}}{\omega\rho_0}\mathrm{curl}_{2D}\,\mathbf{f}\cdot\delta_{j=0} - \frac{\mathrm{i}}{2}\mathrm{curl}_{2D}\,\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{v}^{j-2}.$$

By recursion in j, we obtain an expression of $\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{v}^j$ in terms of $\mathbf{f}$ only (see (2.11) in Schmidt et al.[26])

$$\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\mathbf{v}^j = -\frac{2\mathrm{i}c^2}{\omega^2}\frac{\mathrm{i}\omega}{\rho_0 c^2}\left(-\frac{\mathrm{i}}{2}\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\right)^{(j+1)/2}\mathbf{f}\cdot\delta_{j=0}. \tag{4.3}$$

Using (4.1a) and (4.3), we find by induction in j

$$\begin{aligned}\nabla\mathrm{div}\,\mathbf{v}^j &= -\frac{\omega^2}{c^2}\mathbf{v}^j + \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}\cdot\delta_{j=0} + \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\nabla\mathrm{div}\,\mathbf{v}^{j-2} - \frac{\mathrm{i}\omega^2}{2c^2}\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\mathbf{v}^{j-2}\\ &= -\frac{\omega^2}{c^2}\mathbf{v}^j + \delta_{j\text{is even}}\frac{\mathrm{i}\omega}{\rho_0 c^2}\left(-\frac{\mathrm{i}}{2}\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\right)^{j/2}\mathbf{f} + \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\nabla\mathrm{div}\,\mathbf{v}^{j-2} \in (L^2(\Omega))^2 \cap (H^{m-1}(\Omega_\Gamma))^2,\end{aligned}$$

and so the statement of the lemma. □

Now, we are prepared to prove the asymptotic exactness of the approximative models.

Proof of Lemma. As with the assumption of the lemma and, in particular, the assumptions of Theorem 1 are fulfilled, the approximative solutions $\mathbf{v}_{\mathrm{appr},N}$ and $p_{\mathrm{appr},N}$, $N = 0, 1, 2$, exist and are uniquely defined. By Schmidt et al.,[26, Lemma 2.3] the same holds for the terms of the asymptotic expansions $\mathbf{v}^j, p^j, j = 0,1,2$. Throughout the proof, we use ε for $\sqrt{2\eta/(\omega\rho_0)}$.

Using the operators G_ℓ and H_ℓ defined in (4.2), we find that (2.10b) and (2.11b) are equivalent to

$$\mathbf{v}_{\mathrm{appr},N} \cdot \mathbf{n} - \sum_{\ell=1}^{N} \varepsilon^\ell G_\ell(\partial_\Gamma \mathrm{div}\, \mathbf{v}_{\mathrm{appr},N}) = \sum_{j=1}^{N} \varepsilon^j H_j(\mathbf{f}), \tag{4.4}$$

for $N = 1$ or $N = 2$, respectively. The asymptotic expansion $\mathbf{v}^{\varepsilon,N} = \sum_{j=0}^{N} \varepsilon^j \mathbf{v}^j$ fulfills

$$\mathbf{v}^{\varepsilon,N} \cdot \mathbf{n} = \sum_{j=0}^{N} \varepsilon^j \mathbf{v}^j \cdot \mathbf{n} = \sum_{j=0}^{N} \varepsilon^j \sum_{\ell=1}^{j} G_\ell(\partial_\Gamma \mathrm{div}\, \mathbf{v}^{j-\ell}) + \sum_{j=1}^{N} \varepsilon^j H_j(\mathbf{f}).$$

Resorting the sums in the second term on the right-hand side, we have

$$\begin{aligned}\sum_{j=0}^{N} \varepsilon^j \sum_{\ell=1}^{j} G_\ell(\partial_\Gamma \mathrm{div}\, \mathbf{v}^{j-\ell}) &= \sum_{\ell=1}^{N}\sum_{j=\ell}^{N} \varepsilon^j G_\ell(\partial_\Gamma \mathrm{div}\, \mathbf{v}^{j-\ell}) = \sum_{\ell=1}^{N} \varepsilon^\ell \sum_{j=0}^{N-\ell} \varepsilon^j G_\ell(\partial_\Gamma \mathrm{div}\, \mathbf{v}^{j}) \\ &= \sum_{\ell=1}^{N} \varepsilon^\ell G_\ell(\partial_\Gamma \mathrm{div}\, \mathbf{v}^{\varepsilon,N}) - \varepsilon^{N+1} \sum_{\ell=1}^{N}\sum_{j=0}^{N-\ell} \varepsilon^j G_\ell(\partial_\Gamma \mathrm{div}\, \mathbf{v}^{N+j+1-\ell}).\end{aligned}$$

The latter term simplifies for $N = 1,2$ and therefore $\mathbf{v}^{\varepsilon,N}$, $N = 1,2$ satisfies

$$\mathbf{v}^{\varepsilon,1} \cdot \mathbf{n} - \varepsilon G_1(\partial_\Gamma \mathrm{div}\, \mathbf{v}^{\varepsilon,1} = \varepsilon H_1(\mathbf{f}) + \varepsilon^2 G_1(\partial_\Gamma \mathrm{div}\, \mathbf{v}^1), \tag{4.5}$$

$$\mathbf{v}^{\varepsilon,2} \cdot \mathbf{n} - \sum_{\ell=1}^{2} \varepsilon^\ell G_\ell(\partial_\Gamma \mathrm{div}\, \mathbf{v}^{\varepsilon,2}) = \varepsilon H_1(\mathbf{f}) + \varepsilon^2 H_2(\mathbf{f}) - \varepsilon^3 \left(G_1(\partial_\Gamma \mathrm{div}\, \mathbf{v}^2) + G_2(\partial_\Gamma \mathrm{div}\, \mathbf{v}^1)\right) - \varepsilon^4 G_1(\partial_\Gamma \mathrm{div}\, \mathbf{v}^2). \tag{4.6}$$

Hence, the difference $\delta\mathbf{v}_{\mathrm{appr},N} = \mathbf{v}_{\mathrm{appr},N} - \mathbf{v}^{\varepsilon,N}$ fulfills for $N = 1$

$$\nabla \mathrm{div}\, \delta\mathbf{v}_{\mathrm{appr},1} + \frac{\omega^2}{c^2} \delta\mathbf{v}_{\mathrm{appr},1} = 0, \text{ in } \Omega, \tag{4.7a}$$

$$\delta\mathbf{v}_{\mathrm{appr},1} \cdot \mathbf{n} - \varepsilon G_1(\partial_\Gamma \mathrm{div}\, \delta\mathbf{v}_{\mathrm{appr},1}) = \varepsilon^2 G_1(\partial_\Gamma \mathrm{div}\, \mathbf{v}^1), \text{ on } \partial\Omega, \tag{4.7b}$$

and for $N = 2$

$$\left(1 - \frac{\mathrm{i}\omega(1+\gamma')}{\rho_0 c^2}\varepsilon^2\right) \nabla \mathrm{div}\, \delta\mathbf{v}_{\mathrm{appr},2} + \frac{\omega^2}{c^2} \delta\mathbf{v}_{\mathrm{appr},2} = \varepsilon^3 \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2} \nabla \mathrm{div}\, (\mathbf{v}^1 + \varepsilon \mathbf{v}^2), \qquad \text{in } \Omega, \tag{4.8a}$$

$$\delta\mathbf{v}_{\mathrm{appr},2} \cdot \mathbf{n} - \sum_{\ell=1}^{2} \varepsilon^\ell G_\ell(\partial_\Gamma \mathrm{div}\, \delta\mathbf{v}_{\mathrm{appr},2}) = \varepsilon^3 \left(G_1(\partial_\Gamma \mathrm{div}\, \mathbf{v}^2) + G_2(\partial_\Gamma \mathrm{div}\, \mathbf{v}^1)\right) + \varepsilon^4 G_1(\partial_\Gamma \mathrm{div}\, \mathbf{v}^2), \qquad \text{on } \partial\Omega. \tag{4.8b}$$

Since $\sum_{\ell=1}^{N} \varepsilon^\ell G_\ell(\partial_\Gamma \mathrm{div}\, \delta\mathbf{v}_{\mathrm{appr},N}) = \partial_\Gamma(\beta_\eta^N \partial_\Gamma div \delta\mathbf{v}_{\mathrm{appr},N})$, $N = 1,2$, we can apply Lemma 4 and obtain the estimate

$$\left\|\delta\mathbf{v}_{\mathrm{appr},N}\right\|_{H(\mathrm{div}\ ,\Omega)} \le C\varepsilon^{N+\frac{1}{2}},$$

with a constant $C > 0$ independent of ε. The estimate is unfortunately not optimal as the consistency error is of order $N+1$, and with the stability estimate, we lose half an order in ε.

To obtain an optimal estimate, we consider the asymptotic expansion of $\mathbf{v}_{\mathrm{appr},N}$, $N = 1,2$, with a further term that decreases the residual by one order. For this, we consider formally

$$\begin{aligned}\mathbf{v}_{\mathrm{appr},1} &= \mathbf{v}^0 + \varepsilon\mathbf{v}^1 + \varepsilon^2 \tilde{\mathbf{v}}_1 + o(\varepsilon^2), \\ \mathbf{v}_{\mathrm{appr},2} &= \mathbf{v}^0 + \varepsilon\mathbf{v}^1 + \varepsilon\mathbf{v}^2 + \varepsilon^3 \tilde{\mathbf{v}}_2 + o(\varepsilon^3),\end{aligned}$$

and chose $\tilde{\mathbf{v}}_1$ to solve

$$\begin{aligned}\nabla\text{div }\tilde{\mathbf{v}}_1 + \frac{\omega^2}{c^2}\tilde{\mathbf{v}}_1 &= 0,\\ \tilde{\mathbf{v}}_1\cdot\mathbf{n} &= G_1(\partial_\Gamma\text{div }\mathbf{v}^1),\end{aligned}$$

and $\tilde{\mathbf{v}}_2$ to solve

$$\begin{aligned}\nabla\text{div }\tilde{\mathbf{v}}_2 + \frac{\omega^2}{c^2}\tilde{\mathbf{v}}_2 &= \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\nabla\text{div }\mathbf{v}^1,\\ \tilde{\mathbf{v}}_2\cdot\mathbf{n} &= G_1(\partial_\Gamma\text{div }\mathbf{v}^2) + G_2(\partial_\Gamma\text{div }\mathbf{v}^1).\end{aligned}$$

With Lemma 4.1, the terms $\mathbf{v}^1$ and $\mathbf{v}^2$ of the asymptotic expansion have enough regularity that the right-hand side of these systems and so its solution is well defined.

Now, we find that the difference $\delta\tilde{\mathbf{v}}_{\text{appr},N} = \mathbf{v}_{\text{appr},N} - \mathbf{v}^{\varepsilon,N} - \varepsilon^{N+1}\tilde{\mathbf{v}}_N$ fulfills for $N = 1$

$$\begin{aligned}\delta\tilde{\mathbf{v}}_{\text{appr},1} + \frac{\omega^2}{c^2}\delta\tilde{\mathbf{v}}_{\text{appr},1} &= 0, &&\text{in } \Omega,\\ \delta\tilde{\mathbf{v}}_{\text{appr},1}\cdot\mathbf{n} - \varepsilon G_1(\partial_\Gamma\text{div }\delta\tilde{\mathbf{v}}_{\text{appr},1}) &= 0, &&\text{on } \partial\Omega,\end{aligned}$$

and Lemma 4 implies $\delta\tilde{\mathbf{v}}_{\text{appr},1} = 0$ in Ω, and for $N = 2$,

$$\begin{aligned}\left(1 - \frac{\mathrm{i}\omega(1+\gamma')}{\rho_0 c^2}\varepsilon^2\right)\delta\tilde{\mathbf{v}}_{\text{appr},2} + \frac{\omega^2}{c^2}\delta\tilde{\mathbf{v}}_{\text{appr},2} &= 0, &&\text{in } \Omega,\\ \delta\tilde{\mathbf{v}}_{\text{appr},2}\cdot\mathbf{n} - \sum_{\ell=1}^{2}\varepsilon^\ell G_\ell(\partial_\Gamma\text{div }\delta\tilde{\mathbf{v}}_{\text{appr},2}) &= \varepsilon^4 G_1(\partial_\Gamma\text{div }\mathbf{v}^2), &&\text{on } \partial\Omega.\end{aligned}$$

Applying Lemma 4, it holds $\delta\tilde{\mathbf{v}}_{\text{appr},2} \le C\varepsilon^{\frac{7}{2}}$. As $\tilde{\mathbf{v}}_N$, $N = 1, 2$, do not depend on ε by definition, we find using the triangle inequality

$$\begin{aligned}\left\|\delta\mathbf{v}_{\text{appr},1}\right\|_{H(div,\Omega)} &\le \left\|\delta\widetilde{\mathbf{v}}_{\text{appr},1}\right\|_{H(div,\Omega)} + \varepsilon^2\|\tilde{\mathbf{v}}_1\|_{H(div,\Omega)} \le C\varepsilon^2,\\ \left\|\delta\mathbf{v}_{\text{appr},2}\right\|_{H(div,\Omega)} &\le \left\|\delta\widetilde{\mathbf{v}}_{\text{appr},2}\right\|_{H(div,\Omega)} + \varepsilon^3\|\tilde{\mathbf{v}}_2\|_{H(div,\Omega)} \le C\varepsilon^3,\end{aligned}$$

and with $\varepsilon = \sqrt{2\eta/(\omega\rho_0)}$, this is the estimate (2.22) for the velocity.

Moreover, with the definition (2.12) of the pressure approximation and the definition (4.1c) of the terms of the asymptotic expansion of the pressure, the same bound follows for the $L^2(\Omega)$-norm of the pressure. Finally, the $H^1(\Omega)$-bound follows from the Equations (2.14) and (2.15) for the approximative pressure and respective equations for the terms p^j of the asymptotic pressure expansion that is derived using (4.1a) and (4.1c). That finishes the proof.

5 | NUMERICAL RESULTS

For a torus domain with omitted disk, see Figure 1B, we have performed numerical simulations for the exact model (2.1) and the approximative pressure models (2.13)–(2.16). We consider the problem in dimensionless quantities. The domain is the rectangle $[0, 1] \times [0, 2]$, where the left and right sides are identified with each other, and the disk of diameter 0.30 is centered at $(0.25, 1.5)$. As source $\mathbf{f}$, we use the gradient of the Gaussian $\exp(-|\mathbf{x} - \mathbf{x}_0|^2/0.005)$ with $\mathbf{x}_0 = (0.75, 0.5)^\top$. The source is curl_{2D} free, which has no influence to any of the numerical experiments. Furthermore, we choose for the speed of sound $c = 1$, the (mean) air density as $\rho_0 = 1$, and simplify by neglecting η' that has a minor influence and is not relevant when interpreting the result (despite it might be slightly unphysical).

For the simulation, we have used high-order finite elements within the numerical C++ library *Concepts*[41] to push the discretization error below the modeling error. We use C^0-continuous finite elements for the (approximative)

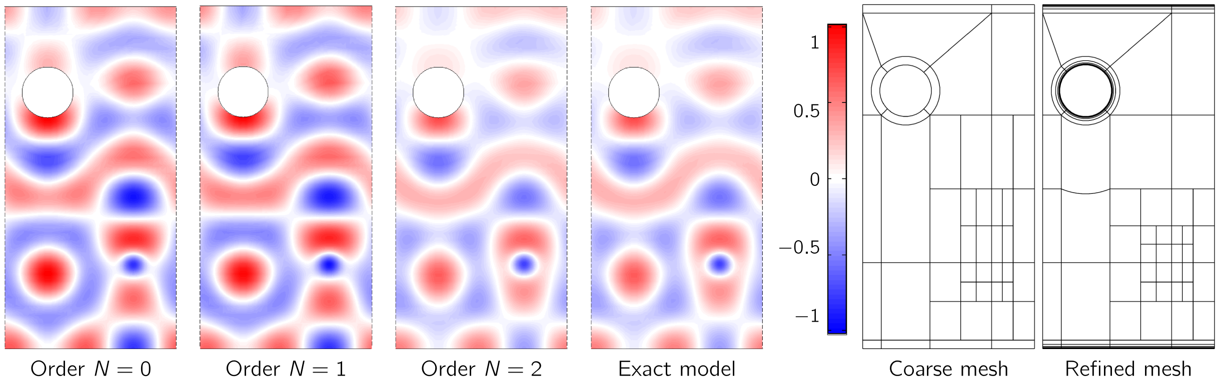

FIGURE 2 Comparison of the real part of the pressure offer the approximate models of order $N = 0, 1, 2$ to the exact pressure ($\eta = 4 \cdot 10^{-6}$, $\omega = 15$). For the approximate models, a FEM discretization on a coarse mesh can be used (second from right), where the boundary layers of the exact model shall be resolved by a FEM mesh refined towards the boundary (right) [Colour figure can be viewed at wileyonlinelibrary.com]

pressure and both components of the (exact) velocity. Note that the classical choice for the approximative velocity models is $H(div, \Omega)$-conforming finite elements like Raviart–Thomas elements. Here, we restrict the numerical experiments to the models of the approximative pressure which provides the greatest simplification.

To resolve the boundary layers in the (exact) velocity, we refine the mesh geometrically towards the boundary; see the right picture in Figure 2. The high gradients of the source term are considered in a further (geometric) mesh refinement towards the point $\mathbf{x}_0$. We have chosen the polynomial degree to be 11 to obtain low enough discretization errors such that the modeling errors become visible. The computation of the FE solution of the exact model for one frequency has taken about 25 s on a Intel i7-7500U with 2.70 GHz, where it has taken about 3.5 s for the FE solution of one of approximative models. The far-field solution of the approximative models could be computed to a high precision on a rather coarse mesh (see Figure 2, second from right) as no boundary layer has to be resolved, which would lead to lower computation times. Anyhow, we have computed the far-field solution on the mesh illustrated in Figure 2, which allowed us firstly a straightforward evaluation of norms of the error functions and secondly a representation of the sum of far field and near field on the same mesh.

For $\eta = 4 \cdot 10^{-6}$ and $\omega = 15$, we have illustrated the exact pressure and its approximations $p_{\text{appr},0}$, $p_{\text{appr},1}$, and $p_{\text{appr},2}$ of orders 0, 1, and 2, respectively, in the first four subfigures of Figure 2. The color scaling in all the four subfigures matches to allow for a direct comparison. In this example, the approximations of order 0 and order 1 provide a coarse-field description, where the pressure amplitude is overestimated. The approximation of order 2, however, predicts the exact quite well. For this example, just with a viscosity $\eta = 1.6 \cdot 10^{-3}$, we have illustrated the boundary layer in the tangential velocity component in Figure 3, both for the exact model and the approximation of order 2. The boundary layer thickness is $d_{\text{BL}} = \sqrt{2\eta/\omega\rho_0} = 1.46 \cdot 10^{-2}$. Here, the approximative far-field velocity $\mathbf{v}_{\text{appr},2}$ and the respective near field were computed from the pressure approximation $p_{\text{appr},2}$. The representation of the velocity is in a side view for $x_1 = 0$, for which the first component is tangential to the lower boundary at $x_2 = 0$. The approximate solution is the sum of the far field, which does not fulfill a homogeneous Dirichlet boundary condition, and a correcting near field. The far-field solution approximates the exact one away from the boundary very well; see Figure 3A. In its turn, Figure 3B shows the near-field correction and the behavior of the solutions close to the wall.

To analyze the modeling error in dependence of the viscosity and, hence ε, we have performed numerical simulations on the simple rectangular torus domain $\Omega = [0, 1]\times[0, 1]$ (i.e., without the hole of the previous problem), for which the left and right sides are again identified with each other. The other parameters are identical to those of the previous problem. The studied frequency $\omega = 15$ is not a Neumann eigenfrequency of $-\Delta$, the closest eigenfrequencies are $\sqrt{20}\pi \approx 14.05$ and $5\pi \approx 15.71$. We compute the error functions on the subdomain $\Omega_\delta = [0, 1]\times[0.2, 0.8]$, which has a distance of $\delta = 0.2$ to the boundary of Ω. This distance is large enough such that in Ω_δ for the studied viscosities, the contribution of the exponentially decaying near fields can be neglected. In Figure 4A, we have shown the relative modeling error

$$\|p - p_{\text{appr},N}\|_{H^1(\Omega_\delta)}/\|p\|_{H^1(\Omega_\delta)} + \|\mathbf{v} - \mathbf{v}_{\text{appr},N}\|_{H(div,\Omega_\delta)}/\|\mathbf{v}\|_{H(div,\Omega_\delta)}$$

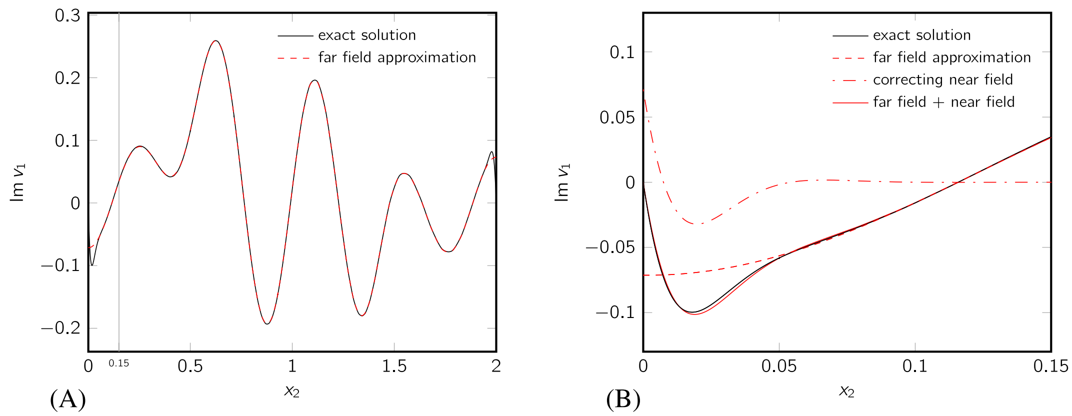

FIGURE 3 Imaginary part of first velocity component in side view for $x_1 = 0$ with $\sqrt{\eta} = 4 \cdot 10^{-2}$, which is at $x_2 = 0$ tangential to the bottom wall. The exact solution v_1 and the approximate (far field) solution $(v_{\text{appr},2})_1$ of order 2, the corresponding near field $(v^{BL}{}_{\text{appr},2})_1$, and the sum of both are shown, in (A) for the whole line $x_1 = 0$ and in (B) close to the wall [Colour figure can be viewed at wileyonlinelibrary.com]

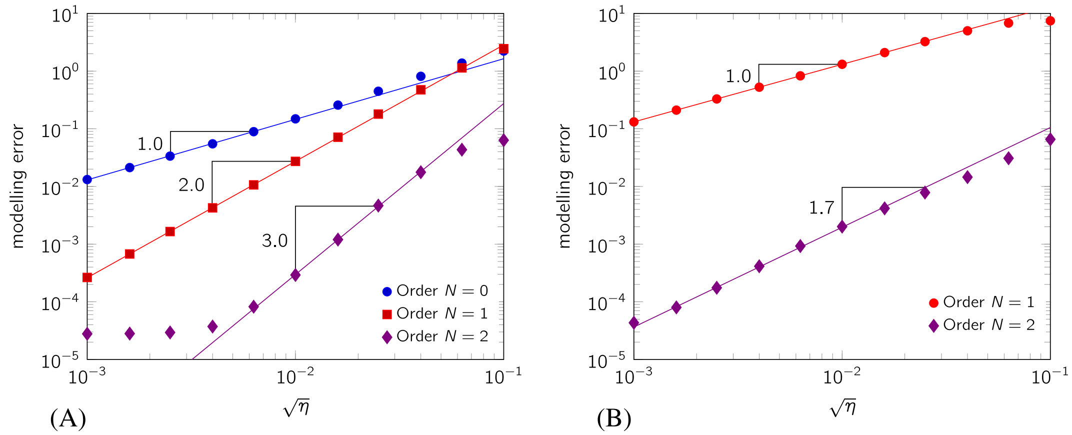

FIGURE 4 The relative modeling error $\|p - q_{\text{appr},N}\|_{H^1(\Omega)}/\|p\|_{H^1(\Omega)} + \|\mathbf{v} - \mathbf{v}_{\text{appr},N}\|_{H(div,\Omega)}/\|\mathbf{v}\|_{H(div,\Omega)}$ for $N = 0, 1, 2$ w.r.t. square root of viscosity for (A) a dimensionless frequency value $\omega = 15$ and (B) an eigenfrequency $\omega = \sqrt{20}\pi$ [Colour figure can be viewed at wileyonlinelibrary.com]

for the approximative solutions of orders 0, 1, and 2 in dependence of the (square root of the) viscosity. We observe linear convergence in $\sqrt{\eta}$ for the approximative solution of order 0, quadratic convergence for that of order 1, and convergence of order 3 for the approximative solution of order 2. These results verify that the estimates in Theorem 2 are sharp. The error is computed on the above mesh with polynomial degree 14 and included indeed a small discretization error which becomes visible for small viscosities ($\sqrt{\eta} < 5 \cdot 10^{-3}$) and the approximative model of order 2.

The theoretical estimates are for nonresonant frequencies, and the constants may blow up if the frequency tends to a resonant one, that is, a Neumann eigenfrequency of $-\Delta$. The eigenfrequencies for the studied example are $\omega_{k,m} = \pi\sqrt{k^2 + 4m^2}$, for $k \in \mathbb{N}, m \in \mathbb{N}_0$. In addition, we analyze the modeling error in dependence of the viscosity for an eigenfrequency value $\omega_0 = \omega_{2,2} = \omega_{4,1} = \sqrt{20}\pi$; see Figure 4B. The convergence in this case looses in order, that is, linear convergence in $\sqrt{\eta}$ for the approximative solution of order 1, convergence of order 1.7 for order 2, and the approximative solution of order 0 explodes and is not represented in the picture.

Furthermore, we analyze the modeling errors of the three approximative solutions in dependence of the frequency for the rectangular domain and $\eta = 1.6 \cdot 10^{-3}$; see Figure 5. The approximate solution of order 0 and so the modeling error blows up close to the eigenfrequencies. However, the approximate solution of order 1 blows up only close to the eigenfrequency values $\omega_{k,0} = k\pi$ for $k \in \mathbb{N}$. That could be explained by the fact that for $m = 0$ in this example, the velocity and so its divergence are constant in x_1, and the additional term in the boundary condition of order 1 disappears. In this case, the order 1 approximation at that frequencies becomes identical to that of order 0. Conversely, the error of the approximate solution of order 2, due to the additional term in the domain, always stays lower than $3 \cdot 10^{-2}$ and, as it

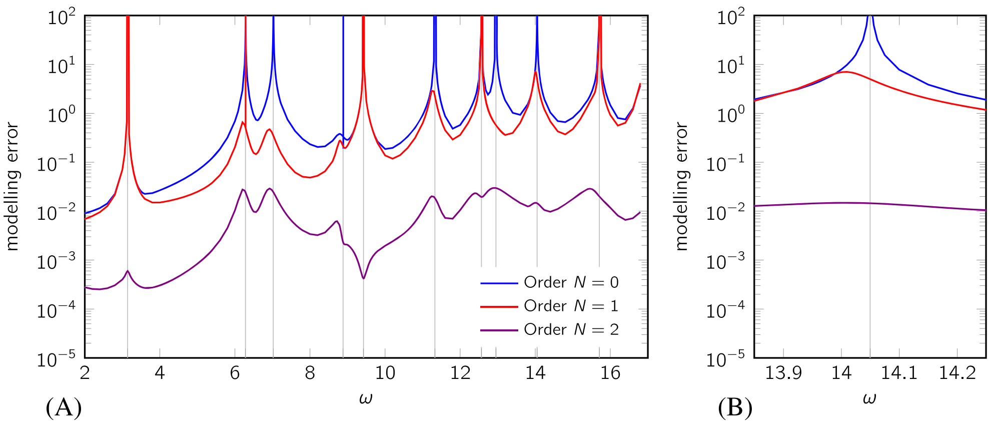

FIGURE 5 The modeling error $\|p - q_{\mathrm{appr},N}\|_{H^1(\Omega)}/\|p\|_{H^1(\Omega)} + \|\mathbf{v} - \mathbf{v}_{\mathrm{appr},N}\|_{H(div,\Omega)}/\|\mathbf{v}\|_{H(div,\Omega)}$ for $N = 0, 1, 2$ w.r.t. dimensionless frequency ω for $\eta = 1.6 \cdot 10^{-3}$ [Colour figure can be viewed at wileyonlinelibrary.com]

was shown earlier, converges w.r.t. viscosity even at the resonance. Yet, in this work, we will leave that sentence without a proof, and the numerical results are presented for illustration reason only.

Note that the above simulation corresponds for dimensionful quantities, for example, to a rectangular domain of size $4\,\mathrm{cm} \times 8\,\mathrm{cm}$, where the hole has a diameter of $1.2\,\mathrm{cm}$, a frequency $\omega = 5.146\,\mathrm{kHz}$, a speed of sound in air $c = 343\,\mathrm{m/s}$, and a mean density of air $\rho_0 = 1.2\,\mathrm{kg/m}^3$. Then, a dynamic viscosity of air $\eta = 17.1\,\mathrm{mPa\,s}$ corresponds to a dimensionless viscosity of $1.04 \cdot 10^{-6}$ (dimensionless value of $\sqrt{\eta}$ would be $1.02 \cdot 10^{-3}$), which is close to the lowest viscosity value studied in the above experiments.

6 | CONCLUSION

In this article, the acoustic wave propagation in viscous gases inside a bounded two-dimensional domain has been studied as a solution of the compressible linearized Navier–Stokes equation. In frequency domain, the governing equations are decoupled in equations for the velocity and pressure, where the pressure equation lacks boundary conditions.

The velocity exhibits a boundary layer on rigid walls, whose extend scales with the square root of the viscosity and the finite element discretization requires a heavy mesh refinement in the neighborhood of the wall. Using the technique of multiscale expansion for small viscosities, impedance boundary conditions separately for acoustic velocity and acoustic pressure are derived up to second order. The derivation and presented analysis is based on a previous work by the Schmidt et al,[26] where the complete asymptotic expansion of velocity and pressure has been derived. It has be shown that the velocity is represented as a sum of a far-field expansion, which does not exhibit a boundary layer, and a correcting near-field expansion close to the wall. For the pressure, which does not exhibit a boundary layer, there is only a far-field expansion, and a near-field expansion is absent.

Using boundary conditions for the pressure presented in this work and respective partial differential equations, pressure approximations are defined independently of respective velocities. The zeroth-order boundary conditions are well known to be of Neumann type for rigid walls, and the conditions of first or second order are of Wentzel type take into account absorption inside the boundary layer. The velocity boundary condition is for a far-field approximation, whose finite element discretization does not need a special mesh refinement close to walls. Here, a boundary layer contribution depending on the far-field velocity can be added to obtain an overall highly accurate description of the velocity. The derivation of the boundary conditions for either pressure or velocity include curvature effects, where the curvature becomes present in the boundary conditions of order 2.

The approximative models with their impedance boundary conditions are justified by a stability and error analysis. The results of the numerical experiments have been provided to illustrate the stability and error estimates. Although throughout the article the frequency is assumed to be not an eigenfrequency of the limit problem for vanishing viscosity, we showed by numerical computations that the second-order model provides accurate approximations for all frequencies and the first-order model except some of the above-mentioned eigenfrequencies. This results give a foundation for future studies for the case of resonances of the limit problem in bounded domains.

Similarly to the presented study, approximative models with impedance boundary conditions might be derived and mathematically justified for nonlinear acoustic models or acoustic models with viscothermal boundary layers[25] and in three dimensions.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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