Information Design with Equilibrium Selection

Abstract

This paper extends the solution concept in information design problems, in which a designer aims to implement a particular game outcome by controlling the structure of signals that the players receive. Specifically, we consider settings in which an equilibrium is implementable only if it satisfies an exogenous selection criterion. We focus on optimal information design in a common-interest two-player game with binary actions, requiring the selected equilibrium to satisfy risk dominance. We provide a method for the designer to maximize the probability of making the best equilibrium risk dominant, and show how to extend our approach to other settings.

Keywords—Bayesian persuasion, information design, Neyman-Pearson Lemma

JEL Classifications—C70, C72, D82, D83

In an information design problem, an individual, called the *designer*, can affect the beliefs of players in a game of imperfect information. She has a preferred equilibrium, and cannot change the basic structure of the game. However, she can control the signals the players receive in different states of nature. One aspect that studies of information design typically do not address is whether the designer can control which equilibrium the players coordinate on, if the game has multiple equilibria. Extending the solution concept to incorporate equilibrium selection criteria is the main focus of this article.

The standard solution concept, Bayes Correlated Equilibrium, requires only that the designer's preferred outcome is a Bayesian Nash equilibrium. The designer chooses the signal structure, and then the players receive the signals and play the game (which we call the *continuation game*), and are typically treated as willing to obey the designer's recommendation as long as it is part of an equilibrium. The players are willing to put their faith in the invisible hand.

In some problems, however, the players might require an equilibrium to satisfy a refinement. A canonical example is a coordination game, such as a stag hunt. Hunting for a stag is in the players' common interest, but experimental evidence and evolutionary game-theoretic studies suggest players may be willing to do so only if it is also risk dominant (examples and discussion of risk dominance include van Huyck, Battalio and Beil, 1990; Crawford, 1991; Cooper et al., 1992; Kandori, Mailath and Rob, 1993; Battalio, Samuelson and van Huyck, 2001; Schmidt et al., 2003; Binmore, 2007; Anctil et al., 2010). The designer needs to reassure the players that the strategic risk of coordination failure is not excessive, because the players seek guidance from the invisible paw.[1] This constrains the designer, and she must consider the risk dominance criterion when deciding what information to commit to providing.

In an economic context, consider a coordination game between a solvent but illiquid borrower and two creditors. The borrower is the designer and the continuation game consists of the creditors choosing between rolling their debts over or seizing

[1] To our disappointment, someone else thought of the invisible paw metaphor first. See Wolgast (1984).

their collateral. To focus on the information and coordination issue, assume for now that the creditors are symmetric; we relax this assumption below. Further assume that the players have shared prior beliefs that are commonly known, so that the designer's task is not complicated by (her or the players') uncertainty about how anyone would interpret the signals she designs. The borrower prefers to stay in business, and would like her chosen signal structure to implement an outcome in which the creditors roll over. This may be an equilibrium, but for familiar reasons, seizing the collateral may also be an equilibrium. No one benefits from runs; however, the creditors might need reassurance that rolling over is a risk dominant equilibrium.

The designer's problem is to choose the signal structure that maximizes the probability that rolling over is a risk dominant equilibrium, conditional on the information the borrowers receive. In a symmetric (2×2) game, the designer's problem has a particularly enchanting solution. The risk dominance constraint collapses into a condition in which she can focus on the payoffs in each state separately, attach prices to each state, and rank the states by the price (on a technical level, her solution is analogous to applying the Neyman-Pearson Lemma). As we show, this approach is quite robust, working even if the designer's objective function changes. We show this by considering a case in which the designer aims to maximize the expected payments to the players, rather than maximizing the probability of the players rolling over.

If the players' payoffs are asymmetric, the designer's problem is generally more complicated, as she cannot necessarily price the states independently. As we show, however, the designer can use a fully constructive algorithm, which would map the asymmetric game to an equivalent symmetric game, if such a symmetric game exists, and would indicate to her whether there is no such equivalent symmetric game. If the algorithm succeeds, she can solve the corresponding symmetric game, pricing the states one by one, and know that she has solved her original problem. Otherwise, she can use Kuhn-Tucker to solve the (generally harder) constrained optimization problem.

By allowing for settings in which the players may not select the designer's preferred equilibrium, we relax the solution concept from Bayes Correlated Equilibrium. This extends work by Mathevet, Perego and Taneva (2020), Morris, Oyama and Takahashi (2020), and Inostroza and Pavan (2022) that addresses the extreme case of adversarial preferences between the players and designer. Examples of cases with misaligned but not fully adversarial preferences are in Candogan (2020). Good overviews of the standard solution concept in information design and prior extensions are in Bergemann and Morris (2019); Taneva (2019). Our approach builds on work exploiting the linear programming interpretation of Bayesian persuasion (Kolotilin, 2018; Dworczak and Kolotilin, 2019; Dworczak and Martini, 2019), which has recently been extended to information design (Doval and Skreta, 2021). In contrast to Bayesian persuasion (Kamenica and Gentzkow, 2011), our information design setting has the challenge of a sender addressing multiple receivers, who may have different payoffs and who interact strategically.

The structure of the rest of this article is as follows. In Section 1 we study information design in a symmetric, common interest coordination game. Section 2 allows for heterogeneity in the payoffs in the continuation game. Section 3 discusses robustness. Section 4 concludes.

1 A Symmetric Coordination Game

We begin by considering a game with three actors: a designer (she) and two players (both he) of a symmetric (2×2) common interest game of imperfect information, referred to as the continuation game. Label the players as $\{1, 2\}$. We assume throughout that all parties have rational expectations, that they share a common prior, and that all communication is public. In the context of a rollover application, allowing for private communication would raise additional issues (e.g., legal or litigation questions) that are beyond the scope of our paper; for discussion of privately informed receivers, see Kolotilin et al. (2017).

The two players move simultaneously. Each player's action set is $\{L, R\}$. Let a_i be the action of player $i \in \{1, 2\}$. The payoffs, described below, depend on an unknown state. Uncertainty is characterized by a probability space $(\Omega, \mathcal{F}, \psi)$ in which ψ is a common prior. If Ω is discrete and $\omega_k \in \Omega$, we write $\psi(\omega_k)$ or ψ_k as a shorthand for $\psi(\{\omega_k\})$. Let r and ℓ be ($\mathcal{F}$-measurable) nonnegative random variables. Assume that $r(\cdot)$ weakly statewise dominates $\ell(\cdot)$; we relax this assumption below in Section 3.

Given state $\omega \in \Omega$ and action profile $(a_1, a_2) \in \{L, R\}^2$, let $i, j \in \{1, 2\}$ with $i \neq j$. Player i's payoff is

$$u_i(a_i, a_j; \omega) = \begin{cases} \ell(\omega), & \text{if } a_i = L \\ r(\omega), & \text{if } a_i = a_j = R \\ 0, & \text{if } a_i = R, a_j = L \end{cases} \tag{1}$$

The players choose their actions simultaneously after observing a public signal, which the designer can influence as described below, but before learning the state. There are two pure strategy equilibria in the continuation game: (R, R), which is Pareto dominant, and (L, L).[2] See Figure 1 for the payoffs; expectations are conditional on players' information.

Player 1 \ Player 2	L	R
L	$\mathrm{E}[\ell]$, $\mathrm{E}[\ell]$	$\mathrm{E}[\ell]$, 0
R	0, $\mathrm{E}[\ell]$	$\mathrm{E}[r]$, $\mathrm{E}[r]$

Figure 1: Payoff matrix of the (2×2) game.

[2] There is a mixed strategy equilibrium, which is never risk dominant and payoff equivalent to (L, L). We do not mention it further.

The designer's objective is to maximize the probability that the players select the Pareto-dominant equilibrium. Her payoff is

$$u_D(a_1, a_2) = \begin{cases} 1, & \text{if } a_1 = a_2 = R \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Strategic risk may induce the players to coordinate on the Pareto inferior equilibrium (L, L). Each player may consider how much he has to lose by guessing a different equilibrium from the one the other player has in mind. From Figure 1, a player who chooses R when the other player believes the (L, L) equilibrium is being selected would lose $\mathrm{E}[\ell]$. One who chooses L when the other player believes the (R, R) equilibrium is being selected would lose $\mathrm{E}[r] - \mathrm{E}[\ell]$.

Harsanyi and Selten (1988) use this intuition to propose an equilibrium refinement called *risk dominance*. Under this refinement, the selected equilibrium maximizes the product of the deviation losses to each player. Thus, if $\mathrm{E}[\ell]^2 \geq (\mathrm{E}[r] - \mathrm{E}[\ell])^2$, the (L, L) equilibrium is weakly risk dominant, and if this inequality is reversed, the (R, R) equilibrium is weakly risk dominant. By the assumptions that r statewise dominates ℓ and that r and ℓ are nonnegative, the Pareto-dominant equilibrium (R, R) is weakly risk dominant if and only if

$$\mathrm{E}[r] \geq 2\mathrm{E}[\ell] \tag{3}$$

An easy but important consequence of (3) is that when the payoffs are symmetric, the effects of each state on risk dominance can be analyzed in isolation. For instance, if the state space Ω is finite and equal to $\{\omega_1, \ldots, \omega_n\}$, (3) becomes

$$\sum_{k=1}^{n} [2\ell(\omega_k) - r(\omega_k)]\psi_k \leq 0 \tag{4}$$

Inequality (4) means that the contribution of ω_k to whether (R, R) is risk dominant does not depend on ω_j for any $j \neq k$.

If the players in the continuation game are willing to select an equilibrium only if it is risk dominant, then the designer's objective is to choose a signal structure that maximizes the probability that the (R, R) equilibrium is risk dominant in expectation conditional on the signals. As is standard in information design, we assume the designer can commit ex ante to release a signal about the state, along with a recommended action profile that the players would be willing to obey given the signal.

If (3) or, equivalently, (4) holds ex ante, the designer's problem is trivial (release no information about the state and recommend that both players choose R). Otherwise, to solve her problem, the designer partitions Ω into three $\mathcal{F}$-measurable subsets. In one, G_0, she recommends the (R, R) equilibrium by releasing public signal g. In another, B, she recommends playing (L, L), by releasing public signal b. She may also need a third set, G_1, on which she mixes, releasing public signal g or b according to an independent randomization device.

The players observe public signal g or b and then make their decisions. The partition and the designer's recommended actions are common knowledge. We summarize the sequence of events in Figure 2.

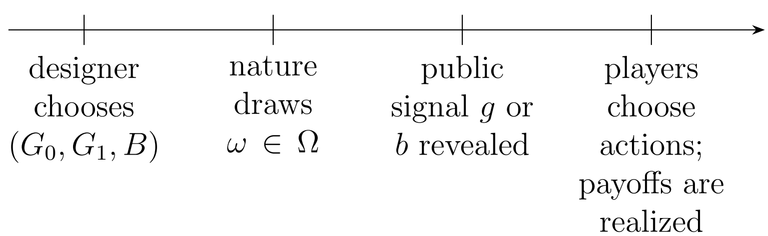

Figure 2: Timeline

1.1 A three-state example

Consider the following three-state example.

Example 1.1. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, let $(\psi_1, \psi_2, \psi_3) = (0.5, 0.2, 0.3)$, and let the corresponding payoffs be $r(\omega) = (9, 4, 3)$ and $\ell(\omega) = (5.5, 0, 2)$.

Consider the ex ante expected payoffs to each player:

$$\begin{aligned} \mathrm{E}[r] &= 0.5 \cdot 9 + 0.2 \cdot 4 + 0.3 \cdot 3 = 6.2 \\ \mathrm{E}[\ell] &= 0.5 \cdot 5.5 + 0.2 \cdot 0 + 0.3 \cdot 2 = 3.35 \end{aligned}$$

By (3), the (R, R) equilibrium is not risk dominant ex ante because $\mathrm{E}[r] = 6.2 < 2\mathrm{E}[\ell] = 6.7$. Therefore, without additional information, the players would select the equilibrium (L, L).

The designer can clearly do better. An easy improvement would be to disclose the state. Equivalently, she could let $G_0 = \{\omega_2\}, G_1 = \varnothing$, and $B = \{\omega_1, \omega_3\}$, telling the players whether, conditional on the state, (R, R) is risk dominant. Coordination is successful with probability $\psi_2 = 0.2$, and each player's expected payoff increases by $4 \cdot 0.2 = 0.8$. This strategy is suboptimal, but a step in the right direction.

To find the designer's optimal strategy, we write her problem as follows:

$$\max_{q_1, q_2, q_3} \sum_{k=1}^{3} q_k \tag{5}$$

subject to

$$(\forall k \in \{1, \ldots, 3\}) \quad 0 \leq q_k \leq \psi_k \qquad \text{(feasibility)} \tag{6}$$

$$2\mathrm{E}[\ell | g; q_1, q_2, q_3] - \mathrm{E}[r | g; q_1, q_2, q_3] \leq 0 \qquad \text{(risk dominance constraint)} \tag{7}$$

Note that this objective function straightforwardly generalizes to an n-state setting for any positive integer n.

In (5), q_k represents the joint probability of state ω_k and the good signal g. Thus, if $q_k = \psi_k$, the designer issues signal g if state ω_k occurs (so that $\omega_k \in G_0$). If $q_k \in (0, \psi_k)$, then state $\omega_k \in G_1$, and the designer mixes. In this case, if ω_k occurs, she reports g with probability q_k/ψ_k. If $q_k = 0$ and ω_k occurs, she reports b.

For the risk dominance constraint, we can use the same reasoning by which we rewrite (3) as (4), and replace the conditional expectation formulation of (7) with

$$\sum_{k=1}^{3} [2\ell(\omega_k) - r(\omega_k)]\, q_k \leq 0 \tag{8}$$

This formulation has the same form as a budget constraint, in which

$$p_k := 2\ell(\omega_k) - r(\omega_k)$$

is the price associated with state ω_k. Thus we refer to (8) as the designer's budget constraint. From the statewise values of ℓ and r in Example 1.1, we have $p = (p_1, p_2, p_3) = (2, -4, 1)$.

Viewed in this light, the designer's problem is equivalent to a consumer's problem, with her objective as (5), which she maximizes subject to her feasibility constraint (6) and her budget constraint (8). Intuitively, $[2\ell(\omega_k) - r(\omega_k)]q_k$ is the amount that consuming q_k (i.e., incorporating q_k units of ω_k in G) moves the desirable equilibrium toward (or below) the risk dominance constraint.

Some states may have a negative price, such as ω_2 in the example. In these states, the (R, R) equilibrium is risk dominant, corresponding to the designer's endowment in the consumer problem interpretation.

We solve the consumer's problem in steps. First, she consumes all states with nonpositive prices to full capacity. In the current example, state ω_2 has a negative price, so she sets $q_2^* = \psi_2 = 0.2$. This creates slack of $-4 \cdot 0.2 = -0.8$ in her budget constraint (8).

Next, among the states with positive prices, she considers the ratio of the marginal utility of each q_k (which we denote by MU_k) to its price. In the example,

$$\frac{MU_1}{p_1} = \frac{1}{2} < \frac{1}{1} = \frac{MU_3}{p_3}$$

Therefore, she always gets more marginal utility from spending on q_3 than from spending on q_1. She consumes q_3 until either her feasibility constraint or her budget constraint binds. If she consumes q_3 to full capacity ψ_3, the amount of slack left in her budget constraint is

$$p_2\psi_2 + p_3\psi_3 = -4 \cdot 0.2 + 1 \cdot 0.3 = -0.5 < 0 \tag{9}$$

leaving her with resources to spend on q_1. As q_1 has positive marginal utility, she consumes it until either there is no more of it available or she runs out of resources.

The designer does not have sufficient slack left in her budget to consume all of ψ_1, so she spends the rest of her budget on q_1. From (8) and (9),

$$p_1 q_1^* - 0.5 = 0$$

so that

$$q_1^* = \frac{0.5}{p_1} = 0.25$$

We can think of the designer's optimal information structure as follows: if the state is in $\{\omega_2, \omega_3\}$, she always reports g, so these two states form her set G_0. If the state is ω_1, she mixes, so $G_1 = \{\omega_1\}$. As $\psi_1 = 0.5$ but $q_1^* = 0.25$, we can see that the designer mixes with probability 0.5 if state ω_1 is realized. The maximized probability of coordination on (R, R) is 0.75, compared with the 0.2 she would achieve by fully disclosing the state. The expected payoff to each player increases from 4.15 if she fully discloses the state to 5.325.

1.2 A general state-space: results and interpretation

The general structure of the designer's problem is that of finding a subset of a state space with maximal measure, subject to a constraint:

$$\max_{G \in \mathcal{F}} \int_{\omega \in G} d\psi(\omega) \tag{10}$$

$$\text{s.t.} \int_{\omega \in G} [2\ell(\omega) - r(\omega)]\, d\psi(\omega) \leq 0 \tag{11}$$

As in the discrete example above, the designer solves this problem by evaluating each state individually (the solution is an application of the Neyman-Pearson Lemma). In particular, the designer forms a cutoff c, and reports g when state ω occurs only if $2\ell(\omega) - r(\omega) \leq c$.

Theorem 1. *Suppose the payoff dominated equilibrium (L, L) is ex ante risk dominant, but that the Pareto dominant equilibrium (R, R) has positive probability of being risk dominant ex post. Then the designer's optimal strategy is to pick a cutoff $c > 0$ such that, for each $\omega \in \Omega$*

$$\begin{aligned}
&\textit{If } 2\ell(\omega) - r(\omega) < c, \textit{ then } \omega \in G_0 \\
&\textit{If } 2\ell(\omega) - r(\omega) > c, \textit{ then } \omega \in B \\
&\textit{If } 2\ell(\omega) - r(\omega) = c, \textit{ then } \omega \in G_1
\end{aligned}$$

with the mixing probability for states in G_1 chosen to make (R, R) weakly risk dominant, i.e., to make

$$\mathrm{E}[2\ell(\omega) - r(\omega)|g] = 0$$

Note that if the Pareto dominant equilibrium (R, R) is ex ante weakly risk dominant, the designer can optimally give a degenerate signal (such as setting $G_0 = \Omega$, i.e., always reporting g). Similarly, if the dominated equilibrium (L, L) is ex post risk dominant with probability 1, then the expected payoff to the designer is 0 regardless of her strategy.

As in the discrete example above, the designer's problem is analogous to a consumer's problem, with (10) as the designer's objective and (11) having the form of a budget constraint, analogous to (4). For a given $\omega \in \Omega$, we write $p(\omega) := 2\ell(\omega) - r(\omega)$ as the price associated with consuming ω. Because the designer receives constant marginal utility from any increase in the probability of reporting g, the solution to her problem has the simple boundary condition: states ω, ω' are both on her boundary (i.e., both in G_1) if and only if

$$\frac{\partial U/\partial\omega}{p(\omega)} = \frac{\partial U/\partial\omega'}{p(\omega')} \quad \Leftrightarrow \quad \frac{1}{2\ell(\omega) - r(\omega)} = \frac{1}{2\ell(\omega') - r(\omega')} \tag{12}$$

Equation (12) shows the intuition of Theorem 1. Any states on the boundary must have the same price, because they generate the same marginal utility. Call this price c. Any state ω'' with a lower price than c would always be included in G_0, and any state with a higher price than c is too costly and therefore is in B.

Her boundary satisfies $2\ell(\omega) - r(\omega) = c$, i.e., it has the same slope of 2 as her risk dominance constraint. In other words, she solves her problem by shifting her risk dominance line to the right, without changing the slope. Figure 3 illustrates.

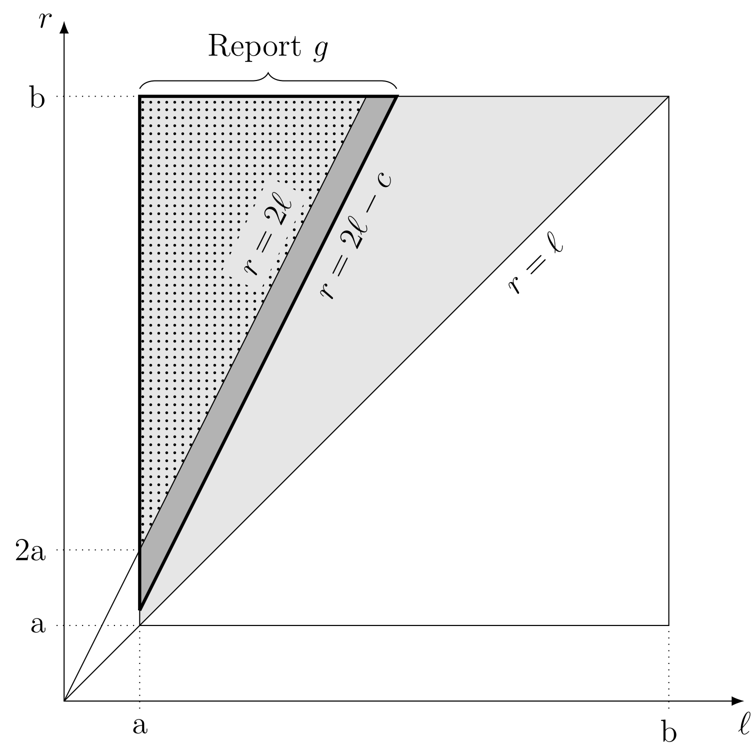

Figure 3: The designer's optimal disclosure region (dotted triangle and dark gray trapezoid)

In the figure, the support of (r, ℓ) is $[a, b]^2$. Because r statewise dominates ℓ, the set of possible ex post values of r and ℓ is the gray triangle with the 45°-line as its hypotenuse. The set of states in which (R, R) is risk dominant is shown as a dotted right triangle, on and above the $r = 2\ell$ line. This region combined with the interior of the dark gray trapezoid bounded on the right by the $r = 2\ell - c$ line defines the set G_0. The $r = 2\ell - c$ line defines G_1. The designer chooses c so that $(\mathrm{E}[\ell|g], \mathrm{E}[r|g])$ lies on the $r = 2\ell$ line. The light gray irregular quadrilateral defines the set B.

2 Asymmetric Payoffs

In the symmetric game discussed above, an important result is that we can analyze risk dominance on a statewise basis. This enables us to treat each state separately and associate a price with including each state in the set $G = G_0 \cup G_1$ of states generating the good signal g.

We now consider a game with asymmetric payoffs. As above, assume Player $i \in \{1, 2\}$ in the continuation game has action set $\{L, R\}$ and payoff function

$$u_i(a_i, a_j; \omega) = \begin{cases} \ell_i(\omega), & \text{if } a_i = L \\ r_i(\omega), & \text{if } a_i = a_j = R \\ 0, & \text{if } a_i = R, a_j = L \end{cases} \tag{13}$$

In contrast to the analysis in Section 1, we no longer require $\ell_1 \equiv \ell_2$ or $r_1 \equiv r_2$, though we continue to require that for $i \in \{1, 2\}$ and for every $\omega \in \Omega$, $0 \leq \ell_i(\omega) \leq r_i(\omega)$. Risk dominance requires that the product of deviation losses is maximized, so the (R, R) equilibrium is weakly risk dominant given signal g if and only if

$$\mathrm{E}[\ell_1|g] \cdot \mathrm{E}[\ell_2|g] - \mathrm{E}[r_1 - \ell_1|g] \cdot \mathrm{E}[r_2 - \ell_2|g] \leq 0 \tag{14}$$

The designer's problem is still a constrained optimization problem, which she can solve using Kuhn-Tucker. The constraints, however, are more difficult to analyze. For instance, in an n-state world, assume as before that designer chooses

$q_k \in [0, \psi_k]$ as the probability of report g and state ω_k. Then (14) becomes

$$\left(\frac{\sum_{k=1}^n \ell_1(\omega_k) q_k}{\sum_{k=1}^n q_k}\right)\left(\frac{\sum_{k=1}^n \ell_2(\omega_k) q_k}{\sum_{k=1}^n q_k}\right) - \left(\frac{\sum_{k=1}^n [r_1(\omega_k) - \ell_1(\omega_k)] q_k}{\sum_{k=1}^n q_k}\right)\left(\frac{\sum_{k=1}^n [r_2(\omega_k) - \ell_2(\omega_k)] q_k}{\sum_{k=1}^n q_k}\right) \leq 0$$

Multiplying both sides by $(\sum_{k=1}^n q_k)^2$ and rearranging terms, we obtain

$$\begin{aligned} &\sum_{k=1}^n [\ell_1(\omega_k) r_2(\omega_k) + \ell_2(\omega_k) r_1(\omega_k) - r_1(\omega_k) r_2(\omega_k)] q_k^2 \\ &\quad + \sum_{j=1}^{n-1} \sum_{k=j+1}^n [\ell_1(\omega_j) r_2(\omega_k) + \ell_1(\omega_k) r_2(\omega_j) + \ell_2(\omega_j) r_1(\omega_k) \\ &\qquad + \ell_2(\omega_k) r_1(\omega_j) - r_1(\omega_j) r_2(\omega_k) - r_1(\omega_k) r_2(\omega_j)] q_j q_k \leq 0 \end{aligned} \tag{15}$$

in which effects of each state do not appear separable. Nevertheless, under some circumstances which we now describe, it remains possible for the designer to proceed as before, assigning a price to including each state in the set G and solving the corresponding consumer problem.

The designer's procedure works as follows. For a given asymmetric game, she searches for a hypothetical corresponding symmetric game. Denote the available actions of the corresponding symmetric game, if one exists, by $\{L^s, R^s\}$ and the payoffs by ℓ^s and r^s. The corresponding symmetric game must have the following property: the equilibrium (R^s, R^s) is risk dominant if and only if the equilibrium (R, R) in the original game is risk dominant, i.e., if and only if (15) holds. If the designer can find a corresponding symmetric game, then she solves her problem exactly as in Section 1.

As the designer manipulates the q_k, she needs the equivalence of risk dominance in both games to be maintained. From (15), for $j, k \in \{1, \ldots, n\}$ with $j < k$, she has the following collection of $n(n+1)/2$ coefficients:

$$\begin{aligned}\lambda_k &:= \ell_1(\omega_k)r_2(\omega_k) + \ell_2(\omega_k)r_1(\omega_k) - r_1(\omega_k)r_2(\omega_k)\\ \mu_{jk} &:= \ell_1(\omega_j)r_2(\omega_k) + \ell_1(\omega_k)r_2(\omega_j) + \ell_2(\omega_j)r_1(\omega_k) + \ell_2(\omega_k)r_1(\omega_j)\\ &\quad - r_1(\omega_j)r_2(\omega_k) - r_1(\omega_k)r_2(\omega_j)\end{aligned}$$

Thus, λ_k is the coefficient on the q_k^2 term in (15), and μ_{jk} is the coefficient on the $q_j q_k$ term. Plugging ℓ^s into (15) for ℓ_1 and ℓ_2, and plugging r^s in for r_1 and r_2, we obtain, for $j, k \in \{1, \ldots, n\}$ with $j < k$,

$$\lambda_k = 2\ell^s(\omega_k)r^s(\omega_k) - [r^s(\omega_k)]^2 \tag{16}$$

$$\frac{\mu_{jk}}{2} = \ell^s(\omega_j)r^s(\omega_k) + \ell^s(\omega_k)r^s(\omega_j) - r^s(\omega_j)r^s(\omega_k) \tag{17}$$

The conditions in (16), along with the requirement that r^s statewise dominates ℓ^s, can be rewritten as

$$r^s(\omega_k) = \ell^s(\omega_k) + \sqrt{[\ell^s(\omega_k)]^2 - \lambda_k} \tag{18}$$

which in turn requires that $\ell^s(\omega_k) \geq \sqrt{\lambda_k}$ if $\lambda_k \geq 0$. In addition, plugging (18) into (17) and rearranging, we obtain

$$\frac{\mu_{jk}}{2} = \ell^s(\omega_j)\ell^s(\omega_k) - \sqrt{([\ell^s(\omega_j)]^2 - \lambda_j)([\ell^s(\omega_k)]^2 - \lambda_k)} \tag{19}$$

Example 2.1 shows a case in which an equivalent symmetric game exists.

Example 2.1. Let $\Omega = \{\omega_1, \omega_2\}$ with $\psi_1 = \psi_2 = 0.5$. Suppose the payoffs are as follow:

$$(\ell_1(\omega), r_1(\omega)) = \begin{cases}(1,3) & \text{if } \omega = \omega_1\\ (2,4) & \text{if } \omega = \omega_2\end{cases} \qquad (\ell_2(\omega), r_2(\omega)) = \begin{cases}(4,7) & \text{if } \omega = \omega_1\\ (3,5) & \text{if } \omega = \omega_2\end{cases}$$

Then $\lambda_1 = -2, \lambda_2 = 2$, and $\mu_{12} = 1$.

The designer's optimal strategy turns out to be $(q_1, q_2) = (0.5, 0.39)$. We now show that the designer can find this solution by constructing an equivalent symmetric game.

The designer can find an equivalent symmetric game by substituting for λ_1, λ_2, and μ_{12} in (19) and solving for $\ell^s(\omega_2)$ as a function of $\ell^s(\omega_1)$. Because $\lambda_2 < 0$, $\ell^s(\omega_1)$ is restricted only by nonnegativity. The solution is

$$\ell^s(\omega_2) = \frac{-\ell^s(\omega_1) + \sqrt{17[\ell^s(\omega_1)]^2 + 34}}{4}$$

This value is strictly monotone and increases approximately linearly in $\ell^s(\omega_1)$, so it is unbounded above. Therefore, the designer can always choose a value of $\ell^s(\omega_1)$ for which $\ell^s(\omega_2) > \sqrt{\lambda_2} = \sqrt{2}$. For example, at $\ell^s(\omega_1) = 2$, she has $\ell^s(\omega_2) \approx 2.02$. With these values, she can use (18) to find $r^s(\omega_1) \approx 4.45$ and $r^s(\omega_2) \approx 3.46$.

Using the corresponding symmetric game, the designer finds that the price of state ω_1 is $p_1 = 2\ell^s(\omega_1) - r^s(\omega_1) = -0.45$. The price is negative, so she consumes all the ω_1 available by setting $q_1 = \psi_1 = 0.5$. That gave her a negative cost of $-0.45 \cdot 0.5 = -0.225$, so she can consume state ω_2 at a price of $p_2 = 2\ell^s(\omega_2) - r^s(\omega_2) = 0.58$ until she exhausts her endowment. That is, she sets $q_2 = 0.225/0.58 \approx 0.39$.

On the other hand, not every asymmetric game has a corresponding symmetric game. For instance, suppose for some j and k, $\mu_{jk} = 0$ and both λ_j and λ_k are negative. Then (19) is not satisfiable. Example 2.2 illustrates.

Example 2.2. Let $\Omega = \{\omega_1, \dots, \omega_n\}$ for $n \geq 3$, and assume that each state occurs with positive probability (i.e., $\psi_i > 0$). For $j, k \in \{1, \dots, n\}$ with $j < k$, suppose the payoffs for states ω_j, ω_k are as follow:

$$(\ell_1(\omega), r_1(\omega)) = \begin{cases} (1, 3) & \text{if } \omega = \omega_j \\ (2, 4) & \text{if } \omega = \omega_k \end{cases} \qquad (\ell_2(\omega), r_2(\omega)) = \begin{cases} \left(4, \frac{25}{4}\right) & \text{if } \omega = \omega_j \\ \left(3, \frac{25}{4}\right) & \text{if } \omega = \omega_k \end{cases}$$

Assume the payoffs in the other states guarantee that (R, R) is not ex ante risk dominant (i.e., (R, R) is not risk dominant if we set $q_i = \psi_i$ at each state). Then

$\lambda_j = \lambda_k = -1/2$ and $\mu_{jk} = 0$. By (19), a solution would require

$$0 = \ell^s(\omega_j)\ell^s(\omega_k) - \sqrt{\left([\ell^s(\omega_j)]^2 + \frac{1}{2}\right)\left([\ell^s(\omega_k)]^2 + \frac{1}{2}\right)}$$

By the nonnegativity of $\ell^s(\cdot)$, the right-hand side is always negative. Therefore, there can be no symmetric game that is equivalent to the original asymmetric game.

In the special case in which the asymmetries are a question of scaling, there is always a corresponding symmetric game. Therefore, the risk dominance condition in this case can always be reduced to a linear constraint, and the designer can price each state individually. We state this result as follows:

Proposition 1. *Let $\alpha > 0$ be an arbitrary constant. Suppose that $\forall \omega \in \Omega$, $\ell_1(\omega) = \alpha\ell_2(\omega)$ and $r_1(\omega) = \alpha r_2(\omega)$. Then the solution to the designer's problem is identical to that of the symmetric game in which $\alpha = 1$.*

In the rollover game interpretation, Proposition 1 says that the relative size of the creditors does not matter, as long as their support is required for the borrower's survival.[3]

3 Robustness

Throughout the analysis so far, we have taken for granted the designer's objective of maximizing the probability of the payoff dominant equilibrium, and that this is in the interest of all parties. We now address two possible objections to this viewpoint.

The first is that the designer's preferred equilibrium could be payoff dominant in ex ante expectation, but may not always be statewise payoff dominant. For example, in the rollover game interpretation, there could be some states in which the

[3] Alternatively, if $\ell_1 = \alpha\ell_2$ and $r_1 = \alpha r_2$, then the designer can pick $\ell^s = \sqrt{\alpha}\ell_2$ and $r^s = \sqrt{\alpha}r_2$. Then the λ_k and the μ_{jk} are unchanged, and both sums on the left-hand side of (15) are multiplied by the positive constant α, which is irrelevant to the inequality.

creditors are better off if the firm is liquidated. It is natural to ask if there are circumstances under which the designer recommends rollover when liquidation is efficient, while simultaneously recommending liquidation when rollover is efficient. Proposition 2 shows the answer is yes. The requirement is that any state $\hat{\omega}$ inefficiently included in G (i.e., for which the $r(\hat{\omega}) < \ell(\hat{\omega})$) must have low stakes, in the sense that $r(\hat{\omega})$ and $\ell(\hat{\omega})$ have to be sufficiently small.

Proposition 2. *Let $\omega_1, \omega_2 \in \Omega$, with $r(\omega_1) > \ell(\omega_1)$ and $r(\omega_2) < \ell(\omega_2)$. The following are necessary for $\omega_2 \in G_0 \cup G_1$ and $\omega_1 \in B$:*

1. $2\ell(\omega_2) - r(\omega_2) \leq 2\ell(\omega_1) - r(\omega_1)$, *and*
2. $r(\omega_2) < \ell(\omega_2) < \ell(\omega_1) < r(\omega_1)$.

That is, the payoffs in state ω_2 must be small compared with those in ω_1.

Strictness of the first inequality in Proposition 2 is not enough for sufficiency, because the designer might report g or b on both ω_1 and ω_2. However, if the first inequality is strict, then $\omega_1 \in G_0 \cup G_1$ only if $\omega_2 \in G_0$. Nothing changes in the proof of Theorem 1, so the result still holds.

A second concern we address is that the players in the continuation game can do better than maximizing the probability of selecting the payoff dominant equilibrium. If the players could choose the designer's strategy, they would prefer to have the designer maximize their expected payoffs. Addressing this concern is a larger departure from the Bayesian persuasion tradition in information design, and has more of the feel of mechanism design.

We first return to the discrete setting of Section 1.1. If the designer's preferences are perfectly aligned with those of the players, her constraints are as before and her objective function is

$$\max_{q_1,\ldots,q_n} U(q_1, \ldots, q_n) = \sum_{i=k}^{n} q_k \left(r(\omega_k) - \ell(\omega_k) \right), \tag{20}$$

so that her marginal utility of q_k is now $r(\omega_k) - \ell(\omega_k)$.

As before, the designer first consumes all of the states with a nonpositive price. In the three-state example of Subsection 1.1, she sets $q_2 = 0.2$, and her budget constraint does not bind, leaving slack of -0.8.

Next, she ranks states by their marginal utility to price ratio:

$$\frac{\partial U/\partial \omega}{p(\omega)} = \frac{r(\omega) - \ell(\omega)}{2\ell(\omega) - r(\omega)}$$

This ratio is $4.5/2$ for state ω_1 and 1 for state ω_3, so the designer focuses on q_1 next. Her budget constraint binds if

$$\begin{aligned} p_1 q_1 + p_2 q_2 &= 0 \\ &\Leftrightarrow 2q_1 = 0.8 \Leftrightarrow q_1^* = 0.4 \end{aligned}$$

This is less than the full capacity of ψ_1, so the designer stops here. If state ω_1 occurs, the designer randomizes and reports g with probability 4/5. She always reports g if the state is ω_2, and she always reports b if the state is ω_3.

Overall, the probability of successfully coordinating on the (R, R) equilibrium is 0.6, compared with 0.75 in the example of Subsection 1.1. However, the expected payoffs to the players are 5.55, compared with 5.325 above.

The general case is similar: because the marginal utility of adding a state to G is now the payoff from the good equilibrium, rather than a constant, the designer shifts the slope rather than the intercept of the risk dominance boundary line. See Figure 4. We state this precisely in Proposition 3. For the proposition, we restrict attention to the case in which ψ is atomless, as extensions to general cases are similar to Theorem 1.

Proposition 3. *The designer maximizes the players' expected payoffs as follows: if* $\mathrm{E}[r] \geq 2\mathrm{E}[\ell]$, *then set* $G_0 = \Omega$. *If* $\psi(\{\omega \in \Omega | r(\omega) > 2\ell(\omega)\}) = 0$, *then set* $G_0 = \{\omega \in \Omega | r(\omega) \geq 2\ell(\omega)\}$ *and* $G_1 = \varnothing$. *Otherwise, for some* $\alpha \in (1, 2)$, *set* $G_0 = \{\omega \in \Omega | r(\omega) > \alpha\ell(\omega)\}$ *and* $G_1 = \{\omega \in \Omega | r(\omega) = \alpha\ell(\omega)\}$.

The slope α *is chosen to make* $\mathrm{E}[r|g] = 2\mathrm{E}[\ell|g]$.

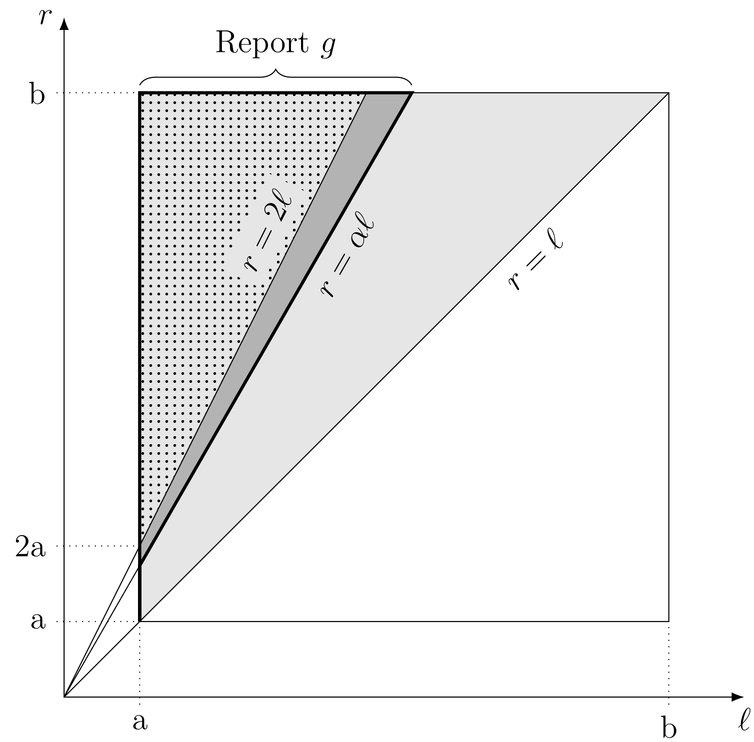

Figure 4: The designer's optimal disclosure region (dotted triangle and dark gray triangle), where $\alpha \in (1, 2)$.

4 Conclusion

In information design problems, the designer cannot always take for granted that the players in the continuation game will follow the designer's advice. If the players require any recommended equilibrium to satisfy a refinement, their requirement becomes a constraint. Incorporating this constraint into the designer's problem therefore requires a change in the main solution concept.

As we show, in the case of risk dominance, the equilibrium refinement constraint is often reducible to a linear inequality, which has the form of a budget constraint. The designer's problem then becomes analogous to a consumer's utility maximization problem, and the optimal consumption choice corresponds the designer's optimal information structure.

A Proofs

Proof of Theorem 1. Given that (L, L) is ex ante risk dominant but that there is positive probability of (R, R) being risk dominant, there is an interior solution to the designer's problem

$$\max_{G \in \mathcal{F}} \int_{\omega \in G} d\psi(\omega)$$
$$\text{s.t.} \int_{\omega \in G} [2\ell(\omega) - r(\omega)]\, d\psi(\omega) \leq 0$$

By the Neyman-Pearson Lemma and its extension to allow for randomization (Kadane, 1968), the solution has the properties that, for some $c > 0$,

$$\omega \in G \text{ if } 2\ell(\omega) - r(\omega) < c$$
$$\omega \notin G \text{ if } 2\ell(\omega) - r(\omega) > c, \text{ and}$$
$$\mathrm{E}[r - 2\ell | \omega \in G] = 0$$

Rearranging, it follows that G_0 (the set of states that are in G with probability 1) is $\{\omega \in \Omega | r(\omega) > 2\ell(\omega) - c\}$ and that $B = \{\omega \in \Omega | r(\omega) < 2\ell(\omega) - c\}$. The designer can randomize only on the boundary set $\{\omega \in \Omega | r(\omega) = 2\ell(\omega) - c\}$, and by the conclusion of the Neyman-Pearson Lemma, she does so in order to make the risk dominance constraint bind. $\square$

Proof of Proposition 1. For a given measurable $G \in \mathcal{F}$, (14) holds if and only if

$$\begin{aligned}
(\mathrm{E}[r_1 - \ell_1|G])(\mathrm{E}[r_2 - \ell_2|G]) &\geq \mathrm{E}[\ell_1|G]\mathrm{E}[\ell_2|G] \\
\Leftrightarrow (\mathrm{E}[\alpha r_2 - \alpha \ell_2|G])(\mathrm{E}[r_2 - \ell_2|G]) &\geq \mathrm{E}[\alpha \ell_2|G]\mathrm{E}[\ell_2|G] \\
\Leftrightarrow (\alpha \mathrm{E}[r_2 - \ell_2|G])(\mathrm{E}[r_2 - \ell_2|G]) &\geq \alpha \mathrm{E}[\ell_2|G]\mathrm{E}[\ell_2|G] \\
\Leftrightarrow (\mathrm{E}[r_2 - \ell_2|G])^2 &\geq \mathrm{E}^2[\ell_2|G]
\end{aligned}$$

The last line brings us back to the quadratic case in Section 1, i.e., the symmetric case (where $\alpha = 1$). □

Proof of Proposition 2. The first condition comes from the associated consumer's problem: each state has the same marginal utility, so if the designer, viewed as a consumer, includes ω_2 and does not include ω_1, then ω_2 must be no more expensive, i.e., $2\ell(\omega_2) - r(\omega_2) \leq 2\ell(\omega_1) - r(\omega_1)$.

To get the second condition, rewrite the first one as follows:

$$\begin{aligned}
2\ell(\omega_2) - r(\omega_2) &\leq 2\ell(\omega_1) - r(\omega_1) \\
\Rightarrow \ell(\omega_2) + (\ell(\omega_2) - r(\omega_2)) &< \ell(\omega_1) + (\ell(\omega_1) - r(\omega_1))
\end{aligned}$$

By hypothesis, the term in parentheses on the left-hand side is positive, and the term in parentheses on the right-hand side is negative. Therefore,

$$\ell(\omega_2) < \ell(\omega_1)$$

as desired. □

Proof of Proposition 3. As in the proof of Theorem 1, the corner cases of nondisclosure ($G_0 = \Omega$) and full disclosure ($G_0 = \{\omega \in \Omega | r(\omega) \geq 2\ell(\omega)\}$ and $G_1 = \varnothing$) are immediate.

Otherwise, the designer's problem is

$$\max_{G \in \mathcal{F}} \int_{\omega \in G} [r(\omega) - \ell(\omega)] d\psi(\omega)$$

subject to (11). We can again apply the Neyman-Pearson Lemma, obtaining for some $d > 0$,

$$\begin{aligned}
&\omega \in G \text{ if } \frac{r(\omega) - \ell(\omega)}{2\ell(\omega) - r(\omega)} < d \\
&\omega \notin G \text{ if } \frac{r(\omega) - \ell(\omega)}{2\ell(\omega) - r(\omega)} > d, \text{ and} \\
&\mathrm{E}[r - 2\ell | \omega \in G] = 0
\end{aligned}$$

with d chosen to make the constraint bind. Rearranging, the boundary condition for $\omega \in G_1$ is

$$r(\omega) - \ell(\omega) = c[2\ell(\omega) - 2r(\omega)] \quad \Leftrightarrow \quad r(\omega) - \frac{2c+1}{c+1}\ell(\omega)$$

Letting $\alpha = (2c+1)/(c+1)$ and noting that $c > 0$, we see that $1 < \alpha < 2$. $\square$

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