DYNAMICAL SYSTEMS AND MATCHING SYMMETRY IN β-EXPANSIONS

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Karl Zieber

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COMMITTEE MEMBERSHIP

TITLE:	Dynamical Systems and Matching Symmetry in β-Expansions
AUTHOR:	Karl Zieber
DATE SUBMITTED:	June 2022
COMMITTEE CHAIR:	Erin Pearse, Ph.D. Professor of Mathematics
COMMITTEE MEMBER:	Matthew White, Ph.D. Professor of Mathematics
COMMITTEE MEMBER:	Ryan Tully-Doyle, Ph.D. Associate Professor of Mathematics

ABSTRACT

Dynamical Systems and Matching Symmetry in β-Expansions

Karl Zieber

Symbolic dynamics, and in particular β-expansions, are a ubiquitous tool in studying more complicated dynamical systems. Applications include number theory, fractals [11], information theory, and data storage [1].

In this thesis we will explore the basics of dynamical systems with a special focus on topological dynamics. We then examine symbolic dynamics and β-transformations through the lens of sequence spaces. We discuss observations from recent literature about how matching (the property that the itinerary of 0 and 1 coincide after some number of iterations) is linked to when $T_{\beta,\alpha}$ generates a subshift of finite type. We prove the set of α in the parameter space for which $T_{\beta,\alpha}$ exhibits matching is symmetric and analyze some examples where the symmetry is both apparent and useful in finding a dense set of α for which $T_{\beta,\alpha}$ generates a subshift of finite type.

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CHAPTER 1: PRELIMINARIES

The study of dynamical systems can be traced back to Poincaré and his work on the three-body problem [7]. Though Poincaré was not the first to explore the motion of several bodies in space, he established some of the techniques that have become ubiquitous in the study of modern dynamical systems, including his famous recurrence theorem. Today, dynamical systems is an active field of mathematical research that draws upon many different fields of mathematics, including differential geometry, measure theory, topology, and group theory, though the fundamental question still remains: how does a system, under a fixed set of rules, behave over long periods of time?

1.1 Basic Definitions

At the most fundamental level, a *dynamical system* is a set X coupled with a map

$$f : X \to X \tag{1.1}$$

We are primarily interested in dynamical systems where X has some additional structure and f has certain restrictions. The most common set-ups are:

- **Topological dynamics:** X is a topological space and f is a continuous map.
- **Ergodic theory:** X is a measure space and f is a measurable function.
- **Differentiable dynamics:** X is a manifold and f is a differentiable map.
- **Metric dynamics:** X is a metric space and f is an isometry.

The study of dynamical systems is concerned primarily with the behavior of points in X when considered under the family of maps $\{f^n : X \to X\}$, where f^n denotes the n-fold composition $f^n = f \circ \cdots \circ f$ of f. We let f^0 denote the identity function I on X. If f is invertible, we let $f^{-n} = f^{-1} \circ \cdots \circ f^{-1}$. Note that $f^{n+m} = f^n \circ f^m$ and so if f is invertible the family of maps forms a group under composition.

If our family is indexed by $\mathbb{N}$ or $\mathbb{Z}$ as above we have a *discrete-time dynamical system.* However, we can also index our family $\{f^t : X \to X\}$ with $t \in \mathbb{R}$ or $t \in \mathbb{R}_0^+$ for a *continuous-time dynamical system*, which is still a semigroup under composition. Note that in this case f^t does not necessarily denote composition but instead corresponds to the member of the family described by the parameter t. In the case where $t \in \mathbb{R}$, we typically refer to the system as a *flow*, and as a *semiflow* if $t \in \mathbb{R}_0^+$. The index set is usually clear from context and is explicitly defined otherwise.

In mathematics it is often helpful to consider when two objects are effectively "the same." In our context, two systems (X, f) and (Y, g) are said to be equivalent if there exists a surjective map $\pi : Y \to X$, called a *semiconjugacy*, such that $f^t \circ \pi = \pi \circ g^t$ for all t. That is, the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{g^t} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f^t} & X \end{array}$$

If π is invertible, it is referred to simply as a *conjugacy.* In this way, properties of one system can be studied by understanding a different and possibly simpler conjugate system.

Now if we consider an arbitrary but fixed $x \in X$ we can define the following:

Definition 1.1.1. (*Orbit*)

For $x \in X$, the *positive semiorbit* of x, denoted $\mathcal{O}_f^+(x)$, is the set of all points $f^t(x)$ for some $t \geq 0$. Symbolically:

$$\mathcal{O}_f^+(x) = \bigcup_{t \geq 0} f^t(x)$$

Similarly, the *negative semiorbit* of x is $\mathcal{O}_f^-(x) = \bigcup_{t \leq 0} f^t(x)$, and together these form the *orbit* of x:

$$\mathcal{O}_f(x) = \mathcal{O}_f^+(x) \cup \mathcal{O}_f^-(x) = \bigcup_t f^t(x)$$

Usually, f is clear from context and the orbit is denoted simply as $\mathcal{O}(x)$.

A point $x \in X$ is a *periodic point* if $f^{n \cdot t}(x) = (f^t \circ \cdots \circ f^t)(x) = x$ for some t and for all $n \in \mathbb{N}$; equivalently, $f^t(x) = x$ for some t. In the special case where $f^t(x) = x$ for all t, then x is a *fixed point* of the system. On the other hand, a point x is called *eventually periodic* if there exists some r such that $f^r(x)$ is periodic. It is worth mentioning that in an invertible system all eventually periodic points are periodic, but if f is not invertible, you can have eventually periodic points that are not periodic (see example 1.2.4).

1.2 Basic Examples

Many of the introductory texts on dynamical systems begin their discussion with examples that illustrate key concepts of the theory. In the interest of acquainting ourselves with these systems, we shall uphold this tradition and examine some basic examples.

Example 1.2.1. (Rotations of the Unit Circle) Let $X = S^1$ denote the unit circle, i.e., $S^1 = [0,1]/\sim$, where $\sim$ identifies 0 and 1. When working in S^1, we use the following metric:

$$d(x,y) = \min\{|x-y|, 1-|x-y|\}$$

Our map f will be the rotation map $R_\alpha : S^1 \to S^1$ defined by

$$R_\alpha(x) = x + \alpha \bmod 1$$

R_α rotates the point x around the unit circle by a distance of $2\pi\alpha$. Iterating R_α gives us $R_\alpha^t(x) = x + t\alpha \bmod 1$. Note that, if α is rational, then every $x \in S^1$ is periodic. To see why, let $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ and note that

$$R_\alpha^q(x) = x + q\alpha \bmod 1 = x + p \bmod 1 = x \bmod 1 = x$$

The case where α is irrational is far more interesting. In fact, if α is irrational, then the positive semiorbit of any $x \in S^1$ is dense in S^1. To see why, first note that no point is periodic when α is irrational, as otherwise there would exist some $n \in \mathbb{N}$ such that $R_\alpha^n(x) = x + n\alpha \bmod 1 = x$, i.e., $n\alpha \in \mathbb{Z}$. Now, let $\varepsilon > 0$ be arbitrary, and divide S^1 into $\lceil \frac{1}{\varepsilon} \rceil$ equal parts (so each part has length at most ε). By the Pigeonhole Principle, there must exist $m, n \in \mathbb{Z}$ with $m < n$ such that $R_\alpha^n(x)$ and $R_\alpha^m(x)$ fall into the same $\lceil \frac{1}{\varepsilon} \rceil$-section. That is, $R_\alpha^{n-m}(x)$ represents a

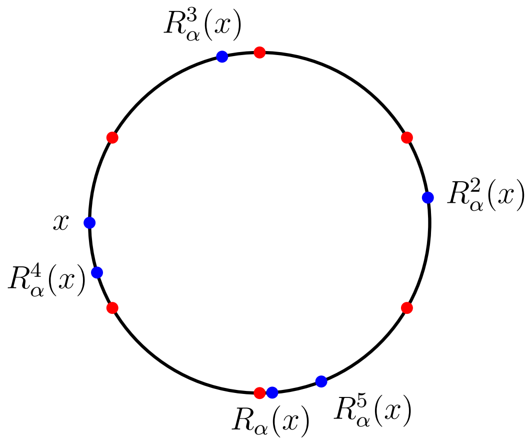

Figure 1.1: Circle Rotations. *The red dots divide S^1 into 6 parts, and the blue dots represent points in $\mathcal{O}^+(x)$.*

rotation by less than ε. Since we can generate a rotation by less than any arbitrary ε, we can bring any point $x \in S^1$ within ε of any other point $y \in S^1$.

Figure 1.1 illustrates this idea when $\varepsilon = \frac{1}{6}$, $\alpha = \frac{\pi}{12}$, $x = \frac{1}{4}$. In this case, inspection tells us we can take $m = 0$ and $n = 4$.

Example 1.2.2. (Hyperbolic Toral Automorphisms - Arnold's Cat Map) If we consider $\mathbb{R}^2$, we can construct the torus $\mathbb{T}^2$ by placing the points $(x, y) \in \mathbb{R}^2$ under the following equivalence:

$$(x, y) \sim (x + 1, y) \sim (x, y + 1)$$

which, among other things, has the effect of identifying all points of the form $(n, m) \in \mathbb{Z} \times \mathbb{Z}$.

Similarly, we can take the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

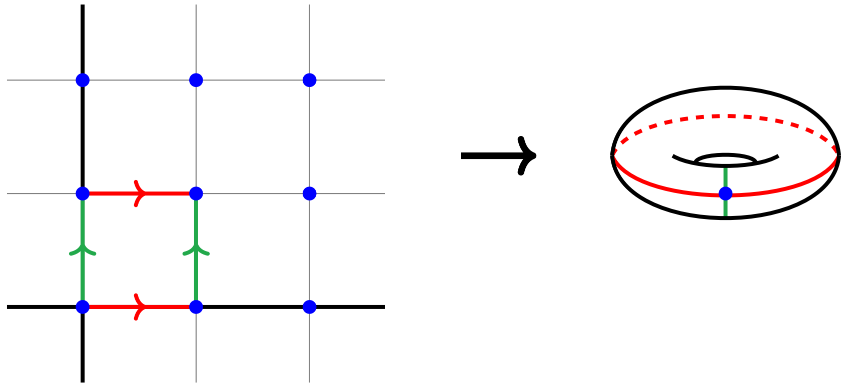

Figure 1.2: Torus Glueing. *The blue dots are points in $\mathbb{Z} \times \mathbb{Z}$, which are all identified. The red sides of the unit square are glued, matching the direction of the arrows. The green sides are similarly glued and we produce the torus $\mathbb{T}^2$.*

and redefine it on $\mathbb{T}^2$ as

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (2x + y) \bmod 1 \\ (x + y) \bmod 1 \end{bmatrix}$$

Here, $A : \mathbb{T}^2 \to \mathbb{T}^2$ is the hyperbolic toral automorphism–colloquially referred to as "Arnold's cat map", after the image that Vladimir Arnold first demonstrated the map's effect on.

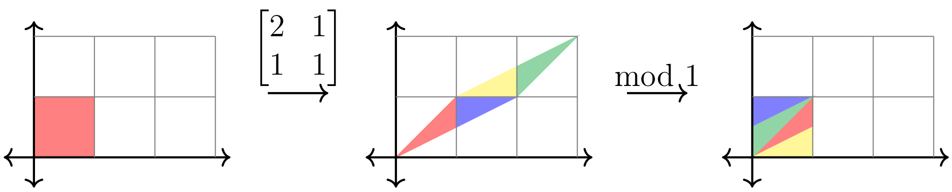

Figure 1.3: Arnold's Cat Map on the Unit Square. *A visualization of Arnold's Cat Map on the unit square.*

The eigenvalues of A are $\lambda = \frac{3+\sqrt{5}}{2} > 1$ and $\frac{1}{\lambda} < 1$ with eigenvectors $v_\lambda = \langle \frac{1+\sqrt{5}}{2}, 1 \rangle$ and $v_{1/\lambda} = \langle \frac{1-\sqrt{5}}{2}, 1 \rangle$ respectively. A expands by a factor of λ along lines parallel to v_λ and A contracts by a factor of λ along lines parallel to $v_{1/\lambda}$. Thus, we call the family of lines parallel to v_λ the set of *unstable foliations*, which we denote W^u, with a particular foliation through $x \in \mathbb{T}^2$ denoted as $W^u(x)$. For the

Figure 1.4: Arnold's Cat Map Example. *An image of a cat under Arnold's Cat Map, from left to right: base image, 1 iteration, 2 iterations, 128 iterations. It can be shown, on a discrete set such as pixels in an image, that Arnold's Cat Map is periodic [4]. For code, see Appendix A.*

family of lines parallel to $v_{1/\lambda}$ we have the *stable foliations* denoted similarly as W^s.

Proposition 1.2.3.

If $\alpha = (x_0, y_0)$ has rational coordinates, α is a periodic point of A.

Proof. Since $x_0, y_0 \in \mathbb{Q}$, let $x_0 = \frac{p}{k}$ and $y_0 = \frac{q}{k}$ be the fraction representations of x_0, y_0 over a common denominator, with $p, q, k \in \mathbb{Z}$. Note that a point of this form, when sent through A, becomes:

$$A \begin{bmatrix} x_0 \\ \\ y_0 \end{bmatrix} = \begin{bmatrix} (2\frac{p}{k} + \frac{q}{k}) \bmod 1 \\ \\ (\frac{p}{k} + \frac{q}{k}) \bmod 1 \end{bmatrix} = \begin{bmatrix} \frac{2p+q}{k} - n \\ \\ \frac{p+q}{k} - m \end{bmatrix} = \begin{bmatrix} \frac{2p+q-nk}{k} \\ \\ \frac{p+q-mk}{k} \end{bmatrix}$$

where $n = \lfloor \frac{2p+q}{k} \rfloor$ and $m = \lfloor \frac{p+q}{k} \rfloor$. Thus any point of the form $\left(\frac{p}{k}, \frac{q}{k}\right)$ is sent to another point of the form $\left(\frac{p'}{k}, \frac{q'}{k}\right)$, and there are at most k^2 points of this form in the unit square. Thus, A is permuting these k^2 points (since A has det $\neq 0$ and thus is injective) and thus these points must be periodic. $\square$

Now, by the density of the rationals in $\mathbb{R}$, points in the unit square of the form $(x_0, y_0) \in \mathbb{Q} \times \mathbb{Q}$ are dense in the torus. Thus periodic points are dense in the torus as well.

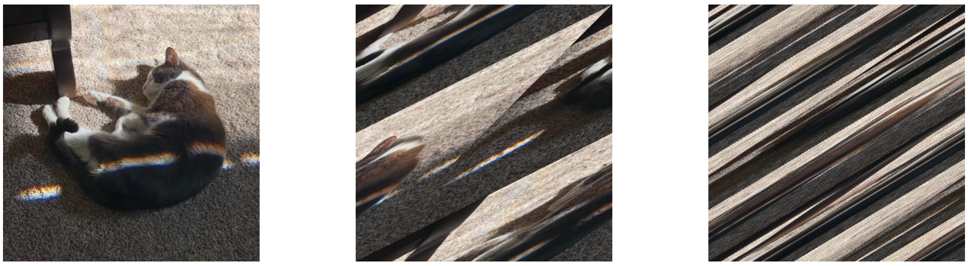

Figure 1.5: Zeno in Arnold's Cat Map. *Another example of Arnold's cat map for the first 2 iterations, featuring Zeno laying in a sunbeam. Image courtesy of Leah Hoogstra.*

Example 1.2.4. (Quadratic Maps) So far, the dynamical systems we've looked at have been linear; let's consider a nonlinear case. The classic example is the quadratic map, which is the system $(\mathbb{R}, q_b)$ where $q_b : \mathbb{R} \to \mathbb{R}$ is defined as

$$q_b(x) = -bx(x-1)$$

where $b > 0$. Since, for $x \in \mathbb{R}\backslash[0,1]$, $q_b(x) \to \infty$, as $x \to \pm\infty$, our interest lies in $[0,1]$, the region between the roots, and thus truly on the system $([0,1], q_b)$. We also restrict $0 \leq b \leq 4$ so the vertex of the parabola is underneath $y = 1$ and $q_b : [0,1] \to [0,1]$ is an interval map of the form 1.1. However, if $4 < b$, then there is a section in the middle that is mapped outside $[0,1]$, in particular, the interval

$$\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{b}}, \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{b}}\right)$$

is sent to $\left(1, \frac{b}{4}\right]$.

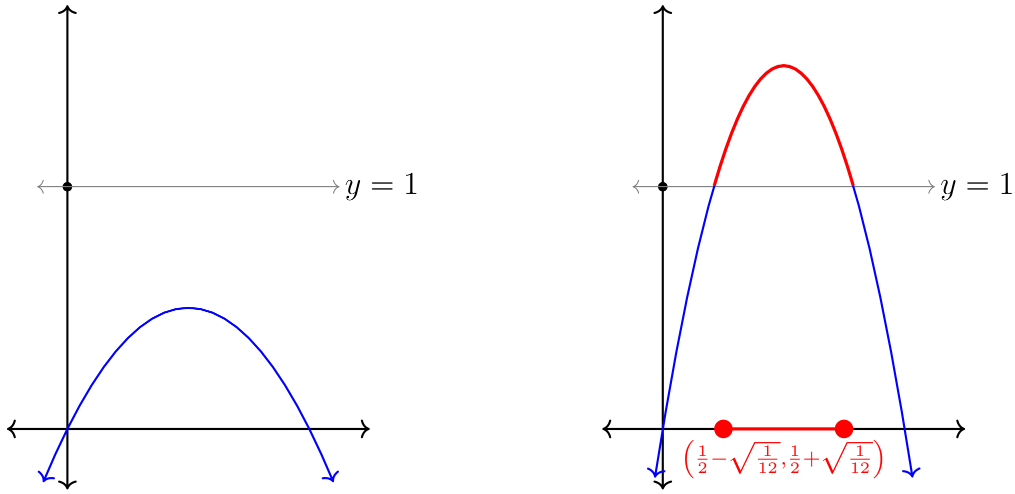

Figure 1.6: Quadratic Map Examples. *On the left,* $b = 2$. *On the right,* $b = 6$.

To compute where iterates go under successive applications of q_b we use what's called a *cobweb diagram*. To construct one, we graph our quadratic map and the line $y = x$ on the same coordinate grid. Then, draw a vertical line from our starting point, x, on the x-axis to the quadratic graph. Then draw a horizontal line from the quadratic graph to the $y = x$ line. Draw another vertical line back to the quadratic function, and repeat. See figure 1.7.

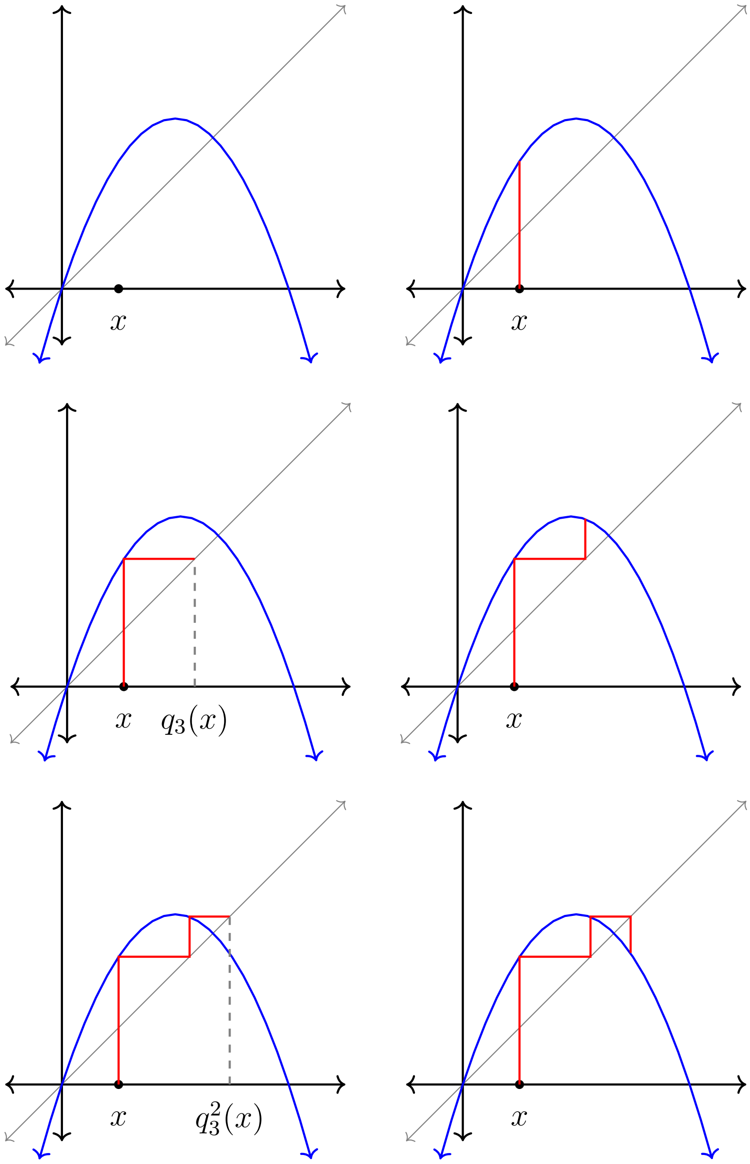

Figure 1.7: Cobweb Diagram Construction. *An example of the construction of a cobweb diagram using q_3 and $x = \frac{1}{4}$.*

Cobweb diagrams are useful because they allow us to visualize the behavior of points under iteration by q_b. By examining some cobweb diagrams, we can uncover some ways of classifying points based on their behavior.

Definition 1.2.5. (*Attracting Point*)

A fixed point x of a dynamical system (f, X) is an *attracting point* if it has a neighborhood U such that the closure, which we will denote $\overline{U}$, is compact, $f(\overline{U}) \subseteq U$, and $\bigcap_{t \geq 0} f^t(U) = \{x\}$. Similarly, a point x is *repelling* if it has a neighborhood U such that $\overline{U}$ is compact, $f(\overline{U}) \supseteq U$, and $\bigcap_{t \geq 0} f^{-t}(U) = \{x\}$.

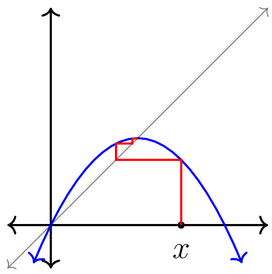

Figure 1.8: An Example of an Attracting Point. *An example of an attracting point.*

For example, if we consider q_2 and $x = \frac{3}{4}$, we can see that the orbit of the point goes to $\frac{1}{2}$ (see figure 1.8). It turns out the fixed points of q_b are 0 and $1 - \frac{1}{b}$.

If $b < 1$ then 0 is an attracting point, and if $b > 1$ it is a repelling point. Meanwhile, if $b \in (1, 3)$, $1 - \frac{1}{b}$ is an attracting point and if $b \notin [1, 3]$ then $1 - \frac{1}{b}$ is a repelling point. For an attracting point, the neighborhood U is referred to as a "trapping region." However, to be consistent with the literature that we will explore in section 4, we use a modified definition.

Definition 1.2.6. (*Trapping Region*)

Let (X, f) be a dynamical system. A compact set $Y \subseteq X$ is a *trapping region* of f if $f^t(Y) \subseteq Y$ for all $t \geq 0$.

Quadratic maps, in addition to hyperbolic toral automorphisms (example 1.2.2), are one of the introductory examples of chaotic maps. That is, quadratic maps are sensitive to small perturbations in initial conditions (starting input value x). While not a focus of this thesis, chaos theory is an important sub-field in dynamical systems that explicitly focuses on the long-term, global behavior of a system. Today, chaos not only attracts interest from mathematicians but is also of interest to physicists, biologists, and economists [3].

CHAPTER 2: TOPOLOGICAL DYNAMICS

Topological dynamics has been successful in generating invariants that help classify and quantify the behavior of systems, which has made it a main approach to dynamics. In addition to having analogues in ergodic theory, these invariants have applications and implications in mathematical physics, information theory, and chaos.

Throughout this section, we will assume our spaces have the following properties:

- Locally compact: for every point $x \in X$ there exists an open set U and a compact set K such that $x \in U \subseteq K$.
- Metrizable: there exists a metric on X.
- Second countable: our topology on X has a countable basis.

2.1 Limit Sets and Recurrence

In dynamical systems, we're often concerned with the long-time behavior of the system. Limit sets are an attempt to capture that long-time behavior local to a point, and this understanding can be generalized to say things about global behavior.

First, recall that a *topological dynamical system* is a topological space X coupled with a continuous map $f : X \to X$. We will be primarily concerned with discrete-time topological dynamical systems, but many of the results here generalize to continuous-time systems. In this context, a semiconjugacy $\pi : X \to Y$ between systems is a continuous map.

Definition 2.1.1. (*ω-Limit Point, Limit Set*)

Let $f : X \to X$ be a topological dynamical system with $x, y \in X$. We say y is an *ω-limit point* of x if there is a sequence $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $n_k \to \infty$ where

$$\lim_{n_k\to\infty} f^{n_k}(x) = y \tag{2.1}$$

We call the set of all such y's the *ω-limit set* of x, which we denote $\omega(x)$.

Definition 2.1.2. (*α-Limit Point, Limit Set*)

If f is invertible, an *α-limit point*, call it y, is similarly defined as

$$\lim_{n_k\to\infty} f^{-n_k}(x) = y$$

with the *α-limit set* denoted as $\alpha(x)$.

Since the ω-limit converges it equals its limit superior and we can use the set-theoretic definition of the limit superior to write the ω-limit set out as an actual set.

$$\omega(x) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \{f^k(x)\}} \quad \text{and similarly} \quad \alpha(x) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} \{f^{-k}(x)\}}$$

Here we require the closure of the unioned sets because we are not guaranteed that our topological space, X, is complete. Also notice the definition is set up so that

$$\omega(x) \subseteq \overline{\mathcal{O}^+(x)} \quad \text{and} \quad \alpha(x) \subseteq \overline{\mathcal{O}^-(x)}$$

is also true, though the reverse subset inclusions may not hold.

Proposition 2.1.3.

The α- and ω-limit sets of a point are closed invariant sets.

Proof. Let y be an element in the ω-limit set of $x \in X$. Note

$$\begin{aligned} f(y) &= f\left(\lim_{n_k\to\infty} f^{n_k}(x)\right) && \text{equation 2.1} \\ &= \lim_{n_k\to\infty} f(f^{n_k}(x)) && \text{since } f \text{ is continuous} \\ &= \lim_{n_k\to\infty} f^{n_k+1}(x) \end{aligned}$$

And thus $f(y)$ is also an ω-limit point of x and the ω-limit set of x is invariant. To show it is closed, let z be a limit point of the ω-limit set. Then there exists a sequence $\{y_j\}$ such that $y_j \to z$ as $j \to \infty$ and $y_j = \lim_{m_k\to\infty} f^{m_k}(x)$. Thus, for any arbitrary $\varepsilon > 0$, there exists some $y_j \in B(z, \frac{\varepsilon}{3})$ and there exists some $f^{m_k}(x) \in B(y_j, \frac{\varepsilon}{3})$. Thus $f^{m_k}(x) \in B(y_j, \varepsilon)$. We can form a sequence of $f^{m_k}(x)$'s that converge to z in this manner. Therefore z is in the ω-limit set and the ω-limit set is closed. The argument for the α-limit set is similar. $\square$

The natural question to ask is when is x in its own ω-limit set?

Definition 2.1.4. (*Recurrent Point*)

A point $x \in X$ is *(positively) recurrent* if $x \in \omega(x)$. We denote the set of all recurrent points of the system (X, f) as $\mathcal{R}(f)$.

Notably, any periodic point is recurrent, since we can simply take our index set under the limit to be $\{n_k\}_{k\in\mathbb{N}} = \{n \cdot t \in \mathbb{N} \mid f^{n\cdot t}(x) = x\}$. Similar logic shows any eventually periodic point is also recurrent. Furthermore, the set of recurrent points $\mathcal{R}(f)$ is f-invariant, since for any $x \in \mathcal{R}(f)$:

$$f(x) = f(\lim_{n_k \to \infty} f^{n_k}(x)) = \lim_{n_k \to \infty} f^{n_k+1}(x) \in \mathcal{R}(f)$$

by the continuity of f.

Example 2.1.5. Recall example 1.2.2, the hyperbolic toral automorphism. Since all periodic points are recurrent and periodic points are dense in $\mathbb{T}^2$ (by Proposition 1.2.4), $\mathcal{R}(A)$ is dense in the torus as well. So, if we consider an open ball B about some point x in $\mathbb{T}^2$, there is some rational point $r \in \mathbb{T}^2$ within B that has some orbit point that falls within this ball. So if we start close to any $x \in \mathbb{T}^2$, we can find a point down the line that is close to x again.

This observation about the hyperbolic toral automorphism generalizes to the following definition.

Definition 2.1.6. (*Non-Wandering Point*)

A point $x \in X$ is *non-wandering* if for any neighborhood U of x and for any $N \in \mathbb{N}$ there exists $n > N$ such that $f^n(U) \cap U \neq \emptyset$. We denote the set of all non-wandering points as $\mathrm{NW}(f)$.

Non-wandering points are a generalization of recurrence, so to speak. In fact, the next proposition confirms that any recurrent point is also a non-wandering point.

Proposition 2.1.7.

The set of non-wandering points is closed, is f-invariant, and contains $\omega(x)$ and $\alpha(x)$ for all $x \in X$.

To prove this result we require a lemma about the image of set intersections.

Lemma 2.1.8.

For some sets A and B

$$f(A \cap B) \subseteq f(A) \cap f(B)$$

Proof. Let $x \in f(A \cap B)$. Then there is some $y \in A \cap B$ such that $f(y) = x$ and note $y \in A$ and $y \in B$. Thus $f(y) \in f(A)$ and $f(y) \in f(B)$, hence $x \in f(A) \cap f(B)$. $\square$

Now for the main result:

Proof. Let $x \in \mathrm{NW}(f)$ and let U be some neighborhood of x. To show that $\mathrm{NW}(f)$ is f-invariant, consider $f(x)$. Let V be an arbitrary neighborhood of $f(x)$. Since f is continuous, the preimage $f^{-1}(V)$ is a neighborhood of x. Thus there exists some $n \in \mathbb{N}$ such that $f^n(f^{-1}(V)) \cap f^{-1}(V) \neq \emptyset$. Therefore

$$\emptyset \neq f(f^n(f^{-1}(V)) \cap f^{-1}(V)) \subseteq f(f^n(f^{-1}(V))) \cap f(f^{-1}(V)) = f^n(V) \cap V$$

Thus $f(x) \in \mathrm{NW}(f)$ and $\mathrm{NW}(f)$ is f-invariant.

To show $\mathrm{NW}(f)$ is closed, let z be a limit point of $\mathrm{NW}(f)$. Let $x_i \to z$ as $i \to \infty$ where $x_i \in \mathrm{NW}(f)$ for all $i \in \mathbb{N}$. Let U be an arbitrary neighborhood of z and note U must contain

$$\{x_i \mid i > k \text{ for some } k \in \mathbb{N}\}$$

I.e., U is a neighborhood for some x_i in the sequence. Thus, there exists some $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$, since x_i is a non-wandering point. Since U was an arbitrary neighborhood of z, $z \in \mathrm{NW}(f)$ as well.

Lastly, we will show $\omega(x) \subseteq \mathrm{NW}(f)$. The proof for $\alpha(x)$ is similar. Let $y \in \omega(x)$, which implies there is a sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ such that $n_k \to \infty$ and $f^{n_k}(x) \to y$. Letting U be an arbitrary neighborhood of y, this implies there exists an n_k such that $f^{n_k}(x) \in U$. Let n_j be such that

$$f^{n_j} \in B(y, d(y, f^{n_k}(x)))$$

which exists by the above as well. Let $m = n_j - n_k$ and note $f^{n_j}(x) \in f^m(U)$ (by applying the continuity of f^m to the fact that $f^{n_k}(x) \in U$) and $f^{n_j}(x) \in U$ (by construction). Thus $f^m(U) \cap U \neq \emptyset$, and $y \in \mathrm{NW}(f)$. □

Corollary 2.1.9.

For a dynamical system (X, f), $\overline{\mathcal{R}(f)} \subseteq \mathrm{NW}(f)$.

Example 2.1.10. This corollary implies that every point of $\mathbb{T}^2$ under the hyperbolic toral automorphism is a non-wandering point, which confirms our informal discussion.

2.2 Minimal Sets

For this section we assume our topological space X is compact.

Sets that have to do with the long-term local behavior in a system, such as $\omega(x)$ and $\mathcal{R}(f)$, are closed and forward f-invariant. Closed, nonempty, forward

f-invariant sets are a natural way to study the anatomy of a dynamical system. Following this line of inquiry we make the following definition:

Definition 2.2.1. (*Minimal Set*)

Let X be compact. We say a set $Y \subseteq X$ is minimal if it is closed, nonempty, and forward f-invariant and contains no proper subset that is also closed, nonempty, and forward f-invariant.

Searching for minimal sets manually would be madness. However, it turns out that such sets are intimately linked with a set we are already familiar with.

Proposition 2.2.2.

Let (X, f) be a dynamical system. Supposing that X is compact and $f : X \to X$ is continuous, $Y \subseteq X$ is minimal if and only if $\omega(y) = Y$ for every $y \in Y$.

Proof. ($\Rightarrow$) Suppose $Y \subseteq X$ is minimal and let $y \in Y$. Since Y is forward f-invariant, $f^n(y) \in Y$ for all $n \in \mathbb{N}$ and since Y is also closed, any $x \in \omega(y)$ is also in Y (since $x = \lim_{n_k \to \infty} f^{n_k}(y)$). Thus $\omega(y) \subseteq Y$. Since Y is minimal and $\omega(y)$ is a closed, nonempty, forward f-invariant set, it must be the case that $\omega(y) = Y$.

($\Leftarrow$) Assume $\omega(y) = Y$ for all $y \in Y$. By Proposition 2.1.3, Y must be closed, non-empty, and forward f-invariant. It remains to show that no proper subset can also have all of these properties. Let $A \subseteq Y$ be a proper, closed, non-empty, and forward f-invariant set and let $a \in A$. Then $f^n(a) \in A$ for all $n \in \mathbb{N}$ (since A is forward f-invariant) and thus $\lim_{n_k \to \infty} f^{n_k}(a) \in A$ for all $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ (since A is closed). So $\omega(a) \subseteq A$ and, since $a \in A \subseteq Y$, $\omega(a) = Y \supseteq A$ as well. So $A = \omega(a) = Y$ and Y is minimal. $\square$

Example 2.2.3. Recall example 1.2.1, rotations of the unit circle. Under an irrational rotation R_α, the forward orbit of every $x \in S^1$ is dense in S^1. So

$$\omega(x) = \overline{\mathcal{O}^+(x)} = S^1$$

and so the circle is a minimal set under R_α. The minimality of S^1 implies that every point in S^1 is nonwandering (since $\omega(x) \subseteq \mathrm{NW}(R_\alpha)$ by Proposition 2.1.7) and recurrent.

Perhaps density of orbits is enough to guarantee minimality? The following proposition confirms this.

Proposition 2.2.4.

Let (X, f) be a dynamical system. Supposing that X is compact and $f : X \to X$ is continuous, Y is minimal if and only if the forward orbit of every point in Y is dense in Y.

Proof. ($\Rightarrow$) Suppose $Y \subseteq X$ is minimal and let $y \in Y$. To show $\mathcal{O}^+(y)$ is dense in Y, it is enough to show $Y = \overline{\mathcal{O}^+(y)}$. Since Y is minimal, we already have $Y = \omega(y) \subseteq \overline{\mathcal{O}^+(y)}$ by Proposition 2.2.2. To show the reverse inclusion, let $x \in \overline{\mathcal{O}^+(y)}$. Thus

$$x = \lim_{k\to\infty} f^{n_k}(y)$$

Either $n_k \to \infty$ or $n_k \not\to \infty$. If $n_k \to \infty$, $x \in \omega(y) = Y$. If $n_k \not\to \infty$, then $x = f^i(y)$ for some $i \in \mathbb{N}$ and since Y is forward-invariant we have $x \in Y$. In either case, $x \in Y$ and $Y = \overline{\mathcal{O}^+(y)}$.

($\Leftarrow$) Suppose $\mathcal{O}^+(y) \subseteq Y$ is dense in Y for any $y \in Y$. Let $z \in Y$ and $\{f^{n_k}(y)\}_{k=1}^{\infty}$ be the natural sequence of elements of $\mathcal{O}^+(y)$ that converges to z.

Note we can force $n_k \to \infty$ and this immediately implies $z \in \omega(y)$ and so $Y \subseteq \omega(y)$. Since Y is compact, and hence closed, and it contains the forward orbit of any $y \in Y$, $\omega(y) \subseteq Y$. That is, for any y, $\omega(y) = Y$ and Proposition 2.2.2 implies Y is minimal. $\square$

Together, Proposition 2.2.2 and Proposition 2.2.4 imply that the forward orbit of every $y \in \omega(x)$ is dense in $\omega(x)$ and any set satisfying this criterion of dense orbits must be an ω-limit set. That is, if X is a minimal system, then $\omega(x) = \overline{\mathcal{O}^+(x)}$ for all $x \in X$.

2.3 Transitivity and Mixing

Minimality and the density of forward orbits are closely linked for compact spaces. What if we were to remove the requirement that X be compact and focus on the orbit of a single point? In pursuit of generalization, we make the following definition:

Definition 2.3.1. (*Topologically Transitive*)

A topological dynamical system $f : X \to X$ is said to be *topologically transitive* if there is some point $x \in X$ whose forward orbit is dense in X.

When working with compactness and the density of *every* forward orbit, the minimal sets were the bridge that allowed us to link dense orbits and ω-limit sets together. Under weaker hypotheses we get an analogous connection.

Proposition 2.3.2.

If X has no isolated points and the forward orbit of x is dense in X then $\omega(x) = X$.

Proof. Let $z \in X$. Since $\overline{\mathcal{O}^+(x)} = X$ we can make a sequence $\{n_k\}_{k\in\mathbb{N}}$ such that

$$z = \lim_{k\to\infty} f^{n_k}(x)$$

since there are no isolated points, we can guarantee that $n_k \to \infty$ by considering successively smaller ε-balls about z. Thus $z \in \omega(x)$, and $\omega(x) = X$. □

It turns out that we can conclude a system is topologically transitive if we can say something about the way any two open sets intersect under successive applications of f. However, in order to prove the result, we require the Baire Category Theorem.

Theorem 2.3.3. (Baire Category Theorem)

Let X be a locally compact Hausdorff space. Then for each countable collection of open dense sets $U_1, U_2, \ldots$, their intersection $\bigcap_{n\in\mathbb{N}} U_n$is dense.

Proposition 2.3.4.

Let $f : X \to X$ be a continuous map of a locally compact Hausdorff space X. Suppose for any two non-empty open sets U and V there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. Then f is topologically transitive.

Proof. Let $U \subseteq X$ be an arbitrary open set. By our hypothesis,

$$\bigcup_{n\in\mathbb{N}} f^{-n}(U)$$

must be dense in X, since it intersects nontrivially with every open set V. Let $\{U_i\}$ be a countable basis for the topology of X and consider the countable family of open dense sets $\{\bigcup_{n\in\mathbb{N}} f^{-n}(U_i)\}$. By the Baire Category Theorem

$$Y = \bigcap_i \bigcup_{n\in\mathbb{N}} f^{-n}(U_i)$$

is dense in X. If we have $y \in Y$, note there must be some $n_i \in \mathbb{N}$ such that $y \in f^{-n_i}(U_i)$, i.e. $f^n(y) \in U_i$, by the construction of y. Since $\{U_i\}$ is a countable basis for X, the forward orbit of y is dense in X. □

It turns out that the converse is not true; topological transitivity does not imply that for any two non-empty open sets U and V there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. We can exploit this more restrictive property to define *topologically mixing*.

Definition 2.3.5. (*Topologically Mixing*)

A topological dynamical system $f : X \to X$ is *topologically mixing* if for any two non-empty open sets U and V there is $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

Note that Proposition 2.3.4 tells us topologically mixing implies topologically transitive. However, one of our examples demonstrates that the converse is not true.

Example 2.3.6. Recall the irrational rotations of S^1 (example 1.2.1). The irrational rotations are minimal, and therefore topologically transitive by Proposition 2.2.4. However, the irrational rotations are not topologically mixing.

To see why, let $x, y \in S^1$ with $x \neq y$. Suppose $d(x, y) = 4\delta$ for the appropriate $\delta > 0$. By construction of δ, the open balls $B(x, \delta)$ and $B(y, \delta)$ do not intersect, i.e., $B(x, \delta) \cap B(y, \delta) = \emptyset$. Let V be some open ball of radius $\frac{\delta}{2}$. Seeking a contradiction, assume R_α is topologically mixing. Then there exist $n_x, n_y \in \mathbb{N}$ such that for all $k > n_x$ and $j > n_y$, $R_\alpha^k(V) \cap B(x, \delta) \neq \emptyset$ and $R_\alpha^j(V) \cap B(y, \delta) \neq \emptyset$. Letting $n = \max\{n_x, n_y\}$, for all $m > n$, $R_\alpha^m(V) \cap B(x, \delta) \neq \emptyset$ and $R_\alpha^m(V) \cap B(y, \delta) \neq \emptyset$ as well. Let $x_0, y_0 \in S^1$ be such that $R_\alpha^m(x_0) \in R_\alpha^m(V) \cap B(x, \delta)$ and $R_\alpha^m(y_0) \in R_\alpha^m(V) \cap B(y, \delta)$. Note that this implies $x_0, y_0 \in V$ and

$$
\begin{aligned}
d(x, y) &\leq d(x, R_\alpha^m(x_0)) + d(R_\alpha^m(x_0), R_\alpha^m(y_0)) + d(R_\alpha^m(y_0), y) && \text{triangle ineq.} \\
&= d(x, R_\alpha^m(x_0)) + d(x_0, y_0) + d(R_\alpha^m(y_0), y) && R_\alpha \text{ an isometry} \\
&\leq d(x, R_\alpha^m(x_0)) + \delta + d(R_\alpha^m(y_0), y) && x_0, y_0 \in V \\
&\leq \delta + \delta + \delta = 3\delta && R_\alpha^m(x_0) \in B(x, \delta), \\
& && R_\alpha^m(y_0) \in B(y, \delta)
\end{aligned}
$$

impossible, as $d(x, y) = 4\delta$.

2.4 Entropy

Topological entropy is, loosely speaking, a measure of complexity. It tracks the asymptotic, exponential growth rate of distinguishable orbits in the system so systems with entropy 0 do not experience more complexity, i.e., more orbits, as time goes on while systems with positive entropy do.

Throughout this section, we assume (X, d) is a compact metric space and $f : X \to X$ is a continuous function. When we say "distinguishable orbits," we mean how far apart (outside ε) the first n iterates of some points x and y are to each other. Naturally, we codify distance with a metric

$$d_n(x,y) = \max_{0\leq k\leq n-1} d(f^k(x), f^k(y))$$

Here, d_n tracks the furthest away x and y get under the first $n-1$ iterates of f. This will be our tool for distinguishing orbits.

There are three pathways to computing topological entropy; all turn out to yield equivalent definitions. Each of the following definitions represents one of those pathways. For all definitions, let $\varepsilon > 0$ be fixed.

Definition 2.4.1. ((n,ε)-*Spanning Set*)

A subset $A \subseteq X$ is an (n,ε)-*spanning set* if for every $x \in X$ there is a $y \in A$ such that $d_n(x,y) < \varepsilon$. Let $\operatorname{span}(n,\varepsilon,f)$ denote the minimum cardinality of an (n,ε)-spanning set.

Definition 2.4.2. ((n,ε)-*Separated Set*)

A subset $A \subseteq X$ is an (n,ε)-*separated set* if any two distinct points $x, y \in A$ are at least ε apart under d_n, i.e., $d_n(x,y) \geq \varepsilon$. Let $\operatorname{sep}(n,\varepsilon,f)$ denote the maximum cardinality of an (n,ε)-separated set.

Definition 2.4.3. ((n,ε)-*Cover*)

A cover $\mathcal{A}$ of X is an (n,ε)-*cover* if all sets $A \in \mathcal{A}$ have d_n-radius less than ε. Let $\operatorname{cov}(n,\varepsilon,f)$ denote the minimum cardinality of such a cover.

At first blush, these ideas seem distinct. However, there is a common thread: if we have an (n,ε)-spanning set A then we can form an (n,ε)-cover by taking the

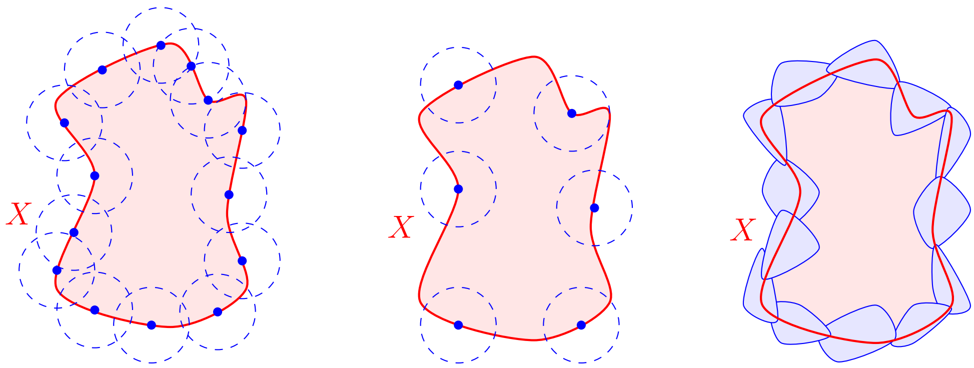

Figure 2.1: span, sep, and cov Depictions. *These are not strictly accurate depictions; here, we illustrate the idea behind* span, sep, *and* cov *with simpler ε-balls (i.e., when $n = 1$). Let X be the boundary of the red set pictured above. In the leftmost image, the blue dots are a spanning set of X with ε-radii about each point to confirm the set is indeed spanning. In the middle image, the blue dots represent the separated set with ε-radii about each point to confirm the set is indeed separated. In the rightmost image, the blue sets represent a cover of X by sets of radius less than ε.*

open ε-balls about each point in A. Similarly, we can take a cover $\mathcal{A}$ and fit each set in the cover inside ε-balls centered at points in X and those center points become our spanning set. Meanwhile, if we fit as many points into an (n, ε)-separated set as possible, the ε-balls "cluster together" and under the limiting process $\varepsilon \to 0^+$ this cluster of balls becomes indistinguishable from a cover and thus a spanning set as well. We shall see that this limiting process will indeed be involved in the definition entropy.

Analogy is not the only link between span, sep, and cov. The next lemma formalizes their relationship.

Lemma 2.4.4.

Let X be some compact metric space and $f : X \to X$ a continuous map. Then

$$\operatorname{cov}(n, 2\varepsilon, f) \leq \operatorname{span}(n, \varepsilon, f) \leq \operatorname{sep}(n, \varepsilon, f) \leq \operatorname{cov}(n, \varepsilon, f)$$

Proof. To demonstrate $\operatorname{cov}(n, 2\varepsilon, f) \leq \operatorname{span}(n, \varepsilon, f)$, let A be an (n, ε)-spanning set of minimum cardinality. Then the set of open balls of radius ε centered at points in A cover X. Let $0 < \varepsilon' < \varepsilon$ and note, by compactness, there is a cover of open balls of radius ε' centered at the points in A. The diameter of these balls is $2\varepsilon' < 2\varepsilon$, and so $\operatorname{cov}(n, 2\varepsilon, f) \leq \operatorname{span}(n, \varepsilon, f)$.

To demonstrate $\operatorname{span}(n, \varepsilon, f) \leq \operatorname{sep}(n, \varepsilon, f)$, let B be a (n, ε)-separated set of maximum cardinality. For any $x \in X \backslash B$ and every $y \in B$, the inequality $\varepsilon \leq d_n(x, y)$ cannot hold by the maximality of B. That is, for all $x \in X$, there is a y in B such that $d_n(x, y) < \varepsilon$. This means B must also be an (n, ε)-spanning set. Thus

$$\operatorname{span}(n, \varepsilon, f) \leq |B| = \operatorname{sep}(n, \varepsilon, f)$$

To prove the last inequality, again let B be a (n, ε)-separated set of maximum cardinality and let C be an (n, ε)-cover of X with minimum cardinality. Seeking a contradiction, suppose $|B| > |C|$. Then there is some set in C that has more than one element from B, by the Pigeonhole Principle. Therefore there exists a set $c \in C$ of d_n-diameter less than ε that contains at least two points from B, impossible as B is an (n, ε)-separated set. Thus the last inequality holds. $\square$

Since X is compact, $\operatorname{cov}(n, \varepsilon, f)$ is finite. Therefore, we can define

$$h_\varepsilon(f) = \limsup_{n\to\infty} \frac{1}{n} \log(\operatorname{cov}(n, \varepsilon, f))$$

where we require the limit superior to guarantee convergence. As ε decreases, $\operatorname{cov}(n, \varepsilon, f)$ increases monotonically, and so

$$\lim_{\varepsilon\to 0^+} h_\varepsilon(f)$$

exists (or is ∞). This limit is the *topological entropy* of f.

Definition 2.4.5. (*Topological Entropy*)

The *topological entropy* of f, denoted $h(f)$, is the limit

$$h(f) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log(\operatorname{cov}(n, \varepsilon, f))$$

Theorem 2.4.6.

cov, sep, and span all define the same topological entropy. That is,

$$\begin{aligned} h(f) &= \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log(\operatorname{cov}(n, \varepsilon, f)) && (2.2) \\ &= \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log(\operatorname{sep}(n, \varepsilon, f)) && (2.3) \\ &= \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log(\operatorname{span}(n, \varepsilon, f)) && (2.4) \end{aligned}$$

Proof. First we shall demonstrate that $\limsup_{n\to\infty} \frac{1}{n} \log(\mathrm{cov}(n, \varepsilon, f))$ is finite. Let U have d_m-diameter less than ε and let V have d_n-diameter less than ε. Note that

$$f^{m+n}(U \cap f^{-m}(V)) \subseteq f^{m+n}(U) \cap f^n(V)$$

and so $U \cap f^{-m}(V)$ has d_{m+n}-diameter less than ε. Therefore any $(n+m, \varepsilon)$ cover of minimum cardinality can be formed from combining sets from (n, ε) and (m, ε) covers of minimum cardinality. That is,

$$\mathrm{cov}(n+m, \varepsilon, f) \leq \mathrm{cov}(n, \varepsilon, f) \cdot \mathrm{cov}(m, \varepsilon, f)$$

Thus

$$\log(\mathrm{cov}(n+m, \varepsilon, f)) \leq \log(\mathrm{cov}(n, \varepsilon, f)) + \log(\mathrm{cov}(m, \varepsilon, f))$$

that is, the sequence $\log(\mathrm{cov}(n, \varepsilon, f))$ is subadditive. By Fekete's Subadditive Lemma [5], the limit $\limsup_{n\to\infty} \frac{1}{n} \log(\mathrm{cov}(n, \varepsilon, f))$ converges to a finite limit as $n \to \infty$.

Lemma 2.4.4 implies the desired result as $\varepsilon \to 0^+$. □

Example 2.4.7. The entropy of the hyperbolic toral automorphism (example 1.2.2) is $h(A) = \log |\lambda|$.

Example 2.4.8. The entropy of an irrational circle rotation (example 1.2.1) is 0. To see why, recall that the irrational circle rotation R_α is an isometry, and so $d_n = d$. So as $n \to \infty$ $\log(\mathrm{cov}(n, \varepsilon, f))$ does not depend on n and $\limsup_{n\to\infty} \frac{1}{n} \log(\mathrm{cov}(n, \varepsilon, f)) = 0$. This result holds in general for all isometries.

Topological entropy is useful because it acts as an invariant for dynamical systems. The next result makes this more formal.

Lemma 2.4.9.

The topological entropy of a continuous map $f : X \to X$ is invariant with respect to the metric generating the topology on X.

Proof. Let d and d' be two metrics on X. For $\varepsilon > 0$, define $\delta_\varepsilon = \sup\{d'(x,y) \mid d(x,y) \leq \varepsilon\}$. If U is some set with d-diameter less than ε, then it will have d'-diameter less than δ_ε. So, letting cov and cov$'$ correspond to d and d' respectively

$$\mathrm{cov}'(n, \delta_\varepsilon, f) \leq \mathrm{cov}(n, \varepsilon, f)$$

Since X is compact, $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ (since the supremum is achieved and to $d(x,y) = 0$ implies $d'(x,y) = 0$). Thus we have

$$\lim_{\delta_\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log(\mathrm{cov}'(n, \delta_\varepsilon, f)) \leq \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log(\mathrm{cov}(n, \varepsilon, f))$$

Swapping the roles of d and d' gives the reverse inequality, thus the entropy must be the same. $\square$

Theorem 2.4.10.

Topological entropy is invariant under topological conjugacy.

Proof. Suppose $f : X \to X$ and $g : Y \to Y$ are topologically conjugate systems with conjugacy $\pi : Y \to X$. If we let d_X be a metric on X and $a, b \in Y$, then $d_Y(a,b) = d_X(\pi(a), \pi(b))$ defines a metric on Y. By (Lemma 2.4.9), the topological entropy of g calculated using d_Y is the same as using the native topology on Y and, since π is an isometry in this case, it follows that $h(f) = h(g)$. $\square$

The following proposition outlines some basic properties of topological entropy. The proof for these facts can be found in [1].

Proposition 2.4.11.

Let $f : X \to X$ be a continuous map on a compact metric space X and let $g : Y \to Y$ be a continuous map on a compact metric space Y. Then

1. $h(f^m) = m \cdot h(f)$ for $m \in \mathbb{N}$.

2. If f is invertible, then $h(f^{-1}) = h(f)$. Consequently, $h(f^m) = |m| \cdot h(f)$ for $m \in \mathbb{Z}$.

3. If A_i, $i = 1, ..., k$ are closed, forward f-invariant subsets of X whose union is all of X, then

$$h(f) = \max_{1 \leq i \leq k} h(f|_{A_i})$$

4. $h(f \times g) = h(f) + h(g)$.

5. If g is a factor of f, then $h(f) \geq h(g)$.

A homeomorphism f is said to be *expansive* with *expansiveness constant* $\delta > 0$ if for any two distinct points $x, y \in X$ there is some $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq \delta$. We can actually use expansiveness to calculate entropy via the following proposition.

Proposition 2.4.12.

Let (X, d) be a compact metric space and $f : X \to X$ an expansive homeomorphism with expansiveness constant δ. Then $h(f) = h_\varepsilon(f)$ for any $\varepsilon < \delta$.

Proof. Fix γ and ε where $0 < \gamma < \varepsilon < \delta$. Since X is compact, so is the set $A = \{(x, y) \in X^2 \mid d(x, y) \geq \gamma\}$. Since $x \neq y$, for each such (x, y) there is some $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > \delta > \varepsilon$. Let k be the supremum of all such $|n|$ and note $k \leq \infty$ by the compactness of A (form a open cover with sets of the form $X_n = \{(x, y) \in X^2 \mid d(f^n(x), f^n(y)) \geq \delta\}$ and take a finite subcover). Thus if S is an (n, γ)-separated set then $f^{-k}(S)$ is $(n + 2k, \varepsilon)$-separated. Lemma 2.4.4 implies $h_\varepsilon(f) \leq h_\gamma(f)$. By monotonicity of cov, $h_\varepsilon(f) \geq h_\gamma(f)$ and so $h_\varepsilon(f) = h_\gamma(f)$. $\square$

As a consequence of this proposition and Lemma 2.4.4 any expansive map f has the property that, for ε sufficiently small,

$$h(f) = h_\varepsilon(f) = \limsup_{n\to\infty} \frac{1}{n} \log(\mathrm{cov}(n, \varepsilon, f)) = \limsup_{n\to\infty} \frac{1}{n} \log(\mathrm{sep}(n, \varepsilon, f))$$

Symbolic dynamics first appeared, in a prototypical fashion, at the very end of the 19th century when Jacques Hadamard was studying geodesic flows on negatively curved surfaces [6]. Several decades later, symbolic dynamics became a field of interest in its own right and interest was heightened by the development of information theory and mathematical communication theory as pioneered by Claude Shannon [12].

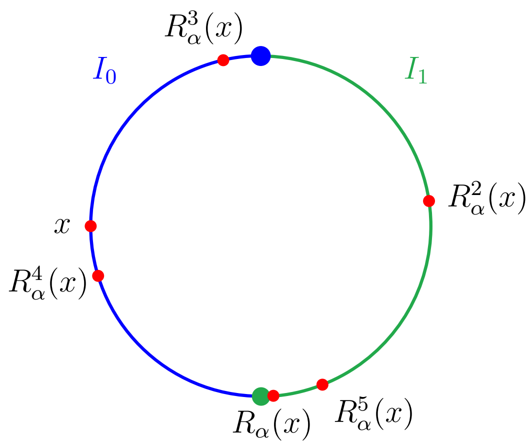

Figure 3.1: Circle Rotations Itinerary. *A depiction of x's itinerary up to $k = 5$.*

There are several approaches to symbolic dynamics but the classical viewpoint (as taken by Hadamard) is by starting with a dynamical system and considering the *itinerary* or *coding* of a point x as it moves through our space under f. Let us consider a relatively straightforward example: the irrational rotations of the unit circle (example 1.2.1). Divide the circle into two halves: $I_0 = [0, \frac{1}{2})$ and $I_1 = [\frac{1}{2}, 1)$. For each iteration of R_α, we will record whether the point falls into I_0 or I_1 with a 0 or a 1, respectively. Via this process, we form a binary sequence $x_0x_1x_2...$ where:

$$x_k = \begin{cases} 0 & \text{if } R^k_\alpha(x) \in I_0 \\ 1 & \text{if } R^k_\alpha(x) \in I_1 \end{cases}$$

For example, if we take $\alpha = \frac{\pi}{12}$ and $x = 0.25$, the first few entries in our sequence are 011001..., as illustrated above. We have a natural map acting on these sequences as a space: R_α itself. If we apply R_α to x first and consider the itinerary of $R_\alpha(x)$ under R_α, we get 11001..., i.e., the sequence shifted over to the left by one entry. From this example, we have uncovered the two basic building blocks of symbolic dynamics–sequences and the shift operator.

3.1 Shifts

We did not need to use just 0 and 1 for our sequences. For some $m \in \mathbb{N}$, we call $\mathcal{A}_m = \{1, 2, ..., m\}$ the *alphabet* of our sequence space and elements of the alphabet are referred to as *symbols*. With our symbols, we can create finite sequences called *words*. However, we are also quite interested in infinite sequences. We let $\Sigma_m^+ = \mathcal{A}_m^{\mathbb{N}_0}$ denote the set of all infinite (one-sided) sequences formed from our alphabet, and similarly we let $\Sigma_m = \mathcal{A}_m^{\mathbb{Z}}$ denote the set of all infinite two-sided sequences.

Now, Σ_m is our set X. Our map is the *shift operator*, $\sigma : \Sigma_m \to \Sigma_m$, defined as

$$...x_{-2}, x_{-1}, x_0, x_1, x_2, ... \overset{\sigma}{\mapsto} ...x_{-1}, x_0, x_1, x_2, x_3, ...$$

or, for $\sigma : \Sigma_m^+ \to \Sigma_m^+$:

$$x_0, x_1, x_2, ... \overset{\sigma}{\mapsto} x_1, x_2, x_3, ...$$

That is, σ shifts everything to the left one spot: $\sigma(x)_i = x_{i+1}$. The systems (Σ_m, σ) and (Σ_m^+, σ) are referred to as the *full two-sided shift* and the *full one-sided shift*, respectively.

Shift space has a product topology (the topology is inherited from the topology on our alphabet $\mathcal{A}_m$). The basis of this topology is *cylinders*, defined as

$$C^{n_1,\ldots,n_k}_{j_1,\ldots,j_k} = \{x = (x_l) \mid x_{n_i} = j_i, 1 \leq i \leq k\}$$

Each cylinder is the set of sequences that have a particular value in a particular slot. For example,

$$(7, 1, \mathbf{1}, \mathbf{2}, 3, \mathbf{5}, 8, \ldots) \in C^{3,4,6}_{1,2,5} = \{x = (x_l) \mid x_3 = 1, x_4 = 2, x_6 = 5\}$$

Note that the preimage of a cylinder under σ is still a cylinder

$$\sigma^{-1}(C^{n_1,\ldots,n_k}_{j_1,\ldots,j_k}) = C^{n_1+1,\ldots,n_k+1}_{j_1,\ldots,j_k}$$

and so σ is a continuous map. On the full two-sided shift space, σ is 1-1 and onto and the inverse map, the right shift, is continuous as well, so σ is a homeomorphism on Σ_m. In addition to a topology, we have a metric:

$$d(x, y) = 2^{-k} \quad \text{where} \quad k = \min\{|i| \mid x_i \neq y_i\}$$

Colloquially, two sequences are close together if they only start to differ far from the 0^{th} slot. Open balls under this metric are cylinders, specifically,

$$B(x, 2^{-k}) = C^{-k,-k+1,\ldots,k-1,k}_{x_{-k},x_{-k+1},\ldots,x_{k-1},x_k}$$

and so the topology and the metric coincide.

Proposition 3.1.1.

The periodic points of (Σ_m, σ) are dense in Σ_m under the product topology.

Proof. Since open balls are cylinders, consider an arbitrary open ball

$$B(x, 2^{-k}) = C^{-k,-k+1,\ldots,k-1,k}_{x_{-k},x_{-k+1},\ldots,x_{k-1},x_k}$$

Let $p = (\ldots, \overline{x_{-k}, x_{-k+1}, \ldots, x_{k-1}, x_k}, \ldots)$ be the sequence that repeats $x_{-k}, x_{-k+1}, \ldots, x_{k-1}, x_k$, with $x_0 = x$ in the zeroth spot. Under a shift by $2k$ places, p is fixed, so $\sigma^{2k}(p) = p$, hence p is periodic. Note $p \in C^{-k,-k+1,\ldots,k-1,k}_{x_{-k},x_{-k+1},\ldots,x_{k-1},x_k}$ by design and, since the cylinder was arbitrary, any open set has a periodic point. □

The following proposition will help us compute the entropy of certain maps on Σ_m.

Proposition 3.1.2.

The full one-sided and the full two-sided shifts are both expansive with expansiveness constant 1.

Proof. Let $x, y \in \Sigma_m^+$ be distinct. Let k be such that $d(x, y) = 2^{-k}$. So, for all $i < k$, $x_i = y_i$. Note $\sigma^k(x)_0 = x_k$ and $\sigma^k(y)_0 = y_k$, thus $d(\sigma^k(x), \sigma^k(y)) = 2^{-k+k} = 1$. The proof for Σ_m works similarly, with $k - 1$ possibly replaced with $-k + 1$. □

3.2 Subshifts

For the remainder of this section, we will be focused on Σ_m. Just as in topological dynamics, we are interested in subsets of Σ_m that are closed and shift-invariant. This motivates the following definition:

Definition 3.2.1. (*Subshift*)

A *subshift* of a shift space (Σ_m, σ) is a subset $X \subseteq \Sigma_m$ that is closed and σ-invariant.

Example 3.2.2. Let $m = 2$ and consider the subset X of Σ_m defined as

$$X = \{x = (x_i = i \bmod 2), y = (y_i = i + 1 \bmod 2)\}$$

That is, X is the set of the two sequences that alternate between 0 and 1 in each slot. Since $\sigma(x) = y$ and $\sigma(y) = x$, X is a subshift. The word "shift" is thrown around a lot in symbolic dynamics and this example is meant to serve as a reminder that a shift is a *map* while a subshift is a *set*.

Example 3.2.3. Let $x \in \Sigma_m$ be some fixed sequence. Then $\overline{\mathcal{O}(x)}$ is a subshift of Σ_m, since it is clearly closed and σ-invariant by construction.

Conjugacy between subshifts (and between shift spaces in general) is given the name *code*. Letting $X \subseteq \Sigma_m$ and $Y \subseteq \Sigma_n$ be subshifts, a code $c : X \to Y$ is a continuous map such that the usual commutativity $\sigma \circ c = c \circ \sigma$ holds. Note that Σ_m is always a compact space (since the farthest apart two points can be is 1) and so if c is bijective we have a topological conjugacy as well.

Having learned about topological dynamics, we may be interested to compute the entropy of the shift space. To that end, let $W_n(X)$ be the set of all words of length n that appear in a subshift X. As is the usual convention, let $|W_n(X)|$ denote the cardinality of this set.

Proposition 3.2.4.

Let $X \subseteq \Sigma_m$ be a subshift. Then the topological entropy of $\sigma|_X$ is

$$h(\sigma|_X) = \lim_{n\to\infty} \frac{1}{n} \log |W_n(X)|$$

Proof. By Proposition 3.1.2, σ (and therefore $\sigma|_X$) is expansive and Proposition 2.4.12 implies that, for ε sufficiently small,

$$h(\sigma|_X) = \limsup_{n\to\infty} \frac{1}{n} \log(\mathrm{sep}(n, \varepsilon, f))$$

I claim that $\mathrm{sep}(n, \varepsilon, f) = |W_n(X)|$. To see why, let $1 > \varepsilon > 0$. Define x_w to be a sequence in X such that the word w occurs starting at the 0^{th} position (such a x_w exists since X is shift invariant). Let $C = \{x_w \mid w \in W_n(X)\}$. I claim C is an (n, ε)-separated set of maximum cardinality. Note that, for any distinct $w, u \in W_n(X)$, x_w and x_u must differ in some position $0 \leq i \leq n$. Thus

$$d_n(x_w, x_u) = \max_{0\leq k\leq n-1} d((\sigma|_X)^k(x_w), (\sigma|_X)^k(x_u)) = 1 > \varepsilon$$

So this is indeed an (n, ε)-separated set. This separated set has cardinality $|W_n(X)|$ and is the biggest such set since if we add one more element to C then we will match up with some x_w in slots $0, 1, ..., n-1$, some $f(x_u)$ in slots $-1, 0, 1, ..., n-2$, and so on. $\square$

3.3 SFTs and Sofic Shifts

Since a subshift $X \subseteq \Sigma_m$ is a closed set, its complement $X^C \subseteq \Sigma_m$ is open and, by our topology, is the union of at most a countable number of cylinders. Since X is invariant under σ and its inverse, $\Sigma_m \backslash X$ must be as well; for any cylinder $C = C^{n_1,\ldots,n_k}_{j_1,\ldots,j_k}$ in $\Sigma_m \backslash X$ and for any $n \in \mathbb{Z}$, $\sigma^n(C) \subseteq \Sigma_m \backslash X$.

Fix a cylinder $C = C^{n_1,\ldots,n_k}_{j_1,\ldots,j_k}$ in this complement and consider the set of words of the form

$$w = (j_1, x_0, ..., x_i, j_2, x_{i+1}, ..., x_n, j_k) \in W_n(C)$$

where the x's are arbitrary elements of our alphabet $\mathcal{A}$ and each j_i is in the n_i^{th} slot. For example

$$C^{4,5,8}_{3,1,2} \to F = \{(3, 1, x_1, x_2, 2) \mid x_1, x_2 \in \mathcal{A}\}$$

If our alphabet is $\mathcal{A} = \{1, 2, 3\}$ then

$$\begin{aligned} F = \{&(3,1,1,1,2), (3,1,1,2,2), (3,1,1,3,2),\\ &(3,1,2,1,2), (3,1,2,2,2), (3,1,2,3,2),\\ &(3,1,3,1,2), (3,1,3,2,2), (3,1,3,3,2)\} \end{aligned}$$

By design, any sequence containing any word in F starting at the 4^{th} entry in the sequence is contained in $C^{4,5,8}_{3,1,2} \subseteq \Sigma_m \backslash X$. Shift invariance tells us that any sequence containing any of these words starting at any point in the sequence is also in $\Sigma_m \backslash X$. Every cylinder $C \subseteq \Sigma_m \backslash X$ has a finite set of words F_C of this form associated to it, and so

$$\mathcal{F} = \bigcup_{\substack{C \text{ is a cylinder} \\ \text{and } C \subseteq \Sigma_m \backslash X}} F_C$$

is countable. We call $\mathcal{F}$ the list of *forbidden words* of X, since if a word is in $\mathcal{F}$, any sequence containing that word must be in $\Sigma_m \backslash X$ and thus cannot appear in X itself. We are forbidden from allowing any sequence with a word $w \in \mathcal{F}$ to appear in X.

Example 3.3.1. Σ_m is a subshift of itself and $\mathcal{F} = \emptyset$.

Example 3.3.2. Let $\mathcal{A} = \{0, 1\}$ and let $\mathcal{F} = \{(0)\}$. Then $X = \{(..., 1, 1, 1, ...)\}$ since the only allowable entry is 1.

Furthermore, any sequence in $\Sigma_m \backslash X$ must have one of these forbidden words since it has to be in one of those cylinders. We can thus conclude that a subshift X is completely determined by its list of forbidden words, which allows us to make the following definition.

Definition 3.3.3. (*Subshift of Finite Type*)

If a subshift X has a list of forbidden words $\mathcal{F}$ that is finite, then X is a *subshift of finite type*, usually abbreviated as SFT.

Example 3.3.4. The system in example 3.3.2 is a subshift of finite type, since $|\mathcal{F}| = 1$.

Subshifts of finite type are closely related to directed graphs and their corresponding adjacency matrices, which makes them more tractable than other dynamical systems. Consequently, subshifts of finite type have been of particular interest to dynamical systems research as a tool for understanding less transparent problems. In fact, Hadamard's original work was with a subshift of finite type [6].

Seeking to understand this connection, let Γ be a finite directed graph on vertices $v_1, ..., v_m$. These vertices correspond to the letters in our alphabet $\mathcal{A} = \{1, ..., m\}$. We can view sequences as infinite walks in our directed graph and we call the resulting space the vertex shift space, denoted Σ_Γ. If all possible edges are in place we have Σ_m; if some are missing we obtain a subshift $X \subseteq \Sigma_m$. Let A_Γ be the adjacency matrix for our graph with entries of the form a_{x_i,x_j}. A word $x = (x_i)$ is said to be *allowed* (i.e., not forbidden) if $a_{x_i,x_{i+1}} \neq 0$ for all i.

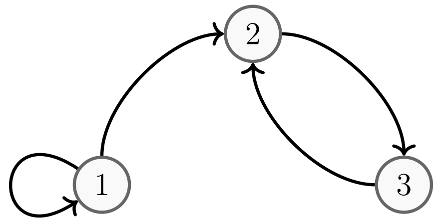

Figure 3.2: An Example of a Finite Directed Graph. *An example of a finite directed graph.*

Example 3.3.5. If we let Γ be the digraph pictured above, then our adjacency matrix is

$$A_\Gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Here, our allowed and forbidden words of length 1, 2, and 3 are

Length	Allowed Words	Forbidden Words
1	$(1), (2), (3)$	None.
2	$(1,1), (1,2), (2,3), (3,2)$	$(2,1), (2,2), (3,3), (1,3), (3,1)$
3	$(1,1,1), (1,1,2), (1,2,3), (2,3,2), (3,2,3)$	No new words.*

(*): *All forbidden words of length 3 have a forbidden word of length 2 as a component.*

Here, $|\mathcal{F}| = 5$, and so we do indeed have an induced subshift of finite type. In fact, all forbidden words of a finite directed graph can be build from a finite number of forbidden words of length 2, each representing a missing edge in the directed graph.

Making this relationship between subshifts of finite type and vertex shift systems more formal:

Proposition 3.3.6.

Every subshift of finite type is isomorphic to a vertex shift.

Proof. Let X be a subshift of finite type with forbidden words length $k+1$ or smaller. Recalling that $W_k(X)$ is the set of all words in X of length k, form the directed graph Γ with vertices the elements of $W_k(X)$ and a directed edge from vertex $(x_1, ..., x_k)$ to vertex $(y_1, ..., y_k)$ if

$$(x_1, ..., x_k, y_k) = (x_1, y_1, ..., y_k) \in W_{k+1}(X) \tag{3.1}$$

Let Σ_Γ denote the shift space of the directed graph. I claim the code $c : X \to \Sigma_\Gamma$ defined as $c(x)_i = x_i, ..., x_{i+k-1}$ gives an isomorphism from X to the vertex shift space. To see why, first note that c is clearly injective, and c is also onto since any arbitrary sequence in Σ_Γ, call it $\gamma = (..., x_0, x_1, x_2, ...) = (x_i)$ is produced by applying c to $x = (..., x_0, x_k, x_{2k}, ...) = (x_i)$. Lastly, c commutes with σ since

$$\begin{aligned}
c(\sigma(..., \underbrace{x_0}_{0}, x_1, x_2, ...)) &= c(..., \underbrace{x_1}_{0}, x_2, x_3, ...) \\
&= (..., \underbrace{x_1}_{0}, x_2, ..., x_k, x_2, x_3, ..., x_{k+1}, ...) \\
&= \sigma(..., \underbrace{x_0}_{0}, x_1, ..., x_{k-1}, x_1, x_2, ..., x_k, ...) \qquad (\bigstar) \\
&= \sigma(c(..., \underbrace{x_0}_{0}, x_1, x_2, ...))
\end{aligned}$$

Where the 0^{th} place is indicated for clarity and ($\bigstar$) follows from the fact that vertices in the shift space are indexed by words of length k and so the shift operator shifts by k letter-entries by the definition of adjacency in our graph (equation 3.1).

For example, let $X \subseteq \Sigma_3$ and let $\mathcal{F} = \{(1), (2, 2, 3)\}$. Then $k = 2$ and Γ is:

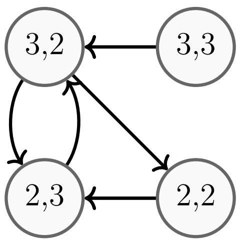

Figure 3.3: Digraph of Γ. *Our digraph* Γ *for our example.*

In this example, we could have $c(3, 2, 3, 3, 2, ...) = (3, 2, 2, 3, 3, 3, 3, 2, ...)$. $\square$

Corollary 3.3.7.

Every subshift of finite type is isomorphic to an edge shift space and every edge shift space is isomorphic to a subshift of finite type.

Now we can comfortably claim that subshifts of finite type and vertex shift spaces on finite directed graphs are inexorably linked: every subshift of finite type is a vertex shift space and every vertex shift space is a subshift of finite type. The following proposition shows how viewing SFT's as finite directed graphs can yield nice results.

Proposition 3.3.8.

Let X be a subshift of finite type. The number of fixed points of (X, σ) is $\text{tr}(A_\Gamma)$, where A_Γ is the adjacency matrix of the associated directed graph.

Proof. Note that fixed points of (X, σ) are $x \in \Sigma_m$ such that $\sigma(x) = x$; that is, the constant sequences. The allowed constant sequences in X are precisely the ones that have a path from their corresponding vertex to back to itself. These are in a one-to-one correspondence with the entries along the diagonal of A_Γ, and this $\text{tr}(A_\Gamma)$ is indeed the number of fixed points. □

Example 3.3.9. Reconsidering example 3.3.5, the trace of A_Γ is 1, and indeed the only fixed element is the infinite sequence of all 1's: $(..., 1, 1, 1, ...)$.

Definition 3.3.10. (*Sofic Shift*)

A *sofic shift* is a factor of a shift of finite type. That is, given subshifts $X, Y \subseteq \Sigma_m$ where X is a subshift of finite type, Y is a sofic shift if there exists an injective code $c : Y \to X$ such that $c \circ \sigma = \sigma \circ c$.

$$\begin{array}{ccccc} Y & \xrightarrow{c} & X & \xhookrightarrow{i} & \Sigma_m \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ Y & \xrightarrow{c} & X & \xhookrightarrow{i} & \Sigma_m \end{array}$$

One way to think about sofic shifts is as finite, labeled, directed graphs like subshifts of finite type, but the vertices of a sofic shift graph are not required to have unique labels.

Example 3.3.11. The classic example of a sofic shift is the even system due to Benjamin Weiss [13]. It is usually presented as an edge shift system:

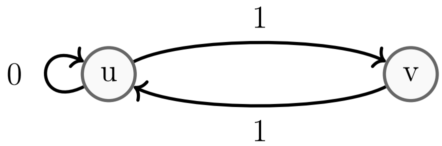

Figure 3.4: A Presentation of the Even Shift. *A presentation of the even shift.*

Here, a sequence is generated by a walk along the edges. For example,

$$(0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, ...) \in X$$

Note that only even-length strings of consecutive 1's can appear, hence the name.

CHAPTER 4: β-EXPANSIONS

As is mentioned in the previous section, symbolic dynamics were originally introduced to simplify larger systems. But which symbolic systems are useful? A subclass of symbolic systems, called β-expansions, have applications to fractals and number theory [10], as well as coding theory [9].

A β-expansion is also called a "non-integer base of enumeration," i.e., an alternative to an integer-based enumeration like base-10 or base-2. In order to understand the mechanics of β-expansions, we will first examine traditional base-2 expansions using the tools that will be helpful to us later. We can view digital expansions as itineraries of a dynamical system, similar to the introductory example in section 3. The system we use is the interval $[0,1]$ coupled with the map $f:[0,1]\to[0,1]$ defined as

$$f(x) = \begin{cases} 2x & \text{if } x < \frac{1}{2} \\ 2x-1 & \text{otherwise} \end{cases}$$

To build our itinerary, we partition our unit interval $[0,1]$ into $[0,\frac{1}{2})$ and $[\frac{1}{2},1]$. If our point lands in $[0,\frac{1}{2})$ we get a 0 in the expansion and if it lands in $[\frac{1}{2},1]$ we get a 1. That is, letting X be the set of all possible sequences found in this way, define the itinerary map $\tau:[0,1]\to X\subseteq\Sigma_2^+$ as

$$\tau(x) = (\omega_0(x), \omega_1(x), ...)$$

where ω_i is defined as

$$\omega_i(x) = \begin{cases} 0 & \text{if } f^i(x) < \frac{1}{2} \\ 1 & \text{otherwise} \end{cases}$$

For example, if we expand 0.15 in this way, our cobweb diagram looks like:

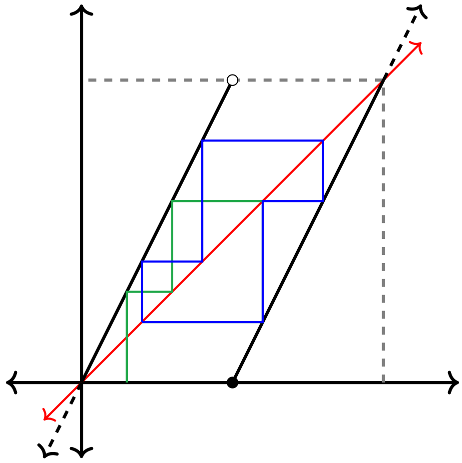

Figure 4.1: Base-2 Transformation. *In this system, 0.15 is an eventually periodic point with period 4. The cycle is indicated in blue.*

So our sequence is $(\omega_0, \omega_1, ...) = (0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, ...)$. We can make this a genuine digital expansion by projecting from the space of possible sequences $X \subseteq \Sigma_2^+$ to $[0, 1]$ via the map π defined as

$$\pi(\omega_0, \omega_1, ...) = \sum_{i=0}^{\infty} \omega_i 2^{-i-1}$$

In a base-2 enumeration, this gives us the binary expansion of 0.15 as 0.001001100110011....

Tying all our maps together, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, bend right, "\pi"'] & X \arrow[d, bend right, "\pi"'] \\ {[0,1]} \arrow[r, "f"] \arrow[u, bend right, "\tau"'] & {[0,1]} \arrow[u, bend right, "\tau"'] \end{tikzcd}$$

These ideas generalize to when we expand about some arbitrary number β, rather than just 2.

4.1 General β-Expansions

When specifying a β-expansion, we actually have two parameters: β, which can be thought of as our "dilation" parameter, and an α, which will be our translation parameter. We usually restrict $\beta \in (1,2)$ and $\alpha \in [0, 2-\beta]$. Our maps, which will be analogous to our f above, will be

$$T^{\pm}_{\beta,\alpha} : \left[\frac{-\alpha}{\beta-1}, \frac{1-\alpha}{\beta-1}\right] \to \left[\frac{-\alpha}{\beta-1}, \frac{1-\alpha}{\beta-1}\right]$$

defined as

$$T^{+}_{\beta,\alpha}(x) = \begin{cases} \beta x + \alpha & \text{if } x < p \\ \beta x + \alpha - 1 & \text{otherwise} \end{cases} \qquad T^{-}_{\beta,\alpha}(x) = \begin{cases} \beta x + \alpha & \text{if } x \leq p \\ \beta x + \alpha - 1 & \text{otherwise} \end{cases}$$

where our cut-off point is $p = \frac{1-\alpha}{\beta}$. These are called our β-*transformations.* Notice the difference between the "$+$" and "$-$" maps is the inequality, where the "$+$" map has a $<$ and the "$-$" map has a $\leq$. In our initial example, $f = T^{+}_{2,0}$. Some more examples of these maps are the "greedy" and "lazy" β-transformations, which are denoted G_β and L_β respectively, and are defined as

$$G_\beta : \left[0, \frac{1}{\beta-1}\right] \to \left[0, \frac{1}{\beta-1}\right]$$

$$G_\beta(x) = T^{+}_{\beta,0}(x) = \begin{cases} \beta x & \text{if } x < \frac{1}{\beta} \\ \beta x - 1 & \text{otherwise} \end{cases}$$

and

$$L_\beta : \left[\frac{\beta - 2}{\beta - 1}, 1\right] \to \left[\frac{\beta - 2}{\beta - 1}, 1\right]$$

$$L_\beta(x) = T^-_{\beta, 2-\beta}(x) = \begin{cases} \beta x + 2 - \beta & \text{if } x \leq 1 - \frac{1}{\beta} \\ \beta x + 1 - \beta & \text{otherwise} \end{cases}$$

Example 4.1.1. For the following, let $\beta = \frac{7}{4}$. Note this implies $\frac{1}{\beta - 1} = \frac{4}{3}$ and $\frac{\beta-2}{\beta-1} = -\frac{1}{3}$, and so the greedy and lazy transformations are defined on $G_{\frac{7}{4}} : \left[0, \frac{4}{3}\right] \to \left[0, \frac{4}{3}\right]$ and $L_{\frac{7}{4}} : \left[-\frac{1}{3}, 1\right] \to \left[-\frac{1}{3}, 1\right]$ as

$$G_{\frac{7}{4}}(x) = T^+_{\frac{7}{4}, 0}(x) = \begin{cases} \frac{7}{4}x & \text{if } x < \frac{4}{7} \\ \frac{7}{4}x - 1 & \text{otherwise} \end{cases}$$

$$L_{\frac{7}{4}}(x) = T^-_{\frac{7}{4}, -\frac{1}{4}}(x) = \begin{cases} \frac{7}{4}x + \frac{1}{4} & \text{if } x \leq \frac{3}{7} \\ \frac{7}{4}x - \frac{3}{4} & \text{otherwise} \end{cases}$$

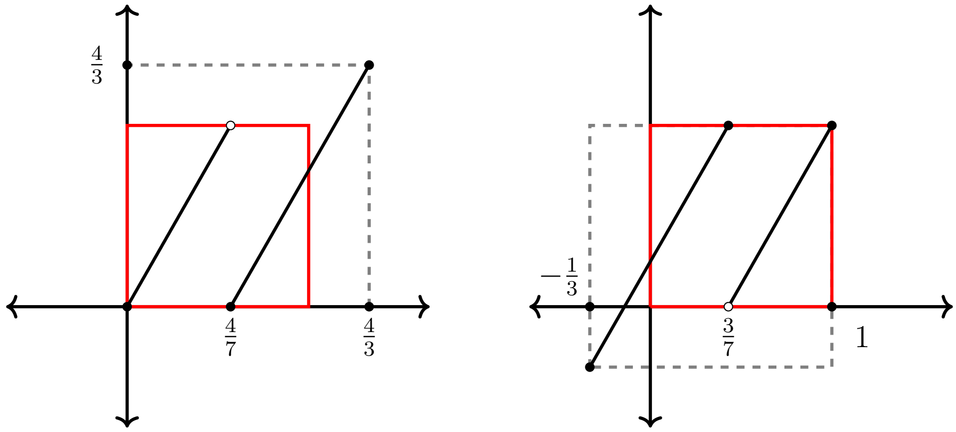

Figure 4.2: Greedy and Lazy Transformations. *We have the greedy (left) and lazy (right) expansions depicted. The gray square represents the domain of the respective systems. The red square represents the trapping region.*

Proposition 4.1.2.

The interval $[0, 1]$ is a trapping region for any β-transformation $T^{\pm}_{\beta,\alpha}$.

Proof. Let $x \in [0, 1]$. If $x = p = \frac{1-\alpha}{\beta}$, then $T^{+}_{\beta,\alpha}(x) = \beta\left(\frac{1-\alpha}{\beta}\right) + \alpha - 1 = 0$ and $T^{-}_{\beta,\alpha}(x) = \beta\left(\frac{1-\alpha}{\beta}\right) + \alpha = 1$, both of which are in $[0, 1]$. If $0 \leq x < p = \frac{1-\alpha}{\beta}$, then

$$0 \leq \alpha \leq \beta x + \alpha < 1$$

If $p = \frac{1-\alpha}{\beta} < x \leq 1$, then

$$0 < \beta x + \alpha - 1 \leq \beta + \alpha - 1$$

and since $\alpha \in [0, 2 - \beta]$, $\alpha + \beta - 1 \leq 1$. $\square$

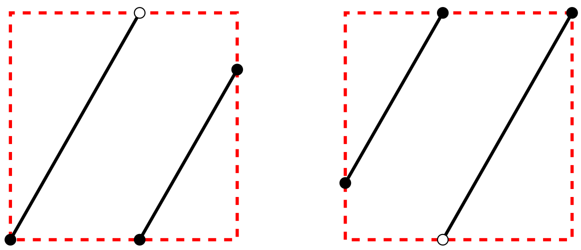

Figure 4.3: Trapping Regions for Greedy and Lazy Transformations. *Looking at the previous example (example 4.1.1), our trapping regions look this for the greedy (left) and lazy (right) β-transformations.*

4.2 β-Shifts

The next step is to look at the itinerary of a point under these transformations.

To that end, just as we did before, we define the *expansion maps* $\tau^{\pm}_{\beta,\alpha}(x)$ as

$$\tau^{\pm}_{\beta,\alpha}(x) : \left[\frac{-\alpha}{\beta-1}, \frac{1-\alpha}{\beta-1}\right] \to \Sigma_2^+$$

$$\tau^{\pm}_{\beta,\alpha}(x) = (\omega_0^{\pm}(x), \omega_1^{\pm}(x), \ldots)$$

where each $\omega_i^{\pm}$ is defined as

$$\omega_i^+(x) = \begin{cases} 0 & \text{if } (T^+_{\beta,\alpha})^i(x) < p \\ 1 & \text{otherwise} \end{cases} \qquad \omega_i^-(x) = \begin{cases} 0 & \text{if } (T^-_{\beta,\alpha})^i(x) \leq p \\ 1 & \text{otherwise} \end{cases}$$

Again, our cut-off point is $p = \frac{1-\alpha}{\beta}$. Note that the image of $\tau^+_{\beta,\alpha}(x)$ and $\tau^-_{\beta,\alpha}(x)$ will be some subsets of Σ_2^+; call them $\Omega^+_{\beta,\alpha}$ and $\Omega^-_{\beta,\alpha}$, respectively, and set $\Omega_{\beta,\alpha} = \Omega^+_{\beta,\alpha} \cup \Omega^-_{\beta,\alpha}$.

Proposition 4.2.1.

$\Omega_{\beta,\alpha}$ is a subshift of Σ_2^+. That is, it is closed and forward-shift invariant.

Proof. First, note that $\Omega_{\beta,\alpha}$ is forward-shift invariant since

$$\sigma(\tau^{\pm}_{\beta,\alpha}(x)) = (\omega^{\pm}_1(x), \omega^{\pm}_2(x), ...) = \tau^{\pm}_{\beta,\alpha}(T^{\pm}_{\beta,\alpha}(x))$$

and $T^{\pm}_{\beta,\alpha}(x) \in \left[\frac{-\alpha}{\beta-1}, \frac{1-\alpha}{\beta-1}\right]$. To show $\Omega_{\beta,\alpha}$ is closed, let $(a_i)_j \in \Omega_{\beta,\alpha}$ be a sequence of sequences in $\Omega_{\beta,\alpha}$. Note

$$\begin{aligned}
\lim_{j\to\infty} (a_i)_j &= \lim_{j\to\infty} (\omega(a_j)_0, \omega(a_j)_1, ...) \\
&= \lim_{j\to\infty} \tau^{\pm}_{\beta,\alpha}(a_j)
\end{aligned}$$

If $x_j \not\to p$, then we're done, since $\tau^{\pm}_{\beta,\alpha}$ is continuous away from p. If $x_j \to p$, then $\tau^{\pm}_{\beta,\alpha}(x_j) \to \tau^{+}_{\beta,\alpha}(p) = (0,0,0,...)$ or $\tau^{\pm}_{\beta,\alpha}(x_j) \to \tau^{-}_{\beta,\alpha}(p) = (1,1,1,...)$. In either case, the limit point is in $\Omega_{\alpha,\beta}$. $\square$

Similar to our prototypical example at the beginning of this section, there is a projection map $\pi_{\beta,\alpha} : \Omega_{\beta,\alpha} \to \left[\frac{-\alpha}{\beta-1}, \frac{1-\alpha}{\beta-1}\right]$ that operates as a left-inverse to $\tau^{\pm}_{\beta,\alpha}$ defined by

$$\pi_{\beta,\alpha}(\omega_0, \omega_1, ...) = \frac{\alpha}{1-\beta} + \sum_{i=0}^{\infty} \omega_i \beta^{-i-1}$$

Again we can construct the following commutative diagram:

$$\begin{tikzcd}
\Omega_{\beta,\alpha} \arrow[r, "\sigma"] \arrow[d, bend right, "\pi_{\beta,\alpha}"'] & \Omega_{\beta,\alpha} \arrow[d, bend right, "\pi_{\beta,\alpha}"'] \\
\left[\frac{-\alpha}{\beta-1}, \frac{1-\alpha}{\beta-1}\right] \arrow[r, "T^{\pm}_{\beta,\alpha}"] \arrow[u, bend right, "\tau^{\pm}_{\beta,\alpha}"'] & \left[\frac{-\alpha}{\beta-1}, \frac{1-\alpha}{\beta-1}\right] \arrow[u, bend right, "\tau^{\pm}_{\beta,\alpha}"']
\end{tikzcd} \tag{4.1}$$

The fact that this diagram commutes follows from the definition of the maps, see [8] for more details. Note as well that the systems $(\Omega^{\pm}_{\beta,\alpha}, \sigma)$ and $\left(\left[\frac{-\alpha}{\beta-1}, \frac{1-\alpha}{\beta-1}\right], T^{\pm}_{\beta,\alpha}\right)$ are topologically conjugate.

4.3 Kneading Invariants and Matching

Since SFT's are of interest in symbolic dynamics, it is natural to ask when $\Omega_{\beta,\alpha}$ is an SFT. There is a characterization of such $\Omega_{\beta,\alpha}$ spaces in terms of its *kneading invariants.*

Definition 4.3.1. (*Kneading Invariants*)

Let $\beta \in (1, 2)$ and $\alpha \in [0, 2-\beta]$. We call $\tau^{+}_{\beta,\alpha}(p)$ and $\tau^{-}_{\beta,\alpha}(p)$, the itineraries of the cut-off point, the *upper* and *lower kneading invariant* of $\Omega_{\beta,\alpha}$, respectively.

The following theorem is due to Ito and Takahashi [8] and explains how kneading invariants and SFTs are connected.

Theorem 4.3.2.

For $\beta \in (1, 2)$ and $\alpha \in [0, 2-\beta]$, the subshift $\Omega_{\beta,\alpha}$ is a subshift of finite type if and only if $\sigma(\tau^{+}_{\beta,\alpha}(p))$ and $\sigma(\tau^{-}_{\beta,\alpha}(p))$ are both periodic.

Since $(\Omega_{\beta,\alpha}, \sigma)$ and $([0, 1], T^{\pm}_{\beta,\alpha})$ are topologically conjugate, if $T^{+}_{\beta,\alpha}(p)$ and $T^{-}_{\beta,\alpha}(p)$ are both periodic then so are $\sigma(\tau^{+}_{\beta,\alpha}(p))$ and $\sigma(\tau^{-}_{\beta,\alpha}(p))$, so when trying to show $\Omega_{\beta,\alpha}$ is a subshift of finite type it is usually easier to demonstrate that $T^{+}_{\beta,\alpha}(p)$ and $T^{-}_{\beta,\alpha}(p)$ are both periodic. Note that

$$T^{+}_{\beta,\alpha}(p) = \beta\left(\frac{1-\alpha}{\beta}\right) + \alpha - 1 = 0 \quad \text{and} \quad T^{-}_{\beta,\alpha}(p) = \beta\left(\frac{1-\alpha}{\beta}\right) + \alpha = 1$$

and so in studying the periodicity of $T^+_{\beta,\alpha}(p)$ and $T^-_{\beta,\alpha}(p)$, we often consider the equivalent periodicity of 0 and 1 under successive applications of $T^{\pm}_{\beta,\alpha}$. Sometimes, after some n iterations, $(T^+_{\beta,\alpha})^n(0)$ and $(T^-_{\beta,\alpha})^n(1)$ will coincide; when that occurs, the periodicity of 0 and 1 are intimately linked. The next definition seeks to formalize this.

Definition 4.3.3. (*Matching*)

If there is some $n \in \mathbb{N}$ such that $(T^+_{\beta,\alpha})^n(0) = (T^-_{\beta,\alpha})^n(1)$, then the systems $([0,1], T^+_{\beta,\alpha})$ and $([0,1], T^-_{\beta,\alpha})$ are said to experience *matching*. The smallest such n where this equality holds is referred to as the *matching exponent*.

Note that other contexts allow unequal matching exponents and define matching equivalently as $(T^+_{\beta,\alpha})^n(0) = (T^-_{\beta,\alpha})^k(1)$ for $n, k \in \mathbb{N}$.

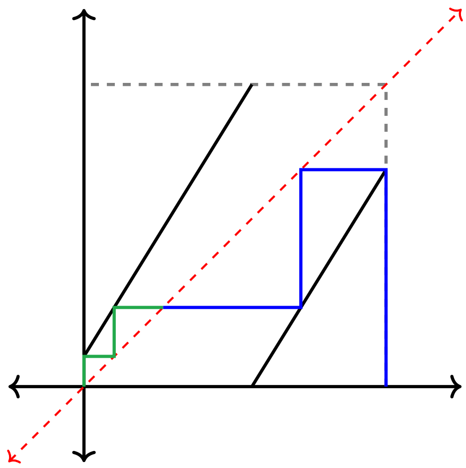

Figure 4.4: Matching Example. *If we let β be the golden ratio and $\alpha = 0.1$, then matching occurs at $n = 2$.*

Matching has recently attracted attention in the study of iterated piecewise maps and has been used to perform entropy calculations and find invariant measures. Our interest in matching is summarized in the following theorem. This idea appears in [11] but is formalized here.

Theorem 4.3.4.

Suppose $([0,1], T^+_{\beta,\alpha})$ and $([0,1], T^-_{\beta,\alpha})$ experience matching. Then $\tau^+_{\beta,\alpha}(p)$ is periodic if and only if $\tau^-_{\beta,\alpha}(p)$ is periodic.

Note that we're proving that $\tau^+_{\beta,\alpha}(p)$ is periodic if and only if $\tau^-_{\beta,\alpha}(p)$ is periodic, and the fact that $\sigma(\tau^+_{\beta,\alpha}(p))$ is periodic if and only if $\sigma(\tau^-_{\beta,\alpha}(p))$ is periodic follows immediately.

Proof. We will show periodicity of $\tau^-_{\beta,\alpha}(p)$ implies $\tau^+_{\beta,\alpha}(p)$ is periodic; the other direction follows almost identically.

By the commutative diagram (4.1), periodicity of $\tau^-_{\beta,\alpha}(p)$ implies that p is periodic under $T^-_{\beta,\alpha}$. That is, for some $n \in \mathbb{N}$, $(T^-_{\beta,\alpha})^n(p) = p$. Let n be the minimal n such that this occurs. Now let k be the matching exponent and let $j \in \mathbb{N}$ be the minimal natural number such that $j \cdot n > k$. Since $T^+_{\beta,\alpha}$ and $T^-_{\beta,\alpha}$ coincide everywhere on $[0,1]$ except at p, $(T^+_{\beta,\alpha})^i(0) = (T^-_{\beta,\alpha})^i(1)$ for $i \in \{k, ..., j \cdot n - 1\}$. We have

$$\begin{aligned}
(T^+_{\beta,\alpha})^{j \cdot n}(p) &= (T^+_{\beta,\alpha})^{j \cdot n - 1}(0) \\
&= (T^-_{\beta,\alpha})^{j \cdot n - 1}(1) && \text{by matching} \\
&= (T^-_{\beta,\alpha})^{j \cdot n}(p) \\
&= p && \text{by periodicity}
\end{aligned}$$

Thus $(T^+_{\beta,\alpha})^{j \cdot n}(p) = p$, so p is periodic under $T^+_{\beta,\alpha}$. Thus $\tau^+_{\beta,\alpha}(p)$ is periodic. $\square$

Now, instead of confirming that both $\sigma(\tau^+_{\beta,\alpha}(p))$ and $\sigma(\tau^-_{\beta,\alpha}(p))$ are both periodic, it is enough to check one and confirm matching. Fixing a β, the proof of theorem 1 in [11] demonstrates that the α for which $\tau^{+}_{\beta,\alpha}(p)$ is periodic is dense in the interval $[0, 2-\beta]$. Thus, if we can show matching occurs for α in some subinterval of $[0, 2-\beta]$, we can conclude that the set of α for which $\Omega_{\beta,\alpha}$ is a subshift of finite type is dense in that subinterval. We will see examples of this in the next sections.

4.4 Matching Symmetry

The plastic number, commonly denoted as ρ, is the unique real solution to $x^3 = x + 1$. It has exact value

$$\rho = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}} \approx 1.32471...$$

Proposition 4.4.1. [Zieber]

Let β be the plastic number. Then for α in the subintervals

$$\left[0, \frac{2-\beta^2}{1+\beta+\beta^2+\beta^3}\right) \quad \text{and} \quad \left(\frac{\beta^2+\beta+1}{1+\beta+\beta^2+\beta^3}, 2-\beta\right]$$

$T_{\beta,\alpha}$ experiences matching, and thus the α for which $\Omega_{\beta,\alpha}$ is a subshift of finite type is dense in these subintervals.

Proof. We will first demonstrate that for any α in the first interval,

$$\tau^{+}_{\beta,\alpha}(0) = (0,0,0,0,0,...) \quad \text{and} \quad \tau^{-}_{\beta,\alpha}(1) = (1,0,0,0,1,...)$$

We begin with the left equality. Note $0 < \frac{1-\alpha}{\beta}$ (since $1 \leq 2-\beta \leq \alpha$), so $\omega_0^+(0) = 0$. This implies the first iterate is of the form α.

For $\omega_1^+(0)$, note

$$(\beta+1)\alpha < (\beta+1)\frac{2-\beta^2}{1+\beta+\beta^2+\beta^3} = 0.0889776... < 1$$

thus $(\beta+1)\alpha < 1$ and $\alpha < p$, so $\omega_1^+(0) = 0$. This implies the second iterate is of the form $\alpha(1+\beta)$.

For $\omega_2^+(0)$, note

$$\alpha(1+\beta)\cdot\beta+\alpha = \alpha(1+\beta+\beta^2) < (1+\beta+\beta^2)\frac{2-\beta^2}{1+\beta+\beta^2+\beta^3} = 0.156145... < 1$$

thus $\alpha(1+\beta)\cdot\beta+\alpha < 1$ and $\alpha(1+\beta) < p$, so $\omega_2^+(0) = 0$. This implies the third iterate is of the form $\alpha(1+\beta+\beta^2)$.

For $\omega_3^+(0)$, note

$$\begin{aligned}\alpha(1+\beta+\beta^2)\cdot\beta+\alpha &= \alpha(1+\beta+\beta^2+\beta^3) \\ &< (1+\beta+\beta^2+\beta^3)\frac{2-\beta^2}{1+\beta+\beta^2+\beta^3} \\ &= 0.245122... < 1\end{aligned}$$

thus $\alpha(1+\beta+\beta^2)\cdot\beta+\alpha < 1$ and $\alpha(1+\beta+\beta^2) < p$, so $\omega_3^+(0) = 0$. This implies the fourth iterate is of the form $\alpha(1+\beta+\beta^2+\beta^3) = \alpha(2+2\beta+\beta^2)$.

For $\omega_4^+(0)$, note

$$\alpha(2+2\beta+\beta^2)\cdot\beta+\alpha = \alpha(1+2\beta+2\beta^2+\beta^3)$$

$$< (1 + 2\beta + 2\beta^2 + \beta^3)\frac{2 - \beta^2}{1 + \beta + \beta^2 + \beta^3}$$

$$= 0.362993... < 1$$

thus $\alpha(2+2\beta+\beta^2)\cdot\beta+\alpha < 1$ and $\alpha(2+2\beta+\beta^2) < p$, so $\omega_4^+(0) = 0$. This implies the fifth iterate is of the form $\alpha(2+2\beta+\beta^2)\beta+\alpha = \alpha(2+3\beta+2\beta^2)$.

Now we will demonstrate the right equality. Note $1 > \frac{1-\alpha}{\beta}$ (since $\beta > 1 > 1-\alpha$), so $\omega_0^-(1) = 1$. This implies the first iterate is of the form $\alpha+\beta-1$.

For $\omega_1^-(1)$, note

$$\beta(\alpha+\beta-1)+\alpha = \alpha(\beta+1)-\beta+\beta^2 < (\beta+1)\frac{2-\beta^2}{1+\beta+\beta^2+\beta^3}-\beta+\beta^2 = 0.519137... < 1$$

thus $\beta(\alpha+\beta-1)+\alpha < 1$ and $\alpha+\beta-1 < p$, so $\omega_1^-(1) = 0$. This implies the second iterate is of the form $\alpha(\beta+1)-\beta+\beta^2$.

For $\omega_2^-(1)$, note

$$\beta(\alpha(\beta+1)-\beta+\beta^2)+\alpha = \alpha(\beta^2+\beta+1)-\beta^2+\beta^3$$

$$< (\beta^2+\beta+1)\frac{2-\beta^2}{1+\beta+\beta^2+\beta^3} - \beta^2+\beta^3$$

$$= 0.725985... < 1$$

thus $\beta(\alpha(\beta+1)-\beta+\beta^2)+\alpha < 1$ and $\alpha(\beta+1)-\beta+\beta^2 < p$, so $\omega_2^-(1) = 0$. This implies the third iterate is of the form $\alpha(\beta^2+\beta+1)-\beta^2+\beta^3 = \alpha(\beta^2+\beta+1)+1+\beta-\beta^2$.

For $\omega_3^-(1)$, note

$$\begin{aligned}\beta(\alpha(\beta^2+\beta+1)+1+\beta-\beta^2)+\alpha &= \alpha(\beta^3+\beta^2+\beta+1)+\beta+\beta^2-\beta^3\\ &= \alpha(\beta^2+2\beta+2)-1+\beta^2\\ &< (\beta^2+2\beta+2)\frac{2-\beta^2}{1+\beta+\beta^2+\beta^3}-1+\beta^2\\ &= 1\end{aligned}$$

Note the strict equality here–thus the interval in the proposition statement is the maximal interval for which the itineraries are of the desired form. Therefore $\beta(\alpha(\beta^2+\beta+1)+1+\beta-\beta^2)+\alpha < 1$ and $\alpha(\beta^2+\beta+1)+1+\beta-\beta^2 < p$, so $\omega_3^-(1) = 0$. This implies the fourth iterate is of the form $\beta(\alpha(\beta^2+\beta+1)+1+\beta-\beta^2)+\alpha = \alpha(\beta^2+2\beta+2)-1+\beta^2$.

Lastly, for $\omega_4^-(1)$, note

$$\begin{aligned}\beta(\alpha(\beta^2+2\beta+2)-1+\beta^2)+\alpha &= \alpha(\beta^3+2\beta^2+2\beta+1)-\beta+\beta^3\\ &= \alpha(2\beta^2+3\beta+2)+1\\ &> 1\end{aligned}$$

thus $\beta(\alpha(\beta^2+2\beta+2)-1+\beta^2)+\alpha > 1$ and $\alpha(\beta^2+2\beta+2)-1+\beta^2 > p$, so $\omega_4^-(1) = 1$. This implies the fifth iterate is of the form $\beta(\alpha(\beta^2+2\beta+2)-1+\beta^2)+\alpha-1 = \alpha(2\beta^2+3\beta+2)$.

The fifth iterates of 0 and 1 are both $\alpha(2\beta^2+3\beta+2)$, and so

$$(T^+_{\beta,\alpha})^4(0) = (T^-_{\beta,\alpha})^4(1)$$

and we have matching. The computations for showing matching occurs in $\left(\frac{\beta^2+\beta+1}{\beta^3+\beta^2+\beta+1}, \beta - 2\right]$ run similarly. $\square$

Notice that these computations involve a significant number of polynomials of β with integer coefficients. Hence it is natural to investigate other values of β that are algebraic over $\mathbb{Z}$. Indeed we can show similar results for other $\beta \in (1, 2)$; in particular, the largest real solution of $x^3 = x^2 + 1$, known as the supergolden ratio ψ, and the largest real solution of $x^5 = x^4 + x^2 + 1$. For context, the supergolden ratio has exact form

$$\psi = \frac{1 + \sqrt[3]{\frac{29+3\sqrt{93}}{2}} + \sqrt[3]{\frac{29-3\sqrt{93}}{2}}}{3} \approx 1.46557...$$

while the largest real solution to $x^5 = x^4 + x^2 + 1$ is approximately 1.570147.

Proposition 4.4.2. [Zieber]

Let β be the supergolden ratio. Then for α in the subintervals

$$\left[0, \frac{1}{\beta^3 + \beta^2 + \beta + 1}\right) \quad \text{and} \quad \left(\frac{\beta^2 + 1}{\beta^3 + \beta^2 + \beta + 1}, 2 - \beta\right]$$

$T_{\beta,\alpha}$ experiences matching, and thus the α for which $\Omega_{\beta,\alpha}$ is a subshift of finite type is dense in these subintervals.

Proposition 4.4.3. [Zieber]

Let β be the largest real solution to $x^5 = x^4 + x^2 + 1$. Then for α in the subintervals

$$\left[0, \frac{1}{1+\beta+\beta^2+\beta^3+\beta^4+\beta^5}\right) \quad \text{and} \quad \left(\frac{\beta^4+\beta^2+1}{1+\beta+\beta^2+\beta^3+\beta^4+\beta^5}, 2-\beta\right]$$

$T_{\beta,\alpha}$ experiences matching, and thus the α for which $\Omega_{\beta,\alpha}$ is a subshift of finite type is dense in these subintervals.

This property that there are intervals of matching for relatively small and large α is not common to all $T_{\beta,\alpha}$. For $\beta = 1.5$, there is no matching for a significant interval of small α in the first 20 iterations (see figure 4.13).

However, there is another generalization that can be made. Notice that in each case we have symmetry; in fact, the two intervals in each pair are the same length. This observation is generalized in the following theorem, which is probably folklore.

Theorem 4.4.4. [Zieber]

$T_{\beta,\alpha}$ experiences matching if and only if $T_{\beta,2-\beta-\alpha}$ experiences matching. That is, the set of $\alpha \in [0, 2-\beta]$ for which $T_{\beta,\alpha}$ experiences matching is symmetric about the midpoint $1 - \frac{\beta}{2}$.

Before we prove the result, we require the following lemma.

Lemma 4.4.5.

$$(T^+_{\beta,\alpha})^n(x) = 1 - (T^-_{\beta,2-\beta-\alpha})^n(1-x) \quad \text{and} \quad (T^-_{\beta,\alpha})^n(x) = 1 - (T^+_{\beta,2-\beta-\alpha})^n(1-x)$$

for all $n \in \mathbb{N}$.

Proof. We will prove the left equality, and the right equality follows by swapping the roles of $\leq$ and $<$.

We proceed by induction. For the base case ($n = 1$):

$$\begin{aligned}
1 - T^-_{\beta,2-\beta-\alpha}(1-x) &= \begin{cases} 1 - (\beta(1-x) + 2 - \beta - \alpha) & 1 - x \leq 1 - \frac{1-\alpha}{\beta} \\ 1 - (\beta(1-x) + 2 - \beta - \alpha - 1) & 1 - x > 1 - \frac{1-\alpha}{\beta} \end{cases} \\
&= \begin{cases} \beta x + \alpha - 1 & x \geq \frac{1-\alpha}{\beta} \\ \beta x + \alpha & x < \frac{1-\alpha}{\beta} \end{cases} \\
&= T^+_{\beta,\alpha}(x)
\end{aligned}$$

Suppose the result holds for $1 \leq k \leq n$. Note

$$\begin{aligned}
(T^+_{\beta,\alpha})^n(x) &= 1 - (T^-_{\beta,2-\beta-\alpha})^n(1-x) \\
T^+_{\beta,\alpha}((T^+_{\beta,\alpha})^n(x)) &= 1 - T^-_{\beta,2-\beta-\alpha}(1 - (1 - (T^-_{\beta,2-\beta-\alpha})^n(1-x))) \\
(T^+_{\beta,\alpha})^{n+1}(x) &= 1 - T^-_{\beta,2-\beta-\alpha}((T^-_{\beta,2-\beta-\alpha})^n(1-x)) \\
(T^+_{\beta,\alpha})^{n+1}(x) &= 1 - (T^-_{\beta,2-\beta-\alpha})^{n+1}(1-x)
\end{aligned}$$

and so the identity holds. □

Now, for the proof of the main theorem:

Proof. For the forwards direction, suppose $T_{\beta,\alpha}$ experiences matching with matching exponent n. Then

$$\begin{aligned} 1-(T^{-}_{\beta,2-\beta-\alpha})^n(1) &= (T^{+}_{\beta,\alpha})^n(0) \\ &= (T^{-}_{\beta,\alpha})^n(1) \\ &= 1-(T^{+}_{\beta,2-\beta-\alpha})^n(0) \end{aligned}$$

So $(T^{-}_{\beta,2-\beta-\alpha})^n(1) = (T^{+}_{\beta,2-\beta-\alpha})^n(0)$, thus $T_{\beta,2-\beta-\alpha}$ experiences matching. This logic is reversible, and so reverse direction holds as well. □

4.5 Methods

One of the main tools in finding examples of symmetric matching was a dynamic cobweb diagram built in Mathematica. We were able to dynamically adjust β, α, and the number of iterations and see the effect on the trajectories of 0 and 1.

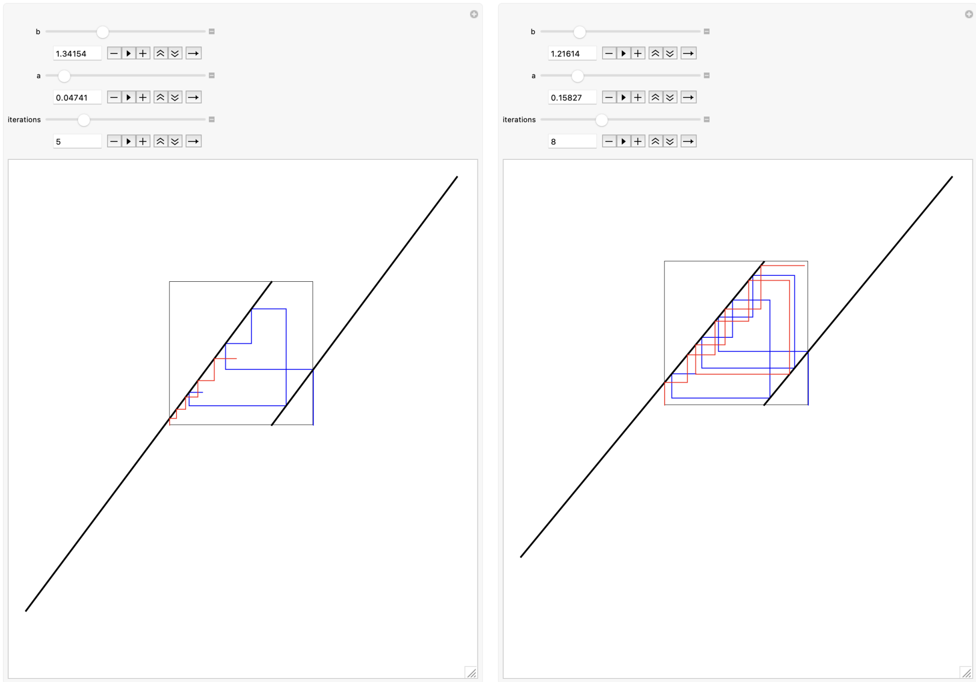

Figure 4.5: Dynamic Cobweb Module Stills. *Some stills of the dynamic cobweb module.*

This tool allowed us to investigate particular values of β and confirm that, when adjusting α, matching occurs in intervals. For the code, see Appendix B.

The other main tool that we developed was a "matching chart", which is a graph created by sampling $\alpha \in [0, 2 - \beta]$ and checking the first N iterations to see if matching occurs. For example, the matching chart for the supergolden ratio is:

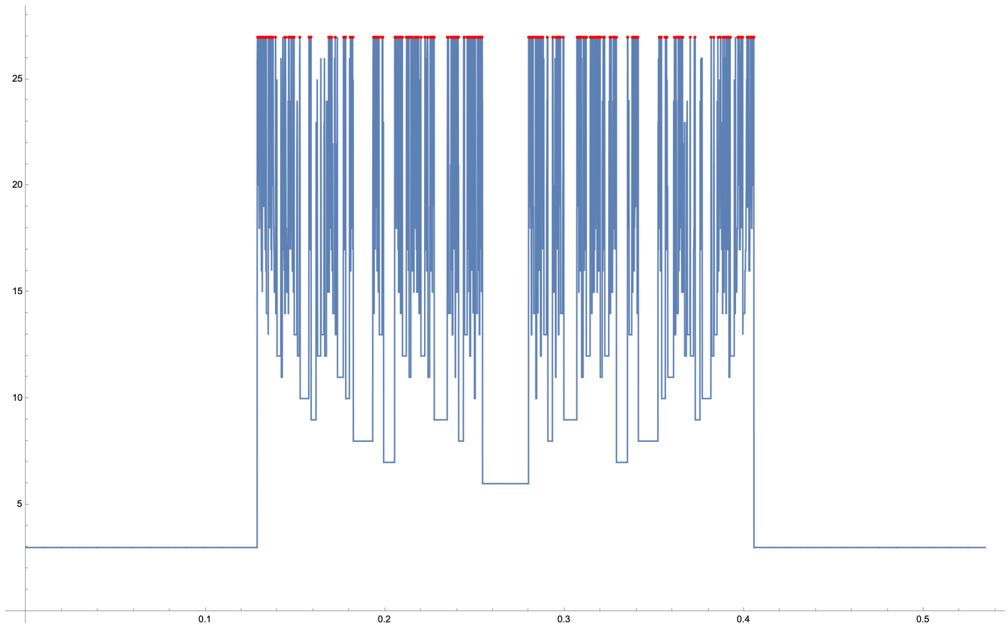

Figure 4.6: Supergolden Ratio Matching Chart. *Matching chart for the supergolden ratio.*

The horizontal axis is the α axis and the vertical axis is the number of iterations it took to match. So the horizontal line at 3 on the left indicates that matching occurred after 2 iterations for values of $\alpha \in [0, \sim 0.13)$. Red points at the top indicate that, for that α, no match was found in the first $N = 25$ iterations. This matching chart for the supergolden ratio clearly reflects the matching symmetry we proved in Theorem 4.4.4 (note that the chart will not be perfectly symmetrical since we are not sampling symmetrically). Similarly, we have the matching chart for the plastic number and the largest real solution to $x^5 = x^4 + x^2 + 1$ below.

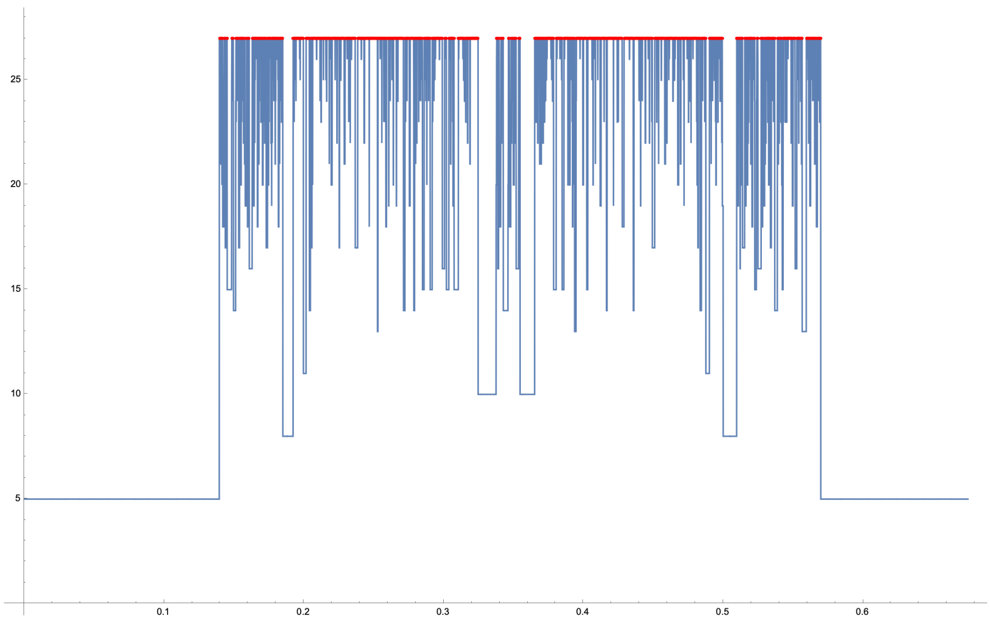

Figure 4.7: Plastic Number Matching Chart. *Matching chart for the plastic number.*

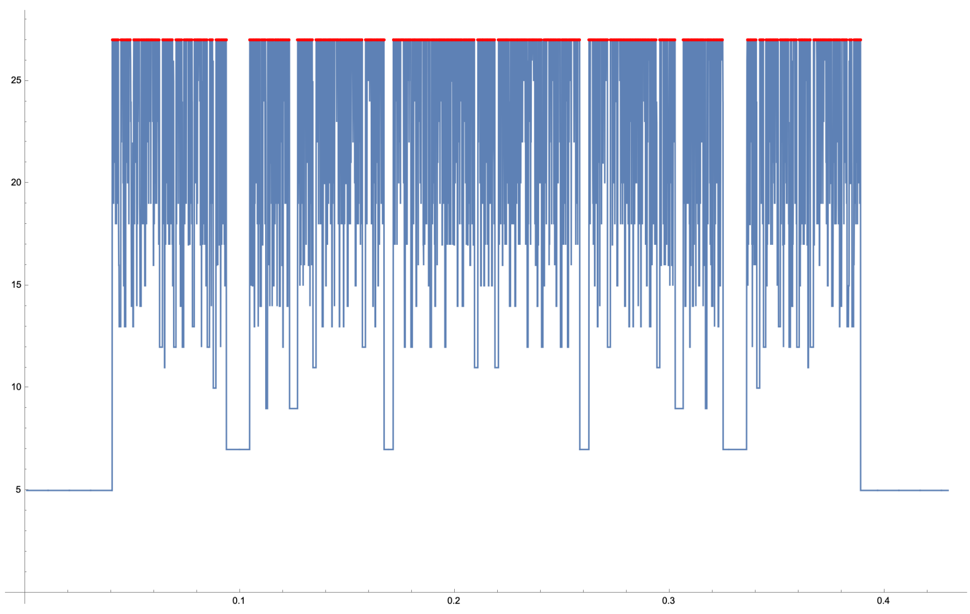

Figure 4.8: Quintic Root Matching Chart. *Matching chart for the largest real solution to* $x^5 = x^4 + x^2 + 1$.

Additionally, it was shown in [2] that for β equal to the golden ratio $T_{\beta,\alpha}$ experiences matching for all α. This is confirmed by the matching chart for the golden ratio.

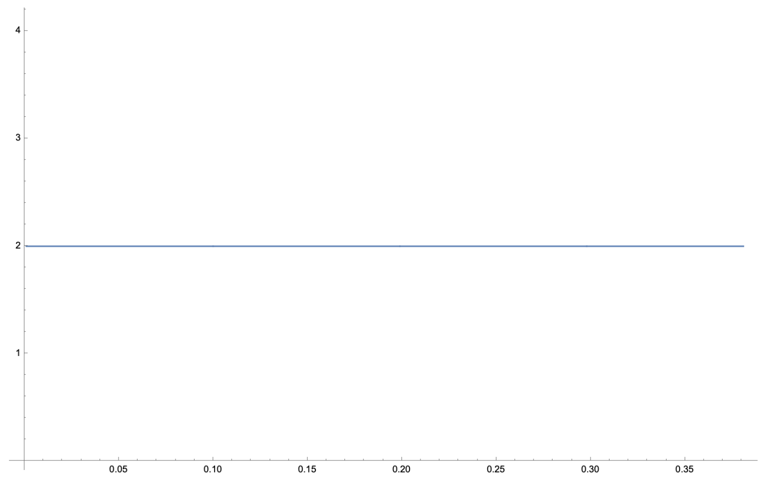

Figure 4.9: Golden Ratio Matching Chart. *Matching chart for the golden ratio.*

The golden ratio is an example of a multinacci number, the largest solution to $x^n = x^{n-1} + x^{n-2} + \cdots + x + 1$. In [11], it was shown that the N^{th} multinacci number experiences matching at step N for all α. For the 3^{rd}, 4^{th}, and 5^{th} multinacci numbers, we have numerical confirmation.

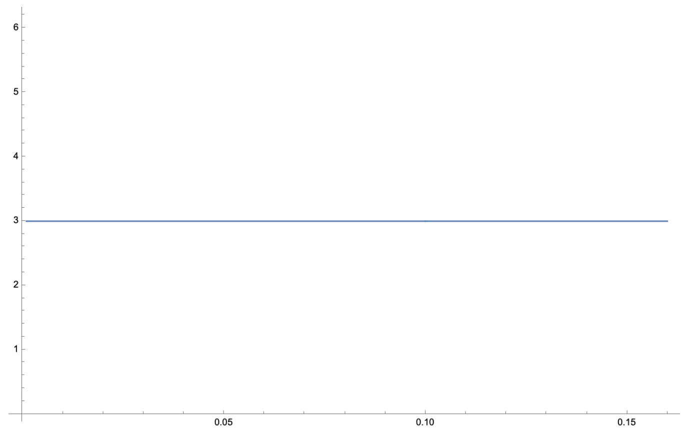

Figure 4.10: Third Multinacci Number Matching Chart. *Matching chart for the third multinacci number.*

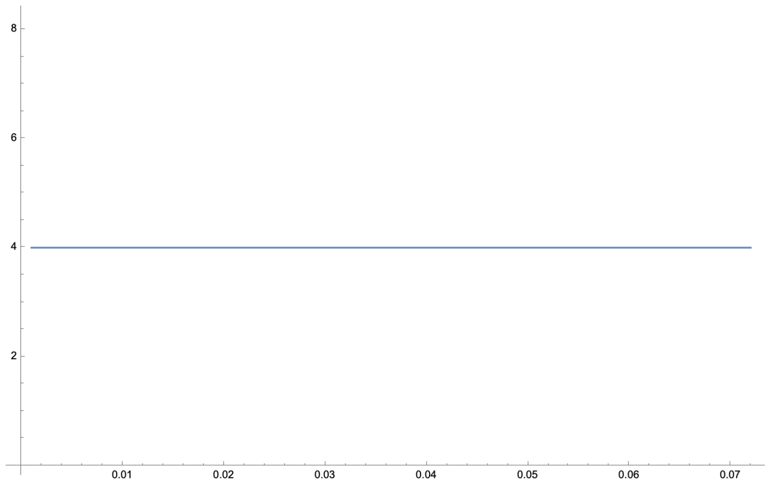

Figure 4.11: Fourth Multinacci Number Matching Chart. *Matching chart for the fourth multinacci number.*

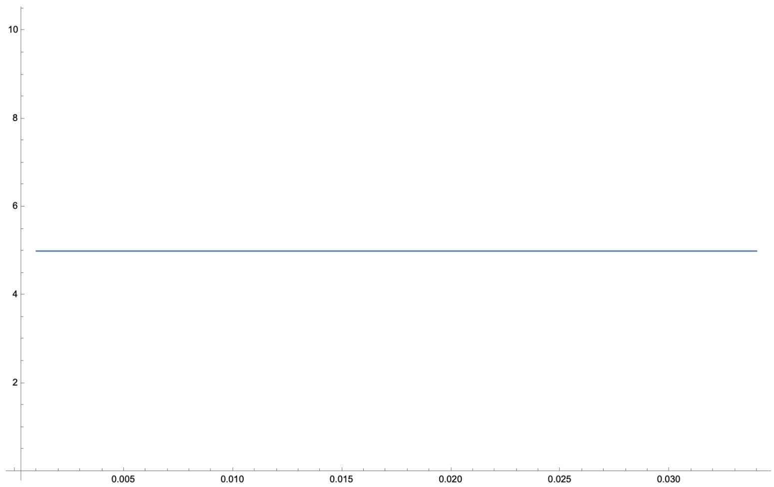

Figure 4.12: Fifth Multinacci Number Matching Chart. *Matching chart for the fifth multinacci number.*

For the code used to generate these charts, see Appendix C.

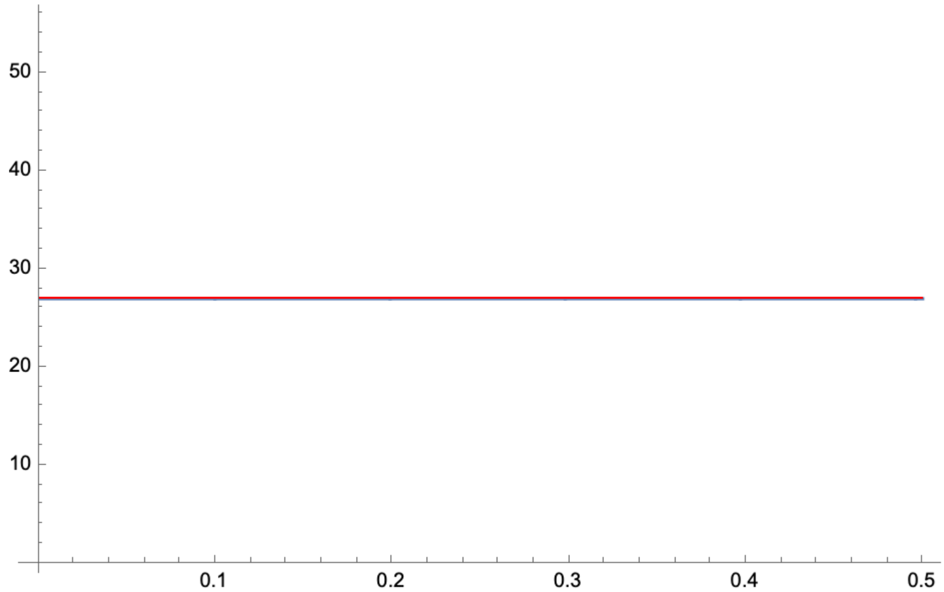

Figure 4.13: $\beta = \mathbf{1.5}$ **Matching Chart.** *Here, we confirm that there is no apparent matching for* $\beta = 1.5$.

4.6 Future Work

Several questions arose during our investigations that have not yet been fully explored. For example, much of the literature (including this thesis) has focused on fixing a β and investigating which values of α produce matching (or an SFT). It would be equally interesting to instead fix α and see which β produce matching. By considering the dynamic cobweb diagrams, it seems probable that the following conjecture is true.

Conjecture 4.6.1.

For a fixed $0 < \alpha < 1$, the β such that $T_{\beta,\alpha}$ experiences matching is dense in $(1, 2-\alpha)$.

Additionally, all of the examples in section 4.4 are Pisot numbers: real algebraic numbers greater than 1 whose Galois conjugates have moduli all strictly less than 1. Results about algebraic numbers and how they produce matching and subshifts of finite type have been largely restricted to Pisot numbers since they have nice properties. Salem numbers are similar to Pisot numbers, but with a twist. They are real algebraic numbers whose Galois conjugates have moduli less than or equal to 1 and at least one conjugate root with modulus equal to 1. Results on how $T_{\beta,\alpha}$ behaves when β is strictly a Salem number are sparse. Below is a matching graph for a Salem number:

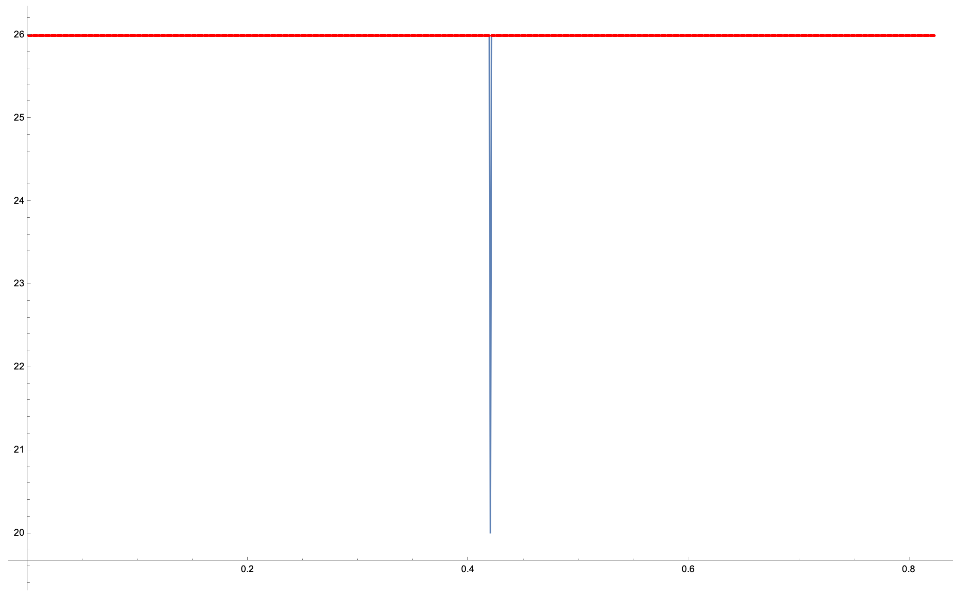

Figure 4.14: Salem Number Matching Chart. *Here, β is the largest real root of Lehmer's polynomial, which is $P(x) = x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$. Matching at smaller exponents for rational α seems much more sparse with Salem numbers than with Pisot numbers.*

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```
import os, sys

from PIL import Image
from PIL.Image import open as load_pic, new as new_pic

def main(path, iterations, keep_all=False, name="arnold_cat-{name}-{index}.png"):
    """
    Params
        path:str
            path to photograph
        iterations:int
            number of iterations to compute
        name:str
            formattable string to use as template for file names
    """
    title = os.path.splitext(os.path.split(path)[1])[0]
    counter = 0
    while counter < iterations:
        with Image.open(path) as image:
            print(type(image))
```

```python
            width = 256
            height = 256
            dim = (width, height)
            with new_pic(mode="RGB", size=dim) as canvas:
                for x in range(width):
                    for y in range(height):
                        nx = (2 * x + y) % width
                        ny = (x + y) % height

                        canvas.putpixel((nx, height-ny-1), image.getpixel((x, height-y-1)))

        if counter > 0 and not keep_all:
            os.remove(path)
        counter += 1
        print(counter, end="\r")
        path = name.format(name=title, index=counter)
        canvas.save(path)

    return canvas


if __name__ == "__main__":
    path = input("Enter the path to an image:\n\t")
```

```
44     while not os.path.exists(path):
45         path = input("Couldn't find your chosen image, please try again:\n\t")
46     result = main(path, 2)
47     result.show()
```

Listing A.1: This code is in Python and for other implementations requires you adjust the width and height be the number of pixels width-wise an height-wise your image has (see lines 21 and 22). Special thanks to Cameron Klig for assistance with debugging.

```
T[x_, beta_, alpha_] := Piecewise[{{beta*x + alpha, x < (1 - alpha)/beta}, {
    beta*x + alpha - 1, (1 - alpha)/beta <= x}}] Options[CobwebPlot] =
    Join[{CobStyle -> Automatic}, Options[Graphics]];
CobwebPlot0[f_, b_?NumericQ, a_?NumericQ, n_, xrange : {xmin_, xmax_},
     opts : OptionsPattern[]] := Module[{cob, x, g1, coor}, f2[x_] := f[x, b, a];
    cob = NestList[f2, 0, n]; (*Change middle input for different initialized
    point*)
    coor = Partition[ Riffle [cob, cob], 2, 1];
    coor [[1, 2]] = 0;
    cobstyle = OptionValue[CobwebPlot, CobStyle];
    cobstyle = If[cobstyle === Automatic, Red, cobstyle];
    g1 = Graphics[{cobstyle, Line[coor ]}];
    Show[{Plot[f2[x], {x, -1, 2}, PlotStyle -> {{Thick, Black}, Black}], g1},
    FilterRules[{opts}, Options[Graphics ]]]]
CobwebPlot1[f_, b_?NumericQ, a_?NumericQ, n_, xrange : {xmin_, xmax_},
     opts : OptionsPattern[]] := Module[{cob, x, g1, coor}, f2[x_] := f[x, b, a];
    cob = NestList[f2, 1, n]; (*Change middle input for different initialized
    point*)
    coor = Partition[ Riffle [cob, cob], 2, 1];
    coor [[1, 2]] = 0;
    cobstyle = OptionValue[CobwebPlot, CobStyle];
    cobstyle = If[cobstyle === Automatic, Blue, cobstyle];
    g1 = Graphics[{cobstyle, Line[coor ]}];
```

```
Show[{Plot[f2[x], {x, −1, 2}, PlotStyle −> {{Thick, Black}, Black}], g1},
FilterRules[{opts}, Options[Graphics]]]]
Manipulate[Show[ Graphics[{EdgeForm[Thin], White, Rectangle[]}],
CobwebPlot1[T[#, b, a] &, b, a, iterations, {−1, 2}, PlotRange −> {
Automatic, {−1, 2}}, Frame −> True, Axes −> False,
CobStyle −> Blue, PlotRangePadding −> None],
CobwebPlot0[T[#, b, a] &, b, a, iterations , {−1, 2},
PlotRange −> {Automatic, {−1, 2}}, Frame −> True, Axes −> False,
CobStyle −> Red, PlotRangePadding −> None]],
{b, 1.0001, 2, 0.00001, AnimationRate −> 0.0004},
{a, 0.0001, 2 − b, 0.00001, AnimationRate −> 0.0004},
{iterations , 1, 20, 1}, ContentSize −> {500, 500}]
```

```
(*Base functions and iterations, iterationslimit specified*)
T[x_,beta_,alpha_]:= Piecewise[{{beta*x+alpha,x<(1-alpha)/beta},{beta*x+
    alpha-1,(1-alpha)/beta<=x}}]
t[x_]:=T[x,\[Beta],\[Alpha]]
Omega[x_]:=If[x<(1-\[Alpha])/\[Beta],0,1]
 iterationslimit =20;
 iterations=iterationslimit +2;
 itinerary [x_]:=Map[Omega,NestList[t, x,iterations]](*tau map*)

(*Code for matching graph of plastic number*)
Clear[a,b,\[Alpha],\[Beta] ]
\[Beta] = 1/3 (27/2-(3 Sqrt[69])/2)^(1/3)+(1/2 (9+Sqrt[69]))^(1/3)/3^(2/3);
plotlistmatching={};
plotlistnomatching ={};
For[\[Alpha]=0.00005,
\[Alpha]<=2-\[Beta],
\[Alpha]=\[Alpha]+0.00005,
    For[x1=a;
    x2=b+a-1;
    i=0,
    i <= iterationslimit,
    i++,
```

```
    x1=If[itinerary[-\[Alpha]][[i+1]]==0,Expand[b*x1+a]/.b^3->b +1 , Expand[b*x1+a-1]/.b^3->b +1 ];
    x2=If[itinerary [\[Beta]-\[Alpha]-1][[i+1]]==0,Expand[b*x2+a]/.b^3->b +1,Expand[b*x2+a-1]/.b^3->b +1];
     If [Expand[x1]==Expand[x2],AppendTo[plotlistmatching,{\[Alpha],i+1}]];If[Expand[x1]==Expand[x2],Break[]];
     If [i==iterationslimit,AppendTo[plotlistnomatching,{\[Alpha],iterationslimit+1}]&&AppendTo[plotlistmatching,{\[Alpha],iterationslimit+1}]]]]
Show[ListLinePlot[plotlistmatching,PlotRange->All],ListPlot[plotlistnomatching,PlotStyle->{Red,PointSize[0.003]}]]

(*Code for matching graph of golden ratio*)
Clear[a,b,\[Alpha],\[Beta] ]
\[Beta] = GoldenRatio;
plotlistmatching={};
plotlistnomatching ={};
For[\[Alpha] =0.001,
\[Alpha]<=2-\[Beta],
\[Alpha]=\[Alpha]+0.001,
    For[x1=a;
    x2=b+a-1;
    i=0,
    i <= iterationslimit,
```

```
    i++,
    x1=If[itinerary [\[Alpha]][[ i+1]]==0,Expand[b*x1+a]/.b^2->b +1 ,
    Expand[b*x1+a-1]/.b^2->b +1 ];
    x2=If[itinerary [\[Beta]+\[Alpha]-1][[i+1]]==0,Expand[b*x2+a]/.b^2->b
    +1,Expand[b*x2+a-1]/.b^2->b +1];
    If [Expand[x1]==Expand[x2],AppendTo[plotlistmatching,{\[Alpha],i+2}]];
    If [Expand[x1]==Expand[x2],Break[]];
    If [ i==iterationslimit,AppendTo[plotlistnomatching,{\[Alpha],
    iterationslimit+2}]&&AppendTo[plotlistmatching,{\[Alpha],iterationslimit
    +2}]]]]
Show[ListLinePlot[plotlistmatching,PlotRange->All],ListPlot[
    plotlistnomatching,PlotStyle->{Red,PointSize[0.003]}]]

(*Code for matching graph of root of quintic*)
Clear[a,b,\[Alpha],\[Beta] ]
\[Beta] = 1.57014731219605436 29;
plotlistmatching={};
plotlistnomatching ={};
For[\[Alpha] =0.001,
\[Alpha]<=2-\[Beta],
\[Alpha]=\[Alpha]+0.0001,
    For[x1=a;
    x2=b+a-1;
    i=0,
```

```
    i <= iterationslimit,
    i++,
    x1=If[itinerary [\[Alpha]][[i+1]]==0,Expand[b*x1+a]/.b^5->b^4+b^2+1
     ,Expand[b*x1+a-1]/.b^5->b^4+b^2+1 ];
    x2=If[itinerary [\[Beta]+\[Alpha]-1][[i+1]]==0,Expand[b*x2+a]/.b^5->b
    ^4+b^2+1,Expand[b*x2+a-1]/.b^5->b^4+b^2+1];
     If [Expand[x1]==Expand[x2],AppendTo[plotlistmatching,{\[Alpha],i+2}]];If[
    Expand[x1]==Expand[x2],Break[]];
     If [i==iterationslimit,AppendTo[plotlistnomatching,{\[Alpha],
    iterationslimit+2}]&&AppendTo[plotlistmatching,{\[Alpha],iterationslimit
    +2}]]]]
Show[ListLinePlot[plotlistmatching,PlotRange->All],ListPlot[
    plotlistnomatching,PlotStyle->{Red,PointSize[0.003]}]]

(*Code for matching graph of supergolden ratio*)
Clear[a,b,\[Alpha],\[Beta] ]
\[Beta] = 1/3 (1+(29/2-(3 Sqrt[93])/2)^(1/3)+(1/2 (29+3 Sqrt[93]))^(1/3));
plotlistmatching={};
plotlistnomatching ={};
For[\[Alpha] =0.0001,
\[Alpha]<=2-\[Beta],
\[Alpha]=\[Alpha]+0.0001,
    For[x1=a;
    x2=b+a-1;
```

```
    i=0,
    i <= iterationslimit,
    i++,
    x1=If[itinerary [\[Alpha]][[i+1]]==0,Expand[b*x1+a]/.b^3->b^2+1 , Expand[b*x1+a-1]/.b^3->b^2+1 ];
    x2=If[itinerary [\[Beta]+\[Alpha]-1][[i+1]]==0,Expand[b*x2+a]/.b^3->b^2+1,Expand[b*x2+a-1]/.b^3->b^2+1];
    If [Expand[x1]==Expand[x2],AppendTo[plotlistmatching,{\[Alpha],i+2}]];
    If [Expand[x1]==Expand[x2],Break[]];If[i==iterationslimit,AppendTo[plotlistnomatching,{\[Alpha],iterationslimit+2}]&&AppendTo[plotlistmatching,{\[Alpha],iterationslimit+2}]]]]
Show[ListLinePlot[plotlistmatching,PlotRange->All],ListPlot[plotlistnomatching,PlotStyle->{Red,PointSize[0.003]}]]
```