ALGEBRAIC AND SEMI-ALGEBRAIC INVARIANTS ON QUADRICS

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For my parents Sunhi Jung and Hyang Huh

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SUMMARY

This thesis mainly concerns the invariants and semi-algebraic invariants on quadrics on varieties. In chapter 1, we discuss bounds on an algebraic invariant of quadratic monomial ideals which inscribes the complexity of the ideals. In chapter 2, we introduce a semi-algebraic invariant which quantifies the structural difference between the cone of sums of squares and the cone of nonnegative quadratic forms on the space of quadrics on varieties. The semi-algebraic invariant is bounded below by an algebraic invariant of the variety. We compare the semi-algebraic invariant and algebraic invariant on rational curves obtained by the projection of rational normal curves away from a point.

1. Bounds on regularity of quadratic monomial ideals

   The minimal graded free resolution of square-free monomial ideals can be investigated combinatorially through the Stanley-Reisner correspondence between the ideals and simplicial complexes. In particular, the square-free quadratic monomial ideals are the Stanley-Reisner ideals of the clique complexes of simple graphs.

   Regarding of the correspondence, we study the bounds on the algebraic invariant, Castelnuovo-Mumford regularity, of edge ideals in terms of properties on the corresponding simple graphs. The main theorem is the graph decomposition theorem that provides a bound on the regularity of a non-edge ideal which corresponds to a graph by the regularity of edge ideals that correspond to proper subgraphs of the graph. By combining the graph decomposition theorem with results in structural graph theory, we proved, improved, and generalized many of the known bounds on regularity of square-free quadratic monomial ideals.

2. Hankel index of non-ACM curves of almost minimal degree.

   The Hankel index of a real variety is an invariant that quantifies the difference between nonnegative quadrics and sums of squares on the variety. Note that the Hankel

index is a semi-algebraic invariant that is difficult to compute and the values are known for only a few cases. The project is motivated by a result [1] that showed an intriguing bound of Hankel index of a variety by an algebraic invariant, Green-Lazarsfeld index, of the variety. In addition, the Hankel index of arithmetrically Cohen-Macaulay variety of almost minimal degree is determined by the Green-Lazarsfeld index of the variety. Therefore, Grigoriy Blekherman, Justin Chen, and I investigated the Hankel index of the non-ACM curves of almost minimal degree.

Since any smooth non-ACM curve of almost minimal degree is a rational curve, obtained by the projection of a rational normal curve away from an outer point, we focus on studying the Hankel index of the rational curves and compared the Hankel index with the Green-Lazarsfeld index of the curves. We found new rank of the center of the projection which detects the Hankel index of the rational curves, and moreover we found the rational curves are the first class of examples that the bound of the Hankel index by the Green-Lazarsfeld index is strict.

CHAPTER 1

BOUNDS ON REGULARITY OF QUADRATIC MONOMIAL IDEALS

Suppose I is a square-free monomial ideal in a polynomial ring. Due to the Stanley-Reisner correspondence, we can study Betti-numbers of the monomial ideal I by the homology of subcomplexes of the simplicial complex Δ_I that corresponds to the ideal I. In particular, if the ideal is generated by square-free quadratic monomials, then there is a correspondence between simple graphs and the ideals. Moreover, we can study the algebraic invariants of ideals through the properties of corresponding graphs.

We consider bounds on Castelnuovo-Mumford regularity of a square-free quadratic monomial ideal over a field of characteristic 0. Many recent papers investigated regularity of such ideals [2][3][4][5][6][7], see also [8] for a survey. In the literature, the quadratic monomial ideals are called edge ideals by associating the generators of the ideal to the edges of the complement of a simple graph.

1.1 Edge ideals, clique complexes of graphs, and Betti numbers

Suppose I is a quadratic square-free monomial ideal in a polynomial ring over a field k of characteristic 0. One can associate the quadratic monomial ideal I to a simple graph G by taking variables as vertices and the quadratic generators of I as non-edges of the graph G. More explicitly, suppose $I = (x_i x_j : i, j \in \{1, \dots, n\})$ is a square-free quadratic monomial ideal in $k[x_1, \dots, x_n]$. Then, the ideal I corresponds to the simple graph $G = (V, E)$ where the vertex set is $V = \{1, \cdots, n\}$ and the edge set is $E = \{ij : x_i x_j \notin I\}$

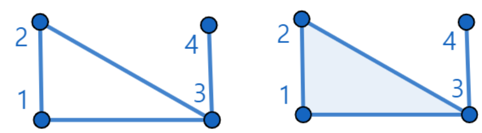
Now, we can associate the quadratic monomial ideal I with the simplicial complex ΔG arised from graphs G. Given a graph G, the clique complex of G, denoted by ΔG, is the simplicial complex that consists of t-simplices $(i_1, i_2, \dots, i_t)$ whenever the induced subgraphs on t vertices are complete graph of size t. Remark that the ideal I is the Stanley-

Reisner ideal of the clique complex ΔG. i.e. $I = \{x_1 \cdots x_r : \{x_1 \ldots x_r\} \notin \Delta G\}$ [9, Chapter 2].

Example 1.1.1. *Suppose I is a square-free quadratic monomial ideal in $S := k[x_1, x_2, x_3, x_4]$ generated by two monomials x_1x_4 and x_2x_4. Then, the ideal $I = I(G)$ is associated with the simple graph $G = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{12, 13, 23, 34\}$. Then, the clique complex ΔG of the graph G is the simplicial complex*

$$\Delta G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{12\}, \{13\}, \{14\}, \{23\}, \{34\}, \{123\}\} \subset 2^V.$$

Remark that the ideal $I(G)$ is the Stanley-Reisner ideal of ΔG since any non-faces of ΔG is contained in the ideal $I(G)$.



$I = \langle x_1x_4, x_2x_4 \rangle$ $I(G)$ $\quad G \quad \Delta G$

Figure 1.1: Stanley-Reisner Correspondence

Since square-free monomial ideals are Stanley-Reisner ideals, we can study the Betti numbers of the monomial ideals through the homologies of subcomplexes of the corresponding simplicial complex. In particular, the subcomplexes of clique complexes of a graph are the clique complexes of induced subgraphs of the graph. Denote by $G[W]$ the induced subgraph on a subset of vertices W of a graph $G = (V, E)$. Then, we obtain a version of Hochster's formula for quadratic monomial ideals.

Theorem 1.1.2 (Hochster)**.** *Suppose $I(G)$ is the non-edge ideal of a graph $G = (V, E)$ in $S = k[V]$. Then for $j \geq i + 1$,*

$$\beta_{i,j}(S/I(G)) = \sum_{|W|=j} \dim_k(\tilde{H}_{j-i-1}(\Delta G[W])),$$

where W runs over all subsets of the vertex set of G of size t.

It means that the total sum of homology of subcomplexes of size $|W|$ is the Betti numbers of the quadratic monomial ideal.

The Castelnuovo-Mumford regularity of the ideal is the maximum degree of entries in differentials of the minimal free resolution of the ideal. (See section A.1 for the explicit definition of Castelnuovo-Mumford regularity of ideals.) Regarding the Stanley-Reisner correspondence between quadratic monomial ideals and simple graphs, we define regularity of a graph G to be the Castelnuovo-Mumford regularity of the corresponding ideal I, denoted by $\mathrm{reg}(G)$. Note that, by Hochster's formula, the regularity of G is the maximum dimension of subcomplexes of the simplicial complex ΔG whose homology is non-zero.

Example 1.1.3. *Suppose I is a square-free quadratic monomial ideal in $S := k[x_1, x_2, x_3, x_4]$ generated by two monomials x_1x_3 and x_2x_4. Then, the ideal $I = I(G)$ is associated with the simple graph $G = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{12, 14, 23, 34\}$ and its clique complex is $\Delta G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{12\}, \{14\}, \{23\}, \{34\}\} \subset 2^V$. i.e. ΔG is a cycle of length four. Therefore, by Theorem 1.1.2, $\beta_{2,4} = 1$ since $\beta_{2,4} = \dim_k(\tilde{H}_1(\Delta G))$ and $\beta_{1,2} = 2$ since there are two subsets of the vertex set of size two in ΔG whose number of connected components is two.*

Indeed, the minimal graded free resolution of S/I over the ring S is

$$0 \longrightarrow S(-4)^{\oplus 1} \longrightarrow S(-2)^{\oplus 2} \longrightarrow S(\longrightarrow S/I \longrightarrow 0)$$

Thus, the Betti diagram of S/I is

Table 1.1: Betti diagram for Example 1.1.3

	0	1	2
0	1	-	-
1	-	2	-
2	-	-	1

In this example, the Castelnuovo-Mumford regularity of S/I is 3 and the Green-Lazarsfeld index of the variety (or the coordinate ring S/I) is 1.

1.2 Graph decomposition theorems

Our main tool for bounding regularity is the following decomposition theorem, which is based on a straightforward application of Hochster's formula [10] and the Mayer-Vietoris sequence [11].

Theorem 1.2.1. *Let G be a graph. Let G_1 and G_2 be subgraphs which cover cliques of G (i.e. any clique of G is a clique in either G_1 or else G_2.) Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_2), \operatorname{reg}(G_1 \cap G_2) + 1\}.$$

Proof. Let W be an induced subgraph of G. Let $W_1 = W \cap G_1$ and $W_2 = W \cap G_2$. We claim that a subcomplex of ΔW is the union of subcomplexes of ΔW_1 and ΔW_2. Let $F = (v_1, \ldots, v_t)$ be a face in ΔW. Then $G(F)$ is a clique in G, and since G_1 and G_2 cover cliques of G, we see that F is a face of either W_1 or W_2, and the claim follows. Additionally, we have $\Delta(W_1 \cap W_2) = \Delta W_1 \cap \Delta W_2$.

Now, we prove the main inequality. Let $m = \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_2), \operatorname{reg}(G_1 \cap G_2) + 1\}$. Given any induced subgraph W, by the Mayer-Vietoris sequence [11, p.149], we have following exact sequence of complexes

$$\begin{aligned} \cdots \to \tilde{H}_i(\Delta(W_1 \cap W_2)) &\to \tilde{H}_i(\Delta W_1) \oplus \tilde{H}_i(\Delta W_2) \to \tilde{H}_i(\Delta W) \\ \to \tilde{H}_{i-1}(\Delta(W_1 \cap W_2)) &\to \tilde{H}_{i-1}(\Delta W_1) \oplus \tilde{H}_{i-1}(\Delta W_2) \to \tilde{H}_{i-1}(\Delta W) \to \cdots \end{aligned}$$

Since regularity of $G_1 \cap G_2$ is at most $m - 1$, we have $\tilde{H}_i(\Delta(W_1 \cap W_2)) = 0$ for all $i \geq m - 2$. Therefore, $\tilde{H}_i(\Delta W) \simeq \tilde{H}_i(\Delta W_1) \oplus \tilde{H}_i(\Delta W_2)$ for all $i \geq m - 1$. Since both G_1 and G_2 have regularity at most m, $\tilde{H}_i(\Delta W_1) = \tilde{H}_i(\Delta W_2) = 0$ for all $i \geq m - 1$. Thus, $\tilde{H}_i(\Delta W) = 0$ for all $i \geq m - 1$ and regularity of G is at most m. $\square$

Our first application deals with the case of defining G_1 and G_2 via a cutset. Suppose G

is a simple graph consisting of the vertex set $V(G)$ and the edge set $E(G)$. For a subgraph G' of G we use $G \setminus G'$ to denote the induced subgraph on $V(G) \setminus V(G')$.

Theorem 1.2.2 (Cut-set/Separator decomposition)**.** *Let T be an induced subraph of G such that the induced graph $G \setminus T$ is disconnected. Let $C_1, \ldots, C_k$ be the connected components of $G \setminus T$ and G_i be induced subgraphs on vertices of C_i and T for $i = 1, \ldots, k$. Then,* $\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G_i)_{i=1,\ldots,k}, \operatorname{reg}(T) + 1\}$.

Proof. Let G_1 be the induced subgraph on vertices of C_1 and T and let G_1' be the induced subgraph on $\cup_{i=2}^{k} V(C_i) \cup V(T)$. In other words, $G_1' = G \setminus C_1$. Then, we can see that G_1 and G_1' cover all cliques of G. Indeed, if a vertex in C_1 and a vertex in $\cup_{i=2}^{k} C_i$ are contained in a clique in G, the induced subgraph on the two vertices must be an edge of G. However, it is not possible because C_1 and $\cup_{i=2}^{k} C_i$ are disjoint. Therefore, two induced subgraphs G_1 and G_1' cover all cliques in G. Then, by Theorem 1.2.1, we have

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_1'), \operatorname{reg}(T) + 1\}.$$

Now, let G_j' be the induced subgraph on vertices of $C_{j+1}, \ldots, C_k$, and T for $j = 2, \ldots, k$. Then, by the same process,

$$\operatorname{reg}(G_{j-1}') \leq \max\{\operatorname{reg}(G_j), \operatorname{reg}(G_j'), \operatorname{reg}(T) + 1\}$$

Thus,

$$\begin{aligned}
\operatorname{reg}(G) &\leq \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_1'), \operatorname{reg}(T) + 1\} \\
&\leq \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_2), \operatorname{reg}(G_2'), \operatorname{reg}(T) + 1\} \\
&\vdots \\
&\leq \max\{\operatorname{reg}(G_i)_{i=1,\ldots,k}, \operatorname{reg}(T) + 1\}.
\end{aligned}$$

∎

We call T in Theorem 1.2.2 a separator of G. Note that Theorem 1.2.2 generalizes a decomposition result used by Dao, Huneke and Schweig in [2, Lemma 3.1].

Recall that an open neighborhood $N_G(v)$ of a vertex v is the induced subgraph on the vertices adjacent to v, and a closed neighborhood $N_G[v]$ of v is the induced subgraph on v and all vertices adjacent to v. Decomposition in [2] arises as a special, but very useful case, where T is the open neighborhood of a vertex v. i.e. an open neighborhood of a vertex of G is a separator of G. An additional simplification comes from the fact that regularity of the open and closed neighborhoods of v are the same. Therefore, we obtain the following Vertex Neighborhood Decomposition theorem.

Corollary 1.2.3 (Vertex Neighborhood Decomposition)**.** *Let v be any vertex of a graph G. Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G \setminus v), \operatorname{reg}(N_G(v)) + 1\}.$$

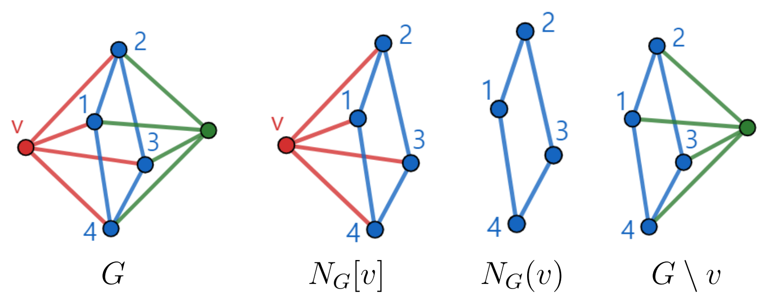


Figure 1.2: Vertex Neighborhood decomposition

Proof. By Theorem 1.2.2, we have $\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G \setminus v), \operatorname{reg}(N_G[v]), \operatorname{reg}(N_G(v)) + 1\}$, where $N_G(v)$ is the open neighborhood of v in G and $N_G[v]$ is the closed neighborhood of v in G. So, it suffices to show that $\operatorname{reg}(N_G[v]) = \operatorname{reg}(N_G(v))$. This follows by a simple application of Hochster's formula, since the clique complex ΔH of an induced subgraph H of $N_G[v]$ with $v \in H$ is contractible. ∎

So far, we have only considered graph decompositions coming from induced subgraphs, but we now define a useful decomposition where this is not the case. Let M be a subgraph of G. Let G_M be the induced subgraph of G on vertices in M and vertices of G which are adjacent to both vertices of some edge of M. Namely, $G_M = G[V(M) \cup W]$ where W is a subset of vertices in G such that $k \in W$ if $ik \in E(G)$ and $jk \in E(G)$ for some $ij \in E(M)$. Also, for a subgraph G' of G, we use $G - G'$ to denote the subgraph of G obtained by deleting all edges of G'. i.e. $G - G' = (V(G), E(G) - E(G'))$. Then, we have following decomposition theorem.

Theorem 1.2.4. *Let M be a subgraph of a graph G. Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G - M), \operatorname{reg}(G_M), \operatorname{reg}(G_M - M) + 1\}$$

Proof. We first claim that $G - M$ and G_M cover all cliques of G. Let F be any clique in G. If F does not contains any edges in M, then $G - M$ contains the clique F. Suppose that F contains some edges of M. If all vertices in F is contained in M, then F is contained in G_M since G_M contains M. If v is any vertex in F outside of M, then $uv, wv \in F$ for some $uw \in E(F \cap M)$. This implies that $v \in V(G_M)$ and so $F \subseteq G_M$ since both F and G_M are induced subgraphs of G. In addition, the intersection of $G-M$ and G_M is $G_M - M$. Indeed, $V(G_M \cap (G - M)) = V(G_M \cap G) = V(G_M)$ and $E(G_M \cap (G - M)) = E(G_M - M)$. Thus, by Theorem 1.2.1, $\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G - M), \operatorname{reg}(G_M), \operatorname{reg}(G_M - M) + 1\}$ □

Similarly to vertex-neighborhood decomposition in Theorem 1.2.3, if we take M to be an edge $e = ij$ in Theorem 1.2.4, then we can bound regularity of G by regularity of two subgraphs.

Corollary 1.2.5 (Edge-neighborhood decomposition)**.** *Let G be a graph and $e = ij$ be an edge in G. Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G - e), \operatorname{reg}(G_e - e) + 1\}.$$

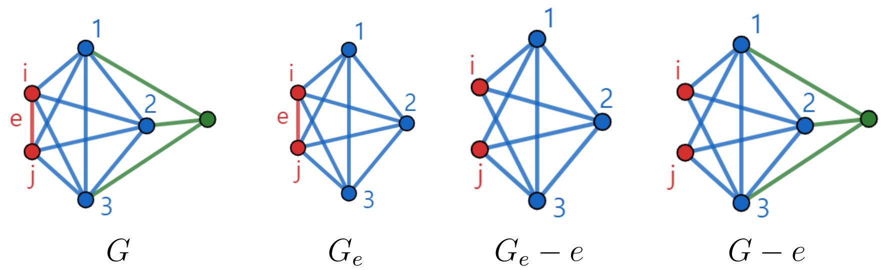


Figure 1.3: Edge Neighborhood decomposition

Proof. By Theorem 1.2.4, it suffices to show that $\operatorname{reg}(G_e) \leq \operatorname{reg}(G_e - e) + 1$ for edge e. Indeed, for any graph G, $\operatorname{reg}(G) \leq \operatorname{reg}(G \setminus v) + 1$ for any vertex v by Corollary Theorem 1.2.3, and so we have

$$\begin{aligned} \operatorname{reg}(G_e) &\leq \operatorname{reg}(G_e \setminus i) + 1 \\ &\leq \operatorname{reg}(G_e - e) + 1, \end{aligned}$$

for the edge $e = ij$ because $G_e \setminus i$ is an induced subgraph of $G_e - e$. $\square$

We will use this decomposition to describe complements of bipartite graphs that have regularity 3 in section 1.4.

1.3 Hereditary Families

Let $\mathscr{G}$ be a family of graphs. We call $\mathscr{G}$ a hereditary family if it is closed under taking induced subgraphs, or equivalently under deleting vertices.

Theorem 1.3.1 (Hereditary theorem)**.** *Let $\mathscr{G}$ be a hereditary family with the following property: there exists $t \in \mathbb{N}$, such that for any $G \in \mathscr{G}$ there is a separator G' of G with* $\operatorname{reg}(G') \leq t$*. Then regularity of any $G \in \mathscr{G}$ is at most $t + 1$.*

Proof. Let $\mathscr{G}$ be a hereditary family with the above property for some $t \in \mathbb{N}$. We will induct on the number of vertices n in graphs of $\mathscr{G}$. The base case $n = 1$ is trivial, since

$t \geq 0$ and $\mathscr{G}$ includes the one vertex graph. Now consider the inductive step. Let $G \in \mathscr{G}$ be a graph on $n+1$ vertices, and let G' be a separator of G. Applying Theorem 1.2.2 with $T = G'$, we get the desired inequality by the induction assumption. □

Chordal graphs form a hereditary family, and it is known in [12] that any chordal graph contains a vertex v such that neighborhood of v is a complete graph. Therefore we immediately obtain the following result of Fröberg:

Corollary 1.3.2. *Let G be a chordal graph. Then regularity of G is at most 2.*

Moreover, we can see that regularity of any hole is at least 3 and therefore chordal graphs are the only graphs of regularity at most 2. On the other hand, by combining Fröberg's result with neighborhood decomposition Theorem 1.2.3 we can give a criterion for graphs that have regularity at most 3:

Corollary 1.3.3. *Let $\mathscr{G}$ be a hereditary family of graphs with the following property: for any $G \in \mathscr{G}$ there is a vertex v of G which has a chordal neighborhood. Then regularity of any $G \in \mathscr{G}$ is at most 3.*

To illustrate the power of the Theorem 1.3.3, we give a quick proof of a generalization of a result by Nevo [13, Section 5]. Let F' be a graph on four vertices consisting of an isolated vertex and a triangle. He showed that if G does not contain F' and a four-cycle as induced subgraphs then regularity of G is at most three. We note that not containing a four-cycle as an induced subgraph corresponds to G satisfying condition $N_{2,2}$. (See Appendix A for details.) Let F be a graph on five vertices consisting of an isolated vertex and two triangles sharing an edge. We show that if G does not contain a four-cycle and F as induced subgraphs, then regularity of G is at most 3, which is a weaker condition on G.

Corollary 1.3.4. *Let $\mathscr{G}$ be the hereditary family of graphs that do not contain F and the four cycle as induced subgraphs. Then regularity of any $G \in \mathscr{G}$ is at most 3.*

Proof. We will show that any $G \in \mathscr{G}$ contains a vertex with a chordal neighborhood. Suppose not, and let $G \in \mathscr{G}$ be a graph such that no vertex of G has a chordal neighborhood. Let v be the vertex of minimal degree in G. Observe that v is not connected to all vertices of G, otherwise G is the complete graph, which is a contradiction. It follows by our assumption that $N_G(v)$ contains a hole C of length at least 5, and there exists $w \in G$ such that v is not connected to w. Since G is F-free we see that w must be connected to two non-adjacent vertices u_1, u_2 of C. But then the induced subgraph on u_1, v, u_2, w is a 4-cycle, which is a contradiction. □

We also generalize Theorem 1.3.4 to the case where G does not contain larger cycles as induced subgraphs. Recall that a graph G not containing an ℓ-hole for $\ell = 4, ..., p+2$ with $p \geq 2$ is equivalent to G satisfying condition $N_{2,p}$. Let a fan F_i for $i \geq 1$ be the graph consisting of an isolated vertex and the graph join of a path on $i+1$ vertices and a distinct vertex. With essentially the same proof as Theorem 1.3.4 we can also show the following:

Corollary 1.3.5. *If for some $i \geq 2$ a graph G is ℓ-hole free for $\ell = 4, ..., i+2$ and does not contain F_i as an induced subgraph, then regularity of G is at most* 3.

It is known that if G is perfect and does not contain 4-holes or if G is even-hole free, then there is a vertex in G whose neighborhood is chordal (for 4-free perfect graphs see [14] and for even-hole free graphs see [15]). Moreover, both 4-hole free perfect graphs and even-hole free graphs form hereditary families. Thus, we obtain another criterion to make graphs to have regularity 3.

Corollary 1.3.6. *If G is perfect and does not contain 4-holes, or if G is even-hole free then regularity of G is at most* 3.

It follows from the Strong Perfect Graph Theorem [16], that G is perfect and 4-hole free if and only if G is 4-hole free and also odd-hole free. Thus Theorem 1.3.6 implies that if G is 4-hole free, and regularity of G is at least 4, then G must contain both even and odd holes. This observation is used for improving a bound on regularity in section 1.6.

1.4 Complements of bipartite graphs whose regularity are three

Fernández-Ramos and Gimenez gave an complete description of bipartite graphs associated to edge ideals that have regularity 3 in [17]. We give an independent proof of their result by using Theorem 1.2.5.

Note that we work with the complements of bipartite graphs since we work over the non-edge ideals, i.e. the complement of graphs of edge ideals. Let G be the complement of a bipartite graph H with bipartition of vertices X and Y. Let B be the subgraph of G with $V(B) = V(G)$ and the edge set consisting of edges of G between vertices in X and vertices in Y. i.e. $B = G(V(G), E(X, Y))$. We call B the bipartite part of G. We recall chordal bipartite graphs [18, Section 12.4].

Definition 1.4.1. *A chordal bipartite graph is a bipartite graph which contains no induced cycles of length greater than four.*

It is shown in [19] that any chordal bipartite graph G with bipartition of vertices X and Y contains an edge ij for $i \in X$ and $j \in Y$ such that the induced subgraph on vertices of $N_G(i)$ and $N_G(j)$ is a complete bipartite graph. Such an edge ij is called a *bisimplicial* edge. Additionally, it is known in [19] that the subgraph $G - ij$ is again a chordal bipartite graph. This implies that subgraphs obtained by deleting a bisimplicial edge from a chordal bipartite graph are also chordal bipartite graphs.

Combining Theorem 1.2.5 with property of chordal bipartite graph, we get an exact description of complements of bipartite graphs of regularity 3.

Theorem 1.4.2. *Let G be the complement of a bipartite graph. Regularity of G is 3 if and only if G contains a hole and the bipartite part B of G is chordal bipartite.*

Proof. Suppose that the complement G of a bipartite graph H has at least one hole and the bipartite part B of G is a chordal bipartite graph. Since G contains at least one hole, regularity of G is at least 3. To show that regularity of G is at most 3 we induct on the number of edges ℓ in B. The base case $\ell = 0$ is simple, since G is then chordal and therefore $\operatorname{reg}(G) \leq 2$. Now we consider the induction step. Let G be the complement of a bipartite graph such that its bipartite part B is a chordal bipartite graph with $\ell + 1$ edges. Then B contains a bisimplicial edge e. By Theorem 1.2.5,

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G - e), \operatorname{reg}(G_e - e) + 1\}.$$

Since e is a bisimplicial edge in B, $G_e - e$ is a chordal graph, and $\operatorname{reg}(G_e - e) \leq 2$. In addition, $\operatorname{reg}(G - e) \leq 3$ by the induction assumption, and the desired result follows.

Conversely, suppose that bipartite part B of G contains a hole of length at least 6. We claim that ΔG contains a subcomplex whose second (reduced) homology is not zero. Let G' be the subgraph of G induced by vertices that form the shortest hole in B. Let X' and Y' be the partitions of vertices G' (induced from the partition of vertices of G). Let v be any vertex of X'. Then, the closed neighborhood $N_{G'}[v]$ and the deletion $G' \setminus v$ of v cover cliques of G'. Observe that $\widetilde{H}_1(\Delta N_{G'}[v]) = \widetilde{H}_1(\Delta(G' \setminus v)) = 0$ since $\Delta N_{G'}[v]$ is contractible, and any hole in $G' - v$ is covered by cliques of size 3, but $\widetilde{H}_1(\Delta N_{G'}(v)) \neq 0$ since $N_{G'}(v)$ contains a hole (of length 4). Since $\widetilde{H}_2(\Delta G') \to \widetilde{H}_1(\Delta N_{G'}(v))$ is surjective by the Mayer-Vietoris sequence, $\widetilde{H}_2(\Delta G') \neq 0$, and this implies that regularity of G is at least 4. $\square$

1.5 Regularity and Genus of graphs

The following bound on regularity is well-known in [20, Lemma 2.1] (or see [21] for a geometric proof), but we provide a short proof for the sake of completeness.

Lemma 1.5.1. *If the number of vertices of G is at most* $2n - 1$*, then regularity of G is at most* n*.*

Proof. We use induction on n. For $n = 1$, regularity is obviously at most 1 since there are no generators in the non-edge ideal of the graph. Assume that any graph with at most $2\ell - 1$

vertices has regularity at most ℓ. Let G be a graph on $2\ell+1$ vertices. Note that by Theorem 1.2.3 we can delete a vertex v without changing regularity if $\operatorname{reg}(G) > \operatorname{reg}(N_G(v))+1$. After deleting such vertices, if possible, let v be the vertex of minimal degree in G. If the degree of v is 2ℓ, then G is a complete graph (which has regularity 1). Therefore, we can assume that degree of v is at most $2\ell-1$. Then, we have

$$\operatorname{reg}(G) \leq \operatorname{reg}(N_G(v)) + 1 \leq \ell + 1,$$

since $N_G(v)$ contains at most $2\ell-1$ vertices. □

In fact, the bound in Theorem 1.5.1 is tight. Let $K_{n(2)}$ be the complete n-partite graph, with each part of size two. Note that the ideal of $K_{n(2)}$ is a complete intersection of n quadrics, so the Koszul complex (with weighting degree of variables by two) is the minimal graded free resolution of the ideal. Thus regularity of $K_{n(2)}$ is $n+1$.

Recall that the *genus* of a graph G is the minimal genus of an orientable surface S_g into which G can be embedded. (See [22] for reference.) Note that any graphs can be embedded into an orientable surface S_g for some genus g and the genus of graphs inscribes a topological complexity of the simple graphs. By using Theorem 1.5.1 we can immediately give an alternative proof of a result in [21] that any planar graphs have regularity at most 4 and it is tight. We note that this is the case of genus 0 and we can provide bounds on regulairty of graphs in terms of arbitrarily genus.

Theorem 1.5.2. *Let g be the genus of a graph G. Then, regularity of G is at most $\lfloor 1 + \sqrt{1+3g} \rfloor + 2$.*

Proof. Let $|V|$ be the number of vertices, $|E|$ be the number of edges, and $|F|$ be the number of (2-dimensional) faces in the embedding of G. By considering the Euler characteristic of the surface S into which G is embedded, we see that $|V|-|E|+|F| = 2-2g$. Recall that $2|E| = \sum_{v\in V} \deg(v) = \sum_{F\in\Delta_2} \ell_F$ where Δ_2 is the set of 2-cells in the embedding and ℓ_F is the number of edges in the face F. In particular, $2|E| = \sum_{F\in\Delta_2} \ell_F \geq 3|F|$ since $\ell_F \geq 3$ for any face F. Let d be the minimal degree of G. Then, $2|E| = \sum_{v\in V} \deg(v) \geq d|V|$. Therefore,

$$\begin{aligned}
2-2g &= |V| - |E| + |F| \\
&\leq |V| - |E| + \frac{2}{3}|E| = |V| - \frac{1}{3}|E| \\
&\leq |V| - \frac{d}{6}|V| = \frac{6-d}{6}|V|.
\end{aligned}$$

Moreover, we can see that $|V| \geq d+2$ since $d \leq \deg(v) \leq |V|-2$. Note that, if $d = |V|-1$, the graph is complete graph, which can be excluded. Thus,

$$6(2g-2) \geq (d-6)|V| \geq (d-6)(d+2) \Rightarrow 0 \geq d^2 - 4d - 12g.$$

This implies that $d \leq 2 + \sqrt{4+12g} = 2 + 2\sqrt{1+3g}$. Let v be the vertex of degree d. Then, $\operatorname{reg}(N_G(v)) \leq \lfloor \frac{1}{2} \lfloor 2 + 2\sqrt{1+3g} \rfloor \rfloor + 1 = \lfloor 1 + \sqrt{1+3g} \rfloor + 1$. By Theorem 1.3.1, $\operatorname{reg}(G) \leq \lfloor 1 + \sqrt{1+3g} \rfloor + 2$. □

Note that this bound is indeed tight. It is known in [23, Section 4.4] that the genus of 2-regular complete n-bipartite graphs $K_{n(2)}(= K_{2,2,\ldots,2})$ is at least $\frac{(n-3)(n-1)}{3}$. Moreover, the genus of $K_{n(2)}$ is exactly $\frac{(n-3)(n-1)}{3}$ if $n \not\equiv 2 \mod 3$ by [24]. In this case, we have $\operatorname{reg}(K_{n(2)}) = n+1$ and the right hand side of inequality in Theorem 1.5.2 is $\lfloor 1 + \sqrt{1 + 3\frac{(n-3)(n-1)}{3}} \rfloor + 2 = n+1$.

1.6 Bounds on regularity of graphs without small holes

Even though regularity of a graph can depend linearly on the number of vertices n, if G does not contain small holes, then regularity of G can be bounded from above by a logarithmic function of n. It was shown in [25] that absence of small holes corresponds to the ideal satisfying property $N_{2,p}$ for some $p \geq 2$.

Theorem 1.6.1. *Let $p \geq 2$ and $I(G)$ be the non-edge ideal corresponding to a graph G. Then, the followings are equivalent.*

1. *The minimal graded free resolution of $I(G)$ is $(p-1)$-step linear.*

2. *The graph G does not contain a hole C_i of length i for $i \leq p+2$.*

3. *$I(G)$ satiesfies $N_{2,i}$ for all $2 \leq i \leq p$.*

It was shown in [2] that if G satisfies $N_{2,p}$ for $p \geq 2$, then

$$\mathrm{reg}(G) \leq \log_{\frac{p+3}{2}} \frac{n-1}{p} + 3.$$

We also provide (a similar and) asymptotically better upper bound on regularity of graphs.

Theorem 1.6.2. *Suppose that G satisfies property $N_{2,p}$ for $p \geq 2$. Then,*

$$\mathrm{reg}(G) \leq \min\{\log_{\frac{p+3}{2}} \left(\frac{n(p+1)}{p(p+3)}\right) + 3, \log_{\frac{p+4}{2}} \left(\frac{n(p+2)}{(p+1)(p+4)}\right) + 4\}.$$

Proof. Given a graph G, there is an induced subgraph G_0 such that $\mathrm{reg}(G) = \mathrm{reg}(G_0) = \mathrm{reg}(N_{G_0}(v)) + 1$ for any vertex v in G_0. Indeed, we can keep deleting vertices y such that $\mathrm{reg}(G) = \mathrm{reg}(G \setminus y)$ until we arrive at a graph G_0, where $\mathrm{reg}(G_0 \setminus v) = \mathrm{reg}(G_0) - 1$ for any vertex v of G. Then, by Theorem 1.2.3 we have $\mathrm{reg}(G_0) = \mathrm{reg}(N_{G_0}(v)) + 1$ for any vertex v in G_0. We call such G_0 a trimming of G. Note that a trimming is not unique.

Let x_0 be a vertex of minimal degree in G_0. Let G_1 be a trimming of the open neighborhood $N_{G_0}(x_0)$ of x_0 in G_0. Now we repeat this process: let x_i be a vertex of minimal degree in G_i and let G_{i+1} be a trimming of the open neighborhood of x_i in G_i. We obtain a sequence of induced subgraphs G_i of G such that

$$\mathrm{reg}(G) = \mathrm{reg}(G_0) = \mathrm{reg}(G_1) + 1 = \cdots = \mathrm{reg}(G_t) + t.$$

Let ℓ be the maximal integer such that G_ℓ contains a hole, and let C_m be the hole in G_ℓ of smallest length m, with $m \geq p+3 \geq 5$. Note that C_m is a hole that is present in all graphs G_i, with $0 \leq i \leq \ell$. We use d_i to denote the degree of x_i in G_i. We claim that for $1 \leq i \leq \ell$ the sum of the degrees of vertices of C_m in $N_{G_{\ell-i}}[x_{\ell-i}]$ is at most

$$md_{\ell-i} - \frac{m^i(m-3)}{2^{i-1}},$$

which we prove by induction on i. The base case is $i = 1$: a vertex of C_m is connected to exactly two vertices of C_m and can be connected to all other vertices in $N_{G_{\ell-1}}[x_{\ell-1}]$. Therefore, the sum of degrees of vertices of C_m is at most $2m + m(d_{\ell-1} + 1 - m) = md_{\ell-1} - m(m-3)$.

For the inductive step, assume that the sum of the degrees of vertices of C_m in $N_{G_{\ell-i+1}}[x_{\ell-i+1}]$ is at most $md_{\ell-i+1} - \frac{m^{i-1}(m-3)}{2^{i-2}}$. Observe that any vertex in $G_{\ell-i+1}$ not connected to $x_{\ell-i+1}$ can be adjacent to at most two vertices of C_m. Otherwise $G_{\ell-i+1}$ is forced to have a 4-hole, which is a contradiction. Since degree of $x_{\ell-i+1}$ in $G_{\ell-i+1}$ is at least the degree of any vertex of C_m is $G_{\ell-i+1}$ we see that there are at least

$$\frac{1}{2}\left(md_{\ell-i+1} - (md_{\ell-i+1} - \frac{m^{i-1}(m-3)}{2^{i-2}})\right) = \frac{m^{i-1}(m-3)}{2^{i-1}} \tag{1.1}$$

vertices in $G_{\ell-i+1} \setminus N_{G_{\ell-i+1}}[x_{\ell-i+1}]$.

Any vertex of $N_{G_{\ell-i}}[x_{\ell-i}]$ belongs to exactly one of $N_{G_{\ell-i}}[x_{\ell-i}] \setminus G_{l-i+1}$, or $G_{l-i+1} \setminus N_{G_{l-i+1}}[x_{l-i+1}]$, or $N_{G_{l-i+1}}[x_{l-i+1}]$. As before, any vertex of $G_{l-i+1} \setminus N_{G_{\ell-i+1}}[x_{\ell-i+1}]$ can be adjacent to at most two vertices in C_m, and a vertex of C_m can be adjacent to all vertices of $N_{G_{\ell-i}}[x_{\ell-i}] \setminus G_{\ell-i+1}$. Therefore,

$$\sum_{v \in C_m} \deg_{N_{G_{\ell-i}}[x_{\ell-i}]}(v) \leq m|N_{G_{\ell-i}}[x_{\ell-i}] \setminus G_{\ell-i+1}| + 2|G_{\ell-i+1} \setminus N_{G_{\ell-i+1}}[x_{\ell-i+1}]|$$
$$+ \sum_{v \in C_m} \deg_{N_{G_{\ell-i+1}}[x_{\ell-i+1}]}(v).$$

Using the induction assumption on $\sum_{v \in C_m} \deg_{N_{G_{\ell-i+1}}[x_{\ell-i+1}]}(v)$ we see that

$$\sum_{v \in C_m} \deg_{N_{G_{\ell-i}}[x_{\ell-i}]}(v) \leq md_{\ell-i} - (m-2)\big(|G_{\ell-i+1} \setminus N_{G_{\ell-i+1}}[x_{\ell-i+1}]|\big) - \frac{m^{i-1}(m-3)}{2^{i-2}}.$$

By (Equation 1.1) we see that

$$(m-2)\big(|G_{\ell-i+1} \setminus N_{G_{\ell-i+1}}[x_{\ell-i+1}]|\big) + \frac{m^{i-1}(m-3)}{2^{i-2}} \geq \frac{m^i(m-3)}{2^{i-1}},$$

and therefore

$$\sum_{v \in C_m} \deg_{N_{G_{\ell-i}}[x_{\ell-i}]}(v) \leq md_{\ell-i} - \frac{m^i(m-3)}{2^{i-1}},$$

as desired. The argument above shows that there are at least $\frac{m^i(m-3)}{2^i}$ vertices in $G_{\ell-i} \setminus N_{G_{\ell-i}}[x_{\ell-i}]$. Since $G_{\ell-i+1}$ is a subgraph of $N_{G_{\ell-i}}[x_{\ell-i}]$, we see that

$$|G_{\ell-i}| - |G_{\ell-i+1}| \geq \frac{m^i(m-3)}{2^i}.$$

Therefore,

$$|G_{\ell-i}| \geq \sum_{t=1}^{i} \frac{m^t(m-3)}{2^t} + m,$$

and by summing the above geometric series we see that

$$|G_{\ell-i}| \geq \frac{m^{i+1}(m-3)}{2^i(m-2)}.$$

Plugging in $i = \ell$, we see that

$$n \geq |G_0| \geq \frac{m^{\ell+1}(m-3)}{2^\ell(m-2)} \geq \frac{p(p+3)^{\ell+1}}{2^\ell(p+1)}.$$

Thus,

$$\operatorname{reg}(G) \leq \operatorname{reg}(G_{\ell+1}) + \ell + 1 \leq \log_{\frac{p+3}{2}}\left(\frac{n(p+1)}{p(p+3)}\right) + 3,$$

which gives us the first upper bound.

For the second upper bound, we observe that if regularity of G is at least four, then G contains both even and odd holes by Theorem 1.3.6. With the same setting above, regularity of $N_{G_{\ell-2}}(x_{\ell-2})$ (or equivalently, $G_{\ell-1}$) is four. Let m be the length of the smallest hole in $N_{G_{\ell-2}}(x_{\ell-2})$. Then $N_{G_{\ell-2}}(x_{\ell-2})$ must also contain a hole of size $m+2\alpha+1$ for some positive integer α. We can now apply the same process as above to bound number of vertices of $G_{\ell-i}$ using this larger hole to obtain

$$|G_{\ell-i}| \geq \frac{(m+2\alpha+1)^i(m+2\alpha-2)}{2^{i-1}(m+2\alpha-1)},$$

for $2 \leq i \leq \ell$. By taking $i=\ell$, we see that

$$n \geq |G_0| \geq \frac{(m+2\alpha+1)^\ell(m+2\alpha-2)}{2^{\ell-1}(m+2\alpha-1)} \geq \frac{(p+4)^\ell(p+1)}{2^{\ell-1}(p+2)}.$$

Thus,

$$\mathrm{reg}(G) \leq \mathrm{reg}(G_{\ell+1}) + \ell + 1 \leq \log_{\frac{p+4}{2}} \frac{n(p+2)}{(p+1)(p+4)} + 4.$$

□

Note that the former term in the bound in Theorem 1.6.2 is slightly better (if $n \geq \frac{p+3}{2}$) than the bound in [2, Theorem 4.9] and the former term will be smaller than the latter term if the size of a graph is relatively small. However, the latter term of the bound is better asymptotically.

CHAPTER 2

HANKEL INDEX, ALMOST REAL RANK, AND SUMS OF SQUARES

2.1 Overview: Hankel index of rational curves

A central problem in real algebraic geometry is understanding the differences between the sums of squares (SOS) cones and the cones of non-negative polynomials (PSD) on varieties. To see the *structural difference* between the SOS cones and the cone of PSD quadrics on varieties, we study the Hankel index of the varieties. The Hankel index of a variety X, denoted by $\eta(X)$, is a representative semi-algebraic invariant that measures the structural gap between the two cones. (See section 2.2 for details.) By [1], the Hankel index is bounded below by an algebraic invariant called the Green-Lazarsfeld index, denoted by $\alpha(X)$. Moreover, the inequality is tight for all known cases such as arithmetically Cohen-Macaulay (ACM) varieties of almost minimal degree. So, we investigated the Hankel index on (smooth) non-ACM curves of almost minimal degree. i.e. the rational curves that is the images of projection of rational normal curves away from points outside of second secant of the rational normal curve [26].

Since now, we assume that X is a rational curve that is the image of the projection of a rational normal curve C_d of degree d away from a point p (outside of C_d). Then, $\alpha(X)$ is obtained by the (complex border) rank of the center p with respect to C_d (cf. [27]). In other words, the Green-Lazarsfeld index of the rational curve X is determined by the smallest number of points in C_d that spans p. However, we found that the Hankel index $\eta(X)$ of X is obtained by a new rank of p, called the almost-real rank of p, which realizes a decomposition of p as a (linear) combination of almost-real points on C_d.

Theorem 2.1.1. *Let C_d be a rational normal curve of degree d in $\mathbb{P}^d$. Let $\pi_p : \mathbb{P}^d \dashrightarrow \mathbb{P}^{d-1}$*

be the projection away from the center $p \in \mathbb{P}^d \setminus \operatorname{Sec}_2(C_d)$ *and* $X = \pi_p(C_d)$*. Then*

$$\eta(X) = \textit{ar-rk}_{C_d}(p) - 2$$

where $\textit{ar-rk}_{C_d}(p)$ *is the minimal number of* almost-real points *in* C_d *that spans* p*.*

Not only did we discover a new rank of the center p which detects the Hankel index $\eta(X)$, but we also found that the rational curves are the first class of examples such that the inequality between the two indexes is strict.

2.2 Hankel index of variety

Recall that both the SOS cone and the cone of PSD quadrics on varieties X are closed cone on space of quadrics on X. Moreover, the facets of each cones correspond to the extreme rays of the dual cones because of duality. Therefore, one can study the differences of the facet structures of the cones by investigating the differences of the extreme rays of dual cones.

Let $P_X^\star$ be the dual cone of non-negative quadratic forms on the variety X and $\Sigma_X^\star$ be the dual cone of sums of squares on the variety. The cone $\Sigma_X^\star$ is a *spectrahedron*, i.e. a slice of the cone of positive semidefinite (PSD) matrices with a linear subspace. We call $\Sigma_X^\star$ the *Hankel spectrahedron* of X. By identifying a linear functional ℓ in $\Sigma_X^\star$ with a PSD matrix we say the rank of ℓ is the matrix rank of the quadratic forms. Rank one extreme rays of Σ_X^* are precisely the extreme rays of $P_X^\star$. Therefore, if $P_X^* \subsetneq \Sigma_X^*$ we can quantitatively measure the difference between these cones by analyzing the ranks of extreme rays of Σ_X^* that are greater than one.

Definition 2.2.1. *(cf. [1, Definition 1]) The* Hankel index *of* X*, denoted* $\eta(X)$*, is defined to be the minimal rank of a(n extreme) ray* $\ell \in \Sigma_X^\star \setminus P_X^\star$*, or* ∞ *if* $\Sigma_X^\star = P_X^\star$*.*

In other words, the Hankel index of a variety X is minimal rank of extreme rays of the dual cone of SOS which is bigger than one.

The Hankel index is a subtle invariant which is often quite hard to compute. Indeed, the Hankel index of varieties is known for only a few cases. For example, varieties of minimal degree, arithmetically Cohen-Macaulay (ACM) varieties of almost minimal degree, varieties defined by quadratic squarefree monomial ideals, some general canonical curves, and Veronese embeddings of $\mathbb{P}^2$ (see [1, Theorem 28] and [28]).

2.3 Waring ranks of binary forms

Since we will work with the rational normal curves, we can identify the points with the binary form $F(p)$. (See Section C.3 for details). Regarding the identification, the ranks of points with respect to rational normal curves are connected to another classical notion on binary forms: *Waring rank* of the binary forms $F(p)$, i.e. shortest length of the decomposition of $F(p)$ as a sum of powers of linear forms. In [27, Theorem 1.1(2)] it was shown that for such curves, the Green-Lazarsfeld index equals the complex Waring border rank of $F(p)$ minus 3. Remark that the complex Waring border rank of $F(p)$ can be investigate through the lowest degree element of apolar ideal of the polynomial $F(p)$. Moreover, the elements in the apolar ideal that can be factored into distinct linear forms generate the decompositions of the binary form $F(p)$ as a linear combination of power sums of linear forms. (See section C.1 for details.)

2.4 Almost reality

We now introduce a central notion for this chapter, which is that of a binary form almost splitting over $\mathbb{R}$, or a univariate polynomial having almost all real roots. For technical reasons we will need to include the possibility of one pair of roots being nondistinct, so that the resulting rank is intermediate between a true rank and a border rank.

Definition 2.4.1. *Let $F \in \mathbb{R}[x, y]_d$. We say that F has* almost real roots *if F has $\geq d - 2$*

simple linear factors over $\mathbb{R}$. Equivalently, F has a factorization over $\mathbb{R}$ of the form

$$F = q \cdot \prod_{i=1}^{d-2} l_i, \quad \{l_1, \dots, l_{d-2}, q\} \textit{ pairwise relatively prime}$$

where l_i are linear and q is quadratic. A polynomial F with almost real roots thus belongs to exactly one of 3 classes: (i) F has all simple real roots, (ii) F has a unique nonreal complex conjugate pair of roots, (iii) F has a unique double real root (note that in cases (ii) and (iii), all other roots are real and simple).

In analogy with Theorem C.2.2, we define the almost real rank *of F as*

$$\operatorname{ar-rk}(F) := \min \left\{ r \;\middle|\; \begin{array}{c} \exists g \in (F)^{\perp}_{r} \textit{ with} \\ \textit{almost real roots} \end{array} \right\}$$

Remark 2.4.2. *One can generalize the definition above to arbitrary (i.e. not necessarily binary) forms. Given a form $F \in \mathbb{R}[x_1, \dots, x_n]$, define the* almost real rank *of F as the minimal length of a zero-dimensional subscheme $Z \subseteq \mathbb{P}^{n-1}_{\mathbb{C}}$ such that $I(Z) \subseteq (F)^{\perp}$ and Z has either (i) all reduced real points, or (ii) exactly 1 nonreal conjugate pair of points, or (iii) exactly 1 double point. In this article though, we will only use the notion of almost real rank for binary forms.*

Note that for any $F \in \mathbb{R}[x, y]_d$, it follows from the definitions that $\mathbb{C}\text{-b.}\operatorname{rk}(F) \leq \operatorname{ar-rk}(F) \leq \mathbb{R}\text{-rk}(F)$. For more properties of almost real rank, see Section 2.7.

The Hankel index of X is determined by the shortest decomposition of $F(p)$ into as a sum of powers of *almost real forms*, (cf. Section 2.4 for precise definitions), which we call the *almost real rank* of $F(p)$.

Theorem 2.4.3. *Let $X = \pi_p(C_d)$ be a projection of a rational normal curve C_d of degree d away from a point $p \in \mathbb{P}^d \setminus C^3_d$, with corresponding binary form $F(p) \in \mathbb{R}[x, y]_d$. Then* *the Hankel index of X is given by*

$$\eta(X) = \text{ar-rk}(F(p)) - 2.$$

This theorem elucidates the semialgebraic nature of the Hankel index, and demonstrates two ways in which it differs from the Green-Lazarsfeld index: the difference between rank and border rank, and the difference between almost real decompositions and complex decompositions.

We note an interesting technical detail of the proof of Theorem 2.4.3. To prove an upper bound on Hankel index we need a construction of rays in $\Sigma_X^\star \setminus P_X^\star$, and for this we use point evaluations at points of X in special position. Such constructions using Cayley-Bacharach relations were used in [29] and more generally in [30] (the idea goes all the way to Hilbert's original proof). Until now these construction only used reduced points of X, but in this paper we use non-reduced 0-dimensional subschemes of X. The use of such non-reduced configurations is necessary, and cannot be replicated by reduced points.

2.5 Construction of rays in $\Sigma_X^\star$

We now turn to the proof of Theorem 2.4.3, which will span the next two sections. In this section, we give a general procedure for constructing elements in $\Sigma_X^\star$ of ranks between $\text{ar-rk}(F(p)) - 2$ and $d - 3$, whose kernels are basepoint-free. By Theorem B.2.3, this shows that if $\text{ar-rk}(F(p)) > 3$, then $\eta(X) \leq \text{ar-rk}(F(p)) - 2$.

Choose r with $\text{ar-rk}(F(p)) \leq r \leq d - 1$, and choose a form $g \in (F(p))_r^\perp$ with almost real roots. We assume that no proper divisor of g is in $(F(p))^\perp$ (which is automatic when $r = \text{ar-rk}(F(p))$, and can be arranged when $r \geq \deg F^\circ$). Then there is a factorization over $\mathbb{C}$

$$g =: \prod_{i=1}^{r} l_i$$

of g into linear forms $l_i =: a_i x + b_i y \in \mathbb{C}[x, y]_1$ where either

1. All l_i's are distinct and real, or
2. All l_i's are distinct, and there is exactly one conjugate pair $l_r = \overline{l_{r-1}}$, or
3. All l_i's are real, and there is exactly one repeated factor $l_r = l_{r-1}$.

For the first two cases, the construction that we give below has appeared before, e.g. in [29, Theorem 6.1 and Theorem 7.1] (for Veronese embeddings of projective spaces) and [30, Proposition 3.2 and Procedure 3.3]. Case (3) however is new, specifically dealing with a non-reduced zero-dimensional scheme.

1. Simple real roots

   By apolarity (Theorem C.1.4), $F(p)$ may be expressed as a linear combination of $(l_1)^d_\perp, \ldots, (l_r)^d_\perp$, i.e. there exist $c_1, \ldots, c_r \in \mathbb{R}$ such that

$$F(p) = \sum_{i=1}^{r} c_i (l_i)^d_\perp \tag{2.1}$$

   Note that since no proper factor of g is in $(F(p))^\perp$, each coefficient c_i in (Equation 2.1) is nonzero.

   We now construct elements in $\Sigma_X^\star$ of rank $r-2$. Let $p_1, \ldots, p_r \in \mathbb{P}^d$ correspond to the r roots of g (explicitly, $p_i = \nu_d([a_i : b_i])$). Consider a linear combination

$$\ell := \sum_{i=1}^{r} d_i \ell_{p_i}^2 \in (R(C_d)_2)^\star \tag{2.2}$$

   with (as yet unspecified) coefficients $d_i \in \mathbb{R}$, where ℓ_{p_i} = evaluation at p_i (note that ℓ_{p_i} corresponds to the binary form $(l_i)^d_\perp \in \mathbb{R}[x,y]_d$). Then as in Theorem C.5.1, ℓ gives rise to a quadratic form Q_ℓ on $R(C_d)_1$, as well as its restriction q_ℓ to $R(X)_1$.

   Next, we claim that if the d_i are chosen so that

$$d_1, \ldots, d_{r-1} > 0, \quad \sum_{i=1}^{r} \frac{c_i^2}{d_i} = 0, \tag{2.3}$$

then $\operatorname{rank}(q_\ell) = r - 2$. To show this, we choose coordinates to reduce to a computation with matrices. Let

$$Z := \{p_1, \ldots, p_r\} \subseteq C_d$$

be the zero-dimensional variety of the points p_i. The coordinate ring $R(Z)$ satisfies $\dim_{\mathbb{R}} R(Z)_1 = r$, with basis $\{e_i\}_{i=1}^r$ given by indicator functions of the points, i.e. $e_i(p_j) = \delta_{ij}$. (One can of course write down explicit polynomial representatives on $\mathbb{P}^d$ for the e_i's via interpolators (with a suitable padding up to degree d), although we will not need such representatives.) If $I(Z)$ is the defining ideal of Z in C_d, then via the isomorphism $R(Z) \cong R(C_d)/I(Z)$, a quadratic form on $R(C_d)_1$ whose kernel contains $I(Z)_1$ (such as Q_ℓ) induces a quadratic form on $R(Z)_1$, which is in turn represented as an $r \times r$ matrix.

The choice of basis $\{e_i\}_{i=1}^r$ then allows for a convenient expression of the matrix of the induced quadratic form $\widetilde{Q_\ell}$ on $R(Z)_1$: namely, $\widetilde{Q_\ell}$ is represented by a diagonal matrix $\operatorname{diag}(d_1, \ldots, d_r)$ in this basis. Note that the conditions (Equation 2.3) imply that $d_r < 0$ (recall that $c_i \neq 0$), so by Theorem C.6.3, $\widetilde{Q_\ell}$ has Lorentz signature (since $r \leq d+1$, any set of r points on C_d are in linearly general position, so the functionals $\ell_{p_1}, \ldots, \ell_{p_r} \in (R(C_d)_2)^\star$ are linearly independent).

On the other hand, we may also consider the quadratic form induced by q_ℓ on the points $\pi(Z) := \{\pi(p_1), \ldots, \pi(p_r)\}$. The key difference is that the points $\pi(p_1), \ldots, \pi(p_r) \in X$ are *not* in linearly general position – indeed, the projection map π can be viewed as a projectivization of the vector space quotient $\mathbb{R}[x, y]_d \twoheadrightarrow \mathbb{R}[x, y]_d / \operatorname{span}\{F(p)\}$, so the linear relation (Equation 2.1) gives a linear dependency

$$0 = \sum_{i=1}^{r} c_i \ell_{\pi(p_i)} \quad \iff \quad \ell_{\pi(p_r)} = -\frac{1}{c_r} \sum_{i=1}^{r-1} c_i \ell_{\pi(p_i)} \tag{2.4}$$

expressing the last point evaluation $\ell_{\pi(p_r)}$ in terms of the others. In particular, on removing the last point $\pi(p_r)$, the coordinate ring $R(\pi(Z \setminus \{p_r\}))$ has a basis $\{e_i\}_{i=1}^{r-1}$

for its degree 1 part (note that $\pi(Z\backslash\{p_r\})$ *is* in linearly general position in $\mathbb{P}^{d-1}$). This gives a quadratic form $\widetilde{q_\ell}$ on $R(\pi(Z \setminus \{p_r\}))_1$ induced by q_ℓ: explicitly, substituting (Equation 2.4) into (Equation 2.2) gives the expression $\sum_{i=1}^{r-1} d_i \ell_{\pi(p_i)}^2 + \frac{d_r}{c_r^2}\Big(\sum_{i=1}^{r-1} c_i \ell_{\pi(p_i)}\Big)^2$ for (the linear functional corresponding to) $\widetilde{q_\ell}$. Setting

$$D := \operatorname{diag}(d_1, \dots, d_{r-1}), \quad \mathbf{c} := \begin{bmatrix} c_1 & \dots & c_{r-1} \end{bmatrix}^T,$$

we see that the matrix of $\widetilde{q_\ell}$ in the basis $\{e_i\}_{i=1}^{r-1}$ is given by $D + \frac{d_r}{c_r^2}\mathbf{c}\mathbf{c}^T$. Finally, observe that the vector $D^{-1}\mathbf{c}$ is in the kernel of $\widetilde{q_\ell}$:

$$\begin{aligned}(D + \frac{d_r}{c_r^2}\mathbf{c}\mathbf{c}^T)(D^{-1}\mathbf{c}) &= \mathbf{c} + \frac{d_r}{c_r^2}\mathbf{c}\mathbf{c}^T D^{-1}\mathbf{c} \\ &= \mathbf{c}\left(1 + \frac{d_r}{c_r^2}\left(\sum_{i=1}^{r-1}\frac{c_i^2}{d_i}\right)\right) = 0\end{aligned}$$

by (Equation 2.3). Theorem C.6.5 then implies that q_ℓ is PSD (which implies that $\ell \in \Sigma_X^\star$) of rank $r-2$.

It remains to show that for any ray ℓ constructed satisfying (Equation 2.2) and (Equation 2.3), $\ker(q_\ell)$ is basepoint-free. The following reasoning will also apply to the cases in Item 2 and Item 3. First, we claim that $\ker(q_\ell)$ can have no basepoints outside of $\pi(Z)$: if not, then $\ker(Q_\ell)$ would have a basepoint outside of Z. However, by Theorem C.6.1, $\ker(Q_\ell) = (g)_d \subseteq (F(p))_d^\perp$ is an $\mathbb{R}$-vector space of dimension $d + 1 - \deg g = d + 1 - r$ which consists of binary forms vanishing at all the points of Z (to orders specified by multiplicities of factors of g in the case of a double root in Item 3), thus cannot have another common zero outside of Z by Theorem C.6.2. It thus suffices to eliminate the possibility of any point of $\pi(Z)$ as a basepoint, but this follows since the vector $D^{-1}\mathbf{c}$ in $\ker(q_\ell)$ has all nonzero entries in the basis $\{e_i\}_{i=1}^{r-1}$.

2. One complex pair

   Assume g has one pair of nonreal roots $l_r = \overline{l_{r-1}}$. The general argument will follow the outline of the first case, so we focus only on the differences (which will mainly be in the last two functionals). Essentially, rather than using two functionals arising from evaluations at complex conjugate points, we use the real and imaginary parts of one complex point evaluation. Over $\mathbb{C}$, there is an expression $F(p) = \sum_{i=1}^{r-2} c_i (l_i)_\perp^d + c_{r-1}(l_{r-1})_\perp^d + c_r(l_r)_\perp^d$, and independence of the forms $\{(l_i)_\perp^d\}_{i=1}^r$ and conjugate-symmetry forces $c_r = \overline{c_{r-1}}$. By rescaling $l_r \in \mathbb{C}[x,y]_1$ we may assume that $c_r = 1$ (so that $c_{r-1} = 1$ as well), and thus write the analogue of (Equation 2.1) in the form

$$F(p) = \sum_{i=1}^{r-2} c_i (l_i)_\perp^d + 2\operatorname{Re}((l_r)_\perp^d) \tag{3$'$}$$

where $c_1, \dots, c_{r-2} \in \mathbb{R}$ are all nonzero, since no proper factor of g is in $(F(p))^\perp$.

We then construct the functional in $\Sigma_X^\star$. As before, choosing $p_1, \dots, p_r \in \mathbb{P}^d$ corresponding to the roots of g (with $p_r = \overline{p_{r-1}}$ a nonreal conjugate pair), we obtain a linear functional $\ell := \sum_{i=1}^{r-2} d_i \ell_{p_i}^2 + d_r \ell_{p_r}^2 + \overline{d_r \ell_{p_r}}^2 \in R(C_d)_2^\star$, which becomes

$$\ell := \sum_{i=1}^{r-2} d_i \ell_{p_i}^2 + 4\alpha(\operatorname{Re}((l_r)_\perp^d)^2 - \operatorname{Im}((l_r)_\perp^d)^2) - 8\beta(\operatorname{Re}((l_r)_\perp^d)\operatorname{Im}((l_r)_\perp^d)) \tag{4$'$}$$

where $\alpha := \frac{\operatorname{Re}(d_r)}{2}$, $\beta := \frac{\operatorname{Im}(d_r)}{2}$. We claim that if the d_i are chosen so that

$$d_1, \dots, d_{r-2} > 0,\ \beta \neq 0,\ \frac{\alpha}{\alpha^2 + \beta^2} + \sum_{i=1}^{r-2} \frac{c_i^2}{d_i} = 0, \tag{5$'$}$$

then q_ℓ has rank $r-2$ and basepoint-free kernel. Indeed, writing $\ell_1, \dots, \ell_r$ for the images of $(l_1)_\perp^d, \dots, (l_{r-2})_\perp^d$, $2\operatorname{Re}((l_r)_\perp^d), 2\operatorname{Im}((l_r)_\perp^d)$ in $R(C_d)_1^\star$, and choosing forms in $R(C_d)_1$ dual to the functionals $\ell_1, \dots, \ell_r$, we see that the matrix of $Q_\ell\Big|_{\operatorname{span}\{e_i\}}$ is given by $\begin{bmatrix} D & 0 \\ 0 & A \end{bmatrix}$ where $D := \operatorname{diag}(d_1, \dots, d_{r-2})$, $A := \begin{bmatrix} \alpha & -\beta \\ -\beta & -\alpha \end{bmatrix}$, so that Q_ℓ

has Lorentz signature (note that $\det(A) < 0$). Expressing (Equation $3'$) in the form

$$\ell_{r-1} = -\sum_{i=1}^{r-2} c_i \ell_i, \tag{$6'$}$$

setting $\mathbf{c} := \begin{bmatrix} c_1 & \dots & c_{r-2} \end{bmatrix}^T$, and substituting (Equation $6'$) into (Equation $4'$) gives the matrix

$$\widetilde{q_\ell} = \begin{bmatrix} D + \alpha \mathbf{c}\mathbf{c}^T & \beta \mathbf{c} \\ \beta \mathbf{c}^T & -\alpha \end{bmatrix}.$$

Finally, observe that the vector $\begin{bmatrix} D^{-1}\mathbf{c} \\ \frac{-\beta}{\alpha^2+\beta^2} \end{bmatrix}$ is in $\ker \widetilde{q_\ell}$, and has all nonzero entries:

$$\begin{aligned} \begin{bmatrix} D + \alpha \mathbf{c}\mathbf{c}^T & \beta \mathbf{c} \\ \beta \mathbf{c}^T & -\alpha \end{bmatrix} \begin{bmatrix} D^{-1}\mathbf{c} \\ \frac{-\beta}{\alpha^2+\beta^2} \end{bmatrix} &= \begin{bmatrix} \mathbf{c} + \alpha \mathbf{c}\mathbf{c}^T D^{-1}\mathbf{c} - \frac{\beta^2}{\alpha^2+\beta^2}\mathbf{c} \\ \beta \mathbf{c}^T D^{-1}\mathbf{c} + \frac{\alpha\beta}{\alpha^2+\beta^2} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{c}(1 + \alpha \mathbf{c}^T D^{-1}\mathbf{c} - \frac{\beta^2}{\alpha^2+\beta^2}) \\ \beta(\mathbf{c}^T D^{-1}\mathbf{c} + \frac{\alpha}{\alpha^2+\beta^2}) \end{bmatrix} \\ &= \begin{bmatrix} \alpha\mathbf{c}(\mathbf{c}^T D^{-1}\mathbf{c} + \frac{\alpha}{\alpha^2+\beta^2}) \\ \beta(\mathbf{c}^T D^{-1}\mathbf{c} + \frac{\alpha}{\alpha^2+\beta^2}) \end{bmatrix} = 0. \end{aligned}$$

The reasoning that $\ker(q_\ell)$ is basepoint-free was already explained at the end of the first case.

3. One double root

   Assume g has a unique real double root $l_r = l_{r-1}$ (with all other roots real and simple). In this case, the two functionals we use correspond to evaluation at the double point, as well as differentiation followed by evaluation. From apolarity, there is a relation

$$F(p) = \sum_{i=1}^{r-2} c_i (l_i)_\perp^d + c_{r-1}(l_r)_\perp^d + c_r l_r (l_r)_\perp^{d-1} \tag{3''}$$

where as before $c_1, \dots, c_r \in \mathbb{R}$ are all nonzero. Let $\ell_1, \dots, \ell_r \in R(C_d)_1^\star$ be the linear functionals corresponding to $(l_1)_\perp^d, \dots, (l_{r-2})_\perp^d, (l_r)_\perp^d, 2l_r(l_r)_\perp^{d-1}$, and consider the linear functional in $(R(C_d)_2)^\star$ defined by

$$\ell := \sum_{i=1}^{r-1} d_i \ell_i^2 + d_r \ell_{r-1}\ell_r. \tag{4''}$$

We claim that if the d_i are chosen so that

$$d_1, \dots, d_{r-1} > 0,\ d_{r-1} - 2\frac{d_r c_{r-1}}{c_r} - \frac{d_r^2}{c_r^2}\sum_{j=1}^{r-2}\frac{c_j^2}{d_j} = 0 \tag{5''}$$

then q_ℓ has rank $r-2$ and basepoint-free kernel. Indeed, the matrix of Q_ℓ (restricted to the subspace of $R(C_d)_1$ spanned by forms dual to $\ell_1, \dots, \ell_r$) is given by $\begin{bmatrix} D & 0 \\ 0 & A \end{bmatrix}$ where $D := \operatorname{diag}(d_1, \dots, d_{r-2}),\ A := \begin{bmatrix} d_{r-1} & \frac{d_r}{2} \\ \frac{d_r}{2} & 0 \end{bmatrix}$, hence has Lorentz signature (note that $\det(A) < 0$). Writing (Equation $3''$) in the form

$$\ell_r = -\frac{2}{c_r}\sum_{i=1}^{r-1} c_i \ell_i, \tag{6''}$$

setting $\mathbf{c} := \begin{bmatrix} c_1 & \dots & c_{r-2} \end{bmatrix}^T$, and substituting (Equation $6''$) into (Equation $4''$) gives the matrix

$$\widetilde{q_\ell} = \begin{bmatrix} D & -\frac{d_r}{c_r}\mathbf{c} \\ -\frac{d_r}{c_r}\mathbf{c}^T & d_{r-1} - \frac{2d_r c_{r-1}}{c_r} \end{bmatrix}.$$

As before, we exhibit a kernel vector $\begin{bmatrix} D^{-1}\mathbf{c} \\ \frac{c_r}{d_r} \end{bmatrix}$ with all nonzero entries:

$$\begin{bmatrix} D & -\frac{d_r}{c_r}\mathbf{c} \\ -\frac{d_r}{c_r}\mathbf{c}^T & d_{r-1} - \frac{2d_r c_{r-1}}{c_r} \end{bmatrix} \begin{bmatrix} D^{-1}\mathbf{c} \\ \frac{c_r}{d_r} \end{bmatrix} = \begin{bmatrix} \mathbf{c} - \mathbf{c} \\ -\frac{d_r}{c_r}\mathbf{c}^T D^{-1}\mathbf{c} + \frac{c_r}{d_r}(d_{r-1} - \frac{2d_r c_{r-1}}{c_r}) \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ \frac{c_r}{d_r}(d_{r-1} - 2\frac{d_r c_{r-1}}{c_r} - \frac{d_r^2}{c_r^2}\mathbf{c}^T D^{-1}\mathbf{c}) \end{bmatrix} = 0.$$

We remark that the (LHS of the) equation in (Equation $5''$) is precisely the Schur complement $\widetilde{q_\ell}/D$ (this provides another proof that $\widetilde{q_\ell}$ is PSD but not PD). As a quadratic in $\frac{d_r}{c_r}$, this equation always has 2 real solutions (as the discriminant $(2c_{r-1})^2 + 4d_{r-1}\mathbf{c}^T D^{-1}\mathbf{c}$ is > 0 by (Equation $5''$)).

As before, the reasoning that $\ker(q_\ell)$ is basepoint-free was given at the end of the first case.

2.6 Lower bound

In this section, we prove a lower bound on the Hankel index in terms of the almost real rank of the center of projection, showing that our construction in Section 2.5 of rays of minimal rank is sharp. Throughout, let $X = \pi_p(C_d)$ be a projection with center p of a rational normal curve $C_d \subseteq \mathbb{P}^d$. We assume that the center p is not contained in C_d^3 (which implies $d \geq 6$), and as in Section C.3, we associate to p a binary form $F(p) \in \mathbb{R}[x, y]_d$.

Theorem 2.6.1. *We have the following bound on the Hankel index of a projected rational normal curve X with center of projection $F(p)$:*

$$\eta(X) \geq \text{ar-rk}(F(p)) - 2.$$

Proof. Fix a ray $\ell \in \Sigma_X^\star$. By Theorem C.5.1(v), we get $L \in \mathbb{R}[x, y]_{2d}$ such that $\ell(\cdot) =$

$\langle \cdot, L \rangle$, and quadratic forms Q_ℓ on $\mathbb{R}[x, y]_d$ and $q_\ell := Q_\ell \Big|_H$ where $H = (F(p))^\perp_d \cong R(X)_1$. Note that q_ℓ is PSD since ℓ was an element of $\Sigma^\star_X$, which implies by Theorem C.6.4 that Q_ℓ has at most one negative eigenvalue.

We now further assume that $\ker q_\ell$ is basepoint-free. By Theorem C.6.3, this implies that Q_ℓ is not PSD, as otherwise q_ℓ would be a sum of point evaluations, contradicting Theorem B.2.3. It follows that Q_ℓ has Lorentz signature $(+, \ldots, +, -)$.

Consider the apolar ideal $(L)^\perp = (L_\perp, L^\circ)$, and set $s := \deg L_\perp$ (so that $\deg L^\circ = 2d + 2 - s$). By Theorem C.6.1, $\ker(Q_\ell) = (L)^\perp_d$, and since this space is nonzero (being basepoint-free), one must have $s \leq d$ (in particular, $s < \deg L^\circ$). Write

$$L_\perp := \prod_{i=1}^{t} l_i^{d_i}$$

where $l_i \in \mathbb{R}[x, y]_1$ are distinct linear forms and $\sum d_i = s$.

We next claim that $L_\perp$ has almost real roots, which is the core of this proof. For convenience, say that a form G has a *triple root* if G has a real root of multiplicity 3, and all other roots are real and simple. We first show, via a perturbation argument, that either $L_\perp$ has almost real roots, or $L_\perp$ has a triple root. Then, we show that $L_\perp$ does not have a triple root.

Thus, suppose that $L_\perp$ does not have almost real roots, nor a triple root. The key idea for the perturbation argument is the following: we may approximate $L_\perp$ by a sequence of polynomials, all of which have at least 2 pairs of simple complex roots. Intuitively, each pair of simple complex roots contributes a negative eigenvalue to the signature, and then continuity implies that Q_ℓ has ≥ 2 negative eigenvalues, a contradiction.

To be precise, we consider the following types of replacements of certain factors of $L_\perp$, depending on the way that $L_\perp$ fails to have almost real roots/a triple root:

$$\begin{aligned}
(x-\alpha)^2(x-\overline{\alpha})^2 &\longrightarrow (x-\alpha-\epsilon)(x-\overline{\alpha}-\epsilon)(x-\alpha+\epsilon)(x-\overline{\alpha}+\epsilon)\\
(x-a)^4 &\longrightarrow (x-a)^4+\epsilon^4\\
(x-a)^2(x-b)^2 &\longrightarrow ((x-a)^2+\epsilon^2)((x-b)^2+\epsilon^2)\\
(x-\alpha)(x-\overline{\alpha})(x-a)^2 &\longrightarrow (x-\alpha)(x-\overline{\alpha})((x-a)^2+\epsilon^2)
\end{aligned}$$

(here $\alpha \in \mathbb{C}\setminus\mathbb{R}$, and $a, b \in \mathbb{R}$ are distinct). Then for all sufficiently small $\epsilon > 0$, the polynomial L_ϵ obtained from $L_\perp$ by performing one of the above replacements has ≥ 2 pairs of simple complex roots, and satisfies $L_\epsilon \to L_\perp$ as $\epsilon \to 0$ (if $L_\perp$ already has 2 pairs of simple complex roots, then we may take $L_\epsilon = L_\perp$). Taking apolar ideals of the form (L_ϵ, L°) gives a sequence of degree $2d$ forms converging to L, and with this associated quadratic forms $Q_\epsilon \to Q_\ell$. Then each L_ϵ has ≥ 2 pairs of simple complex roots, so Q_ϵ has ≥ 2 negative eigenvalues. Furthermore, $\dim\ker(Q_\epsilon) = \dim(L_\epsilon)_d = d - s + 1$ is constant in ϵ. Then continuity of eigenvalues implies that Q_ℓ has ≥ 2 negative eigenvalues (as no negative eigenvalue can become positive without crossing zero, and the number of zero eigenvalues stays constant), contradicting the fact that Q_ℓ has Lorentz signature.

To conclude that $L_\perp$ has almost real roots, it remains to eliminate the possibility that $L_\perp$ has a triple root. We will show that if $L_\perp$ has a triple root, then $\ker q_\ell$ is not basepoint-free. Suppose the roots of $L_\perp$ have multiplicities $(d_1, \ldots, d_{s-2}) = (1, \ldots, 1, 3)$. Setting $l := l_{s-2}$, by apolarity we may write

$$L = \sum_{i=1}^{s-3} d_i (l_i)_\perp^{2d} + d_{s-2}(l_\perp)^{2d} + d_{s-1} l (l_\perp)^{2d-1} + d_s l^2 (l_\perp)^{2d-2} \tag{2.5}$$

for some $d_1, \ldots, d_s \in \mathbb{R}$. Write $\ell_1, \ldots, \ell_s$ for the functionals in $R(C_d)_1^\star$ corresponding to

$(l_1)^d_\perp, \ldots, (l_{s-3})^d_\perp$, $(l_\perp)^d, l(l_\perp)^{d-1}, l^2(l_\perp)^{d-2}$. Then (Equation 2.5) may be expressed as

$$\ell = \sum_{i=1}^{s-2} d_i \ell_i^2 + d_{s-1}\ell_{s-2}\ell_{s-1} + d_s \ell_{s-2}\ell_s \tag{2.6}$$

(note that $\ell_{s-2}\ell_s = \ell_{s-1}^2$). Since $L_\perp \in H = (F(p))^\perp$ (shown below), there is also a relation

$$0 = \sum_{i=1}^{s} c_i \ell_i$$

with $c_i \in \mathbb{R}$. Note that since a proper factor of $L_\perp$ may lie in $(F(p))^\perp$, we cannot say a priori whether any particular c_i is nonzero. We thus consider cases depending on whether c_s is nonzero.

If $c_s \neq 0$, then substituting $\ell_s = -\frac{1}{c_s}\sum_{i=1}^{s-1} c_i \ell_i$ into (Equation 2.6) gives a matrix for $\widetilde{q_\ell}$ (with respect to $\{\ell_1, \ldots, \ell_{s-1}\}$) whose last diagonal entry (= coefficient of ℓ_{s-1}^2)) is 0. If $c_s = 0$, then substituting $\ell_i = -\frac{1}{c_i}\sum_{j\neq i}^{s-1} c_j \ell_j$ (for some $1 \leq i \leq s-1$) into (Equation 2.6) gives a matrix for $\widetilde{q_\ell}$ (with respect to $\{\ell_1, \ldots, \hat{\ell_i}, \ldots, \ell_s\}$) whose last diagonal entry (= coefficient of ℓ_s^2)) is 0. Thus in any case $\widetilde{q_\ell}$ can be represented by a matrix with last diagonal entry 0, and since $\widetilde{q_\ell}$ is PSD, this implies that the entire last column of $\widetilde{q_\ell}$ must be 0. Then $\ker(\widetilde{q_\ell})$ is generated by the vector $\begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^T$, but this implies that $\ker(q_\ell)$ is not basepoint-free (as each of the roots of $l_1, \ldots, l_{s-1}$ would be basepoints).

This shows that $L_\perp$ has almost real roots. Next, we show that $L_\perp$ is contained in the apolar ideal of the center $(F(p))^\perp$. If $L_\perp$ has simple roots, then the points on X corresponding to these roots cannot be in linearly general position: if they were, then Theorem C.6.3 implies that ℓ would be a sum of point evaluations, contradicting Theorem B.2.3(iii). This means precisely that $F(p)$ can be written as a linear combination of d^{th} powers of roots of $L_\perp$, so by apolarity $L_\perp \in (F(p))^\perp$.

Next, suppose that $L_\perp$ does not have simple roots, and define the following "reduction of order" polynomial

$$\widetilde{L_\perp} := \prod_{i=1}^{t} l_i^{\lceil d_i/2 \rceil}$$

with the key property that $L_\perp$ divides $\widetilde{L_\perp}^2$. We claim that

$$(\widetilde{L_\perp})_d \cap H = \ker(q_\ell) = (L_\perp)_d \cap H. \tag{2.7}$$

To see this, note that for $f \in H$, one has $f \in \ker(q_\ell) \iff q_\ell(f) = 0$ (as q_ℓ is PSD on H – this need not be the case if q_ℓ were indefinite). Together with Theorem C.6.1, this gives the second equality. For the first equality, note that $L_\perp \in (\widetilde{L_\perp}) \implies (L_\perp)_d \cap H \subseteq (\widetilde{L_\perp})_d \cap H$. Conversely, any $f \in (\widetilde{L_\perp})_d \cap H$ is of the form $f := g\widetilde{L_\perp}$ (for some $g \in \mathbb{R}[x,y]_{d-\deg \widetilde{L_\perp}}$), hence satisfies $q_\ell(f) = \langle g^2(\widetilde{L_\perp})^2, L\rangle = 0$, since $L_\perp$ divides $\widetilde{L_\perp}^2$, and $\langle L_\perp, L\rangle = 0$.

In view of (Equation 2.7): given that $L_\perp \neq \widetilde{L_\perp}$, one has $(L_\perp)_d \subsetneq (\widetilde{L_\perp})_d$, but since the intersections of these subspaces with the hyperplane H coincide, it must be the case that $\dim(L_\perp)_d$, $\dim(\widetilde{L_\perp})_d$ differ by exactly 1 (note that dimension decreases by at most 1 when intersecting with a hyperplane, and does not change precisely when the subspace is already contained in the hyperplane). From this we deduce that $(L_\perp)_d \subseteq H$, hence $L_\perp \in (F(p))^\perp$ by Theorem C.1.5. (Note that this argument also shows that $\deg L_\perp \leq 1 + \deg \widetilde{L_\perp}$, which gives another proof that $L_\perp$ has at most one multiple real root, which must be of multiplicity ≤ 3).

Putting the above results together, we see that $L_\perp \in (F(p))^\perp$ has almost real roots, so $\text{ar-rk}(F(p)) \leq \deg L_\perp = s$. Now $\dim\ker(Q_\ell) = \dim(L)_d^\perp = \dim(L_\perp)_d = d - s + 1$, so $\operatorname{rank}(Q_\ell) = d + 1 - \dim\ker(Q_\ell) = s$, and by Theorem C.6.5, $\operatorname{rank}(q_\ell) = \operatorname{rank}(Q_\ell) - 2$. Thus $\operatorname{rank}(\ell) = \operatorname{rank}(q_\ell) = s - 2 \geq \text{ar-rk}(F(p)) - 2$. Since this holds for any ray ℓ with $\ker(q_\ell)$ basepoint-free, in particular it holds for any extreme ray of $\Sigma_X^\star$ which is not a point evaluation, so $\eta(X) \geq \text{ar-rk}(F(p)) - 2$ as desired. $\square$

2.7 Studies on almost real rank

As shown by our main result Theorem 2.4.3, the almost real rank of a form is an interesting quantity to study. In this section, we investigate almost real rank of binary forms in general. To begin, the following proposition characterizes some cases where the almost real rank is small.

Proposition 2.7.1. *Let $d \geq 3$ and $F \in \mathbb{R}[x, y]_d$, with apolar ideal $(F)^\perp = (F_\perp, F^\circ)$ of type (d_1, d_2).*

1. $\operatorname{ar-rk}(F) = d_1 \iff F_\perp$ *has almost real roots.*
2. *If* $\operatorname{ar-rk}(F) > d_1$*, then* $\operatorname{ar-rk}(F) \geq d_2$.
3. $\operatorname{ar-rk}(F) = 1 \iff d_1 = 1 \iff \mathbb{R}\text{-rk}(F) = 1$.
4. $\operatorname{ar-rk}(F) = 2 \iff d_1 = 2 \iff \mathbb{C}\text{-b.}\operatorname{rk}(F) = 2$.
5. $\operatorname{ar-rk}(F) = 3 \iff d_1 = 3$ *and $F_\perp$ is not a cube (of a linear form).*

(If $d_1 = d_2$, we interpret "$F_\perp$ has almost real roots" to mean "there exists a form in $(F)^\perp_{d_1}$ with almost real roots", and similarly in (5)).

Proof. Omitted. □

Remark 2.7.2. *One can stratify all degree d binary forms by almost real rank as follows: write $V_i := H^0(\mathcal{O}_{\mathbb{P}^1}(i))$ for the vector space of (real) degree i binary forms. Let $\varphi_{1,d} : \mathbb{P}(V_1) \to \mathbb{P}(V_d)$ be the d^{th} Veronese map, and for $r \geq 2$, define the map*

$$\varphi_{r,d} : \mathbb{P}(V_2) \times (\mathbb{P}(V_1))^{r-2} \times \mathbb{P}^{r-1} \to \mathbb{P}(V_d)$$
$$(q, (l_j)_{j=0}^{r-3}, [c_0 : \ldots : c_{r-1}]) \mapsto \sum_{j=0}^{r-3} c_j l_j^d + c_{r-2} q_1 + c_{r-1} q_2$$

(here q_1, q_2 are the degree d forms corresponding to the complex linear factors of the quadric q as in Section 2.5, e.g. if $q = l^2$, then $q_1 = l^d$, $q_2 = l_\perp(l)^{d-1}$). By Theorem C.1.4, the image of $\varphi_{r,d}$ is precisely the set of degree d binary forms of almost real rank $\leq r$. Restricting $\varphi_{r,d}$ to the (open) subset where $q, l_0, \ldots, l_{r-3}$ are relatively prime, and removing the image of $\varphi_{r-1,d}$, gives the set of degree d binary forms of almost real rank $= r$.

From this description, one can deduce various structural properties of the set of forms of a given almost real rank. For instance, $\varphi_{2,d}$ is injective (for $d \geq 3$), so the set of forms with almost real rank ≤ 2 has dimension 3. Also, when $r = \lfloor \frac{d+2}{2} \rfloor = \lfloor \frac{d}{2} \rfloor + 1$, $\varphi_{r,d}$ is dominant, corresponding to the fact that the generic type is $(r, d+2-r)$, and among forms of degree r, those with almost real roots are typical. For dimension reasons, this is the least value of r for which $\varphi_{r,d}$ can be dominant, with general fibers of dimension 0 (resp. 1) when d is odd (resp. even).

It is natural to ask what the maximal almost real rank is for binary forms of degree d. This is answered by the next theorem:

Theorem 2.7.3. *For any $d \geq 3$ and $F \in \mathbb{R}[x, y]_d$,* $\text{ar-rk}(F) \leq d - 1$.

Proof. First we reduce to the case that $1 < \mathbb{C}\text{-rk}(F) < d$. If $\mathbb{C}\text{-rk}(F) = d$, then the apolar ideal $(F)^\perp$ is of type $(2, d)$ (cf. Theorem C.2.2), so $\text{ar-rk}(F) = 2 \leq d - 1$ by Theorem 2.7.1(4). Additionally, if $\mathbb{C}\text{-rk}(F) = 1$, then $\text{ar-rk}(F) = 1$ as well. Thus we may assume $2 \leq \mathbb{C}\text{-rk}(F) \leq d - 1$.

We now induct on d. For the base case $d = 3$, the apolar ideal is of type $(2, 3)$, so again $\text{ar-rk}(F) \leq 2$. For the inductive step, choose any direction $u = (u_1, u_2) \in \mathbb{R}^2$, corresponding to a linear form $l_u(x, y) := u_1 x + u_2 y$. Then by induction, the apolar ideal of the directional derivative $D_u(F) = \langle l_u, F \rangle$ contains a form with almost real roots of degree $\leq d - 2$ (note that $D_u(F) \neq 0$, since $\mathbb{C}\text{-rk}(F) > 1$ by assumption $\implies l_u \notin (F)^\perp$). By multiplying an additional factor if necessary, we may choose $G \in (D_u(F))^\perp$ of degree $= d-2$ with almost real roots. Then $G \cdot l_u \in (F)^\perp$ is of degree $d-1$. Since $\mathbb{C}\text{-rk}(F) \leq d-1$, we may also choose $H \in (F)^\perp$ of degree $= d - 1$ with simple complex roots.

We claim that for sufficiently small $\epsilon \in \mathbb{R}$, the form $G_\epsilon := G \cdot l_u + \epsilon H \in (F)^\perp$ has almost real roots. First, observe that there are only finitely many ϵ such that G_ϵ does not have simple roots: these are given by the roots of the discriminant of G_ϵ, viewed as a polynomial in ϵ (note that this polynomial is nonzero, since H has simple roots). Thus by avoiding these finitely many choices of ϵ, we may assume that G_ϵ has simple roots, and so it suffices to show that G_ϵ has at most 1 pair of complex roots.

For $|\epsilon|$ sufficiently small, any simple root of $G \cdot l_u$ gives a simple root of G_ϵ (by dehomogenizing we may consider a simple root of a univariate real polynomial, which is e.g. negative to the left of the root and positive to the right, and this is stable under small perturbation). Thus we need only consider the following cases: (i) $G \cdot l_u$ has a triple root, and (ii) $G \cdot l_u$ has 2 double roots. In case (i), since G_ϵ has simple roots, the triple root of $G \cdot l_u$ induces either 3 distinct real roots of G_ϵ, or 1 real root and 1 complex pair, and since all other roots of $G \cdot l_u$ are real and simple in this case, we get at most 1 pair of complex roots of G_ϵ.

In case (ii), suppose $G \cdot l_u$ has 2 double roots, and let p be one of these. If p is a root of H, then p is also a root of G_ϵ for any ϵ, so the double root p of $G \cdot l_u$ induces 2 real roots of G_ϵ (one of which is p, which implies that the other root must be real). Otherwise, if p is not a root of H, then $G \cdot l_u$ will either be nonnegative or nonpositive in a neighborhood of p while $H(p)$ is nonzero, so by choosing the sign of ϵ appropriately, the double root p of $G \cdot l_u$ will again induce distinct real roots of G_ϵ. Hence in either case the other double root of $G \cdot l_u$ gives at most 1 complex pair of roots of G_ϵ. $\square$

Remark 2.7.4. *There are some instances in which the type of the apolar ideal determines the almost real rank. Some cases of this are listed in Theorem 2.7.1. Another example of this occurs in degree 6: if a real binary sextic F has an apolar ideal of type $(4, 4)$, then* $\text{ar-rk}(F) = 4$. *To see this, note that if $F_\perp, F^\circ$ were both 4^{th} powers, then $F_\perp - F^\circ$ has almost real roots. Moreover, if both $F_\perp$ and F° have two pairs of complex roots, then $F_\perp, F^\circ$ is globally positive, in which case a suitable $\mathbb{R}$-linear combination of $F_\perp$, F° has at least a* *pair of real roots. Thus without loss of generality $F_\perp$ has at most 1 root of multiplicity 3, or 2 double roots, or 1 double root and 1 complex pair of roots, and by the reasoning in the proof of Theorem 2.7.3, there exists a form in $(F)^\perp_4$ with almost real roots.*

We next characterize when the maximal almost real rank of $d-1$ is achieved, which serves as a converse of Theorem 2.7.3:

Theorem 2.7.5. *Let $d \geq 5$ and $F \in \mathbb{R}[x, y]_d$. Then $\operatorname{ar-rk}(F) = d - 1 \iff F_\perp$ is a cube of a linear form $\iff (F)^\perp$ contains a cube of a linear form (but no quadratic forms).*

Proof. If $F_\perp$ is a cube of a linear form, then $(F)^\perp$ is of type $(3, d-1)$ and $\operatorname{ar-rk}(F) \geq d-1$ by Theorem 2.7.1(2, 5). Conversely, we show that if $d \geq 5$ and $(F)^\perp$ contains no cubes, then $\operatorname{ar-rk}(F) \leq d - 2$, by induction on d.

We first rule out small types: let (d_1, d_2) be the type of $(F)^\perp$. If $d_1 \leq 3$, then (with the assumptions of no cubes) $\operatorname{ar-rk}(F) \leq d-2$ by Theorem 2.7.1. This is enough to cover the base case $d = 5$, and by Theorem 2.7.4, this also covers the case $d = 6$. Thus we assume for the remainder of the proof that $d_1 \geq 4$.

Now suppose F is a form of degree $d \geq 7$. Note that either $(D_x(F))^\perp$ or $(D_y(F))^\perp$ does not contain a cube of a linear form: if not, say $l_1^3 \in (D_x(F))^\perp$ and $l_2^3 \in (D_y(F))^\perp$, then $(F)^\perp$ would contain 2 independent quartics xl_1^3, yl_2^3, which can only happen if $d_1 \leq 3$ (since $d_1 = 4 \implies d_2 = d - 2 \geq 5$), which has already been covered. Without loss of generality we may assume $(D_x(F))^\perp$ does not contain a cube of a linear form. By induction, there is a form $g \in (D_x(F))^\perp$ of degree $\leq d - 3$ with almost real roots. Then $xg \in (F)^\perp$ is of degree $\leq d - 2$, and since $F^\circ \in (F)^\perp$ has degree $\leq d - 2$ as well, the reasoning in the proof of Theorem 2.7.3 shows that there exists a form in $(F)^\perp_{d-2}$ with almost real roots. $\square$

The characterization above yields sharp bounds on the Hankel index for the curves studied in this paper:

Corollary 2.7.6. *Let $X = \pi_p(C_d)$ be a projection of a rational normal curve C_d away from a point $p \in \mathbb{P}^d \setminus C_d^3$. Then $2 \le \eta(X) \le d-4$. In particular, if $d = 6$, then $\eta(X) = 2$.*

Proof. If $\text{ar-rk}(F(p)) = d-1$, then $(F(p))^\perp$ contains a cube by Theorem 2.7.5. By apolarity, this implies that $p \in C_d^3$, a contradiction. Thus $\text{ar-rk}(F(p)) \le d-2$, so $\eta(X) \le d-4$ by Theorem 2.4.3. □

As preparation for determining the typical almost real ranks, it is useful to know explicit forms which attain a given almost real rank. We thus compute the various ranks of monomials $x^{d-i}y^i \in \mathbb{R}[x,y]_d$. When $i = 0$, $x^{d-i}y^i = x^d$ is a power of a linear form, hence has real (and complex) [border] rank 1. By symmetry, we may therefore assume $1 \le i \le \lfloor \frac{d}{2} \rfloor$. In general, the apolar ideal is

$$(x^{d-i}y^i)^\perp = (y^{i+1}, x^{d-i+1}).$$

From this we see that $\mathbb{C}\text{-b.rk}(x^{d-i}y^i) = i+1$ and $\mathbb{C}\text{-rk}(x^{d-i}y^i) = d-i+1$ (cf. Theorem C.2.2). Since $x^{d-i}y^i$ has all real roots, we also have $\mathbb{R}\text{-rk}(x^{d-i}y^i) = d$.

Proposition 2.7.7. *For $d \ge 1$ and $0 \le i \le \lfloor \frac{d}{2} \rfloor$,*

$$\text{ar-rk}(x^{d-i}y^i) = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } i = 1 \\ d-1 & \text{if } i = 2 \\ d-2 & \text{otherwise.} \end{cases}$$

Proof. The cases $i = 0, 1$ follow from Theorem 2.7.1; the case $i = 2$ is covered by Theorem 2.7.5. This includes all cases with $d \le 5$.

It thus suffices to show that if $d \ge 6$ and $3 \le i \le \lfloor \frac{d}{2} \rfloor$, then $\text{ar-rk}(x^{d-i}y^i) > d-3$. The cases $d = 6$ (resp. $d = 7$) are covered by Theorem 2.7.4 (resp. Theorem 2.7.1). Now suppose $d \geq 8$. Every form of degree $d-3$ in $(x^{d-i}y^i)^{\perp}$ can be expressed as

$$a_0x^{d-3} + \ldots + a_{i-4}x^{d-i+1}y^{i-4} + b_{d-i-4}x^{d-i-4}y^{i+1} + \ldots + b_0y^{d-3}$$

with $(i-3)+(d-i-3) = d-6$ coefficients $a_0, \ldots, a_{i-4}, b_{d-i-4}, \ldots, b_0 \in \mathbb{R}$, where we take no a_i's if $i = 3$ (so that the support of this polynomial has a gap of size 4). By the Descartes' Rule of Signs, the number of distinct nonzero real roots of this polynomial is at most the number of sign changes betwen adjacent coefficients, hence is $\leq d-7$. Thus $\text{ar-rk}(x^{d-i}y^i) > d-3$, and so Theorems Theorem 2.7.3 and Theorem 2.7.5 imply that $\text{ar-rk}(x^{d-i}y^i) = d-2$. $\square$

In particular, we see that for monomial projections, the almost real rank is essentially independent of i (and depends only on whether X_i is contained in the rational normal surface scroll $S(1, d-3)$), and is much larger than the complex border rank (with a gap of at least $\lceil \frac{d}{2} \rceil - 3$, hence the gap is unbounded as $d \to \infty$).

An amusing corollary of Theorem 2.7.7 is the existence, in any degree ≥ 4, of univariate real polynomials with almost real roots whose supports have a gap of size 3, i.e. the Rule of Signs bound is sharp for these polynomials (athough the existence of such polynomials is not sufficient to prove Theorem 2.7.7). For more on the sharpness of the Rule of Signs bound, cf. [31].

Finally, we consider the problem of determining which almost real ranks are typical. Our presentation follows that of [32]. Recall that a property P of degree d forms is said to be *typical* if, on identifying the set of degree d forms with $\mathbb{R}^{d+1}$, there is a nonempty Euclidean open set of degree d forms all of which have property P. We say that an almost real rank r is typical if the property "has almost real rank $= r$" is typical. For $F \in \mathbb{R}[x,y]_d$, we say that F is a *typical form of almost real rank* r if F lies in an open set of $\mathbb{R}[x,y]_d$ which consists of forms of almost real rank r.

Note that the condition "$(F)^{\perp}$ contains a cube" in Theorem 2.7.5 is equivalent to saying that F has a real root of multiplicity $\geq d-2$, which is not a typical property. It follows that $d-1$ is not a typical almost real rank. Moreover, Theorem 2.7.2 implies that any $r < \lfloor \frac{d+2}{2} \rfloor$ cannot be a typical almost real rank. It turns out that these are the only obstructions for an almost real rank to be typical, as will be shown in Theorem 2.7.9. To this end, we first characterize the typical forms of a given almost real rank:

Lemma 2.7.8. *Let $F \in \mathbb{R}[x,y]_d$ with $(F)^\perp$ of generic type, and set $r = \operatorname{ar-rk} F$. Then F is a typical form of almost real rank r if and only if all forms in $(F)^\perp_{r-1}$ have at least two pairs of complex roots (counted with multiplicity).*

Proof. Suppose that F is typical of almost real rank r, and there exists $g \in (F)^\perp_{r-1}$ such that g has at most one pair of complex roots. In any ϵ-neighborhood of g there exists a form g_ϵ such that g_ϵ has almost real roots. For any $\epsilon > 0$ we have $\dim(g_\epsilon)_d = \dim(g)_d = d-r+2$, and as ϵ approaches 0, $(g_\epsilon)_d$ approaches $(g)_d$. Therefore the orthogonal complement of $(g_\epsilon)_d$ also approaches the orthogonal complement of $(g)_d$ as ϵ goes to 0. We conclude that in any neighborhood of F there exist forms of almost real rank at most $r-1$, which is a contradiction.

Conversely, let $F \in \mathbb{R}[x,y]_d$ with $(F)^\perp$ of generic type and $\operatorname{ar-rk} F = r$. Suppose that all forms in $(F)^\perp_{r-1}$ have at least two pairs of complex roots. For $\epsilon > 0$ sufficiently small, the ϵ-neighborhood of F contains only forms with apolar ideals of generic type (as having non-generic type is a Zariski-closed condition). For such ϵ, fix F_ϵ in the ϵ-neighborhood of F. Within this neighborhood, the ideal $(F_\epsilon)^\perp$ (i.e. the sequence of graded components of $(F_\epsilon)^\perp$) depends continuously on the coefficients of F_ϵ. Now both conditions "all forms in $(F)^\perp_{r-1}$ have at most one pair of complex roots" and "there exists a form in $(F)^\perp_r$ with almost real roots" are stable under sufficiently small perturbation, which shows that F is typical of almost real rank r. $\square$

Theorem 2.7.9. *For $d \geq 5$, any r with $\lfloor \frac{d+2}{2} \rfloor \leq r \leq d-2$ is a typical almost real rank.*

Proof. We first show that $d-2$ is always a typical almost real rank. By Theorem 2.7.3 and

Theorem 2.7.5, it suffices to show that for each $d \geq 5$, there exists a nonempty open set of degree d forms with almost real rank $> d-3$. For $5 \leq d \leq 7$, we may verify this directly: if $d=5$, then a general form (which is of type $(3,4)$) has almost real rank 3; the case $d=6$ is covered by Theorem 2.7.4; and for $d=7$, there is an nonempty open set of forms F of type $(4,5)$ for which $F_\perp$ has only complex roots (i.e. is a product of 2 strictly positive quadrics). For $d \geq 8$, it follows from Theorem 2.7.8 and the proof of Theorem 2.7.7 that the "balanced" monomial $x^{\lceil \frac{d}{2} \rceil} y^{\lfloor \frac{d}{2} \rfloor}$ (which is of generic type) is a typical form of almost real rank $d-2$.

For the remaining ranks, we induct on the degree d. For the base cases $d = 5, 6$, we have that $d - 2 = \lfloor \frac{d+2}{2} \rfloor$ is a typical almost real rank by the above. For the inductive step, fix the following data:

1. a rank $\lceil \frac{d+2}{2} \rceil \leq r \leq d-2$,

2. a typical form $F \in \mathbb{R}[x,y]_d$ of almost real rank r (by perturbing F if necessary, we may assume that $(F)^\perp = (F_\perp, F^\circ)$ is of generic type),

3. a nonzero form $S := C_1 F_\perp + C_2 F^\circ \in (F)^\perp_r$ with almost real roots.

We will exhibit a form H of degree $d+1$ such that $(H)^\perp$ is of generic type, $(H)^\perp \subseteq (F)^\perp$, and $S \in (H)^\perp$. By Theorem 2.7.8 this shows that H is a typical form of almost real rank r, so r is a typical almost real rank in degree $d+1$. This is enough for the induction, since we already know that $(d+1)-2$ is a typical almost real rank in degree $d+1$ (note also that $\lceil \frac{d+2}{2} \rceil = \lfloor \frac{(d+1)+2}{2} \rfloor$). We consider two cases depending on the parity of d, namely $d = 2k$ for $k \geq 3$, or $d = 2k-1$ for $k \geq 4$.

First, suppose $d = 2k-1$ is odd, so that $\deg F_\perp = k$, $\deg F^\circ = k+1$. We claim that there exists a linear form $L \in \mathbb{R}[x,y]_1$ such that $C_1 - LC_2$ has a real root which is not a root of $LF_\perp + F^\circ$. If not, then for every linear form L, we have that every root of $C_1 - LC_2$ is a root of $LF_\perp + F^\circ$. Now for any $(a,b) \in \mathbb{R}^2$ with $F_\perp(a,b) \neq 0$ and $C_2(a,b) \neq 0$, there exists a linear form L such that $L(a,b) = \frac{C_1(a,b)}{C_2(a,b)}$, i.e. (a,b) is a root of $C_1 - LC_2$. By assumption (a,b) is also a root of $LF_\perp + F^\circ$, so $L(a,b) = \frac{-F^\circ(a,b)}{F_\perp(a,b)}$. Varying over such (a,b), we see that the two rational functions C_1/C_2 and $-F^\circ/F_\perp$ agree at infinitely many points, hence must be equal. But this implies that $S = C_1F_\perp + C_2F^\circ = 0$, a contradiction. We conclude that such an L exists. For such L, set $G := LF_\perp + F^\circ$, write $C_1 - LC_2 = L_1K$, where $L_1 \in \mathbb{R}[x,y]_1$ does not divide G, and take H to be the unique form of degree $d+1$ with apolar ideal generated by $(L_1F_\perp, G)$. Then $(H)^\perp \subseteq (F)^\perp$, and $S = (C_1 - LC_2)F_\perp + C_2G = K(L_1F_\perp) + C_2G \in (H)^\perp$ as desired.

The reasoning in the case $d = 2k$ is similar: here $\deg(F_\perp) = \deg(F^\circ) = k+1$. We claim that there exists $\alpha \in \mathbb{R}$ such that $C_1 - \alpha C_2$ has a real root which is not a root of $\alpha F_\perp + F^\circ$. This follows from the same reasoning as in the case $d = 2k-1$ (in fact even simpler, since there is no choice involved in the scalar α, as opposed to a linear form). Having obtained such an α, we set $G := \alpha F_\perp + F^\circ$, write $C_1 - \alpha C_2 = L_0K$, where $L_0 \in \mathbb{R}[x,y]_1$ does not divide G, and take H to be the unique form of degree $d+1$ with apolar ideal generated by $(L_0F_\perp, G)$. Then as before, $(H)^\perp \subseteq (F)^\perp$ and $S = (C_1 - \alpha C_2)F_\perp + C_2G = K(L_0F_\perp) + C_2G \in (H)^\perp$. □

Appendices

APPENDIX A

MINIMAL GRADED FREE RESOLUTIONS

We begin with this chapter by introducing basics on the minimal graded free resolutions of ideals. For general settings and statements, consult with [33].

Let S be a standard graded polynomial ring in n variables over field k. i.e. $S = k[x_1, \ldots, x_n]$ with $\deg(x_i) = 1$ for each $1 \leq i \leq n$. Let I be a finitely generated homogeneous ideal of the graded ring S. Note that I is a finitely generated S-module.

Definition A.0.1. *A minimal graded free resolution $\mathbb{F}$ of S/I over S is a sequence of homomorphisms of free S-modules*

$$\mathbb{F} : \ldots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0$$

such that

1. $d_{i-1} \circ d_i = 0$ *for* $i \in \mathbb{Z}$*. i.e.* $\mathbb{F}$ *is a complex.*
2. *The homologies* $\ker(d_i)/\operatorname{Im}(d_{i+1})$ *of the complex* $\mathbb{F}$ *vanish for all* $i \in \mathbb{Z}$*. i.e.* $\mathbb{F}$ *is exact.*
3. $S/I \simeq F_0/Im(d_1)$
4. *The free modules* F_i *are graded. i.e.* $F_i = \oplus_{j \in \mathbb{Z}} S(-j)$ *where* $S(-j)$ *is the graded* S*-module such that* $S(-j)_t = S_{t-j}$*. Also, each* d_i *is a homomorphism of degree* 0*.*
5. $d_{i+1}(F_{i+1}) \subseteq (x_1, \ldots, x_n)F_i$ *for all* $i \geq 0$*. i.e. the resolution is minimal.*

Remark that there is unique minimal graded free resolution of S/I over S up to isomorphisms. (cf. [33, Theorem 7.5]) Therefore, the ranks of each free graded modules in the minimal graded free resolution of S/I is an invariant and we define the *graded Betti* *number* $\beta_{i,j} = \beta^S_{i,j}(S/I)$ by the rank of degree j modules in i-th step of the minimal graded free resolution of S/I over S.

We can create a diagram whose entries are Betti numbers:

Table A.1: Betti diagram

		β_i	
		$\vdots$	
p	$\cdots$	$\beta_{i,i+p}$	$\cdots$
		$\vdots$	

The diagram is called the Betti diagram or computer table. Note that the i-th column contains the data at the i-th step of the minimal graded free resolution. Also, the non-negative integers in p-th row counts the number of copies of the free modules shifted by degree p.

Now, we introduce some algebraic invariants that are read from the minimal graded free resolution. *Castelnuovo-Mumford regularity of a ring(module)* is a representative algebraic invariant that inscribes algebraic complexity of the ring. Indeed, the Castelnuovo-Mumford regularity of the coordinate ring of a finite set of points is the same as the interpolation degree of the set, which is the smallest degree of a polynomial interpolating all points in the set. Moreover, for a homogeneous ideal I, the Hilbert function in d agrees with the Hilbert polynomial in d if d is at least the Castelnuovo-Mumford regularity of I. (See [34, Chapter 4] for details.)

Definition A.0.2. *The Castelnuovo-Mumford regularity of S/I, denoted by $\operatorname{reg}(S/I)$, is defined by*

$$\operatorname{reg}(S/I) := \min\{j - i \,:\, \beta_{i,j}(S/I) = 0 \textit{ for all } i\}$$

Note that the regularity is the height of the Betti diagram of the ring because the height of the diagram is same as the maximal degree of entries of differentials.

Suppose X is a projective variety defined by quadrics. We say the variety X satisfies the $N_{2,p}$ property (for $p \in \mathbb{Z}_{>0}$) if the minimal graded free resolution of the coordinate ring of X is linear for $p-1$ steps from beginning. (If X does not generated by quadrics, we say X does not satisfies $N_{2,1}$) For example, suppose the defining ideal of a projective variety X is generated by quadratic homogeneous polynomials. If the minimal graded free resolution of coordinate ring of X is not linear in first step, the variety X satisfies $N_{2,1}$ property because X is generated by quadrics, but $N_{2,2}$ is failed because the first syzygy of the defining ideal of X is not linear. As the minimal graded free resolution is unique up to isomorphism, we define the Green-Lazarsfeld index of a variety X (or the coordinate ring of X) by the maximum number p such that X satisfies $N_{2,p}$ property.

A.1 Stanley-Reisner theory

The square-free monomial ideals can be studied combinatorially through the correspondence between the monomial ideals and the simplical complexes. This correspondence is called the Stanley-Reisner correspondence, the square-free monomial ideals are called the Stanley-Reisner ideals, and the simplicial complexes are called the Stanley-Reisner complex. We introduce the Hochster's formula that implies the graded Betti numbers of monomial ideals equals to the total dimension of homologies of subcomplexes of the Stanley-Reisner complex. This section is refered the survey [35] and consult with the survey for more general settings and details of Stanley-Reisner theory.

Let Δ be a finite simplicial complex on the vertex set $V=\{x_1,\ldots,x_n\}$. Recall that the simplicial complex Δ is a colleciton of subsets of V such that $F\subseteq G\in\Delta \implies F\in\Delta$ and $\{x_i\}\in\Delta$ for all $x_i\in V$. The elements of Δ are called faces. If F is a face in Δ, then we define (topological) dimension of the face by $\dim F := |F|-1$ and $\dim\Delta := \max\limits_{F\in\Delta}(\dim F)$. Let $d=\dim\Delta+1$. Given any field k we now define the face ring (or Stanley-Reisner ring) $k[\Delta]$ of the simplicial complex Δ.

Definition A.1.1. *$k[\Delta]=k[x_1,\ldots,x_n]/I_\Delta$, where*

$$I_\Delta=(x_{i_1}x_{i_2}\cdots x_{i_r}\mid i_1<i_2<\cdots<i_r,\{x_{i_1}x_{i_2}\cdots x_{i_r}\}\notin\Delta)$$

We refer to [35, Section 7] or [9, Corollary 4.9] for Hochster's formula.

Theorem A.1.2. *Assume k is a field of characteristic 0. Suppose I is a square-free monomial ideal of the polynomial ring $S = k[x_1, \ldots, x_n]$ and Δ_I is the Stanley-Reisner complex associated to I. Also, let $\mathbf{b}$ be a square-free multidegree. i.e. $\mathbf{b} \in \mathbb{N}^n$ whose entries are either 0 or 1. Then*

$$\beta_{i,\mathbf{b}}(S/I) = \dim_k \tilde{H}_{|\mathbf{b}|-i-1}(\Delta_I[\mathbf{b}])$$

where $\tilde{H}_\bullet$ is the reduced simplicial homology, $\Delta_I[\mathbf{b}]$ is the subcomplex of Δ_I induced by the multidegree $\mathbf{b}$.

By the Hochster's formula the regularity of the Stanley-Reisner rings $k[\Delta]$ is the maximal dimension of the subcomplex of the Stanley-Reisner complex that has non-trivial homology.

Example A.1.3. *Let Δ be the following triangulation of 2-sphere by 5 points:*

$$\begin{aligned}\Delta =& \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\} \\ & \cup \{\{12\}, \{13\}, \{14\}, \{23\}, \{24\}, \{25\}, \{34\}, \{35\}, \{45\}\} \\ & \cup \{\{123\}, \{124\}, \{134\}, \{235\}, \{245\}, \{345\}\}\end{aligned}$$

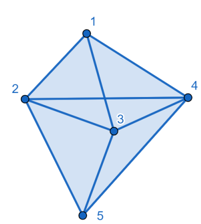


Figure A.1: Triangulation of 2-sphere by 5 points

Then, the Stanley-Reisner ideal is $I = (x_1x_5, x_2x_3x_4)$. Indeed, one can check that any square-free monomials that corresponds to non-faces of Δ is contained in the ideal I. By Theorem A.1.2, $\beta_{2,5}(S/I) = 1$ because $\dim_k \tilde{H}_2(\Delta_I[\{12345\}]) = 1$.

Indeed, the minimal graded free resolution of S/I over the ring S is

$$0 \longrightarrow S(-5) \longrightarrow S(-2) \oplus S(-3) \longrightarrow S(\longrightarrow S/I \longrightarrow 0)$$

Thus, the Betti diagram of S/I is

Table A.2: Betti diagram for Example A.1.3

	0	1	2
0	1	-	-
1	-	1	-
2	-	1	-
3	-	-	1

Here, the Castelnuovo-Mumford regularity of S/I is 4 and the $N_{2,1}$ property is failed because the ideal I does not generated by quadrics.

APPENDIX B

THE CONE OF SUMS OF SQUARES AND THE CONE OF NON-NEGATIVE QUADRATIC FORMS ON VARIETIES

B.1 introduction

The relationship between nonnegative polynomials and sums of squares is a fundamental topic in real algebraic geometry. This subject has received renewed attention in the last twenty years due to its connection with polynomial optimization and many applications [36]. In a foundational paper Hilbert described all the cases in terms of degree and number of variables where any globally nonnegative polynomial can be written as a sum of squares of polynomials [37].

A modern approach to this question is to study nonnegative polynomials and sums of squares on a real projective variety $X \subseteq \mathbb{P}^n_{\mathbb{R}}$. This allows one to restrict to quadrics, since degree $2d$ forms on X are quadrics on the d-th Veronese embedding of X. The two main objects of interest are:

$$P_X := \{f \in R(X)_2 \mid f(x) \geq 0 \text{ for all } x \in X(\mathbb{R})\},$$
$$\Sigma_X := \left\{ f \in R(X)_2 \mid \text{ there exist } l_1, \ldots, l_m \in R(X)_1, f = \sum_{i=1}^{m} l_i^2 \right\}$$

In fact $\Sigma_X \subseteq P_X$ are convex cones in the vector space $R(X)_2$ of all quadrics on X, which facilitates their study via convex geometry (cf. [36]). For instance, as an extension of Hilbert's result, [30, Theorem 1.1] showed that $\Sigma_X = P_X$ if and only if X is a variety of minimal degree, i.e. $\deg X = 1 + \operatorname{codim} X$. However, the structure of these cones is still not well understood in general.

It is sometimes more convenient to work with the dual cones because the structures of the dual cones are well-known. Indeed, the dual cone of non-negative quadratic forms on the variety X is a conical hull of the set of point evaluation on X.

The simplest extreme rays in $\Sigma_X^\star$ are given by point evaluations. For a point $p \in X$, we can pick an affine representative $\tilde{p}$ lying on the line spanned by p, and define a linear functional $\ell_{\tilde{p}}(q) := q(\tilde{p})$ for all $q \in R(X)_2$. Varying the affine representative only rescales the point evaluation functional, and so by a slight abuse of terminology we will talk about point evaluations at a point $p \in X$ and use ℓ_p to denote any of the linear functionals obtained by using an affine representative of p. Point evaluations are precisely the rank 1 quadratic forms in $\Sigma_X^\star$: if $\ell \in \Sigma_X^\star$ has $\operatorname{rank} Q_\ell = 1$, then $\ell = \ell_p$ for some $p \in X$ [30, Lemma 2.3].

Our main object of interest is the Hankel spectrahedron

$$\Sigma_X^\star := \{\ell \in R(X)_2^* \mid \ell(f^2) \geq 0, \text{ for all } f \in R(X)_1\}$$

This is the dual cone to the sums-of-squares cone of X, and is contained in $R(X)_2^\star$, the space of linear functionals on quadrics on X. That it is a spectrahedron can be seen from an alternate description (cf. [29, Lemma 2.1] and Section C.4)

$$\Sigma_X^\star = \mathbb{S}_+ \cap (I(X)_2)^\perp$$

where $\mathbb{S}_+$ is the cone of PSD symmetric matrices (identified with nonnegative quadratic forms) on X, and $(I(X)_2)^\perp$ is the orthogonal complement of the degree 2 part of the ideal of X (which comprises linear equations in $R(X)_2^*$).

B.2 Kernels of rays

Definition B.2.1. *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a convex cone, and $\ell \in \mathcal{K}$. We say that ℓ spans an* extreme ray *of $\mathcal{K}$ if whenever $\ell = \ell_1 + \ell_2$ with $\ell_1, \ell_2 \in \mathcal{K}$, one has $\ell_1 = \lambda_1 \ell$, $\ell_2 = \lambda_2 \ell$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.*

If $\ell \in \mathcal{K}$ spans an extreme ray of $\mathcal{K}$, we will simply say that ℓ is an extreme ray of $\mathcal{K}$ (i.e. we do not distinguish an extreme ray from its nonzero elements). For instance, we can say that every $\ell \in \mathcal{K}$ can be written as a sum of extreme rays.

The dual cone of PSD quadrics on a variety is the conincal hull of point evaluation (cf. [30, Lemma 2.1]). Also, a linear functional on the dual space of quadrics is a point evaluation if and only if the rank of the linear functional is one (cf. [29, Lemma 2.4]). Therefore, the rank of any extreme rays of the dual cone of PSD quadrics on the variety is one.

Since the dual cone of SOS on a variety is a spectrahedron, we obtain a property for being extreme rays of the dual cone of SOS on the variety from convex geometry.

Proposition B.2.2 ([29, Lemma 2.2]). *Let $\mathcal{K} = \mathbb{S}_+ \cap L$ be a spectrahedron, and $\ell \in \mathcal{K}$. Then ℓ is an extreme ray of $\mathcal{K}$ if and only if $\ker \ell$ is maximal, i.e. if $\ker \ell \subseteq \ker \ell'$ for some $\ell' \in L$, then $\ell' = \lambda \ell$ for some $\lambda \in \mathbb{R}$.*

Recall that if $V \subseteq R_d$ is a space of forms, then a point p is called a *basepoint* of V if all forms in V vanish at p. If V has no basepoints, we say that V is *basepoint-free*.

Remark B.2.3. *We take a moment to clarify the relationships between rays with basepoint-free kernels and sums of point evaluations.*

(i) For $\ell_i \in \Sigma_X^\star$, $\ker(\sum q_{\ell_i}) = \bigcap \ker(q_{\ell_i})$: for $v \in R(X)_1$, one has $q_{\ell_i}(v) \geq 0$ with equality if and only if $v \in \ker(q_{\ell_i})$, as q_{ℓ_i} is PSD.

(ii) If ℓ_p is point evaluation at a point $p \in X$, then p is a basepoint of $\ker(q_{\ell_p})$.

(iii) It follows from (i) and (ii) that if $\ell \in \Sigma_X^\star$ is such that $\ker(q_\ell)$ is basepoint-free, then for any decomposition of ℓ as a sum of extreme rays $\ell = \sum \ell_i$ of $\Sigma_X^\star$, each extreme ray ℓ_i has rank > 1, i.e. is not a point evaluation. (In fact the converse holds as well: if p is a basepoint of $\ker(q_\ell)$, then there is a decomposition of ℓ into extreme rays, one of which is ℓ_p. However, note that a sum of extreme rays of rank > 1 may have basepoints.)

B.3 An algebraic invariant that gives a bound of Hankel index

A surprising connection between the Hankel index and homological properties of the minimal free resolution of the ideal of X was found in [1, Theorem 4 and Theorem 6]: namely, there is a lower bound $\eta(X) \geq \alpha(X) + 1$, where $\alpha(X)$ is the Green-Lazarsfeld index of X (here X need not be irreducible). Recall that the Green-Lazarsfeld index of X is defined as follows: $\alpha(X) = 0$ if the ideal of X is not generated by quadrics; otherwise it is equal to one plus the number of steps that the minimal free resolution of the coordinate ring of X is linear, i.e. has only linear syzygies. In all cases where the Hankel index was known, this bound was tight.

APPENDIX C

APOLARITY ON BINARY FORMS

C.1 Apolarity and Ranks

We begin with a brief review of apolarity and the apolar inner product, which is our preferred method of explicitly identifying primal and dual spaces.

Definition C.1.1. *Let k be a field of characteristic 0, and $R = k[x_1, \ldots, x_n]$ a polynomial ring over k. Consider the "differential" pairing on R defined by*

$$\langle f, g \rangle := \partial(f) \bullet g \tag{C.1}$$

where $\partial(f)$ is the differential operator obtained from f by replacing each variable x_i with $\frac{\partial}{\partial x_i}$, and $\bullet$ denotes the action of differential operators on polynomials. For a given degree d, the pairing $\langle \cdot, \cdot \rangle$ restricts to an inner product on R_d, the k-vector space of forms of degree d. For $F \in R$, the apolar ideal *of F is defined as the orthogonal complement of F with respect to the pairing (Equation C.1), i.e.*

$$(F)^\perp := \{f \in R \mid \langle f, F \rangle = 0\}.$$

If $F \in R_d$ is homogeneous, then $(F)^\perp$ is a homogeneous ideal.

Remark C.1.2. *For any form F, the apolar ideal $(F)^\perp$ is an Artinian Gorenstein graded ideal. Conversely, every Artinian Gorenstein graded ideal I is of the form $(F)^\perp$, where F generates the socle of R/I.*

We now specialize to the case of binary forms, i.e. forms in 2 variables x, y. Let $F \in k[x, y]_d$ be a binary form. Then $(F)^\perp$ is Gorenstein of codimension 2, hence is a complete intersection. As this fact will be used repeatedly in the sequel, we introduce some notation for the generators of this complete intersection:

Definition C.1.3. *For $F \in k[x,y]_d$, let $F_\perp, F^\circ \in k[x,y]$ denote forms that satisfy*

$$(F)^\perp = (F_\perp, F^\circ)$$

with $\deg F_\perp \leq \deg F^\circ$. If $d_1 := \deg F_\perp$ and $d_2 := \deg F^\circ$, we say that the apolar ideal $(F)^\perp$ is of type (d_1, d_2). *One always has the relation*

$$d_1 + d_2 = \deg F + 2. \tag{C.2}$$

Note that if $d_1 < d_2$, then $F_\perp$ is uniquely defined by F (up to nonzero scale), while F° is unique modulo the principal ideal $(F_\perp)$.

For example, if $l = ax + by \in k[x,y]_1$ is a binary linear form, then $(l)^\perp$ is of type $(1,2)$, with $l_\perp = bx - ay$, and l° is (the) quadric not in $(l_\perp)$.

We are now ready to state the apolarity lemma for binary forms, which characterizes membership in the apolar ideal:

Lemma C.1.4 (Generalized Apolarity Lemma)**.** *cf.[38, Lemma 1.31] Let $F \in k[x,y]_d$. For a given set $\{l_1, \ldots, l_r\} \subseteq k[x,y]_1$ of linear forms and $d_1, \ldots, d_r \in \mathbb{N}$ with $\sum_{i=1}^r d_i \leq d$, one has $\prod_{i=1}^r l_i^{d_i} \in (F)^\perp$ if and only if there exist $c_{ij} \in k$ ($1 \leq i \leq r$, $0 \leq j \leq d_i - 1$) such that*

$$F = \sum_{i=1}^{r} \sum_{j=0}^{d_i - 1} c_{ij} (l_i)^j (l_i)_\perp^{d-j}.$$

The case $d_1 = \ldots = d_r = 1$ is classically referred to as the *apolarity lemma*, and characterizes squarefree forms in the apolar ideal via a Waring decomposition of F, as a sum of d^{th} powers of linear forms.

Another useful criterion for determining membership in the apolar ideal is:

Lemma C.1.5. *Let $F \in k[x,y]_d$, and $G \in k[x,y]_n$ for some $n \leq d$. Then $G \in (F)^\perp$ if and only if $(G)_d \subseteq (F)^\perp$.*

Proof. If $G \in (F)^\perp$, then certainly $(G)_d \subseteq (F)^\perp$, since $(F)^\perp$ is an ideal. Conversely, suppose $G \notin (F)^\perp$, and set $H := \langle G, F \rangle \in k[x,y]_{d-n} \neq 0$. Since $\langle \cdot, \cdot \rangle$ is a perfect pairing on $k[x,y]_{d-n}$, there exists $0 \neq K \in k[x,y]_{d-n}$ with $\langle K, H \rangle \neq 0$. Then $0 \neq \langle K, \partial(G) \bullet F \rangle = \partial(K) \bullet (\partial(G) \bullet F) = \partial(KG) \bullet F$, so $KG \in (G)_d \setminus (F)^\perp$. $\square$

C.2 Ranks of forms

Classically, it is an important problem to decompose a given form as a linear combination of powers of linear forms. Such decompositions lead various notions of rank of a form, which are sensitive to the underlying field of scalars.

Definition C.2.1. *Let $F \in \mathbb{R}[x,y]_d$. The* real *(resp.* complex*)* rank *of F is the minimal number of real (resp. complex) linear forms $l_1, \ldots, l_r$ such that F is an $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) combination of $l_1^d, \ldots, l_r^d$. The* real *(resp.* complex*)* border rank *of F is the minimal number r such that F is a limit of forms of real (resp. complex) rank r.*

Remark C.2.2. *Via apolarity, we can reinterpret the various ranks in Theorem C.2.1. Indeed, it follows from Theorem C.1.4 that for any $F \in \mathbb{R}[x,y]_d$,*

$$\mathbb{R}\text{-rk}(F) = \min \left\{ r \;\middle|\; \begin{array}{c} \exists g \in (F)_r^\perp \textit{ with } r \textit{ simple} \\ \textit{linear factors over } \mathbb{R} \end{array} \right\}$$

$$\mathbb{C}\text{-rk}(F) = \min \left\{ r \;\middle|\; \begin{array}{c} \exists g \in (F)_r^\perp \textit{ with } r \textit{ simple} \\ \textit{linear factors over } \mathbb{C} \end{array} \right\}$$

$$\mathbb{R}\text{-b.}\operatorname{rk}(F) = \min \left\{ r \;\middle|\; \begin{array}{c} \exists g \in (F)_r^\perp \textit{ which factors} \\ \textit{completely over } \mathbb{R} \end{array} \right\}$$

$$\mathbb{C}\text{-b.}\operatorname{rk}(F) = \min\{r \mid (F)_r^\perp \neq 0\} = \deg(F_\perp)$$

Note that any complex rank is at most the corresponding real rank, and any border rank is at most the corresponding non-border rank. Moreover, if $(F)^{\perp}$ is of type (d_1, d_2), then $\mathbb{C}\text{-rk}(F) = d_1$ if and only if $F_{\perp}$ has distinct factors over $\mathbb{C}$, and equals d_2 otherwise (since $F_{\perp}, F^{\circ}$ form a complete intersection, thus have no common factors).

C.3 Associating forms to points

A crucial identification throughout section 2.2 is that of associating points in projective space to (binary) forms. Let $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ be the d-uple embedding (or d^{th} Veronese map). Let $C_d := \nu_d(\mathbb{P}^1) \subseteq \mathbb{P}^d$ be the image, which is the standard rational normal curve of degree d. Given a point $p \in \mathbb{P}^d$, consider the vector space of linear forms on $\mathbb{P}^d$ vanishing at p (these generate the vanishing ideal of p). Pulling this space back to $\mathbb{P}^1$ via ν_d gives a d-dimensional vector space of degree d binary forms, which is a hyperplane in $k[x, y]_d$ (the space of all degree d binary forms). We set $F(p)$ to be the degree d binary form (unique up to nonzero scale) which is orthogonal to this hyperplane, with respect to the inner product (Equation C.1).

An alternate way to compute $F(p)$ is: under the d-uple embedding, a point $\nu_d([a : b])$ on the rational normal curve is associated to the d^{th}-power $(ax + by)^d \in k[x, y]_d$. Since points on the rational normal curve are in linearly general position, extending additively gives a correspondence between all points in $\mathbb{P}^d$ and binary forms of degree d. Explicitly, for $p \in \mathbb{P}^d$, we may choose an expression of p as a linear combination of $r \leq d + 1$ points on C_d, say $p = \sum_{i=1}^{r} c_i p_i$. Setting $p_i =: \nu_d([a_i : b_i])$, we have

$$F(p) = \sum_{i=1}^{r} c_i (a_i x + b_i y)^d \in k[x, y]_d$$

In this way we may consider the various ranks (defined in Sections 2.4 and C.2) of a point $p \in \mathbb{P}^d$, as the ranks of the associated binary form $F(p)$.

C.4 Quadratic forms vs linear functionals on quadrics

For an embedded nondegenerate projective variety $X \subseteq \mathbb{P}^n$, there is a correspondence between quadratic forms on X and linear functionals on quadrics on X. Let $R = R(X) = \bigoplus_{i \geq 0} R_i$ be the homogeneous coordinate ring of X. A bilinear form on R_1 is a bilinear map $R_1 \times R_1 \to k$, or equivalently a linear map $R_1 \otimes_k R_1 \to k$. The bilinear form is symmetric if and only if this descends to $\mathrm{Sym}^2(R_1) \to k$. Since X is nondegenerate, $\dim R_1 = n + 1$ (i.e. R_1 consists of all linear forms on $\mathbb{P}^d$), so there is a natural surjection $\mathrm{Sym}^2(R_1) \twoheadrightarrow R_2$ with kernel $I(X)_2$, the degree 2 part of the defining ideal of X. This yields a bijection

$$\left\{ \begin{array}{c} \text{symmetric bilinear forms on } R_1 \\ \text{whose kernel contains } I(X)_2 \end{array} \right\} \longleftrightarrow \{ \text{ linear functionals on } R_2 \}$$

Finally, symmetric bilinear forms on R_1 whose kernel contains $I(X)_2$ correspond to quadratic forms on the variety X. Explicitly, given $\ell \in R(X)_2^\star$, we associate to ℓ a quadratic form Q_ℓ on $R(X)_1$ given by $Q_\ell(f) := \ell(f^2)$.

C.5 Curves of almost minimal degree

We now specialize to the main class of varieties of interest in this paper. Since P_X only depends on real points of X, it is natural to restrict to *totally real* varieties (i.e. real varieties whose set of real points is Zariski-dense), and since Σ_X only depends on the quadratic part of the coordinate ring of X, it is important to restrict to varieties defined by quadrics. We consider smooth projective non-ACM curves of almost minimal degree. Such curves arise as projections of the rational normal curve C_d away from a point (cf. [26, Theorem 1.2]). Let $\mathrm{Sec}_2(C_d)$ denote the 3rd secant variety of C_d, i.e. the Zariski closure of the union of all secant 2-planes to C_d in $\mathbb{P}^d$, meeting C_d in 3 distinct points. For $p \in \mathbb{P}^d \setminus \mathrm{Sec}_2(C_d)$, let $\pi_p : \mathbb{P}^d \dashrightarrow \mathbb{P}^{d-1}$ be projection with center p (i.e. away from p). On restriction to C_d, the rational map π_p becomes a morphism, and the image $X := \pi_p(C_d) \subseteq \mathbb{P}^{d-1}$ is a smooth rational curve of almost minimal degree $d = \deg X = \operatorname{codim} X + 2$. Let $R(X) := \mathbb{R}[x_0, \ldots, x_{d-1}]/I(X)$ denote the real coordinate ring of X. The assumption that $p \notin \mathrm{Sec}_2(C_d)$ is equivalent to the statement that $I(X)$ is generated by quadrics, cf. [27, Theorem 1.1(2)]. Since X is projective, $R(X) = \bigoplus_{i=0}^{\infty} R(X)_i$ is naturally $\mathbb{Z}$-graded.

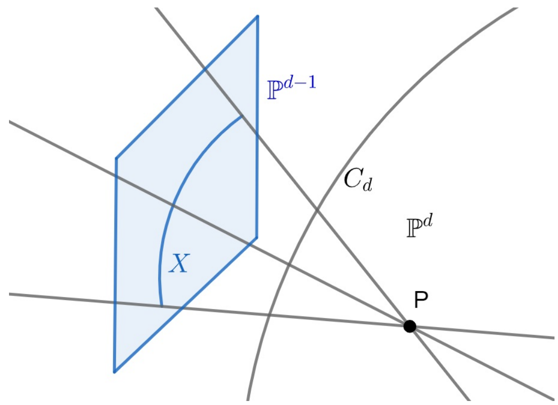


Figure C.1: Projection of the rational normal curve $C_d \subseteq \mathbb{P}^d$ away from a point p

We next spell out a series of basic, but useful, identifications.

Remark C.5.1. *(i) The surjection $\pi_p : C_d \twoheadrightarrow X$ induces an injection of coordinate rings $R(X) \hookrightarrow R(C_d)$, which is naturally graded. In this way $R(X)_1$ is identified with a hyperplane $H \subseteq R(C_d)_1$.*

(ii) Since $p \notin \mathrm{Sec}_2(C_d)$, the quadratic part of the coordinate ring of X can be identified with the quadratic part of the coordinate ring of C_d, i.e. $R(X)_2 = R(C_d)_2$. Equivalently, the Hilbert function of X in degree 2 has value $2d + 1$.

(iii) Via the d-uple embedding $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$, $R(C_d)_1$ can in turn be identified with $R(\mathbb{P}^1)_d = \mathbb{R}[x, y]_d$, the space of all degree d binary forms, and similarly $R(C_d)_2 \cong \mathbb{R}[x, y]_{2d}$.

(iv) The apolar inner product (Equation C.1) on $\mathbb{R}[x, y]_d$, along with (iii), gives an explicit description of the hyperplane H in (i): namely H is the orthogonal complement in $\mathbb{R}[x, y]_d$ of the center $F(p)$ (cf. Section C.3), which is also $(F(p))^{\perp}_d$, the degree d part of the apolar ideal of $F(p)$. Moreover, with respect to the pairing on $\mathbb{R}[x, y]_{2d}$, every functional $\ell \in \mathbb{R}[x, y]^*_{2d}$ can be realized as $\ell(\cdot) = \langle \cdot, L \rangle$ for some $L \in \mathbb{R}[x, y]_{2d}$.

(v) Putting (i) – (iv) together with Section C.4, we may thus associate to any $\ell \in \Sigma_X^\star$ a binary form $L \in \mathbb{R}[x,y]_{2d}$, as well as quadratic forms Q_ℓ acting on $\mathbb{R}[x,y]_d \cong R(C_d)_1$ and q_ℓ acting on $(F(p))_d^\perp \cong R(X)_1$. Note that $q_\ell = Q_\ell\Big|_H$ is the restriction of Q_ℓ to H: when represented as symmetric matrices, Q_ℓ is $(d+1)\times(d+1)$, whereas q_ℓ is $d \times d$.

We briefly review what is known about algebraic invariants of curves of almost minimal degree. First, for any nondegenerate variety $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$, there is a stratification of $\mathbb{P}^n$ by (higher) secant varieties of Y:

$$Y \subsetneq Y^2 \subsetneq Y^3 \subsetneq \cdots \subsetneq Y^{k-1} \subsetneq Y^k = \mathbb{P}^n$$

This gives rise to the notion of Y-border rank: for $p \in \mathbb{P}^n$, the Y-border rank of p is defined as $\operatorname{rk}_Y(p) := \min\{i \mid p \in Y^i\}$ (cf. [39, 40]). For $Y = C_d$, it follows from Section C.3 and apolarity that the C_d-border rank of a point is exactly the complex border rank of the corresponding binary form, i.e. $\operatorname{rk}_{C_d}(p) = \mathbb{C}\text{-b.}\operatorname{rk}(F(p))$.

Next, a fruitful way to study a projected curve $X = \pi_p(C_d)$ is to consider the rational normal scrolls containing X as a divisor. Recall that a rational normal scroll is a variety $S(a_1, \ldots, a_m)$ which is a join of disjoint rational normal curves of degrees $a_1, \ldots, a_m$ in $\mathbb{P}^{\sum_{i=1}^m (a_i+1)-1}$; the tuple $(a_1, \ldots, a_m)$ is called the *type* of the scroll. As $\dim S(a_1, \ldots, a_m) = m$ and $\deg S(a_1, \ldots, a_m) = \sum_{i=1}^m a_i$, every scroll is a variety of minimal degree, and conversely any nondegenerate variety of minimal degree is either a quadric hypersurface, the second Veronese of $\mathbb{P}^2$, or a scroll (cf. [41]). It was shown in [27] that the Green-Lazarsfeld index of X (and even the entire graded Betti table of X) is determined by the types of surface scrolls containing X, which in turn is determined by $\operatorname{rk}_{C_d}(p)$:

Theorem C.5.2 ([27, Theorem 1.1])**.** *Let $C_d \subseteq \mathbb{P}^d$ be a rational normal curve of degree d, $\pi_p : \mathbb{P}^d \dashrightarrow \mathbb{P}^{d-1}$ the projection away from a point $p \in \mathbb{P}^d \setminus C_d^2$, and $X := \pi_p(C_d) \subseteq \mathbb{P}^{d-1}$. Then*

1. *X is contained in a surface scroll $S(a,b)$ with $1 \leq a \leq b$ if and only if $a = \operatorname{rk}_{C_d}(p) -$*

2, *and*

2. *The Green-Lazarsfeld index of X is given by $\alpha(X) = \mathrm{rk}_{C_d}(p) - 3$.*

This implies that

$$\mathbb{C}\text{-b. rk}(F(p)) - 2 = \alpha(X) + 1 \le \eta(X)$$

by [1, Theorems 4, 6].

C.6 Some lemmas

The next lemma connects kernels of quadratic forms to apolar ideals of binary forms, which is key for our main result.

Lemma C.6.1. *Let $d \ge 1$, $L \in \mathbb{R}[x, y]_{2d}$, and Q the quadratic form on $\mathbb{R}[x, y]_d$ associated to the functional $\langle \cdot, L\rangle$ (as in Theorem C.5.1). Then $\ker(Q) = (L)^{\perp}_d$.*

Proof. The matrix A of Q is constructed with respect to a basis $B = \{b_0, \dots, b_d\}$ of $\mathbb{R}[x, y]_d$ as follows: the (i, j) entry of A is $\langle b_i b_j, L\rangle$. Given $f \in \mathbb{R}[x, y]_d$, one has $f \in \ker(Q) \iff Q(b_i f) = 0$ for all $0 \le i \le d \iff f \in (L)^{\perp}$, by Theorem C.1.5. □

We also note that vanishing at points on $\mathbb{P}^1$ with specified multiplicities imposes independent conditions on binary forms.

Proposition C.6.2. *Let $d \ge 0$, $\{P_1, \dots, P_r\} \subseteq \mathbb{P}^1$ and $r_1, \dots, r_r \in \mathbb{N}$ be given. Then the space of degree d binary forms vanishing to order at least r_i at each P_i has codimension $\sum_{i=1}^{r} r_i$ in $k[x, y]_d$ (we interpret the space as empty if $\sum_{i=1}^{r} r_i > d$).*

Proof. Vanishing at $[a_1 : b_1], \dots, [a_r : b_r]$ to orders $r_1, \dots, r_r$ is equivalent to being divisible by $\prod_{i=1}^{r}(b_i x - a_i y)^{r_i}$. □

Also, we collect various results from linear algebra which will be needed in the proof of Theorem 2.4.3.

Lemma C.6.3. *Let* $A = \sum_{i=1}^{k} \lambda_i v_i v_i^T$ *be an* $n \times n$ *symmetric matrix, with* $v_i \in \mathbb{R}^n$*. If* $\{v_1, \ldots, v_k\}$ *are linearly independent, then the signature of* A *is given by the sign pattern of the coefficients* λ_i*.*

Proof. Diagonalize A by extending $\{v_1, \ldots, v_k\}$ to a basis of $\mathbb{R}^n$. □

Lemma C.6.4 (Cauchy interlacing)**.** *Let* A *be a real symmetric matrix. If* B *is any principal submatrix of* A*, then the eigenvalues of* B *interlace the eigenvalues of* A*.*

Proof. Cf. [42, Theorem 4.3.17]. □

Corollary C.6.5. *Let* Q *be a quadratic form on* $\mathbb{R}^n$ *with Lorentz signature* $(n-1, 1) = (+, \ldots, +, -)$*, and* $H \subseteq \mathbb{R}^n$ *a hyperplane. Then the following are equivalent for the restriction* $Q\big|_H$ *of* Q *to* H*:*

1. $\ker(Q\big|_H) \neq 0$
2. $\operatorname{rank} Q\big|_H = n - 2$
3. $Q\big|_H$ *is positive semi-definite, but not positive-definite.*

Proof. Choose a basis of $\mathbb{R}^n$ which arises from extending a basis of H, so that if A is the $n \times n$ symmetric matrix representing Q, then $Q\big|_H$ is represented by a principal $(n-1) \times (n-1)$ submatrix B of A. Now $\ker(Q\big|_H) \neq 0$ implies that 0 is an eigenvalue of B. If B had a negative eigenvalue, then Theorem C.6.4 would imply that A has ≥ 2 negative eigenvalues, contradiction. □

APPENDIX D
MACAULAY 2 PROJECTS

The apolarity for binary forms is very useful when we study the rank of points with respect to a rational normal curve of degree d because of the correspondence between points in the projective space $\mathbb{P}^d$ and binary forms of degree d. In this chapter, we introduce a Macaulay 2 code that produce random points of given C_d-ranks and give the list of C_d-ranks of points in a linear space. The code that we create is available in this link.

D.1 Generating random points of given border rank

Let C_d be a rational normal curve of degree d. i.e. the image of d-Uple map (or d-th Veronese map) of projective line $\mathbb{P}^1$. Recall that the C_d-rank of a point $p \in \mathbb{P}^d$ is the smallest number k such that $\operatorname{Sec}_{k-1}(C_d)$ contains the point p. If we choose a point randomly (without any restrictions), the C_d-rank of the point is $\lfloor \frac{d}{2} \rfloor + 1$ because the collection of points whose rank is less than $\lfloor \frac{d}{2} \rfloor + 1$ forms a proper Zariski closed subset of $\mathbb{P}^d$. However, it is useful if we can choose random points of given rank. For example, random points of the C_d-rank 1 is random points in C_d.

Through the identification of points in $\mathbb{P}^d$ with binary forms of degree d, it suffice to produce a random binary form F such that its complex Waring border rank of F is r to produce a random point having border rank r. Let an integer d be the degree of rational normal curve (embedded in $\mathbb{P}^d$) and an integer $1 \leq r \leq \lfloor \frac{d}{2} \rfloor + 1$ be the desired border rank of a point in $\mathbb{P}^d$.

Theorem D.1.1. *(cf. Theorem 2.2 in [32]) For any (binary) form $F \in \mathbb{R}[x, y]_d$ (with standard grading), the apolar ideal $F^\perp$ of F is complete intersection ideal over $\mathbb{C}$. Conversely, if I is an ideal generated by two homogeneous polynomials g_1 and g_2 such that*

$\deg(g_1) + \deg(g_2) - 2 = d$, *then there is unique binary form F such that $F^{\perp} = I$.*

Suppose g_1 and g_2 are random binary forms such that $\deg g_1 = r$ and $\deg g_2 = d - r + 2$. Set $I = (g_1, g_2)$ the ideal generated by the two random binary forms. Then, by Theorem D.1.1, there exists unique binary form of degree d in the vector space $I_d^{\perp}$ up to constant multiples and we take the binary form F of degree d.

Example D.1.2. *The author of [27] proved that the graded Betti numbers of the rational curve $\pi_p(C_d)$ are uniquely determined by the border rank $\mathrm{rk}_{C_d} p$ of the center p of the projection. In this example, we check the result through Macaulay 2 computer software.*

Suppose $C_6 = \nu_6(\mathbb{P}^1)$ is the rational normal curve of degree 6 and the border rank is $r = 3$. We set the ideal I the defining ideal of C_6 and pt the random point of the border rank r. Then, we run the code that computes the Betti diagram of $\pi_{pt}(C_d)$ 5 times and we include the result in Figure D.1

```
+ M2 --no-readline --print-width 207
Macaulay2, version 1.19.1
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LLLBases, MinimalPrime→

i1 : load "RankofPoints.m2"

i2 : (d,r) = (6,3);

i3 : P1 = QQ[s,t];

i4 : R = QQ[x_0..x_d];

i5 : I = minors(2, matrix table(2, d, (i,j) -> R_(i+j)));

o5 : Ideal of R

i6 : for i to 5 list (
    pt = randomRankPoint (d,r,P1);
    betti res ker projection(I,sub(pt,R))
);

i7 : netList oo

     +--------------------+
     |       0 1  2  3 4 5|
o7 = |total: 1 9 16 13 6 1|
     |    0: 1 .  .  . . .|
     |    1: . 8 12  3 . .|
     |    2: . 1  4 10 6 1|
     +--------------------+
     |       0 1  2  3 4 5|
     |total: 1 9 16 13 6 1|
     |    0: 1 .  .  . . .|
     |    1: . 8 12  3 . .|
     |    2: . 1  4 10 6 1|
     +--------------------+
     |       0 1  2  3 4 5|
     |total: 1 9 16 13 6 1|
     |    0: 1 .  .  . . .|
     |    1: . 8 12  3 . .|
     |    2: . 1  4 10 6 1|
     +--------------------+
     |       0 1  2  3 4 5|
     |total: 1 9 16 13 6 1|
     |    0: 1 .  .  . . .|
     |    1: . 8 12  3 . .|
     |    2: . 1  4 10 6 1|
     +--------------------+
     |       0 1  2  3 4 5|
     |total: 1 9 16 13 6 1|
     |    0: 1 .  .  . . .|
     |    1: . 8 12  3 . .|
     |    2: . 1  4 10 6 1|
     +--------------------+
     |       0 1  2  3 4 5|
     |total: 1 9 16 13 6 1|
     |    0: 1 .  .  . . .|
     |    1: . 8 12  3 . .|
     |    2: . 1  4 10 6 1|
     +--------------------+
```

Figure D.1: Stable Betti diagrams of $\pi_p(C_6)$ with $\mathrm{rk}_{C_d}(p) = 3$

We can see that the Betti diagrams are stable and could obtain same results when we change the pairs (d, r) and increase the number of trials.

D.2 List of ranks of points in linear spaces with respect to rational normal curves

Suppose C_d is a rational normal curve of degree d and L is a linear space in $\mathbb{P}^d$ spanned by a (finite) set of points. Then, we can find the list of all C_d-ranks of points in L.

Recall that, for the non-degenerate variety C_d, there is a stratification of $\mathbb{P}^d$ by secant varieties of C_d:

$$C_d = \mathrm{Sec}_0(C_d) \subsetneq \mathrm{Sec}_1(C_d) \subsetneq \cdots \subsetneq \mathrm{Sec}_{\lfloor \frac{d}{2} \rfloor}(C_d) \subsetneq \mathrm{Sec}_{\lfloor \frac{d}{2} \rfloor + 1}(C_d) = \mathbb{P}^d \,.$$

Also, $\mathrm{rk}_{C_d}(p) = r + 1 \iff p \in \mathrm{Sec}_{r+1}(C_d) \setminus \mathrm{Sec}_r(C_d)$ for $0 \leq r \leq \lfloor \frac{d}{2} \rfloor$. More generally, for a subset T of points in L, $T \cap \mathrm{Sec}_r(C_d) = \emptyset \iff \mathrm{rk}_{C_d}(p) > r$ for any $p \in T$ and $T \subset \mathrm{Sec}_r(C_d) \iff \mathrm{rk}_{C_d}(p) \leq r+1$ for $p \in T$. Therefore, we can collect the list of all ranks of points in the linear space L by checking whether $L \cap (\mathrm{Sec}_r(C_d) \setminus \mathrm{Sec}_{r-1}(C_d)) \neq \emptyset$ for each integer r and finding largest integer r such that $L \subset \mathrm{Sec}_r(C_d)$ (for maximal rank of the list). In other words, the collection of ranks of points in L is obtained by checking whether $\sqrt{I(L) + I(\mathrm{Sec}_r(C_d))} \neq \sqrt{I(L) + I(\mathrm{Sec}_{r-1}(C_d))}$ for each r and finding largest integer r such that $I(L) \supset I(\mathrm{Sec}_r(C_d))$ where $I(Y)$ is the defining ideal of the variety Y and $\sqrt{I}$ is the radical ideal of I.

Example D.2.1. *Suppose we look for the collection of C_{10}-rank of points in a line spanned by two points in $\mathbb{P}^{10}$. Let $pt1$ be a point of C_{10}-rank 3 and $pt2$ be a point of C_{10}-rank 4. Assume L is a line in projective space $\mathbb{P}^{10}$ spanned by $pt1$ and $pt2$. Then, we can find the collection of all ranks of points in L by running the code "allRanks" and include the result in Figure D.2.*

Therefore, the line L contains some points of C_{10}-rank $3, 4, 5,$ and 6.

```
+ M2 --no-readline --print-width 207
Macaulay2, version 1.19.1
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, LLLBases, MinimalPrime→

i1 :  load "RankofPoints.m2"

i2 : (d,rk1,rk2) = (10,3,4);

i3 : P1 = QQ[s,t];

i4 : S = QQ[x_0..x_d];

i5 : pt1 = sub(randomRankPoint(d,rk1,P1),S);

             11       1
o5 : Matrix S   <--- S

i6 : pt2 = sub(randomRankPoint(d,rk2,P1),S);

             11       1
o6 : Matrix S   <--- S

i7 : allRanks line (pt1,pt2)

o7 = {3, 4, 5, 6}

o7 : List
```

Figure D.2: List of all ranks of points in a line L

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