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ASYMPTOTIC EXPANSIONS FOR A CLASS OF FREDHOLM PFAFFIANS AND INTERACTING PARTICLE SYSTEMS

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Motivated by the phenomenon of duality for interacting particle systems, we introduce two classes of Pfaffian kernels describing a number of Pfaffian point processes in the "bulk" and at the "edge." Using the probabilistic method due to Mark Kac, we prove two Szegő-type asymptotic expansion theorems for the corresponding Fredholm Pfaffians. The idea of the proof is to introduce an effective random walk with transition density determined by the Pfaffian kernel, express the logarithm of the Fredholm Pfaffian through expectations with respect to the random walk, and analyse the expectations using general results on random walks. We demonstrate the utility of the theorems by calculating asymptotics for the empty interval and noncrossing probabilities for a number of examples of Pfaffian point processes: coalescing/annihilating Brownian motions, massive coalescing Brownian motions, real zeros of Gaussian power series and Kac polynomials, and real eigenvalues for the real Ginibre ensemble.

CONTENTS

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1. Introduction. The aim of the paper is to prove rigorously, extend, and generalise a number of asymptotic formulae for the empty interval and crossing probabilities for systems of annihilating-coalescing Brownian motions on $\mathbb{R}$ obtained in the 1990s by Derrida and Zeitak [11] and Derrida, Hakim and Pasquier [12]. A very special feature underlying the computation of each of these probabilities is that the corresponding random process is actually a Pfaffian point process for arbitrary deterministic initial conditions. This fact is, by now, well known for coalescing or annihilating Brownian motions on $\mathbb{R}$, [40]. However, it also holds true for mixed coalescing/annihilating systems, for annihilating systems with pairwise immigration, for coelascing systems with branching, and for analogous systems on $\mathbb{Z}$, [21, 22]. Therefore, asymptotics of the probabilities, studied in [11, 12] can be reduced to the asymptotic analysis of Fredholm Pfaffians (introduced in [33], or see the Section 2.1.1 below for a brief review).

Background. For determinantal point processes the empty interval probabilities (also called gap probabilities) are given by Fredholm determinants of the corresponding kernels; see, for example, [2] or [30] for a review. If the kernel is translationally invariant (i.e., depends on the difference of the arguments only), then the Akhiezer–Kac formula [1, 24] gives the asymptotics for a Fredholm determinant as the length of the interval grows. The Akhiezer–Kac formula is an extension of Szegő's theorem [37] for determinants of Toeplitz matrices to the continuous case. Unfortunately, there seems to be no analogous result for Fredholm Pfaffians, even though, via operator manipulations, it is often possible to reduce a Fredholm Pfaffian to a Fredholm determinant of an operator with a scalar kernel; see, for example, [39] for such a reduction in the case of the Gaussian orthogonal and symplectic ensembles of random matrices or [34] for the real Ginibre ensemble. This is probably due to the lack of the general theory of Pfaffian point processes which would require proving an analogue of Soshnikov's theorem [35] for determinantal point processes by classifying all skew-symmetric 2×2 matrix-valued kernels that give rise to a random point process. Therefore, our task is to study the asymptotics of Fredholm Pfaffians for a sufficiently large class of kernels which contains all of the empty-interval probabilities for the Pfaffian point processes in which we are interested.

Classes of Pfaffian kernels. Motivated by the duality ideas often employed in integrable probability to find explicit solutions for exactly solvable interacting particle systems, we find two classes of such processes, for the translationally invariant ("bulk") and noninvariant ("edge") cases. Asymptotics for these two cases (Theorems 1 and 3) are the first main results of the paper. As a byproduct of our analysis, we obtain a generalisation of the Akhiezer–Kac formula for Fredholm determinants for a class of "asymptotically translationally-invariant" kernels; see Theorem 2.

It turns out that a number of well-known Pfaffian point processes fall into one of our classes, including all the reaction diffusion systems listed earlier, the exit measure for annihilating-coalescing Brownian motions (see Section 3.6 below), the law of real eigenvalues for the real Ginibre ensemble, both in the bulk and near the edge discovered in [20] and [6], as well as the law of eigenvalues in the bulk and edge scaling limits for the classical GOE random matrix ensemble; see [2, 30] for a review. Unfortunately, our theorems have nothing to say about the last two cases due to the insufficiently fast decay of the corresponding kernels at infinity.

Approach to the proofs. A notable feature of the class of kernels we consider is the possibility to follow the manipulations carried out in [39] and [34] in order to reduce the corresponding Fredholm Pfaffian to the square root of the product of a Fredholm determinant and a finite dimensional determinant. We are, therefore, left with a pure analysis problem of computing the asymptotic behaviour of each of these determinants in the limit of large gap sizes. In principle, this problem can be solved using a combination of a direct computation and an application of Szegő-type theorems, possibly modified owing to the presence of a Fisher–Hartwig singularity; see [9] for a review of the discrete (Toeplitz) case. Such an approach runs into difficulties due to the dearth of sufficiently general asymptotic results for Fredholm determinants in the nontranslationally invariant case. Fortunately, the analysis problem at hand can be solved using a probabilistic approach. This was used by Mark Kac to formulate the first asymptotic results for Fredholm determinants, thus generalising Szegő's original theorems for Toeplitz matrices; see [24]. It turns out that both the finite dimensional determinant and the Fredholm determinant under consideration can be interpreted as expectations with respect to the law of (time-inhomogeneous) random walk, which can be subsequently analysed using general results concerning random walks, such as Sparre Andersen's formula, Spitzer's formulae, cyclic symmetry, renewal theory, the invariance principle, and optional stopping. The essence of Kac's arguments is very easy to illustrate in the translationally invariant case. Let K_T be an integral operator on $L^2[0,T]$: $K_T f(x) = \int_0^T \rho(x-y) f(y)\,dy$. Assume that ρ is nonnegative and such that $\int_{\mathbb{R}} \rho(x) = 1$, that is, a probability density function. Given some regularity conditions on ρ, for any $0 \le \lambda \le 1$ the Fredholm determinant of the operator $I - \lambda K_T$ can be computed using the trace-log (Plemelj–Smithies) formula,

$$\log \operatorname{Det}(I - \lambda K_T) = -\sum_{n=1}^{\infty} \lambda^n \frac{\operatorname{Tr} K_T^n}{n}.$$

Using the convention $x_{n+1} = x_1$, one can write, for any $n \in \mathbb{N}$,

$$\operatorname{Tr} K_T^n = \int_{[0,T]^n} \prod_{k=1}^{n} \rho(x_{k+1} - x_k)\, dx_1\, dx_2 \dots dx_n = \int_{[0,T]} \mathbb{P}_x[\tau_T > n, X_n \in dx],$$

where $\mathbb{P}_x$ is the law of the $\mathbb{R}$-valued random walk $(X_n : n > 0)$ started at x with the distribution of increments given by ρ, and τ_T is the first exit time from the interval $[0,T]$. Therefore, the problem of computing the Fredholm determinant is reduced to the analysis of random walks constrained not to exit the interval $[0,T]$ killed independently at every step with probability $1 - \lambda$. It was Kac's insight that such an analysis can be carried out for any random walk using probabilistic arguments. Varying the parameter λ is very natural for probabilists: for a point process with (determinantal or Pfaffian) kernel K, the thinned point process, where each particle is removed independently with probability p, has new kernel pK.

The asymptotic results detailed in Theorems 1, 2, 3 are an example of pure analysis statements proved using probabilistic methods. As happened with Kac's theorem for Fredholm determinants, a nonprobabilistic proof of asymptotics for Fredholm Pfaffians will most certainly appear. Kac's assumption of positivity $\rho \geq 0$ was used to allow probabilistic arguments, but the final result was subsequently found to be true without this assumption, and we believe the analogous positivity assumption in our results is unnecessary. Moreover, such a proof is certainly desirable: at the moment the logarithms of both the finite dimensional and Fredholm determinants contain terms proportional to the logarithm of the gap size; see Theorem 2. The logarithmic terms cancel upon taking the final product. In other words, the Fredholm Pfaffian does not have the singularity present in the corresponding Fredholm determinant. It is likely that there is a streamlined proof of Theorems 1 and 3 which avoids reexpressing the Fredholm Pfaffians in terms of determinants and where the logarithmic terms do not appear at the intermediate steps.

Application examples. The motivation for the study of Fredholm Pfaffians came from applications to probability theory. In [11], Derrida and Zeitak study the distribution of domain sizes in the q-state Potts model on $\mathbb{Z}$ for the "infinite temperature" initial conditions (the initial colours are chosen uniformly independently at each site). They show that, as $L/\sqrt{t} \to \infty$,

$$(1) \qquad \mathbb{P}\big[\text{The interval } [0, L] \text{ contains one colour at time } t\big] = e^{-A(q)\frac{L}{\sqrt{t}}+B(q)+o(1)},$$

where A, B are some explicit functions of q. Derrida and Zeitak's formula is valid for all $q \in [1, \infty)$. For Potts models, $q \in 1 + \mathbb{N}$. However, as pointed out by the authors, there is a physical interpretation of (1) for $q \notin 1 + \mathbb{N}$: it gives the probability that the interval $[0, x]$ is contained in a domain of positive spins in zero temperature Glauber model on $\mathbb{Z}$ started with independent homogeneous distribution of spins with the average spin ("magnetisation") equal to $m \in [-1, 1]$. Then, the parameter $q = \frac{2}{1+m}$. The computation method employed in [11] can possibly be made rigorous for $q < 1/2$, but it relies on the assumption of analyticity of the functions A, B in order to obtain the answer for $q \geq 2$. These assumptions seem hard to justify to us. Alternatively, one proves (1) as follows: as is well known, the boundary of monochrome domains in the (diffusive limit of) Potts model behaves as a system of instantaneously coalescing/annihilating Brownian motions on $\mathbb{R}$ (denoted CABM) with the annihilating probability $\frac{1}{q-1}$ and coalescing probability $\frac{q-2}{q-1}$ at each collision; see [11] and [21] for details. Therefore, the distribution of domain sizes in the Potts model corresponds to the empty interval probabilities for coalescing/annihilating Brownian motions which is known to be a Pfaffian point process [21]. Moreover, if $\mathbf{K}$ is the kernel for the purely coalescing case ($q = \infty$), then $\frac{q-1}{q}\mathbf{K}$ is the kernel for CABM. In other words, CABM can be obtained from coalescing Brownian motions by thinning, that is deleting particles independently with probability $p = \frac{q-1}{q}$ (see Section 4 for precise details, including the role of the initial conditions). Therefore, the gap probability is given by the Fredholm Pfaffian $\mathrm{Pf}_{[0,L]}[\mathbf{J} - p\mathbf{K}]$, and applying Theorem 1, one reaches (1). Notice that the interpretation in terms of gap probabilities for CABM uses kernel $p\mathbf{K}$ for $p \geq \frac{1}{2}$ ("weak thinning"). For $p < \frac{1}{2}$ ("strong thinning") one can interpret the answer either in terms of gaps between the boundaries of positive spins in the Glauber model as above, or there is an interpretation in terms of gap sizes for a *massive* coalescence model; see Lemma 21 below.

Derrida and Zeitak's result (1) can be extended as follows. CABM started from half-space initial conditions is still a Pfaffian point process with a translationally noninvariant kernel covered by conditions of Theorem 3. The asymptotics of the corresponding Fredholm Pfaffian can, therefore, be computed yielding the right (left) tail of the fixed time distribution of the leftmost (rightmost) particle. Notice that the value $p = \frac{1}{2}$ of the thinning parameter corresponds to purely annihilating Brownian motions (ABM). It is known [21, 40] that the fixed time law of ABM with full-space (half-space) initial conditions is a Pfaffian point process coinciding with the bulk (edge) scaling limit of the law of real eigenvalues for the real Ginibre random matrix model discovered in [5, 6, 20]. This remark allowed a computation of the tails of the distribution of the maximal real eigenvalue for the edge scaling limit of the real Ginibre ensemble in [32] and [16]. It is the generalisation of the arguments in the last two cited papers that led to the asymptotic Theorems 1, 2, 3.

In another influential investigation, Derrida, Hakim and Pasquier [12] compute the fraction of sites which haven't changed colour up to time t for a q-state Potts model on $\mathbb{Z}$. Due to the translation invariance of the "infinite temperature" initial conditions, this is equivalent to the probability that the colour of the state at the origin hasn't changed up to time t (also known as a "persistence" probability; see [7] for a review of the persistence phenomenon in the context of nonequilibrium statistical mechanics). They show, up to logarithmic precision, that

$$\mathbb{P}\big[\text{The colour at 0 does not change in } [0,t]\big] \sim t^{-\gamma(q)} \quad \text{as } t \to \infty, \tag{2}$$

where

$$\gamma(q) = -\frac{1}{8} + \frac{2}{\pi^2}\left[\cos^{-1}\left(\frac{2-q}{\sqrt{2}q}\right)\right]^2.$$

The authors derive this result by noticing that this event is equivalent for the domain boundaries, which form a system of coalescing/annihilating random walks, to the event that no boundary crosses zero during the time interval $[0,t]$. Motivated by (2), we consider CABM on $[0,\infty)$ started from the "maximal" entrance law supported on (a,∞) for some $a > 0$. Particles hitting the boundary at $x = 0$ are removed from the system, and we record the corresponding exit times. We show that the resulting exit measure is a Pfaffian point process with nontranslationally invariant kernel belonging to our class. Then, Theorem 3 applies, and one can deduce the asymptotics for the probability that the interval $[0,T]$ contains no exiting particles in the limit of large time T. This immediately gives the noncrossing probability for the position of the leftmost particle $(L_t : t \geq 0)$ for the system of CABM on the whole real axis started from every point of (a,∞),

$$\mathbb{P}\Big[\inf_{t\in[0,T]} L_t > 0\Big] = K(q)\left(\frac{T}{a^2}\right)^{-\gamma(q)/2}\big(1+o(1)\big),$$

for a known constant $K(q)$ (independent of a) and the exponent $\gamma(q)$ specified above. If one starts CABM with particles at every point of $(-\infty,-a)\cap(a,\infty)$, the above formula can be used to derive the following continuous counterpart of (2):

$$\mathbb{P}\big[\text{No particle crosses 0 in } [0,T]\big] = K(q)^2\left(\frac{T}{a^2}\right)^{-\gamma(q)}\big(1+o(1)\big). \tag{3}$$

As discussed above, the fixed time law for annihilating Brownian motions is closely related to the real Ginibre ensemble. Similarly, the "bulk" limit of the Pfaffian point process describing the exit measure for ABM is identical, up to a deterministic transformation, to the translationally invariant kernel for the Pfaffian point process giving the law of real roots of the Gaussian power series and also to the large N limit for the real eigenvalues of truncated orthogonal matrices (where one row/column has been removed to obtain a minor); see [18, 29]. Theorem 1 allows one to compute the corresponding empty interval probability, up to the constant term, as the endpoints of the interval approach ± 1. For example,

$$\mathbb{P}\big[\text{The interval } [-1+2\epsilon, 1-2\epsilon] \text{ contains no roots}\big] = \epsilon^{\frac{3}{8}} e^{\kappa_2}\big(1+o(1)\big) \quad \text{as } \epsilon \downarrow 0,$$

where

$$\kappa_2 = \frac{1}{4}\log\left(\frac{\pi^2}{2}\right) - \frac{\gamma}{2} - \frac{1}{4}\int_0^\infty \log(x)\big(\tanh(x) + \tanh(x/2)\big)\big(\operatorname{sech}^2(x) + \frac{1}{2}\operatorname{sech}^2(x/2)\big)\,dx.$$

Here, γ is the Euler–Mascheroni constant. Notice that the exponent $3/8$ coincides with the value $\gamma(2)$ of the persistence exponent (2). This is a reflection of a general property of the class of Pfaffian kernels we consider: the leading order asymptotics of the gap probability for a nontranslationally invariant kernel and its translationally invariant "bulk" limit coincide, while the constant terms differ (but are explicitly known). Notice that the Pfaffian point process describing the edge scaling limit of the Gaussian orthogonal ensemble does not seem to possess this property. Still, it is quite satisfying that there is a rather diverse pool of examples of Pfaffian point processes which can be all treated using Theorems 1, 2, 3.

Moreover, it turns out that our results can be applied to zeros of random polynomials with i.i.d. mean zero, but not necessarily Gaussian, coefficients which are no longer described by a Pfaffian point process. In a remarkable paper [10], Dembo et al. show the probability that such a random polynomial of degree n has no zeros and decays polynomially as n^{-b}. They show that this asymptotic is controlled by the case of Gaussian coefficients and characterise the decay rate in terms of the Gaussian power series. This immediately enables an application of our asymptotic results for Fredholm Pfaffians to evaluate the unknown power b; see Section 3.1 for more details.

Literature review. The analysis of Fredholm Pfaffians is an increasingly active area of research due to applications ranging from interacting particle systems to random matrices. Our interest in the asymptotics of Fredholm Pfaffians was generated by the work of Peter Forrester [19] who used the connection between the ABM and the law of real eigenvalues in the real Ginibre ensemble from [40], and formula (1) of Derrida–Zeitak [11] to calculate the bulk scaling limit for the corresponding gap probability. In [31] Mikhail Poplavskyi and Gregory Schehr use the link between Kac polynomials and the ensemble of truncated orthogonal matrices from [18] to calculate the leading order asymptotic for the gap probability for the real roots of Kac polynomials. It is worth stressing that their work is independent of our own. A common feature of the Pfaffian point processes related to Kac polynomials, truncated orthogonal random matrices and exit measures for interacting particle systems, is the appearing of the scalar sech-kernel; see Section 2 for a review of "derived" Pfaffian point processes and the corresponding scalar kernels. Exploiting the integrability of the sech-kernel, Ivan Dornic analyses the asymptotics of a distinguished solution to Painlevé VI equation to rederive the formula for the empty interval probability for the real roots of Kac polynomials, [13]. Interestingly enough, the integrable structure of kernels associated with the single time law of CABM and the real Ginibre ensemble is not at all obvious. However, in a recent series of papers [3, 4] Jinho Baik and Thomas Bothner establish a link between the Pfaffian kernel for the edge scaling limit of the real Ginibre ensemble and Zakharov–Shabat integrable hierarchy. By analysing the associated Riemann–Hilbert problem using the nonlinear steepest descent method, they manage to obtain the right tails of the distribution for the maximal real eigenvalue for the thinned real Ginibre ensemble up to and including the constant term (Lemma 1.14 of [4]). The answers we obtain for this regime in Section 3 coincide with the answers presented in [3, 4]. It is worth pointing out that, at the moment, there seems to be no link between the integrable structures uncovered in the context of persistence problems and the laws of extreme particles. However, given that these problems can be treated in a unified way using Theorems 1 and 3, it is reasonable to conjecture that there is a universal integrable structure underlying the classes of kernels introduced in the present work. (See also [25] for an extended study of a class Fredholm determinants appearing as solutions of

Zakharov–Shabat system which includes not only the real Ginibre case but also the eigenvalue statistics for Gaussian orthogonal and symplectic ensembles.) As mentioned above, our proof of the asymptotic theorems generalises the probabilistic method of Kac. At the moment there seem to be very little intersection between the approaches to the analysis of Fredholm Pfaffians based on integrable systems and integrable probability. Nevertheless, such a connection might well exist. For example, in a recent paper [26] Alexandre Krajenbrink and Pierre Le Doussal study short time large-deviations behaviour of the solutions to the Kadar–Parisi–Zhang equation. They find that the corresponding critical point equations are closely related to the nonlinear Schrödinger equations and can be solved exactly using the inverse scattering treatment of the corresponding Zakharov–Shabat system. One of the crucial equations appearing is an integral equation with a quadratic nonlinearity under the integral sign. Using the probabilistic interpretation, the authors show that this nonlinear equation is equivalent to a linear integral equation typical of the theory of random walks. Subsequently, they analyse the linear equation using Sparre Anderson's formula which is a cornerstone of our probabilistic argument as well.

Paper organisation. In Section 2 we introduce the two classes of Pfaffian point processes we consider and state, in Theorems 1 and 3, the asymptotic formulae for the corresponding Fredholm Pfaffians. In Section 3 we apply these formulae to calculate gap probabilities for the range of Pfaffian point processes described above. Section 3.6 studies exit measures for systems of coalescing/annihilating Brownian motions, establishes their Pfaffian structure, and determines the corresponding kernels. This is a new result concerning Pfaffian point processes which enables the application of the asymptotic theorems to the study of the law of the leftmost particle for coalescing/annihilating Brownian motions. The rest of the paper is dedicated to the proof of the stated theorems: in Section 4 we prove the asymptotic formula for Fredholm Pfaffians for translationally invariant kernels; in Section 5 we prove the asymptotic statement for a nontranslationally invariant case; in Section 6 we prove the Pfaffian point process structure for exit measures. Finally, Section 7 gives details of proofs for some more technical statements made in the paper, including a Fourier form for the coefficients of the asymptotic expansion for translationally invariant Fredholm Pfaffians and the analytic properties of these coefficients with respect to the thinning parameter.

2. Statement of results.

We will state two theorems on asymptotics for Fredholm Pfaffians for kernels on $\mathbb{R}$, the first for translationally invariant kernels and the second for a class of nontranslationally invariant kernels. We start with recalling basic facts about determinantal and Pfaffian point processes, Fredholm determinants, and Pfaffians and defining two classes of Pfaffian kernels with which the paper deals.

2.1. *Backround.*

2.1.1. *Definitions: Fredholm determinants and Pfaffians.*

A determinantal point process X on $\mathbb{R}$ with kernel $T:\mathbb{R}^2\to\mathbb{C}$ is a simple point process, whose n-point intensities $\rho_n(x_1,\dots,x_n)$ exist for all $n\geq 1$ and are given by

$$\rho_n(x_1,\dots,x_n)=\det\bigl(T(x_i,x_j):1\leq i,j\leq n\bigr)\quad\text{for all } x_1,\dots,x_n\in\mathbb{R}.$$

A Pfaffian point process X on $\mathbb{R}$ with kernel $\mathbf{K}:\mathbb{R}^2\to M_{2\times 2}(\mathbb{C})$ is a simple point process, whose n-point intensities $\rho_n(x_1,\dots,x_n)$ exist for all $n\geq 1$ and are given by

$$\rho_n(x_1,\dots,x_n)=\mathrm{pf}\bigl(\mathbf{K}(x_i,x_j):1\leq i,j\leq n\bigr)\quad\text{for all } x_1,\dots,x_n\in\mathbb{R}.$$

We also meet Pfaffian point processes on intervals $I\subset\mathbb{R}$.

We will consider real Pfaffian kernels $\mathbf{K}:\mathbb{R}^2\to M_{2\times 2}(\mathbb{R})$, written as

$$\mathbf{K}(x,y)=\begin{pmatrix}K_{11}(x,y) & K_{12}(x,y)\\ K_{21}(x,y) & K_{22}(x,y)\end{pmatrix}\quad\text{for all } x,y\in\mathbb{R}. \tag{4}$$

Recall the antisymmetry requirement on Pfaffian kernels,

$$K_{ij}(x,y)=-K_{ji}(y,x)\quad\text{for all } x,y\in\mathbb{R},\ \text{and } i,j\in\{1,2\} \tag{5}$$

which ensures that the matrix $(\mathbf{K}(x_i,x_j):1\le i,j\le n)$ is a $2n\times 2n$ antisymmetric matrix. Recall that the Pfaffian $\mathrm{pf}(A)$ of a $2n\times 2n$ antisymmetric matrix is defined by

$$\mathrm{pf}(A)=\frac{1}{2^n n!}\sum_{\sigma\in\Sigma_{2n}}\mathrm{sgn}(\sigma)\prod_{i=1}^{n}A_{\sigma(2i-1),\sigma(2i)},$$

and recall the basic properties of Pfaffians: $(\mathrm{pf}(A))^2=\det(A)$ and $\mathrm{pf}(BAB^T)=\det(B)\,\mathrm{pf}(A)$ for any B of size $2n\times 2n$. We usually only list the elements $(A_{ij}:i<j\le 2n)$ defining an anti-symmetric matrix.

For a simple point process X, having all intensities $(\rho_n:n\ge 1)$, the gap probabilities, writing $X(a,b)$ as shorthand for $X((a,b))$, are given by

$$\mathbb{P}[X(a,b)=0]=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[a,b]^n}\rho_n(x_1,\dots,x_n)\,dx_1\dots dx_n$$

(where the term $n=0$ is taken to have the value 1) whenever this series is absolutely convergent (see Chapter 5 of Daley and Vere-Jones [8]). For determinantal or Pfaffian point processes, this leads to the following expressions, which can be taken as the definitions of the Fredholm determinant and Fredholm Pfaffian of the kernels T and $\mathbf{K}$, respectively:

$$\mathrm{Det}_{[a,b]}(I-T)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[a,b]^n}\det\bigl(T(x_i,x_j):1\le i,j\le n\bigr)\,dx_1\dots dx_n, \tag{6}$$

$$\mathrm{Pf}_{[a,b]}(\mathbf{J}-\mathbf{K})=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{[a,b]^n}\mathrm{Pf}\bigl(\mathbf{K}(x_i,x_j):1\le i,j\le n\bigr)\,dx_1\dots dx_n. \tag{7}$$

Here, the left-hand side is merely a notation for the right-hand side. However, later (see (49)) we exploit operator techniques to manipulate Fredholm Pfaffians. Then, we treat $\mathbf{J}$ as an operator on $(L^2[a,b])^2$ defined by $\mathbf{J}(f,g)=(g,-f)$. When T or $\mathbf{K}$ are bounded functions (as in all our results), these series converge absolutely (see Chapter 24 of Lax [28]). We will also consider semiinfinite intervals $[a,\infty)$, and then the series converge absolutely under suitable decay conditions on T and $\mathbf{K}$. More generally, we will define $\mathrm{Det}_{[a,\infty)}(I-T)=\lim_{b\to\infty}\mathrm{Det}_{[a,b]}(I-T)$ and $\mathrm{Pf}_{[a,\infty)}(\mathbf{J}-\mathbf{K})=\lim_{b\to\infty}\mathrm{Pf}_{[a,b]}(\mathbf{J}-\mathbf{K})$ whenever these limits exist (as they do when they represent gap probabilities).

2.1.2. *Classes of Pfaffian kernels considered.* We list the hypotheses required on the kernel of the Fredholm Pfaffians for our results. We consider kernels $\mathbf{K}$ in *derived form* (see [2] Definition 3.9.18 for essentially this notion). For us, this means that $\mathbf{K}$ is derived from a single scalar function $K\in C^2(\mathbb{R}^2)$, as follows. There is a simple jump discontinuity along $x=y$, and we let $S(x,y)=\mathrm{sgn}(y-x)$ (with the convention that $S(x,x)=0$) to display this discontinuity. Then, $\mathbf{K}$ has the form

$$\mathbf{K}(x,y)=\begin{pmatrix}S(x,y)+K(x,y) & -D_2K(x,y)\\ -D_1K(x,y) & D_1D_2K(x,y)\end{pmatrix}, \tag{8}$$

where $D_i K$ denotes the derivative of K in the ith coordinate. The symmetry condition (5) then implies that K is antisymmetric (i.e., $K(x,y) = -K(y,x)$ for all x, y). All our applications are Pfaffian point processes with a kernel of the form $p\mathbf{K}$ for some $p \in [0,1]$, where $\mathbf{K}$ is in the above form.

In order to apply probabilistic methods in our analysis, we will assume that K is given in terms of a probability density function ρ, that is $\rho \geq 0$, $\int_{\mathbb{R}} \rho(x)\,dx = 1$. In our translationally invariant examples, K will be given as

$$K(x,y) = -2\int_0^{y-x} \rho(z)\,dz, \quad \text{for even } \rho \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \tag{9}$$

(the symmetry condition (5) requires that ρ must be an even function).

In our nontranslationally invariant examples, K will be given in terms of a probability density function $\rho \in C^1(\mathbb{R}) \cap H^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ as

$$K(x,y) = \int_{-\infty}^0 \begin{vmatrix} \int_{-\infty}^{x-z} \rho(w)\,dw & \int_{-\infty}^{y-z} \rho(w)\,dw \\ \rho(x-z) & \rho(y-z) \end{vmatrix} dz \tag{10}$$

(where $|A|$ stands for the 2×2 determinant). The required symmetry for K holds without any symmetry requirement on ρ. Note that

$$\lim_{c \to -\infty} K(x+c, y+c) = -2\int_0^{y-x} \tilde{\rho}(z)\,dz, \quad \text{where } \tilde{\rho}(z) = \int_{-\infty}^{\infty} \rho(w)\rho(w-z)\,dw \tag{11}$$

so that the kernel is close to the translation invariant form (9) near $-\infty$. Indeed, our examples that fit this nontranslationally framework are from: (i) random matrices that are studied near the right hand edge of their spectrum and (ii) particle systems that are started with "half-space" initial conditions, that is, where particles are initially spread over the half-space $(-\infty, 0]$, and in both these cases the kernels approach the "bulk" form far from the origin.

We remark (see [2] equation (3.9.32)) that the kernel for limiting ($N = \infty$) GOE Pfaffian point process of eigenvalues in the bulk is in the derived form $\frac{1}{2}\mathbf{K}$, with K in the translationally invariant form (9) for $\rho(z) = \frac{1}{\pi}\frac{\sin(z)}{z}$ (which, while not nonnegative, at least satisfies $\int_{\mathbb{R}} \rho = 1$). Moreover (see [2] equation (3.9.41)), the limiting ($N = \infty$) edge kernel for GOE eigenvalues is in the derived form $\frac{1}{2}\mathbf{K}$, with K in the translationally noninvariant form (10) for $\rho(z) = A(z)$ the Airy functions (which again satisfies $\int_{\mathbb{R}} \rho = 1$).

2.2. *Asymptotics for Fredholm Pfaffians: Translationally invariant kernels.*

THEOREM 1. *Let $\mathbf{K}$ be in the derived form (8) using a scalar kernel K in the form (9) for a density function ρ. Then, for $0 < p < 1$ and under the moment assumptions given below, the asymptotic*

$$\log \mathrm{Pf}_{[0,L]}(\mathbf{J} - p\mathbf{K}) = -\kappa_1(p)L + \kappa_2(p) + o(1) \quad \text{as } L \to \infty$$

holds, where (writing ρ^{*n} for the n-fold convolution of ρ with itself):

(i) *for $0 < p < 1/2$, supposing $\int_{\mathbb{R}} |x|\rho(x)\,dx < \infty$,*

$$\kappa_1(p) = \sum_{n=1}^{\infty} \frac{(4p(1-p))^n}{2n} \rho^{*n}(0),$$

$$\kappa_2(p) = \log\left(\frac{\sqrt{1-2p}}{1-p}\right) + \int_0^\infty \frac{x}{2}\left(\sum_{n=1}^\infty \frac{(4p(1-p))^n \rho^{*n}(x)}{n}\right)^2 dx;$$

(ii) *for* $p = 1/2$, *supposing* $\int_{\mathbb{R}} |x|^4 \rho(x)\, dx < \infty$,

$$\kappa_1(1/2) = \sum_{n=1}^{\infty} \frac{1}{2n} \rho^{*n}(0),$$

$$\kappa_2(1/2) = \log 2 - \frac{1}{4} + \frac{1}{2} \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \int_0^{\infty} x \frac{\rho^{*k}(x)\rho^{*(n-k)}(x)}{k(n-k)}\, dx - \frac{1}{2n} \right);$$

(iii) *for* $1/2 < p < 1$, *supposing there exists* $\phi_p > 0$ *so that* $4p(1-p)\int_{\mathbb{R}} e^{\phi_p x} \rho(x)\, dx = 1$ *and for which* $\int_{\mathbb{R}} |x| e^{\phi_p x} \rho(x)\, dx < \infty$,

$$\kappa_1(p) = \phi_p + \sum_{n=1}^{\infty} \frac{(4p(1-p))^n}{2n} \rho^{*n}(0),$$

$$\kappa_2(p) = \log\left(\frac{\sqrt{2p-1}}{8p(1-p)^2} \right) + \int_0^{\infty} \frac{x}{2} \left(\sum_{n=1}^{\infty} \frac{(4p(1-p))^n \rho^{*n}(x)}{n} \right)^2 dx$$
$$- \log\left(\phi_p \int_{\mathbb{R}} x e^{\phi_p x} \rho(x)\, dx \right) - 2 \sum_{n=1}^{\infty} \frac{(4p(1-p))^n}{n} \int_{-\infty}^{0} e^{\phi_p x} \rho^{*n}(x)\, dx.$$

REMARK 1. Motivated by our applications, implementation of this theorem for the densities $\rho(x) = \pi^{-1} \operatorname{sech}(x)$ and $\rho(x) = (4\pi t)^{-1/2} \exp(-x^2/4t)$ will be done in Corollaries 6 and 7, where simpler expressions for κ_1, κ_2 are calculated. In both cases the function $p \to \kappa_1(p)$ for $p \in (0, 1)$ turns out to be analytic. We do not know whether this is the general case for, say, analytic ρ.

REMARK 2. The case $p = 1$ is not covered by our theorem. In this case the Fredholm Pfaffian reduces to the (square root) of a 2×2 determinant (this is evident from the proof in Section 4.1). Then, the asymptotics are easier, and they need not be exponential (see Remark 2 in Section 3.2).

REMARK 3. For numerical or theoretical analysis, it may be useful to rewrite the infinite sums in these formulae in alternate ways. For example, the probabilistic representations (see (68) for $p \in (0, \frac{1}{2})$ and (76) for $p \in (\frac{1}{2}, 1)$) give formulae for $\kappa_i(p)$ when $p \neq \frac{1}{2}$ as expectations. We may also, for suitably good ρ, rewrite some terms in the constants $\kappa_i(p)$ usefully in terms of Fourier transforms. We use the conventions $\hat{\rho}(k) = \int \exp(ikx)\rho(x)\, dx$ with inversion (when applicable) $\rho(x) = (2\pi)^{-1} \int \exp(-ikx)\hat{\rho}(k)\, dk$. The exponents $\kappa_i(p)$ can be expressed using the function

$$L_\rho(p, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx} \log\bigl(1 - 4p(1-p)\hat{\rho}(k)\bigr)\, dk. \tag{12}$$

We do not look for good sufficient conditions but suppose for the formulae (13), (14) below that ρ is in Schwartz class. When $p = \frac{1}{2}$, we assume further that there exists $\mu > 0$ so that $\int_{\mathbb{R}} e^{2\mu|x|} \rho(x)\, dx < \infty$ which justifies certain contour manipulations in the proof,

$$\kappa_1(p) = \begin{cases} -\frac{1}{2} L_\rho(p, 0) & \text{for } p \in \left(0, \frac{1}{2}\right], \\ -\frac{1}{2} L_\rho(p, 0) + \phi_p & \text{for } p \in \left(\frac{1}{2}, 1\right), \end{cases} \tag{13}$$

$$(14)\quad \kappa_2(p) = \begin{cases} \log\Big(\dfrac{\sqrt{1-2p}}{1-p}\Big) + \dfrac{1}{2}\displaystyle\int_0^\infty x L_\rho^2(p,x)\,dx & \text{for } p \in \Big(0, \dfrac{1}{2}\Big), \\ \dfrac{1}{4}\log(2\sigma^2) - \dfrac{\gamma}{2} - \dfrac{1}{2}\displaystyle\int_0^\infty \log(x)\Big(x^2 L_\rho^2\Big(\dfrac{1}{2}, x\Big)\Big)'\,dx & \text{for } p = \dfrac{1}{2}, \\ \log\Big(\dfrac{\sqrt{2p-1}}{8p(1-p)^2}\Big) - \Gamma_p + \dfrac{1}{2}\displaystyle\int_0^\infty x L_\rho^2(p,x)\,dx & \text{for } p \in \Big(\dfrac{1}{2}, 1\Big), \end{cases}$$

where

$$(15)\qquad \Gamma_p = \log\Big(\phi_p \int_{\mathbb{R}} x e^{\phi_p x}\rho(x)\,dx\Big) - \frac{1}{\pi}\int_{\mathbb{R}} \frac{\phi_p}{\phi_p^2 + k^2}\log\big(1 - 4p(1-p)\hat{\rho}(k)\big)\,dk$$

and where $\sigma^2 = \int_{\mathbb{R}} x^2\rho(x)\,dx$ and γ is the Euler Mascheroni constant. When $p = \frac{1}{2}$, the formula for κ_2 encodes the slightly arbitrary compensation by $-\frac{1}{2n}$ used in Theorem 1 in, perhaps, a more natural way using the transform. It is tempting to integrate by parts in the term $\int_0^\infty \log x(x^2 L_\rho^2(\frac{1}{2}, x))'\,dx$ in order to obtain a form closer to those when $p \neq \frac{1}{2}$, but this is not justified (e.g., $x L_\rho(\frac{1}{2}, x) \to -1$ as $x \to \infty$). The proofs of these alternative formulae are in the Section 7.1. Notice that the leading order term $-\frac{1}{2}L_\rho(p,0)L$ for $p < \frac{1}{2}$ is equal to one half of the leading term in the classical Akhiezer–Kac formula for the Fredholm determinant of the operator with the kernel ρ. This is due to the Tracy–Widom map between Fredholm Pfaffians and determinants discussed in Section 4.1.

2.3. *Asymptotics for Fredholm Pfaffians*: *Nontranslationally invariant kernels*. We start with a result on Fredholm determinants, with a kernel in the form that arises in our applications. Indeed, an operator in the form (16) below arises immediately in the analysis of the Fredholm Pfaffian of a derived form kernel **K** in the special nontranslationally invariant form (10) (see (95)).

THEOREM 2. *Suppose that $\rho \in C(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is a probability density function. Define*

$$(16)\qquad T(x,y) = \int_{-\infty}^0 \rho(x-z)\rho(y-z)\,dz,$$

and let $\tilde{\rho}(z) = \int_\infty^\infty \rho(w)\rho(w-z)\,dw$.

For $\beta \in [0,1)$, supposing $\int_{\mathbb{R}} |x|\rho(x)\,dx < \infty$,

$$\log \mathrm{Det}_{[-L,\infty)}(I - \beta T) = -\kappa_1(\beta)L + \kappa_2(\beta) + o(1) \quad \text{as } L \to \infty,$$

where

$$\kappa_1(\beta) = \sum_{n=1}^\infty \frac{\beta^n}{n}\tilde{\rho}^{*n}(0)$$

and

$$\kappa_2(\beta) = \sum_{n=1}^\infty \frac{\beta^n}{n^2}\int_{-\infty}^\infty x\big(\rho^{*n}(x)\big)^2\,dx + \int_0^\infty x\Big(\sum_{n=1}^\infty \frac{\beta^n \tilde{\rho}^{*n}(x)}{n}\Big)^2\,dx.$$

When $\beta = 1$, supposing $\int_{\mathbb{R}} x^4\rho(x)\,dx < \infty$,

$$\log \mathrm{Det}_{[-L,\infty)}(I - T) = -\sum_{n=1}^\infty \frac{1}{n}\tilde{\rho}^{*n}(0)L + \log L + \kappa_2 + o(1) \quad \text{as } L \to \infty,$$

where, *setting* $\tilde{\sigma}^2 = \int_{\mathbb{R}} x^2 \tilde{\rho}(x)\,dx$,

$$\kappa_2 = \frac{3}{2}\log 2 - \frac{1}{2} - \log\tilde{\sigma} + \sum_{n=1}^{\infty}\frac{1}{n^2}\int_{-\infty}^{\infty} x\big(\rho^{*n}(x)\big)^2\,dx + \sum_{n=2}^{\infty}\Bigg(\sum_{k=1}^{n-1}\int_0^{\infty} x\frac{\tilde{\rho}^{*k}(x)\tilde{\rho}^{*(n-k)}(x)}{k(n-k)}\,dx - \frac{1}{2n}\Bigg).$$

We do not discuss any direct applications of Theorem 2 in this paper. However, note that Fredholm determinants of integral operators with kernels of the form (16), given by a Hankel convolution, have recently been linked to integrable hierarchies of partial differential equations, such as the nonlinear Schrodinger equation; see [25]. Moreover, if ρ is Gaussian, the corresponding Fredholm determinant appears in the weak noise theory of Kardar–Parisi–Zhang equation; see [26].

Our main result for nontranslationally invariant Fredholm Pfaffians is as follows.

THEOREM 3. *Let* $\mathbf{K}$ *be in the derived form* (8) *using a kernel in the form* (10) *for a probability density function* $\rho \in C^1(\mathbb{R}) \cap H^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *Define* $\tilde{\rho}(z) = \int_{\mathbb{R}} \rho(w)\rho(w - z)\,dw$. *Then*, *for* $0 < p < 1$ *and under the moment assumptions given below*, *the asymptotic*

$$\log \mathrm{Pf}_{[-L,\infty)}(\mathbf{J} - p\mathbf{K}) = -\kappa_1(p)L + \kappa_2(p) + o(1) \quad \text{as } L \to \infty$$

holds, *where*

(i) *for* $0 < p < 1/2$, *supposing* $\int_{\mathbb{R}} |x|\rho(x)\,dx < \infty$,

$$\kappa_1(p) = \sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{2n}\tilde{\rho}^{*n}(0),$$

$$\kappa_2(p) = \frac{1}{2}\log\bigg(\frac{1-2p}{1-p}\bigg) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{n^2}\int_{-\infty}^{\infty} x\big(\rho^{*n}(x)\big)^2\,dx + \int_0^{\infty}\frac{x}{2}\Bigg(\sum_{n=1}^{\infty}\frac{(4p(1-p))^n\tilde{\rho}^{*n}(x)}{n}\Bigg)^2\,dx;$$

(ii) *for* $p = 1/2$, *supposing* $\int_{\mathbb{R}} |x|^4\rho(x)\,dx < \infty$,

$$\kappa_1(1/2) = \sum_{n=1}^{\infty}\frac{1}{2n}\tilde{\rho}^{*n}(0),$$

$$\kappa_2(1/2) = \frac{1}{2}\log 2 - \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{-\infty}^{\infty} x\big(\rho^{*n}(x)\big)^2\,dx + \frac{1}{2}\sum_{n=2}^{\infty}\Bigg(\sum_{k=1}^{n-1}\int_0^{\infty} x\frac{\tilde{\rho}^{*k}(x)\tilde{\rho}^{*(n-k)}(x)}{k(n-k)}\,dx - \frac{1}{2n}\Bigg);$$

(iii) *for* $1/2 < p < 1$, *supposing there exists* ϕ_p *so that* $4p(1-p)\int_{\mathbb{R}} e^{\phi_p x}\tilde{\rho}(x)\,dx = 1$ *and that the integrals* $\int_{\mathbb{R}} |x|e^{\phi_p x}\tilde{\rho}(x)\,dx$ *and* $\int_{\mathbb{R}} e^{\phi_p |x|}\rho(x)\,dx$ *are finite*,

$$\kappa_1(p) = \phi_p + \sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{2n}\tilde{\rho}^{*n}(0),$$

$$\kappa_2(p) = \log\left(\frac{\sqrt{2p-1}}{16p^{3/2}(1-p)^2}\right) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{n^2}\int_{-\infty}^{\infty} x\big(\rho^{*n}(x)\big)^2\,dx$$
$$+ \int_0^{\infty}\frac{x}{2}\left(\sum_{n=1}^{\infty}\frac{(4p(1-p))^n\tilde{\rho}^{*n}(x)}{n}\right)^2 dx - \log\left(\phi_p\int_{\mathbb{R}} xe^{\phi_p x}\tilde{\rho}(x)\,dx\right)$$
$$- \log\left(\int_{\mathbb{R}} e^{\phi_p x}\rho(x)\,dx\right) - 2\sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{n}\int_{-\infty}^{0} e^{\phi_p x}\tilde{\rho}^{*n}(x)\,dx.$$

REMARK. Implementation of this theorem for the densities $\rho(x) = (2\pi t)^{-1/2}\exp(-x^2/4t)$ and $\rho(x) = \frac{2}{\sqrt{\pi}}\exp(x - e^{2x})$, both arising from applications, is done in Sections 3.3 and 3.4 where simpler expressions for κ_1, κ_2 are calculated.

3. Applications. We repeatedly use the following lemma.

LEMMA 4. *Suppose X is a Pfaffian point process on an interval $A \subseteq \mathbb{R}$ with a kernel $\mathbf{K}$ in the derived form* (8), *with an underlying scalar kernel K. Suppose $\phi : A \to \mathbb{R}$ is C^1 and strictly increasing. Then, the push forward point process X' of X under ϕ, given by $X'(\cdot) = X(\phi^{-1}(\cdot))$, is still a Pfaffian point process on the interval $\phi(A)$ with a kernel $\mathbf{K}'$ in the derived form* (8) *with the underlying scalar kernel $K'(x,y) = K(\phi^{-1}(x), \phi^{-1}(y))$.*

PROOF. The intensities ρ_n' for X' are given by

$$\rho_n'(x_1,\dots,x_n) = \rho_n\big(\phi^{-1}(x_1),\dots,\phi^{-1}(x_n)\big)\prod_{k=1}^{n}\alpha_k \quad \text{where } \alpha_k = (\phi^{-1})'(x_k).$$

Also,

$$\begin{aligned}
&\mathrm{pf}\big(\mathbf{K}'(x_i,x_j) : 1 \le i,j \le n\big)\\
&\quad = \mathrm{pf}_{i,j\le n}\begin{pmatrix} S(x_i,x_j) + K'(x_i,x_j) & -D_2K'(x_i,x_j)\\ -D_1K'(x_i,x_j) & D_1D_2K'(x_i,x_j)\end{pmatrix}\\
&\quad = \mathrm{pf}_{i,j\le n}\begin{pmatrix} S(x_i,x_j) + K\big(\phi^{-1}(x_i),\phi^{-1}(x_j)\big) & -D_2K\big(\phi^{-1}(x_i),\phi^{-1}(x_j)\big)\alpha_j\\ -D_1K\big(\phi^{-1}(x_i),\phi^{-1}(x_j)\big)\alpha_i & D_1D_2K\big(\phi^{-1}(x_i),\phi^{-1}(x_j)\big)\alpha_i\alpha_j\end{pmatrix},
\end{aligned}$$

and the factors of α_k can be extracted, as this is the conjugation with a block diagonal matrix D with blocks $\left(\begin{smallmatrix}1 & 0\\ 0 & \alpha_i\end{smallmatrix}\right)$ for which $\det(D) = \prod_{k=1}^{n}\alpha_k$. □

3.1. *Zeros of Gaussian power series.* Let $(a_k)_{k\ge0}$ be an independent collection of real $N(0,1)$ random variables, and define the *Gaussian power series* $f(z) = \sum_{k=0}^{\infty} a_k z^k$. The series converges almost surely to a continuous function on $|z| < 1$, and we consider the real zeros of f as a point process X on $(-1,1)$. Forrester [18] (see Theorem 2.1 of Matsumoto and Shirai [29]) showed that X is a Pfaffian point process with kernel $\frac{1}{2}\mathbf{K}$, with $\mathbf{K}$ in derived form (8) with the choice

$$K(x,y) = \frac{2}{\pi}\sin^{-1}\left(\frac{\sqrt{(1-x^2)(1-y^2)}}{1-xy}\right) - 1 \quad \text{for } x < y.$$

Using Lemma 4, the push forward of the process X under the mapping $x \mapsto \Phi(x) := \frac{1}{2}\log(\frac{1+x}{1-x})$ is a Pfaffian point process on $\mathbb{R}$, still in the derived form, with the choice (note that $\Phi^{-1}(x) = \tanh(x)$)

$$K\big(\Phi^{-1}(x), \Phi^{-1}(y)\big) = \frac{2}{\pi}\sin^{-1}\big(\mathrm{sech}(y-x)\big) - 1 \quad \text{for } x < y.$$

The problem has now become translationally invariant, and the kernel is in the form (9) with $\rho(z) = \pi^{-1}\operatorname{sech}(z)$. Theorem 1 with $p = \frac{1}{2}$ leads to (see Corollary 6 below)

$$\log \mathbb{P}\big[X(a,b) = 0\big] = -\frac{3}{8}\big(\Phi(b) - \Phi(a)\big) + \kappa_2(1/2) + o(1), \tag{17}$$

where the term $o(1)$ converges to zero whenever $b \uparrow 1$ or $a \downarrow -1$ and $\kappa_2(1/2)$ is given by

$$\frac{1}{4}\log\left(\frac{\pi^2}{2}\right) - \frac{\gamma}{2} - \frac{1}{4}\int_0^\infty \log(x)\big(\tanh(x) + \tanh(x/2)\big)\big(\operatorname{sech}^2(x) + \frac{1}{2}\operatorname{sech}^2(x/2)\big)\,dx.$$

In particular, (17) implies that

$$\lim_{\epsilon\downarrow 0} \epsilon^{-\frac{3}{16}}\mathbb{P}\big[X(0, 1-2\epsilon) = 0\big] = \lim_{\epsilon\downarrow 0} \epsilon^{-\frac{3}{8}}\mathbb{P}\big[X(-1+2\epsilon, 1-2\epsilon) = 0\big] = e^{\kappa_2(\frac{1}{2})}.$$

It is possible that the remaining integral in (24) can be expressed in terms of special functions, but it is also not hard to calculate it numerically which gives $\kappa_2(1/2) \approx 0.0247$.

As an application of the results obtained in this section, let us prove the following statement which completes the theorem of Dembo, Poonen, Shao and Zeitouni [10] concerning the zeros of random polynomials.

PROPOSITION 5. *Let $f_n : x \mapsto \sum_{i=0}^{n-1} a_i x^i$ be a random polynomial on $\mathbb{R}$, where $(a_i)_{i\geq 0}$ is a sequence of i.i.d. random variables with zero mean, unit variance such that moments of all orders exist. Let $p_n = \mathbb{P}[f_n(x) > 0 \quad \forall x \in \mathbb{R}]$ be the probability that f_n stays positive ("persistence probability"). Then,*

$$\lim_{n\to\infty} \frac{\log p_{2n+1}}{\log n} = -\frac{3}{4}. \tag{18}$$

PROOF. All the hard work is done in [10], where strong approximations are used to show the asymptotic will follow from the Gaussian case, and the approximation of the Gaussian polynomial by the Gaussian power series is controlled. We are left with an easy task: by Theorem 1.1 of [10], the limit

$$b := -\lim_{n\to\infty} \frac{\log p_{2n+1}}{\log n} \tag{19}$$

exists and can be characterised in terms of the Gaussian power series, as follows. Let $(Y_t)_{t\in\mathbb{R}}$ be a centered stationary continuous Gaussian process with the correlation $R(t) := \mathbb{E}[Y_0 Y_t]/\mathbb{E}[Y_0^2] = \operatorname{sech}(t/2)$. Then,

$$b = -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big[\sup_{0\leq t\leq T} Y_t \leq 0\Big]. \tag{20}$$

In other words, the constant b is universal.

The process Y can be realised as a rescaling of the Gaussian power series f, followed by pushing it forward to a process on $\mathbb{R}$ by the function $2\Phi^{-1} = 2\tanh^{-1}$,

$$\tilde{Y}_t := \frac{f(\tanh(t/2))}{(\mathbb{E}[f^2(\tanh(t/2))])^{1/2}}, \quad t \in \mathbb{R}.$$

Indeed, $(\tilde{Y}_t)_{t\in\mathbb{R}}$ is a continuous centered Gaussian process on $\mathbb{R}$ with with the correlation function

$$\mathbb{E}[\tilde{Y}_0\tilde{Y}_t]/\mathbb{E}\big[\tilde{Y}_0^2\big] = \mathbb{E}[\tilde{Y}_0\tilde{Y}_t] = \operatorname{sech}(t/2)$$

which also implies the stationarity of $\tilde{Y}$. Therefore, the law of Y coincides with the law of a constant multiple of $\tilde{Y}$, meaning that the laws of real zeros of Y and $\tilde{Y}$ coincide. It follows from the theorem of Forrester above that the law of real zeros of Y is a translationally invariant Pfaffian point process with $\rho(\cdot) = \frac{\operatorname{sech}(\frac{1}{2}\cdot)}{2\pi}$, where the factors of 2 appear because Y is the pushforward of the Gaussian power series of by $2\Phi^{-1}$ rather than Φ^{-1}. As a consequence of the Fourier formula (13) for κ_1 and the $p = 1/2$ statement of Corollary 6 below,

$$\log \mathbb{P}[Y_t \neq 0 \quad \forall 0 \le t \le T] = -\frac{\kappa_1}{2} T + o(T) = -\frac{3}{16} T + o(T). \tag{21}$$

Therefore, using (20),

$$\begin{aligned} b &= -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big[\sup_{0\le t\le T} Y_t \le 0\Big] \\ &= -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big[\sup_{0\le t\le T} Y_t < 0\Big] \\ &= -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}[Y_t \neq 0 \ \forall 0 \le t \le T] = 2\kappa_1 = \frac{3}{4}. \end{aligned}$$

The second equality is due to the fact that zeros of Y are almost surely simple; the third is due to the reflection symmetry of the process Y; the fourth uses (21). □

REMARK. It is worth stressing that Theorem 1.1 is just one of the universality results presented in [10]: the case where $\mathbb{E}[a_i] \neq 0$ was also treated; the probability that random polynomials have exactly k real zeros or the number of real zeros is $o(\log n/\log\log n)$ were analysed. For all of the cases, formulae analogous to (19) were proved (with $b \to b/2$ when means are nonzero). However, the value of the limit b could only be calculated numerically as $b = 0.76 \pm 0.03$ and bounded rigorously as $0.4 \le b \le 2$. For all of these statements, the unknown constant b can now be replaced with 3/4.

We record now the concrete application of Theorem 1 for the specific kernel based on $\rho(x) = \pi^{-1}\operatorname{sech}(x)$ for all $p \in (0, 1)$. The case $p = 1/2$ yields the above application to Gaussian power series. The case $p \in (0, 1/2)$ would correspond to a gap probability for the thinning of the point process formed by the zeros of a Gaussian power series (should this ever be needed). However, the sech kernel arises completely independently (as far as we know) in a later application in Section 3.3, where the problem of a system of coalescing/annihilating particles on $\mathbb{R}$ never crossing the origin by time t is studied. That probability is related to a Fredholm Pfaffian with a nontranslationally invariant kernel but which asymptotically agrees with the kernel based on $\rho(z) = \pi^{-1}\operatorname{sech}(z)$. The corollary below then becomes needed for all $p \in (0, 1)$. It is also our first chance to study the regularity of $p \to \kappa_i(p)$.

COROLLARY 6. *Let* $\mathbf{K}$ *be a derived form kernel, in the translationally invariant form* (9) *with* $\rho(x) = \pi^{-1}\operatorname{sech}(x)$. *Then, for* $p \in [0, 1)$,

$$\log \operatorname{Pf}_{[0,L]}(\mathbf{J} - p\mathbf{K}) = -\kappa_1(p)L + \kappa_2(p) + o(1) \text{ as } L \to \infty,$$

where

$$\kappa_1(p) = \frac{2}{\pi^2}\left(\cos^{-1}\frac{1-2p}{\sqrt{2}}\right)^2 - \frac{1}{8} \tag{22}$$

is real analytic for $p \in [0, 1)$ *and* $\kappa_2(p)$ *is given by*

$$\frac{1}{2}\int_0^\infty x L_\rho^2(p, x)\, dx + \log\left(\frac{\sqrt{1-2p}}{1-p}\right) \quad p < \frac{1}{2}, \tag{23}$$

$$(24)\quad \frac{1}{4}\log\left(\frac{\pi^2}{2}\right)-\frac{\gamma}{2}-\frac{1}{8}\int_0^\infty \log(x)\left((\tanh(x)+\tanh(x/2))^2\right)'\,dx \qquad p=\frac{1}{2},$$

$$\frac{1}{2}\int_0^\infty xL_\rho^2(p,x)\,dx-\log\left(\cos^{-1}(4p(1-p))\right)$$

$$(25)\qquad -\log\left(\frac{\sqrt{(2p-1)(1+4p-4p^2)}}{2p}\right)$$

$$+\frac{1}{\pi}\int_{\mathbb{R}}\frac{1}{1+k^2}\log\left(1-4p(1-p)\operatorname{sech}\left(\cos^{-1}(4p(1-p))k\right)\right)dk \qquad p>\frac{1}{2},$$

where, for $p\neq\frac{1}{2}$ *and* $x\neq 0$,

$$(26)\qquad L_\rho(p,x)=\frac{\cosh(x)-\cosh(\frac{4}{\pi}\cos^{-1}(\frac{|2p-1|}{\sqrt{2}})x)}{2x\sinh(x)\cosh(x)}.$$

PROOF. We calculate $\kappa_1(p),\kappa_2(p)$ from the Fourier transform representations (13), (14), as follows. The Fourier transform of $\rho(x)=\pi^{-1}\operatorname{sech}(x)$ is $\hat{\rho}(k)=\operatorname{sech}(k\pi/2)$. For $|\phi|<1$, the exponential moments are given by $\int_{\mathbb{R}} e^{\phi x}\rho(x)\,dx=\sec(\pi\phi/2)$ so that the solution ϕ_p to $4p(1-p)\int_{\mathbb{R}} e^{\phi x}\rho(x)\,dx=1$ is given by $\phi_p=(2/\pi)\cos^{-1}(4p(1-p))$. The Fourier integral formula (12) can be evaluated using the integral

$$(27)\qquad I_\lambda(x):=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-ikx}\log\left(1-\lambda\operatorname{sech}(k\pi/2)\right)dk, \quad \text{for } x\in\mathbb{R},\ \lambda\in(0,1].$$

Note, $I_\lambda(x)$ is continuous and even in x. Integrating by parts, we find

$$I_\lambda(x)=\frac{1}{2\pi ix}\int_{\mathbb{R}} e^{-2ikx/\pi}\left(\frac{\sinh(k)}{\cosh(k)}-\frac{\sinh(k)}{\cosh(k)-\lambda}\right)dk.$$

For $x>0$, this integral can be computed by closing the integration contour in the lower half plane and applying Cauchy's residue theorem. The only singularities of the first term of the integrand are the first order poles at $z_m^{(1)}=(\frac{\pi}{2}+\pi m)i$ for $m\in\mathbb{Z}$. For $\lambda\in(0,1)$, the singularities of the second term are also first order poles at $z_m^{(2)}=(\alpha+2\pi m)i$ and $z_m^{(3)}=(-\alpha+2\pi m)i$ for $m\in\mathbb{Z}$, where $\alpha\in(0,\pi)$ satisfies $\cos\alpha=\lambda$, $\sin\alpha=\sqrt{1-\lambda^2}$. The corresponding residues are $e^{-2ixz_m^{(k)}/\pi}$. Summing up the three resulting geometric progressions of residues, one finds, for $x>0$,

$$I_\lambda(x)=\frac{1}{2x}\frac{\cosh(x)-\cosh(2x(1-\frac{\alpha}{\pi}))}{\sinh(x)\cosh(x)}.$$

Note also that

$$(28)\qquad 1-\frac{\alpha}{\pi}=1-\frac{\cos^{-1}(\lambda)}{\pi}=\frac{2}{\pi}\cos^{-1}\sqrt{\frac{1-\lambda}{2}}.$$

The value at $\lambda=1$ can be rewritten, for $x>0$, as

$$I_1(x)=\frac{1}{2x}\frac{\cosh(x)-\cosh(2x)}{\sinh(x)\cosh(x)}=-\frac{1}{2x}\left(\tanh(x)+\tanh(x/2)\right).$$

The value at $x=0$, by continuity, is

$$I_\lambda(0)=\frac{1}{4}-\left(1-\frac{\cos^{-1}(\lambda)}{\pi}\right)^2=\frac{1}{4}-\frac{4}{\pi^2}\left(\cos^{-1}\sqrt{\frac{1-\lambda}{2}}\right)^2.$$

Using $L_\rho(p,x) = I_{4p(1-p)}(x)$, the expression (13) leads to the following formulae for κ_1:

$$\kappa_1(p) = \frac{2}{\pi^2}\left(\cos^{-1}\frac{|2p-1|}{\sqrt{2}}\right)^2 - \frac{1}{8} + \mathbb{I}(p > 1/2)\frac{2}{\pi}\cos^{-1}\big(4p(1-p)\big).$$

We can remove the indicator to reveal the analyticity of this formula; indeed we use $\cos^{-1}(x) = \pi - \cos^{-1}(-x)$ and $\cos^{-1}(x) = \frac{1}{2}\cos^{-1}(2x^2-1)$ to see, for $p \in (1/2, 1)$,

$$\begin{aligned}\kappa_1(p) &= \frac{2}{\pi^2}\left(\cos^{-1}\frac{2p-1}{\sqrt{2}}\right)^2 + \frac{2}{\pi}\cos^{-1}\big(4p(1-p)\big) - \frac{1}{8} \\ &= \frac{2}{\pi^2}\left(\cos^{-1}\frac{2p-1}{\sqrt{2}}\right)^2 + 2 - \frac{2}{\pi}\cos^{-1}\big(-4p(1-p)\big) - \frac{1}{8} \\ &= \frac{2}{\pi^2}\left(\cos^{-1}\frac{2p-1}{\sqrt{2}}\right)^2 + 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{2p-1}{\sqrt{2}}\right) - \frac{1}{8} \\ &= \frac{2}{\pi^2}\left(\pi - \cos^{-1}\frac{2p-1}{\sqrt{2}}\right)^2 - \frac{1}{8} = \frac{2}{\pi^2}\left(\cos^{-1}\frac{1-2p}{\sqrt{2}}\right)^2 - \frac{1}{8}\end{aligned}$$

agreeing with the expression for $\kappa_1(p)$ when $p \in (0, 1/2)$. Using the exponential moments, we find, for $p \geq 1/2$,

$$\int_{\mathbb{R}} x e^{\phi_p x}\rho(x)\,dx = \frac{\pi}{2}\frac{\sin(\pi\phi_p/2)}{\cos^2(\pi\phi_p/2)} = \frac{(2p-1)\pi\sqrt{1+4p-4p^2}}{32p^2(1-p)^2},$$

and the formula for κ_2 then follows from (14). □

REMARK. We do not investigate the regularity of κ_2, but the numerics in Figure 1 suggest that it is at least in C^1.

3.2. *Gap probabilities for coalescing/annihilating Brownian motions.* This section discusses the result that arose in Derrida and Zeitak [11] in their study of domain sizes for Potts models. Consider an infinite system of reacting Brownian motions on $\mathbb{R}$, where each colliding pair instantly annihilates with probability θ or instantly coalesces with probability $1-\theta$ (independently at each collision). We will refer to this system as CABM(θ). Suppose the initial positions form a Poisson point process with bounded intensity $\lambda(x)\,dx$. The positions of the particles at time $t > 0$ form a Pfaffian point process X_t with a kernel $(1+\theta)^{-1}\mathbf{K}$ in the derived form (8), where

$$K(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y'}\big(e^{-(1+\theta)\int_{x'}^{y'}\lambda(z)\,dz} - 1\big)\begin{vmatrix} p_t(x,x') & p_t(x,y') \\ p_t(y,x') & p_t(y,y')\end{vmatrix}\,dx'\,dy' \tag{29}$$

where $p_t(x,x')$ is the transition density for Brownian motion on $\mathbb{R}$. When λ is constant, this reduces to

$$K(x,y) = \int_0^{\infty}\big(e^{-\lambda(1+\theta)z} - 1\big)\frac{1}{\sqrt{4\pi t}}\big(e^{-(z-y+x)^2/4t} - e^{-(z+y-x)^2/4t}\big)\,dz. \tag{30}$$

This scalar $K(x,y)$ is in the translationally invariant form (9) with

$$\rho(x) = \int_{\mathbb{R}}\frac{\lambda(1+\theta)}{2}e^{-\lambda(1+\theta)|z|}\frac{1}{\sqrt{4\pi t}}e^{-\frac{|x-z|^2}{4t}}\,dz, \tag{31}$$

that is the density for the convolution of a Gaussian $N(0,2t)$ variable with a two sided Exponential($\lambda(1+\theta)$) variable. One may also let $\lambda \uparrow \infty$, starting the process as an entrance

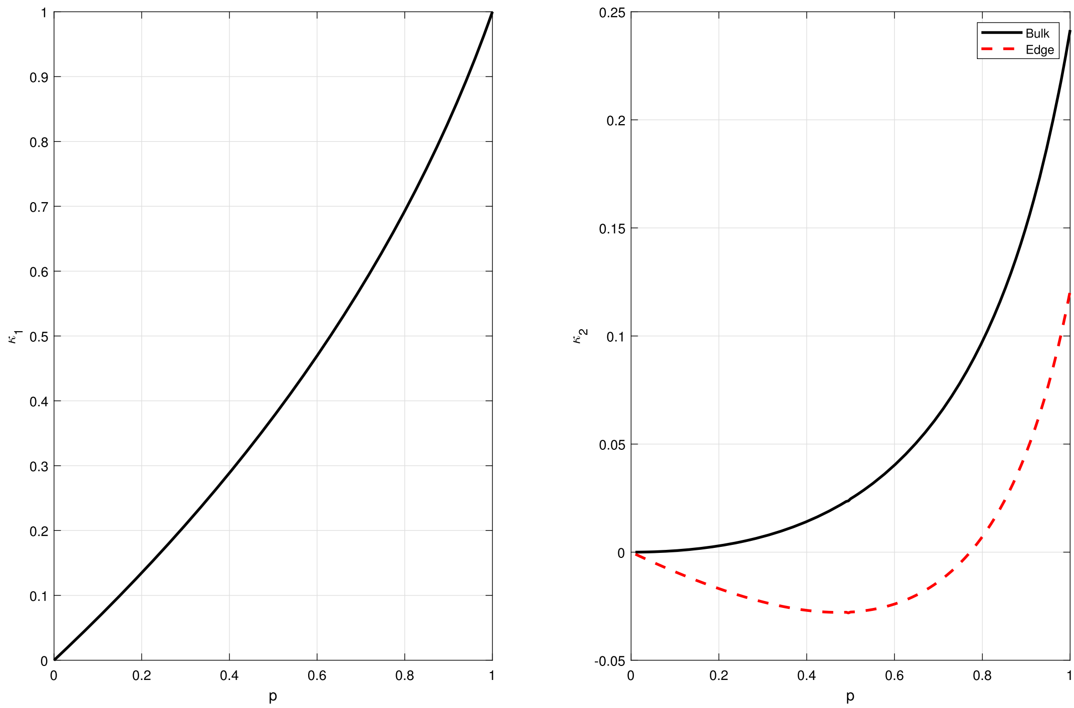

FIG. 1. *Left pane: The leading coefficient* $p \mapsto \kappa_1(p)$ *for the sech kernel from Corollary* 6 *in Section* 3.1. *Right pane: The subleading coefficient* $p \mapsto \kappa_2(p)$ *for the same kernel (labelled "bulk") and for the nontranslationally invariant kernel based on* $\rho(x) = \frac{2}{\sqrt{\pi}} \exp(x - e^{2x})$ *discussed in Corollary* 8 *in Section* 3.4 *(labelled "edge").*

law (which we informally call the maximal entrance law), and where ρ becomes just Gaussian $\mathrm{N}(0, 2t)$ density.

A derivation of the kernel (29) is not quite in the literature. The maximal entrance law and its kernel are derived in [40] for annihilating or coalescing Brownian motions. Discrete analogues of CABM(θ) are discussed in [21], together with the kernels for continuum limits, but for deterministic initial conditions. We go through all the (analogous) steps when deriving the kernel for the novel case of exit measures in Section 6.

Our interest here is to explore the gap probability asymptotics. For constant intensity Poisson(λ) initial conditions, we may apply Theorem 1 to deduce for $\theta > 0$ that, setting $p_\theta = (1+\theta)^{-1}$,

$$\log \mathbb{P}\big[X_t(0, L) = 0\big] = -\kappa_1(p_\theta)L + \kappa_2(p_\theta) + o(1) \text{ as } L \to \infty,$$

where $\kappa_1(p), \kappa_2(p)$ are given in (13) and (14) using $\hat{\rho}(k) = \frac{\lambda^2}{\lambda^2+k^2} \exp(-Tk^2)$. For the maximal entrance law, that is, where $\lambda \uparrow \infty$, the underlying density ρ is Gaussian and the formulae for κ_1 and κ_2 become more tractable, as shown in the upcoming corollary.

Note that, as θ ranges over $(0, 1]$ the value p_θ ranges over $[1/2, 1)$. However, the kernel $p\mathbf{K}$ for $p \in (0, \frac{1}{2})$ also has a use for the study of massive coalescing particles; see Lemma 21. Therefore, we now examine the behaviour of $\kappa_i(p)$ for all $p \in (0, 1)$. Brownian scaling would reduce the two parameters t, L in ρ to one, but we leave both parameters so we can align our results with those in [11].

COROLLARY 7. *Let* $\mathbf{K}$ *be a derived form kernel, in the translationally invariant form* (9) *with* $\rho(x) = (4\pi t)^{-1/2} \exp(-x^2/4t)$. *Then, for* $p \in [0, 1)$,

$$\log \mathrm{Pf}_{[0,L]}(\mathbf{J} - p\mathbf{K}) = -\kappa_1(p)L + \kappa_2(p) + o(1) \textit{ as } L \to \infty,$$

where $\kappa_1(p)$ is given by

$$\kappa_1(p) = \frac{1}{4\sqrt{\pi t}} Li_{3/2}\big(4p(1-p)\big) + \mathbb{I}(p > 1/2)\big(-t^{-1}\log 4p(1-p)\big)^{1/2} \tag{32}$$

using the poly-logarithm function $Li_s(x) = \sum_{n\ge 1} x^n/n^s$, and $\kappa_2(p)$ is given by

$$\log\left(\frac{\sqrt{1-2p}}{1-p}\right) + \frac{1}{4\pi}\sum_{n=2}^{\infty}\frac{(4p(1-p))^n}{n}\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} \quad \textit{for } p \in \left(0, \frac{1}{2}\right), \tag{33}$$

$$\log 2 - \frac{1}{4} + \frac{1}{4\pi}\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} - \pi\right) \quad \textit{for } p = 1/2, \tag{34}$$

$$\begin{aligned} &\frac{1}{2}\log\left(\frac{2p-1}{16(1-p)^2}\right) + \frac{1}{4\pi}\sum_{n=2}^{\infty}\frac{(4p(1-p))^n}{n}\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} \\ &\quad - \log\big(-\log\big(4p(1-p)\big)\big) - \sum_{n=1}^{\infty}\frac{1}{n}\operatorname{erfc}\big(\sqrt{-n\log\big(4p(1-p)\big)}\big) \quad \textit{for } p \in \left(\frac{1}{2}, 1\right). \end{aligned} \tag{35}$$

The function κ_1 is analytic, and the function κ_2 is C^1 for $p \in (0, 1)$.

PROOF. We use (13) to calculate $\kappa_1(p)$. We have $\hat{\rho}(k) = \exp(-tk^2)$ so that

$$L_\rho(p, 0) = \frac{1}{2\pi}\int_{\mathbb{R}} \log\big(1 - 4p(1-p)e^{-tk^2}\big)\, dk = -\frac{1}{\sqrt{4\pi t}}\mathrm{Li}_{3/2}\big(4p(1-p)\big).$$

The factor $\phi_p = (-t^{-1}\log(4p(1-p)))^{1/2}$ and (32) follows from (13). For $\kappa_2(p)$, we use the expressions in Theorem 1, where all the integrals can be evaluated using the explicit Gaussian densities ρ^{*n}.

The regularity of κ_1, κ_2 is not immediately evident from these expressions, but follows after some manipulation which we detail in the Section 7.2. □

REMARK 1. The formulae for $\kappa_2(p)$ are independent of t: (33) agrees with Derrida and Zeitak [11] equation (50); (34) agrees with [11] equation (51); (35) agrees with [11] equation (53). The formulae for $\kappa_1(p)$ depend on t; with the choice $t = p^2/\pi$, we find (32) agrees with [11] equations (49) and (52). This choice of t is also consistent with space scaling used in [11], as it makes the one-point density take the constant value 1.

REMARK 2. Figure 2 plots $p \to \kappa_1(p), \kappa_2(p)$ from Corollary 7 at $t = \frac{1}{2}$. As expected, $\kappa_1(p)$ increases with p which corresponds to weaker thinning. Note that $\kappa_1(p) \to \infty$, as $p \uparrow 1$ (indeed $\kappa_1(p) = (-2\log(4(1-p)))^{\frac{1}{2}} + O(1-p)$.) This is good sense, since at $p = 1$ we are studying coalescing Brownian motions where gap probability have Gaussian tails not exponential tails. Indeed, gap probabilities for $p = 1$ can be read off from the Brownian web in terms of a single pair of dual Brownian motions (see Section 2 of [40]). This simplicity corresponds in the analytic approach to the fact that the Fredholm Pfaffian reduces (see Proposition 10) to a 2×2 determinant.

3.3. *Half-space initial conditions for coalescing/annihilating Brownian motions*. Consider the same system CABM(θ) of reacting Brownian motions on $\mathbb{R}$ as in Section 3.2, but with a a "maximal" entrance law on $(-\infty, 0]$, defined as the limit of Poisson(μ) initial conditions on $(-\infty, 0]$ as $\mu \to \infty$. This example fits into the framework for Theorem 3. Indeed, the positions of the particles at time $t > 0$ form a Pfaffian point process X_t with a kernel

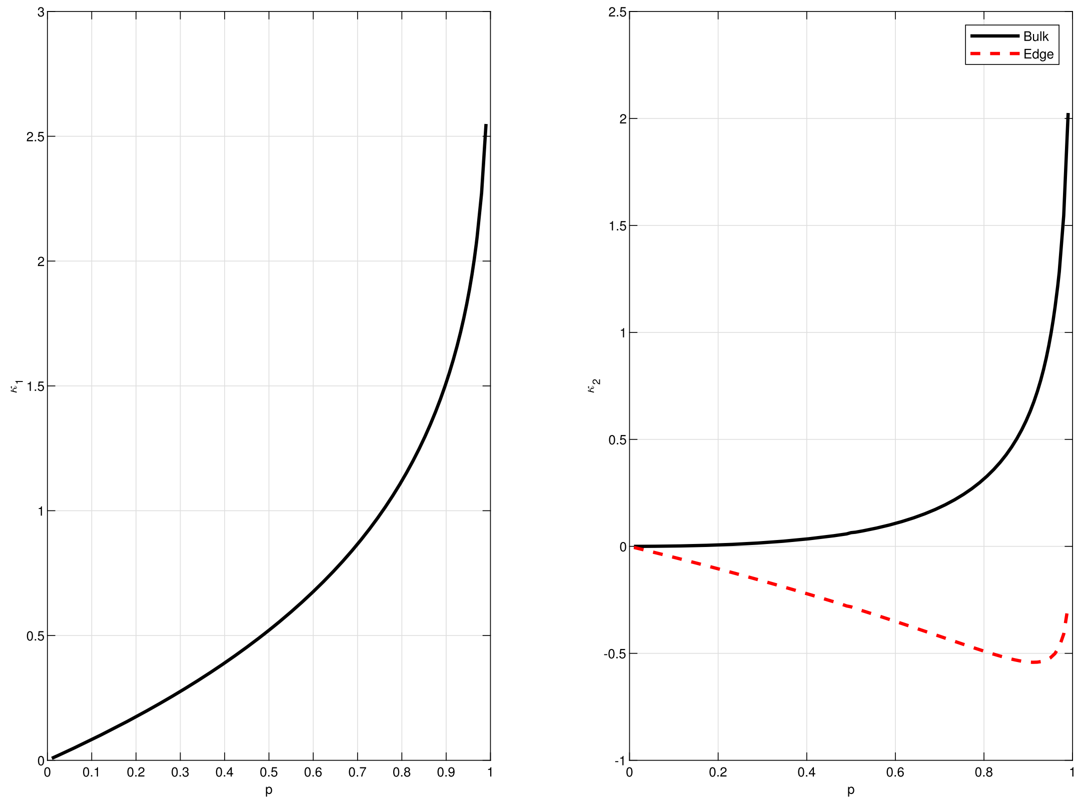

FIG. 2. *Left pane*: *The leading coefficient* $p \mapsto \kappa_1(p)$ *for the Gaussian kernel from Corollary 7 in Section 3.2 at* $t = \frac{1}{2}$. *Right pane*: *The subleading coefficient* $p \mapsto \kappa_2(p)$ *for the same kernel* (*labelled "bulk"*) *and for the nontranslationally invariant kernel discussed in Section 3.3* (*labelled "edge"*).

$(1+\theta)^{-1}\mathbf{K}$ in the derived form (8). Taking $\lambda(z) = \lambda_0 \mathbb{I}(z \le 0)$ in (29) and then letting $\lambda_0 \uparrow \infty$, we find the underlying scalar kernel $K(x, y)$ is given by

$$K(x, y) = \int \int_{x' < y'} \mathbb{I}(x' \le 0) \begin{vmatrix} p_t(x, x') & p_t(x, y') \\ p_t(y, x') & p_t(y, y') \end{vmatrix} \, dx' \, dy'. \tag{36}$$

This scalar $K(x, y)$ is in the nontranslationally invariant form (10) with $\rho(x) = p_t(x)$. Note that $\tilde{\rho}(z) = \int_{\mathbb{R}} \rho(w)\rho(w - z)\, dw = p_{2t}(z)$. Thus, as expected, the kernel for the half-space initial condition converges to the kernel for the full space initial condition near $-\infty$. We, therefore, compare the answers given by Theorem 3 for the half-space maximal initial condition,

$$\log \mathbb{P}[X_t(-L, \infty) = 0] = -\kappa_1^{\text{edge}}(p_\theta)L + \kappa_2^{\text{edge}}(p_\theta) + o(1) \quad \text{as } L \to -\infty, \tag{37}$$

with those for the full space maximal initial condition, given by Theorem 1 in Section 3.2,

$$\log \mathbb{P}[X_t(-L, 0) = 0] = -\kappa_1^{\text{bulk}}(p_\theta)L + \kappa_2^{\text{bulk}}(p_\theta) + o(1) \quad \text{as } L \to \infty \tag{38}$$

(using the random matrix terminology for analogous problems on random spectra). The expression for κ_1 in Theorems 1 and 3 show, as expected, that $\kappa_1^{\text{edge}}(p) = \kappa_1^{\text{bulk}}(p)$. The change in the O(1) constant κ_2 can be evaluated exactly for this Gaussian kernel, and we find

$$\kappa_2^{\text{edge}}(p) = \kappa_2^{\text{bulk}}(p) + \frac{1}{2}\log(1 - p) \quad \text{for all } p \in (0, 1).$$

Thus, the regularity properties of κ_1, κ_2 for $p \in (0, 1)$ are unchanged when switching from the bulk to edge case. Figure 2 plots $\kappa_2^{\text{edge}}(p)$ and $\kappa_2^{\text{bulk}}(p)$. According to Corollary 6, they are at least C^1 functions on [0, 1), which is consistent with the shape of the presented graphs. Near $p=1$, $\kappa_2^{\text{bulk}}(p) = -\log(1-p) - \log(-\log(1-p)) + O((1-p)^0)$, $\kappa_2^{\text{edge}}(p) = -\frac{1}{2}\log(1-p) - \log(-\log(1-p)) + O((1-p)^0)$, so each coefficient approaches $+\infty$ as $p \to 1-$.

REMARK 1. As already mentioned in the Introduction, the answers for κ_1, κ_2 for $p \in [0, 1/2)$ correspond to the thinning of the real Ginibre ensemble with the thinning parameter $\gamma = 2p$ investigated in [4]. Under this substitution the answer for the constant term, given in Lemma 1.14 of the cited paper, coincides with the answers presented above.

REMARK 2. For half-space initial condition it is natural to write the results in terms of the rightmost particle. Let R_t denote the position of the rightmost particle alive at time $t \geq 0$ so that $\mathbb{P}[R_t \leq -L] = \mathbb{P}[X_t(-L,\infty) = 0]$. The limit, as $L \to \infty$, involves events where there are large numbers of annihilations by time t. The easier asymptotic probability $\mathbb{P}[R_t \geq L]$, as $L \to \infty$, involves a particle moving a large distance by time t. Indeed, using $\mathbb{I}(X \geq 1) = X - (X-1)_+$ and ρ_1, it is straightforward to see that

$$\log \mathbb{P}[R_t \geq L] = \log \int_L^\infty \rho_1(x)\,dx + o(1) \quad \text{as } L \to \infty.$$

3.4. *Noncrossing probabilities for coalescing/annihilating Brownian motions.* Here and in Section 3.6, we study the problem by Derrida, Hakim and Pasquier [12], which arose in their study of persistence for Potts models, as discussed in the Introduction. We again consider the system CABM(θ) of reacting Brownian motions on $\mathbb{R}$ as in Section 3.3, started from the "maximal" entrance law on $[0,\infty)$. We denote the position of the leftmost particle by $(L_t : t \geq 0)$. The noncrossing probability

$$\mathbb{P}\big[L_t > -a, \forall t \in [0,T]\big]$$

turns out to be exactly given by a Fredholm Pfaffian. Indeed we believe the entire law of $(L_t : t \geq 0)$ should be determined by Fredholm Pfaffians. This is explained and proved in Section 3.6 where we show that the particles that reach the line $x = -a$ form an exit measure point process that is Pfaffian. Its kernel fits into the hypotheses for Theorem 3, and we will deduce, for all $a > 0$ and $\theta \in [0,1]$, that

$$\log \mathbb{P}\big[L_t > -a, \forall t \in [0,T]\big] = -\frac{1}{2}\kappa_1(p_\theta)\log\big(2T/a^2\big) + \kappa_2(p_\theta) + o(1), \tag{39}$$

as $T/a^2 \to \infty$ (again, Brownian scaling shows that this probability depends only on the combination T/a^2). Here, $p_\theta = (1+\theta)^{-1}$ and the coefficient $\kappa_1(p)$, $\kappa_2(p)$ are given below in Corollary 8. Using an initial condition that is "maximal" entrance law on $(-\infty, a] \cup [a,\infty)$, the probability that no particle crosses the origin is the square of the probability in (39), since on this event the particles to the right and to the left of the origin evolve independently. This confirms the result (3), described in the Introduction, that is closest to those in [12].

Corollary 8 below is a direct application of Theorem 3 to the nontranslationally invariant kernel, based on $\rho(x) = \frac{2}{\sqrt{\pi}}\exp(x - e^{2x})$. Note that $\tilde{\rho}(z) = \int_{\mathbb{R}} \rho(w)\rho(w-z)\,dw = \frac{1}{\pi}\operatorname{sech}(z)$ so that the leading coefficient $\kappa_1(p)$ agrees with that in Corollary 6 for the sech kernel. It is, at the moment, a coincidence that the sech kernel arises in this problem and also for Gaussian power series. The corollary is proved in Section 3.6.

COROLLARY 8. *Let* $\mathbf{K}$ *be a derived form kernel, in the nontranslationally invariant form* (10), *based on the probability density* $\rho(x) = \frac{2}{\sqrt{\pi}}\exp(x - e^{2x})$. *Then, for* $p \in [0,1]$,

$$\log \operatorname{Pf}_{[-L,\infty)}(\mathbf{J} - p\mathbf{K}) = -\kappa_1(p)L + \kappa_2(p) + o(1) \quad \textit{as } L \to \infty,$$

where $\kappa_1(p)$ *is given by*

$$\kappa_1(p) = \frac{2}{\pi^2}\left(\cos^{-1}\frac{|2p-1|}{\sqrt{2}}\right)^2 - \frac{1}{8} + \frac{2}{\pi}\cos^{-1}(4p(1-p))\mathbb{I}(p>1/2), \tag{40}$$

for $p \in [0, 1/2)$,

$$\begin{aligned}\kappa_2(p) &= \frac{1}{2}\log\left(\frac{1-2p}{1-p}\right) + \frac{1}{2}\int_0^\infty x L_\rho^2(p,x)\,dx \\ &\quad + \frac{1}{8\pi}\int_{-\infty}^{\infty}\psi^{(0)}((1+ik)/2)\log\bigl(1-4p(1-p)\operatorname{sech}(k\pi/2)\bigr)\,dk; \\ \kappa_2(1/2) &= \frac{1}{4}\log\left(\frac{\pi^2}{8}\right) - \frac{\gamma}{2} - \frac{1}{8}\int_0^\infty \log(x)\bigl((\tanh(x)+\tanh(x/2))^2\bigr)'\,dx \\ &\quad + \frac{1}{8\pi}\int_{-\infty}^{\infty}\psi^{(0)}((1+ik)/2)\log\bigl(1-\operatorname{sech}(k\pi/2)\bigr)\,dk,\end{aligned}$$

and for $p \in (1/2, 1]$, *using* $\phi_p = \frac{2}{\pi}\cos^{-1}(4p(1-p))$,

$$\begin{aligned}\kappa_2(p) &= \frac{1}{2}\int_0^\infty x L_\rho^2(p,x)\,dx - \log\bigl(\cos^{-1}(4p(1-p))\bigr) \\ &\quad - \log\left(\sqrt{\frac{(2p-1)(1+4p-4p^2)}{\pi p}}\Gamma\bigl((1+\phi_p)/2\bigr)\right) \\ &\quad + \frac{1}{\pi}\int_{\mathbb{R}}\frac{1}{1+k^2}\log\bigl(1-4p(1-p)\operatorname{sech}(\pi\phi_p k/2)\bigr)\,dk \\ &\quad + \frac{1}{8\pi}\int_{-\infty}^{\infty}\psi^{(0)}((1+ik)/2)\log\bigl(1-4p(1-p)\operatorname{sech}(k\pi/2)\bigr)\,dk,\end{aligned}$$

where $L_\rho(p,x)$ *is given in* (26) *and* $\psi^{(0)}(z)$ *is the digamma function.*

REMARK. Figure 1 plots $p \to \kappa_1(p), \kappa_2(p)$ from Corollary 8. When $p = 1$, the exponents correspond to coalescing Brownian motions and take the values $\kappa_1(1) = 1$ and $\kappa_2(1) = \log(2/\sqrt{\pi})$. giving, in (39) that

$$\mathbb{P}[L_t > -a, \forall t \in [0,T]] = \sqrt{\frac{2a^2}{\pi T}}(1+o(1)).$$

The leftmost particle is just a Brownian motion started at 0, and the result is then consistent with the exact formula found from the reflection principle.

Figure 1 also allows a comparison between the coefficients κ_2^{edge} from Corollary 8 with κ_2^{bulk} from Corollary 6 for the sech kernel. An exact computation shows that $\kappa_2^{\text{bulk}}(1) = 2\log(2/\sqrt{\pi}) = 2\kappa_2^{\text{edge}}(1)$, a relation that requires an independent derivation.

3.5. *Real eigenvalues for real Ginibre matrices.* This example is treated in [16] using the techniques that are generalised in this paper. Moreover, it coincides exactly with examples discussed above by considering the purely annihilating case ($\theta = 1$, $p_\theta = \frac{1}{2}$) in Sections 3.2 and 3.3. However, we record the results here again, as examples of both Theorem 1 and Theorem 3 that are of interest to the random matrix community.

A real Ginibre ensemble matrix M_N has i.i.d. real Gaussian $N(0,1)$ entries. Let X_N be the point process created by the positions of the real eigenvalues of M_N. Then, X_N converges to a limit point process X on $\mathbb{R}$ as $N \to \infty$. Also, the shifted point process $\tilde{X}_N(dx) =$

$X_N(N^{1/2}+dx)$ (i.e., shifted to the position of the the right-hand edge of the spectrum) also converge to a limit $\tilde{X}$ on $\mathbb{R}$ as $N \to \infty$. Then,

$$\log \mathbb{P}\bigl[X(0,L)=0\bigr] = -\frac{1}{\sqrt{8\pi}}\zeta(3/2)L + \kappa_2^{\mathrm{bulk}} + o(1) \quad \text{as } L \to \infty,$$

and

$$\log \mathbb{P}\bigl[\tilde{X}(-L,\infty)=0\bigr] = -\frac{1}{\sqrt{8\pi}}\zeta(3/2)L + \kappa_2^{\mathrm{edge}} + o(1) \quad \text{as } L \to \infty,$$

where

$$\kappa_2^{\mathrm{bulk}} = \log 2 + \frac{1}{4\pi}\sum_{n=1}^{\infty}\left(-\pi + \sum_{m=1}^{n-1}\frac{1}{\sqrt{m(n-m)}}\right) = \kappa_2^{\mathrm{edge}} + \frac{1}{2}\log 2.$$

The point is that the Pfaffian kernels for the bulk limit (respectively, the edge limit) for the real eigenvalues in the real Ginibre ensemble coincide with those for annihilating Brownian motions at time $t=\frac{1}{2}$ started from the maximal initial condition (respectively, the half-space maximal initial condition).

3.6. *Exit measures for particle systems*.

3.6.1. *Exit kernels*. To reach the applications above to persistence problems, we will study exit measures for particle systems. We consider particle systems evolving in a region $D \subseteq \mathbb{R} \times [0,\infty)$ where, whenever a particle hits the boundary ∂D, it is frozen at its exit position and plays no further role in the evolution. This leads to a collection of frozen particles on the boundary ∂D, which we call the exit measure. Such exit measures have been used commonly in the study of branching systems, but they are also straightforward to construct for our coalescing and annihilating systems (first for finite systems and then by approximation for certain infinite systems, see the discussion in Section 6.2). We use only the special example of the exit measure from a half-space.

THEOREM 9. *Let X_e be the exit measure for the domain $D=(0,\infty)\times[0,\infty)$ for a system CABM(θ) of coalescing/annihilating particles, as described in example 3.2, started from μ a (deterministic) locally finite simple point measure on $(0,\infty)$. Then, the exit measure X_e on $\{0\}\times[0,\infty)$ is a Pfaffian point process with kernel $(1+\theta)^{-1}\mathbf{K}$, where $\mathbf{K}$ is in the derived form (10), given by, when $s<t$,*

$$K\bigl((0,s),(0,t)\bigr) = \int\int_{0<y_1<y_2}\bigl((-\theta)^{\mu(y_1,y_2)}-1\bigr)\begin{vmatrix} p_s^R(0,y_1) & p_t^R(0,y_1) \\ p_s^R(0,y_2) & p_t^R(0,y_2)\end{vmatrix}\,dy_1\,dy_2, \tag{41}$$

where $p_t^R(x,y)$ is the transition density for reflected Brownian motion on $[0,\infty)$. When the initial condition is Poisson with a bounded intensity $\lambda:(0,\infty)\to\mathbb{R}$, the exit measure X_e remains a Pfaffian point process, as above, with

$$K\bigl((0,s),(0,t)\bigr) = \int\int_{0<y_1<y_2}\bigl(e^{-\int_{y_1}^{y_2}\lambda(x)\,dx}-1\bigr)\begin{vmatrix} p_s^R(0,y_1) & p_t^R(0,y_1) \\ p_s^R(0,y_2) & p_t^R(0,y_2)\end{vmatrix}\,dy_1\,dy_2. \tag{42}$$

REMARK 1. Note that kernel (42) can be obtained by averaging the kernel (41) for deterministic initial conditions, considering μ as Poisson. However, this is not true for all random initial conditions, and the Pfaffian point process structure does not hold in general.

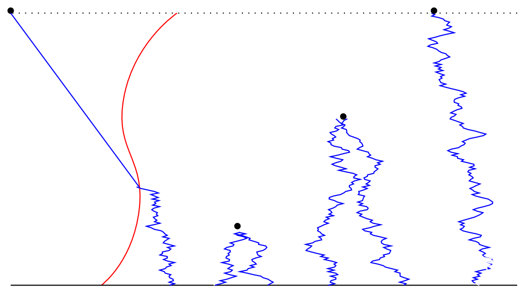

FIG. 3. *The exit measure can be transported by ballistic motion to the halfline* $(-\infty, g(t)) \times \{t\}$.

REMARK 2. We believe that the Pfaffian property holds also for the exit measures on more general regions D, and we explain this informally here. Consider the domain $D = \{(x,s) : x > g(s), s \in [0,t]\}$ for some continuous $g \in C^1([0,t], \mathbb{R})$. We now allow particles that hit ∂D to continue with constant negative drift μ. Choosing $\mu < -\|g'\|_\infty$, the particles can never reenter the region D. This yields a new reacting system on $\mathbb{R} \times [0,t]$ with spatially inhomogeneous motion, where particles to the left of the graph of g move deterministically and never reenter D nor ever again collide (see Figure 3). For coalescing/annihilating spatially inhomogeneous systems on $\mathbb{R}$, we believe the particles at a fixed time $t > 0$ will form a Pfaffian point process (started from suitable initial conditions). Indeed, in [21] a class of interacting particle systems on the lattice $\mathbb{Z}$ are shown to be Pfaffian point processes at fixed times $t \geq 0$. These include spatially inhomogeneous coalescing and annihilating random walks where the right and left jump rates and the coalescence and annihilation parameters may be site dependent. By a continuum approximation, one expects that the analogous systems of continuous diffusions should retain the Pfaffian property. The point process formed at time t by the particles alive on the half-line $(-\infty, g(t)]$ can be mapped (by a deterministic bijection) onto the exit measure of the original system on ∂D, and so this exit measure should itself be a Pfaffian point process. The gap probabilities for this exit measure will coincide (see the discussion in the next section) with $\mathbb{P}[L_s > g_s, s \leq t]$, and by varying the function, g will determine the law of the leftmost particle $\{L_t : t \geq 0\}$ for a process on $\mathbb{R}$.

3.6.2. *Noncrossing probability.* In this section we prove Corollary 8. For specific choices of initial condition, the underlying scalar kernels $K((0,s),(0,t))$ in Theorem 9 can be computed more precisely. They become most tractable for the entrance laws constructed as the limit of Poisson initial conditions with increasing intensities. The existence of these entrance laws is discussed in Section 6.2. Starting with constant Poisson(λ) initial conditions on $(0,\infty)$ and taking the limit as $\lambda \uparrow \infty$, the CABM(θ) starts according to a "maximal" entrance law. The exit measure X_e remains Pfaffian with a kernel, as expected, that is the limit of the corresponding kernels for finite Poisson intensity (this can be checked by passing to the limit in the duality identity (132)). Taking this limit in (42), we find the kernel for X_e under the maximal entrance law on $(0,\infty)$ has underlying scalar kernel

$$
\begin{aligned}
K^{(\infty)}((0,s),(0,t)) &= -\int\int_{0<y_1<y_2} \begin{vmatrix} p_s^R(0,y_1) & p_t^R(0,y_1) \\ p_s^R(0,y_2) & p_t^R(0,y_2) \end{vmatrix} \, dy_1\, dy_2 \\
&= -\int_0^\infty \int_0^\infty \operatorname{sgn}(y_2 - y_1) p_s^R(0,y_1) p_t^R(0,y_2)\, dy_1\, dy_2\, dy_1\, dy_2 \\
&= -\frac{2}{\pi} \int_0^\infty \int_0^\infty \operatorname{sgn}(\sqrt{t} y_2 - \sqrt{s} y_1) e^{-(y_1^2+y_2^2)/2}\, dy_1\, dy_2 \qquad (43) \\
&= -\frac{2}{\pi} \int_0^{\pi/2} \operatorname{sgn}(\sqrt{t}\sin\theta - \sqrt{s}\cos\theta)\, d\theta
\end{aligned}
$$

$$= \frac{4}{\pi} \tan^{-1} \sqrt{\frac{s}{t}} - 1 \quad \text{for } s < t,$$

where in the third equality we have used $p_t^R(0, y) = \sqrt{2/\pi t} \exp(-y^2/2t)$ and in the fourth polar coordinates. Under the map $(0, t) \to \frac{1}{2} \log t$, the exit measure X_e is pushed forward to a translation invariant point process on $\mathbb{R}$, and the kernel (43) is mapped to a translationally invariant kernel in the form (9) with $\rho(z) = \pi^{-1} \operatorname{sech}(z)$. This is exactly the kernel for the zeros of the Gaussian power series in example 3.1, showing that the zeros of the random power series agree in law, after a change of variable, with the exit measure of annihilating Brownian motions. The asymptotics for the probability $\Pr(X_e(\{0\} \times (s, t)) = 0)$ (as $s \downarrow 0$ or $t \uparrow \infty$) can be read off from Corollary 6. Note, however, that the exit measure X_e gives infinite mass to any interval $\{0\} \times [0, \delta)$ if $\delta > 0$.

To study the persistence problem from Section 3.4, we choose $a > 0$ and start the process from Poisson($\lambda_0 \mathbb{I}(a, \infty)$), then let $\lambda_0 \uparrow \infty$, to obtain another entrance law Poisson($\infty \mathbb{I}(a, \infty)$), that is, "maximal on (a, ∞)." Then,

$$\{X_e(\{0\} \times [0, T]) = 0\} = \{L_t > 0, \text{for } t \le T\}, \tag{44}$$

where L_t denotes the position of the leftmost particle at time t, and the probability of this event agrees with the event (39) in Corollary 8 (by translating by a). The parameters a, T are linked, and we choose $a = \sqrt{2}$ and will restore the final answers by Brownian scaling

$$\begin{aligned} &\mathbb{P}[L_t > 0, \forall t \le T] \quad \text{under Poisson}(\infty \mathbb{I}(a, \infty)) \\ &= \mathbb{P}[L_t > 0, \forall t \le 2T/a^2] \quad \text{under Poisson}(\infty \mathbb{I}(\sqrt{2}, \infty)). \end{aligned}$$

Choosing $\lambda(x) = \lambda_0 \mathbb{I}(x > \sqrt{2})$ in (42) and letting $\lambda_0 \uparrow \infty$, we find the kernel under the entrance law Poisson($\infty \mathbb{I}(a, \infty)$) is

$$\begin{aligned} K((0, s), (0, t)) &= -\int\int_{0<y_1<y_2} \mathbb{I}(y_2 > \sqrt{2}) \begin{vmatrix} p_s^R(0, y_1) & p_t^R(0, y_1) \\ p_s^R(0, y_2) & p_t^R(0, y_2) \end{vmatrix} dy_1\, dy_2 \\ &= \int_{\sqrt{2}}^{\infty} \int_0^{y_2} \frac{2}{\pi\sqrt{st}} \left(e^{-\frac{y_2^2}{2s} - \frac{y_1^2}{2t}} - e^{-\frac{y_1^2}{2s} - \frac{y_2^2}{2t}}\right) dy_1\, dy_2 \\ &= \int_{-\infty}^{0} \int_{-\infty}^{-z_2} \frac{4e^{z_1 - z_2}}{\pi\sqrt{st}} \left(e^{-\frac{e^{-2z_2}}{s} - \frac{e^{2z_1}}{t}} - e^{-\frac{e^{2z_1}}{s} - \frac{e^{-2z_2}}{t}}\right) dz_1\, dz_2 \end{aligned}$$

using the substitutions $y_1 = \sqrt{2}\exp(z_1)$ and $y_2 = \sqrt{2}\exp(-z_2)$. Under the map $(0, t) \to -\frac{1}{2} \log t$, the exit measure X_e is pushed forward to a point process $\tilde{X}$ on $\mathbb{R}$. The new kernel for $\tilde{X}$ is $\tilde{K}(x_1, x_2) = K((0, e^{-2x_2}), (0, e^{-2x_1}))$ for $x_1 < x_2$, which becomes

$$\begin{aligned} &\int_{-\infty}^{0} \int_0^{-z_2} \frac{4}{\pi} e^{z_1 - z_2 + x_1 + x_2} \left(e^{-e^{-2(z_2 - x_2)} - e^{2(z_1 + x_1)}} - e^{-e^{2(z_1 + x_2)} - e^{-2(z_2 - x_1)}}\right) dz_1\, dz_2 \\ &= \int_{-\infty}^{0} \begin{vmatrix} \int_{-\infty}^{x_1 - z} \rho(w)\, dw & \int_{-\infty}^{x_2 - z} \rho(w)\, dw \\ \rho(x_1 - z) & \rho(x_2 - z) \end{vmatrix} dz, \end{aligned}$$

for the probability kernel $\rho(x) = \frac{2}{\sqrt{\pi}} \exp(x - e^{2x})$. This is in the nontranslationally invariant form (10) so that we may apply Theorem 3 which gives, for the initial condition Poisson($\infty \mathbb{I}(a, \infty)$),

$$\begin{aligned} \log \mathbb{P}[L_t > 0, \forall t \le T] &= \log \mathbb{P}[\tilde{X}(-\log(2T/a^2)/2, \infty) = 0] \\ &= -\kappa_1(p_\theta) \frac{1}{2} \log(2T/a^2) + \kappa_2(p_\theta) + o(1), \end{aligned} \tag{45}$$

as $T \to \infty$, where $p_\theta = (1+\theta)^{-1}$ and $\kappa(p), \kappa_2(p)$ are given by Theorem 3 using the density $\rho(x)$. To evaluate $\kappa(p), \kappa_2(p)$, we first calculate $\tilde{\rho}(z) = \int_{\mathbb{R}} \rho(w)\rho(w-z)\,dw = \frac{1}{\pi}\operatorname{sech}(z)$. This shows that the leading order asymptotics, that is, $\kappa_1(p)$, will coincide with those for the translationally invariant sech kernel in Corollary 6. This immediately gives the value of $\kappa_1(p)$ in (40) (we need to consider only $p \in [1/2, 1]$ since this is the range of p_θ for $\theta \in [0,1]$). The terms in the formulae for $\kappa_2(p)$ in Theorem 3 that involve $\tilde{\rho}$ have been rewritten using Fourier transforms in (14). We continue this with the terms that involve ρ, so we will use $\hat{\rho}(k) = \frac{1}{\sqrt{\pi}}\Gamma((1+ik)/2)$. Expressing ρ^{*n} via the Fourier inversion formula and then performing the integral in x, we find

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{n^2}\int_{-\infty}^{\infty} x\big(\rho^{*n}(x)\big)^2\,dx \\
&\quad = -\frac{1}{4\pi i}\sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{n^2}\int_{-\infty}^{\infty} n\big(\hat{\rho}(k)\big)^{n-1}\hat{\rho}'(k)\big(\hat{\rho}(-k)\big)^n\,dk \\
&\quad = \frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{\hat{\rho}'(k)}{\hat{\rho}(k)}\log\big(1-4p(1-p)\operatorname{sech}(k\pi/2)\big)\,dk \\
&\quad = \frac{1}{8\pi}\int_{-\infty}^{\infty}\psi^{(0)}\big((1+ik)/2\big)\log\big(1-4p(1-p)\operatorname{sech}(k\pi/2)\big)\,dk,
\end{aligned}
$$

using $\hat{\rho}(k)\hat{\rho}(-k) = \hat{\hat{\rho}}(k) = \operatorname{sech}(k\pi/2)$ and

$$
\frac{\hat{\rho}'(k)}{\hat{\rho}(k)} = \frac{i}{2}\frac{\Gamma'((1+ik)/2)}{\Gamma((1+ik)/2)} = \frac{i}{2}\psi^{(0)}\big((1+ik)/2\big),
$$

where $\psi^{(0)}(z)$ is the digamma function. Finally,

$$
\int_{\mathbb{R}} e^{\phi_p x}\rho(x)\,dx = \hat{\rho}(-i\phi_p) = \frac{1}{\sqrt{\pi}}\Gamma\big((1+\phi_p)/2\big)
$$

which completes all the terms contributing to $\kappa_2(p)$ for $p > \frac{1}{2}$.

4. The proof of Theorem 1. In this section we will derive the asymptotic expressions for Fredholm Pfaffians stated in the translationally invariant case. The proofs consists of the following steps: (i) represent the square of the Fredholm Pfaffian at hand as a product of a Fredholm determinant and a finite dimensional determinant; (ii) interpret each factor as an expectation of a function of a random walk with the transition density determined by the Pfaffian kernel; (iii) calculate each expectation using general theory of random walks.

4.1. *Operator manipulation.* The first step is a calculation that was used by Tracy and Widom [39] in their analysis of the Pfaffian kernels for GOE and GSE. It exploits the special derived form (8) of the Pfaffian kernel.

PROPOSITION 10. *Let* $\mathbf{K}$ *be a kernel in the derived form* (8), *based on kernel* $K \in C^2[a,b]$, *for a finite interval* $[a,b]$. *We suppose that the operator* $I + 2p(1-p)D_2K$ *on* $L^2[a,b]$ *has an inverse* $R = (I + 2p(1-p)D_2K)^{-1}$ *for which* $R - I$ *itself has a* C^1 *kernel. Then,*

$$
\big(\mathrm{Pf}_{[a,b]}(\mathbf{J} - p\mathbf{K})\big)^2 = \mathrm{Det}_{[a,b]}\big(I + 2p(1-p)D_2K\big)\det_2^{a,b}(K), \tag{46}
$$

where $\det_2^{a,b}(K)$ *is the* 2×2 *determinant* $\det\begin{pmatrix} 1+k^{(1)}(a) & k^{(1)}(b) \\ k^{(2)}(a) & 1+k^{(2)}(b) \end{pmatrix}$ *with entries given in terms of the functions*

$$k^{(1)} = (p - p^2)RK(\cdot, a) + p^2 RK(\cdot, b),$$
$$k^{(2)} = (p^2 - p)RK(\cdot, b) - p^2 RK(\cdot, a). \tag{47}$$

This proposition does two things. It represents the square of the Pfaffian in terms of determinants. However, the main point is to exploit the derived form as follows. In the finite Pfaffians that define a Fredholm Pfaffian, there are integrals over $[a, b]$ of products of K, D_1K, D_2K, $D_{12}K$. Each occurrence of a term $K(x_i, x_j)$ can be paired with a term $D_{12}K(x_j, x_k)$, and then integration by parts yields terms that only involve D_1K or D_2K. Moreover, D_1K and D_2K are related by the symmetry conditions. Repeated integration by parts leaves an expression that is mostly expressible only in terms of D_2K. This is all best done at the operator level. Since this is a key starting point for this paper (as it was also for the study for the specific case of the real Ginibre ensemble in [34] and [16]) and since we will also need a modification when we treat the nontranslationally invariant case, we include a proof.

PROOF. The proof exploits results on determinants for trace class operators. We may consider a kernel $(K(x, y) : x, y \in [a, b])$ as an operator on $L^2[a, b]$ via the map $K(f) = \int_a^b K(x, y) f(y)\, dy$ (we need only finite intervals). The references [23, 28] contain most of the results that we need, in particular, that the Fredholm determinant $\mathrm{Det}_{[a,b]}(I + K)$ agrees with the trace class determinant $\mathrm{Det}_{L^2[a,b]}(I + K)$ whenever $K : L^2[a, b] \to L^2[a, b]$ is a trace class operator and that K will be trace class if it is sufficiently smooth.

We need to consider operators $A \in L(H_1, H_2)$ between two different Hilbert spaces. In particular, an operator $A \in L(H_1, H_2)$ is called trace class if it satisfies $\|A\|_{\mathrm{tr}} := \sum_n s_n < \infty$, where (s_n) are the singular values of A, that is, the eigenvalues of $\sqrt{A^*A} : H_1 \to H_1$. For $A \in L(H_1, H_2)$ and $B \in L(H_2, H_1)$ with operator norms $\|A\|$, $\|B\|$, we have

$$\|AB\|_{\mathrm{tr}} \le \|A\|\|B\|_{\mathrm{tr}} \quad \text{and} \quad \|AB\|_{\mathrm{tr}} \le \|A\|_{\mathrm{tr}}\|B\|.$$

Thus if one of the operators A or B is trace class then the compositions AB and BA are trace class. Moreover the Sylvester identity

$$\mathrm{Det}_{H_2}(1 + AB) = \mathrm{Det}_{H_1}(1 + BA) \tag{48}$$

(see [23] Chapter 4 for the case $H_1 = H_2$ where A, B are both trace class) also holds in the case where A is bounded and B is trace class, a result which can, as in [23], be checked by approximating by finite rank operators.

The finite interval $[a, b]$ is fixed throughout this proof. We first suppose that K is smooth. The discontinuity $S(x, y)$ in our kernels means that the entries in $\mathbf{K}$, as in (8), are not trace class operators and, as in Tracy and Widom [39], we first make a smooth approximation. We may choose smooth antisymmetric approximations $S^{(\epsilon)}(x, y)$ that converge pointwise, as $\epsilon \to 0$ to $S(x, y)$, and are uniformly bounded by 1. Then $\mathbf{K}^\epsilon$, defined as in (8) with S, is replaced by $S^{(\epsilon)}$, can be considered a trace class operator on $(L^2_{[a,b]})^2 \to (L^2_{[a,b]})^2$ (and we do this without changing the notation).

For finite dimensional matrices we have $(\mathrm{Pf}(J - K))^2 = \det(J - K) = \det(I + JK)$, where J is block diagonal matrix made from blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (so that $J^2 = -I$ and $\det(J) = 1$). The analogue for us is the relation

$$\big(\mathrm{Pf}_{[a,b]}(\mathbf{J} - p\mathbf{K}^{(\epsilon)})\big)^2 = \mathrm{Det}_{(L^2_{[a,b]})^2}\big(I + p\mathbf{J}\mathbf{K}^{(\epsilon)}\big) \tag{49}$$

where the left-hand side is the Fredholm Pfaffian given by the infinite series (7) and the right-hand side is the trace class determinant on $(L^2[a,b])^2$ and $\mathbf{J}$ is the bounded operator defined by $\mathbf{J}(f,g) = (g,-f)$. To derive the identity (49), it is natural to argue by finite rank approximations. Indeed, K can be approximated by a polynomial $K_N(x,y) = \sum_{n,m\le N} c_{n,m} x^n y^m$ so that K_N converges both uniformly over $[a,b]^2$ and also in trace norm as operators. For the operator K_N the identity reduces to the finite dimensional result.

Tracy and Widom then exploit block manipulations in the operator determinant. Write, in block operator notation,

$$\begin{aligned}\mathbf{JK}^{(\epsilon)} &= \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}\begin{pmatrix} S^{(\epsilon)}+K & -D_2K \\ -D_1K & D_{12}K\end{pmatrix} \\ &= \begin{pmatrix} -D_1K & D_{12}K \\ -S^{(\epsilon)}-K & D_2K\end{pmatrix} \\ &= \begin{pmatrix} 0 & \partial \\ -I & I\end{pmatrix}\begin{pmatrix} S^{(\epsilon)} & 0 \\ -K & D_2K\end{pmatrix},\end{aligned}$$

where $\partial : H^1_{[a,b]} \to L^2_{[a,b]}$ is the derivative operator $\partial(f) = Df$. This expresses $\mathbf{JK}^{(\epsilon)}$ as the composition of two operators $\mathbf{AB}$ where $\mathbf{A} : (H^1_{[a,b]})^2 \to (L^2_{[a,b]})^2$ and $\mathbf{B} : (L^2_{[a,b]})^2 \to (H^1_{[a,b]})^2$. Moreover, A is bounded, and B is trace class, again by the smoothness of the kernels; hence, the compositions $\mathbf{AB}$ and $\mathbf{BA}$ are trace class. Now, we apply the Sylvester identity (48) to find

$$\begin{aligned}\mathrm{Det}_{(L^2_{[a,b]})^2}\big(I + p\mathbf{JK}^{(\epsilon)}\big) &= \mathrm{Det}_{(H^1_{[a,b]})^2}(I + p\mathbf{BA}) \\ &= \mathrm{Det}_{(H^1_{[a,b]})^2}\left(\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} + p\begin{pmatrix} 0 & S^{(\epsilon)}\partial \\ -D_2K & -K\partial + D_2K\end{pmatrix}\right) \\ &= \mathrm{Det}_{H^1_{[a,b]}}\big(I - pK\partial + pD_2K + p^2 D_2KS^{(\epsilon)}\partial\big),\end{aligned} \tag{50}$$

where the last step uses a simple manipulation for determinants of block operators.

Now, we let $\epsilon \to 0$. On the left-hand side of (49) the absolute convergence of the series for the Fredholm Pfaffian allows us to check that $\mathrm{Pf}_{[a,b]}(\mathbf{J} - p\mathbf{K}^{(\epsilon)}) \to \mathrm{Pf}_{[a,b]}(\mathbf{J} - p\mathbf{K})$. On the right-hand side of (50), we rewrite various terms. We have

$$\begin{aligned}K\partial(f)(x) &= \int_a^b K(x,y) f'(y)\,dy \\ &= K(x,b)f(b) - K(x,a)f(a) + \int_a^b D_2K(x,y)f(y)\,dy\end{aligned}$$

so that $K\partial = -D_2K + K(\cdot,b)\otimes\delta_b - K(\cdot,a)\otimes\delta_a$, as an operator mapping $H^1_{[a,b]} \to L^2_{[a,b]}$, where a tensor operator $h\otimes\delta_a$, for $h\in L^2$, acts via $h\otimes\delta_a(f) = f(a)h$. Similarly, again using integration by parts, $S\partial = 1\otimes(\delta_a+\delta_b) - 2I$. Then,

$$\begin{aligned}\big\|S^{(\epsilon)}\partial f + 2f - \big(f(a)+f(b)\big)\big\|^2_{L^2} &= \left\|\int_a^b \big(S^{(\epsilon)}(\cdot,y) - S(\cdot,y)\big)f'(y)\,dy\right\|^2_{L^2} \\ &\le \|f\|^2_{H^1}\int_a^b\int_a^b \big(S^{(\epsilon)}(x,y) - S(x,y)\big)^2\,dx\,dy\end{aligned}$$

showing the convergence $S^{(\epsilon)}\partial \to 1\otimes(\delta_a+\delta_b) - 2I$ in operator norm from $H^1_{[a,b]} \to L^2_{[a,b]}$. Hence, the composition $D_2KS^{(\epsilon)}\partial$ converges in trace norm from $H^1_{[a,b]} \to L^2_{[a,b]}$. Using the continuity of the determinant with respect to the trace norm, the right-hand side of (50) converges, and we reach

$$(51)\qquad \left(\mathrm{Pf}_{[a,b]}(\mathbf{J}-p\mathbf{K})\right)^2=\mathrm{Det}_{H^1_{[a,b]}}\left(I+2p(1-p)D_2K+F\right),$$

where $F: H^1_{[a,b]}\to H^1_{[a,b]}$ is the finite rank operator

$$(52)\qquad \begin{aligned} F &= pK(\cdot,a)\otimes\delta_a - pK(\cdot,b)\otimes\delta_b + p^2D_2K(1)\otimes(\delta_a+\delta_b)\\ &= \left((p-p^2)K(\cdot,a)+p^2K(\cdot,b)\right)\otimes\delta_a + \left((p^2-p)K(\cdot,b)-p^2K(\cdot,a)\right)\otimes\delta_b \end{aligned}$$

(using $D_2K(1)(x)=\int_a^b D_2K(x,z)\,dz = K(x,b)-K(x,a)$). The assumption on the resolvent $R=(I+2p(1-p)D_2K)^{-1}$ now allows us to split this as the product

$$\begin{aligned} \mathrm{Det}_{H^1_{[a,b]}}\left(I+2p(1-p)D_2K+F\right) &= \mathrm{Det}_{H^1_{[a,b]}}\left(I+2p(1-p)D_2K\right)\mathrm{Det}_{H^1_{[a,b]}}(I+RF)\\ &= \mathrm{Det}_{[a,b]}\left(I+2p(1-p)D_2K\right)\det{}_2^{a,b}(K), \end{aligned}$$

where the finite rank determinant $\mathrm{Det}_{H^1_{[a,b]}}(I+RF)$ is evaluated as a 2×2 determinant $\det_2^{a,b}(K)$ by examining the operator RF on its two-dimensional range.

Finally, if K is only C^2, we approximate by smooth anti-symmetric kernels K_ϵ so that the first two derivatives converge uniformly. If $I+2p(1-p)D_2K$ is invertible and $I-R$ has a C^1 kernel, then the same is true for K_ϵ for small ϵ, and one may conclude by passing to the limit in the conclusion (46) for K_ϵ. □

4.2. *Probabilistic representation.* Throughout this section we suppose a kernel $\mathbf{K}$ is in derived form and has the special translationally invariant form (9), based on a probability density ρ. We aim to apply Proposition 10 to the kernel $\mathbf{K}$ on an interval $[a,b]$.

Notation. In this subsection only, we write T for the convolution operator on $L^\infty(\mathbb{R})$ with kernel $\rho(y-x)$, and we write $T_{a,b}$ for the convolution operator restricted to $L^2[a,b]$, that is $T_{a,b}(f)(x)=\int_a^b\rho(y-x)f(y)\,dy$.

Note, from (9), that

$$I+2p(1-p)D_2K = I-\beta_pT,\quad\text{where}\quad \beta_p := 4p(1-p).$$

We first check the resolvent hypothesis for Proposition 10. Since ρ is a probability density, we have $\gamma_0:=\sup_{x\in[a,b]}\int_{[a,b]}\rho(y-x)\,dy\le 1$, and when $\beta_p<1$ (that is when $p\neq\frac12$) or when $\gamma_0<1$, the series

$$\left|\sum_{n=1}^\infty \beta_p^nT_{a,b}^n(x,y)\right|\le\sum_{n=1}^\infty\beta_p^n\gamma_0^{n-1}\|\rho\|_\infty$$

is uniformly convergent, and hence the operator $I-\beta_pT_{a,b}$ has the inverse $R=I+\sum_{k=1}^\infty\beta_p^kT_{a,b}^k$. Similarly, since $\rho\in C^1$, the series for the first derivatives of $(R-I)(x,y)$ also converge uniformly implying that $R-I$ has a C^1 kernel. In the case $p=\frac12$, we may choose $n_0\ge 1$ so that

$$(53)\qquad \gamma_1:=\sup_{x\in[a,b]}\int_{[a,b]}T_{a,b}^{n_0}(x,y)\,dy<1.$$

Repeating the arguments above for $\sum_{k=1}^\infty T_{a,b}^{kn_0}$, we see that $R=I+(I+T_{a,b}+\cdots+T_{a,b}^{n_0-1})\sum_{k=1}^\infty T_{a,b}^{kn_0}$ so that we may apply Proposition 10.

Notation. Let $S = (S_n : n \geq 0)$ be a random walk, with increments distributed according to the law with density $\rho(x)\,dx$, and started at $x \in \mathbb{R}$ under the probability $\mathbb{P}_x$.

We write M_n for the running maximum $M_n = \max_{1 \leq k \leq n} S_k$.

We write $\tau_A = \inf\{n \geq 1 : S_n \in A\}$ for the positive hitting time of $A \subseteq \mathbb{R}$.

We write τ_{a-} as shorthand for $\tau_{(-\infty,a]}$ and τ_{a+} as shorthand for $\tau_{[a,\infty)}$.

We will now rewrite the Fredholm determinant and small determinant $\det_2^{a,b}(K)$ from Proposition 10 as expectations for this random walk. First, we follow Kac's probabilistic representation for the Fredholm determinant from [24] (where the result is established for small β only).

LEMMA 11. *For all* $\beta \in [0, 1]$

$$\log \mathrm{Det}_{[a,b]}(I - \beta T) = -\mathbb{E}_a\big[\beta^{\tau_{a-}} \delta_a(S_{\tau_{a-}})(b - M_{\tau_{a-}})_+\big], \tag{54}$$

where $(z)_+ = \max(z, 0)$ *and* δ_a *stands for the Dirac delta function concentrated at* a.

PROOF. The trace-log formula (sometimes called the Plemelj–Smithies formula)

$$\begin{aligned}
&\log \mathrm{Det}_{[a,b]}(I - \beta T) \\
&\quad = -\sum_{n=1}^{\infty} \frac{\beta^n}{n} \mathrm{Tr}\big(T_{a,b}^n\big) \\
&\quad = -\sum_{n=1}^{\infty} \frac{\beta^n}{n} \int_{[a,b]^n} \rho(x_2 - x_1) \dots \rho(x_n - x_{n-1})\rho(x_1 - x_n)\, dx_1 \dots dx_n
\end{aligned} \tag{55}$$

always holds for $|\beta| > 0$ small (see [23] Theorem 3.1). Also, the Fredholm determinant $\mathrm{Det}_{[a,b]}(I - \beta T)$ is a real-analytic function of $\beta \in \mathbb{R}$. We now show the trace-log expansion is also real-analytic for $\beta \in [0, 1]$ by estimating the growth of the traces. Indeed, the estimate (53) implies that $|\mathrm{Tr}(T_{a,b}^{kn_0+j})| \leq (b-a)^j \|\rho\|_\infty^j \gamma_1^k$, implying the series is analytic for $|\beta| < \gamma_1^{-1}$, so that we may apply (55) for all $\beta \in [0, 1]$.

The derivative below has n equal contributions,

$$\begin{aligned}
\frac{d}{da} \mathrm{Tr}\big(T_{a,b}^n\big) &= -n \int_{[a,b]^{n-1}} \rho(x_2 - a) \dots \rho(x_n - x_{n-1})\rho(a - x_n)\, dx_2 \dots dx_n \\
&= -n\mathbb{E}_a\big[\delta_a(S_n); \tau_{b+} > \tau_{a-} = n\big].
\end{aligned}$$

Subsitituting this into (55), we find

$$\begin{aligned}
\frac{d}{da} \log \mathrm{Det}_{[a,b]}(I - \beta T) &= \mathbb{E}_a\big[\beta^{\tau_{a-}} \delta_a(S_{\tau_{a-}}); \tau_{b+} > \tau_{a-}\big] \\
&= \mathbb{E}_a\big[\beta^{\tau_{a-}} \delta_a(S_{\tau_{a-}}); M_{\tau_{a-}} < b\big].
\end{aligned}$$

Integrating this equality over $[a, b]$ gives

$$\begin{aligned}
\log \mathrm{Det}_{[a,b]}(I - \beta T) &= -\int_a^b \frac{d}{dc} \log \mathrm{Det}_{[c,b]}(I - \beta T)\, dc \\
&= -\mathbb{E}_a\big[\beta^{\tau_{a-}} \delta_a(S_{\tau_{a-}})(b - M_{\tau_{a-}})_+\big]. \qquad \square
\end{aligned}$$

LEMMA 12. *When* K *has the translationally invariant form* (9), *based on a* (*symmetric*) *probability density* ρ, *the factor* $\det_2^{a,b}(K)$ *from Proposition* 10 *has the following probabilistic* *representation*, *recalling* $\beta_p = 4p(1-p)$; *when* $p \neq \frac{1}{2} or 1$,

$$\det_2^{a,b}(K) = \left(1 + \frac{2p}{2p-1}\mathbb{E}_a\big[\beta_p^{\tau_{(a,b)^c}-1} - 1\big]\right) \times \left(1 + \frac{1}{2(1-p)}\big(\mathbb{E}_a\big[\beta_p^{\tau_{b+}}; \tau_{b+} < \tau_{a-}\big] - \mathbb{E}_a\big[\beta_p^{\tau_{a-}}; \tau_{a-} < \tau_{b+}\big]\big)\right).$$

Also, $\det_2^{a,b}(K) = 2\mathbb{P}_a[\tau_{b+} < \tau_{a-}]$, *when* $p = \frac{1}{2}$, *and* $\det_2^{a,b}(K) = 4\mathbb{P}_a[\tau_{b+} = 1]^2$ *when* $p = 1$.

PROOF. We rewrite the functions $k^{(1)}$, $k^{(2)}$ that define $\det_2^{a,b}(K)$ in terms of the kernel $T(x, y) = \phi(y - x)$. Using the form (9) and the symmetry of ρ,

$$\begin{aligned} K(x, a) &= -2\int_0^{a-x} \rho(z)\, dz \\ &= \int_a^\infty \rho(z - x)\, dz - \int_{-\infty}^a \rho(z - x)\, dz \\ &= T\mathbb{I}_{(a,b)}(x) + T\mathbb{I}_{[b,\infty)}(x) - T\mathbb{I}_{(-\infty,a]}(x). \end{aligned}$$

Similarly, $K(x, b) = -T\mathbb{I}_{(-\infty,a]}(x) - T\mathbb{I}_{(a,b)}(x) + T\mathbb{I}_{[b,\infty)}(x)$. Also, for $n \geq 0$,

$$T^n_{a,b} T\mathbb{I}_{(-\infty,a]}(x) = \mathbb{P}_x[\tau_{(a,b)^c} = n + 1, S_{n+1} < a]$$

so that, using $R = \sum_{n=0}^\infty \beta_p^n T^n_{a,b}$,

$$RT\mathbb{I}_{(-\infty,a]}(x) = \mathbb{E}_x\big[\beta_p^{\tau_{(a,b)^c}-1}; S_{(a,b)^c} < a\big] = \mathbb{E}_x\big[\beta_p^{\tau_{a-}-1}; \tau_{a-} < \tau_{b+}\big].$$

Similarly, $RT\mathbb{I}_{[b,\infty)}(x) = \mathbb{E}_x[\beta_p^{\tau_{b+}-1}; \tau_{b+} < \tau_{a-}]$. Also, $T^n_{a,b} T\mathbb{I}_{[a,b]}(x) = \mathbb{P}_x[\tau_{(a,b)^c} \geq n+2]$ so that

$$RT\mathbb{I}_{[a,b]}(x) = \frac{1}{\beta_p - 1}\mathbb{E}_x[\beta_p^{\tau_{(a,b)^c}-1} - 1] \text{ when } p \neq \frac{1}{2}.$$

The symmetry of ρ allows us to rewrite

$$\begin{aligned} \mathbb{E}_b\big[\beta_p^{\tau_{a-}}; \tau_{a-} < \tau_{b+}\big] &= \mathbb{E}_a\big[\beta_p^{\tau_{b+}}; \tau_{b+} < \tau_{a-}\big], \\ \mathbb{E}_b\big[\beta_p^{\tau_{b+}}; \tau_{b+} < \tau_{a-}\big] &= \mathbb{E}_a\big[\beta_p^{\tau_{a-}}; \tau_{a-} < \tau_{b+}\big], \end{aligned}$$

as well as $\mathbb{E}_a[\beta_p^{\tau_{(a,b)^c}}] = \mathbb{E}_b[\beta_p^{\tau_{(a,b)^c}}]$. The lemma follows, after some manipulation, by substituting the above representations into the expressions given in Proposition 10 for $k^{(1)}$, $k^{(2)}$ and then $\det_2^{a,b}(K)$. $\square$

4.3. *Asymptotics*. We will derive the asymptotics in Theorem 1. These rely on some classical results about general random walks, which we recall here. We include some derivations since we will need slight variants in Section 5 for the nontranslationally invariant results. The identities below hold for walks whose steps have a density ρ; we state explicitly when, in addition, they require symmetry and/or continuity of ρ.

4.3.1. *Random walk results*.

Overshoots. Many of the classical results we need follow from the fact that, when the walk starts from the origin, the joint law of $(\tau_{0+}, S_{\tau_{0+}})$ can be calculated in terms of the step distribution. Indeed, supposing only that the step distribution has a density ρ,

$$(56)\qquad 1-\mathbb{E}_0\big[\beta^{\tau_{0+}}e^{ikS_{\tau_{0+}}}\big]=\exp\left(-\sum_{n=1}^{\infty}\frac{\beta^n}{n}\int_0^{\infty}e^{ikx}\rho^{*n}(x)\,dx\right)\quad\text{for } k\in\mathbb{R},\ 0\le\beta<1$$

(see Lemma 1 of XVIII.3 from Feller [15]).

We use various consequences of this joint law. Choosing $k=0$ and letting $\beta\uparrow 1$ yields the entrance probability

$$(57)\qquad \mathbb{P}_0[\tau_{0+}=\infty]=\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}_0[S_n>0]\right).$$

When ρ is in addition symmetric, one has *Sparre Andersen's formula* (Theorem 1 in Section XII.7 of [15]),

$$(58)\qquad \mathbb{E}_0\big[\beta^{\tau_{0+}}\big]=1-\sqrt{1-\beta}\quad\text{for } \beta\in[0,1].$$

When $\mathbb{E}_0[S_1]>0$ and $\mathbb{E}_0[S_1^2]<\infty$, one has

$$(59)\qquad \mathbb{E}_0[S_{\tau_{0+}}]=\mathbb{E}_0[S_1]\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}_0[S_n<0]\right)=\mathbb{E}_0[S_1]/\mathbb{P}_0[\tau_{0-}=\infty].$$

When ρ is symmetric, this is replaced by *Spitzer's formula* (Theorem 1 in Section XVIII.5 of [15]): if $\sigma^2=\mathbb{E}_0[S_1^2]<\infty$, then

$$(60)\qquad \mathbb{E}_0[S_{\tau_{0+}}]=\frac{\sigma}{\sqrt{2}}.$$

We give a derivation of (59) since we do not find it in [15]. We can rewrite (56) as

$$\begin{aligned}1-\mathbb{E}_0\big[\beta^{\tau_{0+}}e^{ikS_{\tau_{0+}}}\big]&=\exp\left(-\sum_{n=1}^{\infty}\frac{\beta^n}{n}\int_{-\infty}^{\infty}e^{ikx}\rho^{*n}(x)\,dx+\sum_{n=1}^{\infty}\frac{\beta^n}{n}\int_{-\infty}^{0}e^{ikx}\rho^{*n}(x)\,dx\right)\\&=\big(1-\beta\mathbb{E}_0\big[e^{ikS_1}\big]\big)\exp\left(\sum_{n=1}^{\infty}\frac{\beta^n}{n}\int_{-\infty}^{0}e^{ikx}\rho^{*n}(x)\,dx\right).\end{aligned}$$

Differentiating in k and then setting $k=0$ yields

$$\mathbb{E}_0[\beta^{\tau_{0+}}S_{\tau_{0+}}]=\exp\left(\sum_{n=1}^{\infty}\frac{\beta^n}{n}\mathbb{P}_0[S_n<0]\right)\left(\beta\mathbb{E}_0[S_1]-(1-\beta)\sum_{n=1}^{\infty}\frac{\beta^n}{n}\int_{-\infty}^{0}x\rho^{*n}(x)\,dx\right).$$

The positive mean $\mathbb{E}_0[S_1]>0$ and finite variance imply that $\frac{1}{n}\int_{-\infty}^{0}x\rho^{*n}(x)\,dx\to 0$, and letting $\beta\uparrow 1$ leads to (59).

Cyclic symmetry. We use several formulae whose proofs exploit cyclic symmetry of the increments of the walk. For these, we suppose ρ is both symmetric and continuous. The first (which can also be derived from (56)) is

$$(61)\qquad \mathbb{E}_0\big[\delta_0(S_n);\,\tau_{0+}=n\big]=\frac{1}{n}\mathbb{E}_0\big[\delta_0(S_n)\big].$$

We give a direct proof using cyclic symmetry since we apply this technique on other similar identities. Let $(\mathcal{X}_0,\dots,\mathcal{X}_{n-1})$ be the first n increments of the walk, that is, $S_k=\sum_{j=1}^{k}\mathcal{X}_{j-1}$. Let $S^{(p)}$, for $p=0,1,\dots,n-1$, be the n-step random walk constructed from the same increments $(\mathcal{X}_0, \dots, \mathcal{X}_{n-1})$ but with a cyclical permutation of the increments: that is, $S_0^{(p)} = 0$, and

$$S_k^{(p)} = \sum_{j=1}^{k} \mathcal{X}_{p\oplus(j-1)} \quad \text{for } 1 \le k \le n,$$

where $p \oplus (j-1)$ is addition modulo n. Note that $(S_k^{(0)})$ coincides with the original walk (S_k). Moreover, $S_n^{(p)} = S_n$ is independent of p. Furthermore,

$$S_k^{(p)} = S_{p\oplus k} - S_p \quad \text{for all } k, p \text{ whenever } S_n = 0. \tag{62}$$

Let $\tau_{0+}^{(p)} = \inf\{k \ge 1 : S_k^{(p)} > 0\}$. The law of each of the $(S_k^{(p)})_{0\le k\le n}$ is identical so that

$$\begin{aligned}\mathbb{E}_0\big[\mathbb{I}(\tau_{0+} = n)\delta_0(S_n)\big] &= \frac{1}{n}\sum_{p=0}^{n-1} \mathbb{E}_0\big[\mathbb{I}\big(\tau_{0+}^{(p)} = n\big)\delta_0\big(S_n^{(p)}\big)\big] \\ &= \frac{1}{n}\sum_{p=0}^{n-1} \mathbb{E}_0\big[\mathbb{I}\big(\tau_{0+}^{(p)} = n\big)\delta_0\big(S_n^{(0)}\big)\big].\end{aligned}$$

The proof of (61) is completed by noting that the sum $\sum_{p=0}^{n-1} \mathbb{I}(\tau_{0+}^{(p)} = n) = 1$ almost surely; this follows from (62) because $\{\tau_{0+}^{(p)} = n\}$ holds if and only if the index p is chosen such that S_p is the global maximum of the random walk $(S_0, S_1, \dots, S_{n-1})$ (this global maximum is almost surely unique since the increments have a continuous density).

Recall that $M_n := \max\{S_k : 1 \le k \le n\}$. A lemma from [24] (or see the short proof, also based on cyclic symmetry, in the Appendix of [16]) states that, for all n,

$$\mathbb{E}_0\big[M_n\delta_0(S_n)\big] = \mathrm{Kac}_\rho(n) := \frac{n}{2}\int_0^\infty x \sum_{k=1}^{n-1} \frac{\rho^{*k}(x)\rho^{*(n-k)}(x)}{k(n-k)}\,dx, \tag{63}$$

where the right-hand side is taken as zero for $n = 0$ or $n = 1$. We call this *Kac*'s formula, as it was originally derived in Kac's work on Fredholm determinants and it enters all our asymptotics.

One final consequence of cyclic symmetry: let $m_n := \min\{S_k : 1 \le k \le n\}$, then

$$\mathbb{E}_0\big[\min\{L, M_n\}\delta_0(S_n); \tau_{0-} = n\big] = \frac{1}{n}\mathbb{E}_0\big[\min\{L, M_n - m_n\}\delta_0(S_n)\big], \tag{64}$$

and therefore, letting $L \uparrow \infty$ and using the symmetry of ρ,

$$\mathbb{E}_0\big[M_n\delta_0(S_n); \tau_{0-} = n\big] = \frac{2}{n}\mathbb{E}_0\big[M_n\delta_0(S_n)\big]. \tag{65}$$

To prove (64), note that

$$\begin{aligned}&\mathbb{E}_0\big[\min\{L, M_n\}\delta_0(S_n)\mathbb{I}(\tau_{0-} = n)\big] \\ &\quad = \mathbb{E}_0\big[\min\{L, M_n - m_n\}\delta_0(S_n)\mathbb{I}(\tau_{0-} = n)\big] \\ &\quad = \frac{1}{n}\sum_{p=0}^{n-1} \mathbb{E}_0\big[\min\{L, M_n^{(p)} - m_n^{(p)}\}\mathbb{I}\big(\tau_{0-}^{(p)} = n\big)\delta_0(S_n)\big]\end{aligned}$$

When $S_n = 0$, then, using (62),

$$M_n^{(p)} - m_n^{(p)} = \max_{1\le k\le n}(S_{p\oplus k} - S_p) - \min_{1\le k\le n}(S_{p\oplus k} - S_p) = M_n - m_n$$

is constant in p and, as above, that there is exactly one value of p with $\{\tau_{0-}^{(p)} = n\}$, establishing (64).

4.3.2. *Asymptotics for* $p \in (0, \frac{1}{2})$. Combining Proposition 10 and Lemmas 11 and 12 for the interval $[0, L]$, we have the probabilistic representation (recall $\beta_p = 4p(1-p)$)

$$\begin{aligned}
&2\log \mathrm{Pf}_{[0,L]}(\mathbf{J} - p\mathbf{K}) \\
&\quad = \log \mathrm{Det}_{[0,L]}\big(I + 2p(1-p)D_2 K\big) + \log \det{}_2^{0,L}(K) \\
&\quad = -\mathbb{E}_0\big[\beta_p^{\tau_{0-}} \delta_0(S_{\tau_{0-}})(L - M_{\tau_{0-}})_+\big] \qquad (66)\\
&\qquad + \log\bigg(1 + \frac{2p}{2p-1}\mathbb{E}_0\big[\beta_p^{\tau_{(0,L)^c}-1} - 1\big]\bigg) \\
&\qquad + \log\bigg(1 + \frac{1}{2(1-p)}\big(\mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] - \mathbb{E}_0\big[\beta_p^{\tau_{0-}}; \tau_{0-} < \tau_{L+}\big]\big)\bigg).
\end{aligned}$$

We split the first term in (66), using the identity $(L - M)_+ = L - \min\{L, M\}$, as follows:

$$\begin{aligned}
&\mathbb{E}_0\big[\beta_p^{\tau_{0-}} \delta_0(S_{\tau_{0-}})(L - M_{\tau_{0-}})_+\big] \\
&\quad = L\mathbb{E}_0\big[\beta_p^{\tau_{0-}} \delta_0(S_{\tau_{0-}})\big] - \mathbb{E}_0\big[\beta_p^{\tau_{0-}} \delta_0(S_{\tau_{0-}}) \min\{L, M_{\tau_{0-}}\}\big] \qquad (67)\\
&\quad = L\mathbb{E}_0\big[\beta_p^{\tau_{0-}} \delta_0(S_{\tau_{0-}})\big] - \mathbb{E}_0\big[\beta_p^{\tau_{0-}} \delta_0(S_{\tau_{0-}}) M_{\tau_{0-}}\big] + o(1),
\end{aligned}$$

as $L \to \infty$, where the $o(1)$ asymptotic follows by monotone convergence, provided that $\mathbb{E}_0[\beta_p^{\tau_{0-}} \delta_0(S_{\tau_{0-}}) M_{\tau_{0-}}]$ is finite, which follows from the explicit finite formula below.

For the second and third terms in (66), we use Sparre Andersen's formula (58) to see that $\mathbb{E}_0[\beta_p^{\tau_{0-}}] = 1 - \sqrt{1 - \beta_p} = 2p$ when $p < \frac{1}{2}$. Thus, we can write

$$\mathbb{E}_0\big[\beta_p^{\tau_{0-}}; \tau_{0-} < \tau_{L+}\big] = \mathbb{E}_0\big[\beta_p^{\tau_{0-}}\big] + o(1) = 2p + o(1).$$

Similarly, $\mathbb{E}_0[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}] = o(1)$ and $\mathbb{E}_0[\beta_p^{\tau_{(0,L)^c}}] = \mathbb{E}_0[\beta_p^{\tau_{0-}}] + o(1)$.

Substituting in all these asymptotics into (66), we reach

$$\begin{aligned}
&2\log \mathrm{Pf}_{[0,L]}(\mathbf{J} - p\mathbf{K}) \qquad (68)\\
&\quad = -L\mathbb{E}_0\big[\beta_p^{\tau_{0-}} \delta_0(S_{\tau_{0-}})\big] + \mathbb{E}_0\big[\beta_p^{\tau_{0-}} \delta_0(S_{\tau_{0-}}) M_{\tau_{0-}}\big] + \log\bigg(\frac{1-2p}{(1-p)^2}\bigg) + o(1)
\end{aligned}$$

which gives a probabilistic formula for the constants $\kappa_1(p)$, $\kappa_2(p)$ when $p < \frac{1}{2}$. Also,

$$\mathbb{E}_0\big[\beta_p^{\tau_{0-}} \delta_0(S_{\tau_{0-}})\big] = \sum_{n=1}^{\infty} \beta_p^n \mathbb{E}_0\big[\delta_0(S_n); \tau_0 = n\big] = \sum_{n=1}^{\infty} \frac{\beta_p^n}{n} \rho^{*n}(0) \qquad (69)$$

using the cyclic symmetry formula (61). This gives the form for $\kappa_1(p)$ stated in Theorem 1. For $\kappa_2(p)$ we use the cyclic symmetry (65) and then Kac's formula (63) as follows:

$$\begin{aligned}
\mathbb{E}_0\big[\beta_p^{\tau_{0-}} M_{\tau_{0-}} \delta_0(S_{\tau_{0-}})\big] &= \sum_{n=1}^{\infty} \beta_p^n \mathbb{E}_0\big[M_n \delta_0(S_n); \tau_{0-} = n\big] \\
&= \sum_{n=1}^{\infty} \frac{2\beta_p^n}{n} \mathbb{E}_0\big[M_n \delta_0(S_n)\big] \qquad (70)\\
&= \sum_{n=1}^{\infty} \beta_p^n \int_0^{\infty} x \sum_{k=1}^{n-1} \frac{\rho^{*k}(x)\rho^{*(n-k)}(x)}{k(n-k)}\, dx \\
&= \int_0^{\infty} x \bigg(\sum_{n=1}^{\infty} \frac{\beta_p^n \rho^{*n}(x)}{n}\bigg)^2 dx.
\end{aligned}$$

This gives the form for $\kappa_2(p)$ stated in Theorem 1. The expression is finite, since we assumed ρ was bounded, so that we may bound $\sup_n \|\rho^{*n}\|_\infty \le \|\rho\|_\infty < \infty$ and that ρ has first moment so that $\int_0^\infty x\rho^{*n}(x)\,dx \le Cn$ for all n.

4.3.3. *Asymptotics for* $p \in (\frac{1}{2}, 1)$. The identity (66) holds for $p \in (\frac{1}{2}, 1)$, and the asymptotic for the Fredholm determinant (67) still holds. The asymptotic for the small determinant $\det_2^{0,L}(K)$ is more complicated and contributes to the leading term $O(L)$. We use Sparre Andersen's formula (58) to see that $\mathbb{E}_0[\beta_p^{\tau_{0-}}] = 1 - \sqrt{1-\beta_p} = 2(1-p)$ when $p > \frac{1}{2}$. This allows us to rewrite the final two terms in (66), using

$$\begin{aligned}
&1 + \frac{1}{2(1-p)}\big(\mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] - \mathbb{E}_0\big[\beta_p^{\tau_{0-}}; \tau_{0-} < \tau_{L+}\big]\big) \\
&\quad = 1 + \frac{1}{2(1-p)}\big(\mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] - \mathbb{E}_0\big[\beta_p^{\tau_{0-}}\big] + \mathbb{E}_0\big[\beta_p^{\tau_{0-}}; \tau_{L+} < \tau_{0-}\big]\big) \qquad (71)\\
&\quad = \frac{1}{2(1-p)}\big(\mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] + \mathbb{E}_0\big[\beta_p^{\tau_{0-}}; \tau_{L+} < \tau_{0-}\big]\big),
\end{aligned}$$

and

$$\begin{aligned}
&1 + \frac{2p}{2p-1}\mathbb{E}_a\big[\beta_p^{\tau_{(0,L)^c}-1} - 1\big] \\
&= \frac{2p}{(2p-1)\beta_p}\big(\mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] + \mathbb{E}_0\big[\beta_p^{\tau_{0-}}\big] - \mathbb{E}_0\big[\beta_p^{\tau_{0-}}; \tau_{L+} < \tau_{0-}\big]\big) - \frac{1}{2p-1} \qquad (72)\\
&= \frac{1}{2(2p-1)(1-p)}\big(\mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] - \mathbb{E}_0\big[\beta_p^{\tau_{0-}}; \tau_{L+} < \tau_{0-}\big]\big).
\end{aligned}$$

Note that

$$\begin{aligned}
0 &\le \mathbb{E}_0\big[\beta_p^{\tau_{0-}}; \tau_{L+} < \tau_{0-}\big] = \mathbb{E}_0\big[\beta_p^{\tau_{L+}}\mathbb{E}_{S_{\tau_L}}\big[\beta_p^{\tau_{0-}}\big]; \tau_{L+} < \tau_{0-}\big] \\
&\le \mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] \sup_{x \ge L} \mathbb{E}_x\big[\beta_p^{\tau_{0-}}\big] \qquad (73)\\
&= \mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] \sup_{x \le 0} \mathbb{E}_x\big[\beta_p^{\tau_{L+}}\big].
\end{aligned}$$

An exact calculation shows that $\mathbb{P}_x[\tau_{L+} = k] \to 0$ as $L \to \infty$ uniformly over $x \le 0$. This implies that $\sup_{x \le 0} \mathbb{E}_x[\beta_p^{\tau_{L+}}] \to 0$ and hence that we need the asymptotics only for one part of the terms (71) and (72). Using this, (66) can be rewritten as

$$\begin{aligned}
&2\log \mathrm{Pf}_{[0,L]}(\mathbf{J} - p\mathbf{K}) \\
&\quad = -\mathbb{E}_0\big[\beta_p^{\tau_{0-}}\delta_0(S_{\tau_{0-}})(L - M_{\tau_{0-}})_+\big] \qquad (74)\\
&\qquad + 2\log \mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] - \log\big(4(2p-1)(1-p)^2\big) + o(1).
\end{aligned}$$

It remains only to find the asymptotics of $\mathbb{E}_0[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}]$ which are as follows.

LEMMA 13. *Suppose there exists $\phi_p > 0$ so that $\beta_p \int e^{\phi_p x}\rho(x)\,dx = 1$ and that $\int |x|e^{\phi_p x}\rho(x)\,dx < \infty$. Let $\mathbb{P}_x^{(p)}$, $\mathbb{E}_x^{(p)}$ be the tilted probability and expectation, where the random walk (S_n) has i.i.d. increments under the tilted density $\rho^{(p)}(x) = \beta_p \exp(\phi_p x) \times \rho(x)\,dx$. Then,*

$$\lim_{L\to\infty} e^{\phi_p L}\mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] = \frac{\sqrt{1-\beta_p}}{\phi_p \mathbb{E}_0^{(p)}[S_1]}\big(\mathbb{P}_0^{(p)}[\tau_{0-} = \infty]\big)^2. \qquad (75)$$

Before the proof we confirm that we have completed part (iii) of Theorem 1. The lemma, combined with (67) and (74), gives

$$
\begin{aligned}
&2\log \mathrm{Pf}_{[0,L]}(\mathbf{J}-p\mathbf{K}) \\
&\quad = -L\mathbb{E}_0\big[\beta_p^{\tau_{0-}}\delta_0(S_{\tau_{0-}})\big] - 2\phi_p L + \mathbb{E}_0\big[\beta_p^{\tau_{0-}}\delta_0(S_{\tau_{0-}})M_{\tau_{0-}}\big] \qquad (76)\\
&\qquad - \log\big(4(2p-1)(1-p)^2\big) + 2\log\bigg(\frac{\sqrt{1-\beta_p}}{\phi_p\mathbb{E}_0^{(p)}[S_1]}\big(\mathbb{P}_0^{(p)}[\tau_{0-}=\infty]\big)^2\bigg) + o(1)
\end{aligned}
$$

which gives the probabilistic representation of $\kappa_1(p)$, $\kappa_2(p)$ when $p \in (\frac{1}{2},1)$. The expressions in the statement of Theorem 1 emerge after using (69), (70) and the exact formula for $\mathbb{P}_0^{(p)}[\tau_{0-}=\infty]$ given in (57).

PROOF OF LEMMA 13. The process $X_n = \beta_p^n \exp(\phi_p S_n)$ is a martingale under $\mathbb{P}_0$ and $d\mathbb{P}_0^{(p)}/d\mathbb{P}_0 = X_n$ on $\mathcal{F}_n = \sigma(S_1,\dots,S_n)$. Since $\mathbb{P}_0[\tau_{L+} < \infty] = 1$, this extends to $d\mathbb{P}_0^{(p)}/d\mathbb{P}_0 = X_{\tau_{L+}}$ on $\mathcal{F}_{\tau_{L+}}$. Hence,

$$
\begin{aligned}
\mathbb{E}_0^{(p)}\big[e^{-\phi_p S_{\tau_{L+}}}; \tau_{L+} < \tau_{0-}\big] &= \mathbb{E}_0\big[e^{-\phi_p S_{\tau_{L+}}} X_{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] \\
&= \mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big]
\end{aligned}
$$

so that

$$
e^{\phi_p L}\mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] = \mathbb{E}_0^{(p)}\big[e^{-\phi_p(S_{\tau_{L+}}-L)}; \tau_{L+} < \tau_{0-}\big]. \qquad (77)
$$

By conditioning on the value of $S_{\tau_{0+}}$, we see that $V(L) = \mathbb{E}_0^{(p)}[\exp(-\phi_p(S_{\tau_{L+}} - L))]$, an exponential moment of the overlap at L, satisfies the renewal equation

$$
V(L) = \int_0^L V(L-z)G(dz) + h(L), \quad \text{for } h(x) = \int_x^\infty e^{-\phi_p(z-x)}G(dz),
$$

where $G(dz)$ is the law of the variable $S_{\tau_{0+}}$ on $[0,\infty)$ under $\mathbb{P}_0^{(p)}$. By the renewal theorem (see [14] Theorem 2.6.12—h is directly Riemann integrable),

$$
V(L) \to C_h := \frac{\int_0^\infty h(x)\,dx}{\int_0^\infty x G(dx)} = \frac{\mathbb{E}_0^{(p)}[1-e^{-\phi_p S_{\tau_{0+}}}]}{\phi_p \mathbb{E}_0^{(p)}[S_{\tau_{0+}}]} \quad \text{as } L \to \infty. \qquad (78)
$$

Conditioning (77) on $\sigma(S_{\tau_{0-}})$, we see, as $L \to \infty$,

$$
\begin{aligned}
e^{\phi_p L}\mathbb{E}_0\big[\beta_p^{\tau_{L+}}; \tau_{L+} < \tau_{0-}\big] &= V(L) - \mathbb{E}_0^{(p)}\big[e^{-\phi_p(S_{\tau_{L+}}-L)}; \tau_{0-} \le \tau_{L+}\big] \\
&= V(L) - \mathbb{E}_0^{(p)}\big[V(L - S_{\tau_{0-}}); \tau_{0-} \le \tau_{L+}\big] \\
&\to C_h - C_h\mathbb{P}_0^{(p)}[\tau_{0-} < \infty] = C_h\mathbb{P}_0^{(p)}[\tau_{0-} = \infty].
\end{aligned}
$$

This shows the existence of the desired limit, and it remains to rewrite this limit in an easier form. The identity (56) holds for all $k \in \mathbb{C}$ with $\Im(k) > 0$, since both sides are analytic there. Using this identity for the density $\rho^{(p)}$, choosing $k = i\phi_p$, and letting $\beta \uparrow 1$, gives

$$
\begin{aligned}
1 - \mathbb{E}_0^{(p)}\big[e^{-\phi_p S_{\tau_{0+}}}\big] &= \exp\bigg(-\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty e^{-\phi_p x}\big(\rho^{(p)}\big)^{*n}(x)\,dx\bigg) \\
&= \exp\bigg(-\frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}\int_{-\infty}^\infty e^{-\phi_p x}\big(\rho^{(p)}\big)^{*n}(x)\,dx\bigg)
\end{aligned}
$$

$$(79)\qquad = \exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{E}_0^{(p)}\left[e^{-\phi_p S_n}\right]\right)$$
$$= \exp(\frac{1}{2}\log(1-\mathbb{E}_0^{(p)}[e^{-\phi_p S_1}])$$
$$= \sqrt{1-\beta_p},$$

where the second equality follows from an explicit calculation using $\rho^{(p)}(x) = \beta_p \times \exp(\phi_p x)\rho(x)\,dx$ and the symmetry of ρ that shows

$$\int_0^{\infty} e^{-\phi_p x}\big(\rho^{(p)}\big)^{*n}(x)\,dx = \int_{-\infty}^{0} e^{-\phi_p x}\big(\rho^{(p)}\big)^{*n}(x)\,dx.$$

Together (79), (59), and (78) lead to the desired form for the limit. □

4.3.4. *Asymptotics for $p = \frac{1}{2}$.* When $p = \frac{1}{2}$, we find, again, by applying Propositions 10 and 11 and Lemma 12 for the interval $[0, L]$ (and noting that the probabilistic representation for the small determinant is different),

$$(80)\quad 2\log \mathrm{Pf}_{[0,L]}\left(\mathbf{J} - \frac{1}{2}\mathbf{K}\right) = -\mathbb{E}_0\big[\delta_0(S_{\tau_{0-}})(L - M_{\tau_{0-}})_+\big] + \log 2 + \log \mathbb{P}_0[\tau_{L+} < \tau_{0-}].$$

The optional stopping theorem is applicable to the martingale $(S_n)_{n\geq 0}$ and the stopping time $\tau_{0-} \wedge \tau_{L+}$ (see [27] Lemma 5.1.3), giving

$$\begin{aligned}
0 &= \mathbb{E}_0[S_{\tau_{L+}\wedge\tau_{0-}}]\\
&= \mathbb{E}_0[S_{\tau_{0-}};\tau_{0-} < \tau_L] + \mathbb{E}_0[S_{\tau_{L+}};\tau_{L+} < \tau_{0-}]\\
&= \mathbb{E}_0[S_{\tau_{0-}}] - \mathbb{E}_0[S_{\tau_{0-}};\tau_{L+} < \tau_{0-}] + \mathbb{E}_0\big[(S_{\tau_{L+}} - L);\tau_{L+} < \tau_{0-}\big] + L\mathbb{P}_0[\tau_{L+} < \tau_{0-}]
\end{aligned}$$

so that

$$(81)\quad \mathbb{P}_0[\tau_{L+} < \tau_{0-}] = \frac{-\mathbb{E}_0[S_{\tau_{0-}}]}{L} + \frac{\mathbb{E}_0[S_{\tau_{0-}} - (S_{\tau_{L+}} - L);\tau_{L+} < \tau_{0-}]}{L} \leq \frac{-\mathbb{E}_0[S_{\tau_{0-}}]}{L}.$$

The overshoots $S_{\tau_{0-}}$ and $S_{\tau_{L+}} - L$ have in general less moments than the underlying step distribution. However, Lemma 5.1.10 from [27] shows that when $\int x^4\rho(x)\,dx < \infty$, then the overshoots have finite second moment (bounded independently of the starting point). Then, for example,

$$\mathbb{E}_0\big[(S_{\tau_{L+}} - L);\tau_{L+} < \tau_{0-}\big] \leq \big(\mathbb{E}_0\big[(S_{\tau_{L+}} - L)^2\big]\big)^{1/2}\big(\mathbb{P}_0[\tau_{L+} < \tau_{0-}]\big)^{1/2},$$

and we deduce from (81) that $\mathbb{P}_0[\tau_{L+} < \tau_{0-}] = -\mathbb{E}_0[S_{\tau_{0-}}]/L + O(L^{-3/2})$. Hence, using Spitzer's formula (60) for $\mathbb{E}_0[S_{\tau_{0-}}]$,

$$(82)\qquad \log \mathbb{P}_0[\tau_{L+} < \tau_{0-}] = -\log L + \log\sigma - \frac{1}{2}\log 2 + o(1).$$

As before, we have

$$(83)\qquad \mathbb{E}_0\big[\delta_0(S_{\tau_{0-}})(L - M_{\tau_{0-}})_+\big] = L\mathbb{E}_0\big[\delta_0(S_{\tau_{0-}})\big] - \mathbb{E}_0\big[\delta_0(S_{\tau_{0-}})\min\{L, M_{\tau_{0-}}\}\big]$$

which, as in the case $p < \frac{1}{2}$, gives the value of the constant $\kappa_1(1/2)$ in the leading order $O(L)$ asymptotic. The following lemma shows that $\mathbb{E}_0[\delta_0(S_{\tau_{0-}})\min\{L, M_{\tau_{0-}}\}]$ is of form $\log L + C_0 + o(1)$, leading to the cancelation of the $-\log L$ term in (82). This lemma, together with (80), (82), (83), completes the $p = \frac{1}{2}$ case of Theorem 1.

LEMMA 14.

$$\begin{aligned}&\mathbb{E}_0\big[\delta_0(S_{\tau_{0-}})\min\{L, M_{\tau_{0-}}\}\big]\\&\quad= \log L + \frac{3}{2}\log 2 - \log\sigma + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\int_0^\infty x\frac{\rho^{*k}(x)\rho^{*(n-k)}(x)}{k(n-k)}\,dx - \frac{1}{2n}\right) + o(1).\end{aligned}$$

PROOF. We follow the strategy used in [16] which considers a walk with Gaussian increments; in this general case the asymptotics differ only in the part of the constant term arising from Kac's formula (63). Write

$$\mathbb{E}_0\big[\delta_0(S_{\tau_{0-}})\min\{L, M_{\tau_{0-}}\}\big] = \sum_{n=1}^{\infty} p(n, L), \tag{84}$$

where, using the cyclic symmetry technique (64),

$$\begin{aligned}p(n, L) &= \mathbb{E}_0\big[\delta_0(S_n)\min\{L, M_n\}\mathbb{I}(\tau_{0-} = n)\big]\\&= \frac{1}{n}\mathbb{E}_0\big[\delta_0(S_n)\min\{L, M_n - m_n\}\big].\end{aligned}$$

While $n \le L^{2-\epsilon}$ (we will soon choose $\epsilon \in (\frac{1}{2}, 2)$), the walk is unlikely to have reached L, and we will approximate $\min\{L, M_n - m_n\} \approx M_n - m_n$. For $n \ge L^{2-\epsilon}$, we will use a Brownian approximation, using a Brownian motion $(W(t) : t \ge 0)$ run at speed σ^2 (that is $[W](t) = \sigma^2 t$) and the running extrema $W^*(t) = \sup_{s\le t} W(s)$ and $W_*(t) = \inf_{s\le t} W(s)$. These approximations lead to

$$p(n, L) = \frac{1}{n}\mathbb{E}_0\big[\delta_0(S_n)(M_n - m_n)\big] + E^{(1)}(n, L), \tag{85}$$

$$p(n, L) = \frac{1}{n}\mathbb{E}_0\big[\delta_0\big(W(n)\big)\min\big\{L, W^*(n) - W_*(n)\big\}\big] + E^{(2)}(n, L), \tag{86}$$

where, for some $\eta > 0$, $C < \infty$

$$E^{(1)}(n, L) \le CnL^{-3} \quad \text{for } n \le L^{2-\epsilon}, \tag{87}$$

$$E^{(2)}(n, L) \le Cn^{-1-\eta} \quad \text{for } n \ge L^{2-\epsilon}. \tag{88}$$

We delay the detailed proof for the error bounds (87), (88) to the Section 7.3.

Kac's formula (63) and the symmetry of ρ give

$$\frac{1}{n}\mathbb{E}_0\big[\delta_0(S_n)(M_n - m_n)\big] = \frac{2}{n}\mathbb{E}_0\big[\delta_0(S_n)M_n\big] = \frac{2}{n}\mathrm{Kac}_\rho(n). \tag{89}$$

The asymptotic

$$\begin{aligned}\mathbb{E}_0\big[\delta_0(S_n)M_n\big] &= \frac{1}{2\pi}\int_{\mathbb{R}}\mathbb{E}_0\big[e^{i\theta S_n}M_n\big]\,d\theta\\&= \frac{1}{2\pi}\int_{\mathbb{R}}\mathbb{E}_0\left[e^{i\theta S_n/\sqrt{n}}\frac{M_n}{\sqrt{n}}\right]d\theta\\&\to \mathbb{E}_0\big[\delta_0\big(W(1)\big)W^*(1)\big] = \frac{1}{4}\end{aligned}$$

(using an explicit calculation with the joint density for $(W^*(t), W(t))$) can be quantified, using the local central limit theorem; see the details in the Section 7.3 to give

$$\mathrm{Kac}_\rho(n) = \frac{1}{4} + O\big(n^{-1/4}\big). \tag{90}$$

Thus, the series $\sum \frac{2}{n}\mathrm{Kac}_\rho(n)$ is divergent, and we will need to compensate the terms to gain a convergent series. This completes all parts of this lemma that are different from the Gaussian case in [16], and we now refer the reader to that paper for some of the subsequent calculations.

The Brownian expectation (86) can be calculated (see Section 3.2.2 of [16]) using the joint distribution of $(W(t), W^*(t), W_*(t))$, yielding

$$\mathbb{E}_0\big[\delta_0\big(W(n)\big)\min\{L, W^*(n) - W_*(n)\}\big] = \frac{1}{2} - \sqrt{\frac{2L^2}{\pi\sigma^2 n}}\,\Omega\Big(\frac{2L^2}{\pi n\sigma^2}\Big), \tag{91}$$

where $\Omega(t) = \sum_{k\ge 1}\exp(-\pi k^2 t)$ is a special function (related to Jacobi's θ-function). From (85), (86) we find, as in [16], the asymptotic for $L \to \infty$,

$$\sum_{n=1}^{\infty} p(n, L) = \sum_{n\le L^{2-\epsilon}} \frac{2}{n}\mathrm{Kac}_\rho(n) + \sum_{n>L^{2-\epsilon}} \frac{1}{2n} - \sqrt{\frac{2L^2}{\pi\sigma^2 n^3}}\,\Omega\Big(\frac{2L^2}{\pi n\sigma^2}\Big) + o(1). \tag{92}$$

The error terms $E^{(i)}(n, L)$ contribute only to the $o(1)$ term, but for this we must choose $\epsilon \in (\frac{1}{2}, 2)$ (in [16] the error term $E^{(1)}(n, L)$ was exponentially small as we dealt with a walk with Gaussian increments; here, we suppose only fourth moments).

We compensate, using (90),

$$\begin{aligned}
\sum_{n\le L^{2-\epsilon}} \frac{2}{n}\mathrm{Kac}_\rho(n) &= \sum_{n\le L^{2-\epsilon}} \frac{2}{n}\Big(\mathrm{Kac}_\rho(n) - \frac{1}{4}\Big) + \sum_{n\le L^{2-\epsilon}} \frac{1}{2n} \\
&= \sum_{n\ge 1} \frac{2}{n}\Big(\mathrm{Kac}_\rho(n) - \frac{1}{4}\Big) + \sum_{n\le L^{2-\epsilon}} \frac{1}{2n} + o(1) \\
&= \sum_{n\ge 1} \frac{2}{n}\Big(\mathrm{Kac}_\rho(n) - \frac{1}{4}\Big) + \frac{1}{2}\log L^{2-\epsilon} + \frac{\gamma}{2} + o(1),
\end{aligned}$$

using the asymptotic $\sum_{n\le N}\frac{1}{n} = \log N + \gamma + O(N^{-1})$, where γ is the Euler–Mascheroni constant. An analysis of the error in a Riemann block approximation implies

$$\begin{aligned}
&\sum_{n>L^{2-\epsilon}} \frac{1}{2n} - \sqrt{\frac{2L^2}{\pi\sigma^2 n^3}}\,\Omega\Big(\frac{2L^2}{\pi n\sigma^2}\Big) \\
&\quad = \int_{L^{-\epsilon}}^{\infty} \Big(\frac{1}{2x} - \sqrt{\frac{2L^2}{\pi\sigma^2 x^3}}\,\Omega\Big(\frac{2L^2}{\pi x\sigma^2}\Big)\Big)\,dx + o(1) \\
&\quad = \int_{1}^{\infty} \Big(\frac{1}{2x} - \sqrt{\frac{2L^2}{\pi\sigma^2 x^3}}\,\Omega\Big(\frac{2L^2}{\pi x\sigma^2}\Big)\Big)\,dx \\
&\qquad - \int_{0}^{1} \sqrt{\frac{2L^2}{\pi\sigma^2 x^3}}\,\Omega\Big(\frac{2L^2}{\pi x\sigma^2}\Big)\,dx + \frac{\epsilon}{2}\log L + o(1).
\end{aligned}$$

The asymptotics $\Omega(t) = \frac{1}{2\sqrt{t}} - \frac{1}{2} + o(1)$, as $t \downarrow 0$, and $\Omega(t) \sim \exp(-\pi t)$, as $t \to \infty$, justify the convergence of these integrals. Substituting these into (92), we find the dependence on ϵ vanishes, and we reach

$$\sum_{n=1}^{\infty} p(n, L) = \log L + \sum_{n\ge 1} \frac{2}{n}\Big(\mathrm{Kac}_\rho(n) - \frac{1}{4}\Big) + C_0 + o(1),$$

where

$$C_0 = \frac{\gamma}{2} + \int_1^\infty \left(\frac{1}{2x} - \sqrt{\frac{2L^2}{\pi\sigma^2 x^3}}\,\Omega\left(\frac{2L^2}{\pi x\sigma^2}\right)\right) dx - \int_0^1 \sqrt{\frac{2L^2}{\pi\sigma^2 x^3}}\,\Omega\left(\frac{2L^2}{\pi x \sigma^2}\right) dx.$$

Amazingly, certain identities for γ and the function $\Omega(t)$ imply that $C_0 = \frac{3}{2}\log 2 - \log\sigma$ (see Section 2 of [16]), and this completes the proof. □

5. The proof of Theorem 3. Throughout this section we suppose we have a kernel $\mathbf{K}$ in the derived form (8), based on a scalar kernel in the form (10), that is,

$$K(x,y) = \int_{-\infty}^0 \begin{vmatrix} \int_{-\infty}^{x-z} \rho(w)\,dw & \int_{-\infty}^{y-z} \rho(w)\,dw \\ \rho(x-z) & \rho(y-z) \end{vmatrix}\, dz \tag{93}$$

for a probability density $\rho \in C^1(\mathbb{R}) \cap H^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. This is sufficient to allow differentiation under the integral and integration by parts showing that K, D_1K, D_2K, $D_{12}K$ are all continuous and, in particular,

$$D_2K(x,y) = -2\int_{-\infty}^0 \rho(x-z)\rho(y-z)\,dz - \rho(y)\int_{-\infty}^x \rho(w)\,dw.$$

We now write T for the integral operator on $L^\infty(\mathbb{R})$ with kernel

$$T(x,y) = \int_{-\infty}^0 \rho(x-z)\rho(y-z)\,dz. \tag{94}$$

Then, recalling $\beta_p = 4p(1-p)$,

$$2p(1-p)D_2K(x,y) = -\beta_p T(x,y) - 2p(1-p)\rho(y)\int_{-\infty}^x \rho(w)\,dw. \tag{95}$$

Note the last term in (95) is a finite rank kernel. The operator T still can be interpreted in probabilistic terms using a random two-step walk, where pairs of increments have the density $\rho(-x)$ and $\rho(x)$. To our surprise, it is possible to follow fairly closely the strategy used in the translationally invariant case, namely: (i) represent the Fredholm Pfaffian in terms of determinants, (ii) represent these in terms of the random two-step walk, and (iii) derive asymptotics from probabilistic results for a two-step walk. Each of these steps requires slight modifications (and becomes slightly messy), due to the different operator T and due to the extra finite rank term above. (The thesis [17] contains an exploration of more general kernels where $T(x,y) = \int_{-\infty}^0 \rho^{(1)}(x-z)\rho^{(2)}(y-z)\,dz$ for two probability densities $\rho^{(1)}$, $\rho^{(2)}$, and shows that many of the steps above go through; however, we have yet to find applications).

5.1. *Operator manipulation.* We again write $T_{a,b}$ for the integral operator restricted to $L^2[a,b]$, that is, $T_{a,b}f(x) = \int_a^b T(x,y)f(y)\,dy$. Our first aim is an analogue of the Tracy Widom manipulations in Proposition 10 for this nontranslationally invariant setting. The reasoning at the start of Section 4.2 extends to this case to show that $1 - \beta_p T_{a,b}$ has the inverse $R = I + \sum_{k=1}^\infty \beta_p^k T_{a,b}^k$ and that $R - I$ has a C^1 kernel.

LEMMA 15. *With the above notation we have*

$$(\mathrm{Pf}_{[a,b]}(\mathbf{J} - p\mathbf{K}))^2 = \mathrm{Det}_{[a,b]}(I - \beta_p T)\,\det_3^{a,b}(K), \tag{96}$$

where $\det_3^{a,b}(K)$ *is the* 3×3 *determinant with entries* $\delta_{ij} + (Re_i, f_j)$, *where* (e, f) *is the dual pairing between* $H^1_{[a,b]}$ *and its dual, for the elements*

$$f_1 = \beta_p \rho, \qquad e_1 = -\frac{1}{2}\Phi_\rho,$$
$$f_2 = \delta_a - \delta_b, \qquad e_2 = pT\mathbb{I}_{[b,\infty)} - pT\mathbb{I}_{(-\infty,a]} + \frac{p}{2}\big(2 - \Phi_\rho(b) - \Phi_\rho(a)\big)\Phi_\rho,$$
$$f_3 = \delta_a + \delta_b, \qquad e_3 = -p(2p-1)T\mathbb{I}_{(a,b)} - \frac{p(2p-1)}{2}\big(\Phi_\rho(b) - \Phi_\rho(a)\big)\Phi_\rho,$$

where $\Phi_\rho(x) = \int_{-\infty}^x \rho(x)$ *is the distribution function for* ρ.

PROOF. We follow the proof of Proposition 10 up to (51), (52). Then, using (95), we have

$$\big(\mathrm{Pf}_{[a,b]}(\mathbf{J} - p\mathbf{K})\big)^2 = \mathrm{Det}_{H^1_{[a,b]}}(I - \beta_p T + F) = \mathrm{Det}_{H^1_{[a,b]}}(I - \beta_p T)\,\mathrm{Det}_{H^1_{[a,b]}}(I + RF),$$

where F is the finite rank operator

$$\begin{aligned} &-\frac{\beta_p}{2}\Phi_\rho \otimes \rho + \big((p-p^2)K(\cdot,a) + p^2K(\cdot,b)\big)\otimes\delta_a \\ &\quad + \big((p^2-p)K(\cdot,b) - p^2K(\cdot,a)\big)\otimes\delta_b \\ &= -\frac{\beta_p}{2}\Phi_\rho\otimes\rho + \frac{p}{2}\big(K(\cdot,b)+K(\cdot,a)\big)\otimes(\delta_a - \delta_b) \\ &\quad + \frac{p(2p-1)}{2}\big(K(\cdot,b) - K(\cdot,a)\big)\otimes(\delta_a+\delta_b). \end{aligned}$$

This gives the rank 3 form for F and the values of f_1, f_2, f_3, e_1, as stated. Again, using (95), we have

$$K(\cdot,b) - K(\cdot,a) = \int_a^b D_2K(\cdot,z)\,dz = -2T\mathbb{I}_{(a,b)} - \int_a^b \rho(w)\,dw\,\Phi_\rho,$$

giving the value of e_3. The limits $K(x,-\infty) = 1$ and $K(x,\infty) = \Phi_\rho(x) - 1$ follow from the definition (93). Then,

$$\begin{aligned} &K(\cdot,b) + K(\cdot,a) \\ &\quad = \big(K(\cdot,a) - K(\cdot,-\infty)\big) - \big(K(\cdot,\infty) - K(\cdot,b)\big) + K(\cdot,\infty) + K(\cdot,-\infty) \\ &\quad = -2T\mathbb{I}_{(-\infty,a]} - \int_{-\infty}^a \rho(w)\,dw\,\Phi_\rho + 2T\mathbb{I}_{[b,\infty)} + \int_b^\infty \rho(w)\,dw\,\Phi_\rho + \Phi_\rho \end{aligned}$$

which gives the value of e_2. □

5.2. *Probabilistic representation.* We need two different two-step random walks. These have independent increments but alternate between a step distributed as $\rho(-x)\,dx$ and then, as $\rho(x)\,dx$, as follows.

Notation. Let $(\mathcal{X}_k)$ and $(\mathcal{Y}_k)$ be two independent families of i.i.d. variables, where $\mathcal{X}_k$ have density $\rho(-x)\,dx$ and $\mathcal{Y}_k$ have density $\rho(x)\,dx$.

Under $\mathbb{P}_x$ the variables $(S_n : n \geq 0)$ and $(\tilde{S}_n : n \geq 1)$ are defined by $S_0 = x$ and

$$\tilde{S}_k = S_{k-1} + \mathcal{X}_k, \qquad S_k = \tilde{S}_k + \mathcal{Y}_k \quad \text{for } k = 1, 2, \ldots.$$

We write $\tau_A = \inf\{n \geq 1 : S_n \in A\}$, $\tilde{\tau}_A = \inf\{n \geq 1 : \tilde{S}_n \in A\}$ and the special cases

$$\tau_{a+} = \inf\{n \geq 1 : S_n > a\}, \qquad \tau_{a-} = \inf\{n \geq 1 : S_n < a\},$$
$$\tilde{\tau}_{a+} = \inf\{n \geq 1 : \tilde{S}_n > a\}, \qquad \tilde{\tau}_{a-} = \inf\{n \geq 1 : \tilde{S}_n < a\},$$

and the running maxima and minima

$$M_n = \max\{S_k : 1 \leq k \leq n\}, \qquad m_n = \min\{S_k : 1 \leq k \leq n\},$$
$$\tilde{M}_n = \max\{\tilde{S}_k : 1 \leq k \leq n\}, \qquad \tilde{m}_n = \min\{\tilde{S}_k : 1 \leq k \leq n\}.$$

Note that, under $\mathbb{P}_x$, the process (S_n) is a random walk whose increments have the density $\tilde{\rho}(z) = \int_{\mathbb{R}} \rho(w)\rho(w-z)\,dw$.

LEMMA 16. *For $\beta \in [0,1]$, when $T(x,y) = \int_{-\infty}^0 \rho(x-z)\rho(y-z)\,dz$ for a probability density $\rho \in C(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$,*

$$\log \operatorname{Det}_{[-L,\infty)}(I - \beta T) = -\mathbb{E}_0\big[\beta^{\tau_{0-}} \delta_0(S_{\tau_{0-}})(L - \tilde{M}_{\tau_{0-}})_+\big]. \tag{97}$$

PROOF. Arguing, as in Lemma 11, the log-trace formula formula

$$\begin{aligned}
&\log \operatorname{Det}_{[a,b]}(I - \beta T) \\
&\quad = -\sum_{n=1}^{\infty} \frac{\beta^n}{n} \operatorname{Tr}\big(T_{a,b}^n\big) \qquad (98)\\
&\quad = -\sum_{n=1}^{\infty} \frac{\beta^n}{n} \int_{[a,b]^n} T(x_1,x_2)\dots T(x_{n-1},x_n)T(x_n,x_1)\,dx_1\dots dx_n
\end{aligned}$$

holds for all $\beta \in [0,1]$. The derivative

$$\begin{aligned}
\frac{d}{da}\operatorname{Tr}\big(T_{a,b}^n\big) &= -n\int_{[a,b]^{n-1}} T(a,x_2)T(x_2,x_3)\dots T(x_n,a)\,dx_2\dots dx_n \\
&= -n\int_{[a,b]^{n-1}} dx_2\dots dx_n \int_{(-\infty,0]^n} dz_1\dots dz_n \\
&\qquad \times \rho(a-z_1)\rho(x_2-z_1)\rho(x_2-z_2)\rho(x_3-z_2)\dots\rho(x_n-z_n)\rho(a-z_n) \\
&= -n\mathbb{P}_a\big[\tilde{S}_1 < 0, S_1 \in (a,b), \dots, \tilde{S}_n < 0, S_n \in da\big] \\
&= -n\mathbb{E}_a\big[\delta_a(S_n); \tau_{(a,b)^c} = n, \tilde{M}_n < 0\big].
\end{aligned}$$

Subsitituting this into (98), we find

$$\frac{d}{da}\log \operatorname{Det}_{[a,b]}(I - \beta T) = \mathbb{E}_a\big[\beta^{\tau_{a-}}\delta_a(S_{\tau_{a-}}); \tau_{b+} > \tau_{a-}, \tilde{M}_{\tau_{a-}} < 0\big].$$

Integrating this equality over $[a,b]$ gives

$$\log \operatorname{Det}_{[a,b]}(I - \beta T) = -\int_a^b \mathbb{E}_c\big[\beta^{\tau_{c-}}\delta_c(S_{\tau_{c-}}); \tau_{b+} > \tau_{c-}, \tilde{M}_{\tau_{c-}} < 0\big]\,dc.$$

Both side of this identity are decreasing in b. Setting $a = -L$ and letting $b \to \infty$, we reach

$$\begin{aligned}
\log \operatorname{Det}_{[-L,\infty)}(I - \beta T) &= -\int_{-L}^{\infty} \mathbb{E}_c\big[\beta^{\tau_{c-}}\delta_c(S_{\tau_{c-}}); \tilde{M}_{\tau_{c-}} < 0\big]\,dc \\
&= -\int_{-L}^{\infty} \mathbb{E}_0\big[\beta^{\tau_{0-}}\delta_0(S_{\tau_{0-}}); \tilde{M}_{\tau_{0-}} < -c\big]\,dc \\
&= -\mathbb{E}_0\big[\beta^{\tau_{0-}}\delta_0(S_{\tau_{0-}})(L - \tilde{M}_{\tau_{0-}})_+\big]. \qquad \square
\end{aligned}$$

LEMMA 17. *The limit* $\det_3^{a,\infty}(K) = \lim_{b\to\infty} \det_3^{a,b}(K)$ *of the finite rank determinant from Lemma 15 exists and is given, when* $p \neq \frac{1}{2}$, *by*

$$\det_3^{a,\infty}(K) = \frac{2}{1-2p} \begin{vmatrix} 1-p-\dfrac{1}{2}\mathbb{E}_a[\beta_p^{\tau_{a-}}; \tilde{\tau}_{0+} > \tau_{a-}] & -\dfrac{1}{2}\mathbb{E}_a[\beta_p^{\tilde{\tau}_{0+}-1}; \tau_{a-} \geq \tilde{\tau}_{0+}] \\ -p\mathbb{E}_a[\beta_p^{\tilde{\tau}_{0+}-1}; \tilde{\tau}_{0+} \leq \tau_{a-}] & \dfrac{1}{2} - p\mathbb{E}_a[\beta_p^{\tau_{a-}-1}; \tau_{a-} < \tilde{\tau}_{0+}] \end{vmatrix}$$

and by $\mathbb{P}_a[\tilde{\tau}_{0+} \leq \tau_{a-}]$ *when* $p = \frac{1}{2}$.

PROOF. We will represent each term (Re_i, f_j) in terms of the two-step walk. All nine terms are somewhat similar, and we detail just a few. For example,

$$\begin{aligned}
(R\Phi_\rho, \delta_{x_0}) = R\Phi_\rho(x_0) &= \sum_{n=0}^{\infty} \beta_p^n T_{a,b}^n \Phi_\rho(x_0) \\
&= \sum_{n=0}^{\infty} \beta_p^n \int_{[a,b]^n} dx_1 \dots dx_n T(x_0,x_1)\dots T(x_{n-1},x_n)\Phi_\rho(x_n) \\
&= \sum_{n=0}^{\infty} \beta_p^n \int_{[a,b]^n} dx_1 \dots dx_n \int_{(-\infty,0]^n} dy_1 \dots dy_n \int_0^\infty dz \\
&\quad \times \rho(x_0-y_1)\rho(x_1-y_1)\rho(x_1-y_2) \\
&\quad \times \rho(x_2-y_2)\dots\rho(x_{n-1}-y_n)\rho(x_n-y_n)\rho(x_n-z) \\
&= \sum_{n=0}^{\infty} \beta_p^n \mathbb{P}_{x_0}[\tilde{S}_1 < 0, S_1 \in (a,b), \dots, \tilde{S}_n < 0, S_n \in (a,b), \tilde{S}_{n+1} > 0] \\
&= \sum_{n=0}^{\infty} \beta_p^n \mathbb{P}_{x_0}[\tilde{\tau}_{0+} = n+1, \tau_{(a,b)^c} \geq n+1] \\
&= \mathbb{E}_{x_0}[\beta_p^{\tilde{\tau}_{0+}-1}; \tau_{(a,b)^c} \geq \tilde{\tau}_{0+}].
\end{aligned} \tag{99}$$

A similar exact calculation shows that, for bounded f,

$$(RTf, \delta_{x_0}) = \sum_{n=0}^{\infty} \beta_p^n \mathbb{E}_{x_0}[f(S_{n+1}); \tilde{\tau}_{0+} \geq n+2, \tau_{(a,b)^c} \geq n+1].$$

Using this for $f = \mathbb{I}_{[b,\infty)}$ gives

$$\begin{aligned}
(RT\mathbb{I}_{[b,\infty)}, \delta_{x_0}) &= \sum_{n=0}^{\infty} \beta_p^n \mathbb{P}_{x_0}[S_{n+1} > b, \tilde{\tau}_{0+} \geq n+2, \tau_{(a,b)^c} \geq n+1] \\
&= \sum_{n=0}^{\infty} \beta_p^n \mathbb{P}_{x_0}[\tilde{\tau}_{0+} \geq n+2, \tau_{b+} = n+1, \tau_{b+} < \tau_{a-}] \\
&= \mathbb{E}_{x_0}[\beta_p^{\tau_{b+}-1}; \tau_{b+} < \tau_{a-} \wedge \tilde{\tau}_{0+}]
\end{aligned} \tag{100}$$

and similarly, using $f = \mathbb{I}_{(-\infty,a]}$,

$$(RT\mathbb{I}_{(-\infty,a]}, \delta_{x_0}) = \mathbb{E}_{x_0}[\beta_p^{\tau_{a-}-1}; \tau_{a-} < \tau_{b+} \wedge \tilde{\tau}_{0+}] \tag{101}$$

and, using $f = \mathbb{I}_{(a,b)}$, when $p \neq \frac{1}{2}$,

$$(RT\mathbb{I}_{(a,b)}, \delta_{x_0}) = \sum_{n=0}^{\infty} \beta_p^n \mathbb{P}_{x_0}[S_{n+1} \in (a,b), \tilde{\tau}_{0+} \geq n+2, \tau_{(a,b)^c} \geq n+1]$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \beta_p^n \mathbb{P}_{x_0}[\tilde{\tau}_{0+} \geq n+2, \tau_{(a,b)^c} \geq n+2] && (102)\\
&= \frac{1}{\beta_p - 1} \mathbb{E}_{x_0}[\beta_p^{(\tau_{(a,b)^c} \wedge \tilde{\tau}_{0+})-1} - 1].
\end{aligned}
$$

The entries of the form $(\mathrm{Re}_i, f_1) = \beta_p(\mathrm{Re}_i, \rho)$ start with an integral against $\rho(w)\,dw$ and need a reflection $x \to -x$ to be written in terms of the two-step walk which start with an increment with density $\rho(-w)\,dw$. For example,

$$
\begin{aligned}
(RTf, \rho) &= \sum_{n=0}^{\infty} \beta_p^n \int_a^b dw \rho(w) T_{a,b}^n T f(w) \\
&= \sum_{n=0}^{\infty} \beta_p^n \int_{[a,b]^{n+1}} dw\, dx_1 \dots dx_n \rho(w) T(w, x_1) \dots T(x_{n-1}, x_n) T f(x_n) \\
&= \sum_{n=0}^{\infty} \beta_p^n \int_{[a,b]^{n+1}} dw\, dx_1 \dots dx_n \int_{(-\infty,0]^{n+1}} dy_1 \dots dy_n\, dz' \int_{-\infty}^{\infty} dz \\
&\quad \times \rho(w)\rho(w - y_1)\rho(x_1 - y_1) \dots \rho(x_{n-1} - y_n) \\
&\quad \times \rho(x_n - y_n)\rho(x_n - z')\rho(z - z') f(z) \\
&= \sum_{n=0}^{\infty} \beta_p^n \int_{[-b,-a]^{n+1}} dw\, dx_1 \dots dx_n \int_{[0,\infty)^{n+1}} dy_1 \dots dy_n\, dz' \int_{-\infty}^{\infty} dz \\
&\quad \times \rho(-w)\rho(y_1 - w)\rho(y_1 - x_1) \dots \rho(y_n - x_{n-1}) \\
&\quad \times \rho(y_n - x_n)\rho(z' - x_n)\rho(z' - z) f(-z) \\
&= \sum_{n=0}^{\infty} \beta_p^n \mathbb{E}_0\big[f(-\tilde{S}_{n+2}); \tilde{S}_1 \in (-b,-a), S_1 > 0, \dots, \tilde{S}_{n+1} \in (-b,-a), S_{n+1} > 0\big] \\
&= \sum_{n=0}^{\infty} \beta_p^n \mathbb{E}_0\big[f(-\tilde{S}_{n+2}); \tau_{0-} \geq n+2, \tilde{\tau}_{(-b,-a)^c} \geq n+2\big] \\
&= \sum_{n=0}^{\infty} \beta_p^n \mathbb{E}_a\big[f(a - \tilde{S}_{n+2}); \tau_{a-} \geq n+2, \tilde{\tau}_{(a-b,0)^c} \geq n+2\big].
\end{aligned}
$$

Using this for $f = \mathbb{I}_{[b,\infty)}$ gives

$$
\begin{aligned}
(RT\mathbb{I}_{[b,\infty)}, \rho) &= \sum_{n=0}^{\infty} \beta_p^n \mathbb{P}_a[\tilde{\tau}_{(a-b)-} = n+2, \tilde{\tau}_{0+} > n+2, \tau_{a-} \geq n+2] \\
&= \mathbb{E}_a\big[\beta_p^{\tilde{\tau}_{(a-b)-}-2}; \tilde{\tau}_{(a-b)-} < \tilde{\tau}_{0+} \wedge (1 + \tau_{a-})\big] - \beta_p^{-1}\mathbb{P}_a[\tilde{\tau}_{(a-b)-} = 1] && (103)\\
&= \mathbb{E}_a\big[\beta_p^{\tilde{\tau}_{(a-b)-}-2}; \tilde{\tau}_{(a-b)-} < \tilde{\tau}_{0+} \wedge (1 + \tau_{a-})\big] - \beta_p^{-1}\big(1 - \Phi_\rho(b)\big),
\end{aligned}
$$

where the final subtracted term emerges since the sum over n does not include the event $\{\tilde{\tau}_{(a-b)-} = 1\}$. Similarly, using $f = \mathbb{I}_{(-\infty,a]}$,

$$
(RT\mathbb{I}_{(-\infty,a]}, \rho) = \mathbb{E}_a\big[\beta_p^{\tilde{\tau}_{0+}-2}; \tilde{\tau}_{0+} < \tilde{\tau}_{(a-b)-} \wedge (1 + \tau_{a-})\big] - \beta_p^{-1}\Phi_\rho(a) \tag{104}
$$

and, using $f = \mathbb{I}_{(a,b)}$, when $p \neq \frac{1}{2}$,

$$
(RT\mathbb{I}_{(a,b)}, \rho) = \sum_{n=0}^{\infty} \beta_p^n \mathbb{P}_a[\tilde{\tau}_{(a-b,0)^c} \geq n+3, \tau_{a-} \geq n+2]
$$

$$
\begin{aligned}
(105)\qquad &= \frac{1}{\beta_p - 1}\mathbb{E}_a\big[\beta_p^{(\tilde\tau_{(a-b,0)^c}-2)\wedge(\tau_{a-}-1)} - 1\big] + \frac{1}{\beta_p}\mathbb{P}_a[\tilde\tau_{(a-b,0)^c} = 1] \\
&= \frac{1}{\beta_p - 1}\mathbb{E}_a\big[\beta_p^{(\tilde\tau_{(a-b,0)^c}-2)\wedge(\tau_{a-}-1)} - 1\big] + \frac{1}{\beta_p}\big(1 + \Phi_\rho(a) - \Phi_\rho(b)\big).
\end{aligned}
$$

The final representation needed is derived in a similar manner,

$$
(106)\qquad (R\Phi_\rho, \rho) = \mathbb{E}_a\big[\beta^{\tau_{a-}-1}; \tilde\tau_{(a-b,0)^c} > \tau_{a-}\big].
$$

The formulae (99),…,(106) can be substituted into the matrix elements (Re_i, f_j), and each has a limit as $b \to \infty$. Before evaluating these limits, it is convenient first to do two row operations to convert e_1, e_2, e_3 to $\hat e_1 = e_1$, $\hat e_2 = e_2 + p(2 - \Phi_\rho(b) - \Phi_\rho(a))e_1$, and $\hat e_3 = e_3 - p(2p-1)(\Phi_\rho(b) - \Phi_\rho(a))e_1$ so that

$$
\begin{aligned}
&\hat e_1 = -\frac{1}{2}\Phi_\rho, \qquad \hat e_2 = pT\mathbb{I}_{[b,\infty)} - pT\mathbb{I}_{(-\infty,a]}, \qquad \hat e_3 = -p(2p-1)T\mathbb{I}_{(a,b)}, \\
&f_1 = \beta_p\rho, \qquad f_2 = \delta_a - \delta_b, \qquad f_3 = \delta_a + \delta_b.
\end{aligned}
$$

Under these operations we change

$$
\det\nolimits_3^{a,b}(K) = \det\big(I + \big((Re_i, f_j) : i, j \le 3\big)\big) = \det\big(\hat I + \big((R\hat e_i, f_j) : i, j \le 3\big)\big),
$$

where

$$
\hat I = \begin{pmatrix} 1 & 0 & 0 \\ p\big(2 - \Phi_\rho(b) - \Phi_\rho(a)\big) & 1 & 0 \\ -p(2p-1)\big(\Phi_\rho(b) - \Phi_\rho(a)\big) & 0 & 1 \end{pmatrix}.
$$

Using (99), we have

$$
\begin{aligned}
(R\hat e_1, f_2) &= -\frac{1}{2}(R\Phi_\rho, \delta_a - \delta_b) \\
&= \frac{1}{2}\mathbb{E}_b\big[\beta_p^{\tilde\tau_{0+}-1}; \tau_{(a,b)^c} \ge \tilde\tau_{0+}\big] - \frac{1}{2}\mathbb{E}_a\big[\beta_p^{\tilde\tau_{0+}-1}; \tau_{(a,b)^c} \ge \tilde\tau_{0+}\big] \\
&\to \frac{1}{2} - \frac{1}{2}\mathbb{E}_a\big[\beta_p^{\tilde\tau_{0+}-1}; \tau_{a-} \ge \tilde\tau_{0+}\big] \quad \text{as } b \to \infty
\end{aligned}
$$

since $\mathbb{P}_a[\tau_{b+} \to \infty] = 1$ and $\mathbb{P}_b[\tilde\tau_{0+} = 1] \to 1$, as $b \to \infty$. The limiting entry for and $(R\hat e_1, f_3)$ differs only by a sign. Using (102), we have, when $p \ne \frac{1}{2}$,

$$
\begin{aligned}
(R\hat e_3, f_2) &= -p(2p-1)(RT\mathbb{I}_{(a,b)}, \delta_a - \delta_b) \\
&= \frac{p}{2p-1}\big(\mathbb{E}_a\big[\beta_p^{(\tau_{(a,b)^c}\wedge\tilde\tau_{0+})-1} - 1\big] - \mathbb{E}_b\big[\beta_p^{(\tau_{(a,b)^c}\wedge\tilde\tau_{0+})-1} - 1\big]\big) \\
&\to \frac{p}{2p-1}\mathbb{E}_a\big[\beta_p^{(\tau_{a-}\wedge\tilde\tau_{0+})-1} - 1\big] \quad \text{as } b \to \infty.
\end{aligned}
$$

The limiting form for the entry $(R\hat e_3, f_3)$ is the same. Using (100) and (101), we have

$$
\begin{aligned}
(R\hat e_2, f_2) &= p(RT\mathbb{I}_{[b,\infty)} - RT_{(-\infty,a]}, \delta_a - \delta_b) \\
&= p\mathbb{E}_a\big[\beta_p^{\tau_{b+}-1}; \tau_{b+} < \tau_{a-} \wedge \tilde\tau_{0+}\big] - p\mathbb{E}_b\big[\beta_p^{\tau_{b+}-1}; \tau_{b+} < \tau_{a-} \wedge \tilde\tau_{0+}\big] \\
&\quad - p\mathbb{E}_a\big[\beta_p^{\tau_{a-}-1}; \tau_{a-} < \tau_{b+} \wedge \tilde\tau_{0+}\big] + p\mathbb{E}_b\big[\beta_p^{\tau_{a-}-1}; \tau_{a-} < \tau_{b+} \wedge \tilde\tau_{0+}\big] \\
&\to -p\mathbb{E}_a\big[\beta_p^{\tau_{a-}-1}; \tau_{a-} < \tilde\tau_{0+}\big], \quad \text{as } b \to \infty
\end{aligned}
$$

and the limiting form for $(R\hat{e}_2, f_3)$ is the same. Using (106) and $\mathbb{P}_a[\tilde{\tau}_{b-} \to \infty] = 1$, as $b \to \infty$, we have

$$(R\hat{e}_1, f_1) = -\frac{1}{2}\mathbb{E}_a\big[\beta_p^{\tau_{a-}}; \tilde{\tau}_{(a-b,0)^c} > \tau_{a-}\big] \to -\frac{1}{2}\mathbb{E}_a\big[\beta_p^{\tau_{a-}}; \tilde{\tau}_{0+} > \tau_{a-}\big].$$

Using (105), we have, when $p \neq \frac{1}{2}$,

$$\begin{aligned}(R\hat{e}_3, f_1) &= -p(2p-1)\beta_p(RT\mathbb{I}_{(a,b)}, \rho) \\ &= -\frac{p(2p-1)}{\beta_p - 1}\mathbb{E}_a\big[\beta_p^{(\tilde{\tau}_{(a-b,0)^c}-1)\wedge\tau_{a-}} - \beta_p\big] - p(2p-1)\big(1 + \Phi_\rho(a) - \Phi_\rho(b)\big) \\ &\to \frac{p}{2p-1}\mathbb{E}_a\big[\beta_p^{(\tilde{\tau}_{0+}-1)\wedge\tau_{a-}}\big] - \frac{4p^2(1-p)}{2p-1} - p(2p-1)\Phi_\rho(a) \quad \text{as } b \to \infty.\end{aligned}$$

Using (103) and (104), we have

$$\begin{aligned}(R\hat{e}_2, f_1) &= p\beta_p(RT\mathbb{I}_{[b,\infty)} - RT\mathbb{I}_{(-\infty,a]}, \rho) \\ &= p\beta_p\mathbb{E}_a\big[\beta_p^{\tilde{\tau}_{(a-b)-}-2}; \tilde{\tau}_{(a-b)-} < \tilde{\tau}_{0+} \wedge (1+\tau_{a-})\big] - p\big(1 - \Phi_\rho(b)\big) \\ &\quad - p\beta_p\mathbb{E}_a\big[\beta_p^{\tilde{\tau}_{0+}-2}; \tilde{\tau}_{0+} < \tilde{\tau}_{(a-b)-} \wedge (1+\tau_{a-})\big] + p\Phi_\rho(a) \\ &\to -p\mathbb{E}_a\big[\beta_p^{\tilde{\tau}_{0+}-1}; \tilde{\tau}_{0+} \le \tau_{a-}\big] + p\Phi_\rho(a) \quad \text{as } b \to \infty.\end{aligned}$$

Combining with the entries in $\hat{I}$, this completes the limiting values for all nine terms for $\det_3^{a,\infty}(K)$ as the determinant, when $p \neq \frac{1}{2}$,

$$\begin{vmatrix} 1 - \frac{1}{2}\mathbb{E}_a[\beta_p^{\tau_{a-}}; \tilde{\tau}_{0+} > \tau_{a-}] & \frac{1}{2} - \frac{1}{2}\mathbb{E}_a[\beta_p^{\tilde{\tau}_{0+}-1}; \tau_{a-} \ge \tilde{\tau}_{0+}] & -\frac{1}{2} - \frac{1}{2}\mathbb{E}_a[\beta_p^{\tilde{\tau}_{0+}-1}; \tau_{a-} \ge \tilde{\tau}_{0+}] \\ p - p\mathbb{E}_a[\beta_p^{\tilde{\tau}_{0+}-1}; \tilde{\tau}_{0+} \le \tau_{a-}] & 1 - p\mathbb{E}_a[\beta_p^{\tau_{a-}-1}; \tau_{a-} < \tilde{\tau}_{0+}] & -p\mathbb{E}_a[\beta_p^{\tau_{a-}-1}; \tau_{a-} < \tilde{\tau}_{0+}] \\ \frac{p}{2p-1}\mathbb{E}_a[\beta_p^{(\tilde{\tau}_{0+}-1)\wedge\tau_{a-}}] - \frac{p}{2p-1} & \frac{p}{2p-1}\mathbb{E}_a[\beta_p^{(\tau_{a-}\wedge\tilde{\tau}_{0+})-1} - 1] & 1 + \frac{p}{2p-1}\mathbb{E}_a[\beta_p^{(\tau_{a-}\wedge\tilde{\tau}_{0+})-1} - 1] \end{vmatrix}.$$

Row and column operations considerably simplify this: after the column operation $C_2 \to C_2 - C_3$ and then the row operation $R_3 \to R_3 + \frac{2p}{2p-1}R_1 + \frac{1}{2p-1}R_2$, we reach, when $p \neq \frac{1}{2}$,

$$\det_3^{a,\infty}(K) = \frac{1}{2p-1}\begin{vmatrix} 1 - \frac{1}{2}\mathbb{E}_a[\beta_p^{\tau_{a-}}; \tilde{\tau}_{0+} > \tau_{a-}] & 1 & -\frac{1}{2} - \frac{1}{2}\mathbb{E}_a[\beta_p^{\tilde{\tau}_{0+}-1}; \tau_{a-} \ge \tilde{\tau}_{0+}] \\ p - p\mathbb{E}_a[\beta_p^{\tilde{\tau}_{0+}-1}; \tilde{\tau}_{0+} \le \tau_{a-}] & 1 & -p\mathbb{E}_a[\beta_p^{\tau_{a-}-1}; \tau_{a-} < \tilde{\tau}_{0+}] \\ 2p & 2 & -1 \end{vmatrix}.$$

After $C_1 \to C_1 - pC_2$ and $C_3 \to C_3 + \frac{1}{2}C_2$, we reach, when $p \neq \frac{1}{2}$,

$$\det_3^{a,\infty}(K) = \frac{1}{2p-1}\begin{vmatrix} 1 - p - \frac{1}{2}\mathbb{E}_a[\beta_p^{\tau_{a-}}; \tilde{\tau}_{0+} > \tau_{a-}] & 1 & -\frac{1}{2}\mathbb{E}_a[\beta_p^{\tilde{\tau}_{0+}-1}; \tau_{a-} \ge \tilde{\tau}_{0+}] \\ -p\mathbb{E}_a[\beta_p^{\tilde{\tau}_{0+}-1}; \tilde{\tau}_{0+} \le \tau_{a-}] & 1 & \frac{1}{2} - p\mathbb{E}_a[\beta_p^{\tau_{a-}-1}; \tau_{a-} < \tilde{\tau}_{0+}] \\ 0 & 2 & 0 \end{vmatrix}$$

which reduces to the form stated in the Lemma for $p \neq \frac{1}{2}$. When $p = \frac{1}{2}$, the calculation is easier and is omitted. □

5.3. *Asymptotics.* We will derive the asymptotics in Theorems 2 and 3.

5.3.1. *Random walk results.* We need some variants of the random walk results in Section 4.3.1 that hold for our two-step walk $(S_n, \tilde{S}_n)$. We use the construction and notation from Section 5.2. Recall that $\tilde{\rho}(z) = \int_{\mathbb{R}} \rho(w)\rho(w-z)\,dw$ which is automatically symmetric.

Two-step Kac's formula. We extend Kac's formula (63) to show that, for all $n \geq 1$,

$$\mathbb{E}_0[\tilde{M}_n \delta_0(S_n)] = \frac{1}{n} \int_{\mathbb{R}} x(\rho^{*n}(x))^2 \, dx + \mathrm{Kac}_{\tilde{\rho}}(n). \tag{107}$$

Note that

$$\mathbb{E}_0[M_n \delta_0(S_n)] = -\mathbb{E}_0[m_n \delta_0(S_n)] = \mathrm{Kac}_{\tilde{\rho}}(n) \tag{108}$$

by a direct application of Kac's formula to (S_n). The extra term in (107) arises due to the maximum $\tilde{M}_n$ being taken over $(\tilde{S}_n)$ rather than (S_n). Indeed,

$$\begin{aligned} \tilde{M}_n \delta_0(S_n) &= \max\{\mathcal{X}_1, \mathcal{X}_1 + \mathcal{Y}_1 + \mathcal{X}_2, \dots, \mathcal{X}_1 + \mathcal{Y}_1 + \cdots + \cdots \mathcal{Y}_{n-1} + \mathcal{X}_n\} \delta_0(S_n) \\ &= \mathcal{X}_1 \delta_0(S_n) + \max\{0, \mathcal{Y}_1 + \mathcal{X}_2, \dots, \mathcal{Y}_1 + \cdots + \cdots \mathcal{Y}_{n-1} + \mathcal{X}_n\} \delta_0(S_n). \end{aligned} \tag{109}$$

Since

$$S_n = (\mathcal{Y}_1 + \mathcal{X}_2) + (\mathcal{Y}_2 + \mathcal{X}_3) + \cdots + (\mathcal{Y}_{n-1} + \mathcal{X}_n) + (\mathcal{Y}_n + \mathcal{X}_1),$$

the second term in (109) involves only the maximum of a one-step walk with increments equal to $\mathcal{X}_k + \mathcal{Y}_k$ and hence has expectation $\mathrm{Kac}_{\tilde{\rho}}(n)$ by Kac's formula (63). The expectation of the first term in (109) equals, by a cyclic symmetry argument,

$$\mathbb{E}_0[\mathcal{X}_1 \delta_0(S_n)] = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_0[\mathcal{X}_k \delta_0(S_n)] = \frac{1}{n} \mathbb{E}_0[\mathcal{X} \delta_0(\mathcal{X} + \mathcal{Y})],$$

where $\mathcal{X} = \sum_{k=1}^n \mathcal{X}_k$ has density $\rho^{*n}(x)\,dx$ and $\mathcal{Y} = \sum_{k=1}^n \mathcal{Y}_k$ has density $\rho^{*n}(-x)\,dx$ which leads to the stated formula.

Cyclic symmetry. We will use cyclic symmetry to show

$$\mathbb{E}_0[\min\{L, \tilde{M}_n\} \delta_0(S_n); \tau_{0-} = n] = \frac{1}{n} \mathbb{E}_0[\min\{L, \tilde{M}_n - m_n\} \delta_0(S_n)] \tag{110}$$

and hence its $L \to \infty$ limit

$$\mathbb{E}_0[\tilde{M}_n \delta_0(S_n); \tau_{0-} = n] = \frac{1}{n} \mathbb{E}_0[(\tilde{M}_n - m_n) \delta_0(S_n)]. \tag{111}$$

The proof is similar to the one-step version (64). Indeed, we consider the n cyclic permutations of the variables $((\mathcal{X}_1, \mathcal{Y}_1), (\mathcal{X}_2, \mathcal{Y}_2), \dots, (\mathcal{X}_n, \mathcal{Y}_n))$. We define, as in Section 5.2, n different two-step walks $(S^{(p)}_\cdot, \tilde{S}^{(p)}_\cdot)$ for $p = 0, 1, \dots, n-1$ as follows: $S^{(p)}_0 = 0$ and

$$\tilde{S}^{(p)}_k = S^{(p)}_{k-1} + \mathcal{X}_{p \oplus k}, \qquad S^{(p)}_k = \tilde{S}^{(p)}_k + \mathcal{Y}_{p \oplus k} \quad \text{for } k = 1, \dots, n,$$

where $p \oplus k$ is addition modulo n. Then, the law of $(S^{(p)}_\cdot, \tilde{S}^{(p)}_\cdot)$ is the same for all p. Moreover, the final value $S^{(p)}_n = \sum_{k \leq n} (\mathcal{X}_k + \mathcal{Y}_k)$ is independent of p.

We define the maxima, minima, and stopping times $\tilde{M}^{(p)}_n, m^{(p)}_n, \tau^{(p)}_{0-}$, as in Section 5.2, but indexed by the superscript p when they are for the two-step walk $(S^{(p)}_\cdot, \tilde{S}^{(p)}_\cdot)$. Then, using $m_n = 0$, whenever $S_n = 0$ and $\tau_{0-} = n$,

$$\begin{aligned} \mathbb{E}_0[\min\{L, \tilde{M}_n\} \delta_0(S_n) \mathbb{I}(\tau_{0-} = n)] &= \mathbb{E}_0[\min\{L, \tilde{M}_n - m_n\} \delta_0(S_n) \mathbb{I}(\tau_{0-} = n)] \\ &= \frac{1}{n} \sum_{p=0}^{n-1} \mathbb{E}_0[\min\{L, \tilde{M}^{(p)}_n - m^{(p)}_n\} \mathbb{I}(\tau^{(p)}_{0-} = n) \delta_0(S^{(p)}_n)] \\ &= \frac{1}{n} \sum_{p=0}^{n-1} \mathbb{E}_0[\min\{L, \tilde{M}_n - m_n\} \mathbb{I}(\tau^{(p)}_{0-} = n) \delta_0(S_n)], \end{aligned}$$

where the final equality comes from the identities

$$S_k^{(p)} = S_{p\oplus k} - S_p \quad \text{and} \quad \tilde{S}_k^{(p)} = \tilde{S}_{p\oplus k} - S_p \quad \text{for all } k,\ p, \text{ whenever } S_n = 0,$$

so that $\tilde{M}_n^{(p)} - m_n^{(p)}$ is independent of p whenever $S_n = 0$. Finally, there is exactly one cyclic permutation with $\{\tau_{0-}^{(p)} = n\}$, establishing (111).

5.3.2. *Proof of Theorem* 2. We note that an assumption that $\int_{\mathbb{R}} |x|^k \rho(x)\,dx < \infty$ implies that $\int_{\mathbb{R}} |x|^k \tilde{\rho}(x)\,dx < \infty$. Suppose first that $\beta \in [0, 1)$. From Lemma 16 we have

$$\begin{aligned}
&\log \mathrm{Det}_{[-L,\infty)}(I - \beta T)\\
&\quad = -\mathbb{E}_0\bigl[\beta^{\tau_{0-}}\delta_0(S_{\tau_{0-}})(L - \tilde{M}_{\tau_{0-}})_+\bigr]\\
&\quad = -L\mathbb{E}_0\bigl[\beta^{\tau_{0-}}\delta_0(S_{\tau_{0-}})\bigr] + \mathbb{E}_0\bigl[\beta^{\tau_{0-}}\delta_0(S_{\tau_{0-}})\min\{L, \tilde{M}_{\tau_{0-}}\}\bigr]\\
&\quad = -L\mathbb{E}_0\bigl[\beta^{\tau_{0-}}\delta_0(S_{\tau_{0-}})\bigr] + \mathbb{E}_0\bigl[\beta^{\tau_{0-}}\tilde{M}_{\tau_{0-}}\delta_0(S_{\tau_{0-}})\bigr] + o(1)
\end{aligned} \tag{112}$$

(where we show below the variable $\beta^{\tau_{0-}}\tilde{M}_{\tau_{0-}}\delta_0(S_{\tau_{0-}})$ is integrable). The first expectation involves only the walk (S_n) and so is given, as in (69), using the increment density $\tilde{\rho}$, giving the desired formula for $\kappa_1(\beta)$. The second expectation is given, using the cyclic symmetry (111), the Kac formulae (107) and (108), and then the argument from (70) by

$$\begin{aligned}
&\mathbb{E}_0\bigl[\beta^{\tau_{0-}}\tilde{M}_{\tau_{0-}}\delta_0(S_{\tau_{0-}})\bigr]\\
&\quad = \sum_{n=1}^{\infty}\beta^n\mathbb{E}_0\bigl[\tilde{M}_n\delta_0(S_n);\ \tau_{0-} = n\bigr]\\
&\quad = \sum_{n=1}^{\infty}\frac{\beta^n}{n}\mathbb{E}_0\bigl[(\tilde{M}_n - m_n)\delta_0(S_n)\bigr]\\
&\quad = \sum_{n=1}^{\infty}\frac{\beta^n}{n^2}\int_{\mathbb{R}} x\bigl(\rho^{*n}(x)\bigr)^2\,dx + 2\sum_{n=1}^{\infty}\frac{\beta^n}{n}\mathrm{Kac}_{\tilde{\rho}}(n)\\
&\quad = \sum_{n=1}^{\infty}\frac{\beta^n}{n^2}\int_{\mathbb{R}} x\bigl(\rho^{*n}(x)\bigr)^2\,dx + \int_0^{\infty} x\left(\sum_{n=1}^{\infty}\frac{\beta^n\tilde{\rho}^{*n}(x)}{n}\right)^2 dx,
\end{aligned}$$

which is finite by the first moment assumption and the boundedness of ρ, completing the formula for $\kappa_2(\beta)$.

When $\beta = 1$, we follow, with small changes, the argument from Lemma 14. Write

$$\mathbb{E}_0\bigl[\delta_0(S_{\tau_{0-}})\min\{L, \tilde{M}_{\tau_{0-}}\}\bigr] = \sum_{n=1}^{\infty}\tilde{p}(n, L), \tag{113}$$

where, using the cyclic symmetry technique (110),

$$\tilde{p}(n, L) := \mathbb{E}_0\bigl[\delta_0(S_n)\min\{L, \tilde{M}_n\};\ \tau_{0-} = n\bigr] = \frac{1}{n}\mathbb{E}_0\bigl[\delta_0(S_n)\min\{L, \tilde{M}_n - m_n\}\bigr],$$

and we approximate

$$\tilde{p}(n, L) = \frac{1}{n}\mathbb{E}_0\bigl[\delta_0(S_n)(\tilde{M}_n - m_n)\bigr] + \tilde{E}^{(1)}(n, L), \tag{114}$$

$$\tilde{p}(n, L) = \frac{1}{n}\mathbb{E}_0\Bigl[\delta_0(W_{n\sigma^2})\min\Bigl\{L, \sup_{t\le n\sigma^2} W_t - \inf_{t\le n\sigma^2} W_t\Bigr\}\Bigr] + \tilde{E}^{(2)}(n, L). \tag{115}$$

We verify in the Section 7.3 that the same error bounds (87) and (88) hold in this case, and then the argument of Lemma 14 goes through, with the only change being the extra term in the two-step Kac's formula (107). This leads to

$$\mathbb{E}_0\big[\delta_0(S_{\tau_{0-}})\min\{L,\tilde{M}_{\tau_{0-}}\}\big]=\log L+\frac{3}{2}\log 2-\log\tilde{\sigma}$$
$$+\sum_{n\ge 1}\frac{2}{n}\Big(\mathrm{Kac}_{\tilde{\rho}}(n)-\frac{1}{4}\Big)+\frac{1}{n^2}\int_{\mathbb{R}}x\big(\rho^{*n}(x)\big)^2\,dx+o(1),$$

where $\tilde{\sigma}^2=\int x^2\tilde{\rho}(x)\,dx$. Using this in (112) completes the proof. □

5.3.3. *Proof for Theorem 3 when $p\in(0,\frac{1}{2})$.* We apply Lemma 15 for the interval $[-L,b]$ and take $b\to\infty$, giving

$$2\log \mathrm{Pf}_{[-L,\infty)}(\mathbf{J}-p\mathbf{K})=\log \mathrm{Det}_{[-L,\infty)}(I-\beta_pT)+\log\det{}_3^{-L,\infty}(K).$$

Lemma 17 gives a probabilistic representation for $\det_3^{-L,\infty}(K)$ where it is simple to let $L\to\infty$. Indeed, in this limit we get

$$\begin{aligned}&\det{}_3^{-L,\infty}(K)\\&=\frac{2}{1-2p}\begin{vmatrix}1-p-\frac{1}{2}\mathbb{E}_{-L}[\beta_p^{\tau_{(-L)-}};\tilde{\tau}_{0+}>\tau_{(-L)-}] & -\frac{1}{2}\mathbb{E}_{-L}[\beta_p^{\tilde{\tau}_{0+}-1};\tau_{(-L)-}\ge\tilde{\tau}_{0+}]\\ -p\mathbb{E}_{-L}[\beta_p^{\tilde{\tau}_{0+}-1};\tilde{\tau}_{0+}\le\tau_{(-L)-}] & \frac{1}{2}-p\mathbb{E}_{-L}[\beta_p^{\tau_{(-L)-}-1};\tau_{(-L)-}<\tilde{\tau}_{0+}]\end{vmatrix}\\&=\frac{2}{1-2p}\begin{vmatrix}1-p-\frac{1}{2}\mathbb{E}_0[\beta_p^{\tau_{0-}};\tilde{\tau}_{L+}>\tau_{0-}] & -\frac{1}{2}\mathbb{E}_0[\beta_p^{\tilde{\tau}_{L+}-1};\tau_{0-}\ge\tilde{\tau}_{L+}]\\ -p\mathbb{E}_0[\beta_p^{\tilde{\tau}_{L+}-1};\tilde{\tau}_{L+}\le\tau_{0-}] & \frac{1}{2}-p\mathbb{E}_0[\beta_p^{\tau_{0-}-1};\tau_{0-}<\tilde{\tau}_{L+}]\end{vmatrix}\\&\to\frac{2}{1-2p}\begin{vmatrix}1-p-\frac{1}{2}\mathbb{E}_0[\beta_p^{\tau_{0-}}] & 0\\ 0 & \frac{1}{2}-p\mathbb{E}_0[\beta_p^{\tau_{0-}-1}]\end{vmatrix}=\frac{1-2p}{1-p},\end{aligned}\tag{116}$$

where we shifted the starting position for the expectations, then used $\mathbb{P}_0[\tilde{\tau}_{L+}\to\infty]=1$ to take the limits, and finally evaluated the expectations using Sparre Andersen's formula (58) (since the stopping times τ_{0-} involve only the walk (S_n) which is a one step random walk with symmetric increments) which gives $\mathbb{E}_0[\beta_p^{\tau_{0-}}]=2p$ when $p\in(0,\frac{1}{2})$. Combined with Theorem 2, which gives the asymptotic for the Fredholm determinant, this gives the values of $\kappa_1(p),\kappa_2(p)$, as required.

5.3.4. *Proof for Theorem 3 when $p\in(\frac{1}{2},1)$.* For $p\in(\frac{1}{2},1)$, Sparre Andersen's formula (58) gives $\mathbb{E}_0[\beta_p^{\tau_{0-}}]=2(1-p)$ which, as in (116), implies that $\lim_{L\to\infty}\det_3^{-L,\infty}(K)=0$. Indeed, as in the translationally invariant case, this small determinant contributes to the leading order asymptotic. To resolve the asymptotic, we use Sparre Andersen's formula via

$$\mathbb{E}_0\big[\beta_p^{\tau_{0-}};\tilde{\tau}_{L+}>\tau_{0-}\big]=\mathbb{E}_0\big[\beta_p^{\tau_{0-}}\big]-\mathbb{E}_0\big[\beta_p^{\tau_{0-}};\tau_{0-}\ge\tilde{\tau}_{L+}\big]$$

and then argue that $\mathbb{E}_0[\beta_p^{\tau_{0-}};\tau_{0-}\ge\tilde{\tau}_{L+}]=o(\mathbb{E}_0[\beta_p^{\tau_{L+}};\tau_{0-}\ge\tilde{\tau}_{L+}])$, as in (73). Using these in (116), we reach

$$\begin{aligned}\det{}_3^{-L,\infty}(K)&=\frac{2}{1-2p}\begin{vmatrix}\frac{1}{2}\mathbb{E}_0[\beta_p^{\tau_{0-}};\tau_{0-}\ge\tilde{\tau}_{L+}] & -\frac{1}{2}\mathbb{E}_0[\beta_p^{\tilde{\tau}_{L+}-1};\tau_{0-}\ge\tilde{\tau}_{L+}]\\ -p\mathbb{E}_0[\beta_p^{\tilde{\tau}_{L+}-1};\tau_{0-}\ge\tilde{\tau}_{L+}] & \frac{p}{\beta_p}\mathbb{E}_0[\beta_p^{\tau_{0-}};\tau_{0-}\ge\tilde{\tau}_{L+}]\end{vmatrix}\\&=\frac{p}{(2p-1)\beta_p^2}\big(\mathbb{E}_0[\beta_p^{\tilde{\tau}_{L+}};\tau_{0-}\ge\tilde{\tau}_{L+}]\big)^2(1+o(1)).\end{aligned}$$

The nontranslationally invariant analogue of Lemma 13 is as follows.

LEMMA 18. *Suppose there exists $\phi_p > 0$ so that $\beta_p \int e^{\phi_p x} \tilde{\rho}(x)\, dx = 1$, and that both $\int |x| e^{\phi_p x} \tilde{\rho}(x)\, dx$ and $\int e^{\phi_p |x|} \rho(x)\, dx$ are finite. Then,*

$$\lim_{L\to\infty} e^{\phi_p L} \mathbb{E}_0\big[\beta_p^{\tilde{\tau}_{L+}}; \tau_{0-} \geq \tilde{\tau}_{L+}\big] = \beta_+ \frac{\sqrt{1-\beta_p}}{\phi_p \mathbb{E}_0^{(p)}[S_1]} \big(\mathbb{P}_0^{(p)}[\tau_{0-} = \infty]\big)^2, \tag{117}$$

where $\beta_+^{-1} = \int_{\mathbb{R}} e^{\phi_p x} \rho(x)\, dx$.

Combined with Theorem 2, which gives the asymptotic for the Fredholm determinant, and the exact formula (57), this lemma leads to the stated forms for $\kappa_1(p)$, $\kappa_2(p)$.

PROOF OF LEMMA 18. Choose β_-, β_+ so that $\beta_- \int_{\mathbb{R}} \rho(-x) e^{\phi_p x}\, dx = \beta_+ \int_{\mathbb{R}} \rho(x) \times e^{\phi_p x}\, dx = 1$. Then, $\beta_p = \beta_- \beta_+$. Let $\mathbb{P}_x^{(p)}$, $\mathbb{E}_x^{(p)}$ be the tilted probability and expectation, where the two-step walk $(\tilde{S}_n, S_n)$ uses increments $\mathcal{X}_i$, $\mathcal{Y}_i$ with the tilted densities $\beta_- \exp(\phi_p x) \rho(-x)\, dx$ and $\beta_+ \exp(\phi_p x) \rho(x)\, dx$. Then, defining

$$\tilde{Z}_n = \beta_- \beta_p^{n-1} e^{\phi_p \tilde{S}_n}, \qquad Z_n = \beta_p^n e^{\phi_p S_n} \quad \text{for } n \geq 1$$

the process $(1, \tilde{Z}_1, Z_1, \tilde{Z}_2, Z_2, \ldots)$ is a martingale. Moreover,

$$\begin{aligned} \mathbb{E}_0^{(p)}\big[e^{-\phi_p \tilde{S}_{\tilde{\tau}_{L+}}}; \tau_{0-} \geq \tilde{\tau}_{L+}\big] &= \mathbb{E}_0\big[e^{-\phi_p \tilde{S}_{\tilde{\tau}_{L+}}} \tilde{Z}_{\tilde{\tau}_{L+}}; \tau_{0-} \geq \tilde{\tau}_{L+}\big] \\ &= \mathbb{E}_0\big[\beta_- \beta_p^{\tilde{\tau}_{L+}-1}; \tau_{0-} \geq \tilde{\tau}_{L+}\big] \end{aligned}$$

so that

$$e^{\phi_p L} \mathbb{E}_0\big[\beta_p^{\tilde{\tau}_{L+}}; \tau_{0-} \geq \tilde{\tau}_{L+}\big] = \beta_+ \mathbb{E}_0^{(p)}\big[e^{-\phi_p (\tilde{S}_{\tilde{\tau}_{L+}} - L)}; \tau_{0-} \geq \tilde{\tau}_{L+}\big]. \tag{118}$$

By conditioning on the value of $\tilde{S}_1$, we see that

$$\tilde{V}(L) := \mathbb{E}_0^{(p)}\big[e^{-\phi_p (\tilde{S}_{\tilde{\tau}_{L+}} - L)}\big] = \int_{-\infty}^{L} \rho(-x) V(L-x)\, dx + \int_L^\infty \rho(-x) e^{-\phi_p (x-L)}\, dx,$$

where $V(L) = \mathbb{E}_0^{(p)}[\exp(-\phi_p (S_{\tau_{L+}} - L))]$. We know from (78) that $V(L) \to C_h$ as $L \to \infty$, and we deduce that $\tilde{V}(L) \to C_h$. Conditioning on $\sigma(S_{\tau_{0-}})$, we see

$$\begin{aligned} \mathbb{E}_0^{(p)}\big[e^{-\phi_p (\tilde{S}_{\tilde{\tau}_{L+}} - L)}; \tau_{0-} \geq \tilde{\tau}_{L+}\big] &= \tilde{V}(L) - \mathbb{E}_0^{(p)}\big[e^{-\phi_p (\tilde{S}_{\tilde{\tau}_{L+}} - L)}; \tau_{0-} < \tilde{\tau}_{L+}\big]. \\ &= \tilde{V}(L) - \mathbb{E}_0^{(p)}\big[\tilde{V}(L - S_{\tau_{0-}}); \tau_{0-} < \tilde{\tau}_{L+}\big] \\ &\to C_h - C_h \mathbb{P}_0^{(p)}[\tau_{0-} < \infty] = C_h \mathbb{P}_0^{(p)}[\tau_{0-} = \infty]. \end{aligned}$$

With (118) this implies that $e^{\phi_p L} \mathbb{E}_0[\beta_p^{\tilde{\tau}_{L+}}; \tau_{0-} \geq \tilde{\tau}_{L+}] \to \beta_+ C_h \mathbb{P}_0^{(p)}[\tau_{0-} = \infty]$, and the desired form for the limit follows from the expression for C_h in Lemma 13. $\square$

5.3.5. *Proof for Theorem 3 when $p = \frac{1}{2}$.* Applying Lemma 15 for the interval $[-L, b]$, taking $b \to \infty$, and then using the probabilitistic representation in Lemma 17 for $\det_3^{-L,\infty}(K)$ gives

$$2 \log \mathrm{Pf}_{[-L,\infty)}(\mathbf{J} - p\mathbf{K}) = \log \mathrm{Det}_{[-L,\infty)}(I - T) + \log \mathbb{P}_0[\tilde{\tau}_{L+} \leq \tau_{0-}]. \tag{119}$$

Comparing with the translationally invariant analogue (80), we see there is a missing $\log 2$. This and the slightly different form for the two-step Kac's formula (107) turn out to be the only differences between the two asymptotics when $p = \frac{1}{2}$.

Let $\mu = \int_{\mathbb{R}} x\rho(-x)\,dx$. Then, the process

$$(0, \tilde{S}_1 - \mu, S_1, \tilde{S}_2 - \mu, S_2, \ldots)$$

is a martingale under $\mathbb{P}_0$. The optional stopping theorem implies

$$\begin{aligned} 0 &= \mathbb{E}_0[S_{\tau_{0-}}; \tau_{0-} < \tilde{\tau}_{L+}] + \mathbb{E}_0[\tilde{S}_{\tilde{\tau}_{L+}} - \mu; \tilde{\tau}_{L+} \le \tau_{0-}] \\ &= \mathbb{E}_0[S_{\tau_{0-}}] - \mathbb{E}_0[S_{\tau_{0-}}; \tilde{\tau}_{L+} \le \tau_{0-}] + \mathbb{E}_0[\tilde{S}_{\tilde{\tau}_{L+}} - L; \tilde{\tau}_{L+} \le \tau_{0-}] + (L+\mu)\mathbb{P}_0[\tilde{\tau}_{L+} \le \tau_{0-}] \end{aligned}$$

(the argument from Lemma 5.1.1 in [27] justifies the optional stopping theorem being valid). Rearranging gives

$$\mathbb{P}_0[\tilde{\tau}_{L+} \le \tau_{0-}] = -\frac{\mathbb{E}_0[S_{\tau_{0-}}]}{L+\mu} + o(1) \quad \text{as } L \to \infty$$

by arguing as in Section 4.3.4. Together with the asymptotics from Theorem 2 for the Fredholm determinant in (119) and Spitzer's formula (60), this finishes the calculation.

6. The proof of Theorem 9.

In this section we will establish the Pfaffian structure for exit measures and find the corresponding kernels. The arguments broadly follow those in [40] and [21] which derive the Pfaffian kernels from duality identities. The new feature here is that the dual process, which is a system of annihilating motions, has immigration of particles.

6.1. *Product ratio moments*. The starting point that shows that the Pfaffian structure still holds is the following Pfaffian formula for product moments for annihilating particle systems with immigration.

Consider the following finite particle system: between reactions particles evolve as independent strong Markov process motions on $\mathbb{R}$; upon collision, any pair of particles instantaneously annihilate. The processes starts from immigrated particles, starting at the space-time points $z_i = (y_i, t_i) \in [0, t] \times \mathbb{R}$ for $i = 1, \ldots, 2n$ for some $n \ge 0$. We list the positions of all surviving particles at time t as $Y_t^1 < Y_t^2 < \cdots$ in increasing order. Note that number of particles alive at time t will be even since the total number of immigrated particles is even, and we remove particles in pairs upon annihilation. Some restriction is needed on the motion process, for example, to ensure no triple collisions occur. Since we need only the two examples of Brownian motions on $\mathbb{R}$ and Brownian motions with reflection on $[0, \infty)$, we restrict to these two cases below (which also makes some of the p.d.e. arguments straightforward), but the proof makes it clear that the result should hold more generally.

We write Y_t for this point process at time t. We write $\mathbb{P}^A_{\mathbf{z}}$, where $\mathbf{z} = (z_1, \ldots, z_{2n})$, for the law of this annihilating process $(Y_t : t \ge 0)$.

LEMMA 19. *Let $g, h : [0, \infty) \to \mathbb{R}$ be bounded and measurable. For the finite annihilating system described above with immigration at $\mathbf{z} = (z_1, \ldots, z_{2n}) \in ([0, t] \times \mathbb{R})^{2n}$, define an alternating product moment by*

$$M_{g,h}(Y_t) = \prod_{i \ge 1} g(Y_t^{2i-1}) \prod_{i \ge 1} h(Y_t^{2i}), \tag{120}$$

where an empty product is taken to have value 1. *Then,*

$$\mathbb{E}^A_{\mathbf{z}}[M_{g,h}(Y_t)] = \mathrm{pf}(\mathbb{E}_{(\mathbf{z_i}, \mathbf{z_j})}[M_{g,h}(Y_t)] : i < j \le 2n).$$

Note that the terms in the Pfaffian use systems with just two particles, and so the double product moment $M_{g,h}(Y_t)$ takes either the value $g(Y_t^1)h(Y_t^2)$ or the value 1, depending on whether the two particles have annihilated.

PROOF. We give the proof in the case of reflected Brownian motions on $\mathbb{R}$ and indicate the slight simplifications for the full space case. The proof follows those in [40] and [21], where the Kolmogorov equation for the expectation is shown to be solved by the Pfaffian. Due to us immigrating particles over the time interval $[0,t]$, we will solve the equation in the intervals between immigration times. For this we need more detailed notation, used only in this proof. The final time t, the number of particles $2n$, and the immigration positions $\mathbf{z}$ are fixed throughout the proof. We suppose the points $z_i = (y_i, t_i)$ are listed so that $y_i \geq 0$ and $t = t_0 > t_1 > \cdots > t_{2n} > 0$; we will establish the result for such $\mathbf{z}$, and when there are one or more equalities between the time points t_i, the result follows by continuity in these variables. We write $\mathbf{z}^p$ for the vector $((y_1,t_1),\ldots,(y_p,t_p))$ when $1 \leq p \leq 2n$ (and $\mathbf{z}^0 = \varnothing$). Also, we take g, h to be continuous, and the measurable case can be established by approximation.

We write $V_k^+ = \{\mathbf{x} = (x_1,\ldots,x_k) : 0 \leq x_1 < \cdots < x_k\}$ for a cell in $\mathbb{R}^k$ and define $\mathbf{x}^q = (x_1,\ldots,x_q) \in V_q^+$ when $q \leq k$. From an element $\mathbf{x} \in V_k^+$, we define a set of space-time points by

$$(\mathbf{x}, s) = \big((x_1,s),\ldots,(x_k,s)\big) \quad \text{for } s \in [0,t].$$

Define the system of functions

$$m_{s,t}^{(p,q)}\big(\mathbf{z}^p, (\mathbf{x},s)\big) = \mathbb{E}^A_{((y_1,t_1),\ldots,(y_p,t_p),(x_1,s),\ldots,(x_q,s))}\big[M_{g,h}(X_t)\big]$$
$$\text{for } p,q \geq 0,\ p+q \leq 2n \text{ and } p+q \text{ even, } \mathbf{x} \in V_q^+, \text{ and } s \in [0,t_p).$$

Thus, $(\mathbf{x}^q, s)$ will describe the positions particles alive at time s, and $\mathbf{z}^p$ describes the remaining positions for particles to be immigrated after time s. Each of these functions satisfies a backward heat equation with reflected boundary condition

$$\left(\partial_s + \frac{1}{2}\sum_{i=1}^q \partial_{x_i}^2\right) m_{s,t}^{(p,q)}\big(\mathbf{z}^p, (\mathbf{x},s)\big) = 0 \quad \text{for } s \in [0,t_p) \text{ and } \mathbf{x} \in V_q^+, \tag{121}$$

$$\partial_{x_1} m_{s,t}^{(p,q)}\big(\mathbf{z}^p, (\mathbf{x},s)\big) = 0 \quad \text{for } s \in [0,t_p) \text{ and } 0 = x_1 < x_2 < \cdots < x_q. \tag{122}$$

(In the case of Brownian motions on $\mathbb{R}$, we remove the boundary condition (122) and allow $\mathbf{x} \in V_k = \{\mathbf{x} = (x_1,\ldots,x_k) : x_1 < \cdots < x_k\}$.)

The function $m_{s,t}^{(p,q)}(\mathbf{z}^p, (\mathbf{x},s))$ extends continuously to $\mathbf{x} \in \overline{V}_k^+$, $s \in [0,t_p)$ and satisfies the boundary conditions, for $i = 1,\ldots,q-1$,

$$\begin{aligned} m_{s,t}^{(p,q)}\big(\mathbf{z}^p, (\mathbf{x},s)\big) &= m_{s,t}^{(p,q-2)}\big(\mathbf{z}^p, (\mathbf{x}^{i,i+1}, s)\big), \\ &\text{when } s \in [0,t_p) \text{ and } 0 < x_1 < \cdots < x_i = x_{i+1} < \cdots < x_q, \end{aligned} \tag{123}$$

where $\mathbf{x}^{i,i+1} = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_q)$ is the vector $\mathbf{x}$ with the coordinates x_i, x_{i+1} "annihilated." (Conditions on other parts of the boundary ∂V_k^+ are not needed to ensure uniqueness.) Finally, they satisfy final conditions

$$\begin{aligned} &\lim_{s\uparrow t} m_{s,t}^{(0,2n)}\big(\varnothing, (\mathbf{x},s)\big) = \prod_{i\geq 1} g(x_{2i-1}) \prod_{i \geq 1} h(x_{2i}) \quad \text{for } \mathbf{x} \in V_{2n}^+, \\ &\lim_{s\uparrow t_p} m_{t_p,t}^{(p,q)}\big(\mathbf{z}^p, (\mathbf{x},s)\big) = m_{t_p,t}^{(p-1,q+1)}\big(\mathbf{z}^{p-1}, (\mathbf{x}|y_p, s)\big) \\ &\qquad \text{when } p \geq 1, \mathbf{x} \in V_q^+, \text{ and } y_p \notin \{x_1,\ldots,x_q\}, \end{aligned} \tag{124}$$

where $\mathbf{x}|y_p$ is the element of V_{q+1}^+ with coordinates $x_1,\ldots,x_q,y_p$ listed in increasing order.

We claim the system (121), (122), (123), (124) of equations for $(m^{(p,q)} : \text{even } p+q \le 2n)$ has unique bounded solutions, where $m^{(p,q)} \in C([0,t_p) \times \overline{V_q^+}) \cap C^{1,2}([0,t_p) \times V_q^+)$. This is an inductive proof, working downward in $p = 2n, 2n-1, \dots, 1, 0$, and for each fixed p working upward in $q = 0, 1, \dots, 2n-p$ (subject to $p+q$ being even), as indicated in the following diagram. The functions to the left of right arrows determine the boundary conditions for the functions on the right; the functions to the northeast of southwest arrows play the role of the final conditions for the functions to the southwest of these arrows. Vertical arrows correspond to the final conditions for the functions $m_{s,t}^{(0,2n)}$ at the top layer:

$$
\begin{array}{ccccccccc}
 & g(x_1)h(x_2) & \cdots & & \prod_{k=1}^{n-1} g(x_{2k-1})h(x_{2k}) & & \prod_{k=1}^{n} g(x_{2k-1})h(x_{2k}) \\
 & \downarrow & & & \downarrow & & \downarrow \\
1 = m_{s,t}^{(0,0)} \mapsto & m_{s,t}^{(0,2)} \mapsto & \cdots & \mapsto & m_{s,t}^{(0,2n-2)} & \mapsto & m_{s,t}^{(0,2n)} \\
 & \swarrow & & & \swarrow & & \swarrow \\
m_{s,t}^{(1,1)} \mapsto & \cdots \mapsto & m_{s,t}^{(1,2n-3)} & \mapsto & m_{s,t}^{(1,2n-1)} & & \\
\swarrow & & \swarrow & & \swarrow & & \\
1 = m_{s,t}^{(2,0)} \mapsto \cdots & & \mapsto m_{s,t}^{(2,2n-4)} & \mapsto & m_{s,t}^{(2,2n-2)} & & \\
 & & \swarrow & & \swarrow & & \\
 & \cdot \quad \cdot & \cdot & \cdot & & &
\end{array}
$$

We will now claim that, when $p+q \le 2n$ is even, that, for $s \in [0,t_p]$ and $\mathbf{x} \in V_q^+$,

$$
m_{s,t}^{(p,q)}\big(\mathbf{z}^p, (\mathbf{x}, s)\big) = \operatorname{pf}\left(\begin{array}{c|c}
\begin{matrix} m_{s,t}^{(2,0)}((y_i,t_i),(y_j,t_j)) \\ 1 \le i < j < p \end{matrix} & \begin{matrix} m_{s,t}^{(1,1)}((y_i,t_i),(x_j,s)) \\ i = 1,\dots,p,\ j = 1,\dots,q \end{matrix} \\
\hline
 & \begin{matrix} m_{s,t}^{(0,2)}((x_i,s),(x_j,s)) \\ 1 \le i < j < q \end{matrix}
\end{array}\right), \tag{125}
$$

where we have listed the upper triangular elements of this $(p+q) \times (p+q)$ antisymmetric block matrix. Specialising to $p = 2n$, $q = 0$ and $s = 0$, we find the conclusion of the lemma.

The claim (125) follows by uniqueness once we verify that the Pfaffian expression on the right-hand side also solves the equations (121), (122), (123), (124). The arguments that the Pfaffian solves the p.d.e (121), (122) and the boundary conditions (123) is the same as for the simpler one time period case in [40]. The final conditions (124) for $m_{t,t}^{(0,2n)}$ require that

$$
\prod_{i \ge 1} g\big(x^{2i-1}\big) \prod_{i \ge 1} h\big(x^{2i}\big) = \operatorname{pf}\big(g(x_i)h(x_j) : i < j \le 2n\big) \quad \text{for } \mathbf{x} \in V_{2n}^+,
$$

which follows since the antisymmetric matrix in the Pfaffian is of the form $D^T J_{2n} D$, where D is the diagonal matrix with entries $(g(x_1), h(x_2), \dots, g(x_{2n-1})h(x_{2n}))$ and J_{2n} is the block diagonal matrix with n blocks of the form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ along the diagonal (so that $\operatorname{pf}(J_{2n}) = 1$). The final conditions for all other $m_{t_p,t}^{(p,q)}$, when $p \ge 1$, follow inductively. □

6.2. *Dualities and thinning*. We now follow the steps used in [40] (to which we will refer for some details) using the Brownian web on a half-space developed in [38]. This gives a duality formula for a coalescing system on $(0,\infty)$ together with its exit measure on $\{0\} \times [0,\infty)$.

Let $(X_t : t \geq 0)$ be a system of (instantaneously) coalescing Brownian motions on $(0, \infty) \times [0, \infty)$, frozen upon hitting the boundary, and let X_e be the exit measure on $\{0\} \times [0, \infty)$. We consider X_t as a locally finite simple point measure on $(0, \infty)$ at each $t \geq 0$. We start by considering the case where the initial condition $X_0 = \mu$ is deterministic and contains a finite number $\mu(0, \infty)$ of particles. We write $\mathbb{P}^C_\mu$ for the law of this process.

The following lemma follows immediately from the noncrossing properties of the half-space Brownian web and dual Brownian web paths (which are reflected Brownian motions). It characterises the joint law of X_t and $X_e|_{\{0\}\times[0,t]}$. We use, from Section 6.1, the law $\mathbb{P}^A_{\mathbf{z}}$ of a finite annihilating system of Brownian motions (Y_y) with reflection on $[0, \infty)$ and with immigrated particles at $\mathbf{z}$.

LEMMA 20 (Toth and Werner [38]). *Let $I_1 = \{0\} \times [a_1, a_2], \dots, I_n = \{0\} \times [a_{2n-1}, a_{2n}]$, for $\mathbf{a} \in V^+_{2n}$, $n \geq 0$ be disjoint intervals inside $\{0\} \times [0, t]$. Let $J_1 = \{0\} \times [b_1, b_2], \dots, J_n = \{0\} \times [b_{2m-1}, b_{2m}]$, for $\mathbf{b} \in V^+_{2m}$, $m \geq 0$, be disjoint intervals inside $[0, \infty)$. Then,*

$$\mathbb{P}^C_\mu[X_e(I_1 \cup \dots \cup I_n) = 0, X_t(J_1 \cup \dots \cup J_m) = 0] = \mathbb{P}^A_{\mathbf{z}}[\mu(S_t) = 0], \tag{126}$$

where $z_i = (0, t - a_i)$ for $i = 1, \dots, 2n$ and $z_i = (b_{i-2n}, 0)$ for $i = 2n + 1, \dots, 2n + 2m$ and where

$$S_t = (Y^1_t, Y^2_t) \cup (Y^3_t, Y^4_t) \cup \dots$$

formed from all remaining annihilating particles at time t (and $S_t = \varnothing$ if there are no particles).

We can obtain a corresponding duality statement for mixed coalescing/annihilating systems CABM(θ) by thinning. We recall a colouring argument. Consider first initial conditions with finitely many particles, that is, $\mu([0, \infty)) < \infty$. Fix an evolution of $(X_t : t \geq 0)$, X_e under $\mathbb{P}^C_\mu$ and colour particles red or blue as follows: at time zero, let each particle independently be red R with probability $1/(1 + \theta)$ and blue B with probability $\theta/(1 + \theta)$; colour the particles at later times by following the colour change rules:

$$B + B \to B, \qquad R + B \to R, \qquad R + R \to \begin{cases} B & \text{with probability } \theta, \\ R & \text{with probability } 1 - \theta, \end{cases}$$

independently at each of the finitely many collisions. The particles coloured red form a CABM(θ) system. Moreover, the particles in X_t alive at time t, together with the frozen particles in $X_e(\{0\} \times [0, t])$, remain independently red R with probability $1/(1 + \theta)$ and blue B with probability $\theta/(1 + \theta)$. This can be shown by checking that, after each collision, this property is preserved.

Write $\Theta(\mu)$ for the thinned random measure created by deleting each particle of μ independently with probability $1/(1 + \theta)$. Writing $\mathbb{P}^{\mathrm{CABM}(\theta)}_{\Xi}$ for the law of the mixed CABM system started at a (possible random) finite initial condition Ξ, the colouring procedure above implies the equality in distribution, for finite μ,

$$(X_t, X_e|_{\{0\}\times[0,t]}) \text{ under } \mathbb{P}^{\mathrm{CABM}(\theta)}_{\Theta(\mu)} \stackrel{\mathcal{D}}{=} (\Theta(X_t), \Theta(X_e|_{\{0\}\times[0,t]})) \text{ under } \mathbb{P}^C_\mu. \tag{127}$$

Thinning a finite set of $n \geq 1$ particles leaves $B(n, (1 + \theta)^{-1})$ a Binomial number of remaining particles. Note, when $\theta \in (0, 1]$, that $\mathbb{E}[(-\theta)^{B(n,(1+\theta)^{-1})}] = 0$ for all $n \geq 1$. Then, we have, writing $\mathbb{E}_\Theta$ for the expectation over the thinning,

$$\begin{aligned} &\mathbb{E}_\Theta \mathbb{E}^{\mathrm{CABM}(\theta)}_{\Theta(\mu)}[(-\theta)^{X_e(I_1\cup\dots\cup I_n)}(-\theta)^{X_t(J_1\cup\dots\cup J_m)}] \\ &\quad = \mathbb{E}_\Theta \mathbb{E}^C_\mu[(-\theta)^{\Theta(X_e)(I_1\cup\dots\cup I_n)}(-\theta)^{\Theta(X_t)(J_1\cup\dots\cup J_m)}], \quad \text{using (127)}, \end{aligned}$$

$$
\begin{aligned}
(128) \qquad &= \mathbb{P}^C_\mu\big[X_e(I_1 \cup \cdots \cup I_n) = 0, X_t(J_1 \cup \cdots \cup J_m) = 0\big] \\
&= \mathbb{P}^A_{\mathbf{z}}\big[\mu(S_t) = 0\big], \quad \text{using (126)}, \\
&= \mathbb{E}_\Theta \mathbb{E}^A_{\mathbf{z}}\big[(-\theta)^{\Theta(\mu)(S_t)}\big].
\end{aligned}
$$

This implies the duality

$$
\mathbb{E}^{\mathrm{CABM}(\theta)}_\mu\big[(-\theta)^{X_e(I_1 \cup \cdots \cup I_n)}(-\theta)^{X_t(J_1 \cup \cdots \cup J_m)}\big] = \mathbb{E}^A_{\mathbf{z}}\big[(-\theta)^{\mu(S_t)}\big] \tag{129}
$$

for finite μ; this can be checked by induction on the number $\mu(0, \infty)$ of initial particles by expanding the identity (128) into the sum over terms where different size subsets of particles in μ remain after the thinning. Note that the duality (129) contains the duality (126) as the limit $\theta \downarrow 0$. Indeed, henceforth we shall take $0^0 = 1$ so that $(-\theta)^k = \mathbb{I}(k = 0)$ when $\theta = 0$ to allow a unified treatment over $\theta \in [0, 1]$.

The extension of (129) to the case of infinite initial conditions μ can be established by approximation arguments. This is (somewhat tersely) sketched in the appendix to [40], and we summarize some points here. We can consider the measure X_t as living in the space $\mathcal{M}$ of locally finite point measures on $[0, \infty)$, which we give the topology of vague convergence. Due to the instantaneous reactions, we restrict to the subset $\mathcal{M}_0$ of simple point measures. The arguments in [40] show that there is a Feller semigroup on this space, allowing us to construct the law $\mathbb{P}^{\mathrm{CABM}(\theta)}_\mu$ for the CABM(θ) process starting from any $\mu \in \mathcal{M}_0$. Moreover, there is an entrance law that is the limit of Poisson(λ) initial conditions as $\lambda \uparrow \infty$, which we informally call the maximal entrance law. (For the case $\theta = 0$ of coalescing particles, this corresponds to the point set process in the Brownian web starting from the set $[0, \infty)$.) The exit measure X_e also exists under $\mathbb{P}^{\mathrm{CABM}(\theta)}_\mu$ and is the limit of the exit measures for any approximating finite system—the point is the formulae (129) characterise the laws of the pair $(X_t, X_e|_{\{0\}\times[0,t]})$ as simple locally finite point measures.

REMARK. We digress here to record a lemma, based on the same tools, that shows thinnings are useful in the study of massive CBMs, where masses add upon coalescence. This process yields a point process in position/mass space $\mathbb{R} \times \mathbb{N}_0$ (in the case of integer masses). The lemma below gives only a very partial description of the process, and a full tractable multi-particle description for this model is still lacking (though, notably, the one point distribution has been obtained [36]).

LEMMA 21. *Consider a system (X_t) of massive coalescing Brownian motions where each particle has an integer mass and masses add upon coalesence. Suppose the masses of particles at $t = 0$ are independent uniform random variable on $\{1, 2, \ldots, q\}$ for a fixed $q \in \{2, 3, \ldots\}$. Let (R_t) be the positions of particles present at time t with labels not divisible by q; let (B_t) be the positions of particles present at time t with labels divisible by q. Then, the process $(R_t, B_t, t \geq 0)$ is a two-species particle system where the evolution of types at a collision time is governed by the following rules: for $\theta = 1/(q - 1)$*

$$
B + B \to B, \qquad B + R \to R, \qquad R + R \xrightarrow{\theta} B, \qquad R + R \xrightarrow{1-\theta} R. \tag{130}
$$

Moreover, at a fixed $t \geq 0$, the positions (B_t) are a $1/q$ thinning, and the positions (R_t) are a $(q - 1)/q$ thinning of the positions of the full system (X_t).

We have not detailed the initial positions or state space for (X_t) which play no part in the simple proof; consider finite systems on $\mathbb{R}$ to be specific. The proof consists of checking that, for two colliding particles whose masses are independent and uniform modulo(q)

on $\{1, \dots, q\}$, the resultant coalesced particle still has mass that is uniform modulo(q) on $\{1, \dots, q\}$. This then implies that the (R_t) system evolves as a CABM(θ) system, as in (130).

The point of including this lemma is to show that both strong thinning ($p < 1/2$) and weak thinning ($p \geq 1/2$) of CBM has an interpretation in terms of interacting particle systems: here, weak thinning with probability $(q-1)/q$ singles out particles with masses not divisible by q, whereas strong thinning with probability $1/q$ singles out particles with masses divisible by q.

6.3. *Pfaffian kernels.* We can now read off the Pfaffian kernels in Theorem 9 from the duality (129) and the alternating product moment formulae in Lemma 19. For $\mu \in \mathcal{M}_0$ and $\theta \in (0, 1]$, the duality (129) in the case where $\mathbf{b} = \varnothing$ so that we are only interested in the exit measure and, when $t = a_{2n}$, gives

$$\mathbb{E}_{\mu}^{\mathrm{CABM}(\theta)}\big[(-\theta)^{X_e(I_1 \cup \cdots \cup I_n)}\big] = \mathbb{E}_{\mathbf{z}}^{A}\big[(-\theta)^{\mu(S_{a_{2n}})}\big],$$

where $z_i = (0, a_{2n} - a_i)$ for $i = 1, \dots, 2n$. The right-hand side is an alternating product moment (120) where $g(x) = (-\theta)^{\mu[0,x]}$ and $h(x) = (-\theta)^{-\mu[0,x]}$. When μ is finite, then g, h are bounded, and Lemma 19 gives

$$\mathbb{E}_{\mu}^{\mathrm{CABM}(\theta)}\big[(-\theta)^{X_e(I_1 \cup \cdots \cup I_n)}\big] = \mathrm{pf}\big(H(a_i, a_j) : i < j \leq 2n\big),$$

where

$$H(a_i, a_j) = \mathbb{E}_{(z_i, z_j)}^{A}\big[(-\theta)^{\mu(S_{a_{2n}})}\big] = \mathbb{E}_{(0,0),(0,a_j - a_i)}^{A}\big[(-\theta)^{\mu(S_{a_j})}\big].$$

The same conclusion holds for $\mu \in \mathcal{M}_0$ by taking limits $\mu|_{[0,n]} \to \mu$. To derive the correlation function, we differentiate in the variables $a_2, \dots, a_{2n}$ and then let $a_1 \uparrow a_2, \dots, a_{2n-1} \uparrow a_{2n}$ (details of this calculation are given in [40]). We reach, writing $\rho_n^{\mathrm{CABM}(\theta)}$ for the n point intensity of X_e under $\mathbb{E}_{\mu}^{\mathrm{CABM}(\theta)}$,

$$\begin{aligned}&\big(-(1+\theta)\big)^n \rho_n^{\mathrm{CABM}(\theta)}(a_2, a_4, \dots, a_{2n}) \\ &\quad = \mathrm{pf}\left(\begin{pmatrix} H(a_{2i}, a_{2j}) & D_2 H(a_{2i}, a_{2j}) \\ D_1 H(a_{2i}, a_{2j}) & D_{12} H(a_{2i}, a_{2j}) \end{pmatrix} : i < j \leq n\right).\end{aligned}$$

To massage this into the stated derived form in Theorem 9, we first conjugate the kernel with the block matrix A with entries ± 1 down the diagonal (using $\mathrm{pf}(A^T B A) = (-1)^n$) and then define $K = H - 1$ to allow for the jump discontinuity in the derived form (8). This leads to $\rho_n^{\mathrm{CABM}(\theta)}(t_1, \dots, t_n) = \mathrm{Pf}(\mathbf{K}(t_i, t_j) : i < j \leq n)$, where

$$K(s,t) = \frac{1}{1+\theta} \mathbb{E}_{(0,0),(0,t-s)}^{A}\big[\big((-\theta)^{\mu(Y_t^2 - Y_t^1)} - 1\big)\mathbb{I}(\tau > t)\big] \quad \text{for } 0 < s < t, \tag{131}$$

where τ is the hitting time of the pair Y^1, Y^2.

For the cases where the initial condition is a Poisson measure Ξ, with bounded intensity $\lambda(x)\,dx$, we restart with the duality (129) which gives

$$\begin{aligned}\mathbb{E}_{\Xi}^{\mathrm{CABM}(\theta)}\big[(-\theta)^{X_e(I_1 \cup \cdots \cup I_n)}\big] &= \mathbb{E}_{\Xi}\mathbb{E}_{\mathbf{z}}^{A}\big[(-\theta)^{X_0(S_t)}\big] \\ &= \mathbb{E}_{\mathbf{z}}^{A}\big[e^{-(1+\theta)\sum_i \int_{Y_t^{2i-1}}^{Y_t^{2i}} \lambda(z)\,dz}\big] \\ &= \mathbb{E}_{\mathbf{z}}^{A}\big[M_{(g,h)}(Y_t)\big]\end{aligned}$$

for $g(x) = \exp(-(1+\theta)\int_0^x \lambda(z)\,dz)$ and $h(x) = 1/g(x)$. When λ is compactly supported, g, h are bounded, and the product moment Lemma 19 gives

$$\mathbb{E}_{\Xi}^{\mathrm{CABM}(\theta)}\big[(-\theta)^{X_e(I_1 \cup \cdots \cup I_n)}\big] = \mathrm{pf}\big(H(a_i, a_j) : i < j \leq 2n\big), \tag{132}$$

where

$$H(a_i, a_j) = \mathbb{E}^A_{(z_i,z_j)}\big[e^{-(1+\theta)\int_{S_t}\lambda(z)\,dz}\big] = \mathbb{E}^A_{(0,0),(0,a_j-a_i)}\big[e^{-(1+\theta)\int_{S_{a_j}}\lambda(z)\,dz}\big].$$

Approximating $\lambda I[0,n] \to \lambda$ allows the same conclusion for bounded λ. Now, we repeat the steps above to extract the kernel showing that, under $\mathbb{E}^{\mathrm{CABM}(\theta)}_{\mu}$, the exit measure X_e is a Pfaffian point process with kernel $\mathbf{K}(s,t)$ in derived form based on the scalar kernel

$$K(s,t) = \frac{1}{1+\theta}\mathbb{E}^A_{(0,0),(0,t-s)}\big[\big(e^{-(1+\theta)\int_{Y_t^1}^{Y_t^2}\lambda(z)\,dz} - 1\big)\mathbb{I}(\tau > t)\big], \tag{133}$$

where τ is the hitting time of the pair Y^1, Y^2. The distribution for (Y_t^1, Y_t^2) is given (e.g., by conditioning at time $t-s$ and then using the Karlin–McGregor formula for noncolliding Markov processes) by

$$\mathbb{P}^A_{(0,0),(0,t-s)}\big[Y_t^1 \in dy_1, Y_t^2 \in dy_2, \tau > t\big] = \begin{vmatrix} p_s^R(0,y_1) & p_t^R(0,y_1) \\ p_s^R(0,y_2) & p_t^R(0,y_2)\end{vmatrix}\,dy_1\,dy_2 \tag{134}$$

(recall $p_t^R(x,y)$ is the transition density for reflected Brownian motion on $[0,\infty)$). Rewriting the kernels using this density gives the form stated in Theorem 9. Note that all derivatives exist of $K(s,t)$ exist and are bounded in the region $0 < s \le t$. The density in (134) is of the form $\phi(s)\psi(t) - \psi(s)\phi(t)$, and this allows one to check that the antisymmetric extension of $K(s,t)$ to $s,t > 0$ is a $\mathcal{C}^2$. For deterministic locally finite μ or Poisson Ξ with a smooth intensity λ, the kernel extends to a $\mathcal{C}^2$ function on $s,t \ge 0$.

7. Further proofs. Here, we collect proofs of more technical statements made in Sections 2 and 3.

7.1. *Proof of Fourier transform formulae for* $\kappa_1(p)$, $\kappa_2(p)$. We derive the formulae (13) and (14). Recall, we are assuming, for simplicity, that ρ is in Schwarz class. Since ρ is symmetric, then $\hat{\rho}$ is symmetric and real valued. For $k \in \mathbb{R}$, $\hat{\rho}(k) < 1$ for all $k \neq 0$ and decays faster than polynomially, and near zero has an expansion

$$\hat{\rho}(k) = 1 - \frac{\sigma^2 k^2}{2} + O\big(|k|^4\big). \tag{135}$$

These imply that the integral (12) defining $L_\rho(p,x)$ is well defined and absolutely integrable for all p.

Fourier inversion and Fubini's theorem imply that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{n}\rho^{*n}(0) &= \sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{2\pi n}\int_{\mathbb{R}}\big(\hat{\rho}(k)\big)^n\,dk \\ &= -\frac{1}{2\pi}\int_{\mathbb{R}}\log\big(1-4p(1-p)\hat{\rho}(k)\big)\,dk = -L_\rho(p,0)\end{aligned}$$

which completes identity (13) for $\kappa_1(p)$. Similarly, when $p \neq \frac{1}{2}$,

$$\int_0^\infty x\left(\sum_{n=1}^{\infty}\frac{(4p(1-p))^n\tilde{\rho}^{*n}(x)}{n}\right)^2 dx = \int_0^\infty x L_\rho^2(p,x)\,dx$$

and

$$\sum_{n=1}^{\infty}\frac{(4p(1-p))^n}{n}\int_{-\infty}^{0} e^{\phi_p x}\rho^{*n}(x)\,dx$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{0}\int_{\mathbb{R}} e^{(\phi_p - ik)x}\log\big(1-4p(1-p)\hat{\rho}(k)\big)\,dk\,dx$$
$$= -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\phi_p}{\phi_p^2+k^2}\log\big(1-4p(1-p)\hat{\rho}(k)\big)\,dk,$$

completing the formula (14) for $\kappa_2(p)$ in the cases $p \neq \frac{1}{2}$.

The case $p = 1/2$ needs some care. Recall that we suppose that ρ has a finite exponential moment. Therefore, $\hat{\rho}(k)$ is analytic in a strip $|k| < 2\mu$, and the small k expansion (135) also holds for $k \in \mathbb{C}$. By (90) the infinite series for $\kappa_2(1/2)$ is absolutely convergent so that

$$\begin{aligned}
\kappa_2\Big(\frac{1}{2}\Big) - \log 2 &= \frac{1}{2}\sum_{n=2}^{\infty}\Bigg(\sum_{k=1}^{n-1}\int_0^\infty x\frac{\rho^{*k}(x)\rho^{*(n-k)}(x)}{k(n-k)}\,dx - \frac{1}{2n}\Bigg) - \frac{1}{4}\\
&= \lim_{\epsilon\downarrow 0}\frac{1}{2}\sum_{n=2}^{\infty}(1-\epsilon)^n\Bigg(\sum_{k=1}^{n-1}\int_0^\infty x\frac{\rho^{*k}(x)\rho^{*(n-k)}(x)}{k(n-k)}\,dx - \frac{1}{2n}\Bigg) - \frac{1}{4}\\
&= \lim_{\epsilon\downarrow 0}\frac{\log\epsilon}{4} + \frac{1}{8\pi^2}\int_0^\infty x\Big(\int_{\mathbb{R}} e^{-ikx}\log\big(1-(1-\epsilon)\hat{\rho}(k)\,dk\big)^2\,dx\\
&= \lim_{\epsilon\downarrow 0}\frac{\log\epsilon}{4} - \frac{(1-\epsilon)^2}{8\pi^2}\int_0^\infty x^{-1}\bigg(\int_{\mathbb{R}} e^{-ikx}\frac{\hat{\rho}'(k)}{1-(1-\epsilon)\hat{\rho}(k)}\,dk\bigg)^2\,dx
\end{aligned}\tag{136}$$

where we have integrated by parts in the dk integral to help understand the divergence in ϵ. Indeed, the function

$$f_\epsilon(x) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-ikx}\frac{\hat{\rho}'(k)}{1-(1-\epsilon)\hat{\rho}(k)}\,dk \tag{137}$$

has an integrand with two poles that approaches the real axis as $\epsilon \downarrow 0$,and this will lead to the cancellation of the term $\frac{1}{4}\log\epsilon$. The asymptotics (135) allow us to fix $\mu > 0$ so that the denominator $1-(1-\epsilon)\hat{\rho}(k)$ has, for small enough ϵ, only two zeros on $|k| \le \mu$, at $\pm r_\epsilon i$, where

$$r_\epsilon = \frac{\sqrt{2\epsilon}}{\sigma} + O(\epsilon). \tag{138}$$

We move the contour defining f_ϵ from the real axis to the curve C_μ, consisting of the segments $(-\infty,-\mu)$, (μ,∞) on the real axis and the half circle $\{-\mu e^{it} : t \in [0,\pi]\}$. This move crosses the the pole at $-r_\epsilon i$ so that, evaluating the residue at $-r_\epsilon i$, we have

$$f_\epsilon(x) = \frac{1}{1-\epsilon}e^{-r_\epsilon x} + \tilde{f}_\epsilon(x), \quad \text{where } \tilde{f}_\epsilon(x) = \frac{1}{2\pi i}\int_{C_\mu} e^{-ikx}\frac{\hat{\rho}'(k)}{1-(1-\epsilon)\hat{\rho}(k)}\,dk. \tag{139}$$

Substituting this into the the expression (136), we find

$$\begin{aligned}
&\kappa_2(1/2) - \log 2\\
&\quad= \lim_{\epsilon\downarrow 0}\frac{\log\epsilon}{4} + \frac{(1-\epsilon)^2}{2}\int_0^\infty x^{-1}f_\epsilon^2(x)\,dx\\
&\quad= \lim_{\epsilon\downarrow 0}\frac{\log\epsilon}{4} - (1-\epsilon)^2\int_0^\infty \log x\, f_\epsilon(x) f_\epsilon'(x)\,dx\\
&\quad= \lim_{\epsilon\downarrow 0}\frac{\log\epsilon}{4} - (1-\epsilon)^2\int_0^\infty \log x\Big(\frac{1}{1-\epsilon}e^{-r_\epsilon x} + \tilde{f}_\epsilon(x)\Big)\Big(\frac{-r_\epsilon}{1-\epsilon}e^{-r_\epsilon x} + \tilde{f}_\epsilon'(x)\Big)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\log\left(\frac{\sigma^2}{8}\right) - \frac{\gamma}{2} \quad \left(\text{recall } \gamma = -\int_0^\infty e^{-x}\log x\,dx\right) \\
&\quad - \lim_{\epsilon\downarrow 0}(1-\epsilon)^2 \int_0^\infty \log x \left(-\tilde{f}_\epsilon(x)\frac{r_\epsilon}{1-\epsilon}e^{-r_\epsilon x} + \left(\frac{1}{1-\epsilon}e^{-r_\epsilon x} + \tilde{f}_\epsilon(x)\right)\tilde{f}'_\epsilon(x)\right)dx \\
&= \frac{1}{4}\log\left(\frac{\sigma^2}{8}\right) - \frac{\gamma}{2} - \int_0^\infty \log x\big(1+\tilde{f}_0(x)\big)\tilde{f}'_0(x)\,dx.
\end{aligned}
$$

To justify passing to the limit in the last equality one can verify (by integrating by parts in the definition of $\tilde{f}_\epsilon$ and $\tilde{f}'_\epsilon$ and noting that $1-\hat{\rho}$ does not vanish on C_μ) that there exists ϵ_0, C so that $|\tilde{f}_\epsilon(x)| \vee |\tilde{f}'_\epsilon(x)| \le C(1+x^2)^{-1}$ for all $x \ge 0$ and all $0 \le \epsilon \le \epsilon_0$.

The last step is to rewrite $\tilde{f}_0(x)$ in terms of L_ρ. We move the contour of integration in $\tilde{f}_0(x) = \frac{1}{2\pi i}\int_{C_\mu} e^{-ikx}\frac{\hat{\rho}'(k)}{1-\hat{\rho}(k)}\,dk$ back to the real line. The integrand $\frac{\hat{\rho}'(k)}{1-\hat{\rho}(k)}$ has a simple at the pole at the origin so that, letting $\mu \downarrow 0$, we get half the residue at the origin and the principle value for the integral around the origin, that is,

$$
\tilde{f}_0(x) = \frac{1}{2\pi i}\int_{C_\mu} e^{-ikx}\frac{\hat{\rho}'(k)}{1-\hat{\rho}(k)}\,dk = 1 + \frac{1}{2\pi i}P.V.\int_{\mathbb{R}} e^{-ikx}\frac{\hat{\rho}'(k)}{1-\hat{\rho}(k)}\,dk.
$$

It is not hard to check that one may integrate by parts to identify

$$
\frac{1}{2\pi i}P.V.\int_{\mathbb{R}} e^{-ikx}\frac{\hat{\rho}'(k)}{1-\hat{\rho}(k)}\,dk = -xL_\rho(1/2, x),
$$

completing the proof.

7.2. *Regularity of $p \to \kappa_i(p)$ for the Gaussian kernel.* We complete the proof of Corollary 7. We recall the expression (32) for $\kappa_1(p)$,

$$
\kappa_1(p) = \frac{1}{4\sqrt{\pi t}}\mathrm{Li}_{3/2}\big(4p(1-p)\big) + \mathbb{I}(p > 1/2)\big(-t^{-1}\log 4p(1-p)\big)^{1/2}. \tag{140}
$$

This shows that $\kappa_1(p)$ is a smooth function of $p \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. To examine the behaviour at $p = \frac{1}{2}$, we use a series representation (Section 9 of [41]) for Li_s, for $s \neq 1, 2, \ldots$,

$$
\mathrm{Li}_s(\beta) = \Gamma(1-s)\big(-\log(\beta)\big)^{s-1} + \sum_{n=0}^\infty \zeta(s-n)\frac{\log^n(\beta)}{n!}, \tag{141}
$$

where ζ is Riemann's zeta function and the infinite series converges for $|\log(\beta)| < 2\pi$. Using this for $s = 3/2$ and taking $p = \frac{1}{2} + \epsilon$ in (140), we reach

$$
t^{1/2}\kappa_1\left(\frac{1}{2}+\epsilon\right) = \frac{1}{2}\sqrt{-\log(1-4\epsilon^2)}\,\mathrm{sgn}(\epsilon) + A(\epsilon),
$$

where A is analytic for $\epsilon \in (-\frac{1}{2}, \frac{1}{2})$ and given by $A(\epsilon) = \frac{1}{4\sqrt{\pi}}\sum_{n=0}^\infty \zeta(\frac{3}{2}-n)\frac{\log^n(1-4\epsilon^2)}{n!}$. Also,

$$
\frac{1}{2}\sqrt{-\log(1-4\epsilon^2)}\,\mathrm{sgn}(\epsilon) = \epsilon\Psi^{1/2}(4\epsilon^2) \quad \text{where } \Psi(z) = -\frac{\log(1-z)}{z}
$$

showing that $\kappa_1(p)$ is analytic for $p \in (0, 1)$.

The infinite series in (33) and (35) for $\kappa_2(p)$, together with their derivatives in p, converge uniformly for p in compacts inside $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. This implies the continuous differentiability of κ_2, except at the point $p = 1/2$. For $p < \frac{1}{2}$, the formula (33) can be rewritten as

$$
\log\frac{1}{1-p} - p(1-p) + \frac{1}{4\pi}\sum_{n=2}^\infty \frac{(4p(1-p))^n}{n}\left(\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}} - \pi\right).
$$

The absolute convergence of the sum over n for all p, using $\sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}} - \pi = O(n^{-1/2})$, implies the left continuity of $\kappa_2(p)$ as $p \uparrow \frac{1}{2}$. A straightforward rearrangement of the terms in (35), constituting $\kappa_2(p)$ for $p > 1/2$, leads to

$$\lim_{p\downarrow\frac{1}{2}} \kappa_2(p) = \kappa_2(1/2) - 2\log 2 - \lim_{\delta\downarrow 0}\left(\log\delta + \sum_{n=1}^{\infty} \frac{1}{n}\operatorname{erfc}(\sqrt{n\delta})\right) \tag{142}$$

(using the complementary error function $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)\,dt = 1 - \operatorname{erf}(x)$). To compute the limit in the right-hand side, we fix $c > 0$ and write

$$\begin{aligned}
&\lim_{\delta\downarrow 0}\left(\log\delta + \sum_{n=1}^{\infty} \frac{1}{n}\operatorname{erfc}(\sqrt{n\delta})\right)\\
&\quad= \lim_{\delta\downarrow 0}\left(\log\delta + \sum_{n\geq c/\delta} \frac{1}{n}\operatorname{erfc}(\sqrt{n\delta}) + \sum_{n<c/\delta}\frac{1}{n}(1-\operatorname{erf}(\sqrt{n\delta}))\right)\\
&\quad= \log c + \gamma + \lim_{\delta\downarrow 0}\left(\sum_{n\geq c/\delta}\frac{1}{n}\operatorname{erfc}(\sqrt{n\delta}) - \sum_{n<c/\delta}\frac{1}{n}\operatorname{erf}(\sqrt{n\delta})\right)\\
&\quad= \log c + \gamma + \int_c^\infty \frac{1}{x}\operatorname{erfc}(\sqrt{x})\,dx - \lim_{\delta\downarrow 0}\mathcal{E}_{c,\delta},
\end{aligned} \tag{143}$$

where we have used $\sum_{n=1}^{N}\frac{1}{n} = \log N + \gamma + O(N^{-1})$ for γ the Euler–Mascheroni constant and we may estimate (using $\operatorname{erf}(x) \leq x$)

$$0 \leq \mathcal{E}_{c,\delta} = \sum_{n<c/\delta}\frac{1}{n}\operatorname{erf}(\sqrt{n\delta}) \leq \sqrt{\delta}\sum_{n<c/\delta} n^{-1/2} \leq 2\sqrt{c}.$$

Integrating by parts,

$$\int_c^\infty \frac{1}{x}\operatorname{erfc}(\sqrt{x})\,dx = \frac{1}{\sqrt{\pi}}\int_c^\infty \frac{\log x}{\sqrt{x}}e^{-x}\,dx - \operatorname{erfc}(\sqrt{c})\log c.$$

Therefore, taking the limit as $c \downarrow 0$ in (143),

$$\lim_{\delta\downarrow 0}\left(\log\delta + \sum_{n=1}^{\infty}\frac{1}{n}\operatorname{erfc}(\sqrt{n\delta})\right) = \gamma + \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\log x}{\sqrt{x}}e^{-x}\,dx = -2\log 2,$$

using the known special value of the digamma function $\psi^{(0)}(\frac{1}{2}) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\log x}{\sqrt{x}}e^{-x}\,dx$. From (142) the right continuity of κ_2 at $1/2$ is proved.

We may directly calculate $\kappa_2'(p)$ for $p \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ from the formulae (33) and (35), leading to (and writing $\beta_p = 4p(1-p)$)

$$\kappa_2'(p) = \begin{cases}
(2p-1) + \dfrac{1}{1-p} + \dfrac{1-2p}{\pi}\left(\dfrac{1}{\beta_p}\mathrm{Li}^2_{\frac{1}{2}}(\beta_p) - \dfrac{\pi\beta_p}{1-\beta_p}\right) & p < \dfrac{1}{2},\\
(2p-1) + \dfrac{1}{1-p} + \dfrac{1-2p}{\pi}\left(\dfrac{1}{\beta_p}\mathrm{Li}^2_{\frac{1}{2}}(\beta_p) - \dfrac{\pi\beta_p}{1-\beta_p}\right) & \\
\quad - \dfrac{1-2p}{\beta_p\sqrt{-\log\beta_p}}\left(\dfrac{1}{\sqrt{\pi}}\mathrm{Li}_{\frac{1}{2}}(\beta_p) - \dfrac{1}{\sqrt{-\log\beta_p}}\right) & p > \dfrac{1}{2}.
\end{cases} \tag{144}$$

Using the series representation (141) for $L_{\frac{1}{2}}(\beta)$ and computing the limits, one finds

$$\lim_{p\downarrow\frac{1}{2}}\kappa_2'(p) = 2 + \frac{2}{\sqrt{\pi}}\zeta(1/2) = \lim_{p\uparrow\frac{1}{2}}\kappa_2'(p)$$

which establishes the continuous differentiability of κ_2 at $\frac{1}{2}$.

7.3. *Proof of error bounds for $p = \frac{1}{2}$ asymptotics.* We give the proofs of the error bound for the asymptotic (90), the error bounds (87), (88), and their analogues needed for the non-translationally invariant case in (114), (115).

We use a local central limit theorem in the form (see Theorem 2 of XVI.2 [15], using the symmetry of ρ to imply the third moment μ_3 is zero)

$$|\rho^{*n}(x) - g_n(x)| \le Cn^{-3/2} \quad \text{for all } n \ge 1,\ x \in \mathbb{R}, \tag{145}$$

for the Gaussian density $g_t(x) = (2\pi\sigma^2 t)^{-1/2} \exp(-x^2/2\sigma^2 t)$. We, therefore, approximate

$$\begin{aligned} \mathrm{Kac}_\rho(n) &= \frac{n}{2}\int_0^\infty x \sum_{k=1}^{n-1} \frac{\rho^{*k}(x)\rho^{*(n-k)}(x)}{k(n-k)}\,dx \\ &= \frac{n}{2}\int_0^\infty x \sum_{k=1}^{n-1} \frac{g_k(x)g_{n-k}(x)}{k(n-k)}\,dx + \mathcal{E}_n \\ &= \frac{1}{4\pi}\sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}} + \mathcal{E}_n. \end{aligned} \tag{146}$$

The error $\mathcal{E}_n$ is bounded by

$$\mathcal{E}_n \le n \sum_{1\le k\le n/2} \int_0^\infty \frac{x}{k(n-k)} |\rho^{*k}(x)\rho^{*(n-k)}(x) - g_k(x)g_{n-k}(x)|\,dx.$$

Letting C depend on σ and vary from line to line, we bound the sum over $k \le n^{1/2}$, using $\rho^{(n-k)*}(x) \le Cn^{-1/2}$ and $g_{n-k}(x) \le Cn^{-1/2}$, by

$$Cn^{1/2} \sum_{1\le k\le n^{1/2}} \int_0^\infty \frac{x}{k(n-k)} (\rho^{*k}(x) + g_k(x))\,dx \le Cn^{-1/2} \sum_{1\le k\le n^{1/2}} \frac{1}{k^{1/2}} = O(n^{-1/4}),$$

and the sum over $n^{1/2} < k \le n/2$, using (145), by

$$\begin{aligned} &Cn \sum_{n^{1/2}<k\le n/2} \int_0^\infty \frac{x}{k(n-k)} (\rho^{(n-k)*}|\rho^{*k} - g_k| + g_k|\rho^{*(n-k)} - g_{n-k}|)(x)\,dx \\ &\quad\le Cn \sum_{n^{1/2}<k\le n/2} \int_0^\infty \frac{x}{k(n-k)} (\rho^{(n-k)*}(x)k^{-3/2} + g_k(x)n^{-3/2})\,dx \\ &\quad\le Cn \sum_{n^{1/2}<k\le n/2} \left(\frac{1}{n^{1/2}k^{5/2}} + \frac{1}{n^{5/2}k^{1/2}}\right) = O(n^{-1/4}). \end{aligned}$$

The sum in (146) is a Riemann approximation to the Beta integral $(4\pi)^{-1}B(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$ with a further error that is $O(n^{-1/2})$. This completes the asymptotic (90).

For the error term bound (13) we start with

$$\begin{aligned} E^{(1)}(n, L) &= p(n, L) - \frac{1}{n}\mathbb{E}_0[\delta_0(S_n)(M_n - m_n)] \\ &= \frac{1}{n}\mathbb{E}_0[\delta_0(S_n)\min\{L, M_n - m_n\}] - \frac{1}{n}\mathbb{E}_0[\delta_0(S_n)(M_n - m_n)] \\ &= -\frac{1}{n}\mathbb{E}_0[\delta_0(S_n)((M_n - m_n) - L)_+] \end{aligned}$$

so that, using $(a+b)_+ \le a_+ + b_+$,

$$
\begin{aligned}
\left|E^{(1)}(n,2L)\right| &\le \frac{1}{n}\mathbb{E}_0\big[\delta_0(S_n)(M_n-L)_+\big] + \frac{1}{n}\mathbb{E}_0\big[\delta_0(S_n)(-m_n-L)_+\big] \\
&= \frac{2}{n}\mathbb{E}_0\big[\delta_0(S_n)(M_n-L)_+\big] \\
&= \frac{2}{n}\mathbb{E}_0\big[\delta_0(S_n)(M_{n-1}-L)_+\big] \\
&\le \frac{2\|\rho\|_\infty}{n}\mathbb{E}_0\big[(M_{n-1}-L)_+\big] \\
&\le \frac{2\|\rho\|_\infty}{n}\mathbb{E}_0\big[(M_n-L)_+\big],
\end{aligned}
$$

where in the penultimate step we bounded the density of the single step $S_n - S_{n-1}$. Then,

$$
\begin{aligned}
\mathbb{E}_0\big[(M_n-L)_+\big] &\le CL^{-3}\mathbb{E}_0\big[|M_n|^4\big] \\
&\le CL^{-3}\mathbb{E}_0\big[|S_n|^4\big] \quad \text{by Doob's inequality,} \\
&\le CL^{-3}n^2 \quad \text{by a Marcinkiewicz-Zygmund inequality,}
\end{aligned}
$$

which completes the proof of (87). The changes needed for $\tilde{E}^{(1)}(n,L)$ in the nontranslationally invariant case are minor: the only new term that arises is

$$
\mathbb{E}_0\big[\delta_0(S_n)(\tilde{M}_n-L)_+\big] \le \|\rho\|_\infty \mathbb{E}_0\big[(\tilde{M}_n-L)_+\big]
$$

(by again averaging over the final step $S_n - \tilde{M}_n = \mathcal{Y}_n$). It is not difficult to check that the bound $\mathbb{E}_0[(\tilde{M}_n-L)_+] \le CL^{-3}n^2$ still holds.

For the second error term $E^{(2)}(n,L)$, we use a Skorokhod embedding of the walk into a Brownian motion $(W(t) : t \ge 0)$ run at speed σ^2. We choose stopping times $(T_1, T_2, \dots)$ so that $(S_1, S_2, \dots) \stackrel{\mathcal{D}}{=} (W(T_1), W(T_2), \dots)$ and so that $T_1, (T_2 - T_1), (T_3 - T_2), \dots$ are an i.i.d. set of nonnegative variables with $\mathbb{E}[T_k] = k$ and $\mathbb{E}[(T_k - T_{k-1})^2] \le 4\mathbb{E}[X_1^4]/\sigma^4 < \infty$. Let $\hat{n} = \lceil n - n^\alpha \rceil$ for some $\alpha \in (\frac{1}{2}, 1)$, and let

$$
\Omega_n = \Big\{\max_{k \le \hat{n}} |T_k - k| \le n^\beta\Big\}
$$

for some $\beta \in (\frac{1}{2}, \alpha)$. Since $(T_k - k)$ is a square integrable martingale, Doob's inequality implies that $P[\Omega_n^c] = O(n^{1-2\beta})$. Then, we make the following approximations:

$$
\begin{aligned}
&\mathbb{E}_0\big[\delta_0(S_n)\min\{L, M_n - m_n\}\big] \\
&\quad = \mathbb{E}_0\big[\delta_0(S_n)\min\{L, M_{\hat{n}} - m_{\hat{n}}\}\big] + \mathcal{E}_1 \\
&\quad = \mathbb{E}_0\big[\delta_0(S_n)\min\{L, M_{\hat{n}} - m_{\hat{n}}\}; \Omega_n\big] + \mathcal{E}_2 \\
&\quad = \mathbb{E}_0\big[\delta_0\big(W(n)\big)\min\{L, M_{\hat{n}} - m_{\hat{n}}\}; \Omega_n\big] + \mathcal{E}_3 \qquad (147) \\
&\quad = \mathbb{E}_0\big[\delta_0\big(W(n)\big)\min\big\{L, W^*(\hat{n}) - W_*(\hat{n})\big\}; \Omega_n\big] + \mathcal{E}_4 \\
&\quad = \mathbb{E}_0\big[\delta_0\big(W(n)\big)\min\big\{L, W^*(\hat{n}) - W_*(\hat{n})\big\}\big] + \mathcal{E}_5 \\
&\quad = \mathbb{E}_0\big[\delta_0\big(W(n)\big)\min\big\{L, W^*(n) - W_*(n)\big\}\big] + \mathcal{E}_6.
\end{aligned}
$$

Thus, we aim to estimate $|E^{(2)}(n,L)| = \frac{1}{n}|\mathcal{E}_6|$. The reason for this slightly messy set of approximations is in order to be able to use the local central limit theorem to estimate the expectation of the delta functions $\delta_0(S_n), \delta_0(W(n))$ by conditioning at an earlier time.

Using first $|\min\{L,x\}-\min\{L,y\}|\le |x-y|$, the symmetry of ρ, and then the simple inequality $|\max\{x,y\}-x|\le y$ for $x,y\ge 0$, we have

$$\begin{aligned}|\mathcal{E}_1| &\le 2\mathbb{E}_0\big[\delta_0(S_n)|M_n-M_{\hat n}|\big]\\ &\le 2\mathbb{E}_0\big[\delta_0(S_n)\max\{S_k:\hat n<k\le n\}\big]\\ &= 2\mathbb{E}_0\big[\delta_0(S_n)\max\{S_k:k\le n-\hat n\}\big]\\ &\le 2\mathbb{E}_0[M_{n-\hat n}]\|\rho^{\hat n *}\|_{\infty},\end{aligned}$$

where we have used time reversal and symmetry of the increments in the equality above and then conditioned at time $n-\hat n$ in the final step. By Doob's inequality $\mathbb{E}_0[M_n]\le Cn^{1/2}$; by (145) we have $|\rho^{*n}(x)|\le Cn^{-1/2}$; using $n-\hat n = O(n^\alpha)$, we conclude that $\mathcal{E}_1 = O(n^{(\alpha-1)/2})$. The bound $|\mathcal{E}_6-\mathcal{E}_5| = O(n^{(\alpha-1)/2})$ is similar.

Next, also by similar steps,

$$\begin{aligned}|\mathcal{E}_2-\mathcal{E}_1| &\le 2\mathbb{E}_0\big[\delta_0(S_n)M_{\hat n}\mathbb{I}(\Omega_n^c)\big]\\ &\le 2\mathbb{E}_0\big[M_{\hat n}\mathbb{I}(\Omega_n^c)\big]\|\rho^{(n-\hat n)*}\|_\infty\\ &\le 2\big(\mathbb{E}_0[M_{\hat n}^2]\big)^{1/2}\big(\mathbb{P}[\Omega_n^c]\big)^{1/2}\|\rho^{(n-\hat n)*}\|_\infty = O\big(n^{1-\beta-\frac{\alpha}{2}}\big).\end{aligned}$$

The bound $|\mathcal{E}_5-\mathcal{E}_4| = O(n^{1-\beta-\frac{\alpha}{2}})$ is similar.

Next, recall that we have embedded $M_{\hat n} = \max_{k\le\hat n} W(T_k)$. Conditioning on $\mathcal{F}^W_{T_{\hat n}}$, we see

$$\begin{aligned}|\mathcal{E}_3-\mathcal{E}_2| &= \big|\mathbb{E}_0\big[\min\{L,M_{\hat n}-m_{\hat n}\}\mathbb{I}(\Omega_n)\big(\delta_0\big(W(n)\big)-\delta_0(S_n)\big)\big]\big|\\ &= \big|\mathbb{E}_0\big[\min\{L,M_{\hat n}-m_{\hat n}\}\mathbb{I}(\Omega_n)\big(g_{n-T_{\hat n}}(S_{\hat n})-\rho^{(n-\hat n)*}(S_{\hat n})\big)\big]\big|\\ &\le 2\mathbb{E}_0\big[M_{\hat n}\mathbb{I}(\Omega_n)\|g_{n-T_{\hat n}}-\rho^{(n-\hat n)*}\|_\infty\big]\\ &\le 2\mathbb{E}_0\big[M_{\hat n}\mathbb{I}(\Omega_n)\big(\|g_{n-T_{\hat n}}-g_{n-\hat n}\|_\infty+\|g_{n-\hat n}-\rho^{(n-\hat n)*}\|_\infty\big)\big].\end{aligned}$$

On the set Ω_n, we have $|\hat n - T_{\hat n}|\le n^\beta$ and then $\|g_{n-T_{\hat n}}-g_{n-\hat n}\|_\infty \le Cn^{\beta-\frac{3}{2}\alpha}$. Combined with (145), we find $|\mathcal{E}_3-\mathcal{E}_2| = O(n^{\frac12+\beta-\frac32\alpha})$.

Finally, we use the modulus of continuity for a Brownian motion showing, for $\epsilon>0$, there is a variable H_ϵ with finite moments so that $|W(t)-W(s)|\le H_\epsilon n^\epsilon|t-s|^{\frac12-\epsilon}$ for all $0\le s,t\le n$, almost surely. The last errror is

$$\begin{aligned}|\mathcal{E}_4-\mathcal{E}_3| &\le 2\mathbb{E}_0\Big[\delta_0\big(W(n)\big)\Big|\max_{k\le\hat n}W(T_k)-W^*(\hat n)\Big|;\Omega_n\Big]\\ &\le Cn^{-\frac{\alpha}{2}}\mathbb{E}_0\Big[\Big|\max_{k\le\hat n}W(T_k)-W^*(\hat n)\Big|;\Omega_n\Big] \qquad (148)\\ &\le Cn^{-\frac{\alpha}{2}}\mathbb{E}_0\Big[\Big|\max_{k\le\hat n}W(T_k)-W^*(T_{\hat n})\Big|+\big|W^*(T_{\hat n})-W^*(\hat n)\big|;\Omega_n\Big].\end{aligned}$$

On the set Ω_n, using the modulus of continuity, we have $|W^*(T_{\hat n})-W^*(\hat n)|\le H_\epsilon n^{(\frac12-\epsilon)\beta+\epsilon}$. Also,

$$\begin{aligned}\Big|\max_{k\le\hat n}W(T_k)-W^*(T_{\hat n})\Big| &\le H_\epsilon n^\epsilon\max_{k<\hat n}|T_k-T_{k+1}|^{\frac12-\epsilon}\\ &\le H_\epsilon n^\epsilon\Big(1+2\max_{k\le\hat n}|T_k-k|\Big)^{\frac12-\epsilon}\\ &\le CH_\epsilon n^{(\frac12-\epsilon)\beta+\epsilon}\end{aligned}$$

(by the triangle inequality $|T_k - T_{k+1}| \le |T_k - k| + 1 + |(k+1) - T_{k+1}|$). Using these estimates in (148), we reach $|\mathcal{E}_4 - \mathcal{E}_3| = O(n^{-\frac{\alpha}{2}+(\frac{1}{2}-\epsilon)\beta+\epsilon})$. Collecting all error estimates and choosing $\alpha = \frac{5}{6}$ and $\beta = \frac{2}{3}$ leads to an overall error $|\mathcal{E}_6| = O(n^{-\frac{1}{12}+\epsilon})$, completing the proof of (88).

The changes needed for $|\tilde{E}^{(2)}(n, L)|$ in the nontranslationally invariant case are again small. We leave the chain of approximations (147) exactly as before, except that it starts with the expectation $\mathbb{E}_0[\delta_0(S_n)\min\{L, \tilde{M}_n - m_n\}]$. This implies that we only need to reestimate the error in the first step which requires a bound on the new term

$$\mathbb{E}_0\big[\delta_0(S_n)|\tilde{M}_n - M_{\hat{n}}|\}\big] \le \mathbb{E}_0\big[\delta_0(S_n)|\tilde{M}_n - \tilde{M}_{\hat{n}}|\}\big] + \mathbb{E}_0\big[\delta_0(S_n)|\tilde{M}_{\hat{n}} - M_{\hat{n}}|\}\big]. \tag{149}$$

The second term in (149) is estimated as $O(n^{(\alpha-1)/2})$, using the local central limit theorem to bound the density of $S_n - S_{\hat{n}}$ as before. For the first term we use time reversal again,

$$\begin{aligned}\mathbb{E}_0\big[\delta_0(S_n)|\tilde{M}_n - \tilde{M}_{\hat{n}}|\}\big] &\le \mathbb{E}_0\big[\delta_0(S_n)\max\{\tilde{S}_k : \hat{n} < k \le n\}\big] \\ &= \mathbb{E}_0\big[\delta_0(S_n)\max\{\tilde{S}_k : 1 \le k \le n - \hat{n}\}\big] \\ &\le Cn^{-1/2}\mathbb{E}_0\big[\max\{\tilde{S}_k : 1 \le k \le n - \hat{n}\}\big] = O\big(n^{(\alpha-1)/2}\big),\end{aligned}$$

where in the final inequality we have estimated the density $\tilde{\rho}^{\hat{n}*} * \rho$ of $(S_n - \tilde{S}_{n-\hat{n}})$, again by the local central limit theorem.

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