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Balanced Circular Packing Problems with Distance Constraints

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Abstract: The packing of different circles in a circular container under balancing and distance conditions is considered. Two problems are studied: the first minimizes the container's radius, while the second maximizes the minimal distance between circles, as well as between circles and the boundary of the container. Mathematical models and solution strategies are provided and illustrated with computational results.

Keywords: dense circular packing; sparse circular packing; balance; mathematical models; nonlinear optimization; Shor's r-algorithm

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1. Introduction

The circular packing problem (CPP) consists of placing several circles in a larger object called a container. The circles must be arranged without mutual overlapping (the non-overlapping condition) and completely inside the container (the containment condition). CPPs (including CPPs in non-Euclidean geometry) form a broad category at the boundary between computational geometry and combinatorics [1–3]. The optimized CPP aims to find "the best" layout, e.g., maximizing the space utilization of the container or minimizing the size of the container. Optimized CPPs have multiple applications in logistics and production planning, biology and medicine, chemistry and additive manufacturing [4–6].

Various types of CPP can differ according to circle sizes (different or equal) [4,7–9] and container shapes (circles or polyhedrons) [10–12]. In open-dimension CPPs, the shape of the container is typically fixed, while the size is optimized [13,14]. The problem of packing circles into containers with prohibited zones is considered in [15–18]. Prohibited zones typically result in non-convex, multiconnected and/or disconnected containers.

Additional constraints motivated by applications can be introduced. For example, in a satellite module layout design, equilibrium/balance constraints are imposed to assure the proximity of the gravity center of the container to the corresponding center of the circles [19–23]. In many applications, CPPs are formulated for circles defined in non-Euclidean norms [17,24,25]. Nesting, i.e., the stacking of circular disks one over another, can also be considered [26–30].

Solution approaches to CPPs can be roughly divided into four large classes [4,6,31]. The techniques of the first group are based on nonlinear programming formulations of the CPP, and these approaches apply different exact or approximation methods for this class of optimization problems [5,11,12,15,20,21]. The second class of approaches is based on tessellating or approximating the container with a finite grid [24,26,32–34]. Then, the CPP is approximately reduced to a large-scale discrete programming problem in order to assign the centers of the circles to grid nodes. Using metaheuristic techniques for CPPs forms the third class of approaches [35,36]. The last group is represented by hybrid approaches combining the algorithmic ideas of the previous groups [7,8,10,13,14,19,28,31]. A collection of benchmark problem instances and corresponding best-known solutions is presented on E. Specht's website [37], specifically for the problem of packing circles into various containers.

The objective in most optimized packing problems is to find the densest possible packing arrangement, e.g., maximizing the occupation of the container or finding the minimal container for an open-dimension problem. However, in many up-to-date applications, the dense packing concept must be revised.

In sparse packing [38], the distance between the objects, as well as between the objects and the container, must be sufficiently large. In container-loading applications, these conditions arise to facilitate access to the objects during loading/unloading. Generating lightweight void structures by 3D printing requires sufficient distance between the voids to assure the mechanical strength of the part [39]. Sparse packing arises in thermal deburring, which is a state-of-the-art technology for cleaning 3D-printed parts of non-sintered powder particles [38]. The burrs are removed from 3D objects by exploding gas mixtures in a closed chamber. To achieve a reasonable processing quality and a "uniform" distribution of power and thermal effects, the parts must be placed at sufficient distances from one another.

In this work, the packing of different circles in a circular container under balancing and distance conditions is considered. Dense and sparse formulations are studied. The first problem aims to minimize the container's radius. In the second, the minimal distance between circles, and between the circles and the container's boundary, is maximized.

2. Dense Packing Circular Problem with Distance and Balancing Conditions

In this section, a circular packing problem is studied under balancing and distance conditions. The objective is to obtain the smallest possible circular container (dense packing).

Let S_i, $i = 1, 2, \ldots, n$ be a family of circles with radii r_i and weights w_i. Let S be an external circle (container) centered at the origin and having radius r. By the definition, the center of gravity of the circle S is also located at the origin. Denote by (x_i, y_i) the unknown center coordinates of S_i, and let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

Let $\rho_{ij} > 0$ be the minimum permitted distance between S_i and S_j, $1 \le i < j \le n$. In addition, let $\rho_i > 0$ be the minimum permitted distance between S_i and the boundary of the object $S^* = R^2 \backslash intS$, $i = 1, \ldots, n$. More specifically, let

$$\begin{array}{c} dist(S_i(v_i), S_j(v_j)) \ge \rho_{ij},\ 1 \le i < j \le n, \\ dist(S_i(v_i), S^*) \ge \rho_i,\ i = 1, \ldots, n \end{array} \quad (1)$$

where $dist(A, B) = \min\limits_{a \in A, b \in B} dist(a, b)$ and $dist(a, b)$ refer to the (Euclidean) distance between two points a and b.

A dense circular balanced packing (DCBP) problem can be stated as follows. Pack the circles S_i, $i = 1, 2, \ldots, n$ in a minimum-radius circular container S subject to the distance constraints in (1), such that the gravitation center of the family of circles S_i, $i = 1, 2, \ldots, n$ is located at the origin (i.e., coincides with the gravity center of S).

A trivial lower bound on the minimal radius of S is

$$r_{low} = \max_{i=1,\ldots,n} r_i + \rho, \rho = \max\{\max_{i=1,\ldots,n} \rho_i, 0.5 \max_{1 \le i < j \le n} \rho_{ij}\}$$

The DCBP is stated as a nonlinear optimization problem in R^{2n+1}:

$$r^* = \min_{r,x,y} r \tag{2}$$

subject to

$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq (r_i + r_j + \rho_{ij})^2, \quad 1 \leq i < j \leq n, \tag{3}$$

$$x_i^2 + y_i^2 \leq (r - r_i - \rho_i)^2, \quad i = 1, \ldots, n, \tag{4}$$

$$\sum_{i=1}^{n} \lambda_i x_i = 0, \quad \sum_{i=1}^{n} \lambda_i y_i = 0, \tag{5}$$

$$r \geq r_{low} \tag{6}$$

where $\lambda_i = w_i / \sum_{j=1}^{n} w_j$.

The constraints in (3) guarantee that the distance between the pair of circles S_i and S_j is at least ρ_{ij}, $1 \leq i < j \leq n$, while the constraints in (4) assure that the distance between S_i and the boundary of S is at least ρ_i $i = 1, 2, \ldots, n$. The linear constraints in (5) state that the gravity center of the family S_i, $i = 1, 2, \ldots, n$ coincides with the origin, while the "dummy" constraint in (6) provides the lower bound for the objective value.

2.1. Sequential Algorithm for DCBPs

Using penalty functions, the problem in (1)–(5) can be transformed into the unconstrained minimization of a non-smooth function

$$\min_{r,x,y} \{ f(r,x,y) = r + \Phi_P(r,x,y) \} \tag{7}$$

where the penalty function $\Phi_P(r,x,y)$ has the form

$$\Phi_P(r,x,y) = P_1 F_1(r,x,y) + P_2 F_2(x,y) + P_3 \max\{0, -r + r_{low}\} \tag{8}$$

Here $P = \{P_1, P_2, P_3\}$, where the penalization coefficients P_k, $k = 1, 2, 3$ are positive, and the functions $F_1(r,x,y)$ and $F_2(x,y)$ are as follows:

$$\begin{aligned} F_1(r,x,y) = & \sum_{i=1}^{n} \max\{0, x_i^2 + y_i^2 - (r - r_i - \rho_i)^2\} \\ & + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \max\{0, -(x_i - x_j)^2 - (y_i - y_j)^2 + (r_i + r_j + \rho_{ij})^2\} \end{aligned} \tag{9}$$

$$F_2(x,y) = \max\{0, \sum_{i=1}^{n} \lambda_i x_i, -\sum_{i=1}^{n} \lambda_i x_i\} + \max\{0, \sum_{i=1}^{n} \lambda_i y_i, -\sum_{i=1}^{n} \lambda_i y_i\}, \tag{10}$$

The local minimization in (2)–(6) is equivalent to the local minimization in the unconstrained non-smooth problem in (7)–(10). If $\Phi_P(r,x,y) = 0$ in the local minimum of the function $f(r,x,y)$ for certain values of the coefficients $P = \{P_1, P_2, P_3\}$, then this point is also a local minimum for the problem in (2)–(6). The choice of the penalty coefficients P_1, P_2 and P_3 allows the adjustment of the violation of the constraints in (3)–(6). The coefficient P_1 corresponds to the "total violation" of the quadratic constraints in (3)–(4), P_2 "controls" the linear constraints in (5), while P_3 adjusts the constraint in (6).

The sequential algorithm to obtain the best local solution to the problem in (2)–(6) uses the multistart method with randomly selected starting points and is based on a variant of Shor's r-algorithm (see, e.g., [40–43] and Appendix A) to locally minimize $f(r,x,y)$. The feasible starting points for the problem in (2)–(6) are not necessary. This algorithm can also be used for the particular case of (2)–(6), where constraints on the center of gravity of the system of circles are eliminated. It is sufficient to set $P_2 = 0$; that it is equivalent to excluding the constraints in (4) from the problem in (2)–(6).

To present the sequential algorithm, denote by ntest the number of starting points uniformly distributed inside the circular container of radius r_{up} (the best upper bound for the container's radius). At the beginning of the process, $r_{up} = \sum_{i=1}^{n}(r_i + \rho)$, $\rho = max\{\underset{i=1,\dots,n}{max}\rho_i, 0.5\underset{1\le i<j\le n}{max}\rho_{ij}\}$. If the value of the local minimum decreases, then r_{up} is sequentially updated for all starting points. To obtain the local minimum of $f(r,x,y)$, the r-algorithm with an adaptive step-size and fixed space dilation [41] is used. The best local minimum of the function $f(r,x,y)$, corresponding to the penalty $\Phi_P(r,x,y)$ close to zero, is considered a solution to the problem in (2)–(6). This way, the best value r_{up} of the radius is obtained, together with the coordinates (x^{up}, y^{up}) of the circles.

2.2. Sequential Implementation of a Parallel Algorithm for DCBP

In a parallel algorithm, multiple searches for local solutions are implemented using the modification of the r-algorithm. This differs from a sequential approach that starts only one search. The parallel algorithm is implemented using the $(k+1)$-processor "master–slave" technique. One processor is selected as the "main" (Master), while the others are "subordinate" (Slave).

In the Master processor, k starting points are randomly generated in a circle with a radius $r_{up} = \sum_{i=1}^{n}(r_i + \rho)$, $\rho = max\{\underset{i=1,\dots,n}{max}\rho_i, 0.5\underset{1\le i<j\le n}{max}\rho_{ij}\}$. Then, these points are sent to the Slave processors. Each Slave processor starts the local minimization of the function $f(r,x,y)$ from its starting point. When the r-algorithm terminates on any Slave processor, the results are transferred to the Master processor. As soon as the local minimum of the problem in (2)–(6) is obtained, the radius of the container must be compared with the best value r_{up}. If the radius is smaller than r_{up}, the value of r_{up} is updated correspondingly, and the values (x^{up}, y^{up}) of the coordinates of the centers of the circles are saved. Then, the Master processor selects a new starting point, which is passed to the Slave processor on which the r-algorithm just finished to begin the new search for the local minimum. The process stops either when all starting points have been investigated or the computational time limit has been exceeded.

The parallel algorithm is implemented using the C++ programming language and the MPI parallel programming environment. The standard rand function is used to generate pseudo-random numbers. For local minimization, the module ralgb5 is used, corresponding to the Octave code [44] of the modification of the r-algorithm presented in [41]. The program is coded to work for a cluster in an MPI environment and on a Linux operating system. It can be executed on a single processor or on several processors in parallel.

Here, the cluster SKIT-4 of the V.M. Glushkov Institute of Cybernetics [45] is used. The speedup of the parallel algorithm for the DCBP problem is studied for an instance with 50 circles: 20 circles of radius 10, 15 circles of radius 20, 10 circles of radius 30 and 5 circles of radius 40. It is assumed that $w_i = r_i$, $i = 1, 2, \dots, n$. The multistart strategy is implemented with 5000 starting points, and the number of processors ranges from 1 to 32. The results are presented in Table 1. Here, k denotes the number of processors, t_k is the problem solution time in seconds, C_k is the parallel algorithm speedup and E_k is the parallel algorithm efficiency. The parallel speedup is calculated by the formula $C_k = t_0/t_k$, $k = 1, 4, 8, 12, 16, 24, 32$, where t_0 is the solution time for the serial algorithm with one processor, and t_k is the solution time for the parallel algorithm with k processors. The effect of parallelization is estimated as $E_k = C_k/k$. It follows from Table 1 that for 4 processors the solution time is reduced by 2.95 times, for 8–5.78 times, for 16–13.43 times, for 24–14.31 times and for 32–15 times.

Figure 1 presents the solution time compared to the number of processors used, Figure 2 shows the speedup diagram and Figure 3 provides the efficiency diagram of the parallel algorithm for the problem instance. We may conclude that, once there are 16 processors in use, further increasing the number of processors does not result in a significant impact on the solution time. This is due to the rising cost of information exchange between parallel processors.

Table 1. Solution time, speedup and efficiency for a DCBP problem instance with 50 circles solved on the cluster SKIT-4.

k	t_k	C_k	E_k
1	2159.68	1.00	1.00
4	731.08	2.95	0.74
8	373.85	5.78	0.72
12	209.04	10.33	0.86
16	160.79	13.43	0.84
24	150.93	14.31	0.60
32	144.02	15.00	0.47

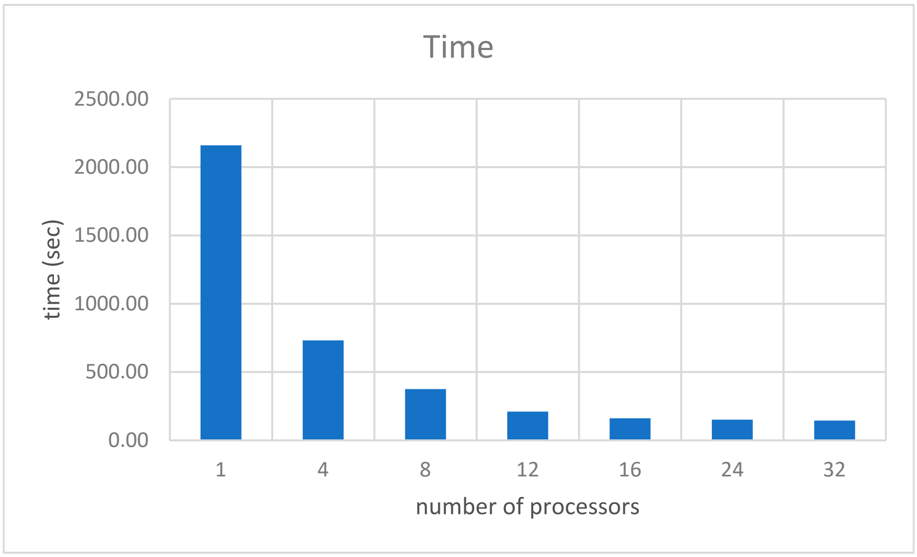


Figure 1. Diagram of solution time/number of processors.

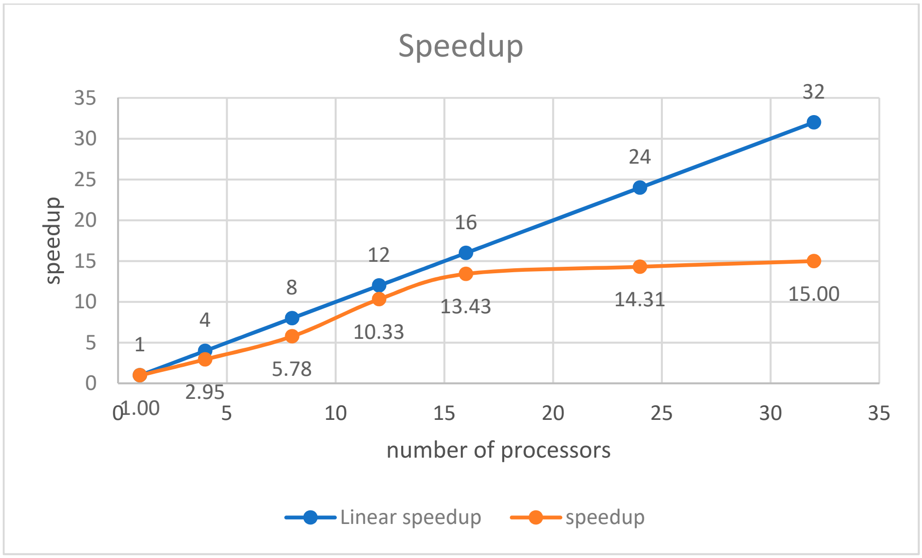


Figure 2. Speedup diagram.

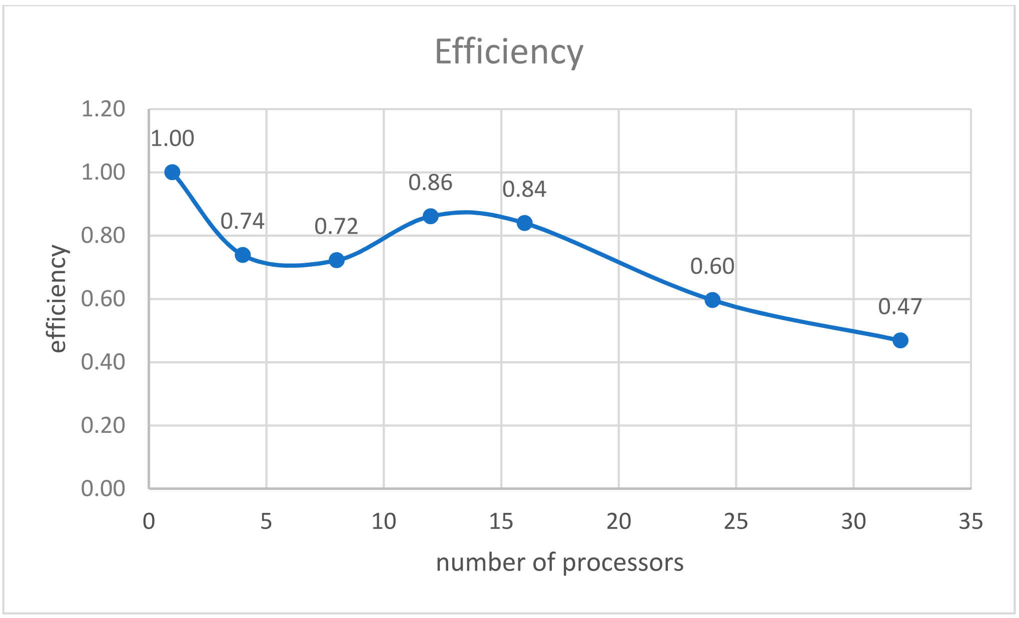


Figure 3. Efficiency diagram.

For a problem instance with 100 circles (40 circles with radius 10, 30 with radius 20, 20 with radius 30 and 10 with radius 40) the sequential algorithm results in $r_{up} = \mathbf{257.35311}$. For the same instance, the parallel algorithm with eight processors and 100,000 starting points gives an improved value $r_{up} = \mathbf{257.1951}$ in 64,649 s (approximately 17 h 58 min). Sequential and parallel solutions are presented in Figures 4 and 5, respectively, while Tables 2 and 3 provide corresponding center coordinates. As can be seen from Figures 4 and 5, the layout for the circles differs significantly.

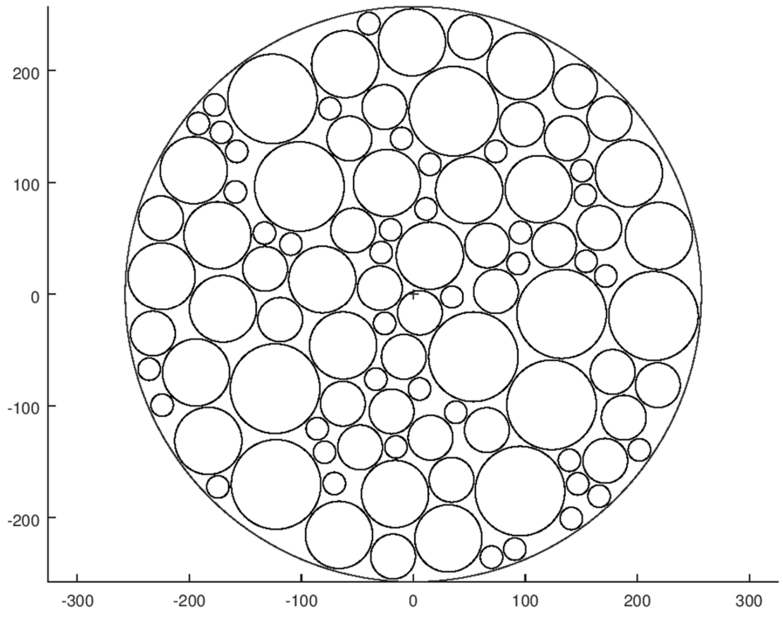


Figure 4. Placement of 100 circles for the sequential solution, $r_{up} = \mathbf{257.35311}$.

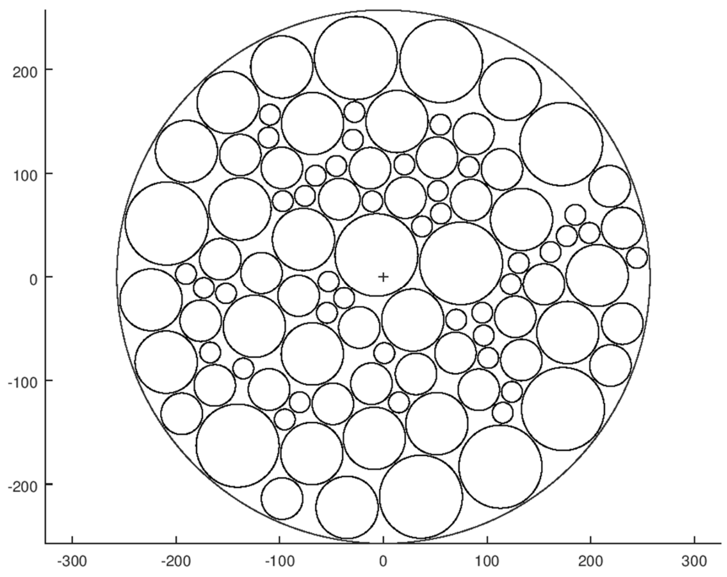


Figure 5. Placement of 100 circles for the parallel solution, r_{up} = **257.1951**.

Table 2. Coordinates of the placement of 100 circles for the sequential solution, r_{up} = **257.35311**.

i	x_i^{up}	y_i^{up}	i	x_i^{up}	y_i^{up}	i	x_i^{up}	y_i^{up}
1	139.4106	−148.7203	35	−192.0113	152.7411	69	−57.0781	139.3253
2	165.9300	−180.7356	36	−20.2820	57.6283	70	65.7095	−121.7164
3	−10.4564	139.3668	37	−33.4277	−76.2563	71	−60.8956	205.8394
4	90.6188	−228.0052	38	−25.6351	−26.4029	72	−66.4087	−215.3460
5	−85.8170	−120.5509	39	153.5649	88.7727	73	95.7131	204.0172
6	73.5905	127.7710	40	14.8656	116.2411	74	−174.8863	52.7062
7	−157.6668	127.7678	41	5.8499	−16.7581	75	31.2989	−218.4935
8	−177.5573	169.3267	42	65.9473	43.6220	76	−23.6507	99.4930
9	−70.5784	−169.3974	43	−25.8560	167.4177	77	−224.7904	15.9151
10	95.7456	55.2866	44	50.5852	229.8526	78	−81.3118	12.9985
11	−109.3281	44.9178	45	175.1576	157.1970	79	−170.0251	−13.1497
12	150.5197	110.5609	46	125.6546	43.9088	80	219.2165	52.2319
13	−74.5636	166.1256	47	178.0610	−69.8027	81	−16.3268	−178.7973
14	−171.4504	144.9147	48	−29.8828	5.3140	82	−196.1507	110.9455
15	−15.5382	−136.8047	49	137.0313	139.5792	83	−62.7066	−46.1440
16	171.8182	16.2934	50	−18.1262	−234.6541	84	14.7585	34.4732
17	−39.7282	242.1153	51	−53.7898	57.1181	85	−1.2195	225.3498
18	141.1580	−200.6803	52	−47.5357	−137.2039	86	111.8985	94.0562
19	−158.9000	91.5448	53	187.9781	−110.6151	87	49.9061	93.0860
20	201.8791	−139.4381	54	34.1020	−166.1113	88	−194.0441	−70.3082
21	37.9330	−105.8272	55	165.9846	59.2811	89	192.4508	108.1568
22	−28.6723	37.2911	56	96.9196	152.0312	90	−183.1233	−131.3388
23	−235.9375	−67.3175	57	15.3707	−128.5195	91	95.2733	−176.2139
24	−132.9352	54.7309	58	171.4069	−149.2078	92	−122.5708	−170.2925
25	93.6021	27.5218	59	−62.9718	−98.1434	93	214.4583	−19.6107
26	146.8699	−169.7488	60	218.3642	−81.6204	94	132.4803	−17.7144
27	11.3898	76.3379	61	−225.3420	67.9122	95	−101.7842	96.3677
28	34.5368	−2.5784	62	73.8224	2.3670	96	53.6401	−56.2562
29	154.2352	29.5161	63	144.4278	185.8271	97	−123.5010	−84.7191
30	−224.2984	−99.4403	64	−8.4552	−56.2468	98	−125.8419	174.7592
31	69.8306	−235.2059	65	−232.6625	−35.4856	99	123.4839	−99.2194
32	−174.5260	−172.4495	66	−132.3921	22.7355	100	36.0347	163.7371
33	5.5154	−85.0361	67	−118.9449	−22.8867			
34	−79.2399	−141.5447	68	−19.4571	−105.0456			

Table 3. Coordinates of the placement of 100 circles for the parallel solution, $r_{up} = \mathbf{257.1951}$.

i	x_i^{up}	y_i^{up}	i	x_i^{up}	y_i^{up}	i	x_i^{up}	y_i^{up}
1	−45.4108	106.9596	35	122.9228	−7.1056	69	84.7426	74.1280
2	70.2790	−41.6277	36	−80.7209	−121.0597	70	127.1753	−38.8218
3	82.4623	106.0466	37	−75.5893	78.1391	71	13.1500	149.8033
4	52.4820	82.9348	38	185.2119	59.8396	72	−224.0971	−22.2112
5	95.4186	−34.8836	39	−65.3203	97.5992	73	−138.3141	65.5081
6	55.4959	61.1422	40	−27.8345	158.9831	74	−35.0326	−222.4535
7	−10.6496	72.8616	41	−110.2915	−108.8296	75	−98.0049	202.3002
8	−37.8368	−20.6103	42	21.1518	76.4216	76	−8.7590	−155.9050
9	−135.0830	−88.5964	43	−12.8746	104.7842	77	−77.0574	36.1647
10	−52.8890	−4.5655	44	−81.7715	−18.3424	78	28.2801	−41.9345
11	−95.1176	−137.6949	45	154.9222	−7.2923	79	−68.9963	−170.5838
12	37.2911	48.7896	46	230.5373	−45.3839	80	−68.5395	−74.5900
13	−151.8092	−16.0172	47	114.3888	103.8789	81	206.2285	1.1733
14	123.9678	−111.0752	48	−97.9383	105.1093	82	−149.7678	168.1739
15	−29.3590	132.2116	49	−176.2423	−42.5563	83	122.4704	180.7840
16	−167.0885	−73.2191	50	86.9860	137.7383	84	−124.5224	−47.9457
17	−190.4386	2.9106	51	−23.2266	−49.0803	85	−68.3349	147.8602
18	115.1268	−131.2206	52	−11.5071	−103.2896	86	−209.5402	−82.4969
19	−109.4713	156.3334	53	−194.4786	−132.2679	87	−189.9519	120.9591
20	101.1534	−78.3999	54	218.3378	87.4377	88	177.8639	−53.9579
21	20.2577	108.4091	55	−42.5723	75.0849	89	133.2627	55.4251
22	−54.4610	−35.0199	56	31.4821	−93.8401	90	51.6126	−141.7886
23	176.9704	39.4416	57	69.5126	−73.6185	91	74.9291	12.9096
24	14.9748	−121.2538	58	51.7187	114.9257	92	−140.5877	−162.9234
25	130.6056	13.5093	59	230.4083	47.2095	93	55.7506	207.8480
26	198.7563	42.5033	60	−157.3516	17.1183	94	171.4856	128.0440
27	96.7626	−56.8425	61	−97.7063	−213.9397	95	173.2629	−127.6280
28	−110.8739	134.3782	62	133.0906	−80.4032	96	−26.5284	210.9667
29	161.2199	24.0819	63	92.0306	−109.0719	97	112.9423	−183.1747
30	244.4979	18.4780	64	−138.0666	117.5075	98	−208.9399	51.5080
31	55.0646	146.7503	65	−48.7620	−122.6821	99	36.2227	−212.1246
32	0.9221	−73.8021	66	218.9989	−85.7685	100	−6.6694	21.0142
33	−97.0099	73.1228	67	−162.6154	−104.9049			
34	−173.1583	−10.7053	68	−117.5900	3.5901			

2.3. *Solving DCBP Using IPOPT*

Several instances of the DCBP problem were solved by the NLP solver IPOPT [46] available at https://projects.coin-or.org/Ipopt (accessed on 4 April 2022). A computer with an AMD Athlon 64 X2 5200+ was used for numerical experiments with the decomposition algorithm [31].

Three problem instances were considered, having the following circle distributions characterized by the indicator (*the number of equal circles*):(*their radius*), as follows: (a) total 50 circles (20:10, 15:20, 10:30 and 5:40); (b) total 150 circles (50:10, 40:20, 30:30, 20:40 and 10:50); (c) total 300 circles, two collections of the 150 circles specified in (b). It was assumed that $w_i = r_i$ for $i = 1, 2, \ldots n$.

Figure 6 shows arrangements of circles corresponding to the local minima of DCBP for cases (a), (b) and (c).

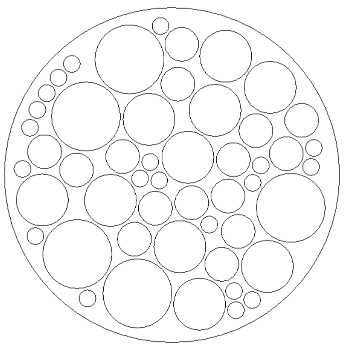
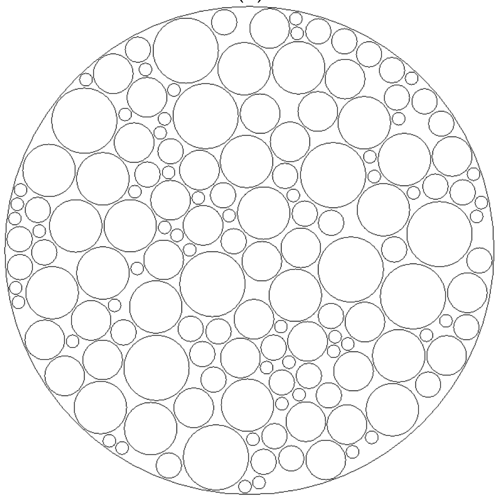
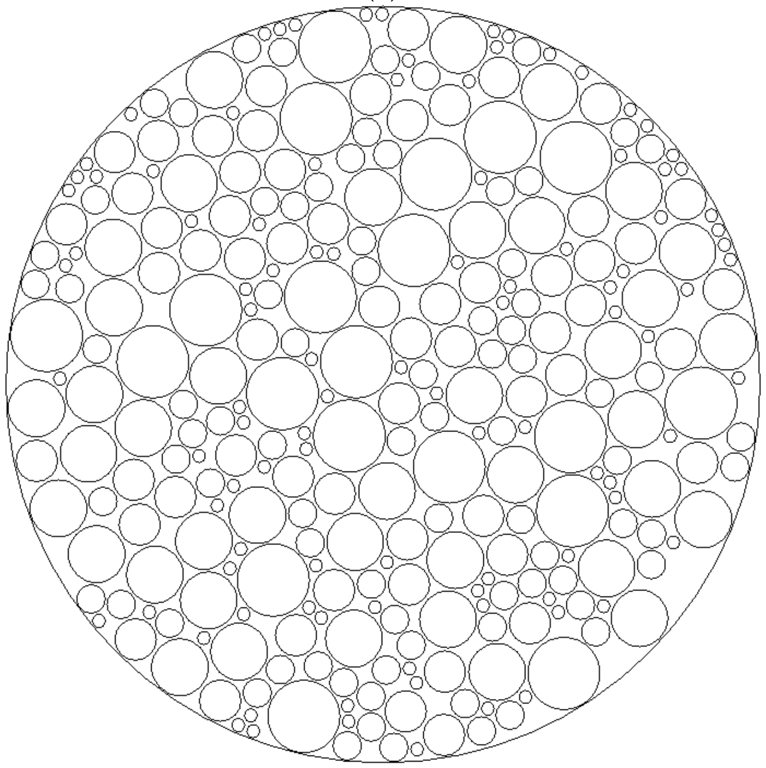

(**a**)

(**b**)

(**c**)

Figure 6. Locally optimal layouts of circles in DCBP: (**a**) n = 50, $\boldsymbol{r}$ * = 182.6996, CPU 1355.85 s for 100 runs; (**b**) n = 150, $\boldsymbol{r}$ * = 368.4018, CPU 45,910.3 s for 50 runs; (**c**) n = 300, $\boldsymbol{r}$ * = 520.5562, CPU 24,185.24 s for 50 runs.

3. Sparse Balanced Packing Problem with Balancing Conditions

In this section, a sparse circular packing problem is considered under balancing conditions. The objective is to place the circles in a fixed circular container maximizing the smallest distance between the circles, as well as the distance between the circles and the boundary of the container.

Denote the Euclidean distance between two circles by d_{ij} for $i > j = 1, 2, \ldots, n$ and the distance between a circle and the boundary of the circular container by d_i, $i = 1, 2, \ldots, n$. Let $d = min\{d_{ij}, i > j = 1, 2, \ldots, n, d_i, i = 1, 2, \ldots, n\}$ be the minimum among all these distances.

A *sparse circular balanced packing* (SCBP) problem is as follows. Pack the circles S_i, $i = 1, 2, \ldots, n$, into a fixed circular container S, maximizing d, subject to a limited deviation between the origin and the gravitation center of all circles S_i, $i = 1, 2, \ldots, n$. The maximal allowed deviation is denoted by ε.

The SCBP problem is stated as follows:

$$d^* = \max_{d>0,x,y} d \tag{11}$$

subject to

$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq (r_i + r_j + d)^2, \quad 1 \leq i < j \leq n, \tag{12}$$

$$x_i^2 + y_i^2 \leq (r - r_i - d)^2, \quad i = 1, \ldots, n, \tag{13}$$

$$-\varepsilon \leq \sum_{i=1}^{n} w_i x_i \leq \varepsilon, \quad -\varepsilon \leq \sum_{i=1}^{n} w_i y_i \leq \varepsilon. \tag{14}$$

The constraints in (12) and (13) assure that the corresponding distances are at least d. The constraint in (14) guarantees that the component-wise difference between the center of gravity and the origin (the center of the container) is at most ε.

The problem in (11)–(14) has $n(n-1)/2$ quadratic nonconvex constraints (12), n quadratic non-convex constraints (13) and $4n$ linear balanced constraints (14).

The following technique is applied for the problem in (11)–(14):

Stage 1. Generate several feasible starting points by homothetic transformations of the circles [21].

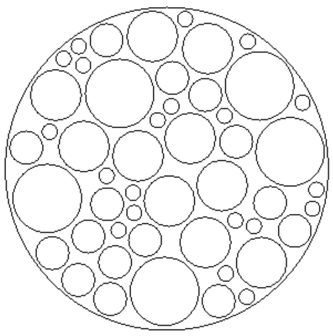
Stage 2. Obtain a local minimum for (11)–(14), starting from the points obtained at Stage 1 and using the optimization procedure proposed in [47]. The procedure substitutes solving the problem in (11)–(14) with $O(n^2)$ nonlinear constraints by analyses of several optimization subproblems with $O(n)$ constraints.

Stage 3. Consider the best local solution from Stage 2 as a solution to the original problem in (11)–(14).

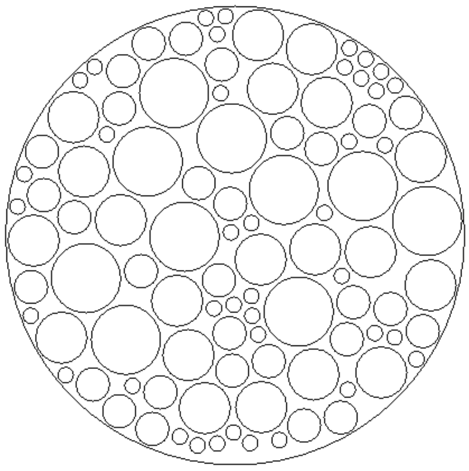
For computational experiments, a computer with an AMD Athlon 64 X2 5200+ was used.

Three problem instances were considered: (a) total 50 circles (20:10, 15: 20, 10:30 and 5:40); (b) total 100 circles (40:10, 30:20, 20:30 and 10:40); (c) total 150 circles (50:10, 40:20, 30:30, 20:40 and 10:50). Note that here, instances (a) and (c) coincide with instances (a) and (b) from Section 2. For all problem instances, $\varepsilon = 0$ and $w_i = r_i$ for $i = 1, 2, \ldots n$. Moreover, the radii of the containers in instances (a) and (b) were chosen as follows: $R_a = 182.6996$, $R_b = 257.3531$, $R_c = 368.4018$, such that R_a and R_c coincide with corresponding radii obtained in Section 2 for the dense packing problem. The radius of container S was found as a solution to the dense packing problem considered in the previous section.

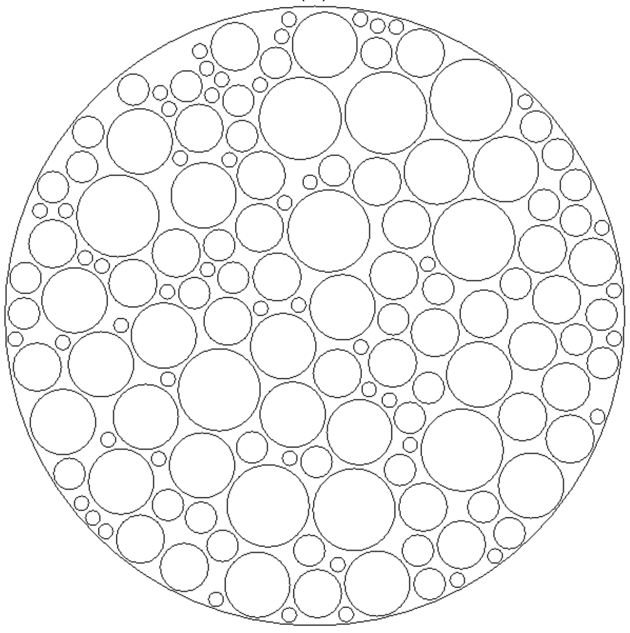
Figure 7 shows arrangements of circles corresponding to the local minima of SCBP for cases (a), (b) and (c).



(**a**)



(**b**)



(**c**)

Figure 7. Locally optimal arrangement of circles in SCBP: (**a**) $n = 50$, $\boldsymbol{d}^* = 2.05510$, CPU 1201.32 s for 100 runs; (**b**) $n = 100$, $\boldsymbol{d}^* = 2.04864$, CPU 8090.16 s for 100 runs; (**c**) $n = 150$, $\boldsymbol{d}^* = 2.05183$, CPU 19,571.66 s for 100 runs, CPU 8090.16 s for 100 runs.

4. Conclusions

Two circular packing problems are considered in this work: dense and sparse. In the first problem, the objective is to find the smallest possible circular container under distance and balancing conditions for the circular objects. In the second problem, the circular container is fixed, and the objective is to locate the circles as distant from one another as possible, as well as from the boundary of the container.

A mathematical model and sequential/parallel algorithms are presented for the dense packing problem. Parallel algorithms use computing technology with distributed memory (clusters). Solution algorithms apply the multistart strategy together with non-smooth minimization methods to search for the local minimum. The results of computational experiments for parallel algorithms on cluster SKIT-4 from the V.M. Glushkov Institute of Cybernetics of the National Academy of Sciences of Ukraine are presented. It is demonstrated that using the parallel approach can significantly reduce the solution time. The parallel solution allows finding good local minima for problem instances with several hundred circles in a reasonable time. The parallel solution also results in an improved value for the container's radius compared with the sequential technique.

The mathematical formulation for the sparse circular packing problem was presented. The corresponding nonlinear optimization problem was solved by the open-source local solver IPOPT [46], combined with the decomposition algorithm in [47] to cope with large dimensions. Numerical results are presented for the arrangement of circular containers obtained by dense packing. The corresponding solutions provide a visually more uniform distribution of the circles compared with dense packing. Obtaining a sparse layout is important in many practical applications, e.g., in thermal deburring [39], where it is used to ensure the uniformity of power and thermal effects in a deburring chamber (container).

Both formulations considered in this work result in large-scale optimization problems. To use the specific structure of the constraints, special-purpose decomposition/aggregation techniques [48,49] can be used to either relax the binding constraints or form an approximate aggregated problem of lower dimension.

An interesting direction for future research is using parallel algorithms for irregular packing problems with distance and balancing conditions. Some results in this direction are on the way [50]. Parallel techniques used in this work demonstrate their efficiency for traditional optimization approaches. We may expect that combining evolution algorithms [33] with parallel frameworks for packing problems may provide more computational savings.

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Appendix A. Shor's $r(\alpha)$-Algorithm

Here, we give a short description of the $r(\alpha)$-algorithm. This algorithm belongs to the family of subgradient methods for minimizing non-smooth convex functions known as Shor's r-algorithms. These algorithms use the steepest possible descent combined with space transformation and ensure monotonic (or almost monotonic) objective function decrees. A space transformation in the direction of the difference of two subsequent subgradients is used in r-algorithms; this improves the behavior of ravine functions in the transformed space of the variables.

Let x be an n-vector of variables and $f(x)$ be a convex function. Assuming $\alpha > 1$, using the $r(\alpha)$-algorithm to minimize $f(x)$ is an iterative technique to obtain vectors $\{x_k\}_{k=0}^{\infty}$ and matrices $\{B_k\}_{k=0}^{\infty}$ such that:

$$x_{k+1} = x_k - h_k B_k \xi_k, \quad B_{k+1} = B_k R_\beta(\tau_k), \quad k = 0, 1, 2, \ldots, \tag{A1}$$

where

$$\xi_k = \frac{B_k^T g_f(x_k)}{\| B_k^T g_f(x_k) \|}, \quad h_k = \underset{h \geq 0}{argmin} f(x_k - hB_k\xi_k), \tag{A2}$$

$$\tau_k = \frac{B_k^T r_k}{\| B_k^T r_k \|}, r_k = g_f(x_{k+1}) - g_f(x_k), \quad \beta = \frac{1}{\alpha} < 1. \tag{A3}$$

In (A1)–(A3), x_0 is an arbitrary starting point, and $B_0 = I_n$ is an identity $n \times n$ -matrix (B_0 also can be defined as a diagonal matrix D_n with positive diagonal elements). The step-size multiplier h_k is obtained by minimizing $f(x)$ in the direction of the subgradient in the transformed space; the coefficient α is used for space dilation, and the operator $R_\beta(\tau) = I_n + (\beta - 1) \cdot \tau \cdot \tau^T$ is used for "pressing" subgradients in the normalized direction τ with a coefficient $\beta = \frac{1}{\alpha} < 1$. Vectors $g_f(x_k)$ and $g_f(x_{k+1})$ denote subgradients of $f(x)$ at x_k and x_{k+1}. The process in (14)–(16) stops if $g_f(x_k) = 0$ and x_k is a minimum point of $f(x)$.

This algorithm is implemented as an Octave program ralgb5. This program can find an approximate value of x_r^*, which is a minimizer of the function $f(x)$. Its name indicates that the recalculation of the matrix B in algorithms (A1)–(A3) needs the $5n^2$ arithmetical operation. The program ralgb5 can find more precise approximations x_r^* than the existing program ralgb4, economizing n^2 arithmetical operations per iteration. Control parameters are the same in both programs.

Adaptive step regulation is related to approximately minimizing the objective function in the subgradient direction and uses h_0, q_1, n_h, and q_2 as parameters. Here, h_0 is an initial step-size (this value is updated after the first iteration), q_1 is a coefficient of the step reduction ($q_1 \leq 1$) if the condition of stopping descent is satisfied in one step and q_2 is a coefficient of the step increasing ($q_2 \geq 1$). The integer n_h denotes the number of steps in a one-dimensional descent ($n_h > 1$). After n_h steps, the step size is q_2 times greater.

The choice of the coefficient of space transformation and the parameters of the adaptive step-size regulation aims to improve the accuracy of minimizing the function along a direction, with the number of steps along a direction being moderate.

Stopping rules for the $r(\alpha)$-algorithm are defined by the parameters ε_x and ε_g: iterations are terminated at x_{k+1} if $\| x_{k+1} - x_k \| \leq \varepsilon_x$ (criterion in the space of variables) or if $\| g_f(x_{k+1}) \| \leq \varepsilon_g$ (criterion in the objective space, used for smooth functions). For values of $f(x)$ not bounded below, or for too-small values of the initial step h_0, the algorithm may terminate in an abnormal manner. In the latter case, the initial step-size must be enlarged.

The following parameter values are recommended for minimizing non-smooth functions: $\alpha = 2 \div 3$, $h_0 = 1.0$, $q_1 = 1.0$, $q_2 = 1.1 \div 1.2$, $n_h = 2 \div 3$. If a priory estimation of the distance between a starting point x_0 and a minimum point x^* is available, then a reasonable choice for the initial step h_0 is approximately $\| x_0 - x^* \|$.

Basically, the same parameters can be used for minimizing smooth functions. However, $q_1 = 0.8 \div 0.95$ is recommended. The reason is that the additional size decomposition provides a more accurate directional minimum of the function; for smooth functions, this provides a better convergence rate. For these parameters, the number of descents is usually less than two, and after n steps, the accuracy in terms of the objective is three to five times better. Choosing $\varepsilon_x, \varepsilon_g \sim 10^{-6} \div 10^{-5}$ for the minimization of a convex function gives good approximations of the optimal objective.

According to numerous tests and applied calculations, for non-smooth functions, we usually have $\frac{f(x_r^*) - f(x^*)}{|f(x^*)|+1} \sim 10^{-6} \div 10^{-5}$ ($\sim 10^{-12} \div 10^{-10}$ for smooth functions).

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