NON-SEPARATING PATHS IN GRAPHS

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NON-SEPARATING PATHS IN GRAPHS

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To my parents and grandparents.

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SUMMARY

Motivated by Tutte's result and Lovász's conjecture, there is a series of work on non-separating paths in graphs and their applications. Let G be a graph and a_1, a_2, b_1, b_2 be distinct vertices of G, we give a structural characterization for G not containing a path A from a_1 to a_2 and avoiding b_1 and b_2 such that removing A from G results in a 2-connected graph. Using this structure theorem, we construct a 7-connected such graph. We will also discuss potential applications to other problems, including the 3-linkage conjecture made by Thomassen in 1980. This is based on joint work with Shijie Xie and Xingxing Yu.

CHAPTER 1

INTRODUCTION

1.1 Notation and terminology

In this section, we give notation and terminology. For some (well-known) graph concepts that are omitted, we refer the readers to Graph Theory textbook by Bondy and Murty [2] and Diestel [5].

1.1.1 Graph operations

Let $G = (V(G), E(G))$ be a graph where $V(G)$ is its vertex set and $E(G)$ is its edge set. For all $x \in V(G)$, $d_G(x)$ (or $d(x)$ if G is understood) denotes the degree of x in G, i.e., $d_G(x) = |\{y \in V(G) : xy \in E(G)\}|$. For any $S \subseteq V(G)$, $N_G(S)$ is the *neighborhood* of S in G, i.e., $N_G(S) = \{v \in V(G) \setminus S : \exists u \in S$ such that $uv \in E(G)\}$. We use $G[S]$ to denote the subgraph of G induced by S, i.e., $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) : \forall u, v \in S\}$. We also use $G - S$ to denote $G[V(G) \setminus S]$. When $S = \{s\}$, we write $G - s$ for $G - \{s\}$.

For two graphs G and H, let $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$, $G \cap H = (V(G) \cap V(H), E(G) \cap E(H))$, and $G - H$ be the graph obtained from G by deleting vertices of H and all edges of G incident with H. We call H a *subgraph* of G, denoted as $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Let G be a graph. For any subgraph $H \subseteq G$, and for any $S_1 \subseteq V(G)$ and $S_2 \in \binom{V(H)\cup S_1}{2}$ (i.e., S_2 is a set of 2-element subsets of $V(H) \cup S_1$), define $H + S_1 + S_2 = (V(H) \cup S_1, E(H) \cup S_2)$. For subgraphs $G_1, G_2 \subseteq G$, we say (G_1, G_2) is a *separation* of G if $E(G_1) \cap E(G_2) = \emptyset$, $G = G_1 \cup G_2$, and for $i = 1, 2$, $E(G_i) \setminus E(G_{3-i}) \neq \emptyset$ or $V(G_i) \setminus V(G_{3-i}) \neq \emptyset$.

1.1.2 Paths

We call a path P with ends a, b an *a-b path*. For $v_1, v_2 \in V(P)$, we define $P[v_1, v_2]$ to be the subpath of P with ends v_1, v_2. Let $P(v_1, v_2] = P[v_1, v_2] - v_1$, $P[v_1, v_2) = P[v_1, v_2] - v_2$ and $P(v_1, v_2) = P[v_1, v_2] - \{v_1, v_2\}$.

We call two paths P_1, P_2 *disjoint* if $V(P_1) \cap V(P_2) = \emptyset$. A collection of paths $P_1, \ldots, P_k$ are *independent* if no vertex of any path is an internal vertex of any other path in the collection. For any a-b path P in a graph G and for any subgraph H of G, P is *internally disjoint* from H if $(V(P) \setminus \{a, b\}) \cap V(H) = \emptyset$. For $A, B \subseteq V(G)$, A-B paths in G are paths in G from A to B and internally disjoint from $A \cup B$.

1.1.3 Connectivity

A graph is *connected* if there is a path from any vertex to any other vertex in the graph, and a graph that is not connected is *disconnected*.

We call a set $T \subseteq V(G)$ a *cut* of a graph G if $G - T$ is disconnected; and if $|T| = k$, we call T a *k-cut*. Note that for any separation (G_1, G_2) of G, $V(G_1 \cap G_2)$ is a cut of G if $V(G_i - G_{3-i}) \neq \emptyset$ for both $i \in [2]$.

For graph G and its subgraph H, we call C a *component* of $G - H$ if C is a subgraph of $G - H$, C is connected, and for any $C' \subseteq G - H$ such that C' is connected and $C \subseteq C'$, $C = C'$.

Let k be a positive integer. We call a graph G *k-connected* if $|V(G)| \geq k + 1$ and for any $S \subseteq V(G)$ with $|S| < k$, $G - S$ is connected. For any set $A \subseteq V(G)$, we say G is (k, A)-connected if for any cut $S \subseteq V(G)$ with $|S| < k$ and for every component C of $G - S$, $|V(C) \cap A| \geq k - |S|$.

A subgraph B of a graph G is called a *block* if it is isomorphic to K_2 or 2-connected, and for any $B' \subseteq G$ such that B' is isomorphic to K_2 or 2-connected, $B \subseteq B'$ implies $B = B'$. A block is *non-trivial* if $|V(B)| \geq 3$.

1.1.4 Bridges

Let G be a graph and $H \subseteq G$, we call $X \subseteq G$ an H-*bridge* of G, if either

(1) X is induced by some edge $e = uv \in E(G) \setminus E(H)$ with $u, v \subseteq V(H)$, or

(2) $X = C + S$ where C is a component of $G - H$ and $S = \{e, v : e = uv \in E(G), u \in V(C), v \in V(H)\}$.

When (1) holds, X is said to be *trivial*, and when (2) holds, X is *non-trivial*. The vertices in $V(X \cap H)$ are called *attachments* of X on H.

1.1.5 Plane graphs

A graph G is *planar* if it can be drawn in the plane with no edge crossing. Such a drawing is called a *plane* graph. Let G be a plane graph. The *faces* of G are the connected open regions of the complement of G in the plane. The *boundary* of a face F consists of vertices and edges incident with F. The boundary of the unbounded (or infinite) face is called the *outerwalk* of G. Two vertices of G are *cofacial* if they belong to the boundary of a common face. Note that if G is 2-connected, then all its faces are bounded by cycles. A *triangular* face in G is a face of G bounded by a triangle.

1.1.6 Lexicographic ordering

For any positive integer k, we denote $[k] = \{1, 2, \ldots, k\}$.

Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ be real numbers. We say that the sequence $(\alpha_1, \cdots, \alpha_n)$ is *larger* than the sequence $(\beta_1, \cdots, \beta_m)$ with respect to the lexicographic ordering, denoted by $(\alpha_1, \ldots, \alpha_n) > (\beta_1, \ldots, \beta_m)$, if either

(i) $n > m$ and $\alpha_i = \beta_i$ for $i = 1, \cdots, m$, or

(ii) there exists $j \in [min(m, n)]$ with $\alpha_j > \beta_j$ and $\alpha_i = \beta_i$ for all $i < j$.

1.2 Background on non-separating paths

When developing a theory of 3-connected graphs, Tutte [25] showed that

Theorem 1.2.1 ([25])**.** *For any* 3*-connected graph* G *and any distinct vertices* a_1, a_2, b *of* G*,* $G - b$ *has an* a_1*-*a_2 *path* P *such that* $G - P$ *is connected.*

We call such a path *non-separating*. The "3-connectedness" condition cannot be relaxed; for instance, when $\{a_1, a_2\}$ is a 2-cut if G is allowed to be 2-connected. Lovász [15] made a conjecture which would generalize Tutte's result.

Conjecture 1.2.2 (Lovász, 1975)**.** *For each natural number* k*, there exists a least natural number* $\beta(k)$ *such that, for any two vertices* a, b *in any* $\beta(k)$*-connected graph* G*, there exists a path* P *between* a *and* b *such that* $G - P$ *is* k*-connected.*

Thus, Tutte's result showed that $\beta(1) = 3$. Chen, Gould and Yu [3], and, independently, Kriesell [13] showed $\beta(2) = 5$. Moreover, Kawarabayashi, Lee and Yu [11] showed that $\beta(2) = 4$ except for double wheels. Conjecture 1.2.2 for $k \geq 3$ is still open.

For $m \geq 0$ and $k \geq 1$, let $\alpha(m, k)$ be the minimum connectivity such that for any $\alpha(m, k)$-connected graph G and distinct $a_1, a_2, b_1, \ldots, b_m \in V(G)$, there exists an a_1-a_2 path P, such that $b_1, \ldots, b_m \notin V(P)$ and $G - P$ is k-connected.

Note that $\alpha(0, k) = \beta(k)$. See the first column of Table 1.1 for the discussion above on $\alpha(0, k) = \beta(k)$ for $k \in [2]$.

Now, let us look at the first row of Table 1.1. Theorem 1.2.1 also proved $\alpha(1, 1) = 3$. One can also deduce Theorem 1.2.1 from the following result of Tutte.

Theorem 1.2.3 ([25])**.** *For any* 3*-connected graph* G *and any distinct vertices* a_1, a_2 *of* G*,* G *has independent* a_1*-*a_2 *paths* P_1, P_2 *such that* $G - P_i$ *is connected for* $i \in [2]$*.*

Similarly, one can deduce from the following result of Chen, Gould and Yu [3] that finds a non-separating path avoiding arbitrarily m vertices in any $(22m + 24)$-connected graph, and thus, $\alpha(m, 1) \leq 22m + 24$.

Table 1.1: Connectivity for non-separating paths avoiding m vertices

$\alpha(m,k)$-connected G \ Avoiding m vertices \ $G-P$ is k-connected	$m=0$	1	2	3	$\cdots m \cdots$
$k=1$	3	3	6	6	$\leq 22m+24$
2	5	5	≥ 8	$\alpha(m,2)$	
3 ⋮ ⋮	Lovász's Conjecture open for $k \geq 3$	$\alpha(m,k)$			

Theorem 1.2.4 ([3])**.** *For any* $(22m+24)$*-connected graph* G *and any distinct vertices* a_1, a_2 *of* G*,* G *has* $m+1$ *independent* a_1*-*a_2 *paths* P_i *such that* $G-P_i$ *is connected for all* $i \in [m+1]$ *.*

It is worth mentioning that with higher connectivity, Wollan [26] showed that one can remove a subset of paths without disconnecting the graph.

Theorem 1.2.5 ([26])**.** *For any* $83(m+1)$*-connected graph* G *and any distinct* a_1, a_2 *of* G*, there exist independent* a_1*-*a_2 *paths* $P_1, \ldots, P_m$ *such that for any subset* $I \subseteq [m]$*,* $G - (\bigcup_{i \in I} V(P_i))$ *is connected.*

Note that the above results (other than Theorem 1.2.1) involve graphs with high connectivity. In applications, one often needs to find a non-separating path that avoids specific vertices in graphs. For example, when proving the Kelmans-Seymour conjecture, He, Wang and Yu [6, 7, 8, 9] needed non-separating paths in 4-connected graphs that avoids two vertices.

The result on 2-linked (defined later) graphs by Jung [10], Seymour [19], Shiloach [20], Thomassen [24], and Chakravarti and Robertson [17] showed that $\alpha(2,1)=6$. Thomas, Xie, and Yu [23] showed that $\alpha(3,1)=6$. One can easily deduce $\alpha(1,2)=5$ from a result of Chen, Gould and Yu [3] and Kriesell [13], and we will present it as Corollary 2.1.4. We are primarily interested in a structural characterization of graphs not containing non-separating paths between two given vertices and avoiding two other given vertices. Such a characterization should help determine $\alpha(2,2)$, and we believe $\alpha(2,2) = 8$.

1.3 Structure theorem

Given a graph G and distinct vertices a_1, a_2, b_1, b_2 of G. We say that (G, a_1, a_2, b_1, b_2) is feasible if $G - \{b_1, b_2\}$ contains an a_1-a_2 path A such that $G - A$ is 2-connected. We say (G, a_1, a_2, b_1, b_2) is infeasible if G is not feasible.

Our aim is to provide structural information about (G, a_1, a_2, b_1, b_2) when it is not feasible. We show that if (G, a_1, a_2, b_1, b_2) is infeasible then G is the edge disjoint union of three graphs A_1, A_2 and H, where $a_i \in V(A_i) \setminus V(A_{3-i} \cup H)$, A_i is planar, and H can be further decomposed into graphs of simple structures. See Figure 1.1 for an illustration.

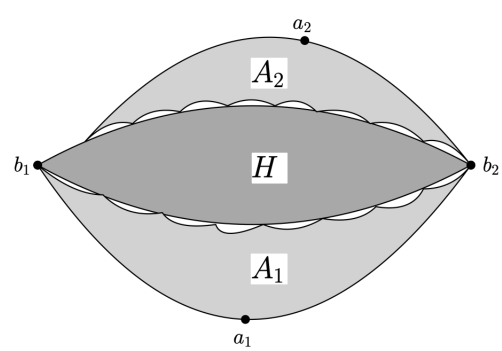

Figure 1.1: Decomposition into edge disjoint subgraphs A_1, A_2 and H.

Theorem 1.3.1. *Let G be an 8-connected graph and let $a_1, a_2, b_1, b_2 \in V(G)$ be distinct. Suppose (G, a_1, a_2, b_1, b_2) is infeasible. Then, the following statements hold:*

(i) $G - \{a_1, a_2\}$ contains three independent induced b_1-b_2 paths B_1, B_2, B_3 such that, for $i \in [2]$, the $(B_1 \cup B_2 \cup B_3)$-bridge of G containing a_i, denoted as $A_i(B_1 \cup B_2 \cup B_3)$, satisfy the following properties (up to relabeling):

- *$A_1(B_1 \cup B_2 \cup B_3)$ has all its attachments on B_3,*
- *$A_1(B_1 \cup B_2 \cup B_3) \cup B_3$ has a plane representation in which B_3 and a_1 are on the boundary of the infinite face,*
- *$A_2(B_1 \cup B_2 \cup B_3)$ has attachments on both B_1 and B_2.*

(ii) There exists $w \in V(A_1(B_1 \cup B_2 \cup B_3)) \cap V(B_3)$ such that $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - w - a_2$ has three independent b_1-b_2 paths P_1, P_2, P_3, and the $(P_1 \cup P_2 \cup P_3)$-bridge of G containing a_2, denoted as $A_2(P_1 \cup P_2 \cup P_3)$, satisfies the following properties:

- *$A_2(P_1, P_2, P_3)$ has all its attachments on P_3,*
- *$A_2(P_1, P_2, P_3) \cup P_3$ has a plane representation in which P_3 and a_2 are on the boundary of its infinite face.*

(iii) $H := G - (A_1(B_1 \cup B_2 \cup B_3) - (B_3 - w)) - (A_2(P_1 \cup P_2 \cup P_3) - P_3)$ is the edge disjoint union of subgraphs $H_1, \ldots, H_{m+1}$, such that $V(H_i \cap H_{i+1}) = \{u_i, v_i, w_i\}$ is a 3-cut of H separating b_1 from b_2, $b_1, u_1, \ldots, u_m, b_2$ occur on P_3 in order, $b_1, v_1, \ldots, v_m, b_2$ occur on P_2 in order, and $b_1, w_1, \ldots, w_m, b_2$ occur on P_1 in order.

(iv) For each vertex $u \in V(A_2(P_1 \cup P_2 \cup P_3)) \cap V(P_3)$, $u = u_i$ for some i.

(v) For each $i \in [m] \setminus \{1\}$, $H_i = (J_i, L_i)$, where J_i is a plane graph and L_i is a ladder consisting of rungs of simple structure.

See Figure 1.2 for an illustration of H in the above theorem. The concept of ladders and rungs will be described in Chapter 2.

Note that "8-connected" cannot be replaced by "7-connected", as we have an example (see Chapter 6) on 7-connected infeasible graph.

We believe Theorem 1.3.1 will be enough to show that 8-connected graphs are feasible, i.e., $\alpha(2, 2) = 8$, which is work in progress.

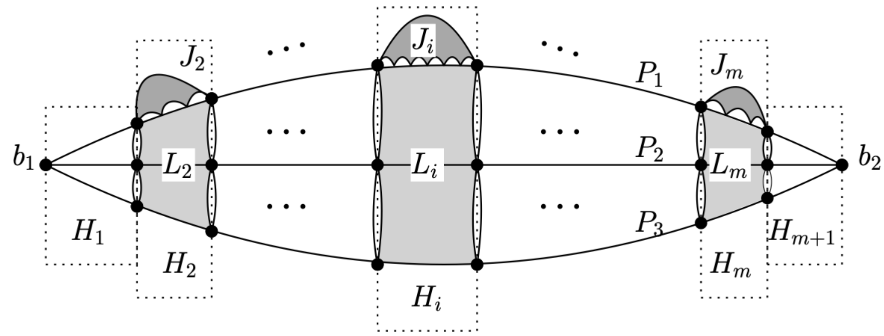

Figure 1.2: H is a union of subgraphs $H_1, \ldots, H_{m+1}$.

1.4 Related problems

1.4.1 Linkage problem

Theorem 1.3.1 should serve as a step towards the following conjecture of Thomassen [24].

Conjecture 1.4.1 (Thomassen, 1980)**.** *Let G be an 8-connected graph and let $a_1, a_2, b_1, b_2, c_1, c_2 \in V(G)$ be distinct. Then, G contains disjoint paths from a_1, b_1, c_1 to a_2, b_2, c_2, respectively.*

More generally, a graph G is *k-linked* if, for any k disjoint pairs of vertices $\{s_i, t_i\}$, $i \in [k]$, in G, G has pairwise disjoint paths from s_i to t_i for $i \in [k]$. Note that if (G, a_1, a_2, b_1, b_2) is infeasible then G is not 3-linked as can be seen by taking $c_i \in N_G(b_i) \setminus \{a_1, a_2, b_1, b_2)$ for both $i \in [2]$.

Thomassen [24] initially conjectured that every $(2k+2)$-connected graph is k-linked, but this is false for $k \geq 4$: the graph obtained from the complete graph K_{3k-1} minus a matching of size k is a counterexample. Robertson and Seymour [18] showed that there is a polynomial time algorithm for deciding whether a graph is k-linked (when k is fixed). Bollobás and Thomason [1] showed that every $(22k)$-connected graph is k-linked. Thomas and Wollan [21] improved this further to that every $(2k)$-connected graph with average degree at least $10k$ is k-linked.

Conjecture 1.4.1 states that 8-connected graphs are 3-linked, which is still open. The best result on this conjecture is due to Thomas and Wollan [22].

Theorem 1.4.2 ([22])**.** *Every 6-connected graph on n vertices with $5n - 14$ edges is 3-linked.*

As a consequence, every 10-connected graph is 3-linked. Theorem 1.4.2 combined with a result of Chen, Gould and Yu [3] (see Lemma 2.1.2) gives the following.

Corollary 1.4.3. *For every 6-connected graph G on n vertices with $5n - 14$ edges and distinct $a_1, a_2, b_1, b_2 \in V(G)$, (G, a_1, a_2, b_1, b_2) is feasible.*

Corollary 1.4.4. *For every 10-connected graph G and $a_1, a_2, b_1, b_2 \in V(G)$, (G, a_1, a_2, b_1, b_2) is feasible.*

Note that the k-linked notion was further extended by Kostochka and G.Yu [12] to H-linked graphs for any fixed graph H. Recent work of Liu, Rolek, Stephens, Ye and G.Yu [14] shows that every 7-connected graph is kite-linked, where a kite is a graph obtained from K_4 by deleting two adjacent edges.

1.4.2 Signed graphs

A *signed graph* is a triple $(V(G), E(G), f)$ where $f : E(G) \to \{1, -1\}$. The *sign* of a cycle is the product of the signs of its edges. We call a signed graph G *balanced* if every cycle is positive and *imbalanced* if G is not balanced.

Theorem 1.2.1 has a signed graph version by Tutte in [25], and we state it here.

Theorem 1.4.5 ([25])**.** *Let G be a 3-connected signed graph and $b \in V(G)$. Suppose $G - b$ is imbalanced, then G has a negative cycle C such that $b \notin V(C)$ and $G - C$ is connected.*

Note that Theorem 1.4.5 implies Theorem 1.2.1: For any 3-connected graph G and distinct $a_1, a_2, b \in V(G)$, let $G' = G + a_1a_2$. We assign $f : E(G') \to \{1, -1\}$ such that

$f(a_1a_2) = -1$ and $f(e) = 1$ for all $e \in E(G') \setminus \{a_1a_2\}$. Then, Theorem 1.2.1 follows from Theorem 1.4.5.

Similarly, the following signed graph version of Corollary 2.1.4, by Devos, Nurse, Qian and Wollan [4], also implies Corollary 2.1.4.

Theorem 1.4.6 ([4]). *Let G be a 5-connected signed graph and $b \in V(G)$. Suppose $G-b$ is imbalanced, then G has a negative cycle C such that $b \notin V(C)$ and $G - C$ is 2-connected.*

It is natural to ask the following:

Question 1.4.7. *Can we extend other results in Table 1.1 to signed graphs?*

The above known signed graph results, Theorem 1.4.5 and Theorem 1.4.6, imply Theorem 1.2.1 and Corollary 2.1.4.

Question 1.4.8. *Can we find an example on other results in Table 1.1 whose signed graph version does not hold? A positive answer would imply that signed graph version could be strictly stronger than the graph version.*

1.4.3 A general conjecture

Recall Table 1.1 and definition of $\alpha(m, k)$. When $m = 0$, it centers around Lovász's conjecture which is open for $k \geq 3$. For $k = 1$, $\alpha(m, k)$ exists by Chen, Gould and Yu [3], and we have exact values when $m \leq 3$. Wollan [1] conjectured that $\alpha(m, 2) = 2m + C$ for some constant C.

It is also natural to formulate a more general conjecture on non-separating paths avoiding more vertices.

Conjecture 1.4.9 (Qian, Xie, Yu). *For each natural number k and m, there exists a least natural number $\alpha(m, k)$ such that, for any two vertices a_1, a_2 in any $\alpha(m, k)$-connected graph G, there exists an a_1-a_2 path P avoiding a given set of m vertices such that $G - P$ is k-connected.*

[1] Paul Wollan: Private communication

The rest of the thesis is organized as follows:

In Chapter 2, we state previous results on disjoint paths that we will use in the thesis. We first state and prove feasibility for 5-connected graphs with given conditions. The result also provides us with an equivalent condition for feasibility that is convenient to use. Then, we introduce Seymour's characterization of 2-linked graphs and Yu's characterization of graphs with special three paths.

In Chapter 3, for an infeasible 8-connected graph G, we use three special paths B_1, B_2, B_3 to give a decomposition of G into three edge disjoint subgraphs A_1, A_2 and H. We will show A_i is planar for both $i \in [2]$ and H can be further decomposed into graphs with simple structures, called rungs.

Structure of H is further explored in Chapter 4 and Chapter 5. In Chapter 4, we show that most rungs will avoid at least one of the special paths B_i for all $i \in [3]$. In Chapter 5, we consider those rungs intersecting at most two B_i's.

In Chapter 6, using the structure theorem we proved, we construct examples of A_1 and H, and we use them to form a 7-connected graph with special vertices a_1, a_2, b_1, b_2 such that (G, a_1, a_2, b_1, b_2) is infeasible. Thus, $\alpha(2, 2) \geq 8$.

CHAPTER 2

PREVIOUS RESULTS ON DISJOINT PATHS

In this chapter, we state and prove some known results on disjoint paths that we will use in the thesis.

First in section 2.1, we state and prove a result on feasibility for 5-connected graphs. That result gives Corollary 2.1.3, providing us with a convenient working condition on disjoint paths which is equivalent to feasibility. One other consequence is Corollary 2.1.4, which reproves $\alpha(1,2) = \alpha(0,2) = 5$.

In section 2.2, we introduce the concept of "3-planar" graphs and state Seymour's characterization of 2-linked graphs.

In section 2.3, we introduce definitions of "rungs" and "ladders", and state Yu's characterization of graphs containing certain types of three disjoint paths.

2.1 Feasibility for 5-connected graphs

The following well-known result of Menger [16] is often used to find independent paths in graphs.

Theorem 2.1.1 ([16]). *For any positive integer k and any k-connected graph G, and for any $A, B \subseteq V(G)$ with $|A| \geq k$ and $|B| \geq k$, there are at least k disjoint A-B paths.*

Chen, Gould and Yu [3] proved a result that implies the following result. We give a proof for the sake of completeness.

Lemma 2.1.2 ([3]). *For any 5-connected graph G and any distinct vertices a_1, a_2, b_1, b_2 of G, if there exist three independent paths A, B_1, B_2 such that A is from a_1 to a_2 and B_i is from b_1 to b_2 for both $i \in [2]$, then (G, a_1, a_2, b_1, b_2) is feasible.*

Proof. We may assume A is induced. Let C_1 be the component of $G - A$ containing $\{b_1, b_2\}$ and B be the block in C_1 containing $B_1 \cup B_2$. Let $B^1, B^2, \ldots, B^n$ denote the B-bridges of C_1, and let $C_2, \ldots, C_m$ be the other components of $G - A$. We may assume $|V(B^{i-1})| \geq |V(B^i)|$ for $2 \leq i \leq n$ and $|V(C_{i-1})| \geq |V(C_i)|$ for $2 \leq i \leq m$. Now, we further choose A, B_1, B_2 such that $(|V(B)|, |V(B^1)|, \ldots, |V(B^n)|, |V(C_1)|, \ldots, |V(C_m)|)$ is maximal with respect to the lexicographic ordering.

Suppose $m \geq 2$. Since G is 5-connected, by Theorem 2.1.1, there exist 5 disjoint paths from $V(C_m)$ to $V(G - C_m)$. Since $V(C_i) \cap N_G(C_m) = \emptyset$ for all $i < m$, $|V(A) \cap N_G(C_m)| \geq 5$. Let $x, y \in V(A) \cap N_G(C_m)$ such that $A[a_1, x) \cap N_G(C_m) = \emptyset$ and $A(y, a_2] \cap N_G(C_m) = \emptyset$. Since $\{x, y\}$ is not a cut in G separating $A(x, y)$ from $G - C_m$, there exists $z \in V(A(x, y))$ such that $N_G(z) \cap V(C_j) \neq \emptyset$ for some $j < m$. Choose minimum such j. Let P be an induced x-y path in $G[V(C_m) \cup \{x, y\}]$. Take $A' = A[a_1, x] \cup P \cup A[y, a_2]$. Note that $C_1, C_2, \ldots, C_{j-1}$ are components of $G - A'$, and if $j = 1$, the block in $G - A'$ containing $\{b_1, b_2\}$ still contains B. However, $|V(C_j')| > |V(C_j)|$, contradicting the choice of A that $(|V(B)|, |V(B^1)|, \ldots, |V(B^n)|, |V(C_1)|, \ldots, |V(C_m)|)$ is maximal with respect to the lexicographic ordering.

So $m = 1$. If $n = 0$, we are done. So assume $n \geq 1$. Let $\{z\} = V(B) \cap V(B^n)$. Since G is 5-connected, $|N_G(B^n - z) \cap V(A)| \geq 2$. Let $x, y \in V(A) \cap N_G(B^n - z)$ such that $A[a_1, x) \cap N_G(B^n - z) = \emptyset$ and $A(y, a_2] \cap N_G(B^n - z) = \emptyset$, and let P be an induced x-y path in $G[V(B^n - z) \cup \{x, y\}]$. Take $A' = A[a_1, x] \cup P \cup A[y, a_2]$ and B' be the block of $G - A'$ containing $\{b_1, b_2\}$.

Suppose G has edges from distinct vertices of B to $A(x, y)$. Then, $G - A'$ has block containing B and a subpath of $A(x, y)$. So A' contradicts the choice of A.

Hence, since G is 5-connected, G has an edge from $A(x, y)$ to B^i for some $i \in [n-1]$. We choose minimum such i. Then, either (1) $G - A'$ has a block containing B and part of $B^i \cup A(x, y)$, or (2) B is a block of $G - A'$, $B^1, \ldots, B^{i-1}$ are B-bridges of $G - A'$, and B^i is properly contained in a B-bridge of $G - A'$. Thus, A' contradicts the choice of A. $\square$

On the other hand, it is straightforward to see that feasibility implies the existence of such three paths in 5-connected graphs.

Corollary 2.1.3. *For any 5-connected graph G and any distinct vertices a_1, a_2, b_1, b_2 of G, the following statements are equivalent:*

(i) *There exist three pairwise independent paths A, B_1, B_2 such that A is from a_1 to a_2 and B_i is from b_1 to b_2 for both $i \in [2]$.*

(ii) *(G, a_1, a_2, b_1, b_2) is feasible.*

Hence, for the rest of the thesis, we also call (G, a_1, a_2, b_1, b_2) *feasible* if G is 5-connected and one can find three pairwise independent paths A, B_1, B_2 such that A is from a_1 to a_2 and B_i is from b_1 to b_2 for both $i \in [2]$.

Another consequence of Lemma 2.1.2 is the following result that $\alpha(1, 2) = 5$ (see Table 1.1).

Corollary 2.1.4. *For any 5-connected graph G and any distinct vertices a_1, a_2, b of G, $G - b$ contains an a_1-a_2 path P such that $G - P$ is 2-connected.*

Proof. Since G is 5 connected, by Menger's Theorem, there exist two independent a_1-a_2 paths P_1, P_2 in $G - b$. By Menger's Theorem again, there exist 5 paths from b to $V(P_1 \cup P_2)$, with only b in common. By Pigeonhole Principle, two of the paths, say Q_1, Q_2, are from b to $P_i(a_1, a_2)$ for some $i \in [2]$. Let B be the block of $G - P_{3-i}$ containing $Q_1 \cup Q_2$. By the same proof in Lemma 2.1.2, G contains an a_1-a_2 path P' such that $G - P'$ is 2-connected and $G - P'$ contains $Q_1 \cup Q_2$. Since $b \in V(Q_1 \cup Q_2)$, $P' \subseteq G - b$ and we are done. □

2.2 Characterization of 2-linked graphs

A result we use often is a characterization of 2-linked graphs, proved independently by Seymour [19], Shiloach [20], Thomassen [24], and Chakravarti and Robertson [17].

A more general result on finding k disjoint paths can be found in [18] by Robertson and Seymour in their monumental project on graph minors over a series of papers.

To state Seymour's version on 2-linked graphs, we introduce several concepts.

A 3-*planar graph* $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, ..., A_k\}$ of pairwise disjoint subsets of $V(G)$ (let $\mathcal{A} = \varnothing$ when $k = 0$) such that

(i) for $i \neq j$, $N_G(A_i) \cap A_j = \varnothing$,

(ii) for $1 \leq i \leq k$, $|N_G(A_i)| \leq 3$, and

(iii) if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding edges joining every pair of distinct vertices in $N_G(A_i)$, then $p(G, \mathcal{A})$ can be drawn in the plane without crossing edges.

If, in addition, $b_1, b_2, \ldots, b_n$ are vertices in G such that $b_i \notin A$ for $i \in [n]$ and $A \in \mathcal{A}$, $p(G, \mathcal{A})$ can be drawn in a closed disk with no edge crossings, and $b_1, b_2, \ldots, b_n$ occur on the boundary of the disk in this cyclic order, then we say that $(G, \mathcal{A}, b_1, b_2, \ldots, b_n)$ is 3 -*planar*. If there is no need to specify $\mathcal{A}$, we may simply say that $(G, b_1, b_2, \ldots, b_n)$ is 3-planar. If $\mathcal{A} = \emptyset$, we say that $(G, b_0, b_1, ..., b_n)$ is planar. If G is planar and is drawn in a closed disk with no edge crossings, for any subgraph $H \subseteq G$, we say (G, H) is planar if all vertices and edges of H are contained in the boundary of the disk, in which case H needs to be the union of disjoint paths.

Now, we can state Seymour's characterization on 2-linked graphs.

Lemma 2.2.1 (Seymour, 1980)**.** *Let G be a graph with distinct vertices x_1, x_2, x_3, x_4. Then either (G, x_1, x_2, x_3, x_4) is 3-planar, or G has disjoint paths from x_1, x_2 to x_3, x_4, respectively.*

2.3 Characterization of graphs with special three paths

While there is no known generalization of the above result to three paths with fixed ends (see Conjecture 1.4.1 of Thomassen), Yu [27, 28, 29] characterized graphs G in which any three disjoint paths from $\{a,b,c\} \subseteq V(G)$ to $\{a',b',c'\} \subseteq V(G)$ must contain a path from b to b'. To state this result, we need to describe *rungs* and *ladders*.

Let G be a graph, $\{a,b,c\} \subseteq V(G)$, and $\{a',b',c'\} \subseteq V(G)$. (Here, a,b,c are pairwise distinct, and a',b',c' are pairwise distinct.) Suppose $\{a,b,c\} \neq \{a',b',c'\}$, and assume that G has no separation (G_1,G_2) such that $|V(G_1 \cap G_2)| \leq 3$, $\{a,b,c\} \subseteq V(G_1)$, and $\{a',b',c'\} \subseteq V(G_2)$. We say that $(G,(a,b,c),(a',b',c'))$ is a *rung* if one of the following holds up to symmetry between $\{a,b,c\}$ and $\{a',b',c'\}$, relabeling a and c, and relabeling a' and c':

(1) $b = b'$ or $\{a,c\} = \{a',c'\}$.

(2) $a = a'$ and $(G - a, c, c', b', b)$ is 3-planar.

(3) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$ and (G, a', b', c', c, b, a) is 3-planar.

(4) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, G has a separation (G_1,G_2) such that $V(G_1 \cap G_2) = \{x\}$, and $\{a,a',b,b'\} \subseteq V(G_1)$, $\{c,c'\} \subseteq V(G_2)$, and (G_1,a,a',b',b) is 3-planar.

(5) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, and G has a separation (G_1,G_2) such that $V(G_1 \cap G_2) = \{z,b\}$, and $(G_1 + bz, a, a', b', b)$ is 3-planar, $\{a,a',b,b'\} \subseteq V(G_1)$, $\{c,c'\} \subseteq V(G_2)$, and (G_2, c, c', z, b) is 3-planar.

(6) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs G_a, G_c, M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u,z\}$, $V(G_c \cap M) = \{p,q\}$, $V(G_a \cap G_c) = \emptyset$, and $\{a,a',b'\} \subseteq V(G_a)$, $\{c,c',b\} \subseteq V(G_c)$, and (G_a,a,a',b',z,u) and (G_c,c',c,b,p,q) are 3-planar.

(7) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, and there are pairwise edge disjoint subgraphs G_a, G_c, M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b,b',q\}$, $V(G_c \cap M) = \{b,b',p\}$, $V(G_a \cap G_c) = \{b,b'\}$, $\{a,a'\} \subseteq V(G_a)$, $\{c,c'\} \subseteq V(G_c)$, and (G_a,a,a',b',q,b) and (G_c,c',c,b,p,b') are 3-planar.

See Figure 2.1 for illustration of all types of rungs.

Let L be a graph and let $R_1,\dots,R_m$ be edge disjoint subgraphs of L such that

(i) $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i))$ is a rung for each $i \in [m]$,

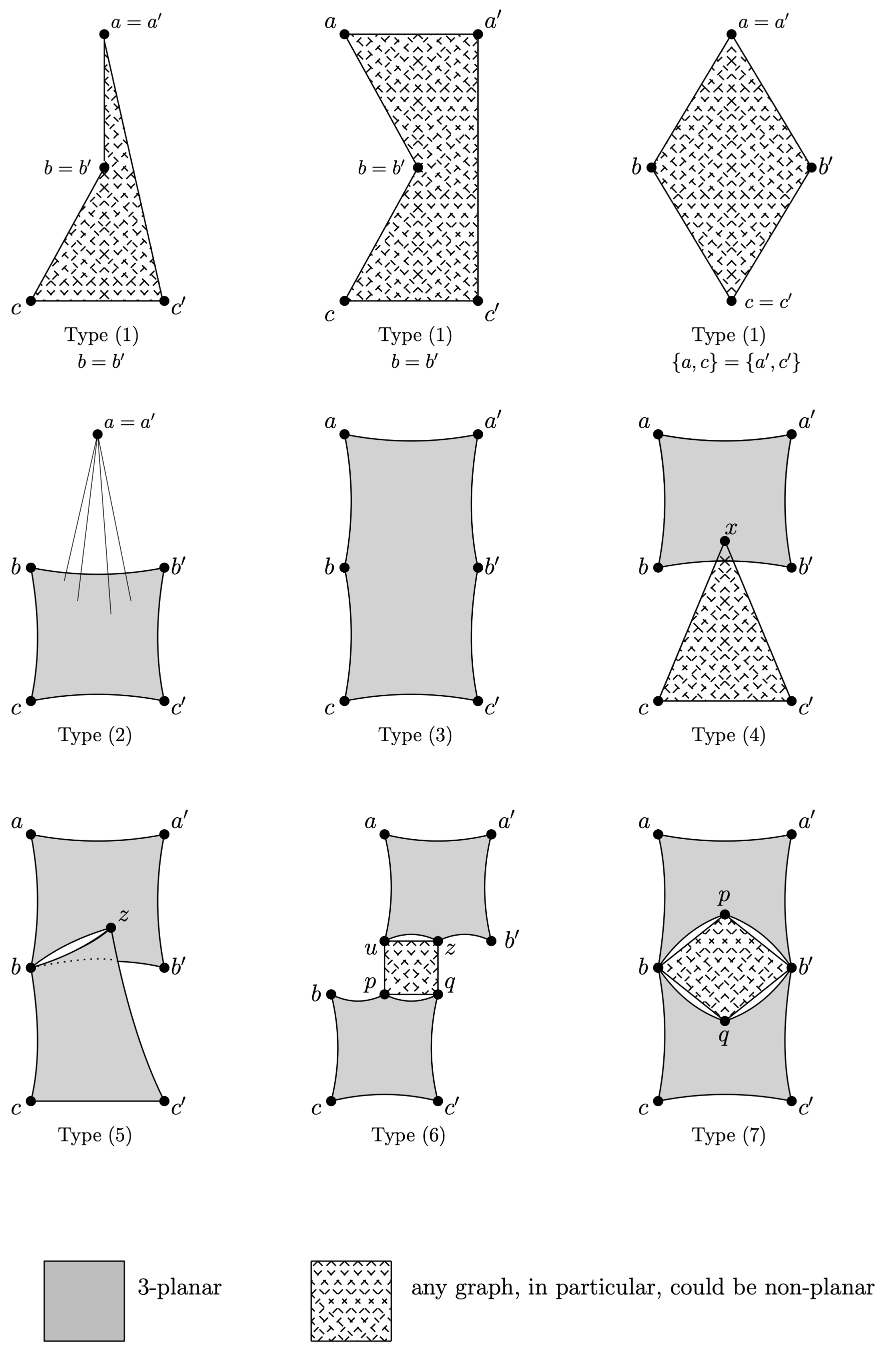

Figure 2.1: All types of rungs

(ii) $V(R_i \cap R_j) = \{x_i, v_i, y_i\} \cap \{x_{j-1}, v_{j-1}, y_{j-1}\}$ for $i, j \in [m]$ with $i < j$,

(iii) for any $i, j \in [m] \cup \{0\}$, if $x_i = x_j$ then $x_k = x_i$ for all $i \leq k \leq j$, if $v_i = v_j$ then $v_k = v_i$ for all $i \leq k \leq j$, and if $y_i = y_j$ then $y_k = y_i$ for all $i \leq k \leq j$, and

(iv) $L = (\bigcup_{i=1}^{m} R_i) + S$, where S consists of those edges of L each of which has both ends in $\{x_i, v_i, y_i\}$ for some $i \in [m] \cup \{0\}$.

Then $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$ is a ladder with rungs $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$, $i \in [m]$, or simply, a *ladder along* $v_0 \ldots v_m$. See Figure 2.2 for an example of ladder L. Note that in this example, edge $x_j v_j$ and edge $x_j y_j$ are in S.

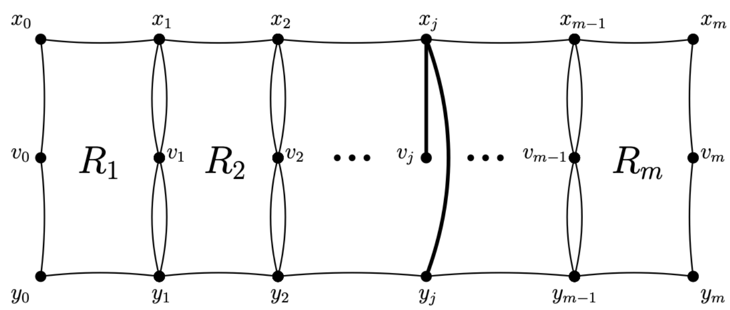

Figure 2.2: Example of ladder L

By definition, for any rung $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$, R_i has three disjoint paths from $\{x_{i-1}, v_{i-1}, y_{i-1}\}$ to $\{x_i, v_i, y_i\}$. So for any ladder $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$, L has three disjoint paths from $\{x_0, v_0, y_0\}$ to $\{x_m, v_m, y_m\}$.

For a sequence W, the *reduced sequence* of W is the sequence obtained from W by removing all but one consecutive identical elements. For example, the reduced sequence of $aaabcca$ is $abca$. We can now state the main result in [27, 28, 29].

Lemma 2.3.1 ([27, 28, 29]). *Let G be a graph, $\{a, b, c\} \subseteq V(G)$, and $\{a', b', c'\} \subseteq V(G)$ such that $\{a, b, c\} \neq \{a', b,' c'\}$. Then any three disjoint paths in G from $\{a, b, c\}$ to $\{a', b', c'\}$ must include a path from b to b' if, and only if, one of the following statements holds:*

(i) *G has a separation (G_1, G_2) of order at most 2 such that $\{a, b, c\} \subseteq V(G_1)$ and $\{a', b', c'\} \subseteq V(G_2)$.*

(ii) *$(G, (a, b, c), (a', b', c'))$ is a ladder.*

(iii) *G has a separation (J, L) such that $V(J \cap L) = \{w_0, \dots, w_n\}$, $(J, w_0, \dots, w_n)$ is 3-planar, $\{a, b, c\} \cup \{a', b', c'\} \subseteq V(L)$, $(L, (a, b, c), (a', b', c'))$ is a ladder along a sequence $v_0 \dots v_m$, where $v_0 = b$, $v_m = b'$, and $w_0 \dots w_n$ is the reduced sequence of $v_0 \dots v_m$.*

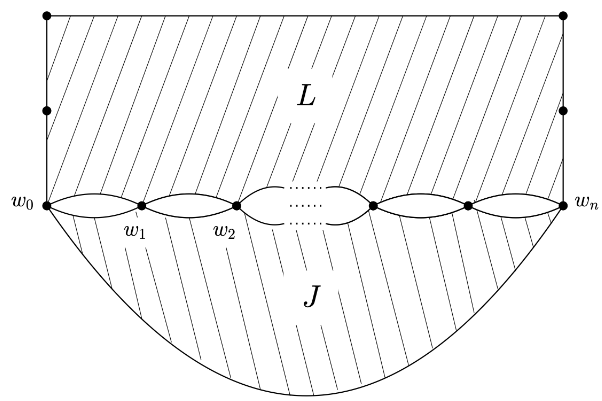

Figure 2.3: Structure (iii) of Yu's characterization for graph G

See Figure 2.3 for structure (iii) of Lemma 2.3.1, where L is a ladder (see Figure 2.2). Note that structure (ii) of the theorem is when $J = \emptyset$.

To help readers familiarize with the above concepts and for later applications, we prove the following properties of rungs.

Proposition 2.3.2. *For any rung $(G, (a, b, c), (a', b', c'))$, the following statements hold:*

(i) *$\{a, b, c\}$ and $\{a', b', c'\}$ are independent sets in G.*

(ii) *For any $x \in \{a, b, c\} \triangle \{a', b', c'\}$, $N_G(x) \neq \emptyset$. When $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$, $|N_G(x)| \geq 2$.*

(iii) *Suppose* $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$ *or* $|\{a,b,c\}\cup\{a',b',c'\}|=5$ *and* $b\neq b'$*. Then, for any* $x\in\{b,b'\}$*,* $N_G(x)\cap\{a,c,a',c'\}=\emptyset$*. Moreover, any three disjoint paths in* G *from* $\{a,b,c\}$ *to* $\{a',b',c'\}$ *must be from* a,b,c *to* a',b',c'*, respectively.*

Proof. Suppose (i) fails and without loss of generality, let $e\in E(G[\{a,b,c\}])$. Let $G_1=(\{a,b,c\},\{e\})$ and $G_2=G-e$. Then (G_1,G_2) is a separation in G contradicting the definition of a rung. Hence, (i) holds.

To prove (ii), let $x\in\{a,b,c\}\triangle\{a',b',c'\}$ and, without loss of generality, assume $x\in\{a,b,c\}\setminus\{a',b',c'\}$. Then, $N_G(x)\neq\emptyset$; otherwise $\{a,b,c\}\setminus\{x\}$ is a 2-cut in G separating $\{a,b,c\}$ from $\{a',b',c'\}$, contradicting the definition of a rung. Now suppose $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$. If $|N_G(x)|=1$ then $(\{a,b,c\}\setminus\{x\})\cup N_G(x)$ is a 3-cut in G separating $\{a,b,c\}$ from $\{a',b',c'\}$, contradicting the definition of a rung. So $|N_G(x)|\geq 2$.

We now prove (iii). First, suppose $|\{a,b,c\}\cup\{a',b',c'\}|=5$ and $b\neq b'$. By symmetry, we may assume $a=a'$ and $(G-a,b,b',c',c)$ is 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in G from $\{a,b,c\}$ to $\{a',b',c'\}$ must be from a,b,c to a',b',c', respectively. Now, $ba'\notin E(G)$ by (i) as $a=a'$, and $bc'\notin E(G)$ as $\{a,b,c'\}$ cannot be a cut in G separating $\{a,b,c\}$ from $\{a',b',c'\}$. Similarly, $b'c,b'a\notin E(G)$.

It remains to consider the case when $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$. Then $(G,(a,b,c),(a',b',c'))$ is a rung of type (3)-(7).

First, assume that $(G,(a,b,c),(a',b',c'))$ is of Type (3). Then (G,a,b,c,c',b',a') is 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in G from $\{a,b,c\}$ to $\{a',b',c'\}$ must be from a,b,c to a',b',c', respectively. Now, $ba',bc',b'a,b'c\notin E(G)$. For, otherwise, by symmetry, assume $bc'\in E(G)$. Then, $\{a,b,c'\}$ is a 3-cut in G separating $\{a,b,c\}$ from $\{a',b',c'\}$, a contradiction.

Next, assume $(G,(a,b,c),(a',b',c'))$ is of Type (4). Then G has a separation (G_1,G_2) such that $V(G_1\cap G_2)=\{x\}$, $\{a,a',b,b'\}\subseteq V(G_1)$, $\{c,c'\}\subseteq V(G_2)$, and (G_1,a,a',b',b) is 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in G from

$\{a,b,c\}$ to $\{a',b',c'\}$ must be from a,b,c to a',b',c', respectively. Now, we prove $ba', bc', b'a, b'c \notin E(G)$. By structure of G, $bc', b'c \notin E(G)$. So, by symmetry, suppose $ba' \in E(G)$. Then, $\{a',b,c\}$ is a 3-cut in G separating $\{a,b,c\}$ from $\{a',b',c'\}$, a contradiction.

Suppose $(G,(a,b,c),(a',b',c'))$ is of Type (5). Then G has a 2-separation (G_1,G_2) such that $V(G_1 \cap G_2) = \{x,b\}$, $\{a,a',b,b'\} \subseteq V(G_1)$, $\{c,c'\} \subseteq V(G_2)$, and $(G_1 + xb, a, a', b', b)$ and (G_2, c, c', x, b) are 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in G from $\{a,b,c\}$ to $\{a',b',c'\}$ must be from a,b,c to a',b',c', respectively. Now, we prove $ba', bc', b'a, b'c \notin E(G)$. By structure of G, $b'c \notin E(G)$. If $bc' \in E(G)$, then $\{a,b,c'\}$ is a 3-cut in G separating $\{a,b,c\}$ from $\{a',b',c'\}$, a contradiction. So, by symmetry, assume $ba' \in E(G)$. Then, $\{a',b,c\}$ is a 3-cut in G separating $\{a,b,c\}$ from $\{a',b',c'\}$, a contradiction.

Now assume $(G,(a,b,c),(a',b',c'))$ is of Type (6). Then there are pairwise edge disjoint subgraphs G_a, G_c, M of G such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{u,z\}$, $V(G_c \cap M) = \{p,q\}$, $V(G_a \cap G_c) = \emptyset$, $\{a,a',b'\} \subseteq V(G_a)$, $\{c,c',b\} \subseteq V(G_c)$, and (G_a, a, a', b', z, u) and (G_c, c', c, b, p, q) are 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in G from $\{a,b,c\}$ to $\{a',b',c'\}$ must be from a,b,c to a',b',c', respectively. Now, we prove $ba', bc', b'a, b'c \notin E(G)$. By structure of G, $ba', b'c \notin E(G)$. So, by symmetry, suppose $bc' \in E(G)$. Then, $\{a,b,c'\}$ is a 3-cut in G separating $\{a,b,c\}$ from $\{a',b',c'\}$, a contradiction.

Finally, assume $(G,(a,b,c),(a',b',c'))$ is of Type (7). Then there are pairwise edge disjoint subgraphs G_a, G_c, M of R such that $G = G_a \cup G_c \cup M$, $V(G_a \cap M) = \{b,b',q\}$, $V(G_c \cap M) = \{b,b',p\}$, $V(G_a \cap G_c) = \{b,b'\}$, $\{a,a'\} \subseteq V(G_a)$, $\{c,c'\} \subseteq V(G_c)$, and (G_a, a, a', b', q, b) and (G_c, c', c, b, p, b') are 3-planar. By applying Lemma 2.2.1, we see that any three disjoint paths in G from $\{a,b,c\}$ to $\{a',b',c'\}$ must be from a,b,c to a',b',c', respectively. Now, we prove $ba', bc', b'a, b'c \notin E(G)$. For, otherwise, by symmetry, assume $bc' \in E(G)$. Then, $\{a,b,c'\}$ is a 3-cut in G separating $\{a,b,c\}$ from $\{a',b',c'\}$, a contradiction. □

CHAPTER 3

FRAMES AND CONSTRAINTS

Let G be a graph and $a_1, a_2, b_2, b_2 \in V(G)$ be distinct. Recall that by Corollary 2.1.3 in Chapter 2, (G, a_1, a_2, b_1, b_2) is *feasible* if G contains three pairwise independent paths A, B_1, B_2, such that A is from a_1 to a_2, and B_i is from b_1 to b_2 for $i \in [2]$.

Our main Theorem 1.3.1 gives a structural result on infeasible 8-connected graphs. In this chapter, we give the decomposition of G into edge disjoint subgraphs A_1, A_2 and H. Suppose (G, a_1, a_2, b_1, b_2) is infeasible.

In section 3.1, we find the subgraphs A_1, A_2, H in G, and prove that A_1 and A_2 are both planar by applying Lemma 2.2.1 on 2-linked graphs.

In section 3.2, by choosing favorite A_1 and A_2 and applying Lemma 2.3.1 on three special paths, we show that there exists $w \in V(H)$ such that $H - w$ is a ladder of rungs.

We give an illustration of the structure of G in Figure 3.1.

3.1 Frame and its properties

For any three independent b_1-b_2 paths B_1, B_2, B_3 in $G - \{a_1, a_2\}$, we use $A_i(B_1 \cup B_2 \cup B_3)$, for $i \in [2]$, to denote the $(B_1 \cup B_2 \cup B_3)$-bridge of G containing a_i.

We say that B_1, B_2, B_3 form a *frame* in (G, a_1, a_2, b_1, b_2), if they satisfy (C1)-(C4), up to relabeling a_1 and a_2 and relabeling b_1 and b_2.

(C1) B_1, B_2, B_3 are independent induced b_1-b_2 paths in $G - \{a_1, a_2\}$,

(C2) $A_1(B_1 \cup B_2 \cup B_3)$ has all its attachments on B_3,

(C3) $A_2(B_1 \cup B_2 \cup B_3)$ has attachments on both $B_1(b_1, b_2)$ and $B_2(b_1, b_2)$, and

(C4) subject to (C1)-(C3), $A_1(B_1 \cup B_2 \cup B_3)$ is maximal.

In this section, we prove the existence of such a frame in 8-connected infeasible graphs, as well as some related properties. Since G is 8-connected, by Theorem 2.1.1, there exist three independent b_1-b_2 paths in $G - \{a_1, a_2\}$. Take such three paths B_1, B_2, B_3 to be induced; so (C1) holds.

Now, we show that (C2) holds for any three independent b_1-b_2 paths B_1, B_2, B_3 in $G - \{a_1, a_2\}$.

Lemma 3.1.1. *Suppose (G, a_1, a_2, b_1, b_2) is infeasible and B_1, B_2, B_3 are three independent b_1-b_2 paths in $G - \{a_1, a_2\}$. Then there exist $i \in [2]$ and $j \in [3]$ such that $A_i(B_1 \cup B_2 \cup B_3)$ has all its attachements contained in B_j.*

Proof. For, suppose such i, j do not exist. Then there exists some $k \in [2]$ such that, for $s \in [2]$, $A_s(B_1 \cup B_2 \cup B_3)$ has an attachement $a'_s \in V(B_k(b_1, b_2))$. Let Q_s denote an a_s-a'_s path in $A_s(B_1 \cup B_2 \cup B_3)$ internally disjoint from $B_1 \cup B_2 \cup B_3$. Without loss of generality, let $k = 1$. Then $B_2, B_3, Q_1 \cup B_1[a'_1, a'_2] \cup Q_2$ show that (G, a_1, a_2, b_1, b_2) is feasible, a contradiction. □

Next, we show that if B_1, B_2, B_3 satisfy (C1)-(C3), then $(A_1(B_1 \cup B_2 \cup B_3)) \cup B_3, B_3 + a_2)$ is planar.

Lemma 3.1.2. *Suppose (G, a_1, a_2, b_1, b_2) is infeasible and G is 4-connected, and suppose B_1, B_2, B_3 are independent b_1-b_2 paths in $G - \{a_1, a_2\}$. For any $i \in [2]$ and $j \in [3]$, if $A_i(B_1 \cup B_2 \cup B_3) \cap (B_1 \cup B_2 \cup B_3) \subseteq B_j$ and $A_{3-i}(B_1 \cup B_2 \cup B_3)$ intersects $B_k(b_1, b_2)$ for both $k \in [3] \setminus \{j\}$, then $(A_i(B_1 \cup B_2 \cup B_3) \cup B_j, B_j + a_i)$ is planar.*

Proof. Without loss of generality, we may assume $i = 1$ and $j = 3$. Let H be the graph obtained from G by contracting $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$ to a single vertex w.

Suppose there exist disjoint paths P_1, P_2 in H from b_1, a_1 to b_2, w, respectively. Let $w' \in N(w) \cap V(P_2) \subseteq V(B_3)$. By symmetry between B_1 and B_2, we may assume that G has a path Q from w' to B_1 and internally disjoint from $A_1(B_1 \cup B_2 \cup B_3) \cup B_1 \cup B_2 \cup B_3$. Since $A_2(B_1 \cup B_2 \cup B_3)$ has attachments on $B_1(b_1, b_2)$, it contains an a_2-w' path, say P, internally disjoint from $B_1 \cup B_2 \cup B_3$. Now $(P_2 - w) \cup Q \cup B_1(b_1, b_2) \cup P$ contains an a_1-a_2 path independent of P_1 and B_2, which, together with P_1 and B_2, shows that (G, a_1, a_2, b_1, b_2) is feasible, a contradiction.

So such paths P_1, P_2 do not exist in H. By Lemma 2.2.1, $(H, \mathcal{A}, \{w, b_1, a_1, b_2\})$ is 3-planar, where $\mathcal{A}$ is a collection of disjoint subsets of $V(H)\backslash\{w, b_1, a_1, b_2\}$. If $\mathcal{A} = \emptyset$, we are done. Hence we may assume there exists $A \in \mathcal{A}$. Since $|N_H(A)| \leq 3$ and G is 4-connected, $V(B_3) \cap A \neq \emptyset$. Therefore, $w \in N_H(A)$ and, thus, $|N_H(A) \cap V(B_3)| = 2$. Hence, $H[A] \subseteq B_3$ by definition of $A_1(B_1 \cup B_2 \cup B_3)$ and B_3. This implies $(H[A \cup N_H(A)], N_H(A))$ is planar for all $A \in \mathcal{A}$. Hence, $(A_i(B_1 \cup B_2 \cup B_3) \cup B_j, B_j + a_i)$ is planar. □

Before we prove the existence of a frame, we need the following lemma for 8-connected graphs when $(A_1(B_1 \cup B_2 \cup B_3), B_3 + a_1)$ is planar.

Lemma 3.1.3. *Suppose (G, a_1, a_2, b_1, b_2) is infeasible and G is 8-connected. Let B_1, B_2, B_3 be three independent induced b_1-b_2 paths in $G - \{a_1, a_2\}$ such that $(A_1(B_1 \cup B_2 \cup B_3), B_3 + a_1)$ is planar. Then there exists $w \in V(A_1(B_1 \cup B_2 \cup B_3)) \cap V(B_3(b_1, b_2))$ such that w is not contained in any 3-cut of $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - \{a_2\}$ separating b_1 from b_2.*

Proof. For convenience, let $A_1 := A_1(B_1 \cup B_2 \cup B_3)$ and $H = G - (A_1 - B_3) - \{a_2\}$. Suppose such w does not exist. Then every vertex in $V(A_1) \cap V(B_3(b_1, b_2))$ is contained in a 3-cut of H separating b_1 from b_2. Let $V(A_1) \cap V(B_3(b_1, b_2)) = \{w_1, \ldots, w_m\}$ such that $b_1, w_1, \ldots, w_m, b_2$ occur on B_3 in order. For $i \in [m]$, let $u_i \in V(B_1(b_1, b_2)), v_i \in V(B_2(b_1, b_2))$ such that $T_i := \{u_i, v_i, w_i\}$ is a 3-cut of H separating b_1 from b_2. We may assume that

(1) for all $i \in [m-1]$, b_1, u_i, u_{i+1}, b_2 occur on B_1 in order and b_1, v_i, v_{i+1}, b_2 occur on B_2 in order.

To see this, we choose T_i such that the T_i-bridge of H containing b_1, denoted by H_i, is minimal. Suppose (1) fails. Then by symmetry between B_1 and B_2, we may assume that for some i, b_1, u_{i+1}, u_i, b_2 are in order on B_1.

First, suppose b_1, v_i, v_{i+1}, b_2 occur on B_2 in order. By the choice of $\{u_i, v_i, w_i\}$, $\{u_{i+1}, v_i, w_i\}$ is not a cut in H separating b_1 from b_2. Hence, there exists a b_1-u_i path P in $H_i - \{u_{i+1}, v_i, w_i\}$. But then $P \cup B_1[u_i, b_2]$ is a b_1-b_2 path in $H - T_{i+1}$, a contradiction.

Now assume that b_1, v_{i+1}, v_i, b_2 are in order on B_2. By the choice of $\{u_i, v_i, w_i\}$, $\{u_{i+1}, v_{i+1}, w_i\}$ is not a cut in H separating b_1 from b_2. So there exists a b_1-w_{i+1} path Q in $H_{i+1} - \{u_{i+1}, v_{i+1}, w_i\}$. But again, $Q \cup B_1[w_{i+1}, b_2]$ is a b_1-b_2 path in $H - T_i$, a contradiction.

(2) $V(H) = \{b_1, b_2\} \cup \left(\bigcup_{i\in[m]} T_i\right)$.

Otherwise suppose there exists $x \in V(H)$ such that $x \notin \{b_1, b_2\} \cup \left(\bigcup_{i\in[m]} T_i\right)$. Then, x is not contained in the T_1-bridge of H containing b_1; as otherwise, $T_1 \cup \{b_1, a_2\}$ is a 5-cut in G separating x from b_2, a contradiction. Similarly, x is not contained in the T_m-bridge of H containing b_2. Hence, there exists $i \in [m]$ such that x is contained in both the T_{i+1}-bridge of H containing b_1 and the T_i-bridge of H containing b_2. Now $T_i \cup T_{i+1} \cup \{a_2\}$ is a cut in G of order at most 7 and separates x from $\{a_1, a_2\}$, a contradiction.

Since $d_G(b_i) \geq 8$ for both $i \in [2]$, it follows from (2) that

(3) $d_{A_1}(b_i) \geq 5$ for $i \in [2]$.

(4) There exists $i \in [m]$ such that $d_H(w_i) \geq 7$.

Suppose for a contradiction, $d_H(w_i) < 7$ for all $i \in [m]$. Then, since G is 8-connected, $d_{A_1}(w_i) \geq 2$ for all $i \in [m]$.

Let H' be the graph obtained from $A_1 \cup B_3$ by adding a new vertex a and an edge from a to each vertex in B_3. Then, (H', a_1, b_1, a, b_2) is planar. We take an embedding of H' in the plane such that a_1, b_1, a, b_2 occur on the outer cycle of H' in clockwise order. Let $F(H')$ denote the set of faces of H'. For convenience, for the rest proof of the lemma, we write $d(x) := d_{H'}(x)$ for $x \in V(H') \cup F(H')$. When $x \in F(H')$, $d(x)$ is the number of edges incident to x.

Note that $d(a) = |V(B_3)|$, $d(w) \geq 5$ for all $w \in V(B_3)$, and $d(v) \geq 8$ for all $v \in V(A_1 - B_3)$. Moreover, $d(a) \geq 8$; otherwise $V(B_3)$ is a cut of size ≤ 7 in G separating a_1 from a_2, a contradiction.

We now apply the discharging method to H'. First, define $\sigma(x) := d(x) - 4$ as the charge of x for all $x \in V(H') \cup F(H')$. Then, $\sigma(x) \geq -1$ for all $x \in F(H')$, $\sigma(x) \geq 1$ for all $x \in V(B_3)$, and $\sigma(x) \geq 4$ for all $x \in V(A_1 - B_3) \cup \{a\}$. So $\sigma(x) < 0$ only if $x \in F(H')$ is a triangular face of H'. By Euler's formula,

$$\sum_{x \in V(H') \cup F(H')} \sigma(x) = -8.$$

Next, we move charges from vertices to faces as follows: For every $v \in V(H' - B_3)$, we discharge $\frac{d(v)-4}{d(v)} \geq \frac{1}{2}$ (since $d(v) \geq 8$) from v to each of the triangular faces of H' incident to v. So the new charge $\tau(v)$ for each vertex v satisfies

$$\tau(v) \geq \sigma(v) - (d(v) - 4) \geq 0,$$

and the new charge $\tau(f)$ for each triangular face f with at most one vertex on B_3 satisfies

$$\tau(f) \geq \sigma(f) + 2 \cdot \frac{1}{2} \geq 0.$$

For each $w \in V(B_3)$, we perform the discharging as follows. If $d(w) \geq 6$, we discharge $\frac{d(w)-4}{d(w)} \geq \frac{1}{3}$ from w to each of the triangular faces incident to w; the new charge of w is

$$\tau(w) \geq \sigma(w) - (d(w) - 4) \geq 0.$$

If $d(w) = 5$, we discharge $\frac{1}{4}$ from w to each triangular face f incident to w and having two vertices from B_3 (there are at most four such faces); so the new charge of w is

$$\tau(w) \geq \sigma(w) - 4 \cdot \frac{1}{4} = 1 - 1 = 0.$$

Now, consider any traingular face f with two vertices on B_3. f gets at least $\frac{1}{2}$ from its vertex in $V(A_1 - B_3)$ and $\frac{1}{4}$ from each of its vertices in $V(B_3)$. So the new charge of f is

$$\tau(f) \geq \sigma(f) + \frac{1}{2} + 2 \cdot \frac{1}{4} = 0.$$

Note that the infinity face of H', say f_0, is incident to at least 4 vertices, so $\tau(f_0) \geq 0$. Thus, $\sum_{x \in V(H') \cup F(H')} \tau(x) \geq 0$. Since the total charge is preserved, we have

$$0 \leq \sum_{x \in V(H') \cup F(H')} \tau(x) = \sum_{x \in V(H') \cup F(H')} \sigma(x) = -8,$$

a contradiction. So we have (4).

By (4), let $j \in [m]$ be such that $d_H(w_j) \geq 7$. By (2) and the pigeonhole principal, $|V(B_i(b_1, b_2)) \cap N_G(w_j)| \geq 3$ for some $i \in [2]$. By symmetry, assume $|V(B_1(b_1, b_2)) \cap N_G(w_j)| \geq 3$.

(5) $N_G(w_j) \cap V(B_1) = \{u_{j-1}, u_j, u_{j+1}\}$ is disjoint from $\{b_1, b_2\}$ and u_{j-1}, u_j, u_{j+1} are pairwise distinct, $N_G(u_j) \cap V(B_2) \subseteq \{v_{j-1}, v_j, v_{j+1}\}$, $N_G(u_j) \cap V(B_3) \subseteq \{w_{j-1}, w_j, w_{j+1}\}$, and if w_{j-1}, w_j, w_{j+1} are pairwise distinct then $N_G(v_j) \cap V(B_1) \subseteq \{u_{j-1}, u_j, u_{j+1}\}$.

Let $x \in N_G(w_j) \cap V(B_1)$. If $x \in V(B(u_{j+1}, b_2])$ then $w_j x$ contradicts the existence of the 3-cut T_{j+1} of H; and if $x \in V(B_1[b_1, u_{j-1}))$ then $w_j x$ contradicts the existence of the 3-cut T_{i-1} of H. So by (2), $N_G(w_j) \cap V(B_1) = \{u_{j-1}, u_j, u_{j+1}\}$, and u_{j-1}, u_j, u_{j+1} are pairwise distinct.

Now, consider $N_G(u_j)$. Clearly, $b_1, b_2 \notin N_G(u_j)$ as B_1 is induced path in G.

Since u_{j-1}, u_j, u_{j+1} are pairwise distinct, similar arguments in last paragraph shows $N_G(u_j) \cap V(B_2) \subseteq \{v_{j-1}, v_j, v_{j+1}\}$ and $N_G(u_j) \cap V(B_3) \subseteq \{w_{j-1}, w_j, w_{j+1}\}$. Similarly, if w_{j-1}, w_j, w_{j+1} are pairwise distinct then $N_G(v_j) \cap V(B_1) \subseteq \{u_{j-1}, u_j, u_{j+1}\}$.

(6) $a_2 \notin N_G(u_j)$.

Suppose $u_j a_2 \in E(G)$. Then, $N_G(u_j) \cap V(B_3) = \{w_j\}$. Otherwise there exists $w_l \in V(B_3(b_1, b_2)) - \{w_j\}$ such that $u_j w_l \in E(G)$. By symmetry, we may assume $l < j$. Let P be a w_l-a_1 path in A_1 independent of B_3. Then, $P' = P \cup w_l u_j a_2$ is an a_1-a_2 path, and, $B_1[b_1, u_{j-1}] \cup u_{j-1} w_j \cup B_3[w_j, b_2]$ and B_2 are two disjoint b_1-b_2 paths in $G - P'$, showing that (G, a_1, a_2, b_1, b_2) is feasible, a contradiction. But then, by (2), $d_G(u_j) \leq |\{u_{j-1}, u_{j+1}, v_{j-1}, v_j, v_{j+1}, w_j, a_2\}| \leq 7$, a contradiction.

By (5) and (6), $N_G(u_j) = \{u_{j-1}, u_{j+1}, w_{j-1}, w_j, w_{j+1}, v_{j-1}, v_j, v_{j+1}\}$. Note that $a_2 \in N_G(v_j)$, to avoid 7-cut $\{u_{j-1}, u_{j+1}, w_{j-1}, w_j, w_{j+1}, v_{j-1}, v_{j+1}\}$ in G separating $\{u_j, v_j\}$ from $\{b_1, b_2\}$. Since $d_G(v_j) \geq 8$, there exists $w_l \in V(B_3(b_1, b_2)) \setminus \{w_j\}$ such that $v_j w_l \in E(G)$. By symmetry, we may assume $l < j$. Let P be a w_l-a_1 path in A_1 independent of B_3. Then, $P' = P \cup w_l v_j a_2$ is an a_1-a_2 path, and $B_1[b_1, u_{j-1}] \cup u_{j-1} w_j \cup B_3[w_j, b_2]$ and $B_2[b_1, v_{j-1}] \cup v_{j-1} u_j \cup B_1[u_j, b_2]$ are two independent b_1-b_2 paths in $G - P'$. This shows that (G, a_1, a_2, b_1, b_2) is feasible, a contradiction. □

Corollary 3.1.4. *Suppose* (G, a_1, a_2, b_1, b_2) *is infeasible and* G *is* 8*-connected. Then* $G - \{a_1, a_2\}$ *contains three independent induced* b_1-b_2 *paths* B_1, B_2, B_3 *such that for some* $i \in [2]$, $A_i(B_1 \cup B_2 \cup B_3)$ *has all its attachments contained in* B_3 *and* $A_{3-i}(B_1 \cup B_2 \cup B_3)$ *has attachments on both* $B_1(b_1, b_2)$ *and* $B_2(b_1, b_2)$.

Proof. Let B_1, B_2, B_3 be three independent induced b_1-b_2 paths in $G - \{a_1, a_2\}$. Choose B_1, B_2, B_3 so that $A_1(B_1 \cup B_2 \cup B_3)$ is maximal.

We may assume that $A_1(B_1 \cup B_2 \cup B_3)$ has all its attachments on B_3. For, otherwise, since (G, a_1, a_2, b_1, b_2) is infeasible, $A_2(B_1 \cup B_2 \cup B_3)$ has all its attachments on B_j for exactly one $j \in [2]$. Then by relabeling, we see that B_1, B_2, B_3 are desired paths.

Let $H := G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - a_2$,. By the maximality of $A_1(B_1 \cup B_2 \cup B_3)$, we see that each $w \in V(A_1(B_1 \cup B_2 \cup B_3)) \cap V(B_3(b_1, b_2))$ is contained in a 3-cut in H separating b_1 from b_2.

Let G' be obtained from $G - a_2$ by contracting $G - (A_1(B_1 \cup B_2 \cup B_3) \cup B_3)$ to a single vertex a_2'. Suppose G' contains disjoint paths Q_a, Q_b from a_1, b_1 to a_2', b_2, respectively. Then the independent b_1-b_2 paths B_1, B_2, Q_b give the desired paths, as (G, a_1, a_2, b_1, b_2) is infeasible and $A_1(B_1 \cup B_2 \cup Q_b)$ has attachments on both Q_b and $B_1(b_1, b_2) \cup B_2(b_1, b_2)$.

So, such paths do not exist in G'. Hence, by Lemma 2.2.1, $(G', \mathcal{A}, a_1, b_1, a_2', b_2)$ is 3-planar, where $\mathcal{A}$ is a collection of disjoint subsets of $V(G') \setminus \{a_1, b_1, a_2', b_2\}$.

We claim that $(A_1(B_1 \cup B_2 \cup B_3) \cup B_3, B_3 + a_1)$ is planar. If $\mathcal{A} = \emptyset$, we are done. Hence we may assume there exists $A \in \mathcal{A}$. Since $|N_{G'}(A)| \leq 3$ and G is 8-connected, $V(B_3) \cap A \neq \emptyset$. Therefore, $a_2' \in N_{G'}(A)$ and, thus, $|N_{G'}(A) \cap V(B_3)| = 2$. Hence, $G'[A] \subseteq B_3$ by definition of $A_1(B_1 \cup B_2 \cup B_3)$ and B_3. This implies $(G'[A \cup N_{G'}(A)], N_{G'}(A))$ is planar for all $A \in \mathcal{A}$. So $(A_1(B_1 \cup B_2 \cup B_3) \cup B_3, B_3 + a_1)$ is planar.

This is a contradiction to Lemma 3.1.3. □

Hence, by Corollary 3.1.4, we may choose three independent b_1-b_2 paths B_1, B_2, B_3 in $G - \{a_1, a_2\}$ which satisfy (C1)-(C4). Moreover, by Lemma 3.1.2, $(A_1(B_1 \cup B_2 \cup B_3), B_3 + a_1)$ is planar. Let

$$S := V(A_1(B_1 \cup B_2 \cup B_3)) \cap V(B_3).$$

By Lemma 3.1.3, there exists $w \in S \setminus \{b_1, b_2\}$ such that $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - \{a_2, w\}$ has three independent b_1-b_2 paths P_1, P_2, P_3.

3.2 Ladders and rungs

In this section, we show that $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3) - \{a_2, w\}$ can be obtained from a plane graph and a ladder (which consists of rungs as defined in section 2.3) by gluing them along a path.

Let $A_2(P_1 \cup P_2 \cup P_3)$ be the $(P_1 \cup P_2 \cup P_3)$-bridge of G containing a_2. We choose $B_1, B_2, B_3, w, P_1, P_2, P_3$, such that

(C5) subject to (C1)-(C4), $A_2(P_1 \cup P_2 \cup P_3)$ is maximal.

By the maximality of $A_1(B_1 \cup B_2 \cup B_3)$ (see (C4)), all attachments of $A_2(P_1 \cup P_2 \cup P_3)$ are contained in exactly one of P_1, P_2, P_3, as otherwise, if $A_1(P_1 \cup P_2 \cup P_3)$ has attachments on at least two of P_i's for $i \in [3]$, (G, a_1, a_2, b_1, b_2) is feasible; and if $A_1(P_1 \cup P_2 \cup P_3)$ has attachments only on one P_j for some $j \in [3]$, P_1, P_2, P_3, w would contradict the choice of B_1, B_2, B_3, w. So we may assume that

(C6) subject to (C1)-(C5), all attachments of $A_2(P_1 \cup P_2 \cup P_3)$ on $P_1 \cup P_2 \cup P_3$ are contained in P_3.

Let

$$H = G - (A_1(B_1 \cup B_2 \cup B_3) - (B_3 - w)) - (A_2(P_1 \cup P_2 \cup P_3) - P_3).$$

Label the vertices in $V(A_2(P_1 \cup P_2 \cup P_3)) \cap V(P_3)$ as $u_1, \ldots, u_m$ in order from b_1 to b_2. Then by the maximality of $A_2(P_1 \cup P_2 \cup P_3)$ (see (C5)), each u_i is in a 3-cut of H separating b_1 from b_2.

Lemma 3.2.1. *For $i \in [m]$, there are 3-cuts $T_i = \{u_i, v_i, w_i\}$ in H separating b_1 from b_2 such that $b_1, u_1, \ldots, u_m, b_2$ occur on P_3 in order, $b_1, v_1, \ldots, v_m, b_2$ occur on P_2 in order, and $b_1, w_1, \ldots, w_m, b_2$ occur on P_1 in order.*

Proof. The proof is the same as (1) in the proof of Lemma 3.1.3. □

Let H_1 denote the T_1-bridge of H containing b_1, and H_{m+1} denote the T_m-bridge of H containing b_2. Let $Int(H_1) = V(H_1) \setminus (T_1 \cup \{b_1\})$ and $Int(H_{m+1}) = V(H_{m+1}) \setminus (T_m \cup \{b_2\})$. For $i \in [m] \setminus \{1\}$, let H_i denote the union of those $(T_{i-1} \cup T_i)$-bridges of H containing the subpaths of P_j between T_{i-1} and T_i for $j \in [3]$.

Lemma 3.2.2. *For $i \in [m] \setminus \{1\}$, any three disjoint paths in H_i from T_{i-1} to T_i contains a u_{i-1}-u_i path.*

Proof. Suppose for some $i \in [m] \setminus \{1\}$, H_i has three disjoints paths Q_u, Q_v, Q_w from $u_{i-1}, v_{i-1}, w_{i-1}$, respectively, to T_i, with no u_{i-1}-u_i path. Then, $u_i \in V(Q_v \cup Q_w)$. Let P_1', P_2', P_3' be formed by taking the union of Q_u, Q_v, Q_w, respectively, with the subpaths of P_1, P_2, P_3 outside of H_i. We may assume that $P_1' \supseteq Q_u$ and P_2' contains Q_v (if $u_i \in V(Q_v)$) or Q_w (if $u_i \in V(Q_w)$). Then, P_1', P_2', P_3' are three independent b_1-b_2 paths in $G-(A_1(B_1 \cup B_2 \cup B_3) - B_3) - \{a_2, w\}$ such that $A_2(P_1' \cup P_2' \cup P_3')$ has attachments on P_1' and P_2'. Hence, P_1', P_2', P_3', w contradict the choice of B_1, B_2, B_3, w, or (G, a_1, a_2, b_1, b_2) is feasible, a contradiction. □

Thus, by Lemma 2.3.1, $H_i = J_i \cup L_i$, where J_i is planar and L_i is a ladder from $(v_{i-1}, u_{i-1}, w_{i-1})$ to (v_i, u_i, w_i). Let

$$L^* = H_1 \cup H_{m+1} \cup (\bigcup_{i=2}^{m} L_i).$$

We further choose P_1, P_2, P_3 such that

(C7) subject to (C1)-(C6), $(P_1 \cup P_2 \cup P_3) \cap H_i \subseteq L_i$ for $i \in [m] \setminus \{1\}$ (and hence, $P_1 \cup P_2 \cup P_3 \subseteq L^*$), and $A_2'(P_1 \cup P_2 \cup P_3) := A_2(P_1 \cup P_2 \cup P_3) \cup J_2 \cup \ldots \cup J_m$ is maximal.

See the following Figure 3.1 for an illustration for all the above results.

The following observation will be convenient.

Observation 3.2.3. *There exists no path in H from $S \setminus \{b_1, b_2\}$ to $P_3(b_1, b_2)$ disjoint from $P_1 \cup P_2$.*

Proof. For, suppose Q is a path between $s \in S \setminus \{b_1, b_2\}$ and $t \in V(P_3(b_1, b_2))$ internally disjoint from $P_1 \cup P_2$. We may further assume Q is independent of $A_1(B_1 \cup B_2 \cup B_3) \cup$

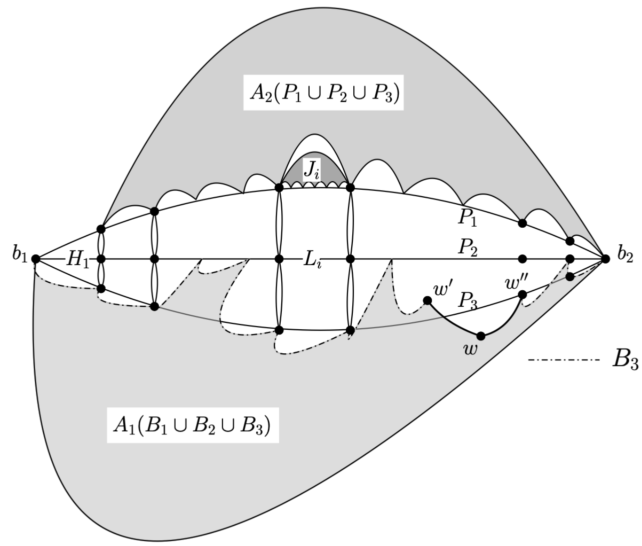

Figure 3.1: Structure of infeasible (G, a_1, a_2, b_1, b_2)

$A_2(P_1 \cup P_2 \cup P_3)$. Note that $A_1(B_1 \cup B_2 \cup B_3)$ contains an a_1-s path independent of H. Let $b \in V(P_3(b_1, b_2)) \cap V(A_2(P_1 \cup P_2 \cup P_3))$. Then $A_2(P_1 \cup P_2 \cup P_3)$ contains a a_2-b path Q_2 independent of P_3. Now $Q_1 \cup Q \cup P_3(t, b) \cup Q_2$ is an a_1-a_2 path disjoint from $P_1 \cup P_2$. This shows that (G, a_1, a_2, b_1, b_2) is feasible, a contradiction. □

We conclude this section with a useful lemma concerning the rungs $(R, (a, b, c), (a', b', c'))$ in L^* with $|\partial R| = 6$ or $|\partial R| = 5$ and $b \neq b'$.

Lemma 3.2.4. *$(R, (a, b, c), (a', b', c'))$ in L^* with $|\partial R| = 6$ or $|\partial R| = 5$ and $b \neq b'$. Then*

(a) any three disjoint paths in R from $\{a, b, c\}$ to $\{a', b', c'\}$ must be from a, b, c to a', b', c', respectively, and

(b) there are disjoint induced paths P_a, P_c in $R - \{b, b'\}$ from a, c to a', c', respectively, such that $R - (P_a \cup P_c)$ is connected and $S \cap V(R) \subseteq V(P_a \cup P_c)$.

Proof. By (iii) of Proposition 2.3.2, we have (a). So there are disjoint induced paths P_a, P_c in $R - \{b, b'\}$ from a, c to a', c', respectively, such that $\{b, b'\}$ is contained in a $(P_a \cup P_c)$-bridge of R, say R_b. Note that $(S \cup N(w)) \cap V(R_b) = \emptyset$ by Observation 3.2.3.

To prove (b), let us assume by symmetry that if $|\partial R| = 5$ then $a = a'$. If R_b is the only component of $R - (P_a \cup P_c)$ then (b) holds; for otherwise (G, a_1, a_2, b_1, b_2) would be feasible as $A_2(P_1 \cup P_2 \cup P_3)$ has attachments on $P_3(b_1, b_2)$. For any component X of $R - (P_a \cup P_c)$ with $X \neq R_b$, it follows from 3-planarity of R or $R - a'$ (when $a = a'$) that we may assume X has neighbors only on P_c unless $a = a'$. Moreover, the two neighbors of X on P_c that are furtherest apart form a cut (with a if $a = a'$) in R, and these two neighbors might be the same.

Hence, let $\{y_i, z_i\}$ be the cut of size at most 2 in R (or $R - a$ when $a = a'$) separating R_b from $P_c[y_i, z_i]$ and at least one vertex of $R - (P_a \cup P_c)$, such that $P_c[y_i, z_i]$ are maximal. Then by planarity, we may assume that $c, y_1, z_1, \ldots, y_t, z_t, c'$ occur on P_c in order. Let X_i denote the union of $P[y_i, z_i]$ and all $(P_a \cup P_c)$-bridges of R with all attachments contained in $P_c[y_i, z_i]$ (or $P_c[y_i, z_i] + a$ if $a = a'$). Let $X_i^* = R[X_i + w]$ and $Int(X_i^*) = V(X_i^*) \setminus \{a, w, y_i, z_i\}$. Note that $X_i^* - (P_a \cup P_c) \neq \emptyset$; so $V(B_3) \cap Int(X_i^*) \neq \emptyset$ (to avoid the cut $\{a, w, y_i.z_i\}$). Let $r_1, r_2 \in V(B_3) \cap \{a, w, y_i, z_i\}$ with $N(r_i) \cap Int(X_i^*) \neq \emptyset$ for $i \in [2]$, such that $B_3[r_1, r_2]$ is maximal.

First, we claim that $\{r_1, r_2\} \neq \{y_i, z_i\}$ for $i \in [t]$. For, suppose $\{r_1, r_2\} = \{y_i, z_i\}$ for some $i \in [t]$. Then $B_i \cap Int(X_i^*) = \emptyset$ for $i \in [2]$. If there exists $s \in (S \cap Int(X_i^*)) \setminus V(P_c[y_i, z_i])$ then letting $B_3' := (B_3 - B_3(y_i, z_i)) \cup P_c[y_i, z_i]$ we see that $A_1(B_1 \cup B_2 \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$, contradicting (C4). So $S \cap Int(X_i^*) \subseteq V(P_c[y_i, z_i])$. Now let Y be a $(P_a \cup P_c)$-bridge of R contained in X_i^* and $y, z \in V(Y) \cap V(P_a \cup P_c)$ with $P_c[y, z]$ maximal, such that no other $(P_a \cup P_c)$-bridges of R has attachments in $P_c(y, z)$. Note Y is well defined because of planarity. Now there exists $s \in S \cap V(P_c(y, z))$ to avoid the cut $\{a, w, y, z\}$. Let B_3' denote the union of $(B_3 - B_3(y, z))$ and an induced y-z path in $Y - V(P_c(y, z))$. Then $A_1(B_1 \cup B_2 \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$, contradicting (C4).

Thus, for any $i \in [t]$, we have $w' \in Int(X_i^*)$, or $a = a'$ and B_3 enters $Int(X_i^*)$ at a; for, otherwise, $B_3 \cap X_i^*$ would be a y_i-z_i path. This, in particular, implies $t \leq 2$.

Case 1. $a \neq a'$.

Then $t = 1$. First, suppose $S \cap Int(R) \subseteq V(X_1^*)$. Let R' be obtained from $R^* - Int(X_1^*)$ by adding edges $\{ab, bc, y_1z_1, c'b', b'a'\}$ (or $\{ab, bc, c'b', b'a'\}$ when $y_1 = z_1$), as well as edges from w to $K := \{a, b, c, y_1, z_1, c', b', a'\}$. Then R' is a planar graph. Let $k = |K|$ and $m = |V(R') \setminus (K \cup \{w\})|$. By the Hand-shaking lemma and Euler's formula, we see that $k \times 4 + k + 8(|V(R')| - k - 1) \leq 6|V(R')| - 12$, which implies $|V(R')| \leq 3k/2 - 2$. So $m \leq 3k/2 - 2 - (k + 1) = k/2 - 3 \leq 1$. This implies that there exists $u \in Int(R)$ such that $N_R(u) = K$. (Note $N(w) \cap V(R_b) = \emptyset$ as (G, a_1, a_2, b_1, b_2) is infeasible.) By planarity of R, $\{a, u, c\}$ is a cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$, a contradiction.

Now, suppose there exists $s \in S \cap Int(R)$ and $s \notin V(X_1^*)$. By symmetry, assume $s \in V(P_c(c, y_1))$. We choose such s with $P_c[c, s]$ minimal. We consider the paths $B_i \cap R$ for $i \in [3]$. If we can find disjoint paths in $R^* - s$ linking the same ends of $B_i \cap R^*$ for $i \in [3]$, then by replacing $B_i \cap Int(R)$ with such paths in $R^* - s$, we obtain independent b_1-b_2 paths B_1', B_2', B_3' such that $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$, contradicting (C4). So such paths do not exist. Hence by 3-planarity of $(R, a, b, c, c', b'a')$ we see that R has a 4-cut $\{s, v_1, v_2, a'\}$ separating $\{a, b, c\}$ from $\{y_1, a', b,' c'\} \cup V(X_1^*)$. Let R' denote the $\{w, s, v_1, v_2, a'\}$-bridge of R^* containing $\{a, b, c\}$ and assume notation is chosen so that $(R' - w, a, b, c, s, v_1, v_2, a')$ is planar. Let R'' be obtained from R' by adding edges in $\{ab, bc, sv_1, v_1v_2, v_2a'\}$, as well as edges from w to all vertices in $K := \{a, b, c, s, v_1, v_2, a'\}$. Let $k := |K|$ and $m := |V(R'') \setminus (K \cup \{w\})|$. By Hand-shaking lemma and Euler's formula, we see that $k \times 4 + k + 8(|V(R'')| - k - 1) \leq 6|V(R'')| - 12$, which implies $|V(R'')| \leq 3k/2 - 3$. So $m \leq 3k/2 - 3 - (k + 1) = k/2 - 3 \leq 1/2$. This leads to a contradiction to (ii) of Proposition 2.3.2.

Case 2. $a = a'$ and $t = 1$.

Suppose $S \cap (Int(R) \setminus V(X_1^*)) = \emptyset$. Let R' be obtained from $R^* - a - (X_1^* - \{y_1, z_1\})$ by adding edges $bc, b'c'$ and y_1z_1 if $y_1 \neq z_1$, as well as edges from w to all vertices in $K := \{b, b', c, c', y_1, z_1\}$. Let $k = |K|$ and $m = |V(R') \setminus (K \cup \{w\})|$. By Hand-shaking lemma and Euler's formula, we see that $4 \times k + k + 7(|V(R')| - k - 1) \leq 6|V(R')| - 12$, which implies $|V(R')| \leq 2k - 5$. Thus, $m \leq k - 6$, Since $k \leq 6$, $V(R') = K$. This leads to a contradiction to (ii) of Proposition 2.3.2.

Now assume there exists $s \in S \cap Int(R)$ and $s \notin inV(X_1^*)$. By symmetry, assume $s \in V(P_c(c, y_1))$. We choose such s with $P_c[c, s]$ minimal. We consider the paths $B_i \cap R^*$ for $i \in [3]$. If we can find disjoint paths in $R^* - s$ linking the same ends of $B_i \cap R^*$ then by replacing $B_i \cap Int(R)$ with such paths in $R^* - s$, we obtain independent b_1-b_2 paths B_1', B_2', B_3' such that $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$, contradicting (C4). So such paths do not exist. Hence by 3-planarity of $(R - a, b, c, c', b')$ we see that $R - a$ has a 3-cut $\{s, v_1, v_2\}$ separating $\{b, c\}$ from $\{y_1, b,' c'\}$. Let R' denote the $\{w, s, v_1, v_2\}$-bridge of $R - a$ containing $\{b, c\}$ and assume notation is chosen so that $(R' - w, b, c, s, v_1, v_2)$ is planar.

Let R'' be obtained from R' by adding edges $\{bc, sv_1, v_1v_2\}$, as well as edges from w to all vertices in $K := \{b, c, s, v_1, v_2\}$. let $k := |K|$ and $m := |V(R'') \setminus (K \cup \{w\})|$. By Hand-shaking lemma and Euler's formula, we see that $k \times 4 + k + 7(|V(R'')| - k - 1) \leq 6|V(R'')| - 12$, which implies $|V(R'')| \leq 2k - 5$. So $m \leq k - 6 < 0$, a contradiction.

Case 3. $a = a'$ and $t = 2$.

Then since $B_3 \cap Int(X_i^*) \neq \emptyset$ cannot be a y_i-z_i path for $i \in [2]$, we see that B_3 enters $Int(X_1^*)$ at a and leaves $Int(X_2^*)$ at w. Thus $S \cap Int(R) \subseteq V(X_1^*) \cup V(P_c[z_1, y_2]) \cup V(X_2^*)$.

Suppose $S \cap Int(R) \subseteq V(X_1^* \cup X_2^*)$. Let R' be obtained from $R^* - a - (X_1^* - \{y_1, z_1\}) - (X_2^* - \{y_2, z_2\})$ by adding edges bc, $b'c'$ and y_iz_i for $i \in [2]$ with $y_i \neq z_i$, as well as edges from w to all vertices in $K := \{b, b', c, c', y_1, z_1, y_2, z_2\}$. Let $k = |K|$ and $m = |V(R') \setminus (K \cup \{w\})|$. By Hand-shaking lemma and Euler's formula, we see that $4 \times k + k + 7(|V(R')| - k - 1) \leq 6|V(R')| - 12$, which implies $|V(R')| \leq 2k - 5$. So $m \leq k - 6 \leq 2$. Using planarity of $R' - w$ and every vertex inside $R' - (K \cup \{w\})$ has degree at least 7, we see that $m = 1$ and the only vertex in $V(R') \setminus (K \cup \{w\})$, say u, is adjacent to both b and b' (and $bb' \in E(G)$) by (ii) of Proposition 2.3.2. Hence, by letting

$P_3' = (P_3 - bb') \cup bub'$, we see that $A_2'(P_1 \cup P_2 \cup P_3')$ contains $A_2'(P_1 \cup P_2 \cup P_3) + bb'$. Hence, $B_1, B_2, B_3, w, P_1, P_2, P_3'$ contradict (C7).

Now assume there exists $s \in S \cap V(P_c(z_1, y_2))$. We choose such s with $P_c[z_1, s]$ minimal. We consider the paths $B_i \cap R^*$ for $i \in [3]$. If we can find disjoint paths in $R^* - s$ linking the same ends of $B_i \cap R^*$, then by replacing $B_i \cap Int(R)$ with such disjoint paths in $R^* - s$, we obtain independent b_1-b_2 paths B_1', B_2', B_3' such that $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$, contradicting (C4). So such paths do not exist. Hence by 3-planarity of $(R - a, b, c, c', b')$ we see that $R - a$ has a 3-cut $\{s, v_1, v_2\}$ separating $\{b, c\} \cup V(X_1^*)$ from $\{b,' c'\} \cup V(X_2^*)$. Let R' denote the graph obtained from the $\{w, s, v_1, v_2\}$-bridge of $R^* - a$ containing $\{b, c\}$ by deleting $Int(X_1^*)$, and assume notation is chosen so that $(R' - w, b, c, y_1, z_1, s, v_1, v_2)$ is planar.

Let R'' be obtained from R' by adding edges in $\{bc, sv_1, v_1v_2\}$ and y_1z_1 (if $y_1 \neq z_1$), as well as edges from u to all vertices in $K := \{b, c, y_1, z_1, s, v_1, v_2\}$. Let $k = |K|$ and $m := |V(R'') \setminus (K \cup \{w\})|$. By Hand-shaking lemma and Euler's formula, we see that $k \times 4 + k + 7(|V(R'')| - k - 1) \leq 6|V(R'')| - 12$, which implies $|V(R'')| \leq 2k - 5$. Hence $m \leq k - 6 \leq 1$. By planarity and (ii) of Proposition 2.3.2, we have $m = 1$. So the unique vertex in $V(R') \setminus \{b, c, s, v_1, v_2, u\} \cup \{y_1, z_1\}$, say u, must be adjacent to w. However, this means $N(w) \cap V(R_b) \neq \emptyset$, a contradiction. $\square$

CHAPTER 4

RUNGS INTERSECTING THREE SPECIAL PATHS

For any rung $(R,(a,b,c),(a',b',c'))$, let $\partial R = \{a,b,c,a',b',c'\}$ and $Int(R) = V(R) \setminus \partial R$. In this chapter, we consider the rungs R in L^* such that $Int(R) \cap V(B_i) \neq \emptyset$ for all $i \in [3]$, including H_1 and H_{m+1}, and prove that only H_1 or H_{m+1} could intersect all three paths.

First, in section 4.1, we prove a technical lemma that will be used to deal with such rungs. In section 4.2, we use Lemma 4.1.1 to obtain structure results of H_1 and H_{m+1} in subsection 4.2.1, and that of the other rungs R in subsection 4.2.2. Last in Lemma 4.2.3, we show that such rungs do not exist except when they are contained in $H_1 \cup H_{m+1}$ or when $|\partial R| = 5$ and $b = b'$.

Let $w', w'' \in N(w) \cap V(B_3)$ such that b_1, w', w, w'', b_2 occur on B_3 in order.

4.1 Technical lemma

In this section, we prove a technical lemma to deal with rungs intersecting B_i for all $i \in [3]$.

Lemma 4.1.1. *Let $\{a',b',c'\}$ be a 3-cut of L^* with $b' \in V(P_3)$ and separating $\{b_1,w'\}$ from $\{b_2,w''\}$, let R denote the $\{a',b',c'\}$-bridge of L^* containing $\{b_1,w'\}$, and let $R^* = R + \{w, wx : x \in N(w) \cap V(R)\}$. Suppose there exists $w^* \in S \cap V(B_3(b_1,w'])$ such that $R^* - w^*$ contains three independent paths Q_a, Q_c, Q_w from b_1 to a', c', w, respectively, such that $b' \in V(Q_a)$, or $Q_a \cap P_3$ is a subpath of $P_3[b_1,b']$ and the $(Q_a \cup Q_c \cup Q_w)$-bridge of R^* containing b' has an attachment on Q_a. Suppose $A_2(P_1 \cup P_2 \cup P_3)$ has attachments on both $P_3(b_1,b']$ and $P_3(b',b_2)$.*

Then L^* has a 3-cut $\{a'',b'',c''\}$ with $b'' \in V(P_3)$ separating $\{a',b',c'\} \cup (N(w) \cap$

$V(L^*))$ from b_2, and $A_2(P_1 \cup P_2 \cup P_3)$ has no attachment in $P_3(b', b'')$. Moreover, if R'' denotes the graph obtained from H by deleting the components of $L^* - (\{a'', b'', c''\} \cup \{a', b', c'\})$ containing b_1 or b_2 then $R'' = J'' \cup L''$ with $b' \in V(J'' - L'')$, $(J'', J'' \cap L'')$ planar, $J'' \cap L''$ is an a'-b'' path, and L'' a ladder from $\{a', c', w\}$ to $\{a'', b'', c''\}$ along $J'' \cap L''$.

Proof. Let $a'' = b'' = c'' = b_2$ if L^* has no 3-cut separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from b_2, and otherwise let $\{a'', b'', c''\}$ be a 3-cut of L^* separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from b_2 and let $b'' \in V(P_3)$. Moreover, let R' denote the graph obtained from L^* by deleting the components of $L^* - (\{a'', b'', c''\} \cup \{a', b', c'\})$ containing b_1 or b_2, and choose $\{a'', b'', c''\}$ to minimize R'. By the choice of R', R' has no cut of size at most 3 separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from $\{a'', b'', c''\}$. Let $R_v = R' + \{v, w, va', vb', wx : x \in N(w) \cap V(R')\}$, where v is a new vertex.

Note that R_v contains three independent paths from v, c', w, respectively, to $\{a'', b'', c''\}$. For, otherwise, R_v has a cut T of size at most 2 separating $\{v, c', w\}$ from $\{a'', b'', c''\}$. Then $v \in T$ as, otherwise, T would separate $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from $\{a'', b'', c''\}$, a contradiction. Moreover, $w \notin T$ because of the existence of three independent paths $P_i \cap R'$, $i \in [3]$, in R'. Now $\{b', a'\} \cup (T \setminus \{v\})$ is a 3-cut in L^* contradicting the choice of $\{a'', b'', c''\}$ (i.e., the minimality of R').

We claim that $A_2(P_1 \cup P_2 \cup P_3)$ has no attachment on $P_3(b', b'')$ (and, hence, $\{a'', b'', c''\}$ is a cut in L^*). For, otherwise, there exists $b^* \in V(A_2(P_1 \cup P_2 \cup P_3)) \cap V(P_3(b', b''))$, and we choose b^* so that $P_3[b^*, b'']$ is minimal. Note that b^* is contained in a 3-cut $\{a^*, b^*, c^*\}$ of L^* separating $\{a', b', c'\}$ from $\{a'', b'', c''\}$. Let M denote the graph obtained from L^* by deleting the components of $L^* - (\{a^*, b^*, c^*\} \cup \{a'', b'', c''\})$ containing b_1 or b_2, and let $M^* = M + \{w, wx : x \in N(w) \cap V(M)\}$. By the choice of $\{a'', b'', c''\}$ (minimality of R'), w has a neighbor in $V(M^*) \setminus \{a^*, b^*, c^*\}$. By the choice of $\{a'', b'', c''\}$ again, $M^* - b^*$ contains independent paths P_a, P_c, P_w from a^*, c^*, w, respectively, to $\{a'', b'', c''\}$. Now we obtain three independent b_1-b_2 paths P_1', P_2', P_3' from $Q_a \cup Q_c \cup Q_w \cup P_a \cup P_c \cup P_w$, $(P_1 \cup$

$P_2) \cap (R' - (M - \{a', c'\})$, and three independent paths from b_2 to a'', b'', c'', respectively, in the $\{a'', b'', c''\}$-bridge of L^* containing b_2. Then $B_1, B_2, B_3, w^*, P_1', P_2', P_3'$ contradict (C5), as $A_2(P_1' \cup P_2' \cup P_3')$ contains $A_2(P_1 \cup P_2 \cup P_3) + b^*$.

We further claim that any three disjoint paths in R_v from $\{v, c', w\}$ to $\{a'', b'', c''\}$ must contain a v-b'' path. For, suppose P_v, P_c, P_w are disjoint paths in R_v from v, c', w, respectively, to $\{a'', b'', c''\}$ with no v-b'' path. Then $b'' \in V(P_c)$ or $b'' \in V(P_w)$. If $b'' \in V(P_w)$, let $v' \in \{b', a'\}$ such that $v' \in V(P_v)$. Then, there is an a_1-a_2 path in union of $A_1(B_1 \cup B_2 \cup B_3) - (B_3 - w)$, P_w and $A_2(P_1 \cup P_2 \cup P_3) - P_3[b_1, b']$ (as $A_2(P_1 \cup P_2 \cup P_3)$ has attachment on $P_3(b', b_2)$), which is independent from the two b_1-b_2 paths obtained from two independent paths from b_1 to $\{v', c'\}$ (subpaths of $P_1 \cup P_2 \cup P_3$), $P_v - v, P_c$, and the two independent paths from b_2 to $\{a'', c''\}$ (subpaths of $P_1 \cup P_2 \cup P_3$). So (G, a_1, a_2, b_1, b_2) is feasible. Thus, $b'' \in V(P_c)$. By symmetry, assume $c'' \in V(P_v)$ and $a'' \in V(P_w)$. If $a' \in V(P_v)$, let $Q_a' = Q_a$; otherwise if $b' \in V(P_v)$, let Q_a' be the b_1-b' path in union of Q_a and the $(Q_a \cup Q_c \cup Q_w)$-bridge of R^* containing b'. Then we obtain three independent b_1-b_2 paths B_1', B_2', B_3' in $H - w^*$ from $Q_a' \cup Q_c \cup Q_w \cup (P_v - v) \cup P_c \cup P_w$ and the three independent paths from b_2 to $\{a'', b'', c''\}$ (subpaths of B_1, B_2, B_3), such that, $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + w^*$ and $A_2(B_1' \cup B_2' \cup B_3')$ has attachments on both B_1' and B_2' (by assumption on Q_a). So B_1', B_2', B_3', w^* contradict (C4).

Hence, by applying Lemma 2.3.1 to $(R_v, (w, v, c'), (a'', b'', c''))$, we see that $R_v = J_v \cup L_v$, where L_v is a ladder from (w, v, c') to (a'', b'', c'') and $(J_v, J_v \cap L_v)$ is planar.

Case 1. $J_v \subseteq L_v$.

Then by the choice of R', L_v is a single rung. By relabeling a'' and c'' if necessary, we may assume $c' = a''$ when $c' \in \{a'', c''\}$. Then, since $v, w \notin \{a'', b'', c''\}$, it follows from definition of rungs that either $c' \neq a''$ and $(L_v, w, v, c', a'', b'', c'')$ is 3-planar, or $c' = a''$ and $(L_v - c', w, v, b'', c'')$ is 3-planar.

Hence, because of P_1, P_2, P_3 and the choice of R', $R' - b'$ contains three disjoint paths P_a, P_c, P_w from a', c', w to b'', a'', c'', respectively. Now these three paths, $Q_a \cup Q_c \cup Q_w$, and $(P_1 \cup P_2 \cup P_3) - ((R^* - a'') + Int(R'))$ form three independent b_1-b_2 paths X_1, X_2, X_3 in $H - w^*$ such that $a' \in V(X_1)$, $c' \in V(X_2)$, and $w \in V(X_3)$. Note that $A_1(X_1 \cup X_2 \cup X_3)$ contains $A_1(B_1 \cup B_2 \cup B_3) + w^*$.

If $A_2(X_1 \cup X_2 \cup X_3)$ has attachments on both X_1 and X_2 then X_1, X_2, X_3, w^* contradict (C4), or all attachment of $A_1(X_1 \cup X_2 \cup X_2)$ are on X_3, then (G, a_1, a_2, b_1, b_2) is feasible with X_1, X_2 and an a_1-a_2 path in union of X_3 and $A_1(X_1 \cup X_2 \cup X_2)$. So assume $A_2(X_1, X_2, X_3)$ has all its attachments on X_1. Then the $(P_a \cup P_c \cup P_w)$-bridge of $R_v - v$ containing b', say J'', has all its attachments in P_a. By choosing P_a, P_c, P_w to maximize J'' and by the planarity of L_v (when $|\partial R| = 6$) or $L_v - c'$ when $|\partial R| = 5$), we see that J'' and $L'' := (R_v - v) - (J'' - P_a)$ satisfies the conclusion of the lemma.

Case 2. $J_v - L_v \neq \emptyset$.

By the minimality of R', we see that the boundary of J_v has a path from v to b'' and avoiding $L_v - \{v, b''\}$, which we denote by Q. Note $b' \in V(Q)$ or $a' \in V(Q)$. If $b' \in V(Q)$ then $R'' = R_v - v, J'' = J_v - v$ and $L'' = L_v - v$ satisfy the conclusion. So assume $a \in V(Q)$.

We claim that $R_v - Q - w$ contains disjoint paths B_b, B_c from b', c', respectively, to $\{a'', c''\}$; for otherwise, there is a cut vertex t in $R_v - Q - w$ separating $\{b, c'\}$ from $\{a'', c''\}$. However, this contradicts the existence of the disjoint paths $P_i \cap (R_v - w)$, $i \in [3]$.

Now $(P_1 \cup P_2 \cup P_3) - Int(R')$, and $(Q - v) \cup B_b \cup B_c$ give three independent b_1-b_2 paths B_1', B_2', B_3' in L^*, such that $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + w$ and $A_2(B_1' \cup B_2' \cup B_3')$ attaches to two of B_1', B_2', B_3' (as $A_2(P_1 \cup P_2 \cup P_3)$ has attachments on both $P_3(b_1, b']$ and $P_3(b', b_2)$). Hence, either (G, a_1, a_2, b_1, b_2) is feasible, or B_1', B_2', B_3', w contradict (C4). □

4.2 Structures

In this section, we apply Lemma 4.1.1 to obtain structures for H_1 and H_{m+1} and rungs R in L^* not contained in $H_1 \cup H_{m+1}$. Then, in Lemma 4.2.4, we conclude that only H_1 or

H_{m+1} could intersect all three paths.

4.2.1 H_1 and H_{m+1}

First, consider H_1 and H_{m+1} in L^*.

Lemma 4.2.1. *If $B_i \cap Int(H_1) \neq \emptyset$ for $i \in [3]$ and if $w' \in V(H_1) \setminus T_1$ and $w'' \notin V(H_1)$, then, there exists $w^* \in S \cap (V(H_1) \setminus (T_1 \cup \{b_1\}))$ such that,*

(a) for each $s \in S \cap V(B_3(b_1, w^*])$, s is contained in a 3-cut of $H_1^* := H_1 + \{w, wv : v \in N(w) \cap Int(H_1)\}$ separating b_1 from $T_1 \cup \{w\}$, and

(b) for each $s \in S \cap V(B_3(w^*, w))$, s is contained in a 3-cut of H_1^* separating $\{b_1, x_3\}$ from $\{w, x_1, x_2\}$, where for $i \in [2]$, x_i denotes the end of $B_i \cap H_1$ other than b_1, and $x_3 \in T_1 \setminus \{x_1, x_2\}$.

The same holds for H_{m+1} and b_2.

Proof. By symmetry, we prove the assertion for H_1. By definition, $B_i \cap H_1^*$, $i \in [3]$, are paths in H_1 from b_1 to $\{u_1, v_1, w_1, w\}$ with only b_1 in common. Let $w^* \in S \cap (V(H_1) \setminus (T_1 \cup \{b_1\}))$ such that w^* is contained in some 3-cut T of H_1^* separating b_1 from $T_1 \cup \{w\}$; and if such w^* does not exist we set $w^* = b_1$. We choose w^* such that $B_3[b_1, w^*]$ is maximal. Let H_1' denote the T-bridge of H_1^* containing b_1 (with $V(H_1') = \{b_1\}$ if $w^* = b_1$).

We claim that for any $s \in S \cap V(B_3(b_1, w^*))$, s is contained in some 3-cut of H_1' separating b_1 from T. For, otherwise, $H_1' - s$ contains independent paths from b_1 to T with only b_1 in common. Now these three paths and $B_i - (H_1' - T)$ for $i \in [3]$ form three independent b_1-b_2 paths B_1', B_2', B_3' in $H - s$ such that $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$ and $A_2(B_1' \cup B_2' \cup B_3')$ has attachments on both B_1' and B_2'. Hence, B_1', B_2', B_3' contradict (C4).

Now let $s \in S \cap V(B_3(w^*, w))$ be arbitrary. By the choice of w^*, s is not contained in any 3-cut of H_1^* separating b_1 from $T_1 \cup \{w\}$. For $i \in [2]$, let x_i be the end of $B_i \cap H_1$ other than b_1. Thus, $x_1, x_2 \in T_1$, and let $x_3 \in T_1 \setminus \{x_1, x_2\}$.

If $H_1^* - s$ contains no independent paths from b_1 to x_1, x_2, w, respectively, then s is contained in a 3-cut T' of H_1^* separating b_1 from $\{x_1, x_2, w\}$. Since T' cannot separate b_1 from $T_1 \cup \{w\}$, T' must separate $\{b_1, x_3\}$ from $\{w, x_1, x_2\}$.

So assume that $H_1^* - s$ contains independent paths Q_1, Q_2, Q_3 from b_1 to x_1, x_2, w, respectively. These paths, $B_3[w, b_2]$, and the parts of B_1, B_2 outside H_1 form three independent b_1-b_2 paths B_1', B_2', B_3' in $H - s$. Since $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$ and $A_2(B_1 \cup B_2 \cup B_3)$ has attachments on both B_1 and B_2, it follows from (C4) that $A_2(B_1' \cup B_2' \cup B_3')$ has all its attachments on $Q_i + b_2$ for exactly one $i \in [2]$, and that $V(A_2(B_1 \cup B_2 \cup B_3)) \cap V(B_{3-i}) \subseteq V(H_1)$. So $u_1 \notin V(B_1 \cup B_2 \cup B_3)$, $u_1 \in A_2(B_1' \cup B_2' \cup B_3')$, and $\{x_1, x_2\} = \{v_1, w_1\}$.

Thus, we may apply Lemma 4.1.1 with the cut T_1 as $\{a', b', c'\}$ and u_1 as b'. So L^* has a 3-cut $\{a'', b'', c''\}$ with $b'' \in V(P_3)$ separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from b_2, and $A_2(P_1 \cup P_2 \cup P_3)$ has no attachment in $P_3(b', b'')$. Moreover, if R'' denotes the graph obtained from H by deleting the components of $L^* - (\{a'', b'', c''\} \cup \{a', b', c'\})$ containing b_1 or b_2, then $R'' = J'' \cup L''$ with $b' \in V(J'' - L'')$, where $(J'', J'' \cap L'')$ is planar, $J'' \cap L''$ is an a'-b'' path, and L'' is a ladder from $\{a', c', w\}$ to $\{a'', b'', c''\}$ along $J'' \cap L''$. Let P_1', P_2', P_3' be three independent b_1-b_2 paths in $H - w^*$ obtained from $Q_1 \cup Q_2 \cup Q_3$, three disjoint paths in L'' from $\{v_1, w_1, w\}$ to $\{a'', b'', c''\}$, and the subpaths of P_i, $i \in [3]$, from $\{a'', b'', c''\}$ to b_2. Since $b' = u_1 \in V(A_2(B_1, B_2, B_3))$, we see that $A_2'(P_1' \cup P_2' \cup P_3')$ contains $A_2'(P_1 \cup P_2 \cup P_3) \cup J''$. Thus, $B_1, B_2, B_3, w^*, P_1', P_2', P_3'$ contradict (C7). $\square$

4.2.2 Rungs not in $H_1 \cup H_{m+1}$

Next, consider rungs $(R, (a, b, c), (a', b', c'))$ not contained in $H_1 \cup H_{m+1}$. First, we show results of such rungs R with $w' \in Int(R)$ and $w'' \notin V(R)$. We discuss them in two cases: $|\partial R| = 5$ and $b = b'$ in Lemma 4.2.2, and $|\partial R| = 6$ or $|\partial R| = 5$ and $b \neq b'$ in Lemma 4.2.3.

Lemma 4.2.2. *Suppose* $(R, (a, b, c), (a', b', c'))$ *is a rung in* L^* *such that* $R \nsubseteq H_1 \cup H_{m+1}$

and $|\{w', w''\} \cap Int(R)| = 1 = |\{w', w''\} \cap V(R)|$*. Moreover, assume that* $b = b'$ *and* $V(B_i) \cap Int(R) \neq \emptyset$ *for* $i \in [3]$*. Then, for all* $s \in S \cap Int(R)$*,* s *is contained in a 3-cut of* $R^* = R + \{w, wv : v \in N(w) \cap Int(R)\}$ *separating* $\{a, b, c\}$ *from* $\{a', c', w\}$*, or for all* $s \in S \cap Int(R)$*,* s *is contained in a 3-cut of* R^* *separating* $\{a', b', c'\}$ *from* $\{a, c, w\}$*,*

Proof. By symmetry, let $w' \in Int(R)$ and $w'' \notin V(R)$, and we may assume that b_1, w', w'', b_2 occur on B_3 in order. Note that $b \in V(P_3)$, and we may assume that $a \in V(P_2)$, and $c \in V(P_1)$. Suppose for a contradiction that there exists some $w^1 \in S \cap Int(R)$ such that $R^* - w^1$ contains disjoint paths Q_a, Q_b, Q_c from a, b, c, respectively, to $\{a', c', w\}$.

Observe that $w \notin V(Q_b)$. For otherwise, by replacing $(P_1 \cup P_2) \cap R$ with $Q_a \cup Q_c$, we obtain from P_1, P_2 independent b_1-b_2 paths P_1' and P_2' such that $G - (P_1' \cup P_2')$ contains an a_1-a_2 path. This shows that (G, a_1, a_2, b_1, b_2) is feasible, a contradiction.

Hence, by symmetry, we may assume that $a' \in V(Q_b)$, $c' \in V(Q_a)$, and $w \in V(Q_c)$. Let K denote the $\{a', b', c'\}$-bridge of L^* containing $\{b_1, w'\}$, and let $K^* = K + \{w, wx : x \in N(w) \cap V(K)\}$. Then Q_a, Q_b, Q_c, $P_1[b_1, c], P_2[b_1, a]$, and $P_3[b_1, b]$ form three independent paths Q_a^1, Q_c^1, Q_w^1 in $K^* - w^1$ from b_1 to a', c', w, respectively, with $b \in V(Q_a')$. Hence, $Q_a^1 \cap P_3 = P_3[b_1, b]$.

Since $R \not\subseteq H_1 \cup H_{m+1}$, $A_2(P_1 \cup P_2 \cup P_3)$ has an attachment on both $P_3(b_1, b]$ and $P_3(b, b_2)$. Since $b \in V(Q_a')$, we may apply Lemma 4.1.1 with the paths Q_a^1, Q_c^1, Q_w^1. So L^* has a 3-cut $\{a^2, b^2, c^2\}$ with $b^2 \in V(P_3)$ separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from b_2, and $A_2(P_1 \cup P_2 \cup P_3)$ has no attachment in $P_3(b', b^2)$. Moreover, if R^2 denotes the graph obtained from H by deleting the components of $L^* - (\{a^2, b^2, c^2\} \cup \{a', b', c'\})$ containing b_1 or b_2, then $R^2 = J^2 \cup L^2$ with $b' \in V(J^2 - L^2)$, where $(J^2, J^2 \cap L^2)$ is planar, $J^2 \cap L^2$ is an a'-b^2 path, and L^2 is a ladder from (c', a', w) to (a^2, b^2, c^2) along the path $J^2 \cap L^2$. Note that L^2 contains three disjoint paths P_a^2, P_c^2, P_w^2 from a', c', w, respectively, to $\{a^2, b^2, c^2\}$, with $P_a^2 = J^2 \cap L^2$.

If $N(w) \cap V(R^* \setminus \{a, b, c\}) = \emptyset$ then let P_1^2, P_2^2, P_3^2 be three independent b_1-b_2 paths in $H - w^*$ obtained from $Q_a^1 \cup Q_c^1 \cup Q_w^1$, $P_a^2 \cup P_c^2 \cup P_w^2$, and the subpaths of P_i, $i \in [3]$, from $\{a^2, b^2, c^2\}$ to b_2. We see that $A_2'(P_1^2 \cup P_2^2 \cup P_3^2)$ contains $A_2'(P_1 \cup P_2 \cup P_3) \cup J^2$; so $B_1, B_2, B_3, w^1, P_1^2, P_2^2, P_3^2$ contradict (C7).

So assume $N(w) \cap V(R^* \setminus \{a, b, c\}) \neq \emptyset$.

We may assume that there exists $w^2 \in S \cap Int(R)$ such that $R^* - w^2$ contains disjoint paths Q_a^2, Q_b^2, Q_c^2 from a', b', c', respectively, to $\{a, c, w\}$; otherwise the assertion of the lemma holds. Hence, we may apply the same argument as above with respect to R and b_1, and conclude that L^* has a 3-cut $\{a^1, b^1, c^1\}$ with $b^1 \in V(P_3)$ separating $\{a, b, c\} \cup (N(w) \cap V(L^*))$ from b_1, and $A_2(P_1 \cup P_2 \cup P_3)$ has no attachment in $P_3(b, b^1)$. Moreover, if R^1 denotes the graph obtained from H by deleting the components of $L^* - (\{a^1, b^1, c^1\} \cup \{a, b, c\})$ containing b_1 or b_2 then $R^1 = J^1 \cup L^1$ with $b \in V(J^1 - L^1)$, where $(J^1, J^1 \cap L^1)$ is planar, $J^1 \cap L^1$ is an a-b^1 path, and L^1 is a ladder from (c, a, w) to (a^1, b^1, c^1) along $J^1 \cap L^1$. Note that L^1 contains three disjoint paths P_a^1, P_c^1, P_w^1 from a, c, w, respectively, to $\{a^1, b^1, c^1\}$, with $P_a^1 = J^1 \cap L^1$.

If $R - b$ has disjoint paths from a, c to c', a', respectively, then, by definition of rung, these paths can be chosen to avoid some $s \in S \cap Int(R)$. So these two paths, $P_a^i \cup P_c^i \cup P_w^i$, $i \in [2]$, and subpaths of P_j, $j \in [3]$, from b_i to $\{a^i, b^i, c^i\}$, form three independent b_1-b_2 paths B_1', B_2', B_3'. We can show that (G, a_1, a_2, b_1, b_2) is feasible or there exists $s \in S \cap Int(R)$ such that B_1', B_2', B_3', s contradict (C4).

Thus, $(R - b, a, a', c', c)$ is planar. Let X_a, X_c denote the disjoint paths in $R - b$ from a, c to a', c', respectively, such that $X_a \cup X_c$ is contained in the outer walk of $R - a$. Then $S \cap Int(R) \subseteq V(X_c)$ by (C4). Moreover, (R, a, b, a', c', c) is 3-planar. For, otherwise, there exists $s \in S \cap V(P_c(c, c'))$ such that $R - s$ has disjoint paths from c, s to a', b, respectively, or disjoint paths from c', s to a, b, respectively. The b-s path can be used to find an a_1-a_2 path that is disjoint from two b_1-b_2 paths using the other paths. So (G, a_1, a_2, b_1, b_2) is feasible, a contradiction.

Let P_1', P_2', P_3' be three independent b_1-b_2 paths in H obtained from $X_a \cup X_c$, $P_a^i \cup P_c^i \cup P_w^i$ (for $i \in [2]$), and the subpaths of P_j, $j \in [3]$, from $\{a^i, b^i, c^i\}$ to b_i (for $i \in [2]$). We see that

$A'_2(P'_1 \cup P'_2 \cup P'_3)$ contains $A'_2(P_1 \cup P_2 \cup P_3) \cup J^1 \cup J^2$. Thus, either (G, a_1, a_2, b_1, b_2) is feasible, or for some $s \in S \cap Int(R)$, $B_1, B_2, B_3, s, P'_1, P'_2, P'_3$ contradict (C7). $\square$

Lemma 4.2.3. *Suppose* $(R, (a, b, c), (a', b', c'))$ *is a rung in* L^* *such that* $|\partial R| = 6$ *or* $|\partial R| = 5$ *and* $b \neq b'$*,* $R \not\subseteq H_1 \cup H_{m+1}$*, and* $|\{w', w''\} \cap Int(R)| = 1 = |\{w', w''\} \cap V(R)|$*. Then there exists* $i \in [2]$ *such that* $V(B_i) \cap Int(R) = \emptyset$*.*

Proof. By symmetry, let $w' \in Int(R)$ and $w'' \notin V(R)$, and assume that b_1, w', w'', b_2 occur on B_3 in order. Note that $b, b' \in V(P_3)$ and, since $R \not\subseteq H_1 \cup H_{m+1}$, $A_2(P_1 \cup P_2 \cup P_3)$ has attachments on both $P(b_1, b]$ and $P[b', b_2)$. Since $|\partial R| = 6$ or $|\partial R| = 5$ and $b \neq b'$, it follows from (b) of Lemma 3.2.4 (with appropriate relabeling) that R contains induced paths P_a, P_c from a, c to a', c', respectively, such that $R - (P_a \cup P_c)$ is connected and contains $\{b, b'\}$, and $S \cap Int(R) \subseteq V(P_a \cup P_c)$. Let $R^* = H[R + w]$. We claim that

(1) $N(w) \cap V(R) \subseteq V(P_a \cup P_c)$.

For otherwise, $R^* - (P_a \cup P_c)$ contains a path P_w from w to $\{b, b'\}$. Let P'_1, P'_2 be the b_1-b_2 paths in L^* obtained from P_1, P_2 by replacing $(P_1 \cup P_2) \cap R$ with $P_a \cup P_c$. Since $A_2(P_1 \cup P_2 \cup P_3)$ has attachments on both $P_3(b_1, b]$ and $P_3[b', b_2)$, $(R^* - (P_a \cup P_c)) \cup (A_1(B_1 \cup B_2 \cup B_3) - (B_3 - w)) \cup (A_2(P_1 \cup P_2 \cup P_3) \cup (P_3(b_1, b_2) - R) \cup P_w$ contains an a_1-a_2 path independent of P'_1, P'_2. This shows that (G, a_1, a_2, b_1, b_2) is feasible, a contradiction.

By symmetry, let $w' \in V(P_c)$. Then $c \neq c'$. Suppose the assertion of the lemma fails, i.e., $V(B_i) \cap Int(R) \neq \emptyset$ for $i \in [3]$. Then by planarity of (R, a, b, c, c', b', a') or $(R - a, b, c, c', b')$, $S \cap Int(R) \subseteq V(P_c(c, c'))$. Let $s \in S \cap V(P_c(c, w'))$ with $P_c[c, s]$ minimal.

(2) s is not contained in any cut of R^* of order at most 3 separating $\{a, a', b, c\}$ from $\{b', c', w\}$.

For, suppose R^* has a 3-cut containing s, say $\{s, v_1, v_2\}$, separating $\{a, a', b, c\}$ from $\{b', c', w\}$.

First, assume $a = a'$. Let K denote the $\{s, v_1, v_2\}$-bridge of R^* containing $\{a, b, c\}$. By choosing notation of v_1 and v_2, we may assume that (K, b, c, s, v_1, v_2) is planar. Let K' be obtained from $K + \{bc, sv_1, v_1v_2\}$ by adding a new vertex v and edges from v to all of $\{b, c, s, v_1, v_2\}$. Then by Hand-shaking lemma and Euler's formula, $5\times4+5+7(|V(K')|-6) \leq 6|V(K')|-12$. This implies $|V(K')| \leq 5$, a contradiction.

Now consider the case when $a \neq a'$. Let K denote the $\{s, v_1, v_2\}$-bridge of R^* containing $\{a, a', b, c\}$. By choosing notation of v_1 and v_2, we may assume that $(K, a, b, c, s, v_1, v_2, c')$ is planar. Let K' be obtained from $K + \{ab, bc, sv_1, v_1v_2, v_2a'\}$ by adding a new vertex v and edges from v to all of $\{a, b, c, s, v_1, v_2, a'\}$. Then by Hand-shaking lemma and Euler's formula, $7 \times 4 + 7 + 8(|V(K')| - 8) \leq 6|V(K')| - 12$. This implies $|V(K')| \leq 10$. Hence, $U := V(K) \setminus \{a, b, c, s, v_1, v_2, a'\}$ contains at most two vertices. Since each vertex in U must have degree at least 8 and $U \cap N(w) = \emptyset$ by (1), we have $|U| = 2$. However, this contradicts the planarity of (R, a, b, c, c', b', a').

We claim that

(3) $a = a'$.

For, suppose $a \neq a'$. Then (R, a, b, c, c', b', a') is planar. Thus, $B_3 \cap R^*$ must be a c-w path, and $B_1 \cap R^*$ and $B_2 \cap R^*$ must be an a-$\{a', b'\}$ path and a b-$\{b', c'\}$ path. By (2) and planarity of R, we see that $R^* - s$ contains disjoint induced paths B'_a, B'_b, B'_c connecting the ends of $B_1 \cap R^*, B_2 \cap R^*, B_3 \cap R^*$, respectively. Thus, by replacing $B_1 \cap R^*, B_2 \cap R^*, B_3 \cap R^*$ with B'_a, B'_b, B'_c, we obtain from B_1, B_2, B_3 three independent induced b_1-b_2 paths B'_1, B'_2, B'_3. Since $A_2(B_1 \cup B_2 \cup B_3)$ has attachments on both B_1 and B_2 and has no attachment in $Int(R)$, $A_2(B'_1 \cup B'_2 \cup B'_3)$ has attachments on both B'_1 and B'_2. Clearly, $A_1(B'_1 \cup B'_2 \cup B'_3)$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$. So $B'_1, B'_2B'_3$ contradicts (C4).

By (3), $(R - a, b, c, c', c')$ is planar. Hence, by (2), $(R^* - a) - s$ contains disjoint paths B^2_b, B^2_c from b, c to c', w, respectively. Note that B^2_b, B^2_c and the subpaths of P_i, $i \in [3]$, between b_1 and $\{a, b, c\}$ form three independent induced paths Q^2_a, Q^2_c, Q^2_w from b_1

to a, c', w, respectively. Moreover, we see that $Q_c^2 \cap P_3$ contains $P_3[b_1, b]$ and has has an attachment of $A_2(P_1 \cup P_2 \cup P_3) - b_1$. Note that $b' \in V(Q_c^2)$ or the $(Q_a^2 \cup Q_c^2 \cup Q_w^2) \cap R^*$-bridge of R^* containing b' has an attachment in Q_c^2. We can now apply Lemma 4.1.1 to obtain a 3-cut $\{a^2, b^2, c^2\}$ in L^* separating $\{a', b', c'\} \cup (N(w) \cap V(L^*))$ from b_2. Moreover, if R^2 denotes the graph obtained from H by deleting the components of $L^* - (\{a', b', c'\} \cup \{a^2, b^2, c^2\})$ containing b_1 or b_2, then $R^2 = J^2 \cup L^2$ with $b' \in V(J^2 - L^2)$, where $(J^2, J^2 \cap L^2)$ is planar, $J^2 \cap L^2$ is an c'-b^2 path, and L^2 is a ladder from (a', c', w) to (a^2, b^2, c^2) along $J^2 \cap L^2$. Moreover, $L^2 - J^2$ has disjoint paths from $\{a, w\}$ to $\{a^2, c^2\}$ which, we may assume, are P_a^2, P_w^2 from a, w to a^2, c^2, respectively.

Let L_1 denote the $\{a, b, c\}$-bridge of L^* containing H_1.

Case 1. $N(w) \cap V(L_1 - \{a, b, c\}) = \emptyset$.

Let P_b be the b-c' path in the boundary of $R - a$ containing b' but not c. Suppose P_b is an induced path. Then let $P_3' := P_3[b_1, b] \cup P_b \cup (J^2 \cap L^2) \cup P_3[b^2, b_2]$, and let P_1', P_2' be obtained from P_1, P_2 by replacing $(P_1 \cup P_2) \cap (R \cup R^2)$ with $P_a^2, B_c \cup P_w^2$. We see that $A_2'(P_1' \cup P_2' \cup P_3')$ contains $A_2'(P_1 \cup P_2 \cup P_3) \cup J^2$; so $B_1, B_2, B_3, s, P_1', P_2', P_3'$ contradict (C7).

Hence, P_b is not an induced path. Thus, let $xy \in E(G) \setminus E(P_b)$ with $x, y \in V(P_b)$. Choose x, y with $P_b[x, y]$ maximal. To avoid the cut set $\{x, y, w, a, b\}$ in G, we may assume that $xb', b'y \in E(P_b)$ and x, b', y occur on P_b in this order. Let $P_3' := P_3[b_1, b] \cup (P_b[b, x] \cup xy \cup P_b[y, c']) \cup (J^2 \cap L^2) \cup P_3[b^2, b_2]$. Let P_1', P_2' be defined as above.

If $b'a \notin E(G)$, then we see that $A_2'(P_1' \cup P_2' \cup P_3')$ contains $A_2'(P_1 \cup P_2 \cup P_3) \cup J^2$; so $B_1, B_2, B_3, s, P_1', P_2', P_3'$ contradict (C7). Thus, $b'a \in E(G)$. By symmetry between a^2 and c^2, let P_c', P_w' be disjoint induced paths in $L^2 - \{a, b^2\}$ from c, w to a^2, c^2, respectively. Let B_1', B_2', B_3' be obtained from $(P_1 \cup P_2 \cup P_3) - ((R \cup R^2) - \{a, b, c, a^2, b^2, c^2\}$ by adding $(P_b[b, x] \cup xy \cup P_b[y, c']) \cup P_c', ab' \cup (J^2 - (L^2 - b^2)), B_c \cup P_w'$. By choosing notation, we may assume $w \in V(B_3')$, $P_3[b_1, b] \subseteq B_1'$, and $P_3[b^2, b_2] + a \subseteq B_2'$. Now $A_2(B_1' \cup B_2' \cup B_3')$ has attachments on both B_1' and B_2'. Clearly, $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$;

so B_1', B_2', B_3' contradict (C4).

Case 2. $N(w) \cap V(L_1 - \{a, b, c\}) = \emptyset$.

Then consider disjoint paths B_b^1, B_c^1 in $(R^* - a) - s$ from b', c' to c, w, respectively, which exists by planarity of $(R - a, b, c, c', b')$. Now B_b^1, B_c^1 and the subpaths of P_i, $i \in [3]$, between b_2 and $\{a', b', c'\}$ form three independent induced paths Q_a^1, Q_c^1, Q_w^1 from b_2 to a, c, w, respectively. Moreover, we see that $Q_c^1 \cap P_3$ contains $P_3[b_2, b']$ and has an attachment of $A_2(P_1 \cup P_2 \cup P_3) - b_2$. We can now apply Lemma 4.1.1 to obtain a 3-cut $\{a^1, b^1, c^1\}$ in L^* separating $\{a, b, c\} \cup (N(w) \cap V(L^*))$ from b_1. Moreover, if R^1 denotes the graph obtained from H by deleting the components of $L^* - (\{a, b, c\} \cup \{a^1, b^1, c^1\})$ containing b_1 or b_2, then $R^1 = J^1 \cup L^1$ with $b \in V(J^1 - L^1)$, where $(J^1, J^1 \cap L^1)$ is planar, $J^1 \cap L^1$ is an c-b^1 path, and L^1 is a ladder from (a, c, w) to (a^1, b^1, c^1) along $J^1 \cap L^1$. Note, $L^1 - J^1$ has disjoint paths from $\{a, w\}$ to $\{a^1, c^1\}$ which, we may assume, are P_a^1, P_w^1 from a, w to a^1, c^1, respectively.

Let Q denote an induced c-c' path with $V(Q)$ contained in the boundary of $R - a$ disjoint from $P_c(c, c')$. Let $P_3' = P_3[b_1, b^1] \cup (J^1 \cap L^1) \cup Q \cup (J^2 \cap L^2) \cup P_3[b^2, b_2]$, and let P_1', P_2' be the b_1-b_2 paths obtained from $P_1 \cup P_2$ by replacing $(P_1 \cup P_2) \cap (R^1 \cup R^2)$ with $P_a^1 \cup P_a^2$ and $P_w^1 \cup P_w^2$. We see that $A_2'(P_1' \cup P_2' \cup P_3')$ contains $A_2'(P_1 \cup P_2 \cup P_3) \cup J^1 \cup J^2$; so $B_1, B_2, B_3, s, P_1', P_2', P_3'$ contradict (C7). □

We conclude this section with the following result.

Lemma 4.2.4. *Let* $(R, (a, b, c), (a', b', c'))$ *be a rung in* L^* *with* $R \not\subseteq H_1 \cup H_{m+1}$ *and* $b \neq b'$. *Then there exists* $i \in [2]$ *such that* $Int(R) \cap V(B_i) = \emptyset$.

Proof. Suppose $Int(R) \cap V(B_i) \neq \emptyset$ for $i \in [2]$. Then, since G is 8-connected, $S \cap Int(R) \neq \emptyset$ and, hence, $V(B_3) \cap Int(R) \neq \emptyset$. Since $R \not\subseteq H_1 \cup H_{m+1}$, it follows from Lemma 4.2.3 that $w', w'' \in V(R)$ or $\{w', w''\} \cap Int(R) = \emptyset$. Hence, $|\partial R| = 6$. By Lemma 3.2.4, let P_a, P_c be the induced paths in R from a, c to a', c', respectively, such that $R - (P_a \cap P_c)$ is connected and contains $\{b, b'\}$. Note that $N(w) \cap Int(R) \subseteq V(P_a \cup P_c)$;

as otherwise (G, a_1, a_2, b_1, b_2) would be feasible.

Suppose $\{w', w''\} \cap Int(R) = \emptyset$ or $B_3 \cap R \subseteq P_a$ or $B_3 \cap R \subseteq P_c$. Then $B_i \cap R$, $i \in [3]$, are $\{a, b, c\}$-$\{a', b', c'\}$ paths. By Lemma 3.2.4, we may assume that $B_1 \cap R = P_a$ and $B_3 \cap R = P_c$. So there exists $s \in S \cap V(P_c(c, c'))$. By definition of rung, R has no 3-cut separating $\{a, b, c\}$ from $\{a', b', c'\}$. Hence, $R - s$ has three disjoint paths Q_a, Q_b, Q_c from a, b, c, respectively, to $\{a', b', c'\}$. By Lemma 3.2.4 again, $a' \in V(Q_a)$, $b' \in V(Q_b)$, and $c' \in V(Q_c)$. For each $i \in [3]$, let B_i' be obtained from B_i by replacing $B_i \cap R$ with one of Q_a, Q_b, Q_c. Now $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$, and $A_2(B_1' \cup B_2' \cup B_3')$ has attachments on both B_1' and B_2' (as $R \not\subseteq H_1 \cup H_{m+1}$). So B_1', B_2', B_3', w contradict (C4).

Hence, we may assume $w' \in V(P_a)$, and $w'' \in V(P_c)$. Choose $w_a \in N(w) \cap V(P_a)$ with $P_a[a, w_a]$ minimal, and choose $w_c \in N(w) \cap V(P_c)$ with $P_c[w_c, c']$ minimal. Note that $S \cap Int(R) \subseteq V(P_a[a, w_a]) \cup V(P_c[w_c, c'])$. For otherwise, we could modify B_3 by replacing $B_3(w_a, w_c)$ with $w_a w w_c$ to obtain a new b_1-b_2 path B_3'. Now $A_1(B_1 \cup B_2 \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3)$ and some vertex $s \in S \cap Int(R)$. Moreover, $A_2'(P_1 \cup P_2 \cup P_3)$ has attachments on both B_1' and B_2'. So B_1, B_2, B_3', s contradict (C4).

Suppose there exists $s \in S$ with $s \in V(P_a(a, w_a)) \cup V(P_c(w_c, c'))$. By symmetry, assume $s \in V(P_c(w_c, c'))$. Since R has no 3-cut separating $\{a, b, c\}$ from $\{a', b', c'\}$, $R - (P_a[w_a a'] \cup P_c[c, w_c]) - s$ contains two disjoint paths Q_a, Q_b from a, b to b', c', respectively. Without loss of generality, we may assume $a, a' \in V(P_1)$ and $c, c' \in V(P_2)$. Let $B_1' = P_1[b_1, a] \cup Q_a \cup P_3[b', b_2]$, $B_2' = P_3[b_1, b] \cup Q_b \cup P_1[c', b_2]$, $B_3' = P_2[b_1, c] \cup P_c[c, w_c] \cup w_c w w_a \cup P_a[w_a, a'] \cup P_1[a', b_2]$. Now B_1', B_2', B_3' are independent b_1-b_2 paths. Moreover, $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$, and $A_2'(P_1 \cup P_2 \cup P_3)$ has attachments on both B_1 and B_2 (as $R \not\subseteq H_1 \cup H_{m+1}$), contradicting (C4).

Thus, $S \cap (V(P_a(a, w_a)) \cup V(P_c(w_c, c'))) = \emptyset$. This implies that $S \cap Int(R) \subseteq \{w_a, w_c\}$. Let R^* be the plane graph obtained from $G[R + w]$ by adding $ba, bc, b'a', b'c'$ and all edges from w to $V(P_a \cup P_c) \cup \{b, b'\}$. Now $|E(R^*)| \geq 8(|R^*| - 8) + 6 \times 4 + 2 \times 5 = 8|R^*| - 30$. So $8|R^*| - 30 \leq 6|R^*| - 12$. This implies $|R^*| \leq 9$, a contradiction as

$|N(b) \cap Int(R)| \geq 2$ by (ii) of Proposition 2.3.2. $\square$

CHAPTER 5

STRUCTURE OF OTHER RUNGS

In this chapter, we consider rungs R in L^* such that $Int(R) \cap B_i = \emptyset$ for some $i \in [2]$.

First, in section 5.1, we prove technical lemmas for separation (G', G'') of $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$ in which $B_i \cap (G' - G'') = \emptyset$ for some $i \in [2]$. Then, we deal with H_1, H_{m+1} in subsection 5.2.1 and all other rungs in subsection 5.2.2.

For $x, y \in V(B_j)$ for some $j \in [3]$, we denote $x \preceq y$ if $B_j[b_1, x] \subseteq B_j[b_1, y]$; and $x \prec y$ if $x \preceq y$ and $x \neq y$.

5.1 Technical lemmas

We begin by showing that for any rung R in L^* or for H_1, H_{m+1}, if neither B_1 nor B_2 intersects $Int(R)$ or $Int(H_1)$, or $Int(H_{m+1})$, then $Int(R), Int(H_1), Int(H_{m+1}) \subseteq S$. For convenience, we prove a more general statement in terms of separations in $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$.

Lemma 5.1.1. *Suppose (G', G'') is a separation of $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$ such that $|V(G' \cap G'')| \leq 7$, $V(G' - G'') \neq \emptyset$, and $V(G'' - G') \neq \emptyset$. Suppose $V(G' - G'') \cap V(B_1 \cup B_2) = \emptyset$. Then $V(G' - G'') \subseteq S$.*

Proof. Note $S \cap V(G' - G'') \neq \emptyset$; otherwise $V(G' \cap G'')$ is a cut of G contradicting the $(8, S)$-connectivity. Let $r_1, r_2 \in V(B_3) \cap V(G' \cap G'')$ be such that $B_3[r_1, r_2]$ is maximal. Note that it is possible $B_3[r_1, r_2] \nsubseteq G'$.

Suppose $V(G' - G'') \nsubseteq S$ and let X be an S-bridge of $G' - \big(V(G' \cap G'') \setminus \{r_1, r_2\}\big)$ with $X - S \neq \emptyset$. Let $x_1, x_2 \in V(X) \cap \big(S \cup \{r_1, r_2\}\big)$ such that $B_3[x_1, x_2]$ is maximal. Then $|V(X) \cap (S \cup \{r_1, r_2\})| \geq 3$; otherwise, $\big(V(X) \cap \big(S \cup \{r_1, r_2\}\big)\big) \cup \big(V(G' \cap G'') \setminus \{r_1, r_2\}\big)$

is a cut of G separating $V(X) \setminus \big(S \cup \{r_1, r_2\}\big)$ from $V(G'' - G')$, a contradiction to the $(8, S)$-connectivity of G.

Hence, there exists $s \in V(B_3(x_1, x_2)) \cap S$. Let A be any induced x_1-x_2 path in $X - s$, and $B_3' = \big(B_3 - B_3(x_1, x_2)\big) \cup Q$. Then B_1, B_2, B_3' are independent b_1-b_2 paths in G such that $A_1(B_1 \cup B_2 \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + w$ and $A_2(B_1 \cup B_2 \cup B_3')$ attaches to both B_1 and B_2, a contradiction. □

Next, we consider rungs R when$Int(R) \cap B_i = \emptyset$ for exactly one $i \in [2]$. Again we prove statements in terms of separations in $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$. We first show that all internal vertices are in $V(B_i) \cup S$ in Lemma 5.1.2. Then, we give structural results of such rungs in Lemma 5.1.3.

Lemma 5.1.2. *Suppose (G', G'') is a separation of $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$ such that $|V(G' \cap G'')| \leq 7$, $V(G' - G'') \neq \emptyset$, and $V(G'' - G') \neq \emptyset$. Let $i \in [2]$ such that $V(G' - G'') \cap V(B_i) \neq \emptyset$, and $V(G' - G'') \cap V(B_{3-i}) = \emptyset$. Then $V(G' - G'') \subseteq V(B_i) \cup S$.*

Proof. Note $S \cap V(G' - G'') \neq \emptyset$; otherwise $V(G' \cap G'')$ is a cut of G contradicting the $(8, S)$-connectivity of G. Let $r_1, r_2 \in V(B_3) \cap V(G' \cap G'')$ and $t_1, t_2 \in V(B_i) \cap V(G' \cap G'')$ be such that $B_3[r_1, r_2]$ and $B_i[t_1, t_2]$ are maximal. For convenience, let $G^* = G' - \big(V(G' \cap G'') \setminus \{r_1, r_2, t_1, t_2\}\big)$. We may assume $r_1 \prec r_2$ and $t_1 \prec t_2$.

Suppose for a contradiction, $V(G' - G'') \setminus (V(B_i) \cup S) \neq \emptyset$. Then G^* has a $((B_i \cup S) \cap G^*)$-bridge X such that $V(X) \setminus (V(B_i) \cup S) \neq \emptyset$. Choose X and modify $B_i \cap G^*$ (if necessary) so that

(1) $|V(X) \cap (S \cup \{r_1, r_2\})|$ is maximal, and

(2) subject to (1), X is maximal.

Let $x_1, x_2 \in V(X) \cap (S \cup \{r_1, r_2\})$ with $B_3[x_1, x_2]$ maximal, and let $x_1 \prec x_2$. We claim that

(3) $|V(X) \cap (S \cup \{r_1, r_2\})| \leq 2$.

For, otherwise, there exists $s \in V(X) \cap V(B_3(x_1, x_2)) \cap S$. Let Q be any induced x_1-x_2 path in $X - (B_i + (S \setminus \{x_1, x_2\}))$, and let $B_3' = \big(B_3 - B_3(x_1, x_2)\big) \cup Q$. Then B_1, B_2, B_3' are independent b_1-b_2 paths in $G - \{a_1, a_2\}$ such that $A_1(B_1 \cup B_2 \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s$ and $A_2(B_1 \cup B_2 \cup B_3')$ attaches to both B_1 and B_2, a contradiction.

By (3), $V(B_3(x_1, x_2)) \cap S = \emptyset$; so we may choose B_3 such that

(4) $B_3[x_1, x_2] \subseteq X$.

Then, $|V(X \cap B_i)| \geq 2$; otherwise by (3), $V(X \cap B_i) \cup \big(V(X) \cap (S \cup \{r_1, r_2\})\big) \cup \big(V(G' \cap G'') \setminus \{r_1, r_2, t_1, t_2\}\big)$ is a cut in G of size ≤ 7 separating $V(X - (B_i \setminus S)$ from $G'' - G'$, contradicting the $(8, S)$ connectivity of G. Let $y_1, y_2 \in V(X \cap B_i)$ with $y_1 \prec y_2$ such that $B_i[y_1, y_2]$ is maximal. Then

(5) G^* has no path from $B_i(y_1, y_2)$ to $B_i - B_i[y_1, y_2]$ and internally disjoint from $B_i \cup B_3$.

For otherwise, let Q be an induced path in G^* from $z_1 \in V(B_i(y_1, y_2))$ to $z_2 \in V(B_i - B_i[y_1, y_2])$, and let B_i' be an induced b_1-b_2 path in $\big(B_i - B_i(z_1, z_2)\big) \cup Q$. Then, the $((B_i' \cup S) \cap G^*)$-bridge of G^* containing X also contains z_2, contradicting (2).

(6) $|V(X) \cap (S \cup \{r_1, r_2\})| \geq 1$.

For, suppose $V(X) \cap (S \cup \{r_1, r_2\}) = \emptyset$. Then by (1), no $((B_i \cup S) \cap G^*)$-bridge of G^* has attachment in $S \cup \{r_1, r_2\}$. Hence by (5) and since $S \cap V(G^*) \subseteq V(B_3)$, there exists an induced path Q' in G^* from some vertex $y \in V(B_i(y_1, y_2))$ to some vertex $s \in (S \cup \{r_1, r_2\}) \cap V(G^*)$, internally disjoint from $X \cup B_i + S$. Let Q'' be an induced y_1-y_2 path in $X - B_i(y_1, y_2)$ and $B_i'' := \big(B_i - B_i(y_1, y_2)\big) \cup Q''$. Then, the $((B_i'' \cup S) \cap G^*)$-bridge of G^* containing Q' also contains s, contradicting (1).

By (3) and (6), we have two cases.

Case 1. $|V(X) \cap (S \cup \{r_1, r_2\})| = 2$.

Since $\{x_1, x_2, y_1, y_2\} \cup \big(V(G' \cap G'') \setminus \{r_1, r_2, t_1, t_2\}\big)$ is not a cut in G separating $X - \{x_1, x_2, y_1, y_2\}$ from $G'' - G'$, it follows from (5) that there is a y_3-x_3 path Q in G^* internally disjoint from $X \cup B_i + S$, with $y_3 \in V(B_i(y_1, y_2))$ and $x_3 \in V(B_3)$. Since $B_3 \cap B_i \subseteq \{b_1, b_2\}$, if $B_3 \cap Q \neq \emptyset$ then x_3 may be chosen so that $x_3 \in (S \cup \{r_1, r_2\}) \setminus \{x_1, x_2\}$.

Note that $x_j \in V(B_3(x_{3-j}, x_3))$ for some $j \in [2]$, and thus, $x_j \in S$. By symmetry, we may assume $j = 2$ and $x_1 \prec x_2 \prec x_3$, and that there exists $z \in V(A_2(B_1 \cup B_2 \cup B_3) \cap B_i[b_1, y_2))$. Let Q' be any x_1-y_2 path in $X - y_3$ internally disjoint from $B_i \cup S$.

Then, the following paths show that (G, a_1, a_2, b_1, b_2) is feasible: B_{3-i}, $B_3[b_1, x_1] \cup Q' \cup B_i[y_2, b_2]$, and an a_1-a_2 path in the union of $A_1(B_1 \cup B_2 \cup B_3) - (B_3 - x_3)$, $B_i[z, y_3] \cup Q \cup B_3[x_3, x_2]$, and $A_2(B_1 \cup B_2 \cup B_3) - ((B_1 \cup B_2) - z)$.

Case 2. $|V(X) \cap (S \cup \{r_1, r_2\})| = 1$.

So $x_1 = x_2$. Since $\{x_1, y_1, y_2\} \cup \big(V(G' \cap G'') \setminus \{r_1, r_2, t_1, t_2\}\big)$ is not a cut in G separating $X - \{x_1, y_1, y_2\}$ from $G'' - G'$, it follows from (5) that there exist disjoint paths Q_1, Q_2 from $z_1, z_2 \in V(B_i(y_1, y_2))$ to $x_2, x_3 \in V(B_3 - x_1)$, respectively, internally disjoint from $X \cup B_i + S$. We may choose $x_2, x_3 \in (S \cup \{r_1, r_2\}) \setminus \{x_1\}$. (If Q_1, Q_2 intersect $B_3 - S$ then we obtain a new bridge contradicting (1).) Since the order of z_1, z_2 will not matter in the rest of our argument, we may assume $x_1 \prec x_2 \prec x_3$ or $x_2 \prec x_1 \prec x_3$.

First, suppose $x_1 \prec x_2 \prec x_3$. Let Q be an induced x_1-y_2 path in X independent of B_i, and let $B_i' = B_i[b_1, z_2] \cup Q_2 \cup B_3[x_3, b_2]$ and $B_3' = B[b_1, x_1] \cup Q \cup B_i[y_2, b_2]$. Note that $A_2(B_{3-i} \cup B_i' \cup B_3')$ attaches to B_{3-i} as well as B_i' or B_3'. If $A_2(B_{3-i} \cup B_i' \cup B_3')$ attaches to B_3' then, since $x_1 \in S$, we see that (G, a_1, a_2, b_1, b_2) is feasible. If $A_2(B_{3-i} \cup B_i' \cup B_3')$ attaches to B_i' then B_{3-I}, B_i', B_3', x_1 contradict (C4) as $A_1(B_{3-i} \cup B_i' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + x_1$.

Now suppose $x_2 \prec x_1 \prec x_3$. Let Y be an induced y_1-y_2 path in $X - x_1$ independent of B_i, let $B_3' = B_3[b_1, x_2] \cup Q_1 \cup B_i[z_1, z_2] \cup Q_2 \cup B_3[x_3, b_2]$ and let $B_i' = B_i[b_1, y_1] \cup Y \cup B_i[y_2, b_2]$. Then $A_1(B_{3-i} \cup B_i' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + x_1$. So B_{3-i}, B_i', B_3', x_1 contradict (C4). □

Lemma 5.1.3. *Suppose (G', G'') is a separation of $G - (A_1(B_1 \cup B_2 \cup B_3) - B_3)$ such that $|V(G' \cap G'')| \leq 7$, $V(G' - G'') \neq \emptyset$ and $V(G'' - G') \neq \emptyset$. Suppose for some $i \in [2]$, $V(G' - G'') \cap V(B_i) \neq \emptyset$, and $V(G' - G'') \cap V(B_{3-i}) = \emptyset$. Let $r_1, r_2 \in V(G' \cap G'' \cap B_3)$ and $t_1, t_2 \in V(G' \cap G'' \cap B_i)$ such that $B_3[r_1, r_2]$ and $B_i[t_1, t_2]$ are maximal, and $N_{G'-G''}(r_j) \cap S \neq \emptyset$ for both $j \in [2]$. Let $V' = V(G' \cap G'') \setminus (\{r_1, r_2\} \cup V(B_i))$.*

Then, for some e with $e = \emptyset$ or $e \in E(G' - V')$ incident to either r_1 or r_2, if $x_j y_j \in E(G' - V') \setminus (E(B_i \cup B_3) \cup \{e\})$ with $x_j \in V(B_i)$ and $y_j \in V(B_3)$ for $j \in [2]$, then $x_1 \preceq x_2$ implies $y_1 \preceq y_2$.

Proof. By Lemma 5.1.2, $V(G' - G'') \subseteq V(B_i) \cup S$. Thus, $G^* := G' - V'$ is obtained from $G^* \cap (B_i[t_1, t_2] \cup B_3[r_1, r_2])$ by adding edges with one end in B_i and the other end in B_3. For any distinct $x_1, x_2 \in V(B_i \cap G^*)$ and distinct $y_1, y_2 \in V(B_3 \cap G^*)$, we say (x_1, x_2, y_1, y_2) is a *cross* if $x_1 \prec x_2$, $y_1 \prec y_2$ and $x_1 y_2, x_2 y_1 \in E(G^*)$. If there is no cross, lemma holds with $e = \emptyset$. So assume there is a cross.

For any cross (x_1, x_2, y_1, y_2), we have $S \cap V(B_3[b_k, y_k]) = \emptyset$ for some $k \in [2]$. For otherwise, both b_1-b_2 paths $B_i' := B_i[b_1, x_1] \cup \{x_1 y_2\} \cup B_3[y_2, b_2], B_3' := B_3[b_1, y_1] \cup \{y_1 x_2\} \cup B_i[x_2, b_2]$ have an internal vertex in $A_2(B_1 \cup B_2 \cup B_3)$. Since $A_2(B_1 \cup B_2 \cup B_3)$ attaches to B_{3-i} and one of B_i' or B_3', we see that (G, a_1, a_2, b_1, b_2) is feasible.

Thus, since $V(G' - G'') \subseteq V(B_i) \cup S$, we have, for any cross (x_1, x_2, y_1, y_2), $y_j = r_j \notin S$ for some $j \in [2]$. For convenience, let $t_1 \prec t_2$ and $r_1 \prec r_2$.

Next, we show that, for any cross (x_1, x_2, y_1, y_2), if $y_1 = r_1$ then $B_i(t_1, x_2) = \emptyset$, and if $y_2 = r_2$ then $B_i(t_2, x_1) = \emptyset$. For, otherwise, suppose $y_1 = r_1$ and there exists $x \in V(B_i(t_1, x_2))$. Since B_i is induced and G is 8-connected, $|N_{G^*}(x) \cap (S \cup \{r_1, r_2\})| \geq 3$; so let $s_1, s_2, s_3 \in N_{G^*}(x) \cap (S \cup \{r_1, r_2\})$ with $s_1 \prec s_2 \prec s_3$. Then $s_2 \in S \setminus \{r_1, r_2\}$. Let $B_{3-i}' = B_{3-i}$, $B_i' := B_i[b_1, x] \cup \{x s_3\} \cup B_3[s_3, b_2]$, and $B_3' := B_3[b_1, y_1] \cup \{y_1 x_2\} \cup B_i[x_2, b_2]$. Then we see that $A_2(B_1' \cup B_2' \cup B_3')$ has attachments on B_{3-i}' and one of B_i' and B_3', and $A_1(B_1' \cup B_2' \cup B_3')$ contains $A_1(B_1 \cup B_2 \cup B_3) + s_2$. It is easy to see that (G, a_1, a_2, b_1, b_2) is feasible or B_1', B_2', B_3', s_2 contradict (C4).

Now let (x_1, x_2, y_1, y_2) be a cross with $y_1 = r_1 \notin S$, and we further choose this cross to maximize $B_i[t_1, x_2]$. By above, we see that $V(B[t_1, x_2]) = \{x_1, x_2\}$. If all crosses use the edge y_1x_2, then the assertion of the lemma holds with $e = y_1x_2$. So assume there is a cross (x'_1, x'_2, y'_1, y'_2) with $y'_1x'_2 \neq y_1x_2$. Then $y'_1 \neq y_1$. Hence, $y'_1 \in S$ and $y'_2 = r_2 \notin S$. This implies that $V(B_i[x'_1, t_2]) = \{x'_1, x'_2\}$. Note that $x'_1 \neq x_1$ and $x'_2 \neq x_2$ (as $V(B_i) \cap V(G' - G'') \neq \emptyset$). Thus, $x_2 \prec x'_1$. By the maximality of $B_i[t_1, x_2]$, we see that $y_2 \neq y'_1$.

Let $B'_i := B_i[b_1, x_1] \cup \{x_1y_2\} \cup B_3[y_2, y'_1] \cup \{y'_1x'_2\} \cup B_i[x'_2, b_2]$ and $B'_3 := B_3[b_1, r_1] \cup \{r_1x_2\} \cup B_i[x_2, x'_1] \cup \{x'_1r_2\} \cup B_3[r_2, b_2]$. Then, both $B'_i(t_1, t_2)$ and $A_2(B_1 \cup B_2 \cup B_3)$ contains y'_1 and y_2; so $G - (B'_3 \cup B_{3-i})$ has an a_1-a_2 path, showing that (G, a_1, a_2, b_1, b_2) is feasible. □

5.2 Structures

In this section, we use technical lemmas from previous section to give structural results of H_1, H_{m+1} and rungs $R \in L^*$ not contained in $H_1 \cup H_{m+1}$.

5.2.1 H_1 and H_{m+1}

First, we consider H_1, H_{m+1} when $Int(H_1), Int(H_{m+1})$ intersects B_i for at most one $i \in [2]$.

Lemma 5.2.1. *If $B_i \cap Int(H_1) \neq \emptyset$ for at most one $i \in [2]$, then one of the following holds:*

(a) $Int(H_1) \subseteq S$.

(b) $Int(H_1) \subseteq V(B_i) \cup S$ for some $i \in [2]$ and the following holds:

- *Let $G' := G[V(H_1) \cup \{w\}]$; let $r_1, r_2 \in V(B_3) \cap (T_1 \cup \{w, b_1\})$ and $t_1, t_1 \in V(B_i) \cap (T_1 \cup \{w, b_1\})$ such that $N(r_j) \cap S \cap Int(H_1) \neq \emptyset$ for both $j \in [2]$ and subject to this, $B_3[r_1, r_2]$ and $B_i[t_1, t_2]$ are maximal; let $V' = (T_1 \cup \{w, b_1\}) \setminus$*

$(\{r_1, r_2\} \cup V(B_i))$. Then, there exists e with $e = \emptyset$ or e incident to either r_1 or r_2, such that, if $x_j y_j \in E(G' - V' - e) \setminus E(B_i \cup B_3)$ with $x_j \in V(B_i)$ and $y_j \in V(B_3)$ for $j \in [2]$, then $x_1 \preceq x_2$ implies $y_1 \preceq y_2$.

The same holds for H_{m+1} and b_2.

Proof. If $Int(H_1) = \emptyset$ then (a) holds. So assume $Int(H_1) \neq \emptyset$. Then $|S \cap Int(H_1)| \geq 3$; otherwise $T_1 \cup \{w, b_1\} \cup (S \cap Int(H_1))$ is a cut in G of size ≤ 7 separating $Int(H_1)$ from b_2, contradicting the $(8, S)$-connectivity of G.

Suppose $V(B_1 \cup B_2) \cap Int(H_1) = \emptyset$. Let $G' := G[V(H_1) \cup \{w\}]$ and $G'' := G - Int(H_1) - E(G[T_1 \cup \{w, b_1\}])$. Then, by Lemma 5.1.1, $Int(H_1) = V(G' - G'') \subseteq S$, and thus, (a) holds.

So $V(B_i) \cap Int(H_1) \neq \emptyset$ for some $i \in [2]$ and $V(B_{3-i}) \cap Int(H_1) = \emptyset$. By Lemma 5.1.2 with $G' := G[V(H_1) \cup \{w\}]$ and $G'' := G - Int(H_1) - E(G[T_1 \cup \{w, b_1\}])$, $Int(H_1) \subseteq V(B_i) \cup S$.

Let $r_1, r_2 \in V(B_3) \cap (T_1 \cup \{w, b_1\})$ and $t_1, t_1 \in V(B_i) \cap (T_1 \cup \{w, b_1\})$ such that $N(r_j) \cap S \cap Int(H_1) \neq \emptyset$ for both $j \in [2]$ and subject to this, $B_3[r_1, r_2]$ and $B_i[t_1, t_2]$ are maximal. Let $V' = (T_1 \cup \{w, b_1\}) \setminus \{r_1, r_2, t_1, t_2\}$. Then, $G' - V' = G[V(B_3[r_1, r_2] \cup B_i[t_1, t_2])]$, and (b) follows from Lemma 5.1.3. □

5.2.2 Rungs not in $H_1 \cup H_{m+1}$

We now consider rungs R in L^* such that $R \not\subseteq H_1 \cup H_{m+1}$. First, we show that if a rung R is 3-planar then R is planar, except in a very special situation which can occur in at most twice in all rungs of L^*.

Corollary 5.2.2. *Suppose (R', R'') is a separation of rung R in L^* such that $|V(R' \cap R'')| \leq 3$ and $\partial R \subseteq V(R')$. Then,*

(a) $(R'', V(R' \cap R''))$ is planar, or

(b) $\{w', w''\} \cap V(R'' - R')) \neq \emptyset$ *and* $\{w', w''\} \not\subseteq V(R'')$, $|V(R' \cap R'')| = 3$, $R'' - R' \neq \emptyset$, $V(R'' - R') \subseteq B_i \cup S$ *for some* $i \in [2]$ *and there exists* $e = \emptyset$ *or* e *has one end in* $V(R' \cap R'')$ *such that* $(R'' - e, V(R' \cap R''))$ *is planar.*

Proof. Note that $V(R'' - R') \cap S \neq \emptyset$ as G os 8-connected. Suppose $|V(R' \cap R'') \cap V(B_3)| \geq 2$. Since $|V(R' \cap R'')| \leq 3$, $V(R'' - R') \cap V(B_1 \cup B_2) = \emptyset$. Let $G' := R''$ and $G'' := G[V(G) \setminus V(R'' - R')]$. Then, $|V(G' \cap G'')| \leq 3$ and $V(G' - G'') \cap V(B_1 \cup B_2) = \emptyset$. By Lemma 5.1.1, $V(R'' - R') = V(G' - G'') \subseteq S$, and thus, $R'' - R'$ is a subpath of B_3. Now, $R'' - B_3 = \emptyset$ or is a single vertex, and hence, $(R'', V(R' \cap R''))$ is planar and (a) holds.

Now, assume $|V(R' \cap R'' \cap B_3)| = 1$. Then, $\{w', w''\} \cap V(R'' - R') \neq \emptyset$ and $\{w', w''\} \not\subseteq V(R'')$. Let $G' := G[V(R'') \cup \{w\}]$ and $G'' := G - (R'' - R') - E\big(G[V(R' \cap R'') \cup \{w\}]\big)$.

If $V(R'' - R') \cap V(B_1 \cup B_2) = \emptyset$, then by Lemma 5.1.1, $V(R'' - R') = V(G' - G'') \subseteq S$, and thus, $R'' - R'$ is a subpath of B_3. Now, $V(R'' - B_3)$ is a set of two vertices (in $V(R' \cap R'')$) and, hence, $(R'', V(R' \cap R''))$ is planar.

So $V(R'' - R') \cap V(B_1 \cup B_2) \neq \emptyset$. Indeed, there exists unique $i \in [2]$ such that $V(R'' - R') \cap V(B_i) \neq \emptyset$. Then, $|V(G' \cap G'')| = |V(R' \cap R'') \cup \{w\}| = 4$, $V(G' - G'') \cap V(B_i) \neq \emptyset$, and $V(G' - G'') \cap V(B_{3-i}) = \emptyset$. By Lemma 5.1.2, $V(G' - G'') = V(R'' - R') \subseteq (V(B_i) \cup S) \setminus \{w\}$. Hence, (b) follows from Lemma 5.1.3. □

Next, we make the following observation to be used.

Observation 5.2.3. $(N(w) \cup S) \cap V(P_3(b_1, b_2)) = \emptyset$.

Proof. For, suppose there exists $v \in (N(w) \cup S) \cap V(P_3(b_1, b_2))$. If $v \in S$, then let Q_1 be an a_1-a_2 path in the union of $A_1(B_1 \cup B_2 \cup B_3) - (B_3 - v)$ and $A_2(P_1 \cup P_2 \cup P_3) \cup P_3(b_1, b_2)$; and if $v \in N(w)$ then let Q_2 be an a_1-a_2 path in the union of $A_1(B_1 \cup B_2 \cup B_3) - (B_3 - w)$, $\{wv\}$ and $A_2(P_1 \cup P_2 \cup P_3) \cup P_3(b_1, b_2)$. Now, P_1, P_2 and Q_1 or Q_2 show that (G, a_1, a_2, b_1, b_2) is feasible. □

Now, we show structures for all rungs R in L^* with $Int(R) \cap B_j = \emptyset$ for some $j \in [2]$ in Lemma 5.2.4.

Lemma 5.2.4. *For any rung $(R, (a, b, c), (a', b', c'))$ in L^* with $R \not\subseteq H_1 \cup H_{m+1}$ and $Int(R) \cap B_j \neq \emptyset$ for at most one $j \in [2]$, $|\partial R| \leq 5$ and one of the following holds:*

(a) $Int(R) \subseteq S$, and if $|\partial R| = 5$ then $b = b'$, or

(b) $b = b'$ and, for some $i \in [2]$, $V(B_i) \cap Int(R) \neq \emptyset$ and $Int(R) \subseteq V(B_i) \cup S$. Moreover, let $r_1, r_2 \in (V(B_3) \cap \partial R) \cup \{w\}$ with $N_{Int(R)}(r_j) \cap S \neq \emptyset$ for $j \in [2]$, and let $t_1, t_2 \in V(B_i) \cap \partial R$ such that $B_3[r_1, r_2]$ and $B_i[t_1, t_2]$ are maximal. Let $R^* = R + \{w, wv : v \in V(R)\}$ and $V' = \partial R \setminus (\{r_1, r_2\} \cup V(B_i))$. Then, there exists e with $e = \emptyset$ or $e \in E(R^*)$ incident to either r_1 or r_2, such that, if $x_j y_j \in E(R^* - V') \setminus (E(B_i \cup B_3) \cup \{e\})$ with $x_j \in V(B_i)$ and $y_j \in V(B_3)$, for $j \in [2]$, then $x_1 \preceq x_2$ implies $y_1 \preceq y_2$.

Proof. Suppose $S \cap Int(R) = \emptyset$. Then $Int(R) = \emptyset$ to avoid the cut $\partial R \cup \{w\}$ in G (of size ≤ 7). By (ii) and (iii) of Proposition 2.3.2, if $|\partial R| = 6$ or $|\partial R| = 5$ and $b \neq b'$, $N_{Int(R)}(b) \neq \emptyset$. So for $Int(R) = \emptyset$, $|\partial R| \leq 5$ and if $|\partial R| = 5$ then $b = b'$.

Now, assume $Int(R) \neq \emptyset$. First, suppose $V(B_1 \cup B_2) \cap Int(R) = \emptyset$. Let $G' := G[V(R) \cup \{w\}]$ and $G'' := G - Int(R) - E(G[\partial R])$. Then, by Lemma 5.1.1, $Int(R) = V(G' - G'') \subseteq S$. Assume (a) fails. Then, $|\partial R| = 5$ and $b \neq b'$ or $|\partial R| = 6$. By (b) of Lemma 3.2.4, let P_a, P_c be disjoint paths in $R - \{b, b'\}$ from a, c to a', c', respectively, such that $R - (P_a \cup P_c)$ is connected and contains $\{b, b'\}$. Then, $Int(R) \subseteq P_a \cup P_c$; otherwise by replacing $(P_1 \cup P_2) \cap R$ with $P_a \cup P_c$, we obtain from P_1, P_2 independent b_1-b_2 paths P_1' and P_2' such that $G - (P_1' \cup P_2')$ contains an a_1-a_2 paths, which shows that (G, a_1, a_2, b_1, b_2) is feasible, a contradiction. By (ii) and (iii) of Proposition 2.3.2, $N(b) \cap Int(R) \neq \emptyset$. So by symmetry, assume there exists $s \in N_{Int(R)}(b) \cap V(P_c)$. Then, $\{a, b, s\}$ is a 3-cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$, contradicting the definition of rung.

So $V(B_i) \cap Int(R) \neq \emptyset$ for some $i \in [2]$. Hence $V(B_{3-i}) \cap Int(R) = \emptyset$. By Lemma 5.1.2, with $G' := G[V(R) \cup \{w\}]$ and $G'' := G - Int(R) - E(G[\partial R])$, we obtain $Int(R) \subseteq V(B_i) \cup S$. By Observation 5.2.3, we see that $\{b, b'\}$ has no neighbors in $S \cap Int(R)$. Thus, $\{b, b'\} \cap \{r_1, r_2\} = \emptyset$ by definition of r_1 and r_2. By Lemma 5.1.3, to prove (b), we need to show $b = b'$. Suppose for a contradiction $b \neq b'$.

First, consider that case when $|\partial R| = 4$. Then, $\{b, b'\} \cap \{t_1, t_2\} \neq \emptyset$ and at least one of the vertices in $\{b, b'\} \cap \{t_1, t_2\}$, say $b = t_1$, has a neighbor v such that $v \in V(B_i) \cap Int(R)$ and $vb \in E(B_i)$. Let $s \in N(v) \cap Int(R) \cap S$, which exists since B_i is induced and G is 8-connected. Now, there is an a_1-a_2 path in the union of $A_1(B_1 \cup B_2 \cup B_3)$, bvs, $P_3(b_1, b_2)$ and $A_2(P_1 \cup P_2 \cup P_3)$, which is disjoint from b_1-b_2 paths P_1, P_2. So (G, a_1, a_2, b_1, b_2) is feasible, a contradiction.

Now assume $|\partial R| \geq 5$. Then $|\partial R| = 5$ and $b \neq b'$ or $|\partial R| = 6$. When $|\partial R| = 5$, we may assume $(R - a, b, b'c, c')$ is planar. By (b) of Lemma 3.2.4, let P_a, P_c be disjoint paths in $R - \{b, b'\}$ from a, c to a', c', respectively, such that $R - (P_a \cup P_c)$ is connected and contains $\{b, b'\}$. As before, $Int(R) \cap S \subseteq V(P_a \cup P_c)$ (as otherwise, (G, a_1, a_2, b_1, b_2) would be feasible) and $N_{Int(R)}(b) \cap V(P_a \cup P_c) = \emptyset$. Then, by (iii) of Proposition 2.3.2 and by Observation 5.2.3, $N_{Int(R)}(b) \cap V(B_i) \neq \emptyset$. Let $v \in N_{Int(R)}(b) \cap V(B_i)$. Since B_i is induced, $|N(v) \cap V(B_i)| = 2$. Since G is 8-connected and $N(v) \subseteq V' \cup V(B_i) \cup \{r_1, r_2\} \cup (Int(R) \cap S)$, there exists $s \in N(v) \cap Int(R) \cap S$. By 3-planarity and since $Int(R) \subseteq V(B_i) \cup S$, $\{a, v, s\}$ or $\{a', v, s\}$ is a 3-cut in R separating $\{a, b, c\}$ from $\{a', b', c'\}$, contradicting the definition of rung. $\square$

CHAPTER 6

A 7-CONNECTED EXAMPLE

In this chapter, we give a 7-connected graph G with distinct vertices $a_1, a_2, b_1, b_2 \in V(G)$ such that (G, a_1, a_2, b_1, b_2) is infeasible. As shown below, G is obtained by gluing H in Figure 6.1 and A_1 in Figure 6.2 together along the b_1-b_2 path B_3.

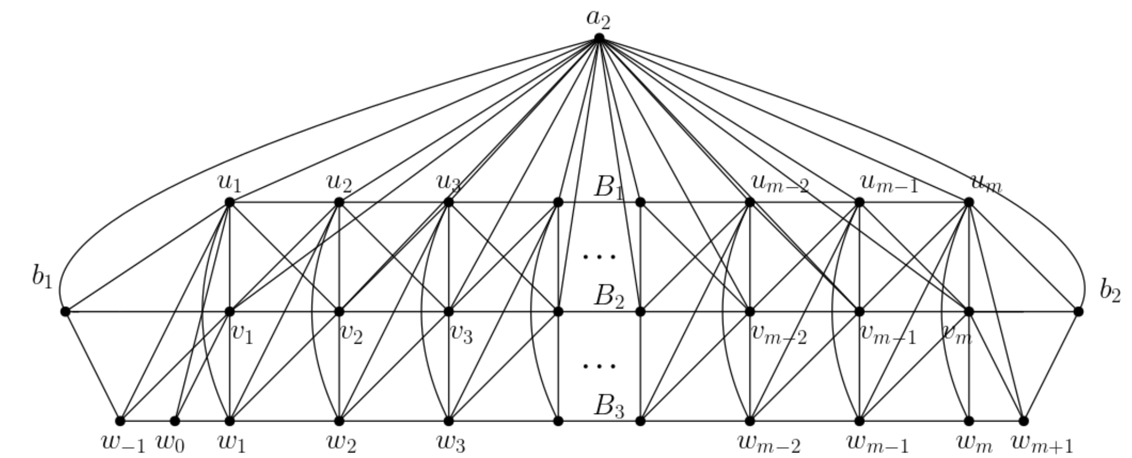

Figure 6.1: H with $m = 7(7^4 - 2) + 3 = 7^5 - 11$

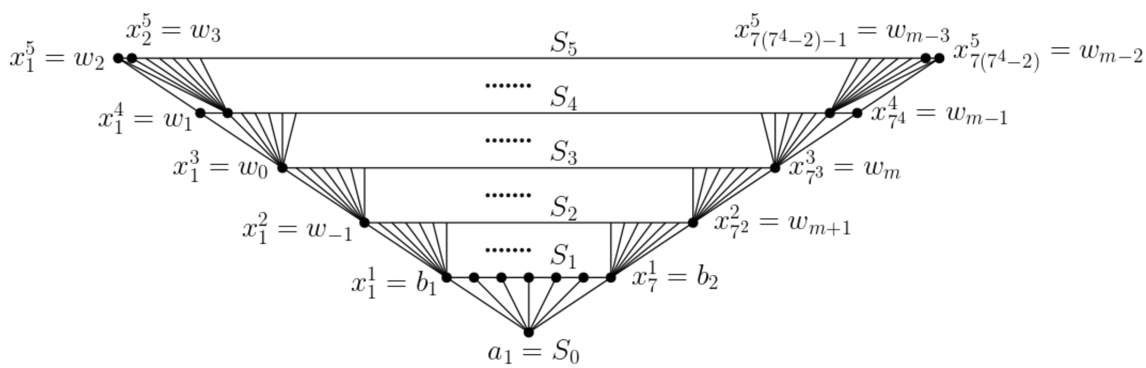

Figure 6.2: A_1

As shown in Figure 6.1, B_1, B_2, B_3 are 3 independent b_1-b_2 paths. H is the graph with $V(H) = \{a_2, b_1, b_2, w_{-1}, w_0, w_{m+1}, u_i, v_i, w_i : i \in [m]\}$ and $E(H) = \bigcup_{i\in[3]} E(B_i) \cup \{a_2x : x \in V(B_1 \cup B_2)\} \cup \{u_jv_j, u_jw_j, v_jw_j, u_jv_{j+1}, v_ju_{j+1}, w_ju_{j+1}, w_jv_{j+1} : j \in [m-1]\} \cup \{w_{-1}u_1, w_{-1}v_1, w_0u_1, w_0v_1, w_{m+1}u_m, w_{m+1}v_m\}$, where $m = 7^5 - 11$.

We construct A_1 as in Figure 6.2, S_i's are the horizontal paths from $b_1, w_{-1}, w_0, w_1, w_2$ to $b_2, w_{m+1}, w_m, w_{m-1}, w_{m-2}$, respectively for $i \in [5]$. So $V(A_1) = \{a_1\} \cup \bigcup_{i\in[5]} V(S_i)$. For each $i \in [5]$, let x^i_j be the j-th vertex from left to right on S_i. For any vertex $x \in V(S_i) \setminus \{w_1, w_{m-1}\}$ where $i \in \{0\} \cup [4]$, $|N_{S_{i+1}}(x)| = 7$.

In the following sections, we show that G is infeasible and 7-connected.

6.1 Infeasibility

Suppose G is feasible and let P be the a_1-a_2 path such that there exist two independent b_1-b_2 paths Q_1, Q_2 in $G - P$. Denote $T_i = \{u_i, v_i, w_i\}$ for $i \in [m]$.

Now, let $w_j \in V(P)$ be such that $V(P[a_2, w_j)) \cap V(B_3) = \emptyset$. Since $P \cap S_i \neq \emptyset$ for all $i \in [5]$, $|(V(Q_1) \cup V(Q_2)) \cap T_k| = 2$ for $k \in \{j, j+1\}$. Let $x \in N_{P[a_2,w_j)}(w_j)$. Then, $x \in \{u_i, v_i | i \in \{j, j+1\}\}$. Suppose $x \in \{u_j, v_j\}$. Then, $|V(P) \cap T_j| = 2$, and thus, $|(V(Q_1) \cup V(Q_2)) \cap T_j| = 1$, a contradiction.

So $x \in \{u_{j+1}, v_{j+1}\}$. Since $w_{j+1}u_j, w_{j+1}v_j \notin E(G)$, $(V(Q_1) \cup V(Q_2)) \cap T_{j+1} = \{u_{j+1}, v_{j+1}\} \setminus \{x\}$, a contradiction.

Hence, (G, a_1, a_2, b_1, b_2) is indeed infeasible. □

6.2 7-connectivity

Suppose not, let T be a minimum cut of G. Then, $|T| \leq 6$. Note that with our construction, $V(H) \cap V(A_1) = V(B_3)$. For simplicity, paths will be represented as sequences of vertices with consecutive vertices adjacent. For path P and $u, v \in V(P)$, we denote uPv be the subpath of P from u to v. For vertices u, v, w such that uv, vw are edges, we use uvw to denote the v-w path of length 2.

Claim 6.2.1. *All components of $G - T$ intersect $V(B_3)$.*

Proof. Suppose for a contradiction, there exists a component C of $G-T$ such that $V(C) \cap V(B_3) = \emptyset$. Then, $V(C) \subseteq V(H - B_3)$ or $V(C) \subseteq V(A_1 - B_3)$.

First, suppose $V(C) \subseteq V(A_1 - B_3)$. Then, there exists $x_j \in V(C) \cap V(S_j)$ for some $0 \leq j \leq 4$. For any $j \leq i \leq 4$ and $x_i \in V(C) \cap V(S_i)$, since $|N_{S_{i+1}}(x_i)| = 7 > |T|$, there exists $x_{i+1} \in V(C) \cap N_{S_{i+1}}(x_i)$. Hence, there exists $x_5 \in V(C) \cap V(S_5)$, a contradiction.

So $C \subseteq V(H - B_3)$. Clearly, $C \neq H - B_3$; otherwise $V(B_3) \subseteq V(T)$, a contradiction. We claim that $a_2 \notin C$. Suppose $a_2 \in C$. Since $a_2 \in N_H(x)$ for all $x \in V(B_1 \cup B_2)$, $|N_C(a_2)| \geq \deg(a_2) - |T| = 2m + 2 - 6 = 2m - 4$. Since $N_C(a_2) \setminus T \subseteq V(B_1 \cup B_2)$, $|N_{B_3}(C)| \geq \frac{|N_C(a_2) \setminus T|}{2} \geq m - 2 > |T|$, a contradiction.

Hence there exists $x \in V(C)$ and $y \in V(H - B_3) \setminus V(C)$. Since $u_iv_i, u_ju_{j+1} \in E(H)$ for all $i \in [m]$ and $j \in [m-1]$, $\{x, y\} \neq \{u_i, v_i\}$ and $\{x, y\} \neq \{u_j, u_{j+1}\}$. By symmetry and without loss of generality, $\{x, y\} = \{u_i, u_j\}$ for some $1 \leq i < i + 1 < j \leq m$. But, there exist the following 7 independent u_i-u_j paths in G: $u_ia_2u_j$, $u_iB_1u_j$, $u_iv_{i+1}B_2v_{j-1}u_j$, $u_iw_iB_3w_{j-1}u_j$, $u_iB_1b_1S_1b_2B_1u_j$, $u_iv_iB_2v_1w_{-1}S_2w_{m+1}v_mB_2v_ju_j$, $u_iw_{i-1}B_3w_0S_3w_mB_3w_ju_j$.

□

By Claim 6.2.1, there exist $x, y \in V(B_3)$ such that x, y belongs to different components of $G - T$. Note that $xy \notin E(B_3)$. But we can find 7 independent x-y paths in all cases as the following, which leads to a contradiction:

Case 1. $x = b_1, y = w_0$.

The 7 independent x-y paths are: $b_1a_2v_2w_1w_0$, $b_1u_1w_0$, $b_1v_1w_0$, $b_1B_3w_0$, $b_1x_2^2x_{2\cdot 7}^3S_3w_0$, $b_1x_3^2x_{3\cdot 7}^3x_{3\cdot 7^2}^4S_4x_3^4w_0$, $b_1S_1b_2B_3x_7^5x_2^4w_0$.

Case 2. $x = b_1, y = w_i$ for $i \in [m-1]$.

The 7 independent x-y paths are: $b_1a_2u_{i+1}w_i$, $b_1B_1u_iw_i$, $b_2B_2v_iw_i$, $b_1B_3w_i$, $b_1x_2^2x_{2\cdot 7}^3x_{2\cdot 7^2}^4S_4sw_i$ where $\{s\} = N_{S_4}(w_i)$, $b_1x_3^2x_{3\cdot 7}^3S_3w_mB_3w_i$, $b_1S_1b_2B_2v_{i+1}w_i$.

Case 3. $x = b_1, y = w_i$ for $i \in \{m, m+1\}$.

The 7 independent x-y paths are: $b_1a_2u_mw_i$, $b_1B_1u_{m-1}w_{m-1}B_3w_i$, $b_1B_2v_mw_i$, $b_1a_1b_2B_3w_i$, $b_1S_1x_6^1x_{6\cdot 7}^2x_{6\cdot 7^2}^3S_3w_m$ or $b_1S_1x_6^1x_{6\cdot 7}^2S_2w_{m+1}$,

$b_1x_3^2x_{3\cdot 7}^3x_{3\cdot 7^2}^4S_4x_{7^4-3}^4w_m$ or $b_1x_3^2x_{3\cdot 7}^3S_3x_{7^3-3}^3w_{m+1}$, $b_1x_2^2x_{2\cdot 7}^3x_{2\cdot 7^2}^4x_{7(2\cdot 7^2-1)}^5S_5w_{m-2}x_{7^4-1}^4w_m$ or $b_1x_2^2x_{2\cdot 7}^3x_{2\cdot 7^2}^4S_4x_{7(7^3-1)}^4x_{7^3-1}^3w_{m+1}$.

Case 4. $x = b_1, y = b_2$.

The 7 independent x-y paths are: $b_1a_2b_2$, $b_1B_1b_2$, $b_1B_2b_2$, $b_1B_3b_2$, $b_1a_0b_2$, $b_1S_1b_2$, $b_1x_2^2S_2x_{7^2-1}^2b_2$.

Case 5. $x = w_i, y = w_j$ where $i \in \{-1, 0\}$ and $j \in [m-1]$.

The 7 independent x-y paths are: $w_iu_1a_2u_{j+1}w_j$, $w_iv_1B_2v_jw_j$, $w_iB_3w_1u_2B_1u_jw_j$, $w_iB_3b_1S_1b_2B_2v_{j+1}w_j$, $w_{-1}S_2w_{m+1}B_3w_j$ or $w_0S_3w_mB_3w_j$, $w_{-1}x_3^3x_{3\cdot 7}^4S_4sw_j$ or $w_0x_3^4S_4sw_j$ where $\{s\} = N_{S_4}(w_j)$, $w_{-1}x_2^3x_{2\cdot 7}^4S_4x_2^4w_2B_3w_j$ or $w_0x_2^4w_2B_3w_j$.

Case 6. $x = w_i, y = w_j$ where $i \in \{-1, 0\}$ and $j \in \{m, m+1\}$.

The 7 independent x-y paths are: $w_iu_1a_2u_mw_j$, $w_iB_3w_1u_2B_1u_{m-1}w_{m-1}B_3w_j$, $w_iv_1B_2v_mw_j$, $w_iB_3b_1S_1b_2B_3w_j$, and the other three paths are X_1, X_2, X_3, where $\{X_1, X_2, X_3\}$ is one of the following: $\{w_{-1}x_4^3S_3w_m, w_{-1}x_3^3x_{3\cdot 7}^4S_4x_{7^4-2}^4w_m, w_{-1}x_2^3x_{2\cdot 7}^4x_{7(2\cdot 7-1)}^5S_5w_{m-2}x_{7^4-1}^4w_m\}$, $\{w_{-1}S_2w_{m+1}, w_{-1}x_4^3S_3x_{7^3-3}^3w_{m+1}, w_{-1}x_3^3x_{3\cdot 7}^4S_4x_{7(7^3-1)}^4x_{7^3-1}^3w_{m+1}\}$, $\{w_0S_3w_m, w_0x_3^4S_4x_{7^4-2}^4w_m, w_0x_2^4x_7^5S_5w_{m-2}x_{7^4-1}^4w_m\}$, or $\{w_0S_3x_{7^3-3}^3w_{m+1}, w_0x_3^4S_4x_{7(7^3-2)}^4x_{7^3-2}^3w_{m+1}, w_0x_2^4x_7^5S_5x_{7(7(7^3-1)-1)}^5x_{7(7^3-1)}^4x_{7^3-1}^3w_{m+1}\}$.

Case 7. $x = w_i, y = b_2$ where $i \in \{-1, 0\}$.

The 7 independent x-y paths are: $w_iu_1a_2b_2$, $w_iv_1B_2b_2$, $w_iB_3w_1u_2B_1b_2$, $w_iB_3b_1a_1b_2$, and the other three paths are X_1, X_2, X_3, where $\{X_1, X_2, X_3\}$ is one of the following: $\{w_{-1}S_2x_{7^2-2}^2b_2, w_{-1}x_3^3S_3x_{7(7^2-1)}^3x_{7^2-1}^2b_2, w_{-1}x_2^3x_{2\cdot 7}^4x_{7(2\cdot 7-1)}^5B_3b_2\}$ or $\{w_0S_3x_{7(7^2-2)}^3x_{7^2-2}^2b_2, w_0x_3^4S_4x_{7^2(7^2-1)}^4x_{7(7^2-1)}^3x_{7^2-1}^2b_2, w_0x_2^4x_7^5B_3b_2\}$.

Case 8. $x = w_i, y = w_j$ for $1 \le i < i+1 < j \le m-1$.

The 7 independent x-y paths are: $w_iu_ia_2u_{j+1}w_j$, $w_iu_{i+1}B_1u_jw_j$, $w_iv_{i+1}B_2v_jw_j$, $w_iB_3w_j$, $w_iv_iB_2b_1S_1b_2B_2v_{j+1}w_j$, $w_iB_3w_0S_3w_mB_3w_j$, $w_isS_4tw_j$ where $\{s\} = N_{S_4}(w_i)$ and $\{t\} = N_{S_4}(w_j)$.

Case 9. $x = w_i, y = w_j$ where $i \in [m-1]$ and $j \in \{m, m+1\}$.

Note that $\{x, y\} \neq \{w_{m-1}, w_m\}$. The 7 independent x-y paths are: $w_iu_ia_2u_mw_j$, $w_iu_{i+1}B_1u_{m-1}w_{m-1}B_3w_j$, $w_iv_{i+1}B_2v_mw_j$, $w_iv_iB_2b_1S_1b_2B_3w_j$, $w_iB_3w_{-1}S_2w_{m+1}(w_m)$, $w_iB_3w_{m-2}x^4_{7^4-1}w_m(w_{m+1})$, $w_isS_4x^4_{7^4-2}w_m$ or $w_isS_4x^4_{7(7^3-2)}x^3_{7^3-2}w_{m+1}$ where $\{s\} = N_{S_4}(w_i)$.

Case 10. $x = w_i, y = b_2$ for $i \in [m-1]$.

The 7 independent x-y paths are: $w_iu_ia_2b_2$, $w_iu_{i+1}B_1b_2$, $w_iv_{i+1}B_2b_2$, $w_iB_3b_2$, $w_iv_iB_2b_1S_1b_2$, $w_iB_3w_{-1}S_2x^2_{7^2-3}b_2$, $w_isS_4x^4_{7^2(7^2-2)}x^3_{7(7^2-2)}x^2_{7^2-2}b_2$ where $\{s\} = N_{S_4}(w_i)$.

Case 11. $x = w_m, y = b_2$.

The 7 independent x-y paths are: $w_mu_mb_2$, $w_mv_mb_2$, $w_mB_3b_2$, $w_mw_{m-1}u_{m-1}a_2b_2$, $w_mx^4_{7^4-1}w_{m-2}B_3b_1S_1b_2$, $w_mx^4_{7^4-2}S_4x^4_{7^2(7^2-2)}x^3_{7(7^2-2)}x^2_{7^2-2}b_2$, $w_mS_3x^3_{7(7^2-1)}x^2_{7^2-1}b_2$. □

Hence, G is a 7-connected infeasible example as desired.

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