ON MULTISTAGE STOCHASTIC AND DISTRIBUTIONALLY ROBUST OPTIMIZATION: NEW ALGORITHMS, COMPLEXITY ANALYSIS, AND PERFORMANCE COMPARISON

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ON MULTISTAGE STOCHASTIC AND DISTRIBUTIONALLY ROBUST OPTIMIZATION: NEW ALGORITHMS, COMPLEXITY ANALYSIS, AND PERFORMANCE COMPARISON

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SUMMARY

Multistage optimization under uncertainty refers to sequential decision-making with the presence of uncertainty information that is revealed partially until the end of planning horizon. Depending on the uncertainty model, it is often studied as multistage stochastic optimization (MSO), where one seeks optimal decisions with minimum mean objective with respect to a certain probabilistic uncertainty model; or more generally multistage distributionally robust optimization (MDRO), where one seeks optimal decisions with respect to a worst-case probability distribution over a candidate set of distributions. Both approaches have found ubiquitous applications such as in energy system and inventory planning.

First, we focus on MSO with possibly integer variables and nonlinear constraints. We develop dual dynamic programming (DDP) type algorithms with nested decomposition, deterministic sampling, and stochastic sampling. Several interesting classes of MSO problems are identified, where the new algorithms are guaranteed to obtain the global optimum without the assumption of complete recourse. We also characterize the iteration complexity of the proposed algorithms, which reveals that the iteration complexity depends polynomially on the number of stages. We further show that the iteration complexity depends linearly on the number of stages T, if all the state spaces are finite sets, or if we allow the optimality gap to scale with T. This complexity study resolves an open question on the iteration complexity of DDP-type algorithms.

Second, we propose a new class of algorithms for solving convex MDRO problems, namely a consecutive dual dynamic programming (DDP) algorithm and a nonconsecutive version. The new algorithms generalize and strengthen existing DDP-type algorithms by introducing an important technique of regularization that enables the algorithms to handle much broader classes of MDRO problems. We then define single stage subproblem oracles (SSSO) and provide a thorough complexity analysis of the new algorithms, proving both upper complexity bounds and a matching lower bound. Numerical examples on in-

ventory problems and hydrothermal power system planning problems are given to show the effectiveness of the proposed regularization technique.

Third, we consider convex MDRO with Wasserstein ambiguity sets constructed from stagewise independent empirical distributions. We show that these data-driven MDRO models have favorable out-of-sample performance guarantees and adjustable levels of in-sample conservatism. Then we extend the DDP algorithms to the data-driven MDRO by proposing two SSSO realizations that are able to handle the Wasserstein ambiguity sets, exploiting either convexity or concavity of the uncertain cost functions, which happens when the uncertainty only appears in the right-hand-side of the constraints or in the objective function. Extensive numerical experiments on inventory problems are then conducted to compare these data-driven MDRO models with the widely used risk-neutral and risk-averse empirical MSO models.

CHAPTER 1
INTRODUCTION

1.1 Background

A multistage stochastic optimization (MSO) is a sequential decision making problem under uncertainty modelled by random vectors in each stage. We allow both continuous and integer decisions and nonconvex nonlinear objective function and constraints for MSO problems, which provides an extremely powerful modeling framework. Special classes of MSO, such as multistage stochastic linear optimization (MSLO) and mixed-integer linear optimization (MSMILO), have already found a wide range of applications in diverse fields such as electric power system scheduling and expansion planning [1, 2, 3, 4], portfolio optimization under risk [5, 6, 7], and production and capacity planning problems [8, 9, 10, 11], just to name a few.

More generally, multistage distributionally robust optimization (MDRO) is an MSO where the probability distributions of the random vectors are not known precisely so we seek an optimal decision considering the worst-case probability distributions in a given ambiguity set in each stage. MDRO provides a unified framework for studying decision-making under uncertainty, which encompasses both MSO and multistage robust optimization (MRO), where in the latter all atomic probability distributions are included in the ambiguity sets. Thus MDRO finds ubiquitous applications in addition to the areas mentioned above (see e.g. [12, 13]).

Distributionally robust optimization (DRO) has received significant research attention, especially in the context of single-stage or two-stage settings. Common choices of the ambiguity sets include the moment-based ambiguity sets [14, 15, 16, 17], discrepancy or distance-based ambiguity sets [18, 19, 20, 21], and others [22, 23]. In particular, DRO con-

structed from Wasserstein distance [24] has both out-of-sample performance guarantee and finite-dimensional tractable reformulation that does not require any approximation, if it is centered at the empirical distributions of the data [25, 26, 27]. Moreover, empirical success has been observed on the data-driven Wasserstein MDRO problems, under the assumption of finite uncertainty supports [28]. In this dissertation, we aim to further study the performance of data-driven Wasserstein MDRO that are solved by algorithms with guaranteed convergence, and have a general uncertainty support set.

In the literature for stochastic optimization, classical algorithmic approaches to tackle two-stage stochastic linear optimization (2SLO) include Benders decomposition [29], Dantzig-Wolfe decomposition [30], and the L-shaped method [31]. Nested decomposition procedures for deterministic models are developed in [32, 33]. Louveaux [34] first generalized the two-stage L-shaped method to multistage quadratic problems. Nested Benders decomposition for MSLO was first proposed in Birge [35] and Pereira and Pinto [36]. However, it is known that multistage problems (MSO and MDRO) are in general extremely challenging to solve, due to the exponentially fast growth of the number of decisions with respect to the number of decision stages [37, 38, 39]. Meanwhile, real-world problems are often endowed with special structures in the uncertainty. In particular, the uncertainty may exhibit stagewise independence (SI), i.e., the uncertainty in different stages are independent from each other. Many uncertainty structures, such as autoregressive stochastic models, can be reformulated to satisfy SI [40]. This versatile modeling capability of SI has great implications on computation. It allows recursive formulation of a cost-to-go function in each stage of an MDRO to be independent of the outcomes in its previous stages, thus making efficient approximations of the cost-to-go functions possible. Therefore, SI has been successfully exploited by various dual dynamic programming (DDP) algorithms, in solving MSO and MRO [40, 41, 39, 42, 43, 44, 45, 46, 47, 42].

Dual dynamic programming (DDP) is a class of recursive cutting plane algorithms that originate from nested Benders decomposition for multistage stochastic linear optimiza-

tion [35, 36]. The earliest form of DDP for MSLO using stochastic sampling method was proposed in [48], known as stochastic dual dynamic programming (SDDP), where in each iteration the scenarios are sampled randomly and solved sequentially before updating the cost-to-go functions recursively. Then the DDP algorithm has been extended to multistage stochastic convex optimization (MSCO) and has been widely adopted in areas such as energy systems scheduling [49, 50, 51]. The deterministic over-approximation was also proposed for an upper bound on the policy value [52], and later incorporated into a deterministic sampling version of the DDP algorithm [43, 44].

Recently, DDP has been further extended to MDRO with promising out-of-sample performance [53, 54, 55, 28]. In particular, [55] uses ambiguity sets defined by the χ^2-distance. In [28], the ambiguity sets are taken to be finitely supported Wasserstein metric balls centered at the empirical distributions, and the algorithm is shown to converge asymptotically with stochastic sampling methods. We comment that many of the above variants of DDP algorithms rely on the assumption of relatively complete recourse, while it is indeed possible to have MSCO without such assumption [56]. Another important constituent of DDP algorithms is known as the robust dual dynamic programming (RDDP), which is proposed for multistage robust linear optimization (MRLO) [39]. Due to its intrinsic difficulty, the uncertainty sets are assumed to be polytopes such that the subproblem in each stage can be solved via a vertex enumeration technique over the uncertainty set. Similar to the deterministic DDP, RDDP constructs both over- and under-approximations to select the worst-case outcome. Moreover, it has the advantage of being able to terminate the algorithm with a guaranteed optimal first stage solution, in contrast to the commonly used decision rules [57, 58, 59], and to handle problems without relatively complete recourse.

Besides the success in MSCO and MRLO, the DDP algorithm class has been extended recently to nonconvex settings [46]. It is observed that the cuts generated from Lagrangian relaxation of the nodal problems in an MSMILO are always tight at the given parent node's state, as long as all the state variables only take binary values and have complete recourse.

From this fact, the algorithm, known as the stochastic dual dynamic integer programming (SDDiP), is proved to find an exact optimal solution in finitely many iterations with probability one. In this way, the SDDiP algorithm makes it possible to solve nonconvex mixed-integer linear MSO problems through binarization of the state variables [3, 60]. In addition, when the value functions of MSMILO with general integer state variables are assumed to be Lipschitz continuous, augmented Lagrangian cuts with an additional reverse norm term to the linear part obtained via augmented Lagrangian duality are proposed in [45]. Despite the rapid development, it remains challenging to generalize DDP to most general MSO because there is no general cutting plane mechanism for generating exact under-approximation of nonconvex, discontinuous, or non-Lipschitzian value functions.

The convergence analysis of the DDP algorithms begins with the MSLO [41, 40, 61, 62, 63], where almost sure finite convergence is established based on the polyhedral nodal problem structures. For MSCO, if the value functions are Lipschitz continuous and the state space is compact, asymptotic convergence of the under-approximation of the value functions leads to asymptotic convergence of the optimal value and optimal solutions [64, 65]. By constructing over-approximations of value functions, the DDP with a deterministic sampling method is proved to converge asymptotically for the MSCO in [43]. There is a recent, independent work [66, 67], which proves iteration complexity upper bounds for MSCO under the assumption that all the value functions and their under-approximations are all Lipschitz continuous. It is shown that for discounted problems, the iteration complexity depends linearly on the number of stages. However, it is not clear whether the proposed iteration complexity bound is sharp, or whether the Lipschitz constants are determined by the problem data or affected by algorithmic choices. This dissertation aims to resolve these issues, as well as extending the complexity results of DDP algorithms to both mixed-integer nonlinear MSO, and convex MDRO.

1.2 Contributions

One motivational conjecture was suggested to us by the late Prof. Shabbir Ahmed.

Conjecture 1.1. *The number of iterations needed for SDDP/SDDiP to find an optimal first-stage solution grows linearly in terms of the number of stages T, while it may depend nonlinearly on other parameters such as the diameter D and the dimension d of the state space.*

In Chapter 2, our study resolves this conjecture by giving a full picture of the iteration complexity of DDP algorithms in the general setting of MSO. In particular, the chapter contains the following contributions to the literature.

1. We provide simple examples to illustrate that the value function of a two-stage stochastic program with complete recourse could be non-Lipschitzian or discontinuous. Existing DDP algorithms and complexity analyses cannot handle such situations. We propose *a regularization approach* to provide a surrogate of the original problem such that the value functions become Lipschitz continuous. In many cases, the regularized problem preserves the set of optimal solutions.

2. We use the theory of generalized conjugacy to develop a cut generation scheme, referred to as *generalized conjugacy cuts*, that are valid for value functions of MSO. Moreover, generalized conjugacy cuts are shown to be tight to the regularized value functions. The generalized conjugacy cuts can be replaced by linear cuts without compromising such tightness when the problem is convex.

3. With the regularization and the generalized conjugacy cuts, we propose three algorithms for MSO based on nested decomposition for general scenario trees, DDP with random sampling as well as deterministic sampling similar to [43] for the convex case and random sampling, both for stagewise independent scenario trees.

4. We obtain upper and lower bounds on the iteration complexity for the proposed SDDP with both sampling methods for MSO problems. The complexity bounds show that in general, Conjecture 1.1 holds if only we seek a $(T\varepsilon)$-optimal solution, instead of an ε-optimal first-stage solution for a $(T+1)$-stage problem, or when all the state spaces are finite sets.

In Chapter 3, we extend the DDP algorithm framework to MDRO, with convexity assumed for simplicity, together with the complexity analysis based on single stage subproblem oracles. Due to the distributional uncertainty in the model, the commonly used statistical upper bound for the policy evaluation in the MSCO literature (e.g., [40, 46]) is no longer valid for the MDRO problems. As a result, the current computational implementations in [55] and [28] choose to terminate at a fixed number of iterations or cuts, without a good guarantee of the solution quality. To overcome the lack of statistical upper bound, we further explore the deterministic upper bound approximations, similar to the one studied for MSO in Chapter 2 and [43], and the one in RDDP [39]. In particular, the chapter contains the following contributions to the literature.

1. We provide a unified framework for studying convex MDRO under SI assumption. For finitely supported convex MDRO in this framework, we construct a novel example to show that the traditional cutting plane method can easily cause the Lipschitz constants of the stage problem to grow with respect to the number of stages. Motivated by this phenomenon, we introduce an important algorithmic technique of regularization to DDP, which can effectively control the growth of Lipschitz constraints and can dispense with the relatively complete recourse assumption.

2. A new class of consecutive DDP algorithms (CDDP) and its nonconsecutive version (NDDP) based on regularization is proposed for solving MDRO Complexity upper bounds based on single stage subproblem oracles are proved for both CDDP and NDDP for the first time.

3. We construct a class of multistage robust convex problems to obtain a complexity lower bound for the new algorithms for the first time, which shows the complexity upper bounds are essentially tight. The complexity bounds can be applied to more general MDRO problems with continuous distributions.

4. Numerical results on a multi-commodity inventory problem and a hydro-thermal power planning problem are given to illustrate the two effects of regularization: capability to solve problems without relatively complete recourse and reduction in the computation time and number of subproblem oracle evaluations.

In Chapter 4, we turn our attention to data-driven MDRO models using Wasserstein ambiguity sets, to address the practical challenge of lacking big data in many MDRO applications. Using the DDP algorithm developed in Chapter 3, we then compare the out-of-sample performance of the Wasserstein MDRO models against the commonly used MSO and MRO models. In particular, the chapter contains the following contributions to the literature.

1. We review the out-of-sample performance guarantee of Wasserstein DRO models and extend it to the multistage settings. We derive the finite dimensional dual recursive formulation for Wasserstein MDRO models under a verifiable assumption, and prove that the in-sample conservatism of Wasserstein MDRO is adjustable given Lipschitz continuity of the value functions.

2. We develop two SSSO implementations by exploiting either the concavity or the convexity of the uncertain local cost functions. Consequently, the DDP algorithms proposed in Chapter 3 can be applied on Wasserstein MDRO models with valid complexity bounds.

3. Extensive numerical experiments on multi-commodity inventory problems are conducted to compare the Wasserstein MDRO models against the nominal risk-neutral

MSO model, the conditional value-at-risk (CVaR) risk-averse MSO models, and the standard MRO model. The numerical results suggest that for the Wasserstein MDRO models are particularly favorable for small data sizes.

The contents in Chapter 2 come from our manuscript [68], which has received honorable mention for the *INFORMS Optimization Society Best Student Paper Prize 2021*. The contents in Chapter 3 come from our manuscript [69], and the contents in Chapter 4 come from our manuscript [70].

CHAPTER 2
ITERATION COMPLEXITY OF DDP ALGORITHMS ON MIXED-INTEGER NONLINEAR MSO

2.1 Problem Formulations

In this section, we first present the extensive and recursive formulations of multistage optimization. Then we characterize the properties of the value functions, with examples to show that they may fail to be Lipschitz continuous even when the constraints are all convex and Lipschitz continuous. With this motivation in mind, we propose a penalty reformulation of the multistage problem through regularization of value functions and show that it is equivalent to the original formulation for a broad class of problems. Finally, we propose generalized conjugacy cuts for under-approximation of value functions.

2.1.1 Extensive and Recursive Formulation

For a multistage stochastic optimization (MSO), let $\mathcal{T} = (\mathcal{N}, \mathcal{E})$ be the scenario tree, where $\mathcal{N}$ is the set of nodes and $\mathcal{E}$ is the set of edges. For each node $n \in \mathcal{N}$, let $a(n)$ denote the parent node of n, $\mathcal{C}(n)$ denote the set of child nodes of n, and $\mathcal{T}(n)$ denote the subtree starting from the node n. Given a node $n \in \mathcal{N}$, let $t(n)$ denote the stage that the node n is in and let $T \coloneqq \max_{n\in\mathcal{N}} t(n)$ denote the last stage of the tree $\mathcal{T}$. A node in the last stage is called a leaf node, otherwise a non-leaf node. The set of nodes in stage t is denoted as $\mathcal{N}(t) \coloneqq \{n \in \mathcal{N} : t(n) = t\}$. We use the convention that the root node of the tree is denoted as $r \in \mathcal{N}$ with $t(r) = 0$ so the total number of stages is $T + 1$. The parent node of the root node is denoted as $a(r)$, which is a dummy node for ease of notation.

For every node $n \in \mathcal{N}$, let $\mathcal{F}_n$ denote the feasibility set in some Euclidean space of decision variables (x_n, y_n) of the nodal problem at node n. We refer to x_n as the state

variable and y_n as the internal variable of node n. Denote the image of the projection of $\mathcal{F}_n$ onto the subspace of the variable x_n as $\mathcal{X}_n$, which is referred to as the state space. Let $x_{a(r)} = 0$ serve as a dummy parameter and thus $\mathcal{X}_{a(r)} = \{0\}$. The nonnegative nodal cost function of the problem at node n is denoted as $f_n(x_{a(n)}, y_n, x_n)$ and is defined on the set $\mathcal{X}_{a(n)} \times \varsigma_n \mathscr{F}_n = \{(z, y, x) : z \in \mathcal{X}_{a(n)}, (x, y) \in \mathcal{F}_n\}$, where $\varsigma_n(x_n, y_n) := (y_n, x_n)$ is the swap map. We allow f_n to take the value $+\infty$ so indicator functions can be modeled as part of the cost. Let $p_n > 0$ for all $n \in \mathcal{N}$ denote the probability that node n on the scenario tree is realized. For the root node, $p_r = 1$. The transition probability that node m is realized conditional on its parent node n being realized is given by $p_{nm} := p_m/p_n$ for all edges $(n, m) \in \mathcal{E}$.

The MSO considered in this chapter is defined in the following extensive form:

$$v^{\text{prim}} := \min_{\substack{(x_n, y_n) \in \mathcal{F}_n, \\ \forall\, n \in \mathcal{N}}} \sum_{n \in \mathcal{N}} p_n f_n(x_{a(n)}, y_n, x_n). \tag{2.1}$$

The recursive formulation of the MSO (2.1) is defined as

$$Q_n(x_{a(n)}) := \min_{(x_n, y_n) \in \mathcal{F}_n} \left\{ f_n(x_{a(n)}, y_n, x_n) + \sum_{m \in \mathcal{C}(n)} p_{nm} Q_m(x_n) \right\}, \tag{2.2}$$

where $n \in \mathcal{T}$ is a non-leaf node and $Q_n(x_{a(n)})$ is the *value function* of node n. At a leaf node, the sum in (2.2) reduces to zero, as the child node $\mathcal{C}(n) = \varnothing$. The problem on the right-hand side of (2.2) is called the *nodal problem* of node n. Its objective function consists of the nodal cost function f_n and the *expected cost-to-go function*, which is denoted as $\mathcal{Q}_n$ for future reference, i.e.

$$\mathcal{Q}_n(x_n) := \sum_{m \in \mathcal{C}(n)} p_{nm} Q_m(x_n). \tag{2.3}$$

To ensure that the minimum in problem (2.1) is well defined and finite, we make the following very general assumption on f_n and $\mathcal{F}_n$ throughout the chapter.

Assumption 2.1. *For every node $n \in \mathcal{N}$, the feasibility set $\mathcal{F}_n$ is compact, and the nodal*

cost function f_n is lower semicontinuous (lsc). The sum $\sum_{n\in\mathcal{N}} p_n f_n$ is a proper function, i.e., there exists $(x_n, y_n) \in \mathcal{F}_n$ for all nodes $n \in \mathcal{N}$ such that $\sum_{n\in\mathcal{N}} p_n f_n(x_{a(n)}, y_n, x_n) < +\infty$.

Note that the state variable $x_{a(n)}$ only appears in the objective function f_n of node n, not in the constraints. Perhaps the more common way is to allow $x_{a(n)}$ to appear in the constraints of node n. It is easy to see that any such constraint can be modeled by an indicator function of $(x_{a(n)}, x_n, y_n)$ in the objective f_n.

We next characterize some important continuity properties of the value function.

2.1.2 Continuity and Convexity of Value Functions

The following proposition presents some basic properties of the value function Q_n under Assumption 2.1.

Proposition 2.1. *Under Assumption 2.1, the value function Q_n is lower semicontinuous (lsc) for all $n \in \mathcal{N}$. Moreover, for any node $n \in \mathcal{N}$,*

1. *if $f_n(z, y, x)$ is Lipschitz continuous in the first variable z with constant l_n, i.e. $|f_n(z, y, x) - f_n(z', y, x)| \le l_n\|z - z'\|$ for any $z, z' \in \mathcal{X}_{a(n)}$ and any $(x, y) \in \mathcal{F}_n$, then Q_n is also Lipschitz continuous with constant l_n;*

2. *if $\mathcal{X}_{a(n)}$ and $\mathcal{F}_n$ are convex sets, and f_n and $\mathcal{Q}_n$ are convex functions, then Q_n is also convex.*

Proof. We show that Q_n is lsc by showing the lower level sets $\mathrm{lev}_a(Q_n) = \{z \in \mathcal{X}_{a(n)} : Q_n(z) \le a\}$ are closed for all $a \in \mathbb{R}$. At any leaf node n, the expected cost-to-go function $\mathcal{Q}_n(x_n)$ is zero, thus z is in $\mathrm{lev}_a(Q_n)$ if and only if z is in the projection of the following set $\{(z, y, x) : (x, y) \in \mathcal{F}_n,\ f_n(z, y, x) \le a\}$. Since f_n is defined on a compact set $\{(z, y, x) : z \in \mathcal{X}_{a(n)}, (x, y) \in \mathcal{F}_n\}$ and lsc by Assumption 2.1, we know that the set $\{(z, y, x) : f_n(z, y, x) \le a\}$ is compact. Moreover, since the projection $(z, y, x) \mapsto z$ is continuous,

the image $\mathrm{lev}_a(Q_n)$ is still compact, hence closed.

At any non-leaf node n, suppose Q_m is lsc for all its child nodes $m \in \mathcal{C}(n)$. Then, $\mathcal{Q}_n$ is lsc since $\mathcal{Q}_n$ is defined in (2.3) and $p_{nm} > 0$ for all m. A point $z \in \mathrm{lev}_a(Q_n)$ if and only if z is in the projection of the set $\{(z, y, x) : (y, x) \in \mathcal{F}_n,\ f_n(z, y, x) + \mathcal{Q}_n(x) \leq a\}$. Similarly, this shows $\mathrm{lev}_a Q_n$ is closed since f_n, $\mathcal{Q}_n$ are lsc and the projection $(z, y, x) \mapsto z$ is continuous. We thus conclude Q_n is lsc for every node n in the scenario tree.

To show claims 1 and 2 in the proposition, take any two points $z_1, z_2 \in \mathcal{X}_{a(n)}$. Suppose $(x_1, y_1),\ (x_2, y_2) \in \mathcal{F}_n$ are the corresponding minimizers in the definition (2.2). Therefore, $Q_n(z_1) = f_n(z_1, y_1, x_1) + \mathcal{Q}_n(x_1)$ and $Q_n(z_2) = f_n(z_2, y_2, x_2) + \mathcal{Q}_n(x_2)$. If f_n is Lipschitz continuous in the first variable, then we have

$$\begin{aligned} Q_n(z_1) - Q_n(z_2) &= f_n(z_1, y_1, x_1) + \mathcal{Q}_n(x_1) - f_n(z_2, y_2, x_2) - \mathcal{Q}_n(x_2) \\ &\leq f_n(z_1, y_2, x_2) + \mathcal{Q}_n(x_2) - f_n(z_2, y_2, x_2) - \mathcal{Q}_n(x_2) \\ &\leq f_n(z_1, y_2, x_2) - f_n(z_2, y_2, x_2) \leq l_n \left\| z_1 - z_2 \right\|. \end{aligned}$$

Likewise, by exchanging z_1 and z_2, we know that $Q_n(z_2) - Q_n(z_1) \leq l_n \left\| z_1 - z_2 \right\|$. This proves that Q_n is Lipschitz continuous with the constant l_n.

To show that Q_n is convex, take any $t \in [0, 1]$. Since $\mathcal{X}_{a(n)}$ is convex, Q_n is defined at $tz_1 + (1-t)z_2$. Thus,

$$\begin{aligned} &Q_n(tz_1 + (1-t)z_2) \\ &\leq f_n(tz_1 + (1-t)z_2, ty_1 + (1-t)y_2, tx_1 + (1-t)x_2) + \mathcal{Q}_n(tx_1 + (1-t)x_2) \\ &\leq t f_n(z_1, y_1, x_1) + (1-t) f_n(z_2, y_2, x_2) + t\mathcal{Q}_n(x_1) + (1-t)\mathcal{Q}_n(x_2) \\ &= tQ_n(z_1) + (1-t)Q_n(z_2). \end{aligned}$$

The first inequality follows from the definition (2.2), while the second inequality follows from the convexity of f_n and $\mathcal{Q}_n$. This shows Q_n is convex. □

When Q_m is lsc for all $m \in \mathcal{C}(n)$, the sum $\sum_{m\in\mathcal{C}(n)} p_{nm}Q_m$ is lsc. Therefore, the minimum in the definition (2.2) is well define, since $\mathcal{F}_n$ is assumed to be compact.

If the objective function $f_n(x_{a(n)}, y_n, x_n)$ is not Lipschitz, e.g., when it involves an indicator function of $x_{a(n)}$, or equivalently when $x_{a(n)}$ appears in the constraint of the nodal problem of $Q_n(x_{a(n)})$, then the value function Q_n may not be Lipschitz continuous, as is shown by the following examples.

Example 2.1. *Consider a deterministic convex problem*

$$v^* := \min_{x,z,w}\Big\{x + z \ : \ (z-1)^2 + w^2 \le 1,\ w = x,\ x \in [0,1]\Big\}.$$

The objective function and all constraints are Lipschitz continuous. The optimal objective value $v^* = 0$*, and the unique optimal solution is* $(x^*, z^*, w^*) = (0, 0, 0)$*. At the optimal solution, the inequality constraint is active. Note that the problem can be equivalently written in two-stage formulation as* $v^* = \min_{0\le x\le 1}\ x + Q(x)$*, where* $Q(x)$ *is defined on* $[0, 1]$ *as* $Q(x) := \min\big\{z : \exists\, w \in \mathbb{R},\ \textit{s.t.}\ (z-1)^2 + w^2 \le 1,\ w = x\big\} = 1 - \sqrt{1-x^2}$*, which is not locally Lipschitz continuous at the boundary point* $x = 1$*. Therefore,* $Q(x)$ *is not Lipschitz continuous on* $[0, 1]$*.*

Example 2.2. *Consider a deterministic mixed-integer linear problem*

$$v^* := \min\Big\{1 - 2x + z \ : \ z \ge x,\ x \in [0,1],\ z \in \{0,1\}\Big\}.$$

The optimal objective value is $v^* = 0$*, and the unique optimal solution is* $(x^*, z^*) = (1, 1)$*. Note that the problem can be equivalently written in a two-stage formulation as* $v^* = \min\{1 - 2x + Q(x) \ : \ 0 \le x \le 1\}$*, where the function* $Q(x)$ *is defined on* $[0, 1]$ *as* $Q(x) := \min\{z \in \{0, 1\} : z \ge x\}$*, which equals* 0 *if* $x = 0$*, and* 1 *for all* $0 < x \le 1$*, i.e.* $Q(x)$ *is discontinuous at* $x = 0$*, therefore, it is not Lipschitz continuous on* $[0, 1]$*.*

These examples show a major issue with the introduction of value functions Q_n, namely

Q_n may fail to be Lipschitz continuous even when the original problem only has constraints defined by Lipschitz continuous functions. This could lead to failure of algorithms based on approximation of the value functions, such as the nested Benders decomposition algorithms or the mixed-integer dynamic approximation scheme (MIDAS) [42]. In the next section, we will discuss how to circumvent this issue without compromise of feasibility or optimality for a wide range of problems.

2.1.3 Regularization and Penalty Reformulation

The main idea of avoiding failure of cutting plane algorithms in multistage dynamic programming is to use some Lipschitz continuous envelope functions to replace the original value functions. We refer to these Lipschitz continuous envelope functions as *regularized value functions*. We derive the form of regularization and discuss its exactness by connecting it to the penalty reformulation of the original problem.

We say a function $\psi : \mathbb{R}^d \to \mathbb{R}_+$ is a *penalty function*, if $\psi(x) = 0$ if and only if $x = 0$, and the diameter of its level set $\mathrm{lev}_a(\psi) := \{x \in \mathbb{R}^d : \psi(x) \leq \alpha\}$ approaches 0 when $a \to 0$. In this dissertation, we focus on penalty functions that are locally Lipschitz continuous, the reason for which will be clear from Proposition 2.2.

For each node n, we introduce a new variable z_n as a local variable of node n and impose the duplicating constraint $x_{a(n)} = z_n$. This is a standard approach for obtaining dual variables through relaxation (e.g. [46]). The objective function can then be written as $f_n(z_n, y_n, x_n)$. Let ψ_n be a penalty function for node $n \in \mathcal{N}$. The new coupling constraint is relaxed and penalized in the objective function by $\sigma_n \psi_n(x_{a(n)} - z_n)$ for some $\sigma_n > 0$. Then the DP recursion with penalization becomes

$$Q_n^{\mathrm{R}}(x_{a(n)}) := \min_{\substack{(x_n, y_n) \in \mathcal{F}_n, \\ z_n \in \mathcal{X}_{a(n)}}} \left\{ f_n(z_n, y_n, x_n) + \sigma_n \psi_n(x_{a(n)} - z_n) + \sum_{m \in \mathcal{C}(n)} p_{nm} Q_m^{\mathrm{R}}(x_n) \right\}, \quad (2.4)$$

for all $n \in \mathcal{N}$, and Q_n^{R} is referred to as the regularized value function. By convention,

$\mathcal{X}_{a(r)} = \{x_{a(r)}\} = \{0\}$ and therefore, penalization $\psi_r(x_{a(r)} - z_r) \equiv 0$ for any $z_r \in \mathcal{X}_{a(r)}$. Since the state spaces are compact, without loss of generality, we can scale the penalty functions ψ_n such that the Lipschitz constant of ψ_n on $\mathcal{X}_{a(n)} - \mathcal{X}_{a(n)}$ is 1. The following proposition shows that Q_n^{R} is a Lipschitz continuous envelope function of Q_n for all nodes n.

Proposition 2.2. *Suppose ψ_n is a 1-Lipschitz continuous penalty function on the compact set $\mathcal{X}_{a(n)} - \mathcal{X}_{a(n)}$ for all $n \in \mathcal{N}$. Then $Q_n^{\mathrm{R}}(x) \leq Q_n(x)$ for all $x \in \mathcal{X}_{a(n)}$ and $Q_n^{\mathrm{R}}(x)$ is σ_n-Lipschitz continuous on $\mathcal{X}_{a(n)}$. Moreover, if the original problem* (2.2) *is convex and ψ_n are convex penalty functions, then $Q_n^{\mathrm{R}}(x)$ is also convex.*

Proof. First we show that the partial inf-convolution

$$f_n \square (\sigma_n \psi_n)(x_{a(n)}, y_n, x_n) \coloneqq \min_{z \in \mathcal{X}_{a(n)}} f_n(z_n, y_n, x_n) + \sigma_n \psi_n(x_{a(n)} - z_n)$$

is σ_n-Lipschitz continuous in the first variable $x_{a(n)}$. Note that the minimum is well-defined since $\mathcal{X}_{a(n)}$ is compact and the functions $f_n, \sigma_n \psi_n$ are lsc. Besides, since $z = x_{a(n)}$ is a feasible solution in the minimization, we know that $f_n \square (\sigma_n \psi_n)(x_{a(n)}, y_n, x_n) \leq f_n(x_{a(n)}, y_n, x_n)$ for all $x_{a(n)} \in \mathcal{X}_{a(n)}$ and $(x_n, y_n) \in \mathcal{F}_n$. Pick any $x_1, x_2 \in \mathcal{X}_{a(n)}$, $(x, y) \in \mathcal{F}_n$, and let $z_1, z_2 \in \mathcal{X}_{a(n)}$ be the corresponding minimizers in the definition of $f_n \square (\sigma_n \psi_n)(x_1, y, x)$ and $f_n \square (\sigma_n \psi_n)(x_2, y, x)$, respectively. By definition,

$$\begin{aligned}
&f_n \square (\sigma_n \psi_n)(x_1, y, x) - f_n \square (\sigma_n \psi_n)(x_2, y, x) \\
&= f_n(z_1, y, x) + \psi(x_1 - z_1) - f_n(z_2, y, x) - \psi(x_2 - z_2) \\
&\leq f_n(z_2, y, x) + \psi(x_1 - z_2) - f_n(z_2, y, x) - \psi(x_2 - z_2) \\
&\leq \sigma_n \left\| x_1 - x_2 \right\|.
\end{aligned}$$

Similarly, we can get $f_n \square (\sigma_n \psi_n)(x_2) - f_n \square (\sigma_n \psi_n)(x_1) \leq \sigma_n \left\| x_1 - x_2 \right\|$ by exchanging x_1, x_2 and z_1, z_2 in the above inequality. Therefore, $f_n \square (\sigma_n \psi_n)$ is σ_n-Lipschitz continuous

in the first variable $x_{a(n)}$.

The regularized problem (2.4) can be viewed as replacing the nodal objective function f_n with the inf-convolution $f_n \square (\sigma_n \psi_n)$. Then by Proposition 2.1, $Q_n^{\mathrm{R}}(x)$ is σ_n-Lipschitz continuous on $\mathcal{X}_{a(n)}$. Moreover, if the original problem (2.2) is convex and ψ_n are convex penalty functions, then $f_n \square (\sigma_n \psi_n)$ is also convex. Proposition 2.1 ensures $Q_n^{\mathrm{R}}(x)$ is also convex on $\mathcal{X}_{a(n)}$. □

The optimal value of the regularized root nodal problem

$$v^{\mathrm{reg}} := \min_{(x_r, y_r) \in \mathcal{F}_r} \left\{ f_r(x_{a(r)}, y_r, x_r) + \sum_{m \in \mathcal{C}(r)} p_{rm} Q_m^{\mathrm{R}}(x_r) \right\} \tag{2.5}$$

is thus an underestimation of v^{prim}, i.e. $v^{\mathrm{reg}} \leq v^{\mathrm{prim}}$. For notational convenience, we also define the regularized expected cost-to-go function for each node n as:

$$\mathcal{Q}_n^{\mathrm{R}}(x_n) := \sum_{m \in \mathcal{C}(n)} p_{nm} Q_m^{\mathrm{R}}(x_n). \tag{2.6}$$

Definition 2.1. *For any $\varepsilon > 0$, a feasible root node solution $(x_r, y_r) \in \mathcal{F}_r$ is said to be* ε-optimal *to the regularized problem* (2.4) *if it satisfies $f_r(x_{a(r)}, y_r, x_r) + \mathcal{Q}_r^{\mathrm{R}}(x_r) \leq v^{\mathrm{reg}} + \varepsilon$.*

Next we discuss conditions under which $v^{\mathrm{reg}} = v^{\mathrm{prim}}$ and any optimal solution $(x_n, y_n)_{n \in \mathcal{N}}$ to the regularized problem (2.4) is feasible and hence optimal to the original problem (2.2). Note that by expanding Q_m^{R} in the regularized problem (2.4) for all nodes, we obtain the extensive formulation for the regularized problem:

$$v^{\mathrm{reg}} = \min_{\substack{(x_n, y_n) \in \mathcal{F}_n, n \in \mathcal{N} \\ z_n \in \mathcal{X}_{a(n)}}} \sum_{n \in \mathcal{N}} p_n \big(f_n(z_n, y_n, x_n) + \sigma_n \psi_n(x_{a(n)} - z_n) \big). \tag{2.7}$$

We refer to problem (2.7) as the *penalty reformulation* and make the following assumption on its exactness.

Assumption 2.2. *We assume that the penalty reformulation* (2.7) *is exact, i.e., there exist finite penalty parameters* $\sigma_n > 0$ *for all* $n \in \mathcal{N}$ *such that any optimal solution of* (2.7) *satisfies* $z_n = x_{a(n)}$ *for all* $n \in \mathcal{N}$.

Assumption 2.2 guarantees the solution of the regularized extensive formulation (2.7) is feasible for the original problem (2.1), then by the fact that $v^{\mathrm{reg}} \leq v^{\mathrm{prim}}$, is also optimal to the original problem, we have $v^{\mathrm{reg}} = v^{\mathrm{prim}}$. In this sense, regularized value functions serve as a surrogate of the original value function, without compromise of feasibility of its optimal solutions.

An important fact following Assumption 2.2 is that the original and regularized value functions coincide at all optimal solutions, as stated in the following lemma.

Lemma 2.1. *Under Assumption 2.2, any optimal solution* $(x_n, y_n)_{n\in\mathcal{N}}$ *to problem* (2.1) *satisfies* $Q^{\mathrm{R}}_n(x_{a(n)}) = Q_n(x_{a(n)})$ *for all* $n \neq r$.

Proof. By definition, we have $Q^{\mathrm{R}}_n(x_n) \leq Q_n(x_n)$ for all $n \in \mathcal{N}$, $n \neq r$. We show the other direction by contradiction. Suppose there exists a node $n \in \mathcal{N}$ such that $Q^{\mathrm{R}}_n(x_{a(n)}) < Q_n(x_{a(n)})$. By definition, there exist $z'_m \in \mathcal{X}_{a(m)}$ and $(x'_m, y'_m) \in \mathcal{F}_m$ for all nodes in the subtree $m \in \mathcal{T}(n)$, such that

$$Q^{\mathrm{R}}_n(x_{a(n)}) = \frac{1}{p_n} \sum_{m\in\mathcal{T}(n)} p_m \left[f_m(z'_m, y'_m, x'_m) + \sigma_m \psi_m(x'_{a(m)} - z'_m) \right].$$

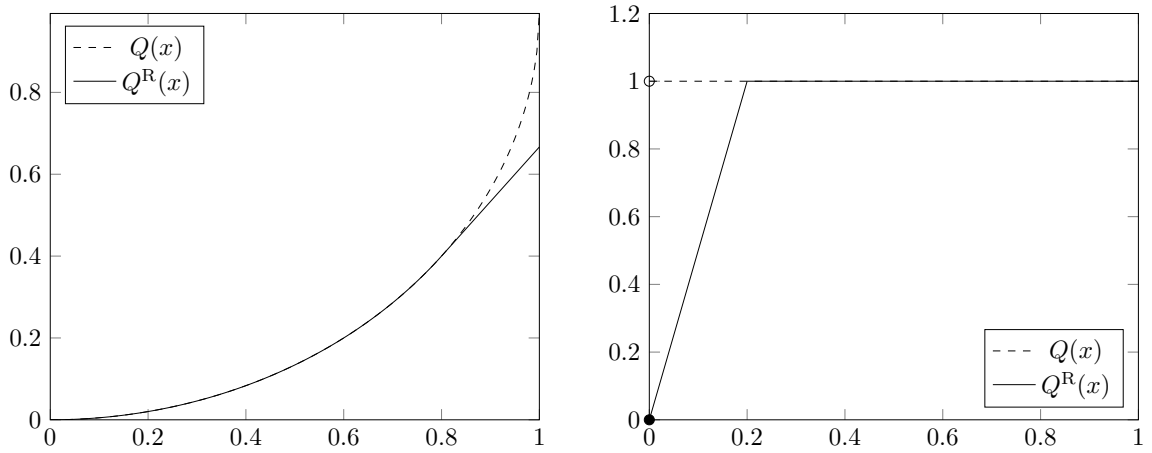
We can extend $(x'_m, y'_m, z'_m)_{m\in\mathcal{T}(n)}$ to a feasible solution $(z'_m, y'_m, x'_m)_{m\in\mathcal{N}}$ of the regular-

ized problem by setting $z'_m = x_{a(m)}$, $y'_m = y_m$, and $x'_m = x_m$ for all $m \notin \mathcal{T}(n)$. Thus

$$\begin{aligned} v^{\mathrm{reg}} &\leq \sum_{m \in \mathcal{T}(n)} p_m f_m(z'_m, y'_m, x'_m) + \sum_{m \notin \mathcal{T}(n)} p_m f_m(z'_m, y'_m, x'_m) \\ &= p_n Q_n^{\mathrm{R}}(x_{a(n)}) + \sum_{m \notin \mathcal{T}(n)} p_m f_m(x_{a(m)}, y_m, x_m) \\ &< p_n Q_n(x_{a(n)}) + \sum_{m \notin \mathcal{T}(n)} p_m f_m(x_{a(m)}, y_m, x_m) \\ &= \sum_{m \in \mathcal{T}(n)} p_m f_m(x_{a(m)}, y_m, x_m) + \sum_{m \notin \mathcal{T}(n)} p_m f_m(x_{a(m)}, y_m, x_m) = v^{\mathrm{prim}}. \end{aligned}$$

This leads to a contradiction with the assumption that $v^{\mathrm{reg}} = v^{\mathrm{prim}}$. Therefore, we conclude that $Q_n^{\mathrm{R}}(x_{a(n)}) = Q_n(x_{a(n)})$ for all $n \in \mathcal{N}$, $n \neq r$. □

We illustrate the regularization on the examples through Figures 2.1a and 2.1b. In



(a) Value Functions in Example 2.1 (b) Value Functions in Example 2.2

Figure 2.1: Value functions in Examples 2.1 and 2.2.

Figure 2.1a, the value function $Q(x)$ derived in Example 2.1 is not Lipschitz continuous at $x = 1$ (plotted with the dashed line). With $\psi(x) = \|x\|$ and $\sigma = 4/3$, we obtain the regularized value function (plotted in the solid line), which coincides with the original one on $[0, 0.8]$ and is Lipschitz continuous on the entire interval $[0, 1]$. In Figure 2.1b, the value function $Q(x)$ derived in Example 2.1 is not continuous at $x = 0$ (plotted with the

dashed line). With $\psi(x) = \|x\|$, $\sigma = 5$, we obtain the regularized value function (plotted in the solid line), which coincides with the primal one on $\{0\} \cup [0.2, 1]$ and is Lipschitz continuous on the entire interval $[0, 1]$. In both examples, it can be easily verified that the penalty reformulation is exact and thus preserves optimal solution.

We comment that Assumption 2.2 holds in various mixed-integer nonlinear optimization problems, including

- convex problems with interior feasible solutions,
- problems with finite state spaces,
- problems defined by mixed-integer linear functions, and
- problems defined by continuously differentiable functions,

if certain constraint qualification is satisfied and proper penalty functions are chosen.

2.1.4 Problem Classes with Exact Penalization

In this section, we discuss the problem classes that allows exact penalty reformulation, as stated in Assumption 2.2. In this section, a penalty function $\psi : \mathbb{R}^d \to \mathbb{R}_+$ is said to be sharp, if $\psi(x) \geq c \|x\|$ for all $x \in V \subset \mathbb{R}^d$, for some open neighborhood $V \ni 0$ and some positive scalar $c > 0$.

Convex problems with interior points

Recall the problem (2.1) is convex if all the feasible sets $\mathcal{F}_n$ and functions f_n are convex for all $n \in \mathcal{N}$. The Slater condition states that the intersection of the domain $\operatorname{dom}\left(\sum_{n\in\mathcal{N}} f_n\right)$ and the feasible sets $\Pi_{n\in\mathcal{N}}\mathcal{F}_n$ has a non-empty interior. Then we have the following proposition on the exactness.

Proposition 2.3. *If the problem* (2.1) *is convex and satisfies the Slater condition, and moreover the penalty functions ψ_n are sharp, then there exist $\sigma_n > 0$ such that the penalty reformulation is exact.*

Proof. Consider a perturbation vector $w = (w_n)_{n\in\mathcal{N}}$ such that $w_n \in \mathcal{X}_{a(n)} - \mathcal{X}_{a(n)}$ for each $n \in \mathcal{N}$, and define the perturbation function

$$\tau(w) := \min_{(z_n,x_n,y_n)\in\mathcal{X}_{a(n)}\times\mathcal{F}_n} \left\{ \sum_{n\in\mathcal{N}} p_n f_n(z_n, y_n, x_n) \;\middle|\; w_n = x_{a(n)} - z_n,\ \forall\, n \in \mathcal{N} \right\}.$$

The function τ is convex and $v^{\text{prim}} = \tau(0)$ by definition. By the Slater condition, $0 \in \text{int}(\text{dom}\,(\tau))$ and hence there exists a vector $\lambda \in \mathbb{R}^{|\mathcal{N}|}$ such that $\tau(w) \geq \tau(0) + \langle \lambda, w\rangle$ for all perturbation w. Since ψ_n are sharp, there exist $\sigma_n > 0$ such that $\sum_{n\in\mathcal{N}} \sigma_n \psi_n(w_n) + \langle \lambda, w\rangle > 0$ for all $w \neq 0$. Consequently the penalty reformulation is exact since $v^{\text{reg}} = \min_w \tau(w) + \sum_{n\in\mathcal{N}} \sigma_n \psi_n(w_n)$ and all optimal solutions must satisfy $w_n = x_{a(n)} - z_n = 0$ for all $n \in \mathcal{N}$. □

Problems with finite state spaces

We say a problem (2.2) has finite state spaces if the state spaces $\mathcal{X}_n$ are finite sets for all nodes n. Such problems appear in multistage integer programming [46], or when the original state spaces can be approximated through finite ones [3, 60]. The following proposition shows the penalty reformulation is exact whenever the state spaces are finite.

Proposition 2.4. *For any penalty functions ψ_n, $n \in \mathcal{N}$, if the state spaces are finite, then there exists a finite $\sigma_n > 0$ such that the penalty reformulation* (2.7) *is exact.*

Proof. Let $d_n := \min_{x\neq z\in\mathcal{X}_{a(n)}} \big|\psi_n(x - z)\big|$ for each $n \in \mathcal{N}$. Since ψ_n is a penalty function and the state space $\mathcal{X}_n$ is finite, we know $d_n > 0$. Define c as

$$c := \min_{(z_n,y_n,x_n)\in\mathcal{X}_{a(n)}\times\mathcal{F}_n} \sum_{n\in\mathcal{N}} p_n f_n(z_n, y_n, x_n). \tag{2.8}$$

Since (2.8) is a relaxation of the original problem (2.1) by ignoring coupling constraint $z_n = x_{a(n)}$, then $c \leq v^{\text{prim}}$. We choose $\sigma_n = 1 + (v^{\text{prim}} - c)/(p_n d_n)$ for all $n \in \mathcal{N}$.

Now let $(x_n, y_n, z_n)_{n\in\mathcal{N}}$ be an optimal solution to the regularized problem (2.4). Then

if there exists $x_{a(m)} \neq z_m$ for some $m \neq r$, then $p_m\sigma_m\psi_m(x_{a(m)} - z_m) > v^{\mathrm{prim}} - c$. Consequently,

$$\begin{aligned} v^{\mathrm{reg}} &\geq c + \sum_{n\in\mathcal{N}} p_n\sigma_n\psi_n(x_{a(n)} - z_n) \\ &\geq c + p_m\sigma_m\psi_m(x_{a(m)} - z_m) \\ &> c + v^{\mathrm{prim}} - c \\ &= v^{\mathrm{prim}}. \end{aligned}$$

This is a contradiction since $v^{\mathrm{reg}} \leq v^{\mathrm{prim}}$. Therefore, any optimal solution to the reformulation (2.7) must have $x_{a(n)} = z_n$ for all $n \neq r$, which means the penalty reformulation is exact. □

Problems defined by mixed-integer linear functions

The problem (2.1) is said to be defined by mixed-integer linear functions, if all the feasible sets $\mathcal{F}_n$ and the epigraphs $\mathrm{epi}f_n$ are representable by mixed-integer variables and non-strict linear inequalities with rational coefficients. Recall that by Assumption 2.1, the primal problem is feasible, $v^{\mathrm{prim}} > -\infty$. We have the following proposition on the exact penalty reformulation.

Proposition 2.5 ([71], Theorem 5)**.** *If problem* (2.1) *is defined by mixed-integer linear functions and the penalty functions ψ_n are sharp for all $n \in \mathcal{N}$, then there exist $\sigma_n > 0$, such that the penalty reformulation is exact.*

Problems defined by C^1-functions

The problem (2.1) is said to be defined by C^1-functions if it is defined by functional constraints using indicator functions in each node $n \in \mathcal{N}$:

$$f_n(x_{a(n)}, y_n, x_n) = \begin{cases} f_{n,0}(x_{a(n)}, y_n, x_n), & \text{if } g_{n,i}(x_{a(n)}, y_n, x_n) \leq 0, i = 1, \ldots, I_n, \\ +\infty \quad \text{otherwise.} \end{cases}$$

with all $f_{n,0}, g_{n,i}, i = 1, \ldots, I_n$ being continuously differentiable. The Karush-Kuhn-Tucker condition at a feasible point $(x_n, y_n)_{n\in\mathcal{N}}$ of (2.1) says that there exist multipliers $\mu_{n,i} \geq 0, i = 1, \ldots, I_n$, such that

$$\nabla_{x_n,y_n} \left\{ \sum_{n\in\mathcal{N}} (f_{n,0}(x_{a(n)}, y_n, x_n) - \mu_{n,i} g_{n,i}(x_{a(n)}, y_n, x_n) \right\} = 0,$$

$$\mu_{n,i} g_{n,i}(x_{a(n)}, y_n, x_n) = 0, \quad i = 1, \ldots, I_n.$$

We have the following proposition on the exactness.

Proposition 2.6. *Suppose the problem* (2.1) *is defined by C^1-functions and the Karush-Kuhn-Tucker condition holds for every local minimum solution of* (2.1). *If the penalty functions ψ_n are sharp for all $n \in \mathcal{N}$, then there exist $\sigma_n > 0$ such that the penalty reformulation is exact.*

We give the proof of Proposition 2.6 below.

Proof for Proposition 2.6

We begin by stating a general exact penalization result for problems defined by C^1-functions. Consider the following perturbation function

$$\begin{aligned} p(u) \coloneqq \min_{x \in \mathbb{R}^d} \quad & f(x,u) \\ \text{s.t.} \quad & g_i(x,u) \leq 0, \quad i = 1, \ldots, I, \\ & h_j(x,u) = 0, \quad j = 1, \ldots, J, \end{aligned} \tag{2.9}$$

Here u is the perturbation vector and $u = 0$ corresponds to the original primal problem. Let ψ be a penalty function on $\mathbb{R}^d$ and $\sigma > 0$ a penalty factor. A penalization of the original primal problem $p(0)$ is given by

$$\begin{aligned} \min_{x \in \mathbb{R}^d} \quad & f(x,u) + \sigma\psi(u) \\ \text{s.t.} \quad & g_i(x,u) \leq 0, \quad i = 1, \ldots, I, \\ & h_j(x,u) = 0, \quad j = 1, \ldots, J. \end{aligned} \tag{2.10}$$

Naturally we could impose some bound on the perturbation as $\|u\| \leq R_u$. We assume that f, g_i, h_j are continuously differentiable in x and u for all i, j. Moreover, the compactness in Assumption 2.1 implies that the feasible region prescribed by the inequality constraints $g_i(x,u) \leq 0$ are compact in x for any u, i.e., $X = \{x \in \mathbb{R}^d : \exists u, \|u\| \leq R_u, \text{s.t.} g_i(x,u) \leq 0, i = 1, \ldots, J\}$ is compact. For example, some of the inequalities are bounds on the variables, $\|x\|_\infty \leq 1$. We will show that there exists a penalty factor $\sigma > 0$ such that any optimal solution to (2.10) is feasible to (2.9).

We first characterize the property of the perturbation function $p(u)$.

Lemma 2.2. *The perturbation function $p(u)$ is lower semicontinuous.*

Proof. Let $X(u) \subset X$ denote the feasible set in x dependent on u. The minimum in the

definition is well defined for every u due to the compactness of $X(u)$.

We show that $p(u)$ is lower semicontinuous (lsc) by showing $\liminf_{v\to u} p(v) \geq p(u)$ for any u. Assume for contradiction that for any $\varepsilon > 0$, there exists a sequence $\{v_k\}_{k=1}^{\infty}$ such that $v_k \to u$ and $p(v_k) \leq p(u) - \varepsilon$. Let $x_k \in \arg\min f(x, v_k)$ and thus $p(v_k) = f(x, v_k)$. Since X is compact, there exists a subsequence x_{k_j} and $z \in X$ such that $x_{k_j} \to z$ as $j \to \infty$. Then by continuity of f, $f(z, u) = \lim_{j\to\infty} f(x_{k_j}, v_{k_j}) \leq p(u) - \varepsilon$. This contradicts with the definition of $p(u)$, since $p(u) = \min_{x\in X(u)} f(x, u) \leq f(z, u) \leq p(u) - \varepsilon$. Therefore $p(u)$ is lsc. □

Now we give the theorem of exact penalization for problems defined by C^1-functions.

Proposition 2.7. *If the Karush-Kuhn-Tucker condition is satisfied at every local minimum solution of* (2.9)*, then the penalty reformulation* (2.10) *is exact for some finite* $\sigma > 0$*.*

Proof. Let $X(u)$ denote the feasible region of x defined by constraints $g_i(x, u) \leq 0,\ i = 1, \ldots, I$ and $h_j(x, u) = 0,\ j = 1, \ldots, J$. Then $X(u)$ is compact for any u by the continuity of the constraint functions. We show that for every optimal solution $x_0 \in X(0)$, there exists a neighborhood $V(x_0) \ni x_0$ in the x space, $U(x_0) \ni u = 0$ in the u space, and constant $L(x_0) > 0$, such that for all $x \in V(x_0)$ and $u \in U(x_0)$, we have

$$f(x, u) \geq f(x_0, 0) - L(x_0) \cdot \|u\| .$$

Then we use this fact together with compactness of $X(0)$ to show the existence of exact penalization. In this proof, the little-o is used to simplify notation, i.e., $o(\|a\|)$ denotes a function $b(a)$ such that

$$\lim_{a\to 0} \frac{|b(a)|}{\|a\|} = 0.$$

Pick any optimal solution $x_0 \in X(0)$. By definition, it is also a local minimum solution. Due to constraint qualification, the KKT condition is satisfied at x_0, that is, there exist

$\lambda_i \in \mathbb{R},\ i = 1, \ldots, I$, and $\mu_j \geq 0,\ j = 1, \ldots, J$ such that

$$\begin{aligned} &\nabla_x f(x_0, 0) + \sum_{i=1}^{I} \lambda_i \nabla_x g_i(x_0, 0) + \sum_{j=1}^{J} \mu_j \nabla_x h_j(x_0, 0) = 0, \\ &h_j(x_0, 0) = 0,\ j = 1, \ldots, J, \\ &g_i(x_0, 0) \leq 0, \quad \lambda_i \cdot g_i(x_0, 0) = 0,\ i = 1, \ldots, I. \end{aligned} \tag{2.11}$$

Since h_j's are continuously differentiable and $h_j(x_0, 0) = 0$, we have

$$\big\langle \nabla_x h_j(x_0, 0), x - x_0 \big\rangle + \big\langle \nabla_u h_j(x_0, 0), u \big\rangle + o(\|x - x_0\| + \|u\|) = 0,\ j = 1, \ldots, J. \tag{2.12}$$

Let $A \subset I$ denote the set of active inequality constraints. Then similarly we have

$$\big\langle \nabla_x g_i(x_0, 0), x - x_0 \big\rangle + \big\langle \nabla_u g_i(x_0, 0), u \big\rangle + o(\|x - x_0\| + \|u\|) \leq 0,\ i \in A. \tag{2.13}$$

For any $i \notin A$, by the continuity of g_i, there exist neighborhoods W_i of x_0 and U_i' of $u = 0$ such that for any $(x, u) \in W_i \times U_i'$, $g_i(x, u) < 0$ remains inactive. Now, from (2.11), (2.12),

(2.13), and f being continuously differentiable, we have

$$
\begin{aligned}
& f(x,u) - f(x_0,0) \\
&= \left\langle \nabla_x f(x_0,0), x - x_0 \right\rangle + \left\langle \nabla_u f(x_0,0), u \right\rangle + o(\|x - x_0\| + \|u\|) \\
&= \left\langle -\sum_{i=1}^{I} \lambda_i \nabla_x g_i(x_0,0) - \sum_{j=1}^{J} \mu_j \nabla_x h_j(x_0,0), x - x_0 \right\rangle \\
&\quad + \left\langle \nabla_u f(x_0,0), u \right\rangle + o(\|x - x_0\| + \|u\|) \\
&= \left\langle -\sum_{i\in A} \lambda_i \nabla_x g_i(x_0,0) - \sum_{j=1}^{J} \mu_j \nabla_x h_j(x_0,0), x - x_0 \right\rangle \\
&\quad + \left\langle \nabla_u f(x_0,0), u \right\rangle + o(\|x - x_0\| + \|u\|) \\
&\geq \left\langle \nabla_u f(x_0,0) + \sum_{i\in A} \lambda_i \nabla_u g_i(x_0,0) + \sum_{j=1}^{J} \mu_j \nabla_u h_j(x_0,0), u \right\rangle + o(\|x - x_0\| + \|u\|) \\
&> -L(x_0) \cdot \|u\| + o(\|x - x_0\| + \|u\|),
\end{aligned}
$$

where $L(x_0) \coloneqq \left\| \nabla_u f(x_0,0) + \sum_{i\in A} \lambda_i \nabla_u g_i(x_0,0) + \sum_{j=1}^{J} \mu_j \nabla_u h_j(x_0,0) \right\| + 1 > 0$. By the definition of the little-o notation, there exists a neighborhood $V(x_0) \subset \cap_{i\notin A} W_i,\ x_0 \in V(x_0)$ and $U(x_0) \subset \cap_{i\notin A} U_i,\ 0 \in U(x_0)$ such that

$$
f(x,u) - f(x_0,0) \geq -L(x_0) \cdot \|u\|, \quad \forall\, (x,u) \in V(x_0) \times U(x_0).
$$

Now, let $X_{\text{opt}}(0)$ denote the set of optimal solutions of x when $u = 0$. Note that $X_{\text{opt}}(0) \subset X(0)$ is closed due to the continuity of f, h_i, g_j, hence compact. The collection of open sets $\{V(x)\}_{x\in X_{\text{opt}}(0)}$ covers $X_{\text{opt}}(0)$. By compactness, there exists a finite subcollection $\{V(x_k)\}_{k=1}^{K}$ such that $X_{\text{opt}}(0) \subset \cup_{k=1}^{K} V(x_k) =: V$. Let $L \coloneqq \max_{k=1,\ldots,K} L(x_k)$ and $U = \cap_{k=1}^{K} U(x_k)$. Let f^* denote the optimal value for $u = 0$. Then we have

$$
f(x,u) \geq f^* - L \cdot \|u\|, \quad \forall\, (x,u) \in V \times U.
$$

To show the inequality for $x \notin V$, define

$$\tilde{p}(u) = \min_{x \in X(u) \setminus V} f(x, u).$$

Note that $\tilde{p}(0) > f^*$ by the definition of $X_{\text{opt}}(0)$. Then by Lemma 2.2, $p(u)$ is lower semicontinuous, and we know that there exists a neighborhood U' of 0 such that $\tilde{p}(u) > f^*$ for all $u \in U'$. Therefore, for all $u \in U \cap U'$, we have

$$f(x, u) \geq f^* - L \cdot \|u\|.$$

Finally, we can show that the penalization is exact. Since ψ is sharp, there exist an open set $\tilde{U} \subset U \cap U'$, and positive constants $c > 0$ such that

$$\psi(u) \geq c \|u\| \text{ on } \tilde{U}.$$

Let $M = \min_{u \in \bar{B}_{R_u}(0) \setminus \tilde{U}} \tilde{p}(u) > f^*$, $m = \min_{u \in \bar{B}_{R_u}(0) \setminus \tilde{U}} \psi(u) > 0$ because ψ is a penalty function. Let $\sigma = (M - f^*)/m + 1$. We have

$$f(x, u) \geq f^* - \sigma \cdot \|u\|, \quad \forall\, u \in \bar{B}_{R_u}(0) \setminus \{0\},\ x \in \cup_u X(u).$$

As a result, any optimal solution to the penalization (2.10) would satisfy $u = 0$. □

Note that our problem (2.7) can be written into the form (2.10) by letting $u = (x_{a(n)}, z_n)_{n \in \mathcal{N}}$, and including the constraints

$$z_n - x_{a(n)} = 0, \quad \forall\, n \neq r \in \mathcal{N}$$

in the equality constraints $h_j(x, u) = 0$. And other constraints $g_i(x, u) \leq 0$ correspond to the functional constraints in the problem (2.1). Since ψ_n are sharp, the aggregate penalty

function defined by

$$\psi(u) = \sum_{n \in \mathcal{N}} p_n \psi_n(x_{a(n)} - z_n),$$

is also sharp. Let σ denote the penalty factor in Proposition 2.7. Proposition 2.6 follows from this by letting $\sigma_n = \sigma / p_n$ for all $n \in \mathcal{N}$.

2.1.5 Generalized Conjugacy Cuts and Value Function Approximation

In this part, we first introduce generalized conjugacy cuts for nonconvex functions and then apply it to the under-approximation of value functions of mixed-integer nonlinear MSO.

Generalized Conjugacy Cuts

Let $Q : \mathcal{X} \to \mathbb{R}_+ \cup \{+\infty\}$ be a proper, lsc function defined on a compact set $\mathcal{X} \subseteq \mathbb{R}^d$. Let $\mathcal{U}$ be a non-empty set for parameters. Given a continuous function $\Phi : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, the Φ-conjugate of Q (see e.g., Chapter 11-L in [72]) is defined as

$$Q^{\Phi}(u) = \max_{x \in \mathcal{X}} \left\{ \Phi(x, u) - Q(x) \right\}. \tag{2.14}$$

The following generalized Fenchel-Young inequality holds by definition for any $x \in \mathcal{X}$ and $u \in \mathcal{U}$,

$$Q(x) + Q^{\Phi}(u) \geq \Phi(x, u).$$

For any $\hat{u} \in \mathcal{U}$ and an associated maximizer $\hat{x}$ in (2.14), we define

$$C^{\Phi}(x \,|\, \hat{u}, \hat{v}) := \hat{v} + \Phi(x, \hat{u}) \tag{2.15}$$

where $\hat{v} := -Q^{\Phi}(\hat{u})$. Then, the following inequality, derived from the generalized Fenchel-Young inequality, is valid for any $x \in \mathcal{X}$,

$$Q(x) \geq C^{\Phi}(x \,|\, \hat{u}, \hat{v}), \tag{2.16}$$

which we call a *generalized conjugacy cut* for the target function Q.

Value Function Approximation

For a nodal problem $n \in \mathcal{N}, n \neq r$ and a point $\bar{x} \in \mathcal{X}_{a(n)}$, define $\Phi_n^{\bar{x}}(x,u) \coloneqq -\langle \lambda, \bar{x} - x \rangle - \rho\psi_n(\bar{x} - x)$, where $u := (\lambda, \rho) \in \mathbb{R}^{d_n+1}$ are parameters. Consider a compact set of parameters $\mathcal{U}_n = \{(\lambda,\rho) : \|\lambda\|_* \leq l_{n,\lambda}, 0 \leq \rho \leq l_{n,\rho}\}$ with nonnegative bounds $l_{n,\lambda}$ and $l_{n,\rho}$, where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$. Consider the following dual problem

$$\hat{v}_n := \max_{(\lambda,\rho)\in\mathcal{U}_n} \left\{ \min_{z\in\mathcal{X}_{a(n)}} \left[Q_n(z) + \langle \lambda, \bar{x} - z \rangle + \rho\psi_n(\bar{x} - z) \right] \right\}. \tag{2.17}$$

Denote $\hat{z}_n$ and $(\hat{\lambda}_n, \hat{\rho}_n)$ as an optimal primal-dual solution of (2.17). The dual problem (2.17) can be viewed as choosing $(\hat{\lambda}_n, \hat{\rho}_n)$ as the value of $\hat{u}$ in (2.15), which makes the constant term $-Q^{\Phi}(\hat{u})$ as large as possible, thus makes the generalized conjugacy cut (2.16) as tight as possible. With this choice of the parameters, a generalized conjugacy cut for Q_n at $\bar{x}$ is given by

$$\begin{aligned} Q_n(x) &\geq C_n^{\Phi_n^{\bar{x}}}(x \mid \hat{\lambda}_n, \hat{\rho}_n, \hat{v}_n) \\ &= -\left\langle \hat{\lambda}_n, \bar{x} - x \right\rangle - \hat{\rho}_n\psi_n(\bar{x} - x) + \hat{v}_n, \quad \forall\, x \in \mathcal{X}_{a(n)}. \end{aligned} \tag{2.18}$$

Proposition 2.8. *Given the above definition of* (2.17)-(2.18)*, if* $(\bar{x}_n, \bar{y}_n)_{n\in\mathcal{N}}$ *is an optimal solution to problem* (2.1) *and the bound* $l_{n,\rho}$ *satisfies* $l_{n,\rho} \geq \sigma_n$ *for all nodes* n*, then for every node* n*, the generalized conjugacy cut* (2.18) *is tight at* $\bar{x}_n$*, i.e.* $Q_n(\bar{x}_n) = C_n^{\Phi_n^{\bar{x}_n}}(\bar{x}_n \mid \hat{\lambda}_n, \hat{\rho}_n, \hat{v}_n)$.

Proof. If $l_{n,\rho} \geq \sigma_n$, then $(\lambda, \rho) = (0, \sigma_n)$ is contained in $\mathcal{U}_n$, and therefore, is a dual feasible

solution for (2.17). Thus, we have

$$
\begin{aligned}
Q_n(\bar{x}_n) &\geq C_n^{\Phi_n^{\bar{x}_n}}(\bar{x}_n \mid \hat{\lambda}_n, \hat{\rho}_n, \hat{v}_n) = \hat{v}_n \qquad (2.19)\\
&\geq \min_{z \in \mathcal{X}_{a(n)}} \{Q_n(z) + \sigma_n \psi_n(\bar{x}_n - z)\} = Q_n^{\mathrm{R}}(\bar{x}_n) = Q_n(\bar{x}_n),
\end{aligned}
$$

where the first inequality is the validity of the generalized conjugacy cut (2.16) and the second and the last equality are due to Lemma 2.1 for $(\bar{x}_n, \bar{y}_n)_{n \in \mathcal{N}}$ being an optimal solution to problem (2.1). This completes the proof. □

The proposition guarantees that, under Assumption 2.2, the generalized conjugacy cuts are able to approximate the value functions exactly at any state associated to an optimal solution.

In the special case where problem (2.2) is convex and $\psi_n(x) = \|x\|$ for all $n \in \mathcal{N}$, the exactness of the generalized conjugacy cut holds even if we set $l_{n,\rho} = 0$, i.e. the conjugacy cut is linear. To be precise, we begin with the following lemma.

Lemma 2.3. *Let $\mathcal{X} \subset \mathbb{R}^d$ be a convex, compact set. Given a convex, proper, lsc function $Q : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, for any $x \in \mathcal{X}$, the inf-convolution satisfies*

$$
Q_{\square}(\sigma \|\cdot\|)(x) \coloneqq \min_{z \in \mathcal{X}} \{Q(z) + \sigma \|x - z\|\} = \max_{\|\lambda\|_* \leq \sigma} \min_{z \in \mathcal{X}} \{Q(z) + \langle \lambda, x - z \rangle\}. \qquad (2.20)
$$

Proof. The minimums in (2.20) are well-defined because of the compactness of $\mathcal{X}$ and lower semicontinuity of Q. Take any $x \in \mathcal{X}$. Since both the primal set $\mathcal{X}$ and the dual set $\{\lambda \in \mathbb{R}^d : \|\lambda\|_* \leq \sigma\}$ are bounded, by strong duality (cf. Theorem 3.1.30 in [73]), we have

$$
\begin{aligned}
\max_{\|\lambda\|_* \leq \sigma} \min_{z \in \mathcal{X}} \{Q(z) + \langle \lambda, x - z \rangle\} &= \min_{z \in \mathcal{X}} \max_{\|\lambda\|_* \leq \sigma} \{Q(z) + \langle \lambda, x - z \rangle\}\\
&= \min_{z \in \mathcal{X}} \{Q(z) + \sigma \|x - z\|\},
\end{aligned}
$$

which completes the proof. □

Next we show the tightness in the convex case similar to Proposition 2.8.

Proposition 2.9. *Suppose* (2.2) *is convex and* $\psi_n(x) = \|x\|$ *for all nodes* n*. Given the above definition of* (2.17)-(2.18)*, if* $(\bar{x}_n, \bar{y}_n)_{n\in\mathcal{N}}$ *is an optimal solution to problem* (2.1) *and the bounds satisfy* $l_{n,\lambda} \geq \sigma_n$*,* $l_{n,\rho} = 0$ *for all nodes* n*, then for every node* n*, the generalized conjugacy cut* (2.18) *is exact at* $\bar{x}_n$*, i.e.* $Q_n(\bar{x}_n) = C_n^{\Phi_n^{\bar{x}_n}}(\bar{x}_n \mid \hat{\lambda}_n, \hat{\rho}_n, \hat{v}_n)$.

Proof. By definition, $Q_n^{\mathrm{R}}(x) = Q_n \square (\sigma_n\psi_n)(x)$. Since $\psi_n(x) = \|x\|$ is convex, by Proposition 2.2, $Q_n^{\mathrm{R}}(x)$ is convex. Then by Lemma 2.3, we have

$$Q_n^{\mathrm{R}}(x) = \max_{\|\lambda\|_* \leq \sigma_n} \min_{z\in\mathcal{X}_{a(n)}} \left\{ Q_n(z) + \langle \lambda, x - z\rangle \right\}.$$

Therefore,

$$\begin{aligned} C_n^{\Phi_n^{\bar{x}_n}}(\bar{x}_n \mid \hat{\lambda}_n, \hat{\rho}_n, \hat{v}_n) = \hat{v}_n &= \max_{\|\lambda\|_* \leq l_{n,\lambda}} \min_{z\in\mathcal{X}_{a(n)}} \left\{ Q_n(z) + \langle \lambda, \bar{x}_n - z\rangle \right\} \\ &\geq \max_{\|\lambda\|_* \leq \sigma_n} \min_{z\in\mathcal{X}_{a(n)}} \left\{ Q_n(z) + \langle \lambda, \bar{x}_n - z\rangle \right\} = Q_n^{\mathrm{R}}(\bar{x}_n). \end{aligned}$$

By Lemma 2.1, $Q_n^{\mathrm{R}}(\bar{x}_n) = Q_n(\bar{x}_n)$ if $(\bar{x}_n, \bar{y}_n)_{n\in\mathcal{N}}$ is an optimal solution to problem (2.1). Therefore, we conclude that $C_n^{\Phi_n^{\bar{x}_n}}(\bar{x}_n \mid \hat{\lambda}_n, \hat{\rho}_n, \hat{v}_n) = Q_n(\bar{x}_n)$ due to the validness of $C_n^{\Phi_n^{\bar{x}_n}}$ by (2.16). □

In this case, the generalized conjugacy reduces to the usual conjugacy for convex functions and the generalized conjugacy cut is indeed linear. This enables approximation of the value function that preserves convexity.

Remark. This proposition can be generalized to special nonconvex problems where Q can be extended to a convex function defined on the convex hull $\mathrm{conv}\mathcal{X}$. This is true if $\mathcal{X}$ is contained in the set of extreme points of a convex set, e.g., $\{0,1\}^d$. From the above discussion, this provides an alternative explanation of the exactness of the Lagrangian cuts in SDDiP [46] when the relatively complete recourse is assumed.

2.2 Nested Decomposition and Dual Dynamic Programming Algorithms

In this section, we introduce a nested decomposition algorithm for general scenario trees, and two dual dynamic programming algorithms for stagewise independent scenario trees. Since the size of the scenario tree could be large, we focus our attention to finding an ε-optimal root node solution x_r^* (see definition (2.5)), rather than an optimal solution $\{x_n^*\}_{n\in\mathcal{T}}$ for the entire tree.

2.2.1 Subproblem Oracles

Before we propose the new algorithms, we first define subproblem oracles, which we will use to describe the algorithms and conduct complexity analysis. A subproblem oracle is an oracle that takes subproblem information together with the current algorithm information to produce a solution to the subproblem. With subproblem oracles, we can describe the algorithms consistently regardless of the problem being convex or not.

We assume three different subproblem oracles in this chapter, corresponding to the forward steps and backward steps of non-root nodes, and the root node step in the algorithms. For non-root nodes, we assume the following two subproblem oracles.

Definition 2.2 (Forward Step Subproblem Oracle for Non-Root Nodes)**.** *Consider the following subproblem for a non-root node n,*

$$\min_{\substack{(x,y)\in\mathcal{F}_n,\\ z\in\mathcal{X}_{a(n)}}} \left\{ f_n(z,y,x) + \sigma_n\psi_n(x_{a(n)} - z) + \Theta_n(x) \right\}. \tag{F}$$

where the parent node's state variable $x_{a(n)} \in \mathcal{X}_{a(n)}$ is a given parameter and $\Theta_n : \mathcal{X}_n \to \bar{\mathbb{R}}$ is a lsc function, representing an under-approximation of the expected cost-to-go function $\mathcal{Q}_n(x)$ defined in (2.3)*. The forward step subproblem oracle finds an optimal solution of* (F) *given $x_{a(n)}$ and Θ_n, that is, we denote this oracle as a mapping $\mathscr{O}_n^{\mathrm{F}}$ that takes $(x_{a(n)}, \Theta_n)$ as input and outputs an optimal solution (x_n, y_n, z_n) of* (F) *for $n \neq r$.*

Definition 2.3 (Backward Step Subproblem Oracles for Non-Root Nodes)**.** *Consider the following subproblem for a non-root node n,*

$$\max_{(\lambda,\rho)\in\mathcal{U}_n} \min_{\substack{(x,y)\in\mathcal{F}_n,\\ z\in\mathcal{X}_{a(n)}}} \left\{ f_n(z,y,x) + \left\langle \lambda, x_{a(n)} - z \right\rangle + \rho\psi_n(x_{a(n)} - z) + \Theta_n(x) \right\}, \tag{B}$$

where the parent node's state variable $x_{a(n)} \in \mathcal{X}_{a(n)}$ is a given parameter and $\Theta_n : \mathcal{X}_n \to \bar{\mathbb{R}}$ is a lsc function, representing an under-approximation of the expected cost-to-go function. The backward step subproblem oracle finds an optimal solution of (B) *for the given $x_{a(n)}$ and Θ_n. Similarly, we denote this oracle as a mapping $\mathscr{O}_n^{\mathrm{B}}$ that takes $(x_{a(n)}, \Theta_n)$ as input and outputs an optimal solution $(x_n, y_n, z_n; \lambda_n, \rho_n)$ of* (B) *for $n \neq r$.*

For the root node, we assume the following subproblem oracle.

Definition 2.4 (Subproblem Oracle for the Root Node)**.** *Consider the following subproblem for the root node $r \in \mathcal{N}$,*

$$\min_{(x,y)\in\mathcal{F}_r} \left\{ f_r(x_{a(r)}, y, x) + \Theta_r(x) \right\}, \tag{R}$$

where $\Theta_r : \mathcal{X}_r \to \bar{\mathbb{R}}$ is a lsc function, representing an under-approximation of the expected cost-to-go function. The subproblem oracle for the root node is denoted as $\mathscr{O}_r$ that takes Θ_r as input and outputs an optimal solution (x_r, y_r) of (R) *for the given function Θ_r.*

All of the subproblem oracles may return any optimal solution to the corresponding nodal subproblem. For numerical implementation, the above defined subproblem oracles are usually handled by subroutines or external solvers.

2.2.2 Under- and Over-Approximations of Cost-to-go Functions

We first show how to iteratively construct under-approximation of expected cost-to-go functions using the generalized conjugacy cuts developed in Section 2.1.5. The under-approximation serves as a surrogate of the true cost-to-go function in the algorithm. Let

$i \in \mathbb{N}$ be the iteration index of an algorithm. Assume $(x_n^i, y_n^i)_{n\in\mathcal{N}}$ are feasible solutions to the regularized nodal problem (2.4) in the i-th iteration. Then the under-approximation of the expected cost-to-go function is defined recursively from leaf nodes to the child nodes of the root node, and inductively for $i \in \mathbb{N}$ as

$$\underline{\mathcal{Q}}_n^i(x) \coloneqq \max\left\{\underline{\mathcal{Q}}_n^{i-1}(x), \sum_{m\in\mathcal{C}(n)} p_{nm} C_m^i(x \mid \hat{\lambda}_m^i, \hat{\rho}_m^i, \underline{v}_m^i)\right\}, \quad \forall x \in \mathcal{X}_n, \tag{2.21}$$

where $\underline{\mathcal{Q}}_n^0 \equiv 0$ on $\mathcal{X}_n$. In the definition (2.21), C_m^i is the generalized conjugacy cut for Q_m at i-th iteration and $\Phi_m^{x_n^i}(x,\lambda,\rho) = -\left\langle \lambda, x_n^i - x\right\rangle - \rho\psi_n(x_n^i - x)$ (cf. (2.17)-(2.18)), that is,

$$C_m^i(x \mid \hat{\lambda}_m^i, \hat{\rho}_m^i, \underline{v}_m^i) \coloneqq -\left\langle \hat{\lambda}_m^i, x_n^i - x\right\rangle - \hat{\rho}_m^i \psi_m(x_n^i - x) + \underline{v}_m^i, \tag{2.22}$$

where $(\hat{x}_m^i, \hat{y}_m^i \hat{z}_m^i; \hat{\lambda}_m^i, \hat{\rho}_m^i) = \mathscr{O}_m^{\mathrm{B}}(x_n^i, \underline{\mathcal{Q}}_m^i)$, and $\underline{v}_m^i$ satisfies

$$\underline{v}_m^i = f_m(\hat{z}_m^i, \hat{y}_m^i, \hat{x}_m^i) + \left\langle \hat{\lambda}_m^i, x_n^i - \hat{z}_m^i\right\rangle + \hat{\rho}_m^i \psi_m(x_n^i - \hat{z}_m^i) + \underline{\mathcal{Q}}_m^i(\hat{x}_m^i). \tag{2.23}$$

The next proposition shows that $\underline{\mathcal{Q}}_n^i$ is indeed an under-approximation of $\mathcal{Q}_n$.

Proposition 2.10. *For any $n \in \mathcal{N}$, and $i \in \mathbb{N}$, $\underline{\mathcal{Q}}_n^i(x)$ is $(\sum_{m\in\mathcal{C}(n)} p_{nm}(l_{m,\lambda} + l_{m,\rho}))$-Lipschitz continuous and*

$$\mathcal{Q}_n(x) \geq \underline{\mathcal{Q}}_n^i(x), \quad \forall x \in \mathcal{X}_n.$$

Proof. Let $L_n \coloneqq \sum_{m\in\mathcal{C}(n)} p_{nm}(l_{m,\lambda} + l_{m,\rho})$ for simplicity. We prove the proposition recursively for nodes $n \in \mathcal{N}$, and inductively for iteration indices $i \in \mathbb{N}$. For leaf nodes and the first iteration, it holds obviously because $\underline{\mathcal{Q}}_n^i(x) = 0$ for any leaf node $n \in \mathcal{N}$ with $\mathcal{C}(n) = \varnothing$, and $\underline{\mathcal{Q}}_n^0(x) = 0$ from the definition (2.21). Now suppose for some $n \in \mathcal{N}$, and $i \in \mathbb{N}$, it holds for all $m \in \mathcal{C}(n)$ that $\underline{\mathcal{Q}}_m^i(x) \leq \mathcal{Q}_m(x)$, $\underline{\mathcal{Q}}_n^{i-1}(x) \leq \mathcal{Q}_n(x)$, and that $\underline{\mathcal{Q}}_n^{i-1}(x)$ is L_n-Lipschitz continuous. Then it follows from (2.18), (2.23), and (B) that

$C_m^i(x \mid \hat{\lambda}_m^i, \hat{\rho}_m^i, \underline{v}_m^i) \leq Q_m(x)$ for all $m \in \mathcal{C}(n)$. Moreover, by (2.22), C_m^i is $(l_{m,\lambda} + l_{m,\rho})$-Lipschitz continuous so $\sum_{m \in \mathcal{C}(n)} p_{nm} C_m^i$ is L_n-Lipschitz continuous. Thus the pointwise maximum of $\underline{\mathcal{Q}}_n^{i-1}(x)$ and $\sum_{m \in \mathcal{C}(n)} p_{nm} C_m^i(x \mid \hat{\lambda}_n^i, \hat{\rho}_n^i, \underline{v}_n^i)$ (cf. (2.21)) is still dominated by $\mathcal{Q}_n(x)$ and L_n-Lipschitz continuous. □

Now, we propose the following over-approximation of the regularized expected cost-to-go functions, which is used in the sampling and termination of the proposed nested decomposition and dual dynamic programming algorithms. For $i \in \mathbb{N}$, at root node r, let $(x_r^i, y_r^i) = \mathscr{O}_r(\underline{\mathcal{Q}}_r^{i-1})$, and, at each non-root node n, let $(x_n^i, y_n^i, z_n^i) = \mathscr{O}_n^{\mathrm{F}}(x_{a(n)}^i, \underline{\mathcal{Q}}_n^{i-1})$. Then the over-approximation of the regularized expected cost-to-go function is defined recursively, from leaf nodes to the child nodes of the root node, and inductively for $i \in \mathbb{N}$ by

$$\overline{\mathcal{Q}}_n^i(x) \coloneqq \begin{cases} \operatorname{conv}\left\{ \overline{\mathcal{Q}}_n^{i-1}(x), \sum_{m \in \mathcal{C}(n)} p_{nm} \left(\bar{v}_m^i + \sigma_m \left\| x - x_n^i \right\| \right) \right\}, & \text{if (2.4) is convex,} \\ \min\left\{ \overline{\mathcal{Q}}_n^{i-1}(x), \sum_{m \in \mathcal{C}(n)} p_{nm} \left(\bar{v}_m^i + \sigma_m \left\| x - x_n^i \right\| \right) \right\}, & \text{otherwise,} \end{cases} \tag{2.24}$$

where $\overline{\mathcal{Q}}_n^0 \equiv +\infty$ for any non-leaf node $n \in \mathcal{N}$, $\overline{\mathcal{Q}}_n^i \equiv 0$ for any iteration $i \in \mathbb{N}$ and any leaf node n, and $\bar{v}_m^i$ satisfies

$$\bar{v}_m^i = f_m(z_m^i, y_m^i, x_m^i) + \sigma_m \psi_m(x_n^i - z_m^i) + \overline{\mathcal{Q}}_m^i(x_m^i). \tag{2.25}$$

Here, the operation $\operatorname{conv}\{f, g\}$ forms the convex hull of the union of the epigraphs of any continuous functions f and g defined on the space $\mathbb{R}^d$. More precisely using convex

conjugacy, we define

$$\begin{aligned}\operatorname{conv}\{f,g\}(x) &:= \big(\min\{f(x),g(x)\}\big)^{**} \\ &= \sup_{\lambda\in\mathbb{R}^d}\inf_{z\in\mathbb{R}^d}\big\{\min\{f(z),g(z)\}+\langle\lambda,x-z\rangle\big\}.\end{aligned} \tag{2.26}$$

The key idea behind the upper bound function (2.24) is to exploit the Lipschitz continuity of the regularized value function $Q_m^R(x)$. In particular, it would follow from induction that $\bar{v}_m^i$ is an upper bound on $Q_m^R(x_n^i)$, and then, by the σ_m-Lipschitz continuity of $Q_m^R(x)$, we have $\bar{v}_m^i + \sigma_m\|x - x_n^i\| \geq Q_m^R(x_n^i) + \sigma_m\|x - x_n^i\| \geq Q_m^R(x)$ for all $x \in X_n$. The next proposition summarizes this property.

Proposition 2.11. *For any non-root node $n \in \mathcal{N}$ and $i \geq 1$, $\overline{\mathcal{Q}}_n^i(x)$ is $(\sum_{m\in\mathcal{C}(n)} p_{nm}\sigma_m)$-Lipschitz continuous. Moreover, we have $\bar{v}_m^i \geq Q_m^{\mathrm{R}}(x_n^i)$ for any node $m \in \mathcal{C}(n)$ and thus*

$$\overline{\mathcal{Q}}_n^i(x) \geq \mathcal{Q}_n^{\mathrm{R}}(x), \quad \forall\, x \in \mathcal{X}_n.$$

Proof. Let $L_n = \sum_{m\in\mathcal{C}(n)} p_{nm}\sigma_m$ for simplicity in this proof. We prove the statement by induction on the number of iterations i. When $i = 1$, $\overline{\mathcal{Q}}_n^1(x) = \min\{+\infty, \sum_{m\in\mathcal{C}(n)} p_{nm}(\bar{v}_m^i + \sigma_m \left\|x - x_n^i\right\|)\} = \sum_{m\in\mathcal{C}(n)} p_{nm}(\bar{v}_m^i + \sigma_m \left\|x - x_n^i\right\|)$ which is clearly L_n-Lipschitz continuous. For any leaf node n, $\overline{\mathcal{Q}}_n^1 \equiv 0 = \mathcal{Q}_n^{\mathrm{R}}$ by definition. Going recursively from leaf nodes to the root node, suppose $\overline{\mathcal{Q}}_m^1 \geq \mathcal{Q}_m^{\mathrm{R}}$ for all $m \in \mathcal{C}(n)$ for some node n, then we have

$$\begin{aligned}\bar{v}_m^1 &= f_m(z_m^1, y_m^1, x_m^1) + \sigma_m\psi_m(x_n^1 - z_m^1) + \overline{\mathcal{Q}}_m^1(x_m^1) \\ &\geq \min\{f_m(z,y,x) + \sigma_m\psi_m(x_n^1 - z) + \overline{\mathcal{Q}}_m^1(x) : (x,y) \in \mathcal{F}_m, z \in \mathcal{X}_n\} \\ &\geq \min\{f_m(z,y,x) + \sigma_m\psi_m(x_n^1 - z) + \mathcal{Q}_m^{\mathrm{R}}(x) : (x,y) \in \mathcal{F}_m, z \in \mathcal{X}_n\} \\ &= Q_m^{\mathrm{R}}(x_n^1).\end{aligned} \tag{2.27}$$

Thus $\overline{\mathcal{Q}}_n^1(x) = \sum_{m\in\mathcal{C}(n)} p_{nm}(\bar{v}_m^1 + \sigma_m \left\|x - x_n^1\right\|) \geq \mathcal{Q}_n^{\mathrm{R}}(x)$ for all $x \in \mathcal{X}_n$ by the σ_m-

Lipschitz continuity of the regularized value functions $Q_m^{\mathrm{R}}(x)$ for all $m \in \mathcal{C}(n)$ shown in Proposition 2.2.

Now assume that the statement holds for all iterations up to $i-1$. For any leaf node n, $\overline{\mathcal{Q}}_n^i \equiv 0 = \mathcal{Q}_n^{\mathrm{R}}$ still holds by definition. For any non-leaf node n, suppose $\overline{\mathcal{Q}}_m^i \geq \mathcal{Q}_m^{\mathrm{R}}$ for all $m \in \mathcal{C}(n)$. Then by the same argument (2.27), we know that $\bar{v}_m^i \geq Q_m^{\mathrm{R}}(x_n^i)$. By induction hypothesis, $\overline{\mathcal{Q}}_n^{i-1}(x) \geq \mathcal{Q}_n^{\mathrm{R}}(x)$ for all $x \in \mathcal{X}_n$. So for the cases without convexity, $\overline{\mathcal{Q}}_n^i(x) = \min\{\overline{\mathcal{Q}}_n^{i-1}(x), \sum_{m\in\mathcal{C}(n)} p_{nm}(\bar{v}_m^i + \sigma_m \left\|x - x_n^i\right\|)\}$ is L_n-Lipschitz continuous and satisfies $\overline{\mathcal{Q}}_n^i(x) \geq \mathcal{Q}_n^{\mathrm{R}}(x)$ since $\sum_{m\in\mathcal{C}(n)} p_{nm}(\bar{v}_m^i + \sigma_m \left\|x - x_n^i\right\|) \geq \mathcal{Q}_n^{\mathrm{R}}(x)$ for all $x \in \mathcal{X}_n$ following Proposition 2.2.

It remains to show that in the convex case $\overline{\mathcal{Q}}_n^i(x)$ is still L_n-Lipschitz continuous and satisfies $\overline{\mathcal{Q}}_n^i(x) \geq \mathcal{Q}_n^{\mathrm{R}}(x)$ for any $x \in \mathcal{X}_n$. Note that $\mathcal{Q}_n^{\mathrm{R}}(x)$ can be naturally extended to the entire space $\mathbb{R}^{d_n} \supset \mathcal{X}_n$ since $\mathcal{Q}_n^{\mathrm{R}} \square (L_n \|\cdot\|)(x) = \mathcal{Q}_n^{\mathrm{R}}(x)$ for any $x \in \mathcal{X}_n$ by the L_n-Lipschitz continuity of $\mathcal{Q}_n^{\mathrm{R}}$. The above argument of the base case $i = 1$ for the nonconvex case can be directly applied to the convex case over $x \in \mathbb{R}^{d_n}$. Now assume that $\overline{\mathcal{Q}}_n^{i-1}(x)$ is L_n-Lipschitz continuous on $\mathbb{R}^{d_n}$ and $\overline{\mathcal{Q}}_n^{i-1}(x) \geq \mathcal{Q}_n^{\mathrm{R}}(x)$ for $x \in \mathbb{R}^{d_n}$ up to $i-1$. Since $Q'_n(x) := \min\{\overline{\mathcal{Q}}_n^{i-1}(x), \sum_{m\in\mathcal{C}(n)} p_{nm}(\bar{v}_m^i + \sigma_m \left\|x - x_n^i\right\|)\}$ is L_n-Lipschitz continuous in $x \in \mathbb{R}^{d_n}$, we claim that the supremum in the definition (2.26) can be attained within the dual norm ball $\mathcal{B}_*(L_n) := \{\lambda \in \mathbb{R}^{d_n} : \|\lambda\|_* \leq L_n\}$. In fact, for any $\lambda \notin \mathcal{B}_*(L_n)$, the infimum

$$\inf_{z\in\mathbb{R}^{d_n}} \{Q'_n(z) + \langle \lambda, z - x\rangle\} \leq \inf_{z\in\mathbb{R}^{d_n}} \{Q'_n(x) + L_n \|z - x\| + \langle \lambda, z - x\rangle\} = -\infty.$$

As a result, $\overline{\mathcal{Q}}_n^i(x)$ is a supremum of L_n-Lipschitz linear functions (of the forms $l(x) := \overline{\mathcal{Q}}_n^i(\hat{z}) + \langle \hat{\lambda}, \hat{z} - x\rangle$ where $\hat{\lambda} \in \mathcal{B}_*(L_n)$ and $\hat{z} \in \mathcal{X}_n$) and thus is also an L_n-Lipschitz continuous function. Therefore, $Q'_n(x) \geq \mathcal{Q}_n^{\mathrm{R}}(x)$ for all $x \in \mathbb{R}^{d_n}$. By (2.26), $\overline{\mathcal{Q}}_n^i(x) = (Q'_n)^{**}(x) \geq (\mathcal{Q}_n^{\mathrm{R}})^{**}(x) = \mathcal{Q}_n^{\mathrm{R}}(x)$ for all $x \in \mathbb{R}^{d_n}$. This completes the proof. □

2.2.3 A Nested Decomposition Algorithm for General Trees

We propose a nested decomposition algorithm in Algorithm 1 for a general scenario tree, i.e. without any stagewise independence assumption on the underlying stochastic process.

In each iteration i, Algorithm 1 carries out a forward step, a backward step, and a root node update step. In the forward step, the algorithm proceeds from $t = 1$ to T by solving all the nodal subproblems with the current under-approximation of their cost-to-go functions in stage t. After all the state variables x_n^i are obtained for nodes $n \in \mathcal{N}$, the backward step goes from $t = T$ back to 1. At each node n in stage t, it first updates the under-approximation of the expected cost-to-go function. Next it solves the dual problem to obtain an optimal primal-dual solution pair $(\hat{x}_n^i, \hat{y}_n^i, \hat{z}_n^i; \hat{\lambda}_n^i, \hat{\rho}_n^i)$, which is used to construct a generalized conjugacy cut using (2.22), together with values $\underline{v}_n^i$ and $\bar{v}_n^i$ calculated with (2.23) and (2.25). Finally the algorithm updates the root node solution using the updated under-approximation of the cost-to-go function, and determines the new lower and upper bounds.

Algorithm 1 solves the regularized problem (2.4) for an ε-optimal root node solution. To justify the ε-optimality of the output of the algorithm, we have the following proposition.

Proposition 2.12. *Given any $\varepsilon > 0$, if* UPPERBOUND $-$ LOWERBOUND $\leq \varepsilon$, *then the returned solution (x_r^*, y_r^*) is an ε-optimal root node solution to the regularized problem* (2.4). *In particular, if $\overline{\mathcal{Q}}_r^i(x_r^{i+1}) - \underline{\mathcal{Q}}_r^i(x_r^{i+1}) \leq \varepsilon$ for some iteration index i, then* UPPERBOUND $-$ LOWERBOUND $\leq \varepsilon$ *and Algorithm 1 terminates after the i-th iteration.*

Proof. From the definition of v^{reg} and Proposition 2.11,

$$v^{\mathrm{reg}} \leq f_r(x_{a(r)}, y_r^*, x_r^*) + \mathcal{Q}_r^{\mathrm{R}}(x_r^*) \leq f_r(x_{a(r)}, y_r^*, x_r^*) + \overline{\mathcal{Q}}_r^i(x_r^*) \leq \text{UPPERBOUND}.$$

Since UPPERBOUND $-$ LOWERBOUND $\leq \varepsilon$, we have

$$f_r(x_{a(r)}, y_r^*, x_r^*) + \overline{\mathcal{Q}}_r^i(x_r^*) \leq f_r(x_{a(r)}, y_r^{i+1}, x_r^{i+1}) + \underline{\mathcal{Q}}_r^i(x_r^{i+1}) + \varepsilon.$$

Algorithm 1 A Nested Decomposition Algorithm for a General Tree

Require: scenario tree $\mathcal{T} = (\mathcal{N}, \mathcal{E})$ with subproblem oracles $\mathscr{O}_r, \mathscr{O}_n^{\mathrm{F}}, \mathscr{O}_n^{\mathrm{B}}, n \neq r$
Require: optimality gap threshold $\varepsilon > 0$
Ensure: an ε-optimal root node solution (x_r^*, y_r^*) to the regularized problem (2.4)

Initialize: $i \leftarrow 1$; $\underline{\mathcal{Q}}_n^0 \leftarrow 0$, $\overline{\mathcal{Q}}_n^0 \leftarrow +\infty \,\forall\, n : \mathcal{C}(n) \neq \varnothing$ and $\overline{\mathcal{Q}}_n^0 \leftarrow 0 \,\forall\, n : \mathcal{C}(n) = \varnothing$
Evaluate $(x_r^1, y_r^1) = \mathscr{O}_r(0)$
Set LowerBound $\leftarrow f_r(x_{a(r)}, y_r^1, x_r^1)$, UpperBound $\leftarrow +\infty$
while UpperBound $-$ LowerBound $> \varepsilon$ **do**
    for $t = 1, \ldots, T-1$ **do** ▷ i-th forward step
        for $n \in \mathcal{N}(t)$ **do**
            Evaluate $(x_n^i, y_n^i, z_n^i) = \mathscr{O}_n^{\mathrm{F}}(x_{a(n)}^i, \underline{\mathcal{Q}}_n^{i-1})$
        end for
    end for
    for $t = T, \ldots, 1$ **do** ▷ i-th backward step
        for $n \in \mathcal{N}(t)$ **do**
            Update $\underline{\mathcal{Q}}_n^i$ and $\overline{\mathcal{Q}}_n^i$ using (2.21) and (2.24)
            Evaluate $(\hat{x}_n^i, \hat{y}_n^i, \hat{z}_n^i; \hat{\lambda}_n^i, \hat{\rho}_n^i) = \mathscr{O}_n^{\mathrm{B}}(x_{a(n)}^i, \underline{\mathcal{Q}}_n^i)$
            Calculate C_n^i, $\underline{v}_n^i$, and $\bar{v}_n^i$ using (2.22), (2.23), and (2.25)
        end for
    end for
    Update $\underline{\mathcal{Q}}_r^i$ and $\overline{\mathcal{Q}}_r^i$ using (2.21) and (2.24) ▷ root node update
    Evaluate $(x_r^{i+1}, y_r^{i+1}) = \mathscr{O}_r(\underline{\mathcal{Q}}_r^i)$
    Update LowerBound $\leftarrow f_r(x_{a(r)}, y_r^{i+1}, x_r^{i+1}) + \underline{\mathcal{Q}}_r^i(x_r^{i+1})$
    if UpperBound $> f_r(x_{a(r)}, y_r^{i+1}, x_r^{i+1}) + \overline{\mathcal{Q}}_r^i(x_r^{i+1})$ **then**
        Update UpperBound $\leftarrow f_r(x_{a(r)}, y_r^{i+1}, x_r^{i+1}) + \overline{\mathcal{Q}}_r^i(x_r^{i+1})$
        Set $(x_r^*, y_r^*) = (x_r^{i+1}, y_r^{i+1})$
    end if
    $i \leftarrow i + 1$
end while

Then, using the optimality of (x_r^{i+1}, y_r^{i+1}) given by $\mathscr{O}_r(\underline{\mathcal{Q}}_r^i)$ and the fact that $\underline{\mathcal{Q}}_r^i(x) \leq \mathcal{Q}_r(x)$, we see that

$$f_r(x_{a(r)}, y_r^{i+1}, x_r^{i+1}) + \underline{\mathcal{Q}}_r^i(x_r^{i+1}) \leq \min_{(x,y)\in\mathcal{F}_r} \left\{ f_r(x_{a(r)}, y, x) + \mathcal{Q}_r(x) \right\} = v^{\mathrm{prim}}.$$

Under Assumption 2.2, $v^{\mathrm{reg}} = v^{\mathrm{prim}}$. Therefore, combining all the above inequalities, we have shown that

$$v^{\mathrm{reg}} \leq f_r(x_{a(r)}, y_r^*, x_r^*) + \overline{\mathcal{Q}}_r^i(x_r^*) \leq v^{\mathrm{reg}} + \varepsilon,$$

which means (x_r^*, y_r^*) is an ε-optimal root node solution to the regularized problem (2.4). Now suppose $\overline{\mathcal{Q}}_r^i(x_r^{i+1}) - \underline{\mathcal{Q}}_r^i(x_r^{i+1}) \leq \varepsilon$ for some iteration index i. Since UPPERBOUND $\leq$ $f_r(x_{a(r)}, y_r^{i+1}, x_r^{i+1}) + \overline{\mathcal{Q}}_r^i(x_r^{i+1})$, we have

$$\begin{aligned}
&\textsc{UpperBound} - \textsc{LowerBound} \\
&\leq f_r(x_{a(r)}, y_r^{i+1}, x_r^{i+1}) + \overline{\mathcal{Q}}_r^i(x_r^{i+1}) - (f_r(x_{a(r)}, y_r^{i+1}, x_r^{i+1}) + \underline{\mathcal{Q}}_r^i(x_r^{i+1})) \\
&= \overline{\mathcal{Q}}_r^i(x_r^{i+1}) - \underline{\mathcal{Q}}_r^i(x_r^{i+1}) \\
&\leq \varepsilon.
\end{aligned}$$

Therefore the algorithm terminates after the i-th iteration. □

2.2.4 A Deterministic Dual Dynamic Programming Algorithm

Starting from this subsection, we focus on the nested decomposition algorithm applied to stagewise independent stochastic problems, which is defined in the following stagewise independence assumption.

Assumption 2.3. *For any $t = 1, \ldots, T-1$ and any $n, n' \in \mathcal{N}(t)$, the state space, the transition probabilities, as well as the data associated with the child nodes $\mathcal{C}(n)$ and $\mathcal{C}(n')$ are identical. In particular, this implies $\mathcal{Q}_n(x) = \mathcal{Q}_{n'}(x) =: \mathcal{Q}_t(x)$ for all $x \in \mathcal{X}_n = \mathcal{X}_{n'} =: \mathcal{X}_t \subseteq \mathbb{R}^{d_t}$.*

We denote $n \sim n'$ for $n, n' \in \mathcal{N}(t)$ for some $t = 1, \ldots, T-1$, if the nodes n, n' are defined by identical data. We then use $\tilde{\mathcal{N}}(t) := \mathcal{N}(t)/\sim$ to denote the set of nodes with size $N_t := |\tilde{\mathcal{N}}(t)|$ that are defined by distinct data in stage t for all $t = 1, \ldots, T-1$, i.e. $\tilde{\mathcal{N}} := \cup_{t=0}^{T} \tilde{\mathcal{N}}(t)$ forms a recombining scenario tree [3]. For each node $m \in \tilde{\mathcal{N}}(t)$, we denote $p_{t-1,m} := p_{nm}$ for any $n \in \tilde{\mathcal{N}}(t-1)$ since $p_{n,m} = p_{n',m}$ for any $n, n' \in \tilde{\mathcal{N}}(t-1)$. Due to stagewise independence, it suffices to keep track of the state of each stage in the algorithm, instead of the state of each node. To be consistent, we also denote the root node solution as (x_0^i, y_0^i) for $i \in \mathbb{N}$. We present the algorithm in Algorithm 2.

Similar to Algorithm 1, each iteration in Algorithm 2 consists of a forward step, a backward step, and a root node update step. The name "deterministic dual dynamic programming" refers to the *deterministic sampling* procedure in the forward step. In particular, at a node $n \in \tilde{\mathcal{N}}(t)$ with $t < T$, the forward step proceeds to a child node $m \in \tilde{\mathcal{N}}(t+1)$, where the *approximation gap* $\gamma_m^i := \overline{\mathcal{Q}}_t^{i-1}(x_m^i) - \underline{\mathcal{Q}}_t^{i-1}(x_m^i)$ is among the largest of all the approximation gaps of states $x_{m'}^i$ of nodes $m' \in \tilde{\mathcal{N}}(t+1)$. Then the state variable of node m is considered the state variable of stage $t(m)$ in the iteration i. Due to stagewise independence, the backward step at each stage t only need to generate cuts for the nodes in the recombining tree $\tilde{\mathcal{N}}$. The optimality of the returned solution (x_0^*, y_0^*) is guaranteed by Proposition 2.12.

2.2.5 A Stochastic Dual Dynamic Programming Algorithm

Now we present a stochastic dual dynamic programming algorithm, which uses stochastic sampling rather than deterministic sampling. So, instead of traversing the scenario tree and finding a path with the largest approximation gap, the stochastic sampling algorithm generates M scenario paths before an iteration begins for some $M \geq 1$. To be precise, we introduce the following notations. Let $\mathcal{P} = \prod_{t=1}^{T} \tilde{\mathcal{N}}(t)$ denote all possible scenario paths from stage 1 to stage T. A scenario path is denoted as a T-element sequence $P = (n_1, \ldots, n_T) \in \mathcal{P}$, where $n_t \in \tilde{\mathcal{N}}(t)$ for each $t = 1, \ldots, T$. In the i-th iteration, we sample

Algorithm 2 Deterministic Sampling Dual Dynamic Programming Algorithm

Require: recombining scenario tree $\tilde{\mathcal{N}}$ with subproblem oracles $\mathscr{O}_r, \mathscr{O}_n^{\mathrm{F}}, \mathscr{O}_n^{\mathrm{B}}, n \neq r$
Require: optimality gap threshold $\varepsilon > 0$
Ensure: an ε-optimal root node solution (x_0^*, y_0^*) to the regularized problem (2.4)

Initialize: $i \leftarrow 1$; $\underline{\mathcal{Q}}_t^0 \leftarrow 0$, $\forall\, t$, $\overline{\mathcal{Q}}_t^0, \leftarrow +\infty \,\forall\, t \leq T-1$ and $\overline{\mathcal{Q}}_T^0 \leftarrow 0$
Evaluate $(x_0^1, y_0^1) = \mathscr{O}_r(0)$
Set LowerBound $\leftarrow f_r(x_{a(r)}, y_0^1, x_0^1)$, UpperBound $\leftarrow +\infty$
while UpperBound $-$ LowerBound $> \varepsilon$ **do**
  for $t = 1, \ldots, T-1$ **do** ▷ i-th forward step
    for $n \in \tilde{\mathcal{N}}(t)$ **do**
      Evaluate $(x_n^i, y_n^i, z_n^i) = \mathscr{O}_n^{\mathrm{F}}(x_{t-1}^i, \underline{\mathcal{Q}}_t^{i-1})$
      Calculate the gap $\gamma_n^i = \overline{\mathcal{Q}}_t^{i-1}(x_n^i) - \underline{\mathcal{Q}}_t^{i-1}(x_n^i)$
    end for
    Select any $n^*(t) \in \{n \in \mathcal{N}(t) : \gamma_n^i \geq \gamma_{n'}^i,\ \forall\, n' \in \tilde{\mathcal{N}}(t)\}$, and let $x_t^i \leftarrow x_{n^*(t)}^i$
  end for
  for $t = T, \ldots, 1$ **do** ▷ i-th backward step
    Update $\underline{\mathcal{Q}}_t^i$ and $\overline{\mathcal{Q}}_t^i$ using (2.21) and (2.24)
    for $n \in \tilde{\mathcal{N}}(t)$ **do**
      Evaluate $(\hat{x}_n^i, \hat{y}_n^i, \hat{z}_n^i; \hat{\lambda}_n^i, \hat{\rho}_n^i) = \mathscr{O}_n^{\mathrm{B}}(x_{t-1}^i, \underline{\mathcal{Q}}_t^i)$
      Calculate $C_n^i, \underline{v}_n^i, \bar{v}_n^i$ using (2.22), (2.23), and (2.25)
    end for
  end for
  Update $\underline{\mathcal{Q}}_0^i$ and $\overline{\mathcal{Q}}_0^i$ using (2.21) and (2.24) ▷ root node update
  Evaluate $(x_0^{i+1}, y_0^{i+1}) = \mathscr{O}_r(\underline{\mathcal{Q}}_0^i)$
  Update LowerBound $\leftarrow f_r(x_{a(r)}, y_0^{i+1}, x_0^{i+1}) + \underline{\mathcal{Q}}_0^i(x_0^{i+1})$
  if UpperBound $> f_r(x_{a(r)}, y_0^{i+1}, x_0^{i+1}) + \overline{\mathcal{Q}}_0^i(x_0^{i+1})$ **then**
    Update UpperBound $\leftarrow f_r(x_{a(r)}, y_0^{i+1}, x_0^{i+1}) + \overline{\mathcal{Q}}_0^i(x_0^{i+1})$
    Set $(x_0^*, y_0^*) = (x_0^{i+1}, y_0^{i+1})$
  end if
  $i \leftarrow i + 1$
end while

M independent scenario paths $\mathscr{P}^i = \{P^{i,1}, \ldots, P^{i,M}\}$, and we use $P_t^{i,j}$ to denote the t-th node in the scenario path $P^{i,j}$, i.e., the node in the t-th stage of the j-th scenario path in the i-th iteration, for $1 \leq j \leq M$ and $1 \leq t \leq T$. Since in each iteration, the solutions and the approximations depend on the scenario path $P^{i,j}$, we use two superscripts i and j for solutions and cuts, where a single superscript i is used in the deterministic sampling algorithm. In addition, for every node $n \in \tilde{\mathcal{N}}(t)$ for some stage t, the under-approximation of the expected cost-to-go function is updated over all scenario path index $j = 1, \ldots, M$, i.e.,

$$\underline{\mathcal{Q}}_t^i(x) \coloneqq \max\left\{\underline{\mathcal{Q}}_t^{i-1}(x), \sum_{m \in \tilde{\mathcal{N}}(t+1)} p_{tm} C_m^{i,j}(x \mid \hat{\lambda}_m^{i,j}, \hat{\rho}_m^{i,j}, \underline{v}_m^{i,j}),\ j = 1, \ldots, M\right\}, \quad (2.28)$$

where $C_m^{i,j}$ is the generalized conjugacy cut generated with $(\hat{x}_m^{i,j}, \hat{y}_m^{i,j}, \hat{z}_m^{i,j}; \hat{\lambda}_m^{i,j}, \hat{\rho}_m^{i,j}) = \mathscr{O}_m^{\mathrm{B}}(x_n^{i,j}, \underline{\mathcal{Q}}_{t+1}^i)$ using formula (2.22). With these notations, the algorithm is displayed in Algorithm 3.

Unlike the preceding two algorithms, Algorithm 3 does not need to construct the over-approximation of the regularized value functions for selecting the child node to proceed with. Instead, it determines the scenario paths before the forward step starts. In the forward step, each nodal problem in the sampled scenario path is solved. Then in the backward step, the dual problems are solved at the nodes that are defined by distinct data, dependent on the parent node's state variable obtained in the forward step. The termination criterion is flexible. In the existing literature [40, 46], statistical upper bounds based on the sampled scenario paths are often used together with the lower bound for terminating the algorithm.

2.3 Upper Bounds on Iteration Complexity of Proposed Algorithms

In this section, we derive upper bounds on the iteration complexity of the three proposed algorithms, i.e. the bound on the iteration index when the algorithm terminates. These upper bounds on the iteration complexity imply convergence of the algorithm to an ε-optimal root node solution for any $\varepsilon > 0$.

Algorithm 3 Stochastic Sampling Dual Dynamic Programming Algorithm

Require: recombining scenario tree $\tilde{\mathcal{N}}$ with subproblem oracles $\mathscr{O}_r, \mathscr{O}_n^{\mathrm{F}}, \mathscr{O}_n^{\mathrm{B}}, n \neq r$

Initialize: $i \leftarrow 1$; $\underline{\mathcal{Q}}_t^0 \leftarrow 0,\ \forall\, t$
Evaluate $(x_0^1, y_0^1) = \mathscr{O}_r(0)$
while some stopping criterion is not satisfied **do**
    Sample M scenario paths $\mathscr{P}^i = \{P^{i,1}, \ldots, P^{i,M}\}$
    for $j = 1, \ldots, M$ **do** ▷ i-th forward step
        for $t = 1, \ldots, T-1$ **do**
            Evaluate $(x_t^{i,j}, y_t^{i,j}, z_t^{i,j}) = \mathscr{O}_n^{\mathrm{F}}(x_{t-1}^{i,j}, \underline{\mathcal{Q}}_t^{i-1})$
        end for
    end for
    for $t = T, \ldots, 1$ **do** ▷ i-th backward step
        Update $\underline{\mathcal{Q}}_t^i$ using (2.28)
        for $j = 1, \ldots, M$ **do**
            for $n \in \mathcal{N}(t)/\sim$ **do**
                Evaluate $(\hat{x}_n^{i,j}, \hat{y}_n^{i,j}, \hat{z}_n^{i,j}; \hat{\lambda}_n^{i,j}, \hat{\rho}_n^{i,j}) = \mathscr{O}_n^{\mathrm{B}}(x_{t-1}^{i,j}, \underline{\mathcal{Q}}_t^i)$
                Calculate $C_n^{i,j}$ and $\underline{v}_n^{i,j}$ using (2.22) and (2.23)
            end for
        end for
    end for
    Update $\underline{\mathcal{Q}}_0^i$ using (2.28) ▷ root node update
    Evaluate $\mathscr{O}_r(\underline{\mathcal{Q}}_0^i) = (x_0^{i+1}, y_0^{i+1})$
    $i \leftarrow i + 1$
end while

2.3.1 Upper Bound Analysis on Iteration Complexity of Algorithm 1

In this section, we discuss the iteration complexity of Algorithm 1. We begin with the definition of a set of parameters used in the convergence analysis. Let ε denote the desired root-node optimality gap ε in Algorithm 1. Let $\delta = (\delta_n)_{n\in\mathcal{N},\mathcal{C}(n)\neq\varnothing}$ be a set of positive numbers such that $\varepsilon = \sum_{n\in\mathcal{N},\mathcal{C}(n)\neq\varnothing} p_n\delta_n$. Since $\varepsilon > 0$, such δ_n's clearly exist. Then, we define recursively for each non-leaf node n

$$\gamma_n(\delta) \coloneqq \delta_n + \sum_{m\in\mathcal{C}(n)} p_{nm}\gamma_m(\delta), \tag{2.29}$$

and $\gamma_n(\delta) = 0$ for leaf nodes n. For $i \in \mathbb{N}$, recall the approximation gap $\gamma_n^i = \overline{\mathcal{Q}}_n^{i-1}(x_n^i) - \underline{\mathcal{Q}}_n^{i-1}(x_n^i)$ for $n \in \mathcal{N}$. For leaf nodes, $\gamma_n^i \equiv 0$ by definition for all $i \in \mathbb{N}$. In addition, we define the sets of indices $\mathcal{I}_n(\delta)$ for each $n \in \mathcal{N}$ as

$$\mathcal{I}_n(\delta) \coloneqq \left\{ i \in \mathbb{N} : \gamma_n^i > \gamma_n(\delta) \text{ and } \gamma_m^i \leq \gamma_m(\delta), \forall\, m \in \mathcal{C}(n) \right\}. \tag{2.30}$$

Intuitively, the index set $\mathcal{I}_n(\delta)$ consists of the iteration indices when all the child nodes of n have good approximation of the expected cost-to-go function at the forward step solution, while the node n itself does not. We show by the following lemma that the backward step for node n in the iteration $i \in \mathcal{I}_n(\delta)$ will reduce the expected cost-to-go function approximation gap at node n to be no more than $\gamma_n(\delta)$.

Lemma 2.4. *If an iteration index $i \in \mathcal{I}_n(\delta)$, i.e., $\overline{\mathcal{Q}}_n^{i-1}(x_n^i) - \underline{\mathcal{Q}}_n^{i-1}(x_n^i) > \gamma_n(\delta)$ and $\overline{\mathcal{Q}}_m^{i-1}(x_m^i) - \underline{\mathcal{Q}}_m^{i-1}(x_m^i) \leq \gamma_m(\delta)$ for all $m \in \mathcal{C}(n)$, then*

$$\overline{\mathcal{Q}}_n^i(x) - \underline{\mathcal{Q}}_n^i(x) \leq \gamma_n(\delta), \quad \forall\, x \in \mathcal{X}_n,\ \|x - x_n^i\| \leq \frac{\delta_n}{2L_n}, \tag{2.31}$$

where $L_n \coloneqq \sum_{m\in\mathcal{C}(n)} p_{nm} \max\{\sigma_m, l_{m,\lambda} + l_{m,\rho}\}$.

Proof. By definition (2.21), $\underline{\mathcal{Q}}_m^i(x) \geq \underline{\mathcal{Q}}_m^{i-1}(x)$ on $\mathcal{X}_m$ for all $m \in \mathcal{C}(n)$. Therefore,

by definition (2.23) and the fact that $(\lambda, \rho) = (0, \sigma_n)$ is a dual feasible solution for the problem (B), we have

$$\begin{aligned}
\underline{v}_m^i &= \max_{(\lambda,\rho)\in\mathcal{U}_m} \min_{\substack{(x,y)\in\mathcal{F}_m,\\ z\in\mathcal{X}_n}} \left\{ f_m(z,y,x) + \left\langle \lambda, x_n^i - z \right\rangle + \rho\psi_n(x_n^i - z) + \underline{\mathcal{Q}}_m^i(x) \right\} \\
&\geq \min_{\substack{(x,y)\in\mathcal{F}_m,\\ z\in\mathcal{X}_n}} \left\{ f_m(z,y,x) + \sigma_n\psi_n(x_n^i - z) + \underline{\mathcal{Q}}_m^i(x) \right\} \\
&\geq \min_{\substack{(x,y)\in\mathcal{F}_m,\\ z\in\mathcal{X}_n}} \left\{ f_m(z,y,x) + \sigma_n\psi_n(x_n^i - z) + \underline{\mathcal{Q}}_m^{i-1}(x) \right\} \\
&= f_m(z_m^i, y_m^i, x_m^i) + \sigma_n\psi_n(x_n^i - z_m^i) + \underline{\mathcal{Q}}_m^{i-1}(x_m^i)
\end{aligned}$$

for all $m \in \mathcal{C}(n)$. The last equality is due to the forward step subproblem oracle $\mathscr{O}_m^{\mathrm{F}}(x_n^i, \underline{\mathcal{Q}}_m^{i-1})$ in the algorithm. Meanwhile, note that $\overline{\mathcal{Q}}_m^i(x) \leq \overline{\mathcal{Q}}_m^{i-1}(x)$ for $x \in \mathcal{X}_m$. By definition (2.25), we have

$$\begin{aligned}
\bar{v}_m^i &= f_m(z_m^i, y_m^i, x_m^i) + \sigma_n\psi_n(x_n^i - z_m^i) + \overline{\mathcal{Q}}_m^i(x_m^i) \\
&\leq f_m(z_m^i, y_m^i, x_m^i) + \sigma_n\psi_n(x_n^i - z_m^i) + \overline{\mathcal{Q}}_m^{i-1}(x_m^i)
\end{aligned}$$

for all $m \in \mathcal{C}(n)$. Note that by definition (2.24), $\overline{\mathcal{Q}}_n^i(x_n^i) \leq \sum_{m\in\mathcal{C}(n)} p_{nm}\bar{v}_m^i$ and by definitions (2.21) and (2.22), $\underline{\mathcal{Q}}_n^i(x_n^i) \geq \sum_{m\in\mathcal{C}(n)} p_{nm} C_m^i(x_n^i \mid \hat{\lambda}_m^i, \hat{\rho}_m^i, \underline{v}_m^i) = \sum_{m\in\mathcal{C}(n)} p_{nm}\underline{v}_m^i$. Therefore,

$$\begin{aligned}
\overline{\mathcal{Q}}_n^i(x_n^i) - \underline{\mathcal{Q}}_n^i(x_n^i) &\leq \sum_{m\in\mathcal{C}(n)} p_{nm}(\bar{v}_m^i - \underline{v}_m^i) \\
&\leq \sum_{m\in\mathcal{C}(n)} p_{nm}[\overline{\mathcal{Q}}_m^{i-1}(x_m^i) - \underline{\mathcal{Q}}_m^{i-1}(x_m^i)] \\
&\leq \sum_{m\in\mathcal{C}(n)} p_{nm}\gamma_m(\delta).
\end{aligned}$$

Note that $\overline{\mathcal{Q}}_n^i(x)$ is $\big(\sum_{m\in\mathcal{C}(n)} p_{nm}\sigma_m\big)$-Lipschitz continuous by Proposition 2.11, and $\underline{\mathcal{Q}}_m^i(x)$ is $\big[\sum_{m\in\mathcal{C}(n)} p_{nm}(l_{n,\lambda}+l_{n,\rho})\big]$-Lipschitz continuous on $\mathcal{X}_n$ by Proposition 2.10. Thus $\overline{\mathcal{Q}}_n^i(x)$

and $\underline{\mathcal{Q}}_n^i(x)$ are both L_n-Lipschitz continuous. Therefore, for any $x \in \mathcal{X}_n$, $\|x - x_n^i\| \leq \delta_n/(2L_n)$, we have

$$\begin{aligned}\overline{\mathcal{Q}}_n^i(x) - \underline{\mathcal{Q}}_n^i(x) &\leq \overline{\mathcal{Q}}_n^i(x_n^i) - \underline{\mathcal{Q}}_n^i(x_n^i) + 2L_n\|x - x_n^i\| \\ &\leq \sum_{m\in\mathcal{C}(n)} p_{nm}\gamma_m(\delta) + \delta_n = \gamma_n(\delta).\end{aligned}$$

This completes the proof. □

In other words, each $i \in \mathcal{I}_n$ would carve out a ball of radius $\delta_n/(2L_n)$ in the state space $\mathcal{X}_n$ such that no point in the ball can be the forward step solution of some iteration i in $\mathcal{I}_n$. This implies that we could bound the cardinality $|\mathcal{I}_n|$ of $\mathcal{I}_n$ by the size of the corresponding state space $\mathcal{X}_n$. Since $\mathcal{X}_n$ can be nonconvex, we consider finite covers of $\mathcal{X}_n$ by norm balls and provide the bound in terms of the number and sizes of the balls. This is made more precise in the following lemma.

Lemma 2.5. *Let $\mathscr{B} = \{\mathcal{B}_{n,k} \subset \mathbb{R}^{d_n}\}_{1\leq k\leq K_n, n\in\mathcal{N}}$ be a collection of balls, each with diameter $D_{n,k} \geq 0$, such that $\mathcal{X}_n \subseteq \bigcup_{k=1}^{K_n} \mathcal{B}_{n,k}$. Then,*

$$\left|\mathcal{I}_n(\delta)\right| \leq \sum_{k=1}^{K_n} \left(1 + \frac{2L_n D_{n,k}}{\delta_n}\right)^{d_n}.$$

Proof. We claim that for any $i, j \in \mathcal{I}_n$, $i \neq j$, then $\|x_n^i - x_n^j\| > \delta_n/(2L_n)$. Assume for contradiction that $\|x_n^i - x_n^j\| \leq \delta_n/(2L_n)$ for some $i < j$ and $i, j \in \mathcal{I}_n(\delta)$. By the definition of $\mathcal{I}_n(\delta)$, $\gamma_m^i \leq \gamma_m(\delta)$ for all $m \in \mathcal{C}(n)$. By Lemma 2.4, $\overline{\mathcal{Q}}_n^i(x) - \underline{\mathcal{Q}}_n^i(x) \leq \gamma_n(\delta)$ for all $x \in \mathcal{X}_n$, $\|x - x_n^i\| \leq \delta_n/(2L_n)$. Since $j > i$ and $\|x_n^i - x_n^j\| \leq \delta_n/(2L_n)$, this implies $\gamma_n^j = \overline{\mathcal{Q}}_n^j(x_n^j) - \underline{\mathcal{Q}}_n^j(x_n^j) \leq \gamma_n(\delta)$, which is a contradiction with $j \in \mathcal{I}_n(\delta)$. Hence we prove the claim.

Let $\mathcal{B}(R), \mathcal{B}(R, x) \subseteq \mathbb{R}^d$ denote the closed balls with radius $R \geq 0$, centered at 0 and x, respectively. It follows from the claim that the closed balls $\mathcal{B}(\delta_n/(4L_n), x_n^i)$ are non-overlapping for all $i \in \mathcal{I}_n(\delta)$, each with the volume $\mathrm{Vol}\mathcal{B}(\delta_n/(4L_n))$. Thus the sum of

the volumes of these balls is $\left|\mathcal{I}_n(\delta)\right| \text{Vol}\mathcal{B}(\delta_n/(4L_n))$. Note that for each index $i \in \mathcal{I}_n(\delta)$, $x_n^i \in \mathcal{X}_n$ and hence $x_n^i \in \mathcal{B}_{n,k}$ for some k. The closed ball $\mathcal{B}(\delta_n/(4L_n), x_n^i) \subseteq \mathcal{B}_{n,k} + \mathcal{B}(\delta_n/(4L_n))$, and therefore

$$\bigcup_{i\in\mathcal{I}_n(\delta)} \mathcal{B}(\delta_n/(4L_n), x_n^i) \subseteq \bigcup_{k=1}^{K_n} (\mathcal{B}_{n,k} + \mathcal{B}(\delta_n/(4L_n))).$$

It follows that

$$\begin{aligned}
\text{Vol}\left(\bigcup_{i\in\mathcal{I}_n(\delta)} \mathcal{B}(\delta_n/(4L_n), x_n^i)\right) &= \left|\mathcal{I}_n(\delta)\right| \cdot \text{Vol}\mathcal{B}(\delta_n/(4L_n)) \\
&\leq \text{Vol}\left(\bigcup_{k=1}^{K_n} (\mathcal{B}_{n,k} + \mathcal{B}(\delta_n/(4L_n))\right) \\
&\leq \sum_{k=1}^{K_n} \text{Vol}\left(\mathcal{B}_{n,k} + \mathcal{B}(\delta_n/(4L_n))\right).
\end{aligned}$$

Therefore,

$$\left|\mathcal{I}_n(\delta)\right| \leq \sum_{k=1}^{K_n} \frac{\text{Vol}\left(\mathcal{B}_{n,k} + \mathcal{B}(\delta_n/(4L_n))\right)}{\text{Vol}\mathcal{B}(\delta_n/(4L_n))} = \sum_{k=1}^{K_n} \left(1 + \frac{2L_n D_{n,k}}{\delta_n}\right)^{d_n}.$$

□

Now we present an upper bound on the iteration complexity of Algorithm 1.

Theorem 2.1. *Given $\varepsilon > 0$, choose values $\delta = (\delta_n)_{n\in\mathcal{N},\mathcal{C}(n)\neq\varnothing}$ such that $\delta_n > 0$ and $\sum_{n\in\mathcal{N},\mathcal{C}(n)\neq\varnothing} p_n\delta_n = \varepsilon$. Let $\mathscr{B} = \{\mathcal{B}_{n,k}\}_{1\leq k\leq K_n, n\in\mathcal{N}}$ be a collection of balls, each with diameter $D_{n,k} \geq 0$, such that $\mathcal{X}_n \subseteq \bigcup_{k=1}^{K_n} \mathcal{B}_{n,k}$ for $n \in \mathcal{N}$. If Algorithm 1 terminates with an ε-optimal root node solution (x_r^*, y_r^*) at the end of i-th iteration, then*

$$i \leq \sum_{\substack{n\in\mathcal{N},\\ \mathcal{C}(n)\neq\varnothing}} \sum_{k=1}^{K_n} \left(1 + \frac{2L_n D_{n,k}}{\delta_n}\right)^{d_n}.$$

Proof. After the i-th iteration, at least one of the following two situations must happen:

i. At the root node, it holds that $\overline{\mathcal{Q}}_r^i(x_r^{i+1}) - \underline{\mathcal{Q}}_r^i(x_r^{i+1}) \leq \gamma_r(\delta)$, where γ_r is defined in (2.29).

ii. There exists a node $n \in \mathcal{N}$ such that $\overline{\mathcal{Q}}_n^i(x_n^{i+1}) - \underline{\mathcal{Q}}_n^i(x_n^{i+1}) > \gamma_n(\delta)$, but all of its child nodes satisfy $\overline{\mathcal{Q}}_m^i(x_m^{i+1}) - \underline{\mathcal{Q}}_m^i(x_m^{i+1}) \leq \gamma_m(\delta)$, $\forall\, m \in \mathcal{C}(n)$. In other words, $i+1 \in \mathcal{I}_n(\delta)$.

Note that $\gamma_r(\delta) = \delta_r + \sum_{m\in\mathcal{C}(r)} p_{rm}\gamma_m(\delta) = \cdots = \sum_{n\in\mathcal{N},\mathcal{C}(n)\neq\varnothing} p_n\delta_n$. If case i happens, then by Proposition 2.12, (x_r^{i+1}, y_r^{i+1}) is an ε-optimal root node solution. Note that case ii can only happen at most $\sum_{n\in\mathcal{N}}\left|\mathcal{I}_n(\delta)\right|$ times by Lemma 2.5. Therefore, we have that

$$i \leq \sum_{\substack{n\in\mathcal{N}\\ \mathcal{C}(n)\neq\varnothing}} \sum_{k=1}^{K_n} \left(1 + \frac{2L_nD_{n,k}}{\delta_n}\right)^{d_n},$$

when the algorithm terminates. □

Theorem 2.1 implies the ε-convergence of the algorithm for any $\varepsilon > 0$. We remark that the form of the upper bound depends on the values δ and the covering balls $\mathcal{B}_{n,k}$, and therefore the right-hand-side can be tightened to the infimum over all possible choices. While it may be difficult to find the best bound in general, in the next section we take some specific choices of δ and $\mathscr{B}$ and simplify the complexity upper bound, based on the stagewise independence assumption.

2.3.2 Upper Bound Analysis on Iteration Complexity of Algorithm 2

Before giving the iteration complexity bound for the deterministic sampling dual dynamic programming algorithm, we slightly adapt the notations in the previous section to the stagewise independent scenario tree. We take the values $\delta = (\delta_n)_{n\in\tilde{\mathcal{N}},\mathcal{C}(n)\neq\varnothing}$ such that $\delta_n = \delta_{n'}$ for all $n, n' \in \tilde{\mathcal{N}}(t)$ for some $t = 1, \ldots, T$. Thus we denote $\delta_t = \delta_n$ for any $n \in \tilde{\mathcal{N}}(t)$, and

$\delta_0 = \delta_r$. The vector of $\gamma_t(\delta)$ is defined recursively for non-leaf nodes as

$$\gamma_t(\delta) := \gamma_{t+1}(\delta) + \delta_t, \quad \text{if } t \leq T-1, \tag{2.32}$$

and $\gamma_T(\delta) = 0$. Let $\gamma_t^i := \overline{\mathcal{Q}}_t^{i-1}(x_t^i) - \underline{\mathcal{Q}}_t^{i-1}(x_t^i)$ and recall that $\gamma_0^i := \gamma_r^i$ for each index i. The sets of indices $\mathcal{I}_t(\delta)$ are defined for $t = 0, \ldots, T-1$ as

$$\mathcal{I}_t(\delta) := \left\{ i \in \mathbb{N} : \gamma_t^i > \gamma_t(\delta) \text{ and } \gamma_{t+1}^i \leq \gamma_{t+1}(\delta) \right\}. \tag{2.33}$$

Note that $\gamma_t^i = \max_{n \in \tilde{\mathcal{N}}(t)} \gamma_n^i$ (line 10 in Algorithm 2). By Lemma 2.4, an iteration $i \in \mathcal{I}_t(\delta)$ implies $\overline{\mathcal{Q}}_t^i(x) - \underline{\mathcal{Q}}_t^i(x) \leq \gamma_t(\delta)$ for all $x \in \mathcal{X}_n$ with $\|x - x_t^i\| \leq \delta_t/(2L_t)$, where $L_t = L_n$ for any $n \in \tilde{\mathcal{N}}(t)$. Moreover, since $\mathcal{X}_n = \mathcal{X}_t$ for $n \in \tilde{\mathcal{N}}(t)$, for any covering balls $\mathcal{B}_{t,k} \subset \mathbb{R}^{d_t}$ with diameters $D_{t,k} \geq 0$, such that $\mathcal{X}_t \subseteq \cup_{k=1}^{K_t} \mathcal{B}_{t,k}$, by the same argument of Lemma 2.5, we know that

$$\left|\mathcal{I}_t(\delta)\right| \leq \sum_{k=1}^{K_t} \left(1 + \frac{2L_t D_{t,k}}{\delta_t}\right)^{d_t}. \tag{2.34}$$

We summarize the upper bound on the iteration complexity of Algorithm 2 in the next theorem, and omit the proof since it is almost a word-for-word repetition with the notation adapted as above.

Theorem 2.2. *Given any $\varepsilon > 0$, choose values $\delta = (\delta_t)_{t=0}^{T-1}$ such that $\delta_t > 0$ and $\sum_{t=0}^{T-1} \delta_t = \varepsilon$. Let $\mathscr{B} = \{\mathcal{B}_{t,k} \subset \mathbb{R}^{d_t}\}_{1 \leq k \leq K_t, 0 \leq t \leq T-1}$ be a collection of balls, each with diameter $D_{t,k} \geq 0$, such that $\mathcal{X}_t \subseteq \bigcup_{k=1}^{K_t} \mathcal{B}_{t,k}$ for $0 \leq t \leq T-1$. If Algorithm 2 terminates with an ε-optimal root node solution (x_0^*, y_0^*) in i iterations, then*

$$i \leq \sum_{t=0}^{T-1} \sum_{k=1}^{K_t} \left(1 + \frac{2L_t D_{t,k}}{\delta_t}\right)^{d_t}.$$

We next discuss some special choices of the values δ and the covering ball collections $\mathscr{B}$. First, since $\mathcal{X}_t$ are compact, suppose $\mathcal{B}_t$ is the smallest ball containing $\mathcal{X}_t$. Then we

have $\operatorname{diam} \mathcal{X}_t \leq D_t \leq 2\operatorname{diam} \mathcal{X}_t$ where $D_t = \operatorname{diam} \mathcal{B}_t$. Moreover, suppose $L_t \leq L$ for some $L > 0$ and $d_t \leq d$ for some $d > 0$. Then by taking $\delta_t = \varepsilon/T$ for all $0 \leq t \leq T-1$, we have the following bound.

Corollary 2.1. *If Algorithm 2 terminates with an ε-optimal root node solution (x_0^*, y_0^*), then the iteration index is bounded by*

$$i \leq T\left(1 + \frac{2LDT}{\varepsilon}\right)^d,$$

where L, d, D are the upper bounds for L_t, d_t, and D_t, $0 \leq t \leq T-1$, respectively.

Proof. Take $\delta_t = \varepsilon/T$ for all $0 \leq t \leq T-1$ and apply Theorem 2.2. □

Note that the iteration complexity bound in Corollary 2.1 grows asymptotically $\mathcal{O}(T^{d+1})$ as $T \to \infty$. Naturally such bound is not satisfactory since it is nonlinear in T with possibly very high degree d. However, by changing the optimality criterion, we next derive an iteration complexity bound that grows linearly in T.

Corollary 2.2. *If Algorithm 2 terminates with a $(T\varepsilon)$-optimal root node solution (x_0^*, y_0^*), then the iteration index is bounded by*

$$i \leq T\left(1 + \frac{2LD}{\varepsilon}\right)^d,$$

where L, d, D are the upper bounds for L_t, d_t, and D_t, $0 \leq t \leq T-1$, respectively.

Proof. Take $\delta_t = \varepsilon$ for all $0 \leq t \leq T-1$ and apply Theorem 2.2. □

The optimality criterion in Corollary 2.2 is reasonable, since practical problems are usually solved in a relative scale with respect to the number of stages. Last, we consider a special case where $\mathcal{X}_t$ are finite for all $0 \leq t \leq T-1$.

Corollary 2.3. *Suppose the cardinality* $|\mathcal{X}_t| \leq K < \infty$ *for all* $0 \leq t \leq T-1$*, for some positive integer* K*. In this case, if Algorithm 2 terminates with an* ε*-optimal root node solution* (x_0^*, y_0^*)*, then the iteration index is bounded by*

$$i \leq TK.$$

Proof. Note that when $\mathcal{X}_t$ is finite, it can be covered by degenerate balls $B_0(x)$, $x \in \mathcal{X}_t$. Thus $D_{t,k} = 0$ for $k = 1, \ldots, K_t$ and $K_t \leq K$ by assumption. Apply Theorem 2.2, we get $i \leq \sum_{t=0}^{T-1} \sum_{k=1}^{K_t} 1 \leq TK$. □

The bound in Corollary 2.3 grows linearly in T and does not depend on the value of ε. In other words, we are able to obtain exact solutions to the regularized problem (2.4) assuming the subproblem oracles.

Remark. All the iteration complexity bounds in Theorem 2.2, Corollary 2.1, Corollary 2.2, and Corollary 2.3 are independent of the size of the scenario tree in each stage N_t, $1 \leq t \leq T$. This can be explained by the fact that Algorithm 2 evaluates $1 + N_T + 2\sum_{t=1}^{T-1} N_t$ times of the subproblem oracles in each iteration.

2.3.3 Upper Bound Analysis on Iteration Complexity of Algorithm 3

In the following we study the iteration complexity of Algorithm 3. For clarity, we model the subproblem oracles $\mathscr{O}_n^{\mathrm{F}}$ and $\mathscr{O}_n^{\mathrm{B}}$ as random functions, that are $\Sigma_i^{\mathrm{oracle}}$-measurable in each iteration $i \in \mathbb{N}$, for any node $n \neq r$, where $\{\Sigma_i^{\mathrm{oracle}}\}_{i=0}^{\infty}$ is a filtration of σ-algebras in the probability space. Intuitively, this model says that the information given by $\Sigma_i^{\mathrm{oracle}}$ could be used to predict the outcome of the subproblem oracles. We now make the following assumption on the sampling step.

Assumption 2.4. *In each iteration* i*, the* M *scenario paths are sampled uniformly with replacement, independent from each other and the outcomes of the subproblem oracles. That is, the conditional probability of the* j*-th sample* $P^{i,j}$ *taking any scenario* $n_t \in \tilde{\mathcal{N}}(t)$

in stage t is almost surely

$$\mathrm{Prob}(P_t^{i,j} = n_t \mid \Sigma_\infty^{\mathrm{oracle}}, \sigma\{P_{t'}^{i',j'}\}_{(i',j',t')\neq(i,j,t)}) = \frac{1}{N_t}, \tag{2.35}$$

where $\Sigma_\infty^{\mathrm{oracle}} := \cup_{i=1}^{\infty}\Sigma_i^{\mathrm{oracle}}$, and $\sigma\{P_{t'}^{i',j'}\}_{(i',j',t')\neq(i,j,t)}$ is the σ-algebra generated by scenario samples other than the j-th sample in stage t of iteration i.

In the sampling step in the i-th iteration, let $\gamma_t^{i,j} := \mathcal{Q}_t^{\mathrm{R}}(x_t^{i,j}) - \underline{\mathcal{Q}}_t^{i-1}(x_t^{i,j})$ for any $t \leq T-1$, which is well defined by Assumption 2.3, and let $\tilde{\gamma}_t^{i,j} := \max\{\mathcal{Q}_t^{\mathrm{R}}(x_n) - \underline{\mathcal{Q}}_t^{i-1}(x_n) : (x_n, y_n, z_n) = \mathscr{O}_n^{\mathrm{F}}(x_{t-1}^{i,j}, \underline{\mathcal{Q}}_n^{i-1}),\ n \in \tilde{\mathcal{N}}(t)\}$ for each scenario path index $1 \leq j \leq M$. Note that by definition, we have $\gamma_t^{i,j} \leq \tilde{\gamma}_t^{i,j}$ for any $t = 1, \ldots, T-1$, everywhere in the probability space. We define the sets of indices $\mathcal{I}_t(\delta)$ for each $t = 0, \ldots, T-1$, similar to those in the deterministic sampling case, as

$$\mathcal{I}_t(\delta) := \bigcup_{j=1}^{M} \left\{ i \in \mathbb{N} : \gamma_t^{i,j} > \gamma_t(\delta) \text{ and } \tilde{\gamma}_{t+1}^{i,j} \leq \gamma_{t+1}(\delta) \right\}. \tag{2.36}$$

With the same argument, we know that the upper bound (2.34) on the sizes of $\mathcal{I}_t(\delta)$ holds everywhere for each $t = 0, \ldots, T-1$. However, since the nodes in the forward steps are sampled randomly, we do not necessarily have $i \in \cup_{t=0}^{T-1}\mathcal{I}_t(\delta)$ for each iteration index $i \in \mathbb{N}$ before Algorithm 3 first finds an ε-optimal root node solution. Instead, we define an event $A_i(\delta) := \{i \in \cup_{t=0}^{T-1}\mathcal{I}_t(\delta)\} \bigcup \cup_{j=1}^{M}\{\gamma_0^{i-1,j} \leq \gamma_0(\delta) = \varepsilon\}$ for each iteration i, that means either some approximation is improved in iteration i or the algorithm has found an ε-optimal root node solution in iteration $i-1$. The next lemma estimates the conditional probability of $A_i(\delta)$ given any oracles outcomes and samplings up to iteration i. For simplicity, we define two σ-algebras $\Sigma_i^{\mathrm{sample}} := \sigma\{P^{i',j'}\}_{i'\leq i, j'=1,\ldots,M}$ and $\Sigma_i := \sigma(\Sigma_i^{\mathrm{oracle}}, \Sigma_i^{\mathrm{sample}})$ for each i.

Lemma 2.6. *Fix any $\varepsilon = \sum_{t=0}^{T-1} \delta_t$. Then the conditional probability inequality*

$$\mathrm{Prob}(A_i(\delta) \,|\, \Sigma_{i-1}) \geq \nu := 1 - (1 - 1/N)^M,$$

holds almost surely, where $N := \prod_{t=1}^{T-1} N_t$ if $T \geq 2$ and $N := 1$ otherwise.

Proof. For each iteration $i \in \mathbb{N}$, the event $\cup_{j=1}^{M} \{\gamma_0^{i-1,j} \leq \gamma_0(\delta) = \varepsilon\}$ is Σ_{i-1}-measurable, so it suffices to prove this inequality for its complement in $A_i(\delta)$. Note that

$$\mathrm{Prob}\{\gamma_t^{i,j} = \tilde{\gamma}_t^{i,j} \,|\, \Sigma_{i-1}\} \geq \mathrm{Prob}\{P_t^{i,j} = n(\tilde{\gamma}_t^{i,j}) \,|\, \Sigma_{i-1}\},$$

where $n(\tilde{\gamma}_t^{i,j})$ is the smallest node index $n \in \tilde{\mathcal{N}}(t)$ such that $\mathcal{Q}_t^{\mathrm{R}}(x_n) - \underline{\mathcal{Q}}_t^{i-1}(x_n) = \tilde{\gamma}_t^{i,j}$ for $(x_n, y_n, z_n) = \mathscr{O}_n^{\mathrm{F}}(x_{t-1}^{i,j}, \underline{\mathcal{Q}}_n^{i-1})$, which is determined given Σ_{i-1}. Using the same argument as in the proof of Theorem 2.1, Lemma 2.4 shows that the event $\cap_{t=1}^{T-1} \{\gamma_t^{i,j} = \tilde{\gamma}_t^{i,j}\}$ implies the event $\{i \in \cup_{t=0}^{T-1} \mathcal{I}_t(\delta)\}$ and hence the event $A_i(\delta)$ for each $j = 1, \ldots, M$. Therefore, since Σ_{i-1} is contained in $\sigma(\Sigma_\infty^{\mathrm{oracle}}, \sigma\{P^{i',j'}\}_{(i',j') \neq (i,j)})$, by the independent, uniform sampling (Assumption 2.4), we have

$$\begin{aligned}
&\mathrm{Prob}(A_i(\delta) \,|\, \Sigma_{i-1}) \\
&\geq \mathrm{Prob}\left(\bigcup_{j=1}^{M} \bigcap_{t=1}^{T-1} \{\gamma_t^{i,j} = \tilde{\gamma}_t^{i,j}\} \;\middle|\; \Sigma_{i-1} \right) \\
&\geq \mathrm{Prob}\left(\bigcup_{j=1}^{M} \bigcap_{t=1}^{T-1} \{\gamma_t^{i,j} = n(\tilde{\gamma}_t^{i,j})\} \;\middle|\; \Sigma_{i-1} \right) \\
&= 1 - \left(1 - \mathrm{Prob}\left(\bigcap_{t=1}^{T-1} \{\gamma_t^{i,j} = n(\tilde{\gamma}_t^{i,j})\} \;\middle|\; \Sigma_{i-1} \right) \right)^M = 1 - (1 - 1/N)^M.
\end{aligned}$$

Here, the last step follows from $\mathrm{Prob}(\bigcap_{t=1}^{T-1} \{\gamma_t^{i,j} = n(\tilde{\gamma}_t^{i,j})\} \mid \Sigma_{i-1}) = \prod_{t=1}^{T-1} \mathrm{Prob}(\{\gamma_t^{i,j} = n(\tilde{\gamma}_t^{i,j})\} \mid \Sigma_{i-1}) = \prod_{t=1}^{T-1} (1/N_t) = N$. □

Now we are ready to present the probabilistic complexity bound of Algorithm 3.

Theorem 2.3. *Let $I = I(\delta, \mathscr{B})$ denote the iteration complexity bound in Theorem 2.2, determined by the vector δ and the collection of state space covering balls $\mathscr{B}$, and ν denote the probability bound proposed in Lemma 2.6. Moreover, let ι be the random variable of the smallest index such that the root node solution $(x_0^{\iota+1}, y_0^{\iota+1})$ is ε-optimal in Algorithm 3. Then for any real number $\kappa > 1$, the probability*

$$\mathrm{Prob}\left(\iota \geq 1 + \frac{\kappa I}{\nu}\right) \leq \exp\left(\frac{-I\nu(\kappa-1)^2}{16\kappa}\right).$$

Proof. Let $a_i := \mathbb{1}_{A_i}$ denote the indicator of the event A_i for $i \in \mathbb{N}$, and $S_i := \sum_{i'=1}^{i} a_i$. Note that the event $\{\iota \geq i\}$ implies the event $\{S_i \leq I\}$, so we want to bound probability of the latter for sufficiently large indices i.

By Lemma 2.6, we see that the adapted sequence $\{S_i - i\nu\}_{i=1}^{\infty}$ is a submartingale with respect to the filtration $\{\Sigma_i\}_{i=1}^{\infty}$, because

$$\mathbb{E}(S_i - i\nu \mid \Sigma_{i-1}) = S_{i-1} - (i-1)\nu + (\mathbb{E}(a_i \mid \Sigma_{i-1}) - \nu) \geq S_{i-1} - (i-1)\nu.$$

Moreover, it has a bounded difference as $S_i + i\nu - (S_{i-1} + (i-1)\nu) = a_i + \nu \leq 2$ almost surely. Now apply the one-sided Azuma-Hoeffding inequality and we get for any $k > 0$ that

$$\mathrm{Prob}(S_i \leq i\nu - k) \leq \exp\left(-\frac{k^2}{8i}\right).$$

For any $\kappa > 1$, take the smallest iteration index i such that $i\nu \geq \kappa I$, and set $k := (\kappa - 1)I$. Since $I \geq \frac{i\nu}{2\kappa}$, the probability bound can be then written as

$$\mathrm{Prob}(\iota \geq i) \leq \mathrm{Prob}(S_i \leq I) \leq \exp\left(-\frac{(\kappa-1)^2 I^2}{8i}\right) \leq \exp\left(-\frac{(\kappa-1)^2 I\nu}{16\kappa}\right)$$

Substitute the left-hand-side with $\mathrm{Prob}(\iota \geq 1 + \frac{\kappa I}{\nu})$ using the definition of i and we have obtained the desired inequality. □

Remark. Theorem 2.3 shows that for a fixed problem (such that $I = I(\delta, \mathscr{B})$ and $N = N_1 \cdots N_{T-1}$ are fixed), given any probability threshold $q \in (0, 1)$, the number of iterations needed for Algorithm 3 to find an ε-optimal root node solution with probability greater than $1 - q$ is $\mathcal{O}(-\ln q/\nu^2)$, which does not depend on I. In particular, if we set $M = 1$, then the number of iterations needed is $\mathcal{O}(-N^2 \ln q)$, which is exponential in the number of stage T if $N_t \geq 2$ for all $t = 1, \ldots, T-1$. It remains unknown to us whether there exists a complexity bound for Algorithm 3 that is polynomial in T in general.

2.4 Lower Bounds on Iteration Complexity of Proposed Algorithms

In this section, we discuss the sharpness of the iteration complexity bound of Algorithm 2 given in Section 2.3. In particular, we are interested in the question whether it is possible that the iteration needed for Algorithm 2 to find an ε-optimal root node solution grows linearly in T when the state spaces are infinite sets. We will see that in general it is not possible, with or without the assumption of convexity. The following lemma simplifies the discussion in this section.

Lemma 2.7. *Suppose $f_n(z, y, x)$ is l_n-Lipschitz continuous in z with for each $n \in \mathcal{N}$. If we choose $\psi_n(x) = \|x\|$ and $\sigma_n \geq l_n$, then $Q_n^{\mathrm{R}}(x) = Q_n(x)$ on $\mathcal{X}_{a(n)}$ for all non-root nodes $n \in \mathcal{N}$.*

Proof. We prove the lemma recursively starting from the leaf nodes. For leaf nodes $n \in \mathcal{N}$, $\mathcal{C}(n) = \varnothing$, $Q_n^{\mathrm{R}}(x) = \min_{z \in \mathcal{X}_{a(n)}} Q_n(z) + \sigma_n \psi_n(x - z) \geq \min_{z \in \mathcal{X}_{a(n)}} Q_n(z) + l_n \|x - z\|$. Since Q_n is l_n-Lipschitz continuous, $Q_n(z) \geq Q_n(x) - l_n \|x - z\|$. Therefore, $Q_n^{\mathrm{R}}(x) \geq Q_n(x)$ and by Proposition 2.2, we know $Q_n^{\mathrm{R}}(x) = Q_n(x)$ for all $x \in \mathcal{X}_{a(n)}$.

Now suppose for a node $n \in \mathcal{N}$, we know that all of its child nodes satisfy $Q_m^{\mathrm{R}}(x) = Q_m(x), \forall\, x \in \mathcal{X}_n$, for all $m \in \mathcal{C}(n)$. Then by definition,

$$Q_n^{\mathrm{R}}(x_{a(n)}) = \min_{(x,y) \in \mathcal{F}_n, z \in \mathcal{X}_{a(n)}} f_n(z, y, x) + \sigma_n \psi_n(x_{a(n)} - z) + \mathcal{Q}_n^{\mathrm{R}}(x).$$

By assumption, we know that $\mathcal{Q}_n^{\mathrm{R}}(x) = \mathcal{Q}_n(x)$ for all $x \in \mathcal{X}_n$. Therefore, $Q_n^{\mathrm{R}}(x_{a(n)}) = \min_{z \in \mathcal{X}_{a(n)}} Q_n(z) + \sigma_n \psi_n(x_{a(n)} - z) \geq \min_{z \in \mathcal{X}_{a(n)}} Q_n(z) + l_n \|x_{a(n)} - z\|$. Then again by l_n-Lipschitz continuity of f_n, we conclude that $Q_n^{\mathrm{R}}(x) = Q_n(x)$ for all $x \in \mathcal{X}_{a(n)}$. □

This lemma shows that for problems that already have Lipschitz continuous value functions, the regularization does not change the function value at any point. Thus the examples in the rest of this section serve the discussion not only for Algorithm 2, but for more general algorithms including SDDP and SDDiP.

2.4.1 General Lipschitz Continuous Problems

We discuss the general Lipschitz continuous case, i.e., the nodal objective functions $f_n(z, y, x)$ are l_n-Lipschitz continuous in z but not necessarily convex. In this case we choose to approximate the value function using $\psi_n(x) = \|x\|$ and assume that $l_{n,\rho} \geq l_n$. We can set $l_{n,\lambda} = 0$ for all $n \in \mathcal{N}$, without loss of exactness of the approximation by inequality (2.19). We begin with the following lemma on the complexity of such approximation.

Lemma 2.8. *Consider a norm ball $\mathcal{X} = \{x \in \mathbb{R}^d : \|x\| \leq D/2\}$ and a finite set of points $\mathcal{W} = \{w_k\}_{k=1}^K \subset \mathcal{X}$. Suppose that there is $\beta > 0$ and an L-Lipschitz continuous function $f : \mathcal{X} \to \mathbb{R}_+$ such that $\beta < f(w_k) < 2\beta$ for $k = 1, \ldots, K$. Define*

- $\underline{Q}(x) \coloneqq \max_{k=1,\ldots,K} \{0, f(w_k) - L\|x - w_k\|\}$ *and*
- $\overline{Q}(x) \coloneqq \min_{k=1,\ldots,K} \{f(w_k) + L\|x - w_k\|\}$.

If $K < \left(\frac{DL}{4\beta}\right)^d$, then $\min_{x \in \mathcal{X}} \underline{Q}(x) = 0$ and $\min_{x \in \mathcal{X}} \overline{Q}(x) > \beta$.

Proof. We claim that if $K < \left(\frac{DL}{4\beta}\right)^d$, then there exists a point $\hat{x} \in \mathcal{X}$ such that $\|\hat{x} - w_k\| \geq \frac{2\beta}{L}$ for all $k = 1, \ldots, K$. We prove the claim by contradiction. Suppose such a point does not exist, or equivalently, for any point $x \in \mathcal{X}$, there exists $w_k \in \mathcal{W}$ such that $\|x - w_k\| < \frac{2\beta}{L}$. This implies that the balls $\mathcal{B}(2\beta/L, w_k)$ cover the set $\mathcal{X}$, which leads

to

$$\mathrm{Vol}\mathcal{X} \leq \mathrm{Vol}\left(\bigcup_{k=1}^{K} \mathcal{B}(2\beta/L, w_k)\right) \leq \sum_{k=1}^{K} \mathrm{Vol}\mathcal{B}(2\beta/L, w_k) = K \cdot \mathrm{Vol}\mathcal{B}(2\beta/L).$$

Therefore, it must hold that $K \geq \mathrm{Vol}\mathcal{X}/\mathrm{Vol}\mathcal{B}(2\beta/L) = \left(\frac{DL}{4\beta}\right)^d$, hence a contradiction. The existence of $\hat{x}$ guarantees that $f(w_k) - L\|\hat{x} - w_k\| \leq f(w_k) - 2\beta < 0$ for each $k = 1, 2, \ldots, K$. Therefore, $0 \leq \min_{x\in\mathcal{X}} \underline{Q}(x) \leq \underline{Q}(\hat{x}) = \max_{1\leq k\leq K}\{0, f(w_k) - L\|\hat{x} - w_k\|\} = 0$. From compactness of $\mathcal{X}$ and the continuity of $\overline{Q}(x)$, we have the inequality $\min_{x\in\mathcal{X}} \overline{Q}(x) \geq \min_{1\leq k\leq K} \overline{Q}(w_k) = \min_{1\leq k\leq K} f(w_k) > \beta$, which completes the proof. □

The lemma shows that if the number of points in $\mathcal{W}$ is too small, i.e., $K < (DL/2\beta)^d$, then the difference between the upper and lower bounds could be big, i.e., $\overline{Q}(\bar{x}) - \underline{Q}(\bar{x}) > \beta$ for some $\bar{x}$. In other words, in order to have a small gap between the upper and lower bounds, we need sufficient number of sample points. This lemma is directly used to provide a lower bound on the complexity of Algorithm 2.

Now we construct a Lipschitz continuous multistage problem defined on a chain, i.e., a scenario tree, where each stage has a single node, $N(t) = 1$ for $t = 1, \ldots, T$. The problem is given by the value functions in each stage as,

$$\begin{cases} Q_r = \min\limits_{x_0\in\mathcal{X}_r} Q_1(x_0), \\ Q_t(x_{t-1}) = \min\limits_{x_t\in\mathcal{X}_t} \left\{ f_t(x_{t-1}) + Q_{t+1}(x_t) \right\}, \quad 1 \leq t \leq T-1, \\ Q_T(x_{T-1}) = f_T(x_{T-1}). \end{cases} \tag{2.37}$$

Here for all $t = 1, \ldots, T$, $f_t : \mathcal{X}_t \to \mathbb{R}_+$ is an L-Lipschitz continuous function that satisfies $\beta < f_t(x) < 2\beta$ for all $x \in \mathcal{X}_t$ with $\beta := \varepsilon/T$, the number of stages $T \geq 1$, and $\varepsilon > 0$ is a fixed constant. The state space $\mathcal{X}_t := \mathcal{B}^d(D/2) \subset \mathbb{R}^d$ is a ball with radius $D/2 > 0$. We remark that ε will be the optimality gap in Theorem 2.4. So for a fixed optimality gap

ε, we construct an instance of multistage problem (2.37) that will prove to be difficult for Algorithm 2 to solve.

Also (2.37) is constructed such that there is no constraint coupling the state variables x_t in different stages.

By Lemma 2.7, if we choose $\psi_n(x) = \|x\|$ for all $n \in \mathcal{N}$ and $l_{n,\rho} = L$ for the problem (2.37), then we have $Q_t^{\mathrm{R}}(x) = Q_t(x)$ for all $t = 1, \ldots, T$. The next theorem shows a lower bound on the iteration complexity of problem (2.37) with this choice of penalty functions.

Theorem 2.4. *For any optimality gap $\varepsilon > 0$, there exists a problem of the form* (2.37) *with subproblem oracles $\mathscr{O}_n^{\mathrm{F}}, \mathscr{O}_n^{\mathrm{B}}$, $n \in \mathcal{N}$, and $\mathscr{O}_r$, such that if Algorithm 2 gives* UPPERBOUND$-$LOWERBOUND $\leq \varepsilon$ *in the i-th iteration, then*

$$i \geq \left(\frac{DLT}{4\varepsilon}\right)^d.$$

Proof. Let us define the forward subproblem oracle $\mathscr{O}_n^{\mathrm{F}}$ in iteration i and stage t as mapping $(x_{t-1}^i, \underline{Q}_{t+1}^{i-1})$ to an optimal solution (x_t^i, z_t^i) of the forward subproblem

$$\min_{x_t, z_t \in \mathcal{X}_t} \left\{ f_t(z_t) + L\|x_{t-1}^i - z_t\| + \underline{Q}_{t+1}^{i-1}(x_t) \right\},$$

and the backward subproblem oracle $\mathscr{O}_n^{\mathrm{B}}$ in iteration i and stage t as mapping $(x_{t-1}^i, \underline{Q}_{t+1}^i)$ to an optimal solution $(\hat{x}_t^i, \hat{z}_t^i; \hat{\lambda}_t^i = 0, \hat{\rho}_t^i = L)$ of the backward subproblem

$$\max_{\substack{\lambda=0 \\ 0 \leq \rho \leq L}} \min_{x_t, z_t \in \mathcal{X}_t} \left\{ f_t(z_t) + \rho\|x_{t-1}^i - z_t\| + \underline{Q}_{t+1}^i(x_t) \right\}. \tag{2.38}$$

Note that in the backward subproblem (2.38), we choose that $l_{n,\lambda} = 0$ and $l_{n,\rho} = L$. It is observed that the objective function in (2.38) is nondecreasing in ρ. Therefore, $\hat{\rho}_t^i = L$ is always an optimal solution for the outer maximization in (2.38). The root-node oracle $\mathscr{O}_r$ in iteration i simply solves $\min_{x_0 \in \mathcal{X}} \underline{Q}_1^i(x_0)$ and outputs x_0^{i+1}.

In the backward step (Algorithm 2, step 16) and c.f. the definition (2.22), the new generalized conjugacy cut in iteration $k \leq i$ is generated by

$$C_t^k(x \,|\, 0, L, \underline{v}_t^k) = \underline{v}_t^k - L\|x - x_{t-1}^k\| = \underline{v}_t^k - L\|x - x_{t-1}^k\|,$$

for node $t \geq 1$, where $\underline{v}_t^k$ is computed and upper bounded as

$$\begin{aligned}
\underline{v}_t^k &= f_t(\hat{z}_t^k) + L\|x_{t-1}^k - \hat{z}_t^k\| + \min_{x_t \in \mathcal{X}_t} \underline{Q}_{t+1}^k(x_t), \\
&\leq f_t(x_{t-1}^k) + \min_{x_t \in \mathcal{X}_t} \underline{Q}_{t+1}^k(x_t), \\
&\leq f_t(x_{t-1}^k) + \min_{x_t \in \mathcal{X}_t} \underline{Q}_{t+1}^i(x_t),
\end{aligned}$$

where the first inequality directly follows from (2.38), as $z = x_{t-1}^i$ is a feasible solution of the inner minimization problem, and the second inequality is due to the monotonicity $\underline{Q}_{t+1}^k(x) \leq \underline{Q}_{t+1}^i(x)$ for $k \leq i$. Therefore,

$$\begin{aligned}
\underline{Q}_t^i(x) &= \max_{k=1,\ldots,i} \left\{0,\ C_t^k(x \,|\, 0, L, \underline{v}_t^k)\right\}, \quad \text{(by (2.21))}, \\
&= \max_{k=1,\ldots,i} \left\{0,\ \underline{v}_t^k - L\|x - x_{t-1}^k\|\right\}, \\
&\leq \max_{k=1,\ldots,i} \left\{0,\ f_t(x_{t-1}^k) - L\|x - x_{t-1}^k\|\right\} + \min_{x_t \in \mathcal{X}_t} \underline{Q}_{t+1}^i(x_t). \qquad (2.39)
\end{aligned}$$

Similarly, by (2.25), the upper approximation of the value function is computed and lower bounded as

$$\begin{aligned}
\bar{v}_t^k &= f_t(z_t^k) + L\|z_t^k - x_{t-1}^k\| + \overline{Q}_{t+1}^k(x_t^k), \\
&\geq f_t(x_{t-1}^k) + \overline{Q}_{t+1}^k(x_t^k), \\
&\geq f_t(x_{t-1}^k) + \min_{x_t \in \mathcal{X}_t} \overline{Q}_{t+1}^k(x_t), \\
&\geq f_t(x_{t-1}^k) + \min_{x_t \in \mathcal{X}_t} \overline{Q}_{t+1}^i(x_t),
\end{aligned}$$

where $\hat{\mathcal{X}}_t^k := \arg\min_{x_t \in \mathcal{X}_t} \underline{Q}_{t+1}^{k-1}(x_t)$. Therefore, the over-approximation satisfies

$$\begin{aligned}
\overline{Q}_t^i(x) &= \min_{k=1,\ldots,i} \left\{ \bar{v}_t^k + L\|x - x_{t-1}^k\| \right\}, \quad \text{(by (2.24))}, \\
&\geq \min_{k=1,\ldots,i} \left\{ f_t(x_{t-1}^k) + L \left\| x - x_{t-1}^k \right\| \right\} + \min_{x_t \in \mathcal{X}_t} \overline{Q}_{t+1}^i(x_t), \\
&> \frac{\varepsilon}{T} + \min_{x_t \in \mathcal{X}_t} \overline{Q}_{t+1}^i(x_t), \qquad (2.40)
\end{aligned}$$

where (2.40) follows from the construction that $f_t(x) > \varepsilon/T$ for all $x \in \mathcal{X}_t$.

Now using (2.39) and (2.40), we can prove the statement of the theorem. Suppose the iteration index $i < \left(\frac{DLT}{4\varepsilon}\right)^d$. Denote $w_k := x_{t-1}^k$ for $k = 1, \ldots, i$. Since $\varepsilon/T < f_t(w_k) < 2\varepsilon/T$ by construction, applying Lemma 2.8, we get

$$\min_{x_t \in \mathcal{X}_t} \left\{ \max_{k=1,\ldots,i} \left\{ 0,\ f_t(x_{t-1}^k) - L\|x_t - x_{t-1}^k\| \right\} \right\} = 0.$$

By (2.39), $\min_{x_{t-1} \in \mathcal{X}_{t-1}} \underline{Q}_t^i(x_{t-1}) \leq \min_{x_t \in \mathcal{X}_t} \underline{Q}_{t+1}^i(x_t)$ for $t = 1, \ldots, T$. Note at stage T, $\underline{Q}_{T+1}^i \equiv 0$. Therefore, $\min_{x_{t-1} \in \mathcal{X}_{t-1}} \underline{Q}_t^i(x_{t-1}) \leq 0$ for all $t = 1, \ldots, T$. But since $\underline{Q}_t^i(x) \geq 0$ for all $x \in \mathcal{X}_{t-1}$, we have $\min_{x \in \mathcal{X}_{t-1}} \underline{Q}_t^i(x) = 0$ for all $1 \leq t \leq T$. Hence we see that LOWERBOUND $= \min_{x_0 \in \mathcal{X}_0} \underline{Q}_1^i(x_0) = 0$ in iteration i.

Since $\mathcal{X}_t$ is a norm ball, it is compact. So by (2.40), we have

$$\min_{x_{t-1} \in \mathcal{X}_{t-1}} \overline{Q}_t^i(x_{t-1}) > \varepsilon/T + \min_{x_t \in \mathcal{X}_t} \overline{Q}_{t+1}^i(x_t), \quad \forall 1 \leq t \leq T.$$

This recursion implies that $\min_{x_0 \in \mathcal{X}_0} \overline{Q}_1^i(x_0) > T(\varepsilon/T) = \varepsilon$. According to Algorithm 2, Steps 22-23, we have that in iteration i,

$$\text{UPPERBOUND} = \min_{k=1,\ldots,i} \{\overline{Q}_1^k(x_0^{k+1})\} \geq \min_{k=1,\ldots,i} \{\overline{Q}_1^i(x_0^{k+1})\} \geq \min_{x_0 \in \mathcal{X}_0} \overline{Q}_1^i(x_0) > \varepsilon.$$

Combining the above analysis, we have UPPERBOUND $-$ LOWERBOUND $> \varepsilon$ in iteration i. Therefore, we conclude that if UPPERBOUND $-$ LOWERBOUND $\leq \varepsilon$ at the i-th iteration,

then we have $i \geq \left(\frac{DLT}{4\varepsilon}\right)^d$. □

The theorem shows that in general Algorithm 2 needs at least $\mathcal{O}(T^d)$ iterations before termination. We comment that this is due to the fact that the approximation using generalized conjugacy is tight only locally. Without convexity, one may need to visit many states to cover the state space to achieve tight approximations of the value functions before the algorithm is guaranteed to find an ε-optimal solution.

2.4.2 Convex Lipschitz Continuous Problems

In the above example for general Lipschitz continuous problem, we see that the complexity of Algorithm 2 grows at a rate of $\mathcal{O}(T^d)$. It remains to answer whether convexity could help us avoid this possibly undesirable growth rate in terms of T. We will show that even by using linear cuts, rather than generalized conjugacy cuts, for convex value functions, the complexity lower bound of the proposed algorithms could not be substantially improved. We begin our discussion on the convex case with a definition.

Definition 2.5. *Given a d-sphere $\mathcal{S}^d(R) = \{x \in \mathbb{R}^{d+1} : \|x\|_2 = R\}$ with radius $R > 0$, a spherical cap with depth $\beta > 0$ centered at a point $x \in \mathcal{S}^d(R)$ is the set*

$$\mathcal{S}^d_\beta(R, x) \coloneqq \{y \in \mathcal{S}^d(R) : \langle y - x, x\rangle \geq -\beta R\}.$$

The next lemma shows that we can put many spherical caps on a sphere, the center of each is not contained in any other spherical cap.

Lemma 2.9. *Given a d-sphere $\mathcal{S}^d(R), d \geq 2$ and depth $\beta < (1 - \frac{\sqrt{2}}{2})R$, there exists a finite set of points $\mathcal{W}$ with*

$$|\mathcal{W}| \geq \frac{(d^2-1)\sqrt{\pi}}{d} \frac{\Gamma(d/2+1)}{\Gamma(d/2+3/2)} \left(\frac{R}{2\beta}\right)^{(d-1)/2},$$

such that, for any $w \in \mathcal{W}$, $\mathcal{S}^d_\beta(R, w) \cap \mathcal{W} = \{w\}$.

Proof. Let v_d denote the d-volume for a d-dimensional unit ball. Recall that the d-volume of $\mathcal{S}^d(R)$ is given by $\mathrm{Vol}_d(\mathcal{S}^d(R)) = (d+1)v_{d+1}R^d = \dfrac{(d+1)\pi^{(d+1)/2}}{\Gamma(\frac{d+1}{2}+1)}R^d$. We next estimate the d-volume for the spherical cap $\mathcal{S}^d_\beta(R,x)$. Let $\alpha \in (0,\pi/2)$ denote the central angle for the spherical cap, i.e., $\cos\alpha = 1-\beta/R$. Since $\beta < (1-\frac{\sqrt{2}}{2})R$, we know that $\alpha < \pi/4$. Then for any $x \in \mathcal{S}^d(R)$, the d-volume of the spherical cap can be calculated through

$$\mathrm{Vol}_d(\mathcal{S}^d_\beta(R,x)) = \int_0^\alpha \mathrm{Vol}_{d-1}(\mathcal{S}^{d-1}(R\sin\theta))R\,\mathrm{d}\theta = dv_dR^d\int_0^\alpha(\sin\theta)^{d-1}\,\mathrm{d}\theta.$$

Note that when $\theta \in (0,\alpha)$, $\sin\theta > 0$ and $\cos\theta/\sin\theta > 1$. Therefore, since $d \geq 2$,

$$\mathrm{Vol}_d(\mathcal{S}^d_\beta(R,x)) \leq dv_dR^d\int_0^\alpha(\sin\theta)^{d-1}\frac{\cos\theta}{\sin\theta}\,\mathrm{d}\theta = dv_dR^d\cdot\frac{(\sin\alpha)^{d-1}}{d-1}.$$

By substituting $\sin\alpha = \sqrt{1-(1-\beta/R)^2}$, we have that

$$\begin{aligned}\frac{\mathrm{Vol}_d(\mathcal{S}^d_\beta(R,x))}{\mathrm{Vol}_d(\mathcal{S}^d(R))} &\leq \frac{d}{d^2-1}\frac{v_d}{v_{d+1}}(\sin\alpha)^{d-1},\\ &= \frac{d}{d^2-1}\frac{v_d}{v_{d+1}}\left(1-\left(1-\frac{\beta}{R}\right)^2\right)^{(d-1)/2},\\ &\leq \frac{d}{d^2-1}\frac{v_d}{v_{d+1}}\left(\frac{2\beta}{R}\right)^{(d-1)/2}.\end{aligned}$$

Now suppose $\mathcal{W} = \{w_i\}_{k=1}^K$ is a maximal set satisfying the assumption, that is, for any $w \in \mathcal{S}^d(R), w \notin \mathcal{W}$, there exists $w_k \in \mathcal{W}$ such that $w \in \mathcal{S}^d_\beta(R,w_k)$. Then, $\bigcup_{k=1}^K \mathcal{S}^d_\beta(R,w_k) \supseteq \mathcal{S}^d(R)$, therefore

$$\mathrm{Vol}_d(\mathcal{S}^d(R)) \leq \sum_{k=1}^K \mathrm{Vol}_d(\mathcal{S}^d_\beta(R,w_k)) = |\mathcal{W}|\,\mathrm{Vol}_d(\mathcal{S}^d_\beta(R,w_1)).$$

Therefore we have

$$|\mathcal{W}| \geq \frac{\mathrm{Vol}_d(\mathcal{S}^d(R))}{\mathrm{Vol}_d(\mathcal{S}^d_\beta(R, w_1))} \geq \left[\frac{d}{d^2-1}\frac{v_d}{v_{d+1}}\left(\frac{2\beta}{R}\right)^{(d-1)/2}\right]^{-1}$$
$$= \frac{(d^2-1)\sqrt{\pi}}{d}\frac{\Gamma(d/2+1)}{\Gamma(d/2+3/2)}\left(\frac{R}{2\beta}\right)^{(d-1)/2}. \qquad \square$$

Hereafter, we denote a set of points that satisfies Lemma 2.9 as $\mathcal{W}^d_\beta(R) \subset \mathcal{S}^d(R)$. Next we construct an L-Lipschitz convex function for any $L > 0$, $\varepsilon > 0$ that satisfies certain properties on $\mathcal{W}^d_{\varepsilon/L}(R)$.

Lemma 2.10. *Given positive constants $\varepsilon > 0, L > 0$ and a set $\mathcal{W}^d_{\varepsilon/L}(R)$. Let $K \coloneqq |\mathcal{W}^d_{\varepsilon/L}(R)|$. For any values $v_k \in (\varepsilon/2, \varepsilon)$, $k = 1, \ldots, K$, define a function $F : \mathcal{B}^{d+1}(R) \to \mathbb{R}$ as $F(x) = \max_{k=1,\ldots,K}\{0, v_k + \frac{L}{R}\langle w_k, x - w_k\rangle\}$. Then F satisfies the following properties:*

1. *F is an L-Lipschitz convex function;*
2. *$F(w_k) = v_k$ for all $w_k \in \mathcal{W}^d_{\varepsilon/L}(R)$;*
3. *F is differentiable at all w_k, with $v_k + \left\langle \nabla F(w_k), w_l - w_k\right\rangle < 0$ for all $l \neq k$;*
4. *For any $w_l \in \mathcal{W}^d_{\varepsilon/L}(R)$, $\underline{Q}_l(x) \coloneqq \max_{k\neq l}\{0, v_k + \left\langle \nabla F(w_k), x - w_k\right\rangle\}$ and $\overline{Q}_l(x) \coloneqq \mathrm{conv}_{k\neq l}\{v_k + L\|x - w_k\|\}$ satisfy*

$$\overline{Q}_l(w_l) - \underline{Q}_l(w_l) > \frac{3\varepsilon}{2}.$$

Proof. 1. By construction, F is a convex piecewise linear function. Since each linear piece has a Lipschitz constant $(L/R)\|w_k\| = L$, as $\|w_k\| = R$. Thus, F, as the maximum of these linear functions, is also an L-Lipschitz function.

2. Since $w_k \notin \mathcal{S}^d_{\varepsilon/L}(w_l)$ for $l \neq k$, $\langle w_l, w_k - w_l \rangle < -\varepsilon R/L$. Hence,

$$v_l + L/R \langle w_l, w_k - w_l \rangle < v_l - \varepsilon < 0.$$

Therefore, $F(w_k) = \max\{0, v_k\} = v_k$.

3. Notice that the above maximum for $F(w_k)$ is achieved at a unique linear piece, which implies that F is differentiable at w_k for all k. The gradient $\nabla F(w_k) = (L/R)w_k$. This gives the inequality in property 3 from the proof of property 2.

4. The inequality of property 3 also implies that $\underline{Q}_l(w_l) = 0$. Now we show $\overline{Q}_l(w_l) > 3\varepsilon/2$. Since $w_l \notin \mathcal{S}^d_{\varepsilon/L}(w_k)$ for any $k \neq l$, then $\|w_l - w_k\| > \varepsilon/L$ by the Pythagorean theorem. Also $v_k > \varepsilon/2$. So $q_k := v_k + L\|w_l - w_k\| > 3\varepsilon/2$. Since $\overline{Q}_l(w_l)$ is the convex combination of q_k's, we have $\overline{Q}_l(w_l) > 3\varepsilon/2$. This completes the proof. □

Now we present the multistage convex dual dynamic programming example based on the following parameters: $T \geq 2$ (number of stages), $L > 0$ (Lipschitz constant), $d \geq 3$ (state space dimension), $D = 2R > 0$ (state space diameter), and $\varepsilon > 0$ (optimality gap). Choose any $L_1, \ldots, L_T$ such that $L/2 \leq L_T < L_{T-1} < \cdots < L_1 \leq L$, and then construct finite sets $\mathcal{W}_t := \mathcal{W}^{d-1}_{\varepsilon/((T-1)L_{t+1})}(R) = \{w_{t,k}\}_{k=1}^{K_t}$, $K_t = |\mathcal{W}_t|$ as defined in Lemma 2.9 for $t = 1, \ldots, T-1$. Moreover, define convex L_{t+1}-Lipschitz continuous functions F_t for some values $v_{t,k} \in (\varepsilon/(2T-2), \varepsilon/(T-1))$, $k = 1, \ldots, K_t$, and the finite sets $\mathcal{W}_t$. By Assumption 2.3, we define the stagewise independent scenario tree as follows. There are K_t distinct nodes in each stage $t = 1, \ldots, T-1$, which can be denoted by an index pair $n = (t, k)$ for $k = 1, \ldots, K_t$, and all nodes are defined by the same data in the last stage T. Then we define our problem by specifying the nodal cost functions $f_r \equiv 0$, $f_{1,k}(x_0, y_1, x_1) := L_1\|x_1 - w_{1,k}\|$ for $k = 1, \ldots, K_1$, $f_{t,k}(x_{t-1}, y_t, x_t) := F_{t-1}(x_{t-1}) + L_t\|x_t - w_{t,k}\|$ for $k = 1, \ldots, K_t$ and $t = 2, \ldots, T-1$, and $f_{T,1}(x_{T-1}, y_T, x_T) := F_{T-1}(x_{T-1})$, and state

spaces $\mathcal{X}_t = \mathcal{X} = \mathcal{B}^{d+1}(R)$. Consequently, the value functions can be written as

$$\begin{cases} Q_r = \mathcal{Q}_r, \\ Q_{1,k} = \min\limits_{x_1 \in \mathcal{X}} \left\{ L_1 \left\| x_1 - w_{1,k} \right\| + \mathcal{Q}_1(x_1) \right\}, \ \forall k \leq K_1, \\ Q_{t,k}(x_{t-1}) = \min\limits_{x_t \in \mathcal{X}} \left\{ F_{t-1}(x_{t-1}) + L_t \left\| x_t - w_{t,k} \right\| + \mathcal{Q}_t(x_t) \right\}, \forall k \leq K_t, \\ Q_{T,1}(x_{T-1}) = F_{T-1}(x_{T-1}), \end{cases} \tag{2.41}$$

where the third equation is defined for all $2 \leq t \leq T-1$, and the expected cost-to-go functions as

$$\mathcal{Q}_t(x_t) \coloneqq \frac{1}{K_t} \sum_{k=1}^{K_t} Q_{t+1,k}(x_t), \quad t = 0, \ldots, T-1.$$

By Lemma 2.9,

$$\begin{aligned} K_t &\geq \frac{((d-1)^2-1)\sqrt{\pi}}{d-1} \frac{\Gamma((d-1)/2+1)}{\Gamma((d-1)/2+3/2)} \left(\frac{RL_t(T-1)}{2\varepsilon} \right)^{(d-2)/2}, \\ &\geq \frac{d(d-2)\sqrt{\pi}}{d-1} \frac{\Gamma((d/2+1/2)}{\Gamma(d/2+1)} \left(\frac{DL(T-1)}{8\varepsilon} \right)^{(d-2)/2}. \end{aligned}$$

Since for each value function $Q_{t,k}$ is L_t-Lipschitz continuous, we choose $\sigma_n = L_t$ with $\psi_n(x) = \|x\|$ for any $n = (t,k) \in \tilde{\mathcal{N}}(t)$ and $t = 1, \ldots, T$ such that by Lemma 2.7 we have $Q_{t,k}(x) = Q^{\mathrm{R}}_{t,k}(x)$ for all $x \in \mathcal{X}$. Moreover, due to convexity, we set $l_{n,\rho} = 0$ for all $n \in \mathcal{N}$ and $l_{n,\lambda} = L_t$ for each $n \in \tilde{\mathcal{N}}(t)$ and $t = 1, \ldots, T$, i.e., the cuts are linear. Following the argument of Proposition 2.9, we know that such linear cuts are capable of tight approximations. With such a choice of regularization we have the following theorem on the complexity of Algorithm 2.

Theorem 2.5. *For any optimality gap $\varepsilon > 0$, there exists a multistage stochastic convex problem of the form* (2.41) *such that, if Algorithm 2 gives* UPPERBOUND$-$LOWERBOUND $<$

ε at i-th iteration, then

$$i > \frac{1}{3}\frac{d(d-2)\sqrt{\pi}}{d-1}\frac{\Gamma(d/2+1/2)}{\Gamma(d/2+1)}\left(\frac{DL(T-1)}{8\varepsilon}\right)^{(d-2)/2}.$$

Proof. First we claim that in any iteration i, for any nodal problem k in stage t, the optimal solution in the forward step (Algorithm 2 line 7) must be $x_t^i = w_{t,k}$. To see this, recall that we set $l_{n,\lambda} = L_t$ and $l_{n,\rho} = 0$ for all $n \in \tilde{\mathcal{N}}(t)$, so by Proposition 2.10, the under-approximation of the cost-to-go function $\underline{\mathcal{Q}}_t^i(x)$ is L_{t+1}-Lipschitz continuous for all iteration $i \in \mathbb{N}$. So consider the forward step subproblem for node $n = (t,k)$ with $t \geq 2$ in iteration i

$$\min_{x_t \in \mathcal{X}}\{F_t(x_{t-1}^i) + L_t\left\|x_t - w_{t,k}\right\| + \underline{\mathcal{Q}}_t^i(x_t)\} = F_t(x_{t-1}^i) + \min_{x_t \in \mathcal{X}}\{L_t\left\|x_t - w_{t,k}\right\| + \underline{\mathcal{Q}}_t^i(x_t)\}. \tag{2.42}$$

Note that by the L_{t+1}-Lipschitz continuity of $\underline{\mathcal{Q}}_t^i$,

$$L_t\left\|x_t - w_{t,k}\right\| + \underline{\mathcal{Q}}_t^i(x_t) \geq \underline{\mathcal{Q}}_t^i(w_{t,k}) + (L_t - L_{t+1})\left\|x_t - w_{t,k}\right\|,$$

which, alongside with the fact that $L_{t+1} < L_t$, implies that $x_t = w_{t,k}$ is the unique optimal solution to the forward step problem (2.42). The above argument also works for any node in the stage $t = 1$ by simply removing the constant term $F_t(x_{t-1}^i)$ in the nodal problem (2.42).

Now we define over- and under-approximations of the value functions for the purpose of this proof. For node $n = (t,k)$, let

$$\underline{Q}_{t,k}^i(x) := \max_{1 \leq j \leq i}\{0, C_{t,k}^j(x \mid \hat{\lambda}_{t,k}^j, 0, \underline{v}_{t,k}^j)\},$$

and

$$\overline{Q}_{t,k}^i(x) := \mathrm{conv}_{1 \leq j \leq i}\{\bar{v}_{t,k}^j + L_t\|x_{t-1}^j - x\|\},$$

where for each j, by formula (2.23),

$$\begin{aligned}\underline{v}^j_{t,k} &:= \max_{\|\lambda\|\le L_t} \min_{z,x\in\mathcal{X}} \{F_{t-1}(z) + \langle \lambda, x^j_{t-1} - z\rangle + L_t \left\|x - w_{t,k}\right\| + \underline{\mathcal{Q}}^j_t(x)\},\\ &= \max_{\|\lambda\|\le L_t} \min_{z\in\mathcal{X}} \{F_{t-1}(z) + \langle \lambda, x^j_{t-1} - z\rangle\} + \min_{x\in\mathcal{X}} \{L_t \left\|x - w_{t,k}\right\| + \underline{\mathcal{Q}}^j_t(x)\}\\ &= F_{t-1}(x^j_{t-1}) + \underline{\mathcal{Q}}^j_t(w_{t,k})\end{aligned}$$

with $-\hat{\lambda}^j_{t,k} \in \partial F_{t-1}(x^j_{t-1})$ being an optimal solution to the outer maximization problem, and by formula (2.25)

$$\begin{aligned}\bar{v}^j_{t,k} &:= F_{t-1}(z^j_{t,k}) + L_t\|x^j_{t-1} - z^j_{t,k}\| + \overline{\mathcal{Q}}^j_t(x^j_{t,k})\\ &= F_{t-1}(x^j_{t-1}) + \overline{\mathcal{Q}}^j_t(x^j_{t,k}).\end{aligned}$$

The last equalities of $\underline{v}^j_{t,k}$ and $\bar{v}^j_{t,k}$ are due to the L_t-Lipschitz continuity of F_{t-1}. So we have by the monotonicity of the under- and over-approximations that for each $j \le i$,

$$\begin{aligned}\underline{Q}^i_{t,k}(x) &= \max_{1\le j\le i}\{0, F_{t-1}(x^j_{t-1}) + \langle \hat{\lambda}^j_{t,k}, x^j_{t-1} - x\rangle + \underline{\mathcal{Q}}^j_t(w_{t,k})\}\\ &\le \max_{1\le j\le i}\{0, F_{t-1}(x^j_{t-1}) + \langle \hat{\lambda}^j_{t,k}, x^j_{t-1} - x\rangle\} + \underline{\mathcal{Q}}^i_t(w_{t,k}),\end{aligned}$$

and

$$\begin{aligned}\overline{Q}^i_{t,k}(x) &= \mathrm{conv}_{1\le j\le i}\{F_{t-1}(x^j_{t-1}) + L_t\|x^j_{t-1} - x\| + \overline{\mathcal{Q}}^j_t(w_{t,k})\}\\ &\ge \mathrm{conv}_{1\le j\le i}\{F_{t-1}(x^j_{t-1}) + L_t\|x^j_{t-1} - x\|\} + \overline{\mathcal{Q}}^i_t(w_{t,k}).\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{Q}^i_{t,k}(x)-\underline{Q}^i_{t,k}(x) \geq \overline{\mathcal{Q}}^i_t(w_{t,k}) - \underline{\mathcal{Q}}^i_t(w_{t,k}) \\ + \operatorname{conv}_{1\leq j\leq i}\{F_{t-1}(x^j_{t-1}) + L_t\|x^j_{t-1} - x\|\} \\ - \max_{1\leq j\leq i}\{0, F_{t-1}(x^j_{t-1}) + \langle \hat{\lambda}^j_{t,k}, x^j_{t-1} - x\rangle\}.\end{aligned} \tag{2.43}$$

Thus we have $\overline{Q}^i_{t,k}(w_{t-1,k'})-\underline{Q}^i_{t,k}(w_{t-1,k'}) \geq \overline{\mathcal{Q}}^i_t(w_{t,k})-\underline{\mathcal{Q}}^i_t(w_{t,k})$ for any $k' = 1, \ldots, K_{t-1}$, since the last two terms on the right hand side of (2.43) are over- and under-approximations of the function F_{t-1}, respectively. Moreover, note that $x^j_{t-1} = w_{t-1,k'}$ for some $k' = 1, \ldots, K_{t-1}$ as it is the unique solution in the forward step. By Lemma 2.10, whenever the node $n' = (t-1, k')$ is never sampled up to iteration i, we further have

$$\overline{Q}^i_{t,k}(w_{t-1,k'}) - \underline{Q}^i_{t,k}(w_{t-1,k'}) > \frac{3\varepsilon}{2(T-1)} + \overline{\mathcal{Q}}^i_t(w_{t,k}) - \underline{\mathcal{Q}}^i_t(w_{t,k}).$$

Recall the definitions (2.21) and (2.24), for any $x \in \mathcal{X}$,

$$\underline{\mathcal{Q}}^i_{t-1}(x) = \max_{1\leq j\leq i}\left\{0, \frac{1}{K_t}\sum_{k=1}^{K_t} C^j_{t,k}(x \mid \hat{\lambda}^j_{t,k}, 0, \underline{v}^j_{t,k})\right\} \leq \frac{1}{K_t}\sum_{k=1}^{K_t}\underline{Q}^i_{t,k}(x),$$

and

$$\overline{\mathcal{Q}}^i_{t-1}(x) = \operatorname{conv}_{1\leq j\leq i}\left\{\frac{1}{K_t}\sum_{k=1}^{K_t}(\bar{v}^j_{t,k} + L_t\|x^j_{t-1} - x\|)\right\} \geq \frac{1}{K_t}\sum_{k=1}^{K_t}\overline{Q}^i_{t,k}(x).$$

Consequently, for any $k' = 1, \ldots, K_{t-1}$,

$$\overline{\mathcal{Q}}^i_{t-1}(w_{t-1,k'}) - \underline{\mathcal{Q}}^i_{t-1}(w_{t-1,k'}) \geq \frac{1}{K_t}\sum_{k=1}^{K_t}[\overline{\mathcal{Q}}^i_t(w_{t,k}) - \underline{\mathcal{Q}}^i_t(w_{t,k})],$$

and in addition, for any node $n' = (t-1, k')$ not sampled up to iteration i,

$$\overline{\mathcal{Q}}^i_{t-1}(w_{t-1,k'}) - \underline{\mathcal{Q}}^i_{t-1}(w_{t-1,k'}) > \frac{3\varepsilon}{2(T-1)} + \frac{1}{K_t}\sum_{k=1}^{K_t}[\overline{\mathcal{Q}}^i_t(w_{t,k}) - \underline{\mathcal{Q}}^i_t(w_{t,k})].$$

Therefore, for any iteration index $i \leq \frac{1}{3}|\mathcal{W}_t|$, $t = 1, \ldots, T-1$, then there are $K_t - i \geq \frac{2}{3}|\mathcal{W}_t|$ nodes not sampled in stage t, which implies

$$\frac{1}{K_{t-1}}\sum_{k'=1}^{K_{t-1}}[\overline{\mathcal{Q}}^i_{t-1}(w_{t-1,k'}) - \underline{\mathcal{Q}}^i_{t-1}(w_{t-1,k'})] > \frac{\varepsilon}{T-1} + \frac{1}{K_t}\sum_{k=1}^{K_t}[\overline{\mathcal{Q}}^i_t(w_{t,k}) - \underline{\mathcal{Q}}^i_t(w_{t,k})].$$

Consequently, $\overline{\mathcal{Q}}^i_r - \underline{\mathcal{Q}}^i_r > (T-1)\cdot\dfrac{\varepsilon}{T-1} = \varepsilon$. Therefore, if UPPERBOUND$-$LOWERBOUND $= \overline{\mathcal{Q}}^i_r - \underline{\mathcal{Q}}^i_r \leq \varepsilon$ in the iteration i, then

$$i > \frac{1}{3}\min_{t=1,\ldots,T-1}|\mathcal{W}_t| \geq \frac{1}{3}\frac{d(d-2)\sqrt{\pi}}{d-1}\frac{\Gamma(d/2+1/2)}{\Gamma(d/2+1)}\left(\frac{DL(T-1)}{8\varepsilon}\right)^{(d-2)/2}.$$

This completes the proof. □

The theorem implies that, even if problem (2.2) is convex and has Lipschitz continuous value functions, the minimum iteration for Algorithm 2 to get a guaranteed ε-optimal root node solution grows as a polynomial of the ratio T/ε, with the degree being $d/2 - 1$.

We remark that Theorems 2.4 and 2.5 correspond to two different challenges of the SDDP type algorithms. The first challenge is that the backward step subproblem oracle may not give cuts that provide the desired approximation in the largest neighborhood, which could happen when the value functions are nonconvex or nonsmooth. Theorem 2.4 results from the worst case that the backward step subproblem oracle is giving the dual variables that approximate the value function in the smallest neighborhood.

The second challenge is that different nodes, or more generally, different scenario paths give different states in each stage, so sampling and solving the nodal problem on one scenario path provides little information to the nodal problem on another scenario path. In

example (2.41), the linear cut obtained in each iteration does not provide any information on the subsequent iteration states (unless the same node is sampled again). From this perspective, we believe that unless some special structure of the problem is exploited, any algorithm that relies on local approximation of value functions will face the "curse of dimensionality," i.e., the growth rate of the iteration complexity is exponential in the dimension of the state spaces.

CHAPTER 3

DDP ALGORITHMS FOR CONVEX MDRO WITH COMPLEXITY ANALYSIS

3.1 Formulations and Recursive Approximation

In this section, we introduce formulations of convex multistage distributionally robust optimization (MDRO). In the case where the MDRO has finite support, we build approximations of the value functions using recursions. We then discuss the regularization technique and its exactness that is used for complexity analysis in Section 3.2.

3.1.1 Problem Formulations

We review the definitions of MSCO and MRCO, before we proceed to the general definition of MDRO. Then we show that MDRO with finite support is a sufficiently general framework to encompass most MSCO and MRCO in the existing literature ([64] and [39]).

Multistage Stochastic Convex Optimization

We first briefly review the definition of multistage stochastic convex optimization under the assumption of stagewise independence (SI). The SI assumption is necessary for efficient algorithmic development and we refer any interested reader to [12] for more general settings.

Let $\mathcal{T} \coloneqq \{1, 2, \ldots, T\}$ denote the set of stage indices. For each $t \in \mathcal{T}$, the decision variable in stage t is denoted as x_t and constrained in a compact convex set $\mathcal{X}_t \subset \mathbb{R}^{d_t}$ with dimension $d_t \in \mathbb{Z}_{\geq 0}$. The uncertainty in stage t is modeled as a random vector ξ_t with its support set denoted as Ξ_t. In particular, we use x_0 and ξ_1 to denote deterministic parameters, known as the initial condition. The cost incurred by the decisions and uncertainty in stage t is modeled by a nonnegative, lower semicontinuous, proper convex

function $f_t(x_{t-1}, x_t; \xi_t)$, that is allowed to take $+\infty$ for infeasibility. For example, given a real-valued cost function $f_t^{<\infty}(x_{t-1}, x_t; \xi_t)$, to model any (continuous) functional feasibility constraints $g_t(x_{t-1}, y_t, x_t; \xi_t) \leq 0$ with an auxiliary decision vector y_t in stage t, we can define $\mathcal{F}_t := \{(x_{t-1}, x_t, \xi_t) \in \mathcal{X}_{t-1} \times \mathcal{X}_t \times \Xi_t : \exists\, y_t \text{ s.t. } g_t(x_{t-1}, y_t, x_t, \xi_t) \leq 0\}$ and set the extended real-valued cost function as

$$f_t(x_{t-1}, x_t; \xi_t) = \begin{cases} f_t^{<\infty}(x_{t-1}, x_t; \xi_t), & \text{if } (x_{t-1}, x_t, \xi_t) \in \mathcal{F}_t, \\ +\infty, & \text{otherwise.} \end{cases}$$

The SI assumption says that the random variables $\xi_1, \ldots, \xi_T$ are mutually independent, and allows the following nested formulation for the MSCO:

$$\begin{aligned} \min_{x_1 \in \mathcal{X}_1} \quad f_1(x_0, x_1; \xi_1) + \mathbb{E}_{\xi_2} \min_{x_2 \in \mathcal{X}_2} \Big[& f_2(x_1, x_2; \xi_2) + \\ & + \mathbb{E}_{\xi_3} \min_{x_3 \in \mathcal{X}_3} \Big[f_3(x_2, x_3; \xi_3) + \cdots \\ & + \mathbb{E}_{\xi_T} \min_{x_T \in \mathcal{X}_T} f_T(x_{T-1}, x_T; \xi_T) \Big] \Big]. \end{aligned} \tag{3.1}$$

Here, we use the notation $\mathbb{E}_{\xi_t}$ to represent the expectation (with the subscript ξ_t emphasizing that the function is ξ_t-measurable) for each $t \in \mathcal{T}$. Alternatively, we can rewrite the MSCO (3.1) as a recursion using the (expected) cost-to-go function defined as follows:

$$\mathcal{Q}_{t-1}(x_{t-1}) := \mathbb{E}_{\xi_t} \left(\min_{x_t \in \mathcal{X}_t} f_t(x_{t-1}, x_t; \xi_t) + \mathcal{Q}_t(x_t) \right), \tag{3.2}$$

where by convention $\mathcal{Q}_T(x_T) \equiv 0$ for any $x_T \in \mathcal{X}_T$. Then the optimal value and optimal first stage decision can be obtained by solving the deterministic problem

$$\min_{x_1 \in \mathcal{X}_1} f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1(x_1). \tag{3.3}$$

If the random vector ξ_t has a finite support set, say $\Xi_t = \{\hat{\xi}_{t,1}, \ldots, \hat{\xi}_{t,N_t}\}$, each with the

realization probability $p_n > 0$ for $n \in \mathcal{N}(t) := \{1, \ldots, N_t\}$, then we can define the cost function associated with the n-th realization in stage t as $f_n(x_{t-1}, x_t) := f_t(x_{t-1}, x_t; \hat{\xi}_{t,n})$. Moreover, the disjoint union of these index sets $\mathcal{N} := \sqcup_{t\in\mathcal{T}}\mathcal{N}(t)$ can be viewed as the node set of a recombining scenario tree (see the definition and discussion in Section 2.2.4 or [46]).

In this case, the MSCO can be simplified as

$$\begin{aligned}\min_{x_1\in\mathcal{X}_1} \quad f_1(x_0, x_1; \xi_1) + \sum_{n_2\in\mathcal{N}(2)} p_{n_2} \cdot \min_{x_2\in\mathcal{X}_2} \Bigg[f_{n_2}(x_1, x_2) + \\ + \sum_{n_3\in\mathcal{N}(3)} p_{n_3} \cdot \min_{x_3\in\mathcal{X}_3} \Bigg[f_{n_3}(x_2, x_3) + \cdots \\ + \sum_{n_T\in\mathcal{N}(T)} p_{n_T} \cdot \min_{x_T\in\mathcal{X}_T} f_{n_T}(x_{T-1}, x_T) \Bigg] \Bigg].\end{aligned} \tag{3.4}$$

Further, the expected cost-to-go function becomes a finite sum of minimization value functions

$$\mathcal{Q}_{t-1}(x_{t-1}) = \sum_{n\in\mathcal{N}(t)} p_n Q_n(x_{t-1}), \text{ with } Q_n(x_{t-1}) := \min_{x_t\in\mathcal{X}_t} f_n(x_{t-1}, x_t) + \mathcal{Q}_t(x_t), \ n \in \mathcal{N}(t), \tag{3.5}$$

for any stage $t \in \mathcal{T} \setminus \{1\}$. We remark that although the problem (3.4) is only a special case of (3.1), it is actually widely used in sample average approximation methods for solving (3.1) and can be solved by the well-studied stochastic dual dynamic programming (SDDP) algorithm [40].

Multistage Robust Convex Optimization

Another important source of our formulation is the multistage robust convex optimization (MRCO) problems with stagewise independent (a.k.a., rectangular) uncertainty sets. With the same notation of decision variables $x_t \in \mathcal{X}_t$ and uncertainty vectors $\xi_t \in \Xi_t$ (which is

not necessarily a random vector in this case), such MRCO can be written as

$$\min_{x_1 \in \mathcal{X}_1} \quad f_1(x_0, x_1; \xi_1) + \sup_{\xi_2 \in \Xi_2} \min_{x_1 \in \mathcal{X}_2} \Bigg[f_2(x_1, x_2; \xi_2) + \sup_{\xi_3 \in \Xi_3} \min_{x_3 \in \mathcal{X}_3} \Bigg[f_3(x_2, x_3; \xi_3) + \cdots + \sup_{\xi_T \in \Xi_T} \min_{x_T \in \mathcal{X}_T} f_T(x_{T-1}, x_T; \xi_T) \Bigg] \Bigg]. \tag{3.6}$$

Analogous to (3.2), we can define the (worst-case) cost-to-go functions in a recursive fashion.

$$\mathcal{Q}_{t-1}(x_{t-1}) \coloneqq \sup_{\xi_t \in \Xi_t} \min_{x_t \in \mathcal{X}_t} f_t(x_{t-1}, x_t; \xi_t) + \mathcal{Q}_t(x_t), \tag{3.7}$$

where $\mathcal{Q}_T(x_T) \equiv 0$ for any $x_T \in \mathcal{X}_T$. Similarly, the goal of the MRCO is to find a first stage optimal solution in the sense of (3.3). We remark that the recursion (3.7) is in general very challenging to solve because it involves finding the supremum over a usually nonconcave function in the uncertainty vectors ξ_t. However, when the cost function f_t is polyhedrally representable and both the uncertainty set Ξ_t and the state space $\mathcal{X}_t$ are polytopes, the MRCO (3.6) has witnessed successful algorithmic applications in [39].

Distributionally Robust Multistage Stochastic Convex Optimization

While MSCO (3.1) and MRCO (3.6) appear different, they can be unified in the framework of MDRO using the same set of notation. Let $\mathcal{P}_t$ denote a set of Borel probability measures on the uncertainty set Ξ_t, which is known as the ambiguity set, for each stage $t \in \mathcal{T} \setminus \{1\}$.

Now we can define the MDRO as follows.

$$\min_{x_1\in\mathcal{X}_1} \quad f_1(x_0,x_1;\xi_1) + \sup_{p_2\in\mathcal{P}_2} \mathbb{E}_{\xi_2\sim p_2} \min_{x_2\in\mathcal{X}_2} \Bigg[f_2(x_1,x_2;\xi_2) + \\ + \sup_{p_3\in\mathcal{P}_3} \mathbb{E}_{\xi_3\sim p_3} \min_{x_3\in\mathcal{X}_3} \Bigg[f_3(x_2,x_3;\xi_3) + \cdots \\ + \sup_{p_T\in\mathcal{P}_T} \mathbb{E}_{\xi_T\sim p_T} \min_{x_T\in\mathcal{X}_T} f_T(x_{T-1},x_T;\xi_T) \Bigg] \Bigg]. \tag{3.8}$$

Here, each expectation $\mathbb{E}_{\xi_t\sim p_t}$ is taken with respect to the given probability measure $p_t \in \mathcal{P}_t$. The (worst-case expected) cost-to-go functions are defined recursively from $t = T$ to $t = 2$ as:

$$\mathcal{Q}_{t-1}(x_{t-1}) \coloneqq \sup_{p_t\in\mathcal{P}_t} \mathbb{E}_{\xi_t\sim p_t} \min_{x_t\in\mathcal{X}_t} f_t(x_{t-1},x_t;\xi_t) + \mathcal{Q}_t(x_t), \tag{3.9}$$

with $\mathcal{Q}_T(x_T) \equiv 0$ for any $x_T \in \mathcal{X}_T$. Similar to MSCO (3.1) and MRCO (3.6), the goal is to find an optimal first stage solution in the sense of (3.3). It is well-known that if the ambiguity set $\mathcal{P}_t$ consists of only one probability measure, then the supremum is redundant and hence MDRO (3.8) reduces to MSCO (3.1); if the ambiguity set $\mathcal{P}_t$ contains all atomic probability measures (i.e., measures $\delta_{\hat{\xi}}$ for all $\hat{\xi} \in \Xi_t$ such that $\mathbb{E}_{\xi_t\sim\delta_{\hat{\xi}}} g(\xi) = g(\hat{\xi})$ for any Borel measurable function $g : \Xi_t \to \mathbb{R}\cup\{+\infty\}$), then MDRO (3.8) reduces to MRCO (3.6). The following proposition checks that minimizations in MDRO (3.8) (and thus also in MSCO and MRCO) are convex and well-defined.

Proposition 3.1. *Suppose the state spaces $\mathcal{X}_t$ are compact convex and the nonnegative cost functions f_t are lower semicontinuous (lsc) and convex for all $t \in \mathcal{T}$. Then the cost-to-go functions $\mathcal{Q}_t$ are lsc and convex for all $t \in \mathcal{T}$.*

Proof. We prove by recursion from $t = T$ to $t = 1$. By definition, $\mathcal{Q}_T \equiv 0$ is lsc and convex. Now assume $\mathcal{Q}_t$ is lsc and convex for some $t \in \mathcal{T}$. Then the sum $f_t + \mathcal{Q}_t$ is also lsc and convex. Since $\mathcal{X}_t$ is compact, we have $Q_t(x_{t-1};\xi_t) := \min_{x_t\in\mathcal{X}_t} f_t(x_{t-1},x_t;\xi_t)+\mathcal{Q}_t(x_t)$ is lsc (see e.g., Lemma 1.30 in [74]) and convex. Now fix any Borel probability measure $p_t \in \mathcal{P}_t$ and take any sequence $\{x^i\} \subset \mathcal{X}_{t-1}$ with $\lim_{i\to\infty} x^i = x_{t-1}$. Note that Q_t is

nonnegative by definition, so by Fatou's lemma (see e.g., Lemma 1.28 in [75]) we have

$$\liminf_{i\to\infty} \mathbb{E}_{\xi_t\sim p_t} Q_t(x^i;\xi_t) \geq \int_{\Xi_t} \liminf_{i\to\infty} Q_t(x^i;\xi_t)\,\mathrm{d}p_t(\xi_t) \geq \int_{\Xi_t} Q_t(x_{t-1};\xi_t)\,\mathrm{d}p_t(\xi_t).$$

The expectation and the integrals are well-defined since Q_t is lsc, hence Borel measurable. This inequality shows that the function $\mathbb{E}_{\xi_t\sim p_t} Q_t(x_{t-1};\xi_t)$ is lsc. It is also convex by the linearity and monotonicity of expectations. Finally, the epigraph of $\mathcal{Q}_{t-1}$ is the intersection of epigraphs of $\mathbb{E}_{\xi_t\sim p_t} Q_t(x_{t-1};\xi_t)$ for all $p_t \in \mathcal{P}_t$, which shows that $\mathcal{Q}_{t-1}$ is lsc and convex.

□

If the uncertainty set $\Xi_t = \{\hat{\xi}_{t,1},\ldots,\hat{\xi}_{t,N_t}\}$ is finite, then the ambiguity set is a subset of a Euclidean space $\mathcal{P}_t \subseteq \Delta^{N_t} := \{p_t \in \mathbb{R}^{N_t}_{\geq 0} : \sum_{n=1}^{N_t} p_{t,n} = 1\}$. With $f_n(x_{t-1},x_t) := f_t(x_{t-1},x_t;\hat{\xi}_{t,n})$ for any $n \in \mathcal{N}(t) := \{1,\ldots,N_t\}$ and $t \in \mathcal{T}$, the MDRO (3.8) can be written as

$$\begin{aligned}\min_{x_1\in\mathcal{X}_1} \quad f_1(x_0,y_1,x_1) + \sup_{p_2\in\mathcal{P}_2} \sum_{n_2\in\mathcal{N}(2)} p_{2,n_2} \cdot \min_{x_2\in\mathcal{X}_2} \Bigg[f_{n_2}(x_1,x_2) + \\ + \sup_{p_3\in\mathcal{P}_3} \sum_{n_3\in\mathcal{N}(3)} p_{3,n_3} \cdot \min_{x_3\in\mathcal{X}_3} \Big[f_{n_3}(x_2,x_3) + \cdots \\ + \sup_{p_T\in\mathcal{P}_T} \sum_{n_T\in\mathcal{N}(T)} p_{T,n_T} \cdot \min_{x_T\in\mathcal{X}_T} f_{n_T}(x_{T-1},x_{n_T}) \Big] \Bigg],\end{aligned} \tag{3.10}$$

and the worst-case expected cost-to-go functions recursively as

$$\mathcal{Q}_{t-1}(x_{t-1}) \coloneqq \sup_{p_t\in\mathcal{P}_t} \sum_{n\in\mathcal{N}(t)} p_{t,n} Q_n(x_{t-1}), \tag{3.11}$$

with $\mathcal{Q}_T(x_T) \equiv 0$ and the value functions

$$Q_n(x_{t-1}) := \min_{x_t\in\mathcal{X}_t} f_n(x_{t-1},x_t) + \mathcal{Q}_t(x_t), \quad \text{for} \quad n \in \mathcal{N}(t). \tag{3.12}$$

While it is clear that the finitely supported MDRO (3.10) includes the finitely supported MSCO (3.4) as a special case, we show by the following proposition that it encompasses an important class of MRCO as well.

Proposition 3.2. *Let $\mathcal{Q}_t$ denote the worst-case cost-to-go functions defined in* (3.7)*, where we assume that the uncertainty sets Ξ_t are polytopes for all $t \in \mathcal{T}$. Then let $\operatorname{ext}\Xi_t := \{\hat{\xi}_{t,1}, \ldots, \hat{\xi}_{t,N_t}\}$ denote the finite set of extreme points of Ξ_{t+1} and $\mathcal{N}(t) := \{1, \ldots, N_t\}$ the index set. We have*

$$\mathcal{Q}_{t-1}(x_{t-1}) = \max_{p_t \in \Delta^{N_t}} \sum_{n \in \mathcal{N}(t)} p_{t,n} \left(\min_{x_t \in \mathcal{X}_t} f_t(x_{t-1}, x_t; \hat{\xi}_{t,n}) + \mathcal{Q}_t(x_t) \right).$$

Proof. By Proposition 3.1, each worst-case cost-to-go function $\mathcal{Q}_t$ is convex for any $t \in \mathcal{T}$. The minimization value function $Q_t(x_{t-1}; \xi_t) := \min_{x_t \in \mathcal{X}_t} \{f_t(x_{t-1}, x_t; \xi_t) + \mathcal{Q}_t(x_t)\}$ is thus also convex. Consequently, we have

$$\mathcal{Q}_{t-1}(x_{t-1}) = \sup_{\xi_t \in \Xi_t} Q_t(x_{t-1}; \xi_t) = \sup_{\xi_t \in \operatorname{ext}\Xi_t} Q_t(x_{t-1}, \xi_t) = \max_{p_t \in \Delta^{N_t}} \sum_{n \in \mathcal{N}(t)} p_{t,n} Q_t(x_{t-1}; \hat{\xi}_{t,n}),$$

due to the linearity of the right-most maximization. □

Motivated by Proposition 3.2, we decide to make the following assumption to restrict our attention in this chapter mainly to a simple yet useful special class of MDRO.

Assumption 3.1. *For each stage $t \in \mathcal{T} \setminus \{1\}$, the uncertainty set $\Xi_t = \{\hat{\xi}_{t,1}, \ldots, \hat{\xi}_{t,N_t}\}$ is finite. Moreover, the ambiguity set $\mathcal{P}_t \subseteq \Delta^{N_t}$ in* (3.10) *is assumed to be closed and convex.*

We remark that the assumption on the closedness and convexity of the ambiguity sets can be made without loss of generality when the MDRO (3.10) has complete recourse, i.e., $Q_n(x_{t-1}) < \infty$ for any $x_{t-1} \in \mathcal{X}_{t-1}$ for any $n \in \mathcal{N}(t)$ in (3.11). This can be seen from the linearity of the maximization in the probability vector p_t, whence we know the supremum over $\mathcal{P}_t$ equals the maximum over the closed convex hull of $\mathcal{P}_t$.

3.1.2 Approximation of Recursions

We now discuss the approximation of functions $\mathcal{Q}_t$ and Q_n in the recursion equations (3.11) and (3.12). Recall the relation $\mathcal{Q}_t(x_t) = \max_{p_{t+1}\in\mathcal{P}_{t+1}} \sum_{n\in\mathcal{N}(t+1)} p_{t+1,n} Q_n(x_t)$. The following lemma relates the Lipschitz continuity of the value functions Q_n for $n \in \mathcal{N}(t+1)$ and that of the cost-to-go function $\mathcal{Q}_t$.

Lemma 3.1. *For each stage $t < T$, if Q_n is l_n-Lipschitz continuous on $\mathcal{X}_t$ for each $n \in \mathcal{N}(t+1)$, then $\mathcal{Q}_t$ is L_t-Lipschitz continuous on $\mathcal{X}_t$ where $L_t := \max_{n\in\mathcal{N}(t+1)} l_n$.*

Proof. Take any two points $x^i \in \mathcal{X}_t$ for $i = 1, 2$. Let

$$p^i_{t+1} \in \operatorname*{arg\,max}_{p_{t+1}\in\mathcal{P}_{t+1}} \sum_{n\in\mathcal{N}(t+1)} p_{t+1,n} Q_n(x^i)$$

denote corresponding maximizers for $i = 1, 2$. We have $\mathcal{Q}_t(x^i) = \sum_{n\in\mathcal{N}(t+1)} p^i_{t+1,n} Q_n(x^i)$, where $p^i_{t+1,n} \geq 0$ and $\sum_{n\in\mathcal{N}(t+1)} p^i_{t+1,n} = 1$ for each $i = 1, 2$. Therefore,

$$\begin{aligned}
\mathcal{Q}_t(x^1) - \mathcal{Q}_t(x^2) &= \sum_{n\in\mathcal{N}(t)} p^1_{t+1,n} Q_n(x^1) - \sum_{n\in\mathcal{N}(t+1)} p^2_{t+1,n} Q_n(x^2) \\
&\leq \sum_{n\in\mathcal{N}(t+1)} p^1_{t+1,n} \left(Q_n(x^1) - Q_n(x^2)\right) \\
&\leq \sum_{n\in\mathcal{N}(t+1)} p^1_{t+1,n} \cdot l_n \left\|x^1 - x^2\right\| \\
&\leq L_t \left\|x^1 - x^2\right\|,
\end{aligned}$$

where $L_t := \max_{n\in\mathcal{N}(t+1)} l_n$. By exchanging the indices $i = 1, 2$, we can similarly derive $\mathcal{Q}_t(x^2) - \mathcal{Q}_t(x^1) \leq \sum_{n\in\mathcal{N}(t+1)} p^2_{t+1,n} \cdot l_n \left\|x^1 - x^2\right\| \leq L_t \left\|x^1 - x^2\right\|$, which completes the proof. □

Combining Lemma 3.1 and Proposition 3.1, we know that if the value functions are convex and Lipschitz continuous, then so are the cost-to-go functions. In such a case, we

can use cutting plane method to build an under-approximation of the cost-to-go functions. To be precise, for each node $n \in \mathcal{N}(t+1)$, let $V_n(x_t)$ denote an affine function such that $Q_n(x_t) \geq V_n(x_t)$ for all $x_t \in \mathcal{X}_t$. Such affine function is referred to as a linear valid inequality or a linear cut for the value function, which is generated in the following way. Let $\underline{\mathcal{Q}}_{t+1}$ denote an under-approximation of the cost-to-go function $\mathcal{Q}_{t+1}$ and $\hat{x}_t \in \mathcal{X}_t$ a feasible state. For each $n \in \mathcal{N}(t+1)$, we can introduce an auxiliary variable $z_n \in \mathbb{R}^{d_t}$ with the complicating constraint $z_n = \hat{x}_t$. Then the Lagrangian dual problem

$$\sup_{\lambda_n \in \mathbb{R}^{d_t}} \min_{x_n \in \mathcal{X}_{t+1}, z_n \in \mathbb{R}^{d_t}} \quad f_n(z_n, x_n) + \underline{\mathcal{Q}}_{t+1}(x_n) + \langle \lambda_n, \hat{x}_t - z_n \rangle \tag{3.13}$$

gives an affine function $V_n(x_t) \coloneqq \underline{v}_n + \langle \hat{\lambda}_n, x_t - \hat{x}_t \rangle$, where $\hat{\lambda}_n$ is a dual solution of (3.13) and $\underline{v}_n \coloneqq \min\{f_n(z_n, x_n) + \underline{\mathcal{Q}}_{t+1}(x_n) + \langle \hat{\lambda}_n, \hat{x}_t - z_n \rangle : x_n \in \mathcal{X}_{t+1}, z_n \in \mathbb{R}^{d_t}\}$ is the associated value to the problem (3.13). Then, by definition (3.12) and weak duality, we have for every $x_t \in \mathcal{X}_t$,

$$\begin{aligned}
Q_n(x_t) = & \sup_{\lambda_n \in \mathbb{R}^{d_t}} \min_{x_n, z_n} \left\{ f_n(z_n, x_n) + \mathcal{Q}_{t+1}(x_n) + \langle \lambda_n, x_t - z_n \rangle : x_n \in \mathcal{X}_{t+1}, z_n \in \mathbb{R}^{d_t} \right\} \\
= & \langle \lambda_n, x_t - \hat{x}_t \rangle \\
& + \sup_{\lambda_n \in \mathbb{R}^{d_t}} \min_{x_n, z_n} \left\{ f_n(z_n, x_n) + \underline{\mathcal{Q}}_{t+1}(x_n) + \langle \lambda_n, \hat{x}_t - z_n \rangle : x_n \in \mathcal{X}_{t+1}, z_n \in \mathbb{R}^{d_t} \right\} \\
\geq & \langle \hat{\lambda}_n, x_t - \hat{x}_t \rangle \\
& + \min_{x_n, z_n} \left\{ f_n(z_n, x_n) + \underline{\mathcal{Q}}_{t+1}(x_n) + \langle \hat{\lambda}_n, x_t - z_n \rangle : x_n \in \mathcal{X}_{t+1}, z_n \in \mathbb{R}^{d_t} \right\} \\
\geq & \langle \hat{\lambda}_n, x_t - \hat{x}_t \rangle + \underline{v}_n.
\end{aligned} \tag{3.14}$$

Therefore, $V_n(x_t)$ is a valid inequality for the value function $Q_n(x_t)$. The next proposition shows that we can combine linear cuts for value functions into a valid inequality for the cost-to-go function.

Proposition 3.3. *Fix a point $x_t^0 \in \mathcal{X}_t$ and a probability vector $p_t^0 \in \mathcal{P}_t$. If for each*

$n \in \mathcal{N}(t+1)$, $V_n(x_t) = \underline{v}_n + \langle \hat{\lambda}_n, x_t - \hat{x}_t \rangle$ is an l'_n-Lipschitz continuous valid inequality, i.e. $\|\hat{\lambda}_n\| \leq l'_n$, then $\mathcal{Q}_t(x_t) \geq \mathcal{V}_t(x_t) \coloneqq \sum_{n \in \mathcal{N}(t+1)} p^0_{t+1,n} V_n(x_t)$ for all $x_t \in \mathcal{X}_t$. Moreover, the valid inequality $\mathcal{V}_t$ is $L'_t \coloneqq (\sum_{n \in \mathcal{N}(t+1)} p^0_{t+1,n} l'_n)$-Lipschitz continuous such that for any $\delta > \gamma \coloneqq \mathcal{Q}_t(x^0_t) - \mathcal{V}_t(x^0_t)$, we have $\mathcal{Q}_t(x_t) \leq \mathcal{V}_t(x_t) + \delta$, for any point $x_t \in \mathcal{X}_t$ with $\left\|x_t - x^0_t\right\| \leq (\delta - \gamma)/(L_t + L'_t)$.

Proof. The first claim follows from the definition

$$\begin{aligned}
\mathcal{Q}_t(x_t) &= \max_{p_{t+1} \in \mathcal{P}_{t+1}} \sum_{n \in \mathcal{N}(t+1)} p_{t,n} Q_n(x_t) \\
&\geq \sum_{n \in \mathcal{N}(t+1)} p^0_{t+1,n} Q_n(x_t) \\
&\geq \sum_{n \in \mathcal{N}(t+1)} p^0_{t+1,n} V_n(x_t) = \mathcal{V}_t(x_t).
\end{aligned}$$

The second claim follows from the Lipschitz continuity of the valid inequality and the cost-to-go function, i.e., for any point $x_t \in \mathcal{X}_t$ with $\left\|x_t - x^0_t\right\| \leq (\delta - \gamma)/(L_t + L'_t)$,

$$\mathcal{Q}_t(x_t) \leq \mathcal{Q}_t(x^0_t) + L_t \|x_t - x^0_t\| \leq \mathcal{V}_t(x_t) + (L_t + L'_t)\|x_t - x^0_t\| + \gamma \leq \mathcal{V}_t(x_t) + \delta,$$

where the first inequality is due to $\mathcal{Q}_t$ being L_t-Lipschitz continuous by Lemma 3.1. □

The proposition suggests that the combined linear cut is close to the cost-to-go function in a neighborhood, given that the gap $\mathcal{Q}_t(x^0_t) - \mathcal{V}_t(x^0_t)$ is small. However, the radius of such neighborhood depends on the Lipschitz constants of the linear cuts V_n, which are not necessarily bounded by the Lipschitz constants l_n of value functions Q_n, as is shown by the following example.

Example 3.1. *Consider a T-stage deterministic problem (i.e., $N_t = 1$ for all $t \in \mathcal{T}$)*

defined as

$$Q_t(x_{t-1}) \coloneqq \min_{y_t, x_t} \quad y_t + Q_{t+1}(x_t)$$
$$\text{s.t.} \quad y_t \geq \max\{0, 1 - 2x_{t-1}\},\ x_t \leq x_{t-1} + \frac{1}{2},\ 0 \leq x_t \leq 1.$$

Here, the convention $Q_{T+1}(x_t) \equiv 0$ is used to simplify the definition. Note that for each stage $t \in \mathcal{T}$, since $x_{t-1} \in [0,1]$, we have a feasible solution $x_t = 1/2$, which implies that $Q_t(x_{t-1}) \leq \min\{y_t : y_t \geq \max\{0, 1 - 2x_{t-1}\}$. By taking $y_t = \max\{0, 1 - 2x_{t-1}\}$ and using the fact that $Q_{t+1}(x_t) \geq 0$ recursively, we conclude that the cost-to-go functions are $Q_{t+1}(x_t) = \max\{0, 1 - 2x_t\}$ for all $t \in \mathcal{T}$. However, if we start our approximation with points $x_t^0 = 0$ for all stages $t \in \mathcal{T}$, then the linear cut $V_t(x_t) \coloneqq v_t + \langle \lambda_t, x_t - x_t^0 \rangle$ can be generated from the following dual problem at stage t:

$$v_t \coloneqq \max_{\lambda_t} \min_{z_t, y_t, x_t} \quad y_t + \lambda_t(0 - z_t) + \underline{Q}_{t+1}^0(x_t)$$
$$\text{s.t.} \quad y_t \geq \max\{0, 1 - 2z_t\},\ x_t \leq z_t + \frac{1}{2},\ 0 \leq x_t \leq 1.$$

Thus the under-approximation of the value functions of stage T will be $\underline{Q}_T^0(x_{T-1}) = 1 - 2x_{T-1}$. Plugging in the under-approximation and the dual problem becomes

$$v_t = \max_{\lambda_t} \min_{z_t, y_t, x_t} \quad y_t + \lambda_t(0 - z_t) + 1 - c_t x_t$$
$$\text{s.t.} \quad y_t \geq \max\{0, 1 - 2z_t\},\ x_t \leq z_t + \frac{1}{2},\ 0 \leq x_t \leq 1,$$

where $c_T = 2$. By solving the dual problem recursively, we conclude that $c_t = c_{t+1} + 2$ and hence $c_t = 2(T - t + 1)$. In other words, the under-approximation obtained after the first iteration has a Lipschitz constant greater than the actual one.

3.1.3 Regularization and Its Exactness

The potential gap between the actual Lipschitz constants of the value functions Q_n and the generated linear cuts V_n (Example 3.1) not only affects the complexity analysis, but may also negatively impact algorithmic performance. For this reason, we consider the regularization of value functions, in which an infimal convolution is conducted to bound the Lipschitz constant of the generated linear cuts. We begin with the recursive definition of regularized value functions and cost-to-go functions. Let $M_t > 0$ denote a regularization factor for each $t \in \mathcal{T} \setminus \{T\}$. In general, the regularized cost-to-go function is defined recursively as

$$\mathcal{Q}_t^{\mathrm{R}}(x_t) \coloneqq \sup_{p_{t+1} \in \mathcal{P}_{t+1}} \mathbb{E}_{\xi_{t+1} \sim p_{t+1}} \min_{x_{t+1} \in \mathcal{X}_{t+1}, z_t \in \mathbb{R}^{d_t}} f_t(z_t, x_{t+1}; \xi_{t+1}) + \mathcal{Q}_{t+1}^{\mathrm{R}}(x_{t+1}) + M_t \left\| x_t - z_t \right\|. \tag{3.15}$$

In our case where Ξ_t is finite, the regularized cost-to-go function can be written as

$$\mathcal{Q}_t^{\mathrm{R}}(x_t) = \max_{p_{t+1} \in \mathcal{P}_{t+1}} \sum_{n \in \mathcal{N}(t+1)} p_{t+1,n} Q_n^{\mathrm{R}}(x_t), \tag{3.16}$$

where the regularized value function Q_n^{R} for each subproblem $n \in \mathcal{N}(t+1)$ is defined as

$$Q_n^{\mathrm{R}}(x_t) \coloneqq \min_{x_n \in \mathcal{X}_{t+1}, z_n \in \mathbb{R}^{d_t}} \quad f_n(z_n, x_n) + \mathcal{Q}_{t+1}^{\mathrm{R}}(x_n) + M_t \left\| x_t - z_n \right\| \tag{3.17}$$

We set the same convention that $\mathcal{Q}_T^{\mathrm{R}}(x_T) \equiv 0$. Note that for each $n \in \mathcal{N}(T)$ in the last stage and any $x_{T-1} \in \mathcal{X}_{T-1}$, since $Q_n^{\mathrm{R}}(x_{T-1}) = \min_{z_n} \left\{ Q_n(z_n) + M_{T-1} \left\| x_{T-1} - z_n \right\| \right\} \leq Q_n(x_{T-1})$ is an infimal convolution of Q_n and the norm function $M_{T-1} \left\| \cdot \right\|$, hence it is M_t-Lipschitz continuous. The next proposition shows that the functions Q_n^{R} and $\mathcal{Q}_t^{\mathrm{R}}$ are Lipschitz continuous envelopes of Q_n and $\mathcal{Q}_t$, respectively, for all $n \in \mathcal{N}(t)$ and $t \in \mathcal{T}$, which implies that our Lipschitz regularization is different from the quadratic regularization heuristic studied in [76].

Proposition 3.4. *For every node $n \in \mathcal{N}(t+1)$, $t \in \mathcal{T} \setminus \{T\}$, the regularized value function Q_n^{R} is M_t-Lipschitz continuous, and $Q_n^{\mathrm{R}}(x_t) \leq Q_n(x_t)$, $\forall\, x_t \in \mathcal{X}_t$. Moreover, if Q_n is M_t'-Lipschitz continuous with $M_t' < M_t$ for all $n \in \mathcal{N}(t+1)$ and $t < T$, then $Q_n(x_t) = Q_n^{\mathrm{R}}(x_t)$, $\forall\, x_t \in \mathcal{X}_t$.*

Proof. For any node $n \in \mathcal{N}(t+1)$ for some $t \in \mathcal{T}$, pick any $x_t^1, x_t^2 \in \mathcal{X}_t$. Let x_n^i, z_n^i denote the solutions in the definition (3.16) associated with x_t^i for $i = 1$ and 2, respectively. Then,

$$\begin{aligned}
Q_n^{\mathrm{R}}(x_t^1) - Q_n^{\mathrm{R}}(x_t^2) &= f_n(z_n^1, x_n^1) + \mathcal{Q}_{t+1}^{\mathrm{R}}(x_n^1) + M_t\|x_t^1 - z_n^1\| \\
&\quad - f_n(z_n^2, x_n^2) - \mathcal{Q}_{t+1}^{\mathrm{R}}(x_n^2) - M_t\|x_t^2 - z_n^2\| \\
&\leq f_n(z_n^2, x_n^2) + \mathcal{Q}_{t+1}^{\mathrm{R}}(x_n^2) + M_t\|x_t^1 - z_n^2\| \\
&\quad - f_n(z_n^2, x_n^2) - \mathcal{Q}_{t+1}^{\mathrm{R}}(x_n^2) - M_t\|x_t^2 - z_n^2\| \\
&= M_t\left(\|x_t^1 - z_n^2\| - \|x_t^2 - z_n^2\|\right) \leq M_t\|x_t^1 - x_t^2\|.
\end{aligned}$$

Similarly by exchanging the indices $i = 1, 2$, we derive that $Q_n^{\mathrm{R}}(x_t^2) - Q_n^{\mathrm{R}}(x_t^1) \leq M_t\left\|x_t^1 - x_t^2\right\|$, which shows that Q_n^{R} is M_t-Lipschitz continuous.

We next prove the inequalities $Q_n^{\mathrm{R}}(x_t) \leq Q_n(x_t)$, $x_t \in \mathcal{X}_t$ recursively. For nodes in the last stage $n \in \mathcal{N}(T)$, we already show that $Q_n^{\mathrm{R}}(x_{T-1}) \leq Q_n(x_{T-1})$ for any $x_{T-1} \in \mathcal{X}_{T-1}$. By definition (3.9), we have $\mathcal{Q}_{T-1}^{\mathrm{R}}(x_{T-1}) \leq \mathcal{Q}_{T-1}(x_{T-1})$. Now suppose $\mathcal{Q}_t^{\mathrm{R}}(x_t) \leq \mathcal{Q}_t(x_t)$ for some $t \in \mathcal{T} \setminus \{T\}$. Then,

$$\begin{aligned}
Q_n^{\mathrm{R}}(x_t) &= \min_{x_n, z_n}\left\{f_n(z_n, x_n) + \mathcal{Q}_{t+1}^{\mathrm{R}}(x_n) + M_t\left\|x_t - z_n\right\| : x_n \in \mathcal{X}_{t+1}, z_n \in \mathbb{R}^{d_t}\right\} \\
&\leq \min_{x_n, z_n}\left\{f_n(z_n, x_n) + \mathcal{Q}_{t+1}(x_n) + M_t\left\|x_t - z_n\right\| : x_n \in \mathcal{X}_{t+1}, z_n \in \mathbb{R}^{d_t}\right\} \\
&\leq \min_{x_n}\left\{f_n(x_t, x_n) + \mathcal{Q}_{t+1}(x_n) : x_n \in \mathcal{X}_{t+1}\right\} = Q_n(x_t), \qquad (3.18)
\end{aligned}$$

where the last inequality is due to the fact that $z_n = x_t$ is a feasible solution to the minimization problem. Now by definitions (3.9) and (3.16), we have $\mathcal{Q}_t^{\mathrm{R}}(x_t) \leq \mathcal{Q}_t(x_t)$ as well. We have thus shown recursively that $Q_n^{\mathrm{R}}(x_t) \leq Q_n(x_t)$ for any $x_t \in \mathcal{X}_t$.

For the last statement, we claim that if $\mathcal{Q}_{t+1}(x_{t+1}) = \mathcal{Q}^{\mathrm{R}}_{t+1}(x_{t+1})$ for any $x_{t+1} \in \mathcal{X}_{t+1}$ and $Q_n(x_t)$ is M'_t-Lipschitz continuous, then $Q^{\mathrm{R}}_n(x_t) = Q_n(x_t)$ for any $x_t \in \mathcal{X}_t$. To see this claim, note that both inequalities in the above argument (3.18) become equalities: the first one follows from the assumption, and the second one is due to that $z_n = x_t$ is the unique solution to the minimization problem. Therefore, we can apply the claim recursively as well to see that $Q^{\mathrm{R}}_n(x_t) = Q_n(x_t)$ for all $x_t \in \mathcal{X}_t$, $n \in \mathcal{N}(t+1)$, and all $t \in \mathcal{T} \setminus \{T\}$. $\square$

We can generate linear cuts using the procedure for the regularized value functions in (3.13) and its validness can be seen from the same argument as in (3.14). An important difference that regularization brings is that all the linear cuts generated for the regularized value functions are all M_t-Lipschitz continuous, as shown by the following proposition.

Proposition 3.5. *Let $\underline{\mathcal{Q}}_{t+1}$ denote any under-approximation of the cost-to-go function $\mathcal{Q}^{\mathrm{R}}_{t+1}$ and $\hat{x}_t \in \mathcal{X}_t$ a feasible state. The linear cut $V_n(x_t) \coloneqq \underline{v}_n + \langle \hat{\lambda}_n, x_t - \hat{x}_t \rangle$ is M_t-Lipschitz continuous, where $\hat{\lambda}_n$ is a dual solution and $\underline{v}_n$ is its associated value to the Lagrangian dual problem*

$$\sup_{\lambda_n \in \mathbb{R}^{d_t}} \min_{x_n \in \mathcal{X}_{t+1}, z_n, w_n \in \mathbb{R}^{d_t}} \quad f_n(z_n, x_n) + \underline{\mathcal{Q}}^{\mathrm{R}}_{t+1}(x_n) + M_t \left\| \hat{x}_t - w_n \right\| + \langle \lambda_n, w_n - z_n \rangle \quad (3.19)$$

Proof. Let $\hat{\lambda}_n$ denote a feasible dual solution, i.e., the associated value $\underline{v}_n > -\infty$. Then using the optimality condition for the variable w_n, we have $-\hat{\lambda}_n \in \partial_{w_n}(M_t \left\| \hat{x}_t - w_n \right\|)$, where all the norms of elements in the subdifferential set are bounded by $M_t > 0$. Therefore we must have $\|\hat{\lambda}_n\|_* \leq M_t$ and $V_n(x_t)$ is M_t-Lipschitz continuous. $\square$

We remark that after regularization, the Lipschitz constant of the linear cut can always be bounded by the regularization factor M_t. In fact, it can be checked using the same argument in Proposition 3.5 that the regularized value function is equal to

$$Q^{\mathrm{R}}_n(x_t) \coloneqq \max_{\|\lambda_n\|_* \leq M_t} \min_{x_n \in \mathcal{X}_{t+1}, z_n \in \mathbb{R}^{d_t}} \quad f_n(z_n, x_n) + \mathcal{Q}^{\mathrm{R}}_{t+1}(x_n) + \langle \lambda_n, x_t - z_n \rangle.$$

for each $n \in \mathcal{N}(t+1)$, so we only need to find an optimal dual variable $\hat{\lambda}_n$ within the dual norm ball $\{\lambda_n \in \mathbb{R}^{d_t} : \|\lambda_n\|_* \leq M_t\}$. Together with Proposition 3.4, this implies that even for those problems with Lipschitz continuous value functions, regularization may help with the approximation by avoiding linear cuts that have larger Lipschitz constant than that of the value function (cp. Example 3.1).

Non-Lipschitz-Continuous Value Function Cases

We spend the rest of this section showing that the regularization can be exact for a broader class of problems than those that already have Lipschitz continuous value functions, which extends our analysis to problems without relatively complete recourse. The main idea here is that if the extensive formulation of the finitely supported MDRO (3.10) on a finite scenario tree satisfies the constraint qualification, and each component of the worst-case transition probability $p_t \in \mathcal{P}_t$ is either zero or strictly positive, then we can use the exact penalization of the extensive formulation would imply the existence of an exact Lipschitz regularization. To make the idea precise, we need to first define the extensive formulation of MDRO (3.10) on a finite scenario tree with the nested structure removed by taking convex dual problems in each stage.

It is known that $\mathcal{P}_t$ can be assumed to be convex without affecting the value functions. Without loss of generality, we write $\mathcal{P}_t = \mathcal{K}_t \cap \Delta^{N_t}$ where $\mathcal{K}_t$ is a closed convex cone contained in the nonnegative orthant, for each $t \in \mathcal{T}$. Then the value function for node $n \in \mathcal{N}(t)$ can be written as

$$\begin{aligned} Q_n(x_{t-1}) = \min_{x_n, c_n} & f_n(x_t, x_n) + c_n, \\ \text{s.t. } & \big(c_n - Q_m(x_n)\big)_{m \in \mathcal{N}(t)} \in \mathcal{K}_t^*, \\ & x_n \in \mathcal{X}_t, \quad \forall\, n \in \mathcal{N}(t), \end{aligned} \tag{3.20}$$

by taking the dual of the maximization over transition probability vectors $p_t \in \mathcal{P}_t$ in the

definition (3.11). To get an extensive formulation, we define $\tilde{\mathcal{N}}(t) \coloneqq \prod_{t'=1}^{t} \mathcal{N}(t')$ as a node set of the scenario tree, such that each $n \in \tilde{\mathcal{N}}(t)$ as a node in the recombining scenario tree is determined by a vector $(n_1, \ldots, n_t)$ where $n_{t'} \in \mathcal{N}(t')$ for each $t' \leq t$. The extended node set $\tilde{\mathcal{N}} \coloneqq \cup_{t=1}^{T} \tilde{\mathcal{N}}(t)$ naturally has a tree structure, so we use $a(n)$, $\mathcal{C}(n)$, and $\mathcal{D}(n)$ to denote the parent node, the set of child nodes, and the set of all descendent nodes of a node $n \in \tilde{\mathcal{N}}$, respectively. For notational convenience, we use $n = 1$ to denote the root node, corresponding to the deterministic first stage, and $t(n)$ to denote the associated stage to a node $n \in \tilde{\mathcal{N}}$. Now by substituting the formulation (3.20) into the recursion (3.11) recursively, we obtain an extensive formulation of the problem (3.10).

$$
\begin{aligned}
v^{\mathrm{ext}} \coloneqq \min \quad & f_1(x_0, x_1; \xi_1) + c_1 && && (3.21)\\
\text{s.t.} \quad & x_n \in \mathcal{X}_{t(n)}, && \forall\, n \in \tilde{\mathcal{N}},\\
& (c_n - q_{nm})_{m \in \mathcal{C}(n)} \in \mathcal{K}^*_{t(n)}, && \forall\, n \in \tilde{\mathcal{N}},\\
& q_{nm} \geq f_m(x_n, x_m) + c_m, && \forall\, n = a(m),\ m \in \tilde{\mathcal{N}}.
\end{aligned}
$$

We can develop exact penalization on the extensive formulation now. Let $\sigma > 0$ denote a penalty factor for non-root nodes $m \neq 1$. Then the penalization value is defined by

$$
\begin{aligned}
v^{\mathrm{pen}} \coloneqq \min \quad & f_1(x_0, y_1, x_1) + c_1 + \sum_{m \neq 1} \sigma \left\| x_{a(m)} - z_m \right\| && && (3.22)\\
\text{s.t.} \quad & x_n \in \mathcal{X}_{t(n)}, && \forall\, n \in \tilde{\mathcal{N}},\\
& (c_n - q_{nm})_{m \in \mathcal{C}(n)} \in \mathcal{K}^*_{t(n)}, && \forall\, n \in \tilde{\mathcal{N}},\\
& q_{nm} \geq f_m(z_m, x_m) + c_m, && \forall\, n = a(m),\ m \in \tilde{\mathcal{N}}.
\end{aligned}
$$

The penalization objective value v^{pen} depends on the choice of the penalty factor σ. We make the following assumption on the exactness of this penalization.

Assumption 3.2. *There exists a penalty factor $\sigma > 0$ such that $v^{\mathrm{pen}} = v^{\mathrm{ext}}$. Moreover, any*

optimal solution to the penalization (3.22) *satisfies* $z_m = x_{a(m)}$ *for all* $m \neq 1 \in \tilde{\mathcal{N}}$.

We remark that this assumption is satisfied given some constraint qualification, e.g., Slater condition for (3.21) assuming convexity. By the introduction of cost-to-go functions $\mathcal{Q}_t$, we can define the recursive formulation as follows.

$$
\begin{aligned}
v^{\text{pen}} = \min_{x_1 \in \mathcal{X}_1} \quad & f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1^{\text{pen}}(x_1), \\
\mathcal{Q}_1^{\text{pen}}(x_1) \coloneqq \min \quad & c_1 + \sum_{m \neq 1} \sigma \left\| x_{a(m)} - z_m \right\| \\
\text{s.t.} \quad & x_m \in \mathcal{X}_{t(m)}, && \forall\, m \neq 1 \in \tilde{\mathcal{N}}, \\
& (c_m - q_{ml})_{l \in \mathcal{C}(m)} \in \mathcal{K}^*_{t(m)}, && \forall\, m \in \tilde{\mathcal{N}}, \\
& q_{ml} \geq f_l(z_l, x_l) + c_l, && \forall\, m = a(l),\ l \in \tilde{\mathcal{N}}, \\
= \min_{z_m} \sum_{m \neq 1} \sigma \left\| x_{a(m)} - z_m \right\| + & \min_{c_1, q_{1m}} \min_{c_m, q_{ml}, x_m} \quad c_1 \\
\text{s.t.} \quad & x_m \in \mathcal{X}_{t(m)}, && \forall\, m \neq 1 \in \tilde{\mathcal{N}}, \\
& (c_m - q_{ml})_{l \in \mathcal{C}(m)} \in \mathcal{K}^*_{t(m)}, && \forall\, m \in \tilde{\mathcal{N}}, \\
& q_{ml} \geq f_l(z_l, x_l) + c_l, && \forall\, m = a(l),\ l \in \tilde{\mathcal{N}}.
\end{aligned}
$$

Note that by the definition of the dual cone $\mathcal{K}_1^*$, the cost-to-go function can be rewritten as

$$\begin{aligned}
\mathcal{Q}_1^{\text{pen}}(x_1) = \min_{z_m} \; & \sum_{m\neq 1} \sigma \left\| x_{a(m)} - z_m \right\| + \max_{p_2\in\mathcal{P}_2} \min_{x_m} \sum_{m\in\mathcal{C}(1)} p_{2,m}(f_m(z_m, x_m) + c_m) \\
\text{s.t.} \quad & x_m \in \mathcal{X}_{t(m)}, && \forall\, m \neq 1 \in \tilde{\mathcal{N}}, \\
& (c_m - q_{ml})_{l\in\mathcal{C}(m)} \in \mathcal{K}^*_{t(m)}, && \forall\, m \neq 1 \in \tilde{\mathcal{N}}, \\
& q_{ml} \geq f_l(z_l, x_l) + c_l, && \forall\, m = a(l) \neq 1,\ l \in \tilde{\mathcal{N}}. \\
= \max_{p_2\in\mathcal{P}_2} \min \quad & \sum_{m\in\mathcal{C}(1)} p_{2,m}(f_m(z_m, x_m) + c_m) + \sum_{m\neq 1} \sigma \left\| x_{a(m)} - z_m \right\| \\
\text{s.t.} \quad & x_m \in \mathcal{X}_{t(m)}, && \forall\, m \neq 1 \in \tilde{\mathcal{N}}, \\
& (c_m - q_{ml})_{l\in\mathcal{C}(m)} \in \mathcal{K}^*_{t(m)}, && \forall\, m \neq 1 \in \tilde{\mathcal{N}}, \\
& q_{ml} \geq f_l(z_l, x_l) + c_l, && \forall\, m = a(l) \neq 1,\ l \in \tilde{\mathcal{N}}.
\end{aligned}$$

The last step of exchanging $\min$ and $\max$ is due to the convexity of the problem and the compactness of the uncertainty set $\mathcal{P}_1$. Now, we define the cost-to-go functions for $m \in \mathcal{C}(1)$:

$$\mathcal{Q}_1^{\text{pen}}(x_1) = \max_{p_2\in\mathcal{P}_2} \min_{x_m, z_m} \sum_{m\in\mathcal{C}(1)} \left[p_{2,m}(f_m(z_m, x_m) + \mathcal{Q}_m^{\text{pen}}(x_m)) + \sigma \left\| x_1 - z_m \right\| \right],$$

$$\begin{aligned}
\text{with } \mathcal{Q}_m^{\text{pen}}(x_m) \coloneqq \min \quad & c_m + \sum_{l\in\mathcal{D}(m)} \sigma \left\| x_{a(l)} - z_l \right\| \\
\text{s.t.} \quad & x_l \in \mathcal{X}_{t(l)}, && \forall\, l \in \mathcal{D}(m), \\
& (c_l - q_{lk})_{k\in\mathcal{C}(l)} \in \mathcal{K}^*_l, && \forall\, l \in \mathcal{D}(m), \\
& q_{lk} \geq f_k(z_k, x_k) + c_k, && \forall\, l = a(k) \in \mathcal{D}(m).
\end{aligned}$$

By repeating the above definition of cost-to-go functions, we can have the recursive formu-

lation of the penalization

$$v^{\text{pen}} = \min_{x_1 \in \mathcal{X}_1} \quad f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1^{\text{pen}}(x_1), \tag{3.23}$$

$$\mathcal{Q}_n^{\text{pen}}(x_n) = \max_{p_{t'} \in \mathcal{P}_{t'}} \min_{x_m \in \mathcal{X}_{t(m)}} \sum_{m \in \mathcal{C}(n)} \left[p_{t',m} (f_m(z_m, x_m) + \mathcal{Q}_m^{\text{pen}}(x_m)) + \sigma \left\| x_n - z_m \right\| \right], \tag{3.24}$$

where $t' := t(n) + 1 \in \mathcal{T}$. Note by this definition, $\mathcal{Q}_n \equiv 0$ for all leaf nodes $n \in \tilde{\mathcal{N}}$ with $\mathcal{C}(n) = \varnothing$.

While the penalization is known to be exact under Assumption 3.2, the evaluation of the cost-to-go function at a given point may be challenging. In the sequel, we show that we can replace the penalization formulation with regularization under the following assumption:

Assumption 3.3. *There exists a constant $0 < c \leq 1$ such that for any $n \in \tilde{\mathcal{N}}$ and $x_n \in \mathcal{X}_n$, there exists a maximizer $\hat{p}_t \in \mathcal{P}_{t(n)}$ in* (3.24) *with each component satisfying either $\hat{p}_{t,m} = 0$ or $\hat{p}_{t,m} \geq c$, $m \in \mathcal{C}(n)$.*

Let $\tau > 0$ denote a regularization factor for all non-root nodes. We now define the regularization cost-to-go functions recursively as

$$v^{\text{reg}} = \min_{x_1 \in \mathcal{X}_1} \quad f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1^{\text{reg}}(x_1), \tag{3.25}$$

$$\mathcal{Q}_n^{\text{reg}}(x_n) = \max_{p_{t'} \in \mathcal{P}_{t'}} \min_{x_m \in \mathcal{X}_{t(m)}} \sum_{m \in \mathcal{C}(n)} p_{nm} \left[f_m(z_m, x_m) + \mathcal{Q}_m^{\text{reg}}(x_m) + \tau \left\| x_n - z_m \right\| \right]. \tag{3.26}$$

We can now state the exactness result for the regularization.

Proposition 3.6. *Given the constant $c > 0$ stated in Assumption 3.3, if $\tau \geq \sigma / c$, then the regularization is exact, i.e., $v^{\text{reg}} = v^{\text{pen}} = v^{\text{ext}}$ and the corresponding set of first stage minimizers are the same.*

Proof. For ease of notation, we denote

$$\hat{\mathcal{F}}_1 := \underset{x_1 \in \mathcal{X}_1}{\arg\min}\{f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1(x_1)\},$$

$$\hat{\mathcal{F}}_1^{\mathrm{pen}} := \underset{x_1 \in \mathcal{X}_1}{\arg\min}\{f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1^{\mathrm{pen}}(x_1)\},$$

and

$$\hat{\mathcal{F}}_1^{\mathrm{reg}} := \underset{x_1 \in \mathcal{X}_1}{\arg\min}\{f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1^{\mathrm{reg}}(x_1)\},$$

respectively. By definition, we have $\mathcal{Q}_n^{\mathrm{reg}}(x_n) \leq \mathcal{Q}_n(x_n)$ for all $n \in \tilde{\mathcal{N}}$ and $x_n \in \mathcal{X}_n$, which implies $v^{\mathrm{reg}} \leq v^{\mathrm{ext}}$. Assumption 3.2 of exact penalization implies that $v^{\mathrm{pen}} = v^{\mathrm{ext}}$ and $\hat{\mathcal{F}}_1^{\mathrm{pen}} = \hat{\mathcal{F}}_1$. We claim that $\mathcal{Q}_n^{\mathrm{reg}}(x_n) \geq \mathcal{Q}_n^{\mathrm{pen}}(x_n)$ for all nodes $n \in \tilde{\mathcal{N}}$ and states $x_n \in \mathcal{X}_n$. Given this claim, it follows that $v^{\mathrm{reg}} \geq v^{\mathrm{pen}}$ and hence $v^{\mathrm{reg}} = v^{\mathrm{pen}} = v^{\mathrm{ext}}$. We thus conclude $\hat{\mathcal{F}}_1 \subseteq \hat{\mathcal{F}}_1^{\mathrm{reg}} \subseteq \hat{\mathcal{F}}_1^{\mathrm{pen}}$, implying the equality of these three sets.

We prove the claim recursively. For any leaf node n, $\mathcal{Q}_n^{\mathrm{reg}}(x_n) = \mathcal{Q}_n^{\mathrm{pen}}(x_n) = 0$ since $\mathcal{C}(n) = \varnothing$. Now assume that the claim $\mathcal{Q}_m^{\mathrm{reg}}(x_m) \geq \mathcal{Q}_m^{\mathrm{pen}}(x_m)$ holds for all descendent nodes $m \in \mathcal{D}(n)$. Let $p_{t'} \in \mathcal{P}_{t'}$ denote a maximizer associated with the state $x_n \in \mathcal{X}_n$ in the definition (3.24) for $t' := t(n) + 1$. Thus

$$\mathcal{Q}_n^{\mathrm{pen}}(x_n) = \sum_{\substack{m \in \mathcal{C}(n): \\ p_{t',m} \neq 0}} \min_{x_m \in \mathcal{X}_{t(m)}} \sum_{m \in \mathcal{C}(n)} p_{nm} \left(f_m(z_m, x_m) + \mathcal{Q}_m^{\mathrm{pen}}(x_m) \right) + \sigma \left\| x_n - z_m \right\|.$$

By Assumption 3.3, we can have $p_{t',m} \geq c$. This implies that

$$\begin{aligned}
\mathcal{Q}_n^{\mathrm{pen}}(x_n) &\leq \sum_{\substack{m\in\mathcal{C}(n):\\ p_{t',m}\neq 0}} p_{t',m} \cdot \min_{x_m\in\mathcal{X}_{t(m)}} f_m(z_m, x_m) + \mathcal{Q}_m^{\mathrm{pen}}(x_m) + \tau \left\|x_n - z_m\right\| \\
&\leq \sum_{\substack{m\in\mathcal{C}(n):\\ p_{t',m}\neq 0}} p_{t',m} \cdot \min_{x_m\in\mathcal{X}_{t(m)}} f_m(z_m, x_m) + \mathcal{Q}_m^{\mathrm{reg}}(x_m) + \tau \left\|x_n - z_m\right\| \\
&\leq \max_{p_{t'}\in\mathcal{P}_{t'}} \min_{x_m\in\mathcal{X}_{t(m)}} \sum_{m\in\mathcal{C}(n)} p_{t',m} \left[f_m(z_m, x_m) + \mathcal{Q}_m^{\mathrm{reg}}(x_m) + \tau \left\|x_n - z_m\right\|\right] \\
&= \mathcal{Q}_n^{\mathrm{reg}}(x_n).
\end{aligned}$$

The first inequality is due to $p_{nm}\tau \geq c\tau \geq \sigma$; the second inequality is due to the recursion hypothesis; the third inequality is due to the definition of maximum in the worst-case probability distribution. Thus we have shown $\mathcal{Q}_n^{\mathrm{reg}}(x_n) \geq \mathcal{Q}_n^{\mathrm{pen}}(x_n)$, which finishes the proof through recursion. □

We remark that although in general Assumption 3.3 is not easy to verify, it holds in some common cases where the uncertainty set does not contain any point with zero component (and thus in particular, MSCO with empirical distributions), or the case where all the uncertainty sets and the subproblems are polyhedral (e.g., the MRCO studied in [39]).

3.2 Algorithms and Complexity Analysis

In this section, we first define single stage subproblem oracles (SSSO) for MDRO (3.8), based on which we define the notion of complexity of the algorithms. A simple implementation of the SSSO is then discussed for the finitely supported problems (3.10). We describe two versions of our proposed dual dynamic programming algorithm with consecutive and nonconsecutive stage selection strategies, respectively. The complexity upper bound for each algorithm is then presented, and finally we provide a complexity lower bound for both algorithms, which shows the upper complexity bounds are nearly tight.

3.2.1 Single Stage Subproblem Oracles

A subproblem oracle is an oracle that gives a solution to the subproblem given its own information as well as the data generated by the algorithm. The single stage subproblem oracles (SSSO) used in this dissertation solve an approximation of the problem given by (3.16) and (3.17) (or given by (3.11) and (3.12) if $M_t = +\infty$, i.e., no regularization is conducted) for some stage $t \in \mathcal{T}$.

Definition 3.1 (Initial stage subproblem oracle)**.** *Let $\underline{\mathcal{Q}}_1, \overline{\mathcal{Q}}_1 : \mathcal{X}_1 \to \bar{\mathbb{R}}$ denote two lsc convex functions, representing an under-approximation and an over-approximation of the cost-to-go function $\mathcal{Q}_1^{\mathrm{R}}$ in* (3.15)*, respectively. Consider the following subproblem for the first stage $t = 1$,*

$$\min_{x_1 \in \mathcal{X}_1} f_1(x_0, x_1; \xi_1) + \underline{\mathcal{Q}}_1(x_1), \tag{I}$$

where x_0 and ξ_1 are given parameters. The initial stage subproblem oracle provides an optimal solution x_1 to (I) *and calculates the approximation gap $\gamma_1 \coloneqq \overline{\mathcal{Q}}_1(x_1) - \underline{\mathcal{Q}}_1(x_1)$ at the solution. We thus define the subproblem oracle formally as the map $\mathscr{O}_1 : (\underline{\mathcal{Q}}_1, \overline{\mathcal{Q}}_1) \mapsto (x_1; \gamma_1)$.*

Definition 3.2 (Noninitial stage subproblem oracle)**.** *Let $\underline{\mathcal{Q}}_t, \overline{\mathcal{Q}}_t : \mathcal{X}_t \to \bar{\mathbb{R}}$ denote two lsc convex functions, representing an under-approximation and an over-approximation of the cost-to-go function $\mathcal{Q}_t^{\mathrm{R}}$ in* (3.15)*, respectively, for some stage $t > 1$. Then given a feasible state $x_{t-1} \in \mathcal{X}_{t-1}$, the noninitial stage subproblem oracle provides a feasible state $x_t \in \mathcal{X}_t$, an M_{t-1}-Lipschitz continuous linear cut $\mathcal{V}_{t-1}(\,\cdot\,)$, and an over-estimate value v_{t-1} such that*

- *they are valid, i.e., $\mathcal{V}_{t-1}(x) \le \mathcal{Q}_{t-1}^{\mathrm{R}}(x)$ for any $x \in \mathcal{X}_{t-1}$ and $v_{t-1} \ge \mathcal{Q}_{t-1}^{\mathrm{R}}(x_{t-1})$;*
- *the gap is controlled, i.e., $v_{t-1} - \mathcal{V}_{t-1}(x_{t-1}) \le \gamma_t \coloneqq \overline{\mathcal{Q}}_t(x_t) - \underline{\mathcal{Q}}_t(x_t)$.*

We thus define the subproblem oracle formally as the map $\mathscr{O}_t : (x_{t-1}, \underline{\mathcal{Q}}_t, \overline{\mathcal{Q}}_t) \mapsto (\mathcal{V}_{t-1}, v_{t-1}, x_t; \gamma_t)$.

The noninitial stage subproblem oracles are different from the initial stage subproblem oracle, in the sense that it does not necessarily provide any optimal solution to some

optimization problem. Instead, it provides some feasible state, which could be used for exploration of the following stages, a linear cut and an estimate value for updating the approximation in the previous stages. This is a much weaker requirement on the oracle itself than assuming that we can get optimal solutions to the MDRO recursion (3.9).

Since the initial stage subproblem oracle in Definition 3.1 represents the solution procedure of the first stage problem (I), its implementation is straightforward. In contrast, implementations for Definition 3.2 may be less intuitive, so we provide some further discussion for the finitely supported MDRO (3.10) below.

SSSO Implementation for Finitely Supported MDRO

We now propose a possible realization of the noninitial stage subproblem oracle based on the linear cut generation scheme discussed in Section 3.1.2. We first consider an under-approximation problem:

$$\max_{p_t \in \mathcal{P}_t} \sum_{n \in \mathcal{N}(t)} p_{t,n} \cdot \left(\min_{x_t \in \mathcal{X}_t} f_n(x_{t-1}, x_t) + \underline{\mathcal{Q}}_t(x_t) \right). \tag{3.27}$$

Recall that the Lagrangian dual for each inner minimization problem in (3.27) gives a primal-dual solution pair $(\hat{x}_n, \hat{z}_n; \hat{\lambda}_n)$ and its associated value $\underline{v}_n$, as in (3.19). The linear cut $V_n(x) = \underline{v}_n + \langle \hat{\lambda}_n, x - x_{t-1} \rangle$ is valid for the value function Q_n^{R} and by Proposition 3.5, it is M_{t-1}-Lipschitz continuous. Now we define a linear cut $\mathcal{V}_{t-1}(x) \coloneqq \sum_{n \in \mathcal{N}(t)} \hat{p}_{t,n} V_n(x)$ where $\hat{p}_t \in \arg\max_{p_t \in \mathcal{P}_t} \sum_{n \in \mathcal{N}(t)} p_{t,n} V_n(x_{t-1})$ is a probability vector maximizer at the current state x_{t-1}. Then by Proposition 3.3, $\mathcal{V}_{t-1}$ is a valid linear cut for $\mathcal{Q}_{t-1}^{\mathrm{R}}$ with a Lipschitz constant M_{t-1}.

For the over-estimate value v_{t-1}, we consider an over-approximation problem:

$$\max_{p_t \in \mathcal{P}_t} \sum_{n \in \mathcal{N}(t)} p_{t,n} \cdot \left(\min_{x_t \in \mathcal{X}_t} f_n(x_{t-1}, x_t) + \overline{\mathcal{Q}}_t(x_t) \right). \tag{3.28}$$

Since by assumption $\overline{\mathcal{Q}}_t(x) \geq \mathcal{Q}_t^{\mathrm{R}}(x)$ for all $x \in \mathcal{X}_t$, the optimal value v_{t-1} of (3.28) satisfies $v_{t-1} \geq \mathcal{Q}_{t-1}^{\mathrm{R}}(x_{t-1})$ by definition. Moreover, suppose the gap at realization $\hat{\xi}_{t,n}$ given by the primal solution $\hat{x}_n$ is $\gamma_n := \overline{\mathcal{Q}}_t(\hat{x}_n) - \underline{\mathcal{Q}}_t(\hat{x}_n)$. We pick the index n^* such that the state $\hat{x}_{n^*}$ has the largest approximation gap γ_{n^*}, and set $x_t = \hat{x}_{n^*}$, $\gamma_t = \gamma_{n^*}$. Consequently, we have

$$
\begin{aligned}
v_{t-1} &= \max_{p_t \in \mathcal{P}_t} \sum_{n \in \mathcal{N}(t)} p_{t,n} \left(\min_{x_n \in \mathcal{X}_t, z_n \in \mathbb{R}^{d_{t-1}}} f_n(z_n, x_n) + \overline{\mathcal{Q}}_t(x_n) + M_{t-1} \left\| x_{t-1} - z_n \right\| \right) \\
&\leq \max_{p_t \in \mathcal{P}_t} \sum_{n \in \mathcal{N}(t)} p_{t,n} \left(f_n(\hat{z}_n, \hat{x}_n) + \overline{\mathcal{Q}}_t(\hat{x}_n) + M_{t-1} \left\| x_{t-1} - \hat{z}_n \right\| \right) \\
&= \max_{p_t \in \mathcal{P}_t} \sum_{n \in \mathcal{N}(t)} p_{t,n} \left(f_n(\hat{z}_n, \hat{x}_n) + \underline{\mathcal{Q}}_t(\hat{x}_n) + \gamma_n + M_{t-1} \left\| x_{t-1} - \hat{z}_n \right\| \right) \\
&\leq \gamma_t + \max_{p_t \in \mathcal{P}_t} \sum_{n \in \mathcal{N}(t)} p_{t,n} \left(f_n(\hat{z}_n, \hat{x}_n) + \underline{\mathcal{Q}}_t(\hat{x}_n) + M_{t-1} \left\| x_{t-1} - \hat{z}_n \right\| \right) \\
&= \gamma_t + \mathcal{V}_{t-1}(x_{t-1}).
\end{aligned}
$$

Therefore, the condition $v_{t-1} - \mathcal{V}_{t-1}(x_{t-1}) \leq \gamma_t$ is satisfied. We summarize the above realization of the noninitial stage subproblem oracles in Algorithm 4.

Assuming that the subproblems (3.27) and (3.28) can be solved for any state $x_{t-1} \in \mathcal{X}_{t-1}$ and approximation of the cost-to-go function $\underline{\mathcal{Q}}_t$ and $\overline{\mathcal{Q}}_t$, Algorithm 4 provides an implementation of Definition 3.2. However, we remark that Algorithm 4 is not the only way to realize the SSSO in Definition 3.2. For example, it is discussed in [39] that a polyhedral single stage subproblem of MRCO (3.7) can be reformulated as mixed-integer linear program, which may then be solved by branch-and-bound type algorithms. Therefore, the introduction of SSSO may benefit our discussion by avoiding restriction of solution methods in each stage, even when the uncertainty sets are finite. Besides, with SSSO, the complexity analysis will better reflect the computation time as the for-loop in Algorithm 4 can be easily parallelized. We also show in the next section that SSSO enables us to introduce a nonconsecutive dual dynamic programming algorithm.

Algorithm 4 A Realization of Noninitial Stage Subproblem Oracle for (3.11)

Require: state $x_{t-1} \in \mathcal{X}_{t-1}$, approximations $\underline{\mathcal{Q}}_t, \overline{\mathcal{Q}}_t : \underline{\mathcal{Q}}_t(x) \leq \mathcal{Q}_t^{\mathrm{R}}(x) \leq \overline{\mathcal{Q}}_t(x),\ \forall\, x \in \mathcal{X}_t, t > 1$

Ensure: a linear cut $\mathcal{V}_{t-1}$, an over-estimate v_{t-1}, a state x_t, and a gap value γ_t as in Definition 3.2

1: **for** $n \in \mathcal{N}(t)$ **do**

2: solve the Lagrangian dual of

$$\min\left\{ f_n(z_n, x_n) + \underline{\mathcal{Q}}_t(x_n) + M_{t-1}\left\|x_{t-1} - z_n\right\| : x_n \in \mathcal{X}_t, z_n \in \mathbb{R}^{d_{t-1}} \right\}$$

3: collect the primal-dual solution pair $(\hat{x}_n, \hat{z}_n; \hat{\lambda}_n)$ and the value $\underline{v}_n$

4: define $V_n(x) \coloneqq \underline{v}_n + \langle \hat{\lambda}_n, x - x_{t-1} \rangle$

5: calculate $\gamma_n \coloneqq \overline{\mathcal{Q}}_t(x_n) - \underline{\mathcal{Q}}_t(x_n)$

6: let $\bar{v}_n \coloneqq \min\left\{ f_n(z_n, x_n) + \overline{\mathcal{Q}}_t(x_n) + M_{t-1}\left\|x_{t-1} - z_n\right\| : x_n \in \mathcal{X}_t, z_n \in \mathbb{R}^{d_{t-1}} \right\}$

7: **end for**

8: construct $\mathcal{V}_{t-1}(x) \coloneqq \sum_{n \in \mathcal{N}(t)} \hat{p}_{t,n} V_n(x)$ where $\hat{p}_t \in \arg\max_{p_t \in \mathcal{P}_t} \sum_{n \in \mathcal{N}(t)} p_{t,n} \underline{v}_n$

9: calculate $v_{t-1} \coloneqq \max_{p_t \in \mathcal{P}_t} \sum_{n \in \mathcal{N}(t)} p_{t,n} \bar{v}_n$

10: find $n^* \in \mathcal{N}(t)$ such that $\gamma_{n^*} \geq \gamma_n$ for all $n \in \mathcal{N}(t)$ and set $x_t \coloneqq \hat{x}_{n^*}$, $\gamma_t \coloneqq \gamma_{n^*}$

3.2.2 Dual Dynamic Programming Algorithms

With the subproblem oracles, we first introduce a consecutive dual dynamic programming (CDDP) algorithm. To ease the notation, we use $\mathrm{conv}\{h_1, h_2\} =: g$ to denote the function corresponding to the closed convex hull of the epigraphs of functions h_1 and h_2. More precisely, using convex bi-conjugacy, we define

$$g(x) \coloneqq \left(\min\{h_1(x), h_2(x)\}\right)^{**} = \sup_{\lambda} \inf_{z} \left\{ \min\{h_1(z), h_2(z)\} + \langle \lambda, x - z \rangle \right\}.$$

Note that if h_1, h_2 are both polyhedral (hence closed and convex), then by linear program strong duality, the function g (assuming it is proper) can be represented as

$$g(x) = \min\left\{ v \in \mathbb{R} : \begin{array}{l} (z, v) = \mu_1(z_1, v_1) + \mu_2(z_2, v_2), \\ v_i \geq h_i(z_i), \mu_i \in \mathbb{R}_{\geq 0}, i = 1, 2, \\ \mu_1 + \mu_2 = 1 \end{array} \right\}.$$

Algorithm 5 Consecutive Dual Dynamic Programming Algorithm

Require: subproblem oracles $\mathscr{O}_t$ for $t \in \mathcal{T}$, optimality gap $\varepsilon > 0$
Ensure: an ε-optimal first stage solution x_1^* to the regularization (3.15)

initialize: $\underline{\mathcal{Q}}_t^0 \leftarrow 0, \overline{\mathcal{Q}}_t^0 \leftarrow +\infty, t \in \mathcal{T}\backslash\{T\}$; $\underline{\mathcal{Q}}_T^j, \overline{\mathcal{Q}}_T^j \leftarrow 0, j \in \mathbb{N}$; $i \leftarrow 1$
evaluate $(x_1^1; \gamma_1^1) \leftarrow \mathscr{O}_1(\underline{\mathcal{Q}}_1^0, \overline{\mathcal{Q}}_1^0)$
set LOWERBOUND $\leftarrow f_1(x_0, x_1^1; \xi_1)$, UPPERBOUND $\leftarrow +\infty$
while UPPERBOUND $-$ LOWERBOUND $> \varepsilon$ **do**
    for $t = 2, \ldots, T$ **do**
        evaluate $(\mathcal{V}_{t-1}^i, v_{t-1}^i, x_t^i; \gamma_t^i) = \mathscr{O}_t(x_{t-1}^i, \underline{\mathcal{Q}}_t^{i-1}, \overline{\mathcal{Q}}_t^{i-1})$ ▷ forward step
    end for
    for $t = T, \ldots, 2$ **do**
        update $\underline{\mathcal{Q}}_{t-1}^i(x) \leftarrow \max\{\underline{\mathcal{Q}}_{t-1}^{i-1}(x), \mathcal{V}_{t-1}^i(x)\}$ ▷ backward step
        update $\overline{\mathcal{Q}}_{t-1}^i(x) \leftarrow \operatorname{conv}\{\overline{\mathcal{Q}}_{t-1}^{i-1}(x), v_{t-1}^i + M_{t-1}\left\|x - x_{t-1}^i\right\|\}$
    end for
    evaluate $(x_1^{i+1}; \gamma_1^{i+1}) \leftarrow \mathscr{O}_1(\underline{\mathcal{Q}}_1^i, \overline{\mathcal{Q}}_1^i)$ ▷ initial stage step
    update LOWERBOUND $\leftarrow f_1(x_0, x_1^{i+1}; \xi_1) + \underline{\mathcal{Q}}_1^i(x_1^{i+1})$
    update UPPERBOUND$' \leftarrow f_1(x_0, x_1^{i+1}; \xi_1) + \overline{\mathcal{Q}}_1^i(x_1^{i+1})$
    if UPPERBOUND$'$ $<$ UPPERBOUND **then**
        set $x_1^* \leftarrow x_1^{i+1}$, UPPERBOUND $\leftarrow$ UPPERBOUND$'$
    end if
    update $i \leftarrow i + 1$
end while

For each iteration $i \in \mathbb{N}$, the main loop of Algorithm 5 consists of three parts. The forward step uses the state x_{t-1}^i in the previous stage and the approximations $\underline{\mathcal{Q}}_t^{i-1}$ and $\overline{\mathcal{Q}}_t^{i-1}$ to produce a new state x_t^i. Then the backward step at stage t uses the cut $\mathcal{V}_{t-1}^i(x)$ and the value v_{t-1}^i to update the approximations $\underline{\mathcal{Q}}_{t-1}^i, \overline{\mathcal{Q}}_{t-1}^i$ in its precedent stage $t-1$. Finally, the initial stage step produces a new first stage solution x_1^{i+1} and updates the lower and upper bounds.

We next show the correctness of Algorithm 5, i.e., the returned solution x_1^* is ε-optimal, while leaving the finiteness proof to Section 3.2.3. From the termination of the while-loop, it suffices to show that the approximations are valid $\underline{\mathcal{Q}}_t^i(x) \le \mathcal{Q}_t^{\mathrm{R}}(x) \le \overline{\mathcal{Q}}_t^i(x)$ for each $t \in \mathcal{T}$ and $i \in \mathbb{N}$. The first inequality follows from the validness of linear cuts $V_t^i(x)$ (cf. Proposition 3.3). The second inequality is due to the M_t-Lipschitz continuity of the

regularized cost-to-go functions $\mathcal{Q}_t^{\mathrm{R}}$. In particular, by Definition 3.2, whenever the input $\overline{\mathcal{Q}}_t^i(x) \geq \mathcal{Q}_t^{\mathrm{R}}(x)$ for $x \in \mathcal{X}_t$, the value $v_{t-1}^i \geq \mathcal{Q}_{t-1}^{\mathrm{R}}(x_{t-1}^i)$. Then $v_{t-1}^i + M_t \left\|x - x_{t-1}^i\right\| \geq \mathcal{Q}_t^{\mathrm{R}}(x)$ for all $x \in \mathcal{X}_{t-1}$. Given that $\overline{\mathcal{Q}}_{t-1}^{i-1}(x) \geq \mathcal{Q}_{t-1}^{\mathrm{R}}(x)$ for $x \in \mathcal{X}_{t-1}$, which is obviously true for $i = 1$, we conclude that

$$\min\{\overline{\mathcal{Q}}_{t-1}^{i-1}(x), v_{t-1}^i + M_{t-1}\|x - x_{t-1}^i\|\} \geq \mathcal{Q}_{t-1}^{\mathrm{R}}(x), \quad \forall\, x \in \mathcal{X}_{t-1}. \tag{3.29}$$

By taking the closed convex hull of the epigraphs on both sides, we have shown that $\overline{\mathcal{Q}}_{t-1}^i(x) \geq \mathcal{Q}_{t-1}^{\mathrm{R}}(x)$ for all $x \in \mathcal{X}_{t-1}$. The above argument shows inductively that for all $i \in \mathbb{N}$, the approximations are valid, which then implies the correctness of the algorithm.

We comment that the linear cut $\mathcal{V}_{t-1}^i$ and the over-estimate value v_{t-1}^i are generated using only the information in the previous iteration $i - 1$. In fact, the subproblem oracles can be re-evaluated in the backward steps to produce tighter approximations. We simply keep the CDDP algorithm in its current form because it is already sufficient for us to provide its complexity bound. At the same time, we propose an alternative nonconsecutive version of the dual dynamic programming (NDDP) algorithm that could possibly conduct more efficient approximation updates.

Algorithm 6 describes the NDDP algorithm. To start the algorithm, it requires an additionally chosen vector of approximation gaps $\delta \coloneqq (\delta_t)_{t=1}^T$ such that $\varepsilon = \delta_1 > \delta_2 > \cdots > \delta_T = 0$, compared with the CDDP algorithm. These predetermined approximation gaps serve as criteria at stage t for deciding the next stage to be solved: the precedent stage $t - 1$ or the subsequent one $t + 1$. If the algorithm decides to proceed to the subsequent stage $t + 1$, then the current state x_t is used; otherwise the generated linear cut $\mathcal{V}_{t-1}$ and over-estimate value v_{t-1} are used for updating the approximations. The above argument of validness of approximations imply that $\underline{\mathcal{Q}}_t^{i_t}(x) \leq \mathcal{Q}_t^{\mathrm{R}}(x) \leq \overline{\mathcal{Q}}_t^{i_t}(x)$ for all $x \in \mathcal{X}_t$ holds for any stage $t \in \mathcal{T}$ and any index $i_t \in \mathbb{N}$. Therefore, when NDDP terminates, the returned solution x_1^* is indeed ε-optimal.

Algorithm 6 Nonconsecutive Dual Dynamic Programming Algorithm

Require: subproblem oracles $\mathscr{O}_t$ for $t \in \mathcal{T}$, opt. and approx. gaps $\varepsilon = \delta_1 > \cdots > \delta_T = 0$
Ensure: an ε-optimal first stage solution x_1^* to the regularization (3.15)

initialize: $\underline{\mathcal{Q}}_t^0 \leftarrow 0, \overline{\mathcal{Q}}_t^0 \leftarrow +\infty, t \in \mathcal{T}\backslash\{T\}$; $\underline{\mathcal{Q}}_T^j, \overline{\mathcal{Q}}_T^j \leftarrow 0, j \in \mathbb{N}$; $i_t \leftarrow 0, t \in \mathcal{T}$
set LowerBound $\leftarrow 0$, UpperBound $\leftarrow +\infty$, $t \leftarrow 1$
while true do
    update $i_t \leftarrow i_t + 1$
    if $t = 1$ **then**
        evaluate $(x_1^{i_1}; \gamma_1^{i_1}) \leftarrow \mathscr{O}_1(\underline{\mathcal{Q}}_1^{i_1}, \overline{\mathcal{Q}}_1^{i_1})$ ▷ initial stage step
        update LowerBound $\leftarrow f_1(x_0, x_1^{i_1}; \xi_1) + \underline{\mathcal{Q}}_1^{i_1}(x_1^{i_1})$
        update UpperBound$'$ $\leftarrow f_1(x_0, x_1^{i_1}; \xi_1) + \overline{\mathcal{Q}}_1^{i_1}(x_1^{i_1})$
        if UpperBound$'$ $<$ UpperBound **then**
            set $x_1^* \leftarrow x_1^{i_1}$, UpperBound $\leftarrow$ UpperBound$'$
        end if
        if UpperBound $-$ LowerBound $\leq \varepsilon$ **then**
            break
        end if
        maintain $\underline{\mathcal{Q}}_2^{i_2+1}(x) \leftarrow \underline{\mathcal{Q}}_2^{i_2}(x)$, $\overline{\mathcal{Q}}_2^{i_2+1}(x) \leftarrow \overline{\mathcal{Q}}_2^{i_2}(x)$
        set $t \leftarrow t + 1$
    else
        evaluate $(\mathcal{V}_{t-1}^{i_t}, v_{t-1}^{i_t}, x_t^{i_t}; \gamma_t^{i_t}) \leftarrow \mathscr{O}_t(x_{t-1}^{i_{t-1}}, \underline{\mathcal{Q}}_t^{i_t}, \overline{\mathcal{Q}}_t^{i_t})$ ▷ noninitial stage step
        if $t < T$ **and** $\gamma_t^{i_t} > \delta_t$ **then**
            maintain $\underline{\mathcal{Q}}_{t+1}^{i_{t+1}+1}(x) \leftarrow \underline{\mathcal{Q}}_{t+1}^{i_{t+1}}(x)$, $\overline{\mathcal{Q}}_{t+1}^{i_{t+1}+1}(x) \leftarrow \overline{\mathcal{Q}}_{t+1}^{i_{t+1}}(x)$
            set $t \leftarrow t + 1$
        else
            update $\underline{\mathcal{Q}}_{t-1}^{i_{t-1}+1}(x) \leftarrow \max\{\underline{\mathcal{Q}}_{t-1}^{i_{t-1}}(x), \mathcal{V}_{t-1}^{i_t}(x)\}$
            update $\overline{\mathcal{Q}}_{t-1}^{i_{t-1}+1}(x) \leftarrow \mathrm{conv}\{\overline{\mathcal{Q}}_{t-1}^{i_{t-1}}(x), v_{t-1}^{i_t} + M_{t-1}\|x - x_{t-1}^{i_{t-1}}\|\}$
            set $t \leftarrow t - 1$
        end if
    end if
end while

3.2.3 Complexity Upper Bounds

In this section, we provide a complexity analysis for the proposed CDDP and NDDP algorithms, which implies that both algorithms terminate in finite time. Our goal is to derive an upper bound on the total number of subproblem oracle evaluations before the termination of the algorithm. To begin with, let $\mathcal{J}_t, t > 1$ denote the set of pair of indices (i_{t-1}, i_t) such that the noninitial stage subproblem oracle is evaluated at the i_t-th time at the state $x_{t-1}^{i_{t-1}}$, i.e., $(\mathcal{V}_{t-1}^{i_t}, v_{t-1}^{i_t}, x_t^{i_t}; \gamma_t^{i_t}) = \mathscr{O}_t(x_{t-1}^{i_{t-1}}, \underline{\mathcal{Q}}_t^{i_t}, \overline{\mathcal{Q}}_t^{i_t})$. For the CDDP algorithm, all stages share the same iteration index $i_t = i$, so $\mathcal{J}_t = \{(i,i) : i \in \mathbb{N}\}$ for all $t > 1$. We define the following sets of indices for each $t \in \mathcal{T} \setminus \{T\}$:

$$\mathcal{I}_t(\delta) \coloneqq \left\{ i_t \in \mathbb{N} : \gamma_t^{i_t} > \delta_t \text{ and } \gamma_{t+1}^{i_{t+1}} \leq \delta_{t+1},\ (i_t, i_{t+1}) \in \mathcal{J}_{t+1} \right\}. \tag{3.30}$$

Here, for NDDP algorithm, δ is the given approximation gap vector, while for CDDP algorithm, $\delta = (\delta_t)_{t=1}^T$ can be any vector satisfying $\varepsilon = \delta_1 > \delta_2 > \cdots > \delta_T = 0$ for the purpose of analysis, since it is not required for the CDDP algorithm. We adopt the convention that the gap for the last stage $\gamma_T^{i_T} \equiv 0$ such that $i_{T-1} \in \mathcal{I}_{T-1}(\delta)$ if and only if $\gamma_{T-1}^{i_{T-1}} > \delta_{T-1}$ and $(i_{T-1}, i_T) \in \mathcal{J}_T$. An important observation is that all these index sets are finite (before algorithm termination) $|\mathcal{I}_t| < \infty$, which is more precisely stated in the following lemma.

Lemma 3.2. *For stage t, suppose the state space $\mathcal{X}_t \subset \mathbb{R}^{d_t}$ is contained in a ball with diameter $D_t > 0$. Then,*

$$\left|\mathcal{I}_t(\delta)\right| \leq \left(1 + \frac{2M_t D_t}{\delta_t - \delta_{t+1}}\right)^{d_t}. \tag{3.31}$$

Proof. We claim that for any $j, k \in \mathcal{I}_t$, $j \neq k$, it holds that $\|x_t^j - x_t^k\| > (\delta_t - \delta_{t+1})/(2M_t)$. Assume for contradiction that $\|x_t^j - x_t^k\| \leq (\delta_t - \delta_{t+1})/(2M_t)$ for some $j < k$, $j, k \in \mathcal{I}_t(\delta)$. By definition of $\mathcal{I}_t(\delta)$, the $t+1$-th subproblem oracle is evaluated at the state x_t^j, and in both the CDDP and the NDDP algorithms, the approximations $\underline{\mathcal{Q}}_t^j$ and $\overline{\mathcal{Q}}_t^j$ are updated since $\gamma_{t+1}^{i_{t+1}} \leq \delta_{t+1}$ for some $i_{t+1} \in \mathbb{N}$ with $(j, i_{t+1}) \in \mathcal{J}_{t+1}$. Then by Definition 3.2

of the noninitial stage subproblem oracle, we have $\overline{\mathcal{Q}}_t^j(x_t^j) - \underline{\mathcal{Q}}_t^j(x_t^j) \leq \delta_{t+1}$. Following Proposition 3.3, for any point $x \in \mathcal{X}_t$ with $\|x - x_t^j\| \leq (\delta_t - \delta_{t+1})/(2M_t)$, we have $\overline{\mathcal{Q}}_t^j(x) - \underline{\mathcal{Q}}_t^j(x) \leq \delta_t$ because of the M_t-Lipschitz continuity of the approximations. By setting $x = x_t^k$, we see a contradiction with the assumption that $k \in \mathcal{I}_t(\delta)$, which proves the claim.

To ease the notation, let $r_t \coloneqq (\delta_t - \delta_{t+1})/(2M_t)$ denote the radius of the d_t-dimensional balls $\mathcal{B}^{d_t}(x_t^j; r_t)$ centered at x_t^j for $j \in \mathcal{I}_t(\delta)$, and let $\mathcal{B}_t \supseteq \mathcal{X}_t$ denote a ball with diameter D_t. From the above claim, we know that $x_t^k \notin \mathcal{B}^{d_t}(x_t^j; r_t)$ for any $j, k \in \mathcal{I}_t(\delta)$ with $j < k$. In other words, the smaller balls $\mathcal{B}^{d_t}(x_t^j; r_t/2)$ are disjoint. Meanwhile, note that each of these smaller balls satisfies $\mathcal{B}^{d_t}(x_t^j; r_t/2) \subset \mathcal{B}_t + \mathcal{B}^{d_t}(0; r_t/2)$ (the Minkowski sum in the Euclidean space $\mathbb{R}^{d_t}$). Therefore, the volumes satisfy the relation

$$\mathrm{Vol}\left(\bigcup_{j \in \mathcal{I}_t(\delta)} \mathcal{B}^{d_t}(x_t^j; r_t/2)\right) = \left|\mathcal{I}_t(\delta)\right| \cdot \mathrm{Vol}\mathcal{B}^{d_t}(0; r_t/2) \leq \mathrm{Vol}\left(\mathcal{B}_t + \mathcal{B}^{d_t}(0; r_t/2)\right),$$

which implies that

$$\left|\mathcal{I}_t(\delta)\right| \leq \frac{\mathrm{Vol}(\mathcal{B}_t + \mathcal{B}^{d_t}(0; r_t/2))}{\mathrm{Vol}\mathcal{B}^{d_t}(0; r_t/2)} = \left(\frac{D_t/2 + r_t/2}{r_t/2}\right)^{d_t} = \left(1 + \frac{2M_t D_t}{\delta_t - \delta_{t+1}}\right)^{d_t}.$$

Thus we complete the proof. □

We prove the following complexity upper bounds for the CDDP algorithm (Theorem 3.1) and the NDDP algorithm (Theorem 3.2).

Theorem 3.1. *Suppose the state spaces $\mathcal{X}_t \subset \mathbb{R}^{d_t}$ are contained in balls, each with diameter $D_t > 0$. Then for the CDDP algorithm (Algorithm 5), the total number of subproblem oracle evaluations $\#\mathtt{Eval}_{\mathrm{CDDP}}$ before termination is bounded by*

$$\#\mathtt{Eval}_{\mathrm{CDDP}} \leq 1 + T \cdot \inf_{\delta} \left\{ \sum_{t=1}^{T-1} \left(1 + \frac{2M_t D_t}{\delta_t - \delta_{t+1}}\right)^{d_t} : \varepsilon = \delta_1 > \delta_2 > \cdots > \delta_T = 0 \right\}.$$

Proof. We prove by showing that for any approximation gap vector δ satisfying $\varepsilon = \delta_1 > \delta_2 > \cdots > \delta_T = 0$, the largest iteration index i is bounded by

$$i \leq \sum_{t=1}^{T-1} \left(1 + \frac{2M_t D_t}{\delta_t - \delta_{t+1}}\right)^{d_t}. \tag{3.32}$$

We claim that each iteration $i \in \mathbb{N}$ must lie in either of the following two cases:

1. the initial stage step has $\gamma_1^i \leq \varepsilon$; or
2. the i-th forward step is in the index set $i \in \mathcal{I}_t(\delta)$ for some stage $t < T$.

To see the claim, suppose that the iteration $i \in \mathbb{N}$ is not in the first case. Then we have $\gamma_1^i > \varepsilon$ and by convention $\gamma_T^i = 0 \leq \delta_T$. Therefore, there exists a stage $t < T$ such that $\gamma_t^i > \delta_t$ while $\gamma_{t+1}^i \leq \delta_{t+1}$, which is the second case. Note that when the first case happens, we have $\textsc{UpperBound} - \textsc{LowerBound} \leq \gamma_1^i \leq \varepsilon$ and thus the CDDP algorithm terminates. By Lemma 3.2, the second case can only happen at most $\sum_{t=1}^{T-1} \left|\mathcal{I}_t(\delta)\right|$ times, proving the bound (3.32). The theorem then follows from the fact that in each CDDP iteration, the subproblem oracle is evaluated T times and one additional evaluation of the initial stage subproblem oracle is needed for checking the termination criterion. □

Theorem 3.2. *Suppose the state spaces $\mathcal{X}_t \subset \mathbb{R}^{d_t}$ are contained in balls, each with diameter $D_t > 0$. Then, for the NDDP algorithm (Algorithm 6) with the predetermined approximation gap vector $(\delta_t)_{t=1}^T$ satisfying $\varepsilon = \delta_1 > \delta_2 > \cdots > \delta_T = 0$, the total number of subproblem oracle evaluations $\texttt{\#Eval}_{\text{NDDP}}$ before termination is bounded by*

$$\texttt{\#Eval}_{\text{NDDP}} \leq 1 + 2 \cdot \sum_{t=1}^{T-1} \left(1 + \frac{2M_t D_t}{\delta_t - \delta_{t+1}}\right)^{d_t}.$$

Proof. For the NDDP algorithm, each time when it decides to go back to the precedent stage $t \leftarrow t - 1$, we must have $\gamma_t^{i_t} \leq \delta_t$ while $\gamma_{t-1}^{i_{t-1}} > \delta_{t-1}$ for some $(i_{t-1}, i_t) \in \mathcal{J}_t$. In this case, we have by definition that $i_t \in \mathcal{I}_t(\delta)$. By Lemma 3.2, such "going back" step can only happen at most $\sum_{t=1}^{T-1} \left|\mathcal{I}_t(\delta)\right|$ times. The theorem then follows from the fact that there are exactly two times such "going back" cases and one additional evaluation of the single

stage subproblem oracle for checking the termination criterion. □

Let us compare the complexity bounds of the two algorithms. If we fix the approximation gap vector δ in Theorem 3.1 to be the same in the NDDP algorithm, then the complexity bound of CDDP is worse than that of NDDP as $T \geq 2$. However, since an optimal choice of the gap vector δ is usually not known, CDDP has the advantage of not requiring an a-priori estimate of these factors for the complexity bound to be valid. We provide below an important simplification of the above complexity bounds that applies to many practical problems.

Corollary 3.1. *Suppose that all the state spaces have the same dimension $d_t = d$ and bounded by a common diameter $D_t \leq D$, and let $M \coloneqq \max\{M_t : t = 1, \ldots, T-1\}$. If for each stage $t \in \mathcal{T}$, the local cost functions are strictly positive for all feasible solutions $f_n(x_{t-1}, x_n) \geq C$, $n \in \mathcal{N}(t)$ for some $C > 0$, then the total number of subproblem oracle evaluations before achieving an α-relative optimal solution x_1^* for CDDP and NDDP are upper bounded respectively by*

$$\#\texttt{Eval}_{\text{CDDP}} \leq 1 + T(T-1)\left(1 + \frac{2MD}{\alpha C}\right)^d,$$
$$\#\texttt{Eval}_{\text{NDDP}} \leq 1 + 2(T-1)\left(1 + \frac{2MD}{\alpha C}\right)^d.$$

Proof. Note that if an solution x_1^* is ε-optimal with $\varepsilon = \alpha C(T-1) < \alpha CT$, then it is also α-relative optimal. The result then follows from Theorems 3.1 and 3.2 by setting $\delta_t = (T-t)\alpha C$. □

Corollary 3.1 shows that for problems that have strictly positive cost in each stage, the proposed complexity bounds for an α-relative optimal solution grow at most *quadratically* for CDDP and *linearly* for NDDP with respect to the number of stages T. This provides an answer to the open question about the iteration complexity of DDP-type algorithms. In the next subsection, we will show this complexity bound is essentially tight by providing a

matching lower bound.

3.2.4 Complexity Lower Bound

Note that if we take $\delta_t = \varepsilon(T-t)/(T-1)$ for $t \in \mathcal{T}$, then the complexity upper bounds in Theorems 3.1 and 3.2 depend on the terms $(T-1)^{d_t}$ where d_t is the state space dimension of stage $t < T$. It is natural to ask whether it is possible for either algorithm to achieve an ε-optimal solution with complexity that is linear or quadratic in T, independent of the state space dimensions (cp. Corollary 3.1). We present a class of convex problems to show that this is indeed *impossible*.

Given a d-sphere $\mathcal{S}^d(r) = \{x \in \mathbb{R}^{d+1} : \|x\|_2 = r\}$ with radius $r > 0$, a spherical cap with depth $\theta > 0$ centered at a point $x \in \mathcal{S}^d(r)$ is the set $\mathcal{S}^d_\theta(r,x) \coloneqq \{y \in \mathcal{S}^d(r) : \langle y - x, x\rangle \geq -\theta r\}$. The next lemma shows that we can put many spherical caps on a sphere such that the center of each is not contained in any other spherical cap. This is a key technical result needed for proving lower complexity bound. Let $\Gamma(\cdot)$ denote the gamma function.

Lemma 3.3. *Given a d-sphere $\mathcal{S}^d(r), d \geq 2$ and depth $\theta < (1 - \frac{\sqrt{2}}{2})r$, there exists a finite set of points $\mathcal{W}$ with*

$$|\mathcal{W}| \geq \frac{(d^2-1)\sqrt{\pi}}{d} \frac{\Gamma(d/2+1)}{\Gamma(d/2+3/2)} \left(\frac{r}{2\theta}\right)^{(d-1)/2},$$

such that for any $w \in \mathcal{W}$, $\mathcal{S}^d_\theta(r,w) \cap \mathcal{W} = \{w\}$.

Proof. Let v_d denote the d-volume for a d-dimensional unit ball. Recall that the d-volume for $\mathcal{S}^d(r)$ is given by $\mathrm{Vol}_d(\mathcal{S}^d(r)) = (d+1)v_{d+1}r^d = \dfrac{(d+1)\pi^{(d+1)/2}}{\Gamma(\frac{d+1}{2}+1)}r^d$. We next estimate the d-volume for the spherical cap $\mathcal{S}^d_\theta(r,x)$. Let $\alpha \in (0, \pi/2)$ denote the central angle for the spherical cap, i.e., $\cos\alpha = 1 - \theta/r$. Since $\theta < (1-\frac{\sqrt{2}}{2})r$, we know that $\alpha < \pi/4$. Then

for any $x \in \mathcal{S}^d(r)$, the d-volume of the spherical cap can be calculated through

$$\mathrm{Vol}_d(\mathcal{S}^d_\theta(r,x)) = \int_0^\alpha \mathrm{Vol}_{d-1}(\mathcal{S}^{d-1}(r\sin\varphi))r\,\mathrm{d}\varphi = dv_d r^d \int_0^\alpha (\sin\varphi)^{d-1}\,\mathrm{d}\varphi.$$

Note that when $\varphi \in (0,\alpha)$, $\sin\varphi > 0$ and $\cos\varphi/\sin\varphi > 1$. Therefore, since $d \geq 2$,

$$\mathrm{Vol}_d(\mathcal{S}^d_\theta(r,x)) \leq dv_d r^d \int_0^\alpha (\sin\varphi)^{d-1}\frac{\cos\varphi}{\sin\varphi}\,\mathrm{d}\varphi = dv_d r^d \cdot \frac{(\sin\alpha)^{d-1}}{d-1}.$$

By substituting $\sin\alpha = \sqrt{1-(1-\theta/r)^2}$, we have

$$\begin{aligned}\frac{\mathrm{Vol}_d(\mathcal{S}^d_\theta(r,x))}{\mathrm{Vol}_d(\mathcal{S}^d(r))} &\leq \frac{d}{d^2-1}\frac{v_d}{v_{d+1}}(\sin\alpha)^{d-1} \\ &= \frac{d}{d^2-1}\frac{v_d}{v_{d+1}}\big(1-\big(1-\frac{\theta}{r}\big)^2\big)^{(d-1)/2} \\ &\leq \frac{d}{d^2-1}\frac{v_d}{v_{d+1}}\left(\frac{2\theta}{r}\right)^{(d-1)/2}.\end{aligned}$$

Now suppose $\mathcal{W} = \{w_i\}_{k=1}^K$ is a maximal set satisfying the assumption, that is, for any $w \in \mathcal{S}^d(r), w \notin \mathcal{W}$, there exists $w_k \in \mathcal{W}$ such that $w \in \mathcal{S}^d_\theta(r,w_k)$. Then, $\cup_{k=1}^K \mathcal{S}^d_\theta(r,w_k) \supseteq \mathcal{S}^d(r)$, and thus $\mathrm{Vol}_d(\mathcal{S}^d(r)) \leq \sum_{k=1}^K \mathrm{Vol}_d(\mathcal{S}^d_\theta(r,w_k)) = |\mathcal{W}|\,\mathrm{Vol}_d(\mathcal{S}^d_\theta(r,w_1))$. Therefore we have

$$\begin{aligned}|\mathcal{W}| \geq \frac{\mathrm{Vol}_d(\mathcal{S}^d(r))}{\mathrm{Vol}_d(\mathcal{S}^d_\theta(r,w_1))} &\geq \left[\frac{d}{d^2-1}\frac{v_d}{v_{d+1}}\left(\frac{2\theta}{r}\right)^{(d-1)/2}\right]^{-1} \\ &= \frac{(d^2-1)\sqrt{\pi}}{d}\frac{\Gamma(d/2+1)}{\Gamma(d/2+3/2)}\left(\frac{r}{2\theta}\right)^{(d-1)/2}.\end{aligned}$$

This completes the proof. □

We denote the set of points associated to the sphere $\mathcal{S}^d(r)$ as $\mathcal{W}^d_\theta(r) = \{w_k\}_{k=1}^K$ in Lemma 3.3. For any constants $\varepsilon > 0$ and $l > 0$, and values $v_k \in (\varepsilon/2, \varepsilon)$, $k = 1,\ldots,K$, we define a function associated with the pair $(\mathcal{W} = \mathcal{W}^d_\theta(r), v = (v_k)_{k=1}^K)$ as $F^{\mathcal{W},v}(x) := \max_k\{0, v_k + \frac{l}{r}\langle w_k, x - w_k\rangle\}$, $x \in \mathcal{B}^{d+1}(r)$, which is obviously convex and l-Lipschitz

continuous. Moreover, if the depth θ and the constants ε, l satisfy $l\theta \geq \varepsilon$, then we have $v_k + \frac{l}{r}\langle w_k, w_{k'} - w_k\rangle < v_k - \frac{l}{r}\theta r \leq v_k - \varepsilon < 0$ for any $w_{k'} \neq w_k \in \mathcal{W}_\theta^d(r)$. This implies that $F^{\mathcal{W},v}(w_k) = v_k$ and the subdifferential $\partial F^{\mathcal{W},v}(w_k) = \{\frac{l}{r}w_k\}$ for all $k = 1, \ldots, K$. Another important observation is that the convex function $\overline{F}^{\neq k}(x) \coloneqq \mathrm{conv}_{k'\neq k}\{v_{k'} + l\,\|x - w_{k'}\|\}$ has the property that $\overline{F}^{\neq k}(w_k) > \varepsilon/2 + l \cdot \mathrm{dist}(w_k, \mathrm{conv}_{k'\neq k}\{w_{k'}\}) \geq \varepsilon/2 + l\theta \geq 3\varepsilon/2$.

We next construct a class of MRCO's using such convex functions, with the following parameters: $T \geq 3$ as the number of stages, $L > 0$ as a prescribed Lipschitz constant, $d \geq 3$ as the state space dimension, $D = 2r > 0$ as the state space diameter, and $\varepsilon > 0$ as the optimality gap. Choose any $l_1, \ldots, l_{T-1}$ such that $L/2 = l_T < l_{T-1} < \cdots < l_1 = L$, and set $\varepsilon_t = 2\varepsilon/(T-2)$. Construct sets of points $\mathcal{W}_t \coloneqq \mathcal{W}_{\theta_t}^{d-1}(r) = \{w_{t,k}\}_{k=1}^{K_t}$, where $\theta_t = \varepsilon_t/l_t$ for $t = 1, \ldots, T-1$. Let $F_t(x) = F^{\mathcal{W}_t, v_t}(x)$ be constructed as above for any values $v_t = (v_{t,k})_{k=1}^{K_t}$, $v_{t,k} \in (\varepsilon_t/2, \varepsilon_t)$, and the Lipschitz constant l_t, $k = 1, \ldots, K_t$, for $t > 1$ and $F_1(x) \equiv 0$. The problem is then constructed as

$$\mathcal{Q}_t(x_t) = \max_{\xi_t \in \mathrm{conv}(\mathcal{W}_{t+1})} \min_{x_{t+1} \in \mathcal{B}^d(r)} \left\{F_t(x_t) + l_t\,\|x_{t+1} - \xi_t\| + \mathcal{Q}_{t+1}(x_{t+1})\right\}, 1 < t < T, \tag{3.33}$$

where $\mathcal{Q}_T(x) \equiv 0$, and the deterministic first stage problem is defined as $\min_{x_1=0} \mathcal{Q}_1(x_1) = \mathcal{Q}_1(0)$. In other words, this class of problems seeks the optimal value corresponding to $x_1 = 0$. We are now at the point to give the lower bound of the complexity of CDDP and NDDP algorithms, assuming the regularization factors $M_t \geq L$, $t \in \mathcal{T}$ for the exactness by Proposition 3.4.

Theorem 3.3. *For the problem* (3.33)*, the number of subproblem oracle evaluations* `#Eval` *for either of Algorithms 5 and 6 before termination has the following lower bound*

$$\texttt{\#Eval} \geq \frac{d}{d-1}\sqrt{\frac{\pi}{2}(d^2-4)}\left(\frac{DL(T-2)}{16\varepsilon}\right)^{d/2-1} = \mathcal{O}(T^{d/2-1}) \text{ as } T \to \infty.$$

Proof. By reformulation of the problem (3.33), we assume that the algorithms only con-

sider the worst-case uncertainty vector $\xi_t \in \mathcal{W}_t$. Note that for $1 < t < T$,

$$\mathcal{Q}_t(x_t) = F_t(x_t) + \max_{\xi_t \in \mathcal{W}_{t+1}} \min_{x_{t+1} \in \mathcal{B}^d(r)} \{l_t \|x_{t+1} - \xi_t\| + \mathcal{Q}_{t+1}(x_{t+1})\} =: F_t(x_t) + c_{t+1}.$$

Therefore, the cost-to-go function $\mathcal{Q}_t$ and any under-approximation $\underline{\mathcal{Q}}_t$ is l_t-Lipschitz continuous, which means $\mathcal{Q}_t^{\mathrm{R}} = \mathcal{Q}_t$ by Proposition 3.4. We further assume that the return of the SSSO satisfies $x_t = \xi_{t-1} \in \mathcal{W}_{t-1}$, which is true for the case of Algorithm 4 since $l_{t-1} > l_t$ implies $x_t = \xi_{t-1}$ is the unique minimizer to the recursion (3.33).

Now let $\underline{\mathcal{Q}}_t^{i_t}, \overline{\mathcal{Q}}_t^{i_t}$ denote the under- and over-approximations of $\mathcal{Q}_t$ at stage index i_t, and $\bar{c}_t^{i_t}, \underline{c}_t^{i_t}$ denote the corresponding under- and over-estimations of the value c_t. Let $\underline{F}_t^{i_t}(x) \coloneqq \max\{0, \max\{v_{t,k} + \frac{l_t}{r}\langle w_{t,k}, x - w_{t,k}\rangle : w_{t,k} = x_t^{i_t}, (i_t, i_{t+1}) \in \mathcal{J}_{t+1}\}\}$ and $\overline{F}_t^{i_t}(x) \coloneqq \mathrm{conv}\{v_{t,k} + M_t \|x - w_{t,k}\| : w_{t,k} = x_t^{i_t}, (i_t, i_{t+1}) \in \mathcal{J}_{t+1}\}$ denote the under- and over-approximations of the function F_t. Note that $\underline{\mathcal{Q}}_t^{i_t}(x) \le \underline{F}_t^{i_t}(x) + \underline{c}_{t+1}^{i_{t+1}}$ and $\overline{\mathcal{Q}}_t^{i_t}(x) \ge \overline{F}_t^{i_t}(x) + \bar{c}_{t+1}^{i_{t+1}}$, for all $t \in \mathcal{T}$. For all $i_t < K_t$, there exists $w_{t,k} \in \mathcal{W}_t$ such that $\overline{F}_t^{i_t}(w_{t,k}) > 3\varepsilon_t/2$ from the discussion following the definition. Therefore, if $i_t < K_t$ for all $t > 1$, then we have

$$\begin{aligned}
\bar{c}_t^{i_t} - \underline{c}_t^{i_t} &= \max_{\xi_{t-1} \in \mathcal{W}_t} \min_{x_t \in \mathcal{B}^d(r)} \{l_{t-1} \|x_t - \xi_{t-1}\| + \overline{\mathcal{Q}}_t^{i_t}(x_t)\} \\
&\quad - \max_{\xi_{t-1} \in \mathcal{W}_t} \min_{x_t \in \mathcal{B}^d(r)} \{l_{t-1} \|x_t - \xi_{t-1}\| + \underline{\mathcal{Q}}_t^{i_t}(x_t)\} \\
&\ge \overline{F}_t^{i_t}(w_{t,k}) - \max_{\xi_{t-1} \in \mathcal{W}_t} F_t(\xi_{t-1}) + \bar{c}_{t+1}^{i_{t+1}} - \underline{c}_{t+1}^{i_{t+1}} \\
&> \frac{\varepsilon_t}{2} + \bar{c}_{t+1}^{i_{t+1}} - \underline{c}_{t+1}^{i_{t+1}}.
\end{aligned}$$

Therefore, UPPERBOUND$-$LOWERBOUND $= \bar{c}_1^{i_1} - \underline{c}_1^{i_1} > \sum_t \varepsilon_t/2 \ge (T-2)\cdot 2\varepsilon/(T-2) = \varepsilon$. Equivalently, when the algorithms terminate, we must have $i_t \ge K_t$ for some $t > 1$,

which implies

$$\begin{aligned}\texttt{\#Eval} \geq K_t &\geq \frac{((d-1)^2-1)\sqrt{\pi}}{d-1}\frac{\Gamma((d-1)/2+1)}{\Gamma((d-1)/2+3/2)}\left(\frac{rL_t(T-2)}{2\varepsilon_t}\right)^{(d-2)/2} \\ &\geq \frac{d(d-2)\sqrt{\pi}}{d-1}\frac{\Gamma(d/2+1/2)}{\Gamma(d/2+1)}\left(\frac{DL(T-2)}{16\varepsilon}\right)^{(d-2)/2} \\ &> \frac{d}{d-1}\sqrt{\frac{\pi}{2}(d^2-4)}\left(\frac{DL(T-2)}{16\varepsilon}\right)^{d/2-1}\end{aligned}$$

by Lemma 3.3 since $L_t \geq L/2$. Here, the last inequality is due to Wendel's bound on the ratio of two gamma functions [77]. This completes the proof. □

Remark. The CDDP and NDDP algorithms (Algorithms 5 and 6) and their complexity analyses depend only on the SSSO (Definitions 3.1 and 3.2). Therefore, while we focus on the MDRO with finite uncertainty sets in this chapter, the CDDP and NDDP algorithms should work for more general MDRO problems (3.8), once we are able to implement the SSSO (esp. the noninitial stage subproblem oracles). Moreover, the complexity upper bounds (Theorems 3.1 and 3.2, Corollary 3.1) and the lower bound (Theorem 3.3) remain valid for them as well.

3.3 Numerical Experiments

In this section, we numerically test the proposed CDDP and NDDP algorithms. The first test problem is a robust multi-commodity inventory problem with customer demand uncertainty. The second test problem is a distributionally robust hydro-thermal power planning problem with stochastic energy inflows. The computation budget consists of 40 2.1-GHz CPU cores and a total of 80 GBytes of RAM. The algorithms are implemented using JuMP package ([78], v0.21) in Julia language (v1.4) with Gurobi 9.0 as its underlying LP solver.

3.3.1 Multi-Commodity Inventory Problem

We consider a multi-commodity inventory problem with uncertain customer demands and deterministic holding and backlogging costs, following the description in [39]. Due to the stagewise independence of the bounded uncertainties and convexity of the problem, we formulate the problem below as an MRCO recursion (3.7). Let $\mathcal{K} \coloneqq \{1, 2, \ldots, K\}$ denote the set of product indices. We first describe the variables in each stage $t \in \mathcal{T}$. We use $l_{t,k}$ to denote the inventory level, $a_{t,k}$ (resp. $b_{t,k}$) to denote the amount of express (resp. standard) order fulfilled in the current (resp. subsequent) stage, of some product $k \in \mathcal{K}$. Let $\xi_t \in \Xi_t$ denote the uncertainty vector controlling the customer demands in stage t. The first stage is assumed to be deterministic, i.e., $\Xi_1 = \{0\}$ without loss of generality. Then, the stage t subproblem can be written as

$$\mathcal{Q}_{t-1}(x_{t-1}) \coloneqq \max_{\xi_t \in \Xi_t} \min \quad \sum_{k \in \mathcal{K}} \Big(c^F + c_k^a a_{t,k} + c_k^b b_{t,k} + c_k^H [l_{t,k}]_+ + c_k^B [l_{t,k}]_- \Big) + \mathcal{Q}_t(x_t) \tag{3.34}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{k \in \mathcal{K}} a_{t,k} \leq B^c, \\ & l_{t,k} - a_{t,k} - b_{t-1,k} = l_{t,k} - d_{t,k}(\xi_t), \quad \forall\, k \in \mathcal{K}, \\ & a_{t,k} \in [0, B_k^a], \quad \forall\, k \in \mathcal{K}, \\ & b_{t,k} \in [0, B_k^b], \quad \forall\, k \in \mathcal{K}, \\ & l_{t,k} \in [-B_k^l, B_k^l], \quad \forall\, k \in \mathcal{K}. \end{aligned}$$

Here in the formulation, c_k^a (resp. c^b) denotes the express (resp. standard) order unit cost, c_k^H (resp. c_k^B) the inventory holding (resp. backlogging) unit cost, B_k^a (resp. B_k^b) the productwise bound for the express (resp. standard) order, and B_k^l the inventory level bound, for the product k, respectively. The first constraint in (3.34) is a cumulative bound B^c on the express orders, the second constraint characterizes the change in the inventory level,

and the rest are bounds on the decision variables with respect to each product. We also put $c^F > 0$ as a fixed cost to ensure the cost function is strictly positive (cf. Corollary 3.1). We use $[l]_+ \coloneqq \max\{l, 0\}$ and $[l]_- \coloneqq -\min\{0, l\}$ to denote the positive and negative part of a real number l. The state variables x_t consist of the inventory levels $(l_{t,k})_{k\in\mathcal{K}}$ and the standard order amounts $(b_{t,k})_{k\in\mathcal{K}}$, while the internal variables are the express order amounts $y_t = (a_{t,k})_{k\in\mathcal{K}}$. The initial state x_0 is given by $l_{0,k} = b_{0,k} = 0$ for all $k \in \mathcal{K}$. The uncertainty set Ξ_t is a E-dimensional box $[-1, 1]^E$, and the customer demand is predicted by the following factor model:

$$d_{t,k}(\xi_t) = \begin{cases} 2 + \sin\left(\dfrac{(t-1)\pi}{5}\right) + \Phi_{t,k}\xi_t, & k \le K/2, \\ 2 + \cos\left(\dfrac{(t-1)\pi}{5}\right) + \Phi_{t,k}\xi_t, & k > K/2, \end{cases} \tag{3.35}$$

where $\Phi_{t,}$ is a E-dimensional vector where each entry is chosen uniformly at random from $[-1/E, 1/E]$. Thus the value $\Phi_{t,k}\xi_t \in [-1, 1]$ and $d_{t,k}(\xi_t) \ge 0$ for all $t \in \mathcal{T}$ and $k \in \mathcal{K}$.

For the following numerical test, we set the number of products $K = 5$, the number of uncertainty factors $E = 4$, $B^a_k = B^b_k = B^l_k = 10$, $c^b_k = 1$ for all $k \in \mathcal{K}$, $c^F = 1$, and $B^c = 0.3K = 1.5$. The costs are generated uniformly at random within $c^a_k \in [1, 3]$, $c^H_k, c^B_k \in [0, 2]$ for all $k \in \mathcal{K}$. Due to lack of relatively complete recourse of the problem (3.34), we use the nonconsecutive dual dynamic programming algorithm (Algorithm 6) with the optimality gap set to be relative $\alpha = 1\%$ and approximation gaps set dynamically by $\delta^{i_t}_t = \textsc{LowerBound} \cdot \alpha(T - t)/(T - 1)$ for $t \in \mathcal{T}$. As a comparison, we implement the same algorithm without regularization which generates linear feasibility cuts for approximation of the feasible sets (see definition of feasibility cuts in, e.g., [79]). For 5 independently generated test cases, we have obtained the following results (Table 3.1) within a time limit of 5 hours and regularization factor of $M_t = 1.0 \times 10^2$ for all $t \in \mathcal{T}$.

In Table 3.1, the inf indicates values of infinity or numerically infinity values (i.e., values greater than 10^9) within the computation time of 5 hours. As we see from the table,

Table 3.1: Comparison of NDDP with and without regularization

	Regularized Problem				Unregularized Problem			
Stage	LB	UB	Time (s)	#Eval	LB	UB	Time (s)	#Eval
	154.66	155.92	203.99	1497	154.70	154.70	96.15	1448
	155.94	157.01	175.12	1502	155.95	inf	18000.29	14151
10	128.31	129.39	230.47	1638	127.56	inf	18002.84	14195
	137.01	138.37	168.36	1405	137.06	137.16	82.40	1565
	120.13	121.01	365.65	1948	120.13	121.16	160.65	2302
	232.46	234.35	473.59	3158	232.46	232.59	265.05	3109
	233.37	235.40	509.25	3107	233.44	235.18	121.63	2752
15	202.48	203.84	413.36	3038	201.89	inf	18001.74	15956
	208.59	209.78	443.00	2930	208.57	208.95	214.33	2992
	195.16	196.97	680.97	4272	187.11	inf	18069.06	26686
	291.87	294.41	1222.93	6189	291.90	292.76	404.01	5175
	292.47	294.87	1248.34	6156	292.84	293.77	437.24	4848
20	256.28	258.53	1026.97	5190	256.30	257.20	401.07	5519
	261.70	264.31	775.68	4770	261.81	261.81	385.10	4765
	249.86	251.32	2135.50	8455	248.12	inf	18072.89	26618
	369.52	370.89	3079.75	11507	369.66	370.31	613.28	7399
	370.01	373.55	2473.70	9814	370.47	370.87	529.98	7422
25	330.01	332.46	1474.33	7674	330.42	332.22	718.88	8291
	333.28	336.27	1009.98	6345	333.34	333.40	578.25	7247
	324.54	326.43	3283.15	12148	311.97	inf	18106.49	33727
	428.94	432.04	4319.78	14394	429.11	429.23	1004.61	10102
	429.48	430.88	3836.59	13707	407.22	inf	18057.45	29520
30	384.21	387.11	2445.64	11142	383.76	inf	18001.47	23233
	386.44	390.29	2006.87	10319	386.54	387.36	922.64	10277
	379.33	382.14	5981.94	16756	379.59	382.24	1615.91	15104

the NDDP algorithm together with feasibility cuts fails to solve two out of five cases even when there is only 10 stages, showing the instability of the performance of feasibility cuts. In contrast, the algorithm with the regularization technique solves all of the cases within a reasonable computation time and number of subproblem oracle evaluations, without any optimality gap on those cases that both formulations are able to solve. This demonstrates the ability of the NDDP handling problems without relatively complete recourses. It is worth mentioning that for cases where the NDDP algorithm converges without regularization, the computation time used is usually smaller than it spends on the regularized problem, which can be explained by better numerical conditions of feasibility cuts and their effect on reducing the effective volumes of the state space.

3.3.2 Hydro-Thermal Power Planning Problem

We next consider the Brazilian interconnected power system described in [80]. By assuming the stagewise independence in the underlying stochastic energy inflow, we formulate the problem below as a MDRO recursion (3.9). Let $\mathcal{K} = \{1, \ldots, K\}$ denote the indices of four regions in the system, and $\mathcal{L} = \cup_{k \in \mathcal{K}} \mathcal{L}_k$ the indices of thermal power plants, where each of the disjoint subsets $\mathcal{L}_k$ is associated with the region $k \in \mathcal{K}$. We first describe the decision variables in each stage $t \in \mathcal{T}$. Let $n \in \mathcal{N}(t)$ denote the index of a sampled outcome in stage t. We use $l_{n,k}$ to denote the stored energy level, $h_{n,k}$ to denote the hydro power generation, and $s_{n,k}$ to denote the energy spillage, of some region $k \in \mathcal{K}$; and $g_{n,l}$ to denote the thermal power generation for some thermal power plant $l \in \mathcal{L}$. For two different regions $k \neq k' \in \mathcal{K}$, we use $e_{n,k,k'}$ to denote the energy exchange from region k to region k', and $a_{n,k,k'}$ to denote the deficit account for region k in region k'. Suppose $(w_{n,k})_{k \in \mathcal{K}}$ is the energy inflow associated with the sampled outcome $n \in \mathcal{N}(t)$, then the stage t subproblem

can be written as

$$\mathcal{Q}_{t-1}(x_{t-1}) \coloneqq \max_{p_{t-1}\in\mathcal{P}_{t-1}} \sum_{n\in\mathcal{N}(t)} p_{t-1,n}\cdot \tag{3.36}$$

$$\left[\min \quad \sum_{k\in\mathcal{K}}\left(c^s s_{n,k} + \sum_{l\in\mathcal{L}_k} c^g_l g_{n,l} + \sum_{k'\in\mathcal{K}}\left(c^e_{k,k'} e_{n,k,k'} + c^a_{k,k'} a_{n,k,k'}\right)\right) + \mathcal{Q}_t(x_n)\right]$$

$$\begin{aligned}
\text{s.t.} \quad & l_{n,k} + h_{n,k} + s_{n,k} = l_{t-1,k} + w_{n,k}, && \forall\, k\in\mathcal{K},\\
& h_{n,k} + \sum_{l\in\mathcal{L}_k} g_{n,l} + \sum_{k'\in\mathcal{K}} (a_{n,k,k'} - e_{n,k,k'} + e_{n,k',k}) = d_{t,k}, && \forall\, k\in\mathcal{K},\\
& l_{n,k} \in [0, B^l_k], && \forall\, k\in\mathcal{K},\\
& h_{n,k} \in [0, B^h_k], && \forall\, k\in\mathcal{K},\\
& g_{n,l} \in [B^{g,-}_l, B^{g,+}_l], && \forall\, l\in\mathcal{L},\\
& a_{n,k,k'} \in [0, B^a_{k,k'}], && \forall\, k,k'\in\mathcal{K},\\
& e_{n,k,k'} \in [0, B^e_{k,k'}], && \forall\, k,k'\in\mathcal{K}.
\end{aligned}$$

Here in the formulation, c^s denotes the unit penalty on energy spillage, c^g_l the unit cost of thermal power generation of plant l, $c^e_{k,k'}$ the unit cost of power exchange from region k to region k', $c^a_{k,k'}$ the unit cost on the energy deficit account for region k in region k', $d_{t,k}$ the deterministic power demand in stage t and region k, B^l_k the bound on the storage level in region k, B^h_k the bound on hydro power generation in region k, $B^{g,-}_l, B^{g,+}_l$ the lower and upper bounds of thermal power generation in plant l, $B^a_{k,k'}$ the bound on the deficit account for region k in region k', and $B^e_{k,k'}$ the bound on the energy exchange from region k to region k'. The first constraint in (3.36) characterizes the change of energy storage levels in each region k, the second constraint imposes the power generation-demand balance for each region k, and the rest are bounds on the decision variables. The state variables x_t (resp. x_n) are the energy storage levels $(l_{t,k})_{k\in\mathcal{K}}$ (resp. $(l_{n,k})_{k\in\mathcal{K}}$), while the internal variables consist of all the rest of decision variables. The initial state x_0 is given by data.

The energy inflow outcomes are sampled from multivariate lognormal distributions that

are interstage independent. Then the distributional uncertainty set is constructed using Wasserstein metric to reduce the effect of overtraining with the sampled outcome, according to [28]. To be precise, suppose $\hat{p}_t \in \Delta^N$ is an empirical distribution of outcomes $l_n \coloneqq (l_{n,k})_{k\in\mathcal{K}}$ for $n \in \mathcal{N}(t)$, where $N = \left|\mathcal{N}(t)\right|$ and often $\hat{p}_t = (1/N, \ldots, 1/N)$. Then, the distributional uncertainty set $\mathcal{P}_t$ is described by

$$\mathcal{P}_t \coloneqq \left\{ p_t \in \Delta^N : \rho(p_t, \hat{p}_t) \leq \sigma \right\}, \tag{3.37}$$

for some radius $\sigma \geq 0$, where the Wasserstein metric ρ for finitely supported distributions is defined by

$$\begin{aligned} \rho(p_t, \hat{p}_t) \coloneqq \min_{u_{m,n}\geq 0} \quad & \sum_{m,n\in\mathcal{N}(t)} \|l_m - l_n\| \, u_{m,n} \\ \text{s.t.} \quad & \sum_{n\in\mathcal{N}(t)} u_{m,n} = p_{t,m}, \qquad \forall\, m \in \mathcal{N}(t), \\ & \sum_{m\in\mathcal{N}(t)} u_{m,n} = \hat{p}_{t,n}, \qquad \forall\, n \in \mathcal{N}(t). \end{aligned} \tag{3.38}$$

Note when the radius $\sigma = 0$, the ambiguity set $\mathcal{P}_t$ becomes a singleton. In our numerical tests, we choose the radius to be relative to the total distances, i.e., $\sigma = \beta \cdot \sum_{m,n\in\mathcal{N}(t)} \|l_m - l_n\|$ for some $\beta \geq 0$. At the same time, we use uniform regularization factors for the tests, i.e., $M_t = M > 0$ for all $t \in \mathcal{T}$. When the relative optimality gap α is smaller than the threshold 5%, we check whether all the active cuts in the recent iterations are strictly smaller than the regularization factor. If they are, then the algorithm is terminated, and otherwise the regularization factor M is increased by a factor of $\sqrt{10} \approx 3.1623$ with all the over-approximations reset to $\underline{\mathcal{Q}}_t^i(x) \leftarrow +\infty$, $t \in \mathcal{T}$. Five scenarios are sampled independently in each stage for the nominal problem $N = 5$ before the distributional robust counterpart is constructed by (3.37). for the 24-stage problem that we consider, the samples already give a total $5^{24} \approx 5.9 \times 10^{16}$ scenario paths, which is practically impossible

to solve via an extensive robust formulation. We have then obtained the following results (Table 3.2) using our CDDP algorithm within a time limit of 5 hours.

Table 3.2: Comparison of CDDP with different regularization factors and ambiguity radii

β	$\log_{10}(M)$	LB ($\cdot 10^7$)	UB ($\cdot 10^7$)	Med. Time (s)	Med. `#Eval`
0.00	2	4.63	4.87	4411.02	35314
	3	4.62	4.86	3635.46	15012
	4	4.60	4.84	5947.57	18377
	5	4.60	4.84	7918.13	17795
0.02	2	4.88	5.14	3676.19	32111
	3	4.84	5.09	2642.95	14663
	4	4.85	5.10	5086.24	16480
	5	4.84	5.10	6333.10	16605
0.04	2	5.11	5.38	2976.62	28469
	3	5.11	5.38	4443.41	23952
	4	5.08	5.35	4757.78	15897
	5	5.09	5.35	5475.20	15465
0.06	2	5.37	5.66	2988.33	29978
	3	5.34	5.62	2000.84	12999
	4	5.33	5.61	3434.14	14401
	5	5.33	5.61	4717.09	14578
0.08	2	5.61	5.90	2642.00	28871
	3	5.59	5.88	1889.54	12243
	4	5.57	5.86	2645.60	13432
	5	5.57	5.86	3338.21	13653
0.10	2	5.85	6.16	2551.69	29052
	3	5.82	6.12	1260.03	10404
	4	5.81	6.11	1713.84	12433
	5	5.81	6.12	2677.05	12660

In Table 3.2, the lower bound (LB), the upper bound (UB) at termination, the computation time (Med. Time) and the number of subproblem oracles (Med. `#Eval`) shown are the median of the five test cases of the hydro-thermal power planning problems. The logarithmic regularization factors $\log_{10}(M)$ listed in the table correspond to the initial regularization factors. We see that for different choices of the relative radii β, the median computation time and number of subproblem oracle evaluations are usually smaller when

$\log_{10}(M) = 3$, without compromising the quality of upper and lower bounds. Moreover, for $\beta \leq 0.08$, the median computation times for $\log_{10}(M) = 2$ are still smaller than those of $\log_{10}(M) = 4$ or 5, despite the larger number of subproblem oracle evaluations. This can be explained by the better numerical conditions for the smaller regularization factors (the cuts have smaller Lipschitz constants), leading to shorter subproblem oracle evaluations times (cf. Algorithm 4). We thus conclude that the regularization technique could lead to smaller number of subproblem oracle evaluations, as well as shorter computation time for a given MDRO problem.

CHAPTER 4

DATA-DRIVEN CONVEX MDRO MODELS AND THEIR PERFORMANCE

4.1 Data-driven Model and Properties

4.1.1 Data-driven Model Formulation

In this section, we present a data-driven model for MDRO and some of its properties. Let $\mathcal{T} := \{1, \ldots, T\}$ denote the set of stage indices. In each stage $t \in \mathcal{T}$, we use $\mathcal{X}_t \subset \mathbb{R}^{d_t}$ to denote the convex state space and x_t its elements, which is known as the state vector. We denote the set of uncertainties before stage t as $\Xi_t \subseteq \mathbb{R}^{\delta_t}$ and its elements as ξ_t. For simplicity, we use the notation $\mathcal{X}_0 = \{x_0\}$ and $\Xi_1 = \{\xi_1\}$ to denote parameter sets of the given initial state. After each stage $t \in \mathcal{T} \setminus \{T\}$, the uncertainty ξ_t is assumed to be distributed according to an unknown probability measure p_t taken from a subset of the probability measures $\mathcal{P}_t \subset \mathcal{M}^{\mathrm{Prob}}(\Xi_t)$. The cost in each stage $t \in \mathcal{T}$ is described through a nonnegative, lower semicontinuous local cost function $f_t(x_{t-1}, x_t; \xi_t)$, that is assumed to be convex in x_{t-1} and x_t for every $\xi_t \in \Xi_t$. We allow f_t taking value of $+\infty$ to model constraints relating the states x_{t-1}, x_t and the uncertainty ξ_t. The MDRO can be written in a nested formulation as follows.

$$\begin{aligned}\inf_{x_1 \in \mathcal{X}_1} \quad f_1(x_0, x_1; \xi_1) + \sup_{p_2 \in \mathcal{P}_2} \mathbb{E}_{\xi_2 \sim p_2} \Bigg[\inf_{x_2 \in \mathcal{X}_2} f_2(x_1, x_2; \xi_2) + \\ + \sup_{p_3 \in \mathcal{P}_3} \mathbb{E}_{\xi_3 \sim p_3} \Bigg[\inf_{x_3 \in \mathcal{X}_3} f_3(x_2, x_3; \xi_3) + \cdots \\ + \sup_{p_T \in \mathcal{P}_T} \mathbb{E}_{\xi_T \sim p_T} \Big[\inf_{x_T \in \mathcal{X}_T} f_T(x_{T-1}, x_T; \xi_T) \Big] \cdots \Bigg] \Bigg].\end{aligned} \tag{4.1}$$

Here, $\mathbb{E}_{\xi_t \sim p_t}$ is the expectation with respect to variable ξ_t distributed according to the probability measure p_t. We remark that both the uncertainty sets Ξ_t and the ambiguity sets $\mathcal{P}_t$

are independent between stages, which are usually referred to as *stagewise independence*.

Based on the stagewise independence in the nested formulation, we can write the following recursion that is equivalent to (4.1) using the (worst-case expected) cost-to-go functions,

$$\mathcal{Q}_{t-1}(x_{t-1}) := \sup_{p_t \in \mathcal{P}_t} \mathbb{E}_{\xi_t \sim p_t} \left[\inf_{x_t \in \mathcal{X}_t} f_t(x_{t-1}, x_t; \xi_t) + \mathcal{Q}_t(x_t) \right], \tag{4.2}$$

for each $t \in \mathcal{T}$ and we set by convention $\mathcal{Q}_T(x_T) := 0$ for any $x_T \in \mathcal{X}_T$. To simplify the notation, we also define the following value functions for each stage $t \in \mathcal{T}$:

$$Q_t(x_{t-1}; \xi_t) := \inf_{x_t \in \mathcal{X}_t} f_t(x_{t-1}, x_t; \xi_t) + \mathcal{Q}_t(x_t). \tag{4.3}$$

Using these value functions, we may write the optimal value of the MDRO (4.1) as $Q_1(x_0; \xi_1)$ and further simplify the recursion (4.2) as

$$\mathcal{Q}_{t-1}(x_{t-1}) = \sup_{p_t \in \mathcal{P}_t} \int_{\Xi_t} Q_t(x_{t-1}; \xi_t) \, \mathrm{d}p_t(\xi_t). \tag{4.4}$$

While there are many different choices of the ambiguity set $\mathcal{P}_t$ for each stage $t \in \mathcal{T}$ (see e.g., [17]), we would like to focus on the data-driven ambiguity sets constructed as follows. Suppose we have the knowledge of n_t samples $\hat{\xi}_{t,1}, \ldots, \hat{\xi}_{t,n_t}$ of the uncertainty ξ_t. We can write $\hat{\nu}_t := \frac{1}{n_t} \sum_{k=1}^{n_t} \delta_{\hat{\xi}_{t,k}}$ for the empirical probability measure, where for each $k = 1, \ldots, n_t$, $\delta_{\hat{\xi}_{t,k}}$ is the Dirac probability measure supported at the point $\hat{\xi}_{t,k} \in \Xi_t$, i.e., $\int_{\Xi_t} f \, \mathrm{d}\delta_{\hat{\xi}_{t,k}} = f(\hat{\xi}_{t,k})$ for any compactly supported function f on Ξ_t. Such empirical probability measure $\hat{\nu}_t$ captures the information from the sample data and is often used to build the sample average approximation for multistage stochastic optimization [12].

Fix any distance function $d_t(\cdot, \cdot)$ on Ξ_t, the Wasserstein (1-)distance (a.k.a, Kantorovich-Rubinstein distance) is defined as

$$W_t(\mu, \nu) := \inf_{\pi \in \mathcal{M}^{\mathrm{Prob}}(\Xi_t \times \Xi_t)} \left\{ \int_{\Xi_t \times \Xi_t} d_t(\xi^1, \xi^2) \, \mathrm{d}\pi(\xi^1, \xi^2) : P^1_*(\pi) = \mu, \, P^2_*(\pi) = \nu \right\}, \tag{4.5}$$

for any two probability measures $\mu, \nu \in \mathcal{M}^{\mathrm{Prob}}(\Xi_t)$, where $P^i_*(\pi)$ is the pushforward measure induced by the projection maps $P^i : \Xi_t \times \Xi_t \to \Xi_t$ by sending $P^i(\xi^1, \xi^2) = \xi^i$, for $i = 1$ or 2. In plain language, the joint probability measure π in (4.5) should have marginal probability measures equal to the given ones μ and ν.

It can be shown that W_t is indeed a distance on the space of probability measures $\mathcal{M}^{\mathrm{Prob}}(\Xi_t)$ [81, Definition 6.1] except that it may take the value of $+\infty$. Thus it is natural to restrict our attention to the convex subset of probability measures with finite distance to a Dirac measure on Ξ_t

$$\mathcal{W}_t := \left\{ \mu \in \mathcal{M}^{\mathrm{Prob}}(\Xi_t) : \int_{\Xi_t} d_t(\bar{\xi}, \xi) \, \mathrm{d}\mu(\xi) < +\infty, \text{ for some } \bar{\xi} \in \Xi_t \right\}. \tag{4.6}$$

Note that any continuous function $g(\xi)$ that satisfies $|g(\xi)| \leq C(1 + d_t(\bar{\xi}, \xi))$ for some $C > 0$ and $\bar{\xi} \in \Xi_t$ would be integrable for any probability measure in $\mathcal{W}_t$. Now given any such continuous functions $g_{t,1}, \ldots, g_{t,m_t}$ on Ξ_t and a real vector $\rho_t := (\rho_{t,j})_{t=0}^{m_t} \in \mathbb{R}^{m_t+1}$, we define the Wasserstein ambiguity set $\mathcal{P}_t$ as

$$\mathcal{P}_t := \left\{ p \in \mathcal{W}_t : W_t(p, \hat{\nu}_t) \leq \rho_{t,0}, \ \langle g_{t,j}, p \rangle \leq \rho_{t,j}, \ j = 1, \ldots, m_t \right\}. \tag{4.7}$$

The first inequality constraint in the definition (4.7) bounds the Wasserstein distance of the probability measure $p \in \mathcal{W}_t$ from the empirical measure $\hat{\nu}_t$, while the rest are structural constraints on p, e.g., bounds on the moments, that are known beforehand. We next show some favorable properties of the Wasserstein ambiguity sets (4.7) that are analogous to the single-stage or the two-stage cases [26, 27].

With the choice of the Wasserstein ambiguity set $\mathcal{P}_t$, the recursion (4.4) can be written

explicitly as

$$\mathcal{Q}_{t-1}(x_{t-1}) = \sup_{p_t \in \mathcal{P}_t} \int_{\Xi_t} Q_t(x_{t-1}; \xi_t) dp_t(\xi_t) \tag{4.8a}$$

$$\text{s.t.} \inf_{\pi \in \mathcal{M}^{\text{Prob}}(\Xi_t \times \Xi_t)} \left\{ \int_{\Xi_t \times \Xi_t} d_t(\xi^1, \xi^2) \, \mathrm{d}\pi(\xi^1, \xi^2) : P_*^1(\pi) = p_t, \; P_*^2(\pi) = \hat{\nu}_t \right\} \leq \rho_{t,0}, \tag{4.8b}$$

$$\int_{\Xi_t} g_{t,j}(\xi_t) dp_t(\xi_t) \leq \rho_{t,j}, \quad \forall j = 1, \ldots, m_t, \tag{4.8c}$$

where (4.8b) can be equivalently written as

$$\begin{aligned} &\forall \varepsilon > 0, \; \exists \, \pi \in \mathcal{M}^{\text{Prob}}(\Xi_t \times \Xi_t), \\ &\int_{\Xi_t \times \Xi_t} d_t(\xi^1, \xi^2) \, \mathrm{d}\pi(\xi^1, \xi^2) \leq \rho_{t,0} + \varepsilon, \; P_*^1(\pi) = p_t, \; P_*^2(\pi) = \hat{\nu}_t. \end{aligned} \tag{4.9}$$

4.1.2 Finite Dimensional Dual Recursion

The arguably most important property of Wasserstein ambiguity sets is that they allow finite dimensional dual reformulation, which often leads to computational tractability in many practical cases. We briefly review the duality result and derive the dual reformulation for the recursion (4.2).

Generalized Slater Condition and Lagrangian Duality

Given an $\mathbb{R}$-vector space $\mathcal{M}$, we consider the following problem.

$$\begin{aligned} v^{\text{P}} := \inf_{\mu \in C} \quad & \varphi_0(\mu) \\ \text{s.t.} \quad & \varphi_j(\mu) \leq 0, \quad j = 1, \ldots, l, \\ & \varphi_j(\mu) = 0, \quad j = l+1, \ldots, m. \end{aligned} \tag{4.10}$$

Here, $C \subset \mathcal{M}$ is a convex subset, the functions $\varphi_j : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ are convex for each $j = 0, 1, \ldots, l$ and $\varphi_j : \mathcal{M} \to \mathbb{R}$ are affine for each $j = l+1, \ldots, m$. Using a vector of multipliers $\lambda \in \mathbb{R}^m$, the Lagrangian dual problem of (4.10) can be written as

$$v^{\mathrm{D}} := \sup_{\lambda \in \Lambda} \inf_{\mu \in C} \left\{ \varphi_0(\mu) + \sum_{j=1}^{m} \lambda_j \varphi_j(\mu) \right\}, \tag{4.11}$$

where the admissible set for the multipliers is defined as $\Lambda := \{\lambda \in \mathbb{R}^m : \lambda_j \geq 0, \forall\, j = 1, \ldots, l\}$. We want to show the strong duality between (4.10) and (4.11), given the following condition.

Definition 4.1. *We say that the problem* (4.10) *satisfies the (generalized) Slater condition if the point $\eta = 0$ is in the relative interior of the effective domain of the convex value function associated with the primal problem* (4.10) *defined for $\eta \in \mathbb{R}^m$:*

$$v(\eta) := \inf_{\mu \in C} \left\{ \varphi_0(\mu) : \varphi_j(\mu) = \eta_j,\ j = 1, \ldots, l,\ \textit{and}\ \varphi_j(\mu) \leq \eta_j,\ j = l+1, \ldots, m \right\}.$$

Recall that the effective domain of a convex function $v : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ is defined as $\operatorname{dom} v := \{\eta \in \mathbb{R}^m : v(\eta) < +\infty\}$, which is clearly a convex set. The affine hull of a convex set $K \subset \mathbb{R}^m$ is defined to be the smallest affine space containing K, and the relative interior of K is the interior of K viewed as a subset of its affine hull (equipped with the subspace topology). By convention, we have $v(\eta) = +\infty$ if there is no $\mu \in C$ such that $\varphi_j(\mu) \leq \eta_j$ for all $j = 1, \ldots, m$.

Proposition 4.1. *Assuming the Slater condition, the strong duality holds $v^{\mathrm{P}} = v^{\mathrm{D}}$ with an optimal dual solution $\lambda^* \geq 0$ (i.e., the supremum in the dual problem* (4.11) *is attained).*

Proof. The weak duality $v^{\mathrm{P}} \geq v^{\mathrm{D}}$ holds with a standard argument of exchanging the $\inf$ and $\sup$ operators, so it suffices to show that $v^{\mathrm{P}} \leq v^{\mathrm{D}}$. If $v^{\mathrm{P}} = -\infty$ then the inequality holds trivially, so we assume that $v^{\mathrm{P}} > -\infty$. Given the Slater condition, the value function $v(\eta)$ of the primal problem (4.10) must be proper $v(\eta) > -\infty$ for all $\eta \in \mathbb{R}^m$ (ref. Theorem

7.2 in [82]) because $\eta = 0$ is in the relative interior of the effective domain of v and $v(0) > -\infty$. Thus it is also subdifferentiable at the point $\eta = 0$ (ref. Theorem 23.4 in [82]), i.e., there exists a subgradient vector $\lambda^* \in \mathbb{R}^m$ such that $v(\eta) \geq v(0) - (\lambda^*)^{\mathsf{T}}\eta$ for any $\eta \in \mathbb{R}^m$. Here, for each $j = 1, \ldots, l$, the multiplier λ_j^* must be nonnegative since the function $v(\eta)$ is not increasing in the j-th component, so we have $\lambda \in \Lambda$. Since the inequality $v(\eta) + (\lambda^*)^{\mathsf{T}}\eta \geq v(0) = v^{\mathrm{P}}$ holds for any $\eta \in \mathbb{R}^m$, we have

$$
\begin{aligned}
v^{\mathrm{P}} &\leq \inf_{\eta \in \mathbb{R}^m} \{v(\eta) + (\lambda^*)^{\mathsf{T}}\eta\} \\
&= \inf_{\mu \in C} \inf_{\eta \in \mathbb{R}^m} \left\{ \varphi_0(\mu) + \sum_{j=1}^{m} \lambda_j^* \eta_j : \begin{array}{l} \varphi_j(\mu) \leq \eta_j,\ j = 1, \ldots, l, \\ \varphi_j(\mu) = \eta_j,\ j = l+1, \ldots, m \end{array} \right\} \\
&= \inf_{\mu \in C} \left\{ \varphi_0(\mu) + \sum_{j=1}^{m} \lambda_j^* \varphi_j(\mu) \right\} \leq v^{\mathrm{D}}.
\end{aligned}
$$

The first equality here results from exchanging two infimum operators, while the second one follows by taking $\eta_j = \varphi_j(\mu)$, due to the nonnegativity of λ_j^* for each $j = 1, \ldots, l$, and replacing η_j with $\varphi_j(\mu)$ for each $j = l+1, \ldots, m$. □

The strong Lagrangian duality guaranteed by the Slater condition is useful for many applications because we do not have to specify the topology on the vector space $\mathcal{M}$. The corollary below summarizes a special case where there is no equality constraint, and all the inequality constraints can be strictly satisfied.

Corollary 4.1. *For problems* (4.10) *and* (4.11) *with $l = m$ (no equality constraints), the strong duality holds if there exists a point $\bar{\mu} \in C$ such that $\varphi_j(\bar{\mu}) < 0$, for each $j = 1, \ldots, m$.*

Proof. Let $\varepsilon_j := -\varphi_j(\mu) > 0$ for $j = 1, \ldots, m$, and $U := \prod_{j=1}^{m} (-\frac{\varepsilon_j}{2}, \frac{\varepsilon_j}{2}) \subset \mathbb{R}^m$ be an open hyperrectangle. Then for any $\eta \in U$, we have $v(\eta) \leq \varphi_0(\bar{\mu}) < +\infty$. Therefore, we know that the Slater condition holds because $0 \in U \subset \operatorname{dom} v$, and the result follows from

Proposition 4.1. □

Another interesting case is summarized below where the Slater condition can be applied to problems with equality constraints, assuming the vector space $\mathcal{M}$, equipped with a norm $\|\cdot\|$, is complete. This condition could be, however, much harder to check. For example, in the space of finite signed regular Borel measures on a set $\Xi \subseteq \mathbb{R}^\delta$, an empirical probability measure is usually not in the relative interior of the cone of nonnegative measures (with respect to the subspace $\{\mu : \mu(\Xi) = 1\}$), for the total variation norm. Thus we do not base our discussion on it.

Corollary 4.2. *Suppose* $(\mathcal{M}, \|\cdot\|)$ *is a Banach space and* φ_j *is continuous for each* $j = 1, \ldots, m$. *For problems* (4.10) *and* (4.11)*, the strong duality holds if there exists a point* $\bar{\mu} \in \operatorname{int} C$ *with* $\varphi_j(\bar{\mu}) < 0$ *for each* $j = 1, \ldots, l$ *and* $\varphi_j(\bar{\mu}) = 0$ *for each* $j = l+1, \ldots, m$.

Proof. Since φ_j with $j = l+1, \ldots, m$ are continuous affine functions, they can be translated to a continuous linear map. More specifically, let $\Phi : \mathcal{M} \to \mathbb{R}^{m-l}$ be a linear map that sends $\mu \mapsto (\varphi_{l+1}(\mu), \ldots, \varphi_m(\mu))$ and $L := \operatorname{im} \Phi \subset \mathbb{R}^{m-l}$ its image subspace, which is complete due its finite dimension. Since Φ is continuous and surjective onto its image L, it is an open map onto L (ref. Theorem 2.6 in [83]). Now let $\varepsilon_j := -\varphi_j(\mu) > 0$ for $j = 1, \ldots, l$, and $U := \prod_{j=1}^{l} (-\frac{\varepsilon_j}{2}, \frac{\varepsilon_j}{2}) \subset \mathbb{R}^l$ be an open hyperrectangle. By the continuity of φ_j for $j = 1, \ldots, l$, there exists an open neighborhood $V \subset \mathcal{M}$ of $\bar{\mu} \in V$ such that $\varphi_j(\mu) < -\frac{\varepsilon_j}{2}$ for all $j = 1, \ldots, l$. Then $U \times \Phi(V)$ is an open subset of the product space $\mathbb{R}^l \times L$, which is the affine hull of the effective domain of the value function v. Thus the Slater condition holds and the result follows from Proposition 4.1. □

Dual Recursion for Wasserstein MDRO

Now recall that by the definition of Wasserstein distance (4.5), the constraint $W_t(p, \hat{\nu}_t) \leq \rho_{t,0}$ is ensured if there exists a probability measure on the product space $\pi_t \in \mathcal{M}^{\mathrm{Prob}}(\Xi_t \times$

Ξ_t) with marginal probability measures $P_*^1(\pi_t) = p$ and $P_*^2(\pi_t) = \hat{\nu}_t$, such that

$$\int_{\Xi_t \times \Xi_t} d_t \, \mathrm{d}\pi_t = \int_{\Xi_t} \int_{\Xi_t} d_t(\xi, \xi') \, \mathrm{d}\pi_t(\xi|\xi') \, \mathrm{d}\hat{\nu}_t(\xi') = \frac{1}{n_t} \sum_{k=1}^{n_t} \int_{\Xi_t} d_{t,k}(\xi) \, \mathrm{d}p_{t,k}(\xi) \leq \rho_{t,0}, \tag{4.12}$$

where we define $p_{t,k}(\cdot) := \pi_t(\cdot|\hat{\xi}_{t,k}) \in \mathcal{W}_t$ to be the probability measure conditioned on $\{\xi' = \hat{\xi}_{t,k}\}$ and $d_{t,k}(\xi) := d(\xi, \hat{\xi}_{t,k})$ for any $\xi \in \Xi_t$. Then by the law of total probability, we have

$$p_t = \frac{1}{n_t} \sum_{k=1}^{n_t} p_{t,k}. \tag{4.13}$$

Due to this condition, we define the following parametrized optimization problem for any $\rho > 0$ and any state $x_{t-1} \in \mathcal{X}_{t-1}$,

$$\begin{aligned} q_t(x_{t-1}; \rho) := \sup_{p_{t,k} \in \mathcal{W}_t} \quad & \frac{1}{n_t} \sum_{k=1}^{n_t} \int_{\Xi_t} Q_t(x_{t-1}; \xi_t) \, \mathrm{d}p_{t,k}(\xi_t) && \text{(4.14a)} \\ \text{s.t.} \quad & \frac{1}{n_t} \sum_{k=1}^{n_t} \int_{\Xi_t} d_{t,k}(\xi_t) \, \mathrm{d}p_{t,k}(\xi_t) \leq \rho, && \text{(4.14b)} \\ & \frac{1}{n_t} \sum_{k=1}^{n_t} \int_{\Xi_t} g_{t,j}(\xi_t) \, \mathrm{d}p_{t,k}(\xi_t) \leq \rho_{t,j}, \quad j = 1, \ldots, m_t. && \text{(4.14c)} \end{aligned}$$

In the following, we establish some simple properties of $q_t(x_{t-1}; \rho)$ and prove the equality $q_t(x_{t-1}; \rho_{t,0}) = \mathcal{Q}_{t-1}(x_{t-1})$ and eventually establish the dual recursion for $\mathcal{Q}_t$ using strong Lagrangian duality. First, we prove the following lemma.

Lemma 4.1. *$\mathcal{Q}_{t-1}(x_{t-1}) \geq q_t(x_{t-1}; \rho_{t,0})$, for any $x_{t-1} \in \mathcal{X}_{t-1}$.*

Proof. Comparing with (4.8) and (4.9), we see that if $p_{t,k}$ is a feasible solution of (4.14) with $\rho = \rho_{t,0}$, then it can be used to form p_t as in (4.13) such that p_t is feasible to (4.8c) and makes the objectives (4.8a) and (4.8b) identical. Moreover, the infimum in (4.8b) can be achieved by $\pi := \frac{1}{n_t} \sum_{k=1}^{n_t} p_{t,k} \delta_{\hat{\xi}_{t,k}}$ and (4.8b) is satisfied due to (4.14b). Therefore, the

desired inequality follows. □

We next show that the reverse inequality, thus the equality, indeed holds under a Slater condition (Assumption 4.1) for (4.14).

Assumption 4.1. *The empirical probability measure $\hat{\nu}_t$ satisfies the constraints $\langle g_{t,j}, \hat{\nu}_t \rangle < \rho_{t,j}$ for all $j = 1, \ldots, m_t$. Equivalently, we assume that $\frac{1}{n_t} \sum_{i=1}^{n_t} g_{t,j}(\hat{\xi}_{t,k}) < \rho_{t,j}$, for all $j = 1, \ldots, m_t$.*

Note that the solution $p_{t,k} := \delta_{\hat{\xi}_{t,k}}$ for $k = 1, \ldots, n_t$ satisfies (4.14b) with $\rho = 0$. Therefore, Assumption 4.1 is equivalent to assuming that (4.14) has a strictly feasible solution, which is sufficient to prove the following strong duality for (4.14).

Lemma 4.2. *Under Assumption 4.1, the Lagrangian dual problem of* (4.14) *satisfies strong duality*

$$
\begin{aligned}
& q_t(x_{t-1}; \rho) \qquad (4.15)\\
&= \min_{\lambda \geq 0} \left\{ \rho\lambda_0 + \sum_{j=1}^{m_t} \rho_{t,j}\lambda_j + \frac{1}{n_t} \sum_{k=1}^{n_t} \sup_{p_{t,k} \in \mathcal{W}_t} \int_{\Xi_t} \left[Q_t(x_{t-1}; \cdot) - \lambda_0 d_{t,k} - \sum_{j=1}^{m_t} \lambda_j g_{t,j} \right] \mathrm{d}p_{t,k} \right\} \\
&= \min_{\lambda \geq 0} \left\{ \rho\lambda_0 + \sum_{j=1}^{m_t} \rho_{t,j}\lambda_j + \frac{1}{n_t} \sum_{k=1}^{n_t} \sup_{\xi_k \in \Xi_t} \left\{ Q_t(x_{t-1}; \xi_k) - \lambda_0 d_{t,k}(\xi_k) - \sum_{j=1}^{m_t} \lambda_j g_{t,j}(\xi_k) \right\} \right\}.
\end{aligned}
$$

Proof. The first equation follows from Corollary 4.1 in the Appendix. The second equality holds because each Dirac measure δ_ξ centered at $\xi \in \Xi_t$ satisfies $\delta_\xi \in \mathcal{W}_t$ and each $p_{t,k} \in \mathcal{W}_t$ is a probability measure. □

We need one more property of $q_t(x_{t-1}; \rho)$ before we can show $\mathcal{Q}_{t-1}(x_{t-1}) = q_t(x_{t-1}; \rho_{t,0})$.

Lemma 4.3. *Under Assumption 4.1, $q_t(x_{t-1}; \rho)$ is a concave function, thus continuous, in $\rho \in (0, +\infty)$, for any fixed $x_{t-1} \in \mathcal{X}_{t-1}$.*

Proof. Fix any $x_{t-1} \in \mathcal{X}_{t-1}$. The concavity of $q_t(x_{t-1}; \rho)$ follows directly from (4.15) in Lemma 4.2, since $q_t(x_{t-1}; \cdot)$ is a minimum of affine functions. Moreover, from the

definition (4.14) we see that $q_t(x_{t-1};\rho) \geq 0$ for any $\rho > 0$ as the measures $p_{t,k} = \delta_{\hat{\xi}_{t,k}}$ satisfy the constraints and $Q_t(x_{t-1};\xi_t) \geq 0$ by the nonnegativity of the cost functions f_t. If $q_t(x_{t-1};\rho_{t,0}) = +\infty$, then the equality holds trivially as we already showed that $\mathcal{Q}_{t-1}(x_{t-1}) \geq q_t(x_{t-1};\rho_{t,0})$. Otherwise, we must have $q_t(x_{t-1};\rho) < +\infty$ for any $\rho > 0$ due to the concavity. Thus $q_t(x_{t-1};\cdot)$ is a continuous function on $(0,+\infty)$. □

Now we can prove $\mathcal{Q}_{t-1}(x_{t-1}) = q_t(x_{t-1};\rho_{t,0})$ and thus give a reformulation of (4.4).

Theorem 4.1. *Under Assumption 4.1, in any stage $t \geq 2$, the expected cost-to-go function* (4.2) *satisfies $\mathcal{Q}_{t-1}(x_{t-1}) = q_t(x_{t-1};\rho_{t,0})$ and thus* (4.4) *can be equivalently rewritten as*

$$\mathcal{Q}_{t-1}(x_{t-1}) = \min_{\lambda\geq 0}\left\{\sum_{j=0}^{m_t}\rho_{t,j}\lambda_j + \frac{1}{n_t}\sum_{k=1}^{n_t}\sup_{\xi_k\in\Xi_t}\left[Q_t(x_{t-1};\xi_k) - \lambda_0 d_{t,k}(\xi_k) - \sum_{j=1}^{m_t}\lambda_j g_{t,j}(\xi_k)\right]\right\}. \tag{4.16}$$

Proof. Due to Lemma 4.1, we only need to prove $\mathcal{Q}_{t-1}(x_{t-1}) \leq q_t(x_{t-1};\rho_{t,0})$. From (4.9), we know that the constraint $W_t(p,\hat{\nu}_t) \leq \rho_{t,0}$ implies that, for any $\varepsilon > 0$, there exists a probability measure $\pi_t \in \mathcal{M}^{\mathrm{Prob}}(\Xi_t \times \Xi_t)$ with marginal probability measures $P^1_*(\pi_t) = p_t$ and $P^2_*(\pi_t) = \hat{\nu}_t$ such that $\int d_t \,\mathrm{d}\pi_t \leq \rho_{t,0} + \varepsilon$. Such π and p_t are feasible for (4.14). Therefore, $\mathcal{Q}_{t-1}(x_{t-1}) \leq q_t(x_{t-1};\rho_{t,0}+\varepsilon)$. Now by the continuity of $q_t(x_{t-1};\cdot)$ proved in Lemma 4.3 under Assumption 4.1, we conclude that $\mathcal{Q}_{t-1}(x_{t-1}) \leq \lim_{\varepsilon\to 0+} q_t(x_{t-1};\rho_{t,0}+\varepsilon) = q_t(x_{t-1};\rho_{t,0})$. Thus, (4.16) follows from Lemma 4.2. □

As a corollary, we can easily prove a special version of the famous Kantorovich-Rubinstein duality formula [81, Remark 6.5].

Corollary 4.3. *Under Assumption 4.1, if the value function $Q_t(x_{t-1};\xi_t)$ is l_t-Lipschitz continuous in the uncertainty $\xi_t \in \Xi_t$ for any $x_{t-1} \in \mathcal{X}_{t-1}$, then we have*

$$\mathcal{Q}_{t-1}(x_{t-1}) \leq \rho_{t,0} l_t + \frac{1}{n_t}\sum_{k=1}^{n_t} Q_t(x_{t-1};\hat{\xi}_{t,k}).$$

Proof. Take a feasible solution $\lambda_0 = l_t$ and $\lambda_j = 0$ for $j = 1, \ldots, m_t$ in Theorem 4.1. Note that by the Lipschitz continuity assumption, the supremum is attained at $\hat{\xi}_{t,k}$ for each $k = 1, \ldots, n_t$. Thus we have the desired inequality. □

4.1.3 Out-of-Sample Performance Guarantee

A major motivation for using distributionally robust models over stochastic models is the out-of-sample performance guarantee, which ensures that the decisions evaluated on the true probability distribution would perform no worse than the in-sample training with high probability. To begin with, we say that a probability measure $\mu \in \mathcal{W}_t$ is sub-Gaussian if $\int_{\Xi_t} \exp(C\xi^2)\,\mathrm{d}\mu(\xi) < +\infty$ for some constant $C > 0$, or it has finite skewness if $\int_{\Xi_t} |\xi|^3\,\mathrm{d}\mu(\xi) < +\infty$. Our discussion is based on the following specialized version of the well-known measure concentration inequality [24, Theorem 2].

Theorem 4.2. *Fix any probability measure $\nu_t \in \mathcal{M}^{\mathrm{Prob}}(\Xi_t)$ in stage t and let $\hat{\nu}_t$ denote the empirical measure from n_t iid samples of ν_t. Then for any $\rho_{t,0} > 0$, we have*

$$\mathbb{P}\left(W_t(\nu_t, \hat{\nu}_t) > \rho_{t,0}\right) \leq \begin{cases} C_t \exp\left(-C_t' n_t \rho_{t,0}^{\max\{\delta_t, 2\}}\right), & \text{if } \nu_t \text{ is sub-Gaussian and } \delta_t \neq 2, \quad (4.17) \\ C_t''(n_t \rho_{t,0}^2)^{-1}, & \text{if } \nu_t \text{ has finite skewness,} \quad (4.18) \end{cases}$$

for some positive constants $C_t, C_t', C_t'' > 0$ that depend only on ν_t.

The measure concentration bound in (4.17) becomes slightly more intricate when the dimension of the uncertainty $\delta_t = 2$ (see the details in [24]), so we focus our discussion below on the other cases that can be adapted to the 2-dimensional case with little effort.

The out-of-sample performance refers to the evaluation of the solutions and policies obtained from solving MDRO (4.1) on the true probability measures ν_t for each $t \in \mathcal{T}$. To be precise, consider the multistage stochastic convex optimization (MSCO) problem

defined by the following recursion (cf. (4.2)) for $t \in \mathcal{T}$

$$\mathcal{Q}_{t-1}^{\mathrm{Stoch}}(x_{t-1}) := \int_{\Xi_t} Q_t^{\mathrm{Stoch}}(x_{t-1}; \xi_t) \, \mathrm{d}\nu_t(\xi_t), \tag{4.19}$$

where

$$Q_t^{\mathrm{Stoch}}(x_{t-1}; \xi_t) := \inf_{x_t \in \mathcal{X}_t} f_t(x_{t-1}, x_t; \xi_t) + \mathcal{Q}_t^{\mathrm{Stoch}}(x_t), \tag{4.20}$$

and $\mathcal{Q}_T^{\mathrm{Stoch}}(x_T) = 0$ for any $x_T \in \mathcal{X}_T$. That is, the MSCO uses singleton ambiguity sets $\mathcal{P}_t^{\mathrm{Stoch}} := \{\nu_t\}$ in the places of $\mathcal{P}_t$ in the MDRO (4.1). The next theorem characterizes the probabilistic bound with which the decisions and policies made according to $\mathcal{Q}_t$ in the MSCO (4.19) would perform no worse than our MDRO (4.2).

Theorem 4.3. *Fix any probability measure $\nu_t \in \mathcal{W}_t$ and let $\hat{\nu}_t$ denote the empirical measure from n_t iid samples of ν_t for all stages $t \in \mathcal{T}$. Assume that $\langle g_{t,j}, \nu_t \rangle \leq \rho_{t,j}$ for any $t \in \mathcal{T}$ and $j = 1, \ldots, m_t$. Then for any $\alpha \in (0,1)$, we have $\mathcal{Q}_t^{\mathrm{Stoch}}(x_t) \leq \mathcal{Q}_t(x_t)$ and $Q_t^{\mathrm{Stoch}}(x_t; \xi_t) \leq Q_t(x_t; \xi_t)$ for any $x_t \in \mathcal{X}_t$, $\xi_t \in \Xi_t$, $t \in \mathcal{T}$ with probability at least α if either of the following conditions holds for each $t \in \mathcal{T}$:*

1. *the probability measure ν_t is sub-Gaussian, $\delta_t \neq 2$, and*

$$n_t \cdot \rho_{t,0}^{\max\{\delta_t, 2\}} \geq \frac{1}{C_t'} \left[\ln C_t - \ln \left(1 - \alpha^{1/(T-1)} \right) \right],$$

2. *the probability measure ν_t has finite skewness and*

$$n_t \cdot \rho_{t,0}^2 \geq \frac{C_t''}{1 - \alpha^{1/(T-1)}},$$

where C_t, C_t', and C_t'' are the positive constants in Theorem 4.2, that depend only on ν_t.

Proof. If either of the conditions is satisfied, then it is straightforward to check from Theorem 4.2 that the probability $\mathbb{P}(W_t(\nu_t, \hat{\nu}_t) > \rho_{t,0}) \leq 1 - \alpha^{1/(T-1)}$. By the assumption on the iid sampling of $\hat{\nu}_t$ and that $\langle g_{t,j}, \nu_t \rangle \leq \rho_{t,j}$ for all $j = 1, \ldots, m_t$, the event

$E := \{\nu_t \in \mathcal{P}_t \text{ for all } t \in \mathcal{T}\}$ has the probability

$$\mathbb{P}(E) = \mathbb{P}\{W_t(\nu_t, \hat{\nu}_t) \leq \rho_{t,0} \text{ for all } t \in \mathcal{T}\} = \prod_{t=2}^{T} \mathbb{P}(W_t(\nu_t, \hat{\nu}_t) \leq \rho_{t,0}) \geq \alpha.$$

Note that on this event E, we have $\mathcal{P}_t^{\text{Stoch}} \subseteq \mathcal{P}_t$ for each $t \in \mathcal{T}$. Thus the theorem follows directly from the recursions (4.2) and (4.19). □

While Theorem 4.3 is a direct consequence of Theorem 4.2, it shows some interesting aspects of MDRO using Wasserstein ambiguity sets. First, to get a certain probabilistic bound for the out-of-sample performance guarantee, we may need to increase the number of samples n_t or the Wasserstein distance bound $\rho_{t,0}$ for a larger number of stages T. Second, for probability measures that are not sub-Gaussian (or more generally those with heavy tails, see [26]), for which we would like to apply the second condition, such increase require the product $n_t \cdot \rho_{t,0}^2$ to grow approximately on the order of $\mathcal{O}(T)$ when α is close to 1. Therefore, it is sometimes useful to take larger values of $\rho_{t,0}$ for out-of-sample performance guarantees, especially when the number of sample n_t is limited, or when the true probability measures ν_t are not sub-Gaussian.

4.1.4 Adjustable In-Sample Conservatism

Another well-studied approach to guarantee out-of-sample performance is the multistage robust convex optimization (MRCO) model [39]. However, MRCO considers only the worst-case outcomes of the uncertainties and thus can be overly conservative. To be precise, we define the nominal MSCO from the empirical measures $\hat{\nu}_t$ by the following recursion

$$\mathcal{Q}_{t-1}^{\text{Nomin}}(x_{t-1}) := \int_{\Xi_t} Q_t^{\text{Nomin}}(x_{t-1}; \xi_t) \, \mathrm{d}\hat{\nu}_t(\xi_t) = \frac{1}{n_t} \sum_{k=1}^{n_t} Q_t^{\text{Nomin}}(x_{t-1}; \hat{\xi}_{t,k}), \tag{4.21}$$

where

$$Q_t^{\text{Nomin}}(x_{t-1}; \xi_t) := \inf_{x_t \in \mathcal{X}_t} f_t(x_{t-1}, x_t; \xi_t) + \mathcal{Q}_t^{\text{Nomin}}(x_t), \tag{4.22}$$

and $\mathcal{Q}_T^{\mathrm{Nomin}}(x_T) = 0$ for any $x_T \in \mathcal{X}_T$. Any MRCO that is built directly from data could have a much larger optimal cost than the nominal MSCO with a probability growing with the numbers of samples n_t and stages T, as illustrated by the following example.

Example 4.1. *Consider an MSCO with local cost functions $f_t(x_{t-1}, x_t; \xi_t) := x_t + \xi_t$ and state spaces $\mathcal{X}_t := [0, 1] \subseteq \mathbb{R}$ for all $t \in \mathcal{T}$. For each $t \geq 2$, the uncertainties are described by a probability measure ν_t on the set $\Xi_t := \mathbb{R}_{\geq 0}$ such that for any $C > 0$, we have $\nu_t(C, +\infty) > 0$. By taking solutions $x_t = 0$ in each stage $t \geq 2$, it is easy to see that the optimal value $Q_1^{\mathrm{Stoch}}(x_0; \xi_1) = x_0 + \xi_1 + \sum_{t=2}^{T} \mathbb{E}[\xi_t]$ assuming the expectations exist.*

Now to approximate this MSCO, suppose we are given iid samples $\hat{\xi}_{t,1}, \ldots, \hat{\xi}_{t,n_t}$ from the probability measure ν_t in each stage $t \geq 2$. Then any MRCO defined by the following recursion of expected cost-to-go functions

$$\mathcal{Q}_{t-1}^{\mathrm{Robust}}(x_{t-1}) := \sup_{\xi_t \in \hat{\Xi}_t} \min_{x_t \in \mathcal{X}_t} f_t(x_{t-1}, x_t; \xi_t) + \mathcal{Q}_t^{\mathrm{Robust}}(x_t), \quad \forall\, t \geq 2,$$

where $\hat{\Xi}_t \supseteq \{\hat{\xi}_{t,1}, \ldots, \hat{\xi}_{t,n_t}\}$ is an uncertainty subset constructed from the data, would have its optimal value $Q_1^{\mathrm{Robust}}(x_0; \xi_1) \geq x_0 + \xi_1 + \sum_{t=2}^{T} \hat{\xi}_t^{\max}$, where $\hat{\xi}_t^{\max} := \max\{\hat{\xi}_{t,1}, \ldots, \hat{\xi}_{t,n_t}\}$. The optimal value of the corresponding nominal MSCO is $Q_1^{\mathrm{Nomin}}(x_0; \xi_1) = x_0 + \xi_1 + \sum_{t=2}^{T} \hat{\xi}_t^{\mathrm{mean}}$ where $\hat{\xi}_t^{\mathrm{mean}} := \frac{1}{n_t} \sum_{k=1}^{n_t} \hat{\xi}_{t,k}$. Therefore, for any constant $C > \max_{t=2,\ldots,T} \mathbb{E}[\xi_t]$,

we have

$$\begin{aligned}
&\mathbb{P}\left\{Q_1^{\mathrm{Robust}}(x_0;\xi_1) - Q_1^{\mathrm{Nomin}}(x_0;\xi_1) > C\right\} \\
&\geq 1 - \prod_{t=2}^{T} \mathbb{P}\left\{\hat{\xi}_t^{\max} - \hat{\xi}_t^{\mathrm{mean}} \leq C\right\} \\
&\geq 1 - \prod_{t=2}^{T} \left(1 - \mathbb{P}\left\{\hat{\xi}_t^{\max} > 2C,\ \hat{\xi}_t^{\mathrm{mean}} \leq C\right\}\right) \\
&\geq 1 - \prod_{t=2}^{T} \left(2 - \mathbb{P}\left\{\hat{\xi}_t^{\max} > 2C\right\} - \mathbb{P}\left\{\hat{\xi}_t^{\mathrm{mean}} \leq C\right\}\right) \\
&= 1 - \prod_{t=2}^{T} \left((\nu_t[0,2C])^{n_t} + \mathbb{P}\left\{\hat{\xi}_t^{\mathrm{mean}} > C\right\}\right),
\end{aligned}$$

where $\nu_t[0,2C] < 1$ *by assumption. Using the law of large numbers, we see that* $\mathbb{P}\{\hat{\xi}_t^{\mathrm{mean}} > C\} \to 0$ *as* $n_t \to \infty$. *In fact, if the probability measures* ν_t *have finite moment generating functions, then by the large deviation theory, the right-hand-side is on the order of* $1 - \mathcal{O}(\exp(-C'n(T-1)))$, *where* $n := \min\{n_2, \ldots, n_T\}$, *for some* $C' > 0$.

In contrast, MDRO using Wasserstein ambiguity sets has an adjustable level of in-sample conservatism. The difference of the optimal values between the MDRO and the nominal MSCO $Q_1(x_0;\xi_1) - Q_1^{\mathrm{Nomin}}(x_0;\xi_1)$ can be bounded by the Wasserstein distances $\rho_{t,0}$, as shown below.

Theorem 4.4. *Suppose the value function* $Q_t(x_{t-1};\xi_t)$ *is* l_t*-Lipschitz continuous in* $\xi_t \in \Xi_t$ *for any* $x_{t-1} \in \mathcal{X}_{t-1}$, *for each stage* $t \in \mathcal{T}$. *Under Assumption 4.1, the difference of optimal values between the MDRO and the nominal MSCO satisfies*

$$Q_1(x_0;\xi_1) - Q_1^{\mathrm{Nomin}}(x_0;\xi_1) \leq \sum_{t=2}^{T} l_t \rho_{t,0}.$$

Proof. We first observe that if there exists $\varepsilon \geq 0$ such that $\sup_{x_t \in \mathcal{X}_t} |\mathcal{Q}_t(x_t) - \mathcal{Q}_t^{\mathrm{Nomin}}(x_t)| \leq \varepsilon$, then the value functions satisfy $Q_t(x_{t-1};\xi_t) - Q_t^{\mathrm{Nomin}}(x_{t-1};\xi_t) \leq \varepsilon$ by the definitions (4.3) and (4.22). Now we prove by recursion that $\mathcal{Q}_t(x_t) - \mathcal{Q}_t^{\mathrm{Nomin}}(x_t) \leq \sum_{s>t} l_s \rho_{s,0}$

for any $x_t \in \mathcal{X}_t$, which holds trivially for $t = T$. For any $t \in \mathcal{T}$, we have

$$\begin{aligned}\mathcal{Q}_{t-1}(x_{t-1}) - \mathcal{Q}_{t-1}^{\text{Nomin}}(x_{t-1}) = &\left(\sup_{p_t \in \mathcal{P}_t} \int_{\Xi_t} Q_t(x_{t-1};\xi_t)\,\mathrm{d}p_t(\xi_t) - \int_{\Xi_t} Q_t(x_{t-1};\xi_t)\,\mathrm{d}\hat{\nu}_t(\xi_t)\right) \\ &+ \int_{\Xi_t} \left(Q_t(x_{t-1};\xi_t) - Q_t^{\text{Nomin}}(x_{t-1};\xi_t)\right)\mathrm{d}\hat{\nu}_t(\xi_t) \\ \leq &l_t\rho_{t,0} + \sum_{s>t} l_s\rho_{s,0} = \sum_{s>t-1} l_s\rho_{s,0},\end{aligned}$$

where the first part before the inequality is bounded by $l_t\rho_{t,0}$ using Corollary 4.3, and the second part is bounded by our observation above. This recursion shows that $\mathcal{Q}_1(x_1) - \mathcal{Q}_1^{\text{Nomin}}(x_1) \leq \sum_{t=2}^{T} l_t\rho_{t,0}$, which completes the proof by the same observation. □

Theorem 4.4 shows that the conservatism of the MDRO can be adjusted linearly with the Wasserstein distance bound $\rho_{t,0}$, assuming the Lipschitz continuity of the value functions in the uncertainties. Together with Theorem 4.3, it shows that the trade-off of the objective value for the out-of-sample performance guarantee is on the order $\mathcal{O}(n^{-1/2})$ for $n := \min\{n_2, \ldots, n_T\}$ (or $\mathcal{O}(n^{-1/\delta})$ if the probability measures are sub-Gaussian and $\delta := \min\{\delta_2, \ldots, \delta_T\} > 2$), for some fixed probability threshold α and the number of stages T. However, as this optimal rate depends on the unknown constants in Theorem 4.2, it is not easy to numerically determine the Wasserstein distance bounds $\rho_{t,0}$. We discuss some practical choices of the bounds $\rho_{t,0}$ in Section 3.3.

4.2 Dual Dynamic Programming Algorithm

In this section, we first review the recursive cutting plane approximations for the expected cost-to-go functions and the dual dynamic programming (DDP) algorithm. Then we focus on different realizations of the single stage subproblem oracles (SSSO) for MDRO with Wasserstein ambiguity sets that would guarantee the convergence of the DDP algorithm.

4.2.1 Recursive Approximations and Regularization

Recall that for any convex function $\mathcal{Q} : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, an affine function $\mathcal{V} : \mathcal{X} \to \mathbb{R}$ is called a (valid) linear cut if $\mathcal{Q}(x) \geq \mathcal{V}(x)$ for all $x \in \mathcal{X}$. A collection of such valid linear cuts $\{\mathcal{V}^j\}_{1 \leq j \leq i}$ defines a valid under-approximation $\underline{\mathcal{Q}}^i(x) := \max_{1 \leq j \leq i} \mathcal{V}^j(x)$ of $\mathcal{Q}(x)$. Similarly by convexity, given a collection of overestimate values $v^j \geq \mathcal{Q}(x^j)$ for $j = 1, \ldots, i$, we can define a valid over-approximation by the convex envelope $\overline{\mathcal{Q}}^i(x) := \mathrm{conv}_{1 \leq j \leq i}(v^j + \iota_{x^j}(x))$, where $\iota_{x^j}(x) = 0$ when $x = x^j$ and $+\infty$ otherwise, is the convex indicator function centered at x^j. The validness of these approximations $\underline{\mathcal{Q}}(x) \leq \mathcal{Q}(x) \leq \overline{\mathcal{Q}}(x)$ for all $x \in \mathcal{X}$ suggests that we may use them in the place of $\mathcal{Q}$ for recursive updates during a stagewise decomposition algorithm.

As discussed in Section 3.1.3, given regularization factors $M_t > 0$, we define the regularized local cost function as

$$f_t^{\mathrm{R}}(x_{t-1}, x_t; \xi_t) := \inf_{z_t \in \mathbb{R}^{d_{t-1}}} f_t(z_t, x_t; \xi_t) + M_t \|x_{t-1} - z_t\|, \tag{4.23}$$

and the regularized value function

$$Q_t^{\mathrm{R}}(x_{t-1}; \xi_t) := \min_{x_t \in \mathcal{X}_t} f_t^{\mathrm{R}}(x_{t-1}, x_t; \xi_t) + \mathcal{Q}_t^{\mathrm{R}}(x_t), \tag{4.24}$$

recursively for $t = T, T-1, \ldots, 2$, where $\mathcal{Q}_t^{\mathrm{R}}$ is the regularized expected cost-to-go function defined as

$$\mathcal{Q}_t^{\mathrm{R}}(x_t) := \sup_{p_{t+1} \in \mathcal{P}_{t+1}} \mathbb{E}_{\xi_{t+1} \sim p_{t+1}} Q_{t+1}^{\mathrm{R}}(x_t; \xi_{t+1}), \tag{4.25}$$

for $t \leq T-1$, and $\mathcal{Q}_T^{\mathrm{R}}(x_T) \equiv 0$ for any $x_T \in \mathcal{X}_T$. It is then straightforward to check that $Q_t^{\mathrm{R}}(x_{t-1}; \xi_t)$ is uniformly M_t-Lipschitz continuous in x_{t-1} for each $\xi_t \in \Xi_t$, and consequently $\mathcal{Q}_{t-1}^{\mathrm{R}}$ is also M_t-Lipschitz continuous. These regularized cost-to-go functions fit into the definitions of SSSO (Definitions 3.1 and 3.2) in Section 3.2.1.

The Lipschitzian regularization in general only gives under-approximations of the true

value and expected cost-to-go functions. We need the following assumption to preserve the optimality and feasibility of the solutions.

Assumption 4.2. *For the given regularization factors $M_t > 0$, $t \in \mathcal{T}$, the optimal value of the regularized MDRO satisfies*

$$\min_{x_1 \in \mathcal{X}_1} f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1(x_1) = \min_{x_1 \in \mathcal{X}_1} f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1^{\mathrm{R}}(x_1)$$

and the set of optimal first-stage solutions

$$\arg\min\{f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1^{\mathrm{R}}(x_1) : x_1 \in \mathcal{X}_1\} = \arg\min\{f_1(x_0, x_1; \xi_1) + \mathcal{Q}_1(x_1) : x_1 \in \mathcal{X}_1\}.$$

We remark by the following proposition that Assumption 4.2 can be satisfied in any problem that already have uniformly Lipschitz continuous value function $Q_t(\cdot; \xi_t)$ for all $\xi_t \in \Xi_t$. We begin with the following technical lemma.

Lemma 4.4. *For any convex function $f : \mathbb{R}^d \to \mathbb{R}$ that is M-Lipschitz continuous on a convex subset $\mathcal{X} \subseteq \mathbb{R}^d$ with $\mathrm{int}\mathcal{X} \neq \varnothing$, we have $f(x) = f \square (M\|\cdot\|)(x) := \inf_{z \in \mathbb{R}^d}\{f(z) + M\|x - z\|\}$ for any $x \in \mathcal{X}$.*

Proof. Assume for contradiction that for some $x \in \mathcal{X}$, there exists $z \in \mathbb{R}^d$ and $\varepsilon > 0$ such that $f(x) > f(z) + (1+\varepsilon)M\|z - x\|$. If $x \in \mathrm{int}\mathcal{X}$, then we can find $z' = x + c(x - z) \in \mathcal{X}$ for some $c > 0$. Thus by convexity, $(f(z') - f(x))/\|z' - x\| \geq (f(x) - f(z))/\|x - z\| > (1 + \varepsilon)M$, which contradicts the M-Lipschitz continuity of f on $\mathcal{X}$.

Otherwise if $x \notin \mathrm{int}\mathcal{X}$, we can find $x' \in \mathrm{int}\mathcal{X}$ with $\|x' - x\| \leq \|z - x\| \cdot \varepsilon/2$, so $\|x' - z\| \leq (1 + \varepsilon/2)\|z - x\|$. Besides, by the M-Lipschitz continuity of f on $\mathcal{X}$, we have $f(x') \geq f(x) - M\|x - x'\| > f(z) + (1 + \varepsilon/2)M\|z - x\|$. Therefore,

$$\frac{f(x') - f(z)}{\|x' - z\|} > \frac{(1 + \varepsilon/2)M\|z - x\|}{(1 + \varepsilon/2)\|z - x\|} = M,$$

which shows contradiction as $x' \in \mathrm{int}\mathcal{X}$. This completes the proof. □

Proposition 4.2. *Suppose each state space $\mathcal{X}_t \subseteq \mathbb{R}^{d_t}$ is full dimensional, i.e.,* $\mathrm{int}\mathcal{X}_t \neq \varnothing$. *Then Assumption 4.2 holds if for each stage $t \geq 2$, the value function $Q_t(\cdot;\xi_t)$ is M_t-Lipschitz continuous for any $\xi_t \in \Xi_t$.*

Proof. We prove the assertion by induction from $t = T$ to $t = 1$ using Lemma 4.4. Suppose that $\mathcal{Q}_t = \mathcal{Q}_t^{\mathrm{R}}$ on $\mathcal{X}_t$ for some $t \in \mathcal{T}$, which holds trivially for $t = T$. We see that $Q_t(\cdot;\xi_t) = Q_t^{\mathrm{R}}(\cdot;\xi_t)$ everywhere on $\mathcal{X}_{t-1}$ for any $\xi_t \in \Xi_t$ by definition (4.24). Thus by definition (4.25), $\mathcal{Q}_{t-1} = \mathcal{Q}_{t-1}^{\mathrm{R}}$ on $\mathcal{X}_{t-1}$ and this finishes the induction. □

There are also other cases for Assumption 4.2 to hold, especially where the uncertainty sets Ξ_t are finite (see Section 3.1.3). In general, we can execute the DDP algorithm even with $M_t = +\infty$.

Now if we can implement the SSSO (Definitions 3.1 and 3.2), we know we can apply Algorithm 5 (or 6) with guaranteed convergence (Theorems 3.1 and 3.2). However, the SSSO implementations are not straightforward. It is well known that the worst-case probability measure may not exist (see e.g., [26, Example 2]). Moreover, the integration with respect to some worst-case probability measure could also be numerically challenging. Therefore, we next provide two possible SSSO implementation methods directly using the finite dimensional recursion (4.16).

4.2.2 Subproblem Oracles: Concave Uncertain Cost Functions

If the following assumption holds, we are able to reformulate the recursion (4.16) into a convex optimization problem using a minimax theorem.

Assumption 4.3. *The local cost function $f_t(x_{t-1}, x_t; \xi_t)$ is concave and upper semicontinuous in the uncertainty ξ_t for any $x_{t-1} \in \mathcal{X}_{t-1}$ and $x_t \in \mathcal{X}_t$.*

A direct consequence of Assumption 4.3 is that the effective domain of the state x_t does not depend on the uncertainty ξ_t, as shown in the following lemma.

Lemma 4.5. *Under Assumption 4.3, we have* $\operatorname{dom} f_t(x_{t-1}, \cdot; \xi_t) = \operatorname{dom} f_t(x_{t-1}, \cdot; \xi_t')$ *for any* $x_{t-1} \in \mathcal{X}_{t-1}$ *and* $\xi_t, \xi_t' \in \Xi_t$.

Proof. Assume for contradiction that there exists some $x_{t-1} \in \mathcal{X}_{t-1}$, $x_t \in \mathcal{X}_t$, and $\xi_t, \xi_t' \in \Xi_t$ such that $f_t(x_{t-1}, x_t; \xi_t) < +\infty$ but $f_t(x_{t-1}, x_t; \xi_t') = +\infty$. Then for $c \in (0, 1)$, we have $f_t(x_{t-1}, x_t; (1-c)\xi_t + c\xi_t') = +\infty$ by the concavity and nonnegativity of f_t. It follows from upper semicontinuity that $f_t(x_{t-1}, x_t; \xi_t) \geq \limsup_{c \to 0+} f_t(x_{t-1}, x_t; (1-c)\xi_t + c\xi_t') = +\infty$, which is a contradiction. □

We are now ready to prove the alternative formulation of the recursion (4.16).

Theorem 4.5. *Under Assumption 4.3, if we further assume that the continuous functions* $g_{t,j}$ *are convex for* $j = 1, \ldots, m$ *and* $d_{t,k}(\xi_{t,k}) = \|\xi_{t,k} - \hat{\xi}_{t,k}\|$ *for* $k = 1, \ldots, n_t$*, then we have*

$$\begin{aligned} \mathcal{Q}_{t-1}(x_{t-1}) = \min \quad & \sum_{j=0}^{m_t} \rho_{t,j} \lambda_{t,j} + \frac{1}{n_t} \sum_{k=1}^{n_t} \left[h_{t,k}(z_t, x_{t,k}, \zeta_{t,k}, \lambda_t) + \mathcal{Q}_t(x_{t,k}) \right] \\ \text{s.t.} \quad & \left\| \zeta_{t,k} \right\|_* \leq \lambda_{t,0}, \\ & z_t = x_{t-1}, \\ & \lambda_t \in \mathbb{R}^{m_t+1}_{\geq 0}, x_{t,k} \in \mathcal{X}_t, \end{aligned} \tag{4.26}$$

where for each $k = 1, \ldots, n_t$*,* $h_{t,k}$ *is defined as*

$$h_{t,k}(x_{t-1}, x_{t,k}, \zeta_{t,k}, \lambda_t) := \sup_{\xi_{t,k} \in \Xi_t} f_t(x_{t-1}, x_{t,k}; \xi_{t,k}) - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_{t,k}) + \zeta_{t,k}^{\mathsf{T}}(\xi_{t,k} - \hat{\xi}_{t,k}).$$

Proof. By Lemma 4.5, for any $x_{t-1} \in \mathcal{X}_{t-1}$, we can define a set

$$\mathcal{X}_t(x_{t-1}) := \operatorname{dom} f_t(x_{t-1}, \cdot; \xi_t) \subseteq \mathcal{X}_t,$$

which is independent of $\xi_t \in \Xi_t$ and closed by the lower semicontinuity of f_t. Note that the norm function has the dual representation $d_{t,k}(\xi_{t,k}) = \|\xi_{t,k} - \hat{\xi}_{t,k}\| = \max_{\|\zeta\|_* \leq 1} \zeta^{\mathsf{T}}(\xi_{t,k} -$

$\hat{\xi}_{t,k}$). Thus by the recursion (4.16), we can write

$$\begin{aligned}\mathcal{Q}_{t-1}(x_{t-1}) = \min_{\lambda_t \in \mathbb{R}^{m_t+1}_{\geq 0}} \sum_{j=0}^{m_t} \rho_{t,j}\lambda_{t,j} + \frac{1}{n_t}\sum_{k=1}^{n_t} \sup_{\xi_{t,k}\in\Xi_t} \Big[-\sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_{t,k}) \\ + \min_{x_{t,k},\zeta_{t,k}} \quad f_t(x_{t-1}, x_{t,k}; \xi_{t,k}) + \mathcal{Q}_t(x_{t,k}) + \zeta_{t,k}^{\mathsf{T}}(\xi_{t,k} - \hat{\xi}_{t,k})\Big] \\ \text{s.t.} \quad \big\|\zeta_{t,k}\big\|_* \leq \lambda_{t,0}, \\ x_{t,k} \in \mathcal{X}_t(x_{t-1}).\end{aligned}$$

Now for any fixed x_{t-1} and λ_t, we see that the sets $\{\zeta_{t,k} : \|\zeta_{t,k}\|_* \leq \lambda_{t,0}\}$ and $\mathcal{X}_t(x_{t-1})$ are compact. Moreover, the function inside the supremum of $\xi_{t,k}$ is concave and upper semicontinuous in $\xi_{t,k}$, while convex and lower semicontinuous in $\xi_{t,k}$ and $\zeta_{t,k}$. Thus the result follows by applying Sion's minimax theorem [84]. □

Remark. The proof remains valid if we replace simultaneously $\mathcal{Q}_t$, $\mathcal{Q}_{t-1}$, and f_t with $\mathcal{Q}_t^{\mathrm{R}}$, $\mathcal{Q}_{t-1}^{\mathrm{R}}$, and f_t^{R} in the theorem. In this case we use $h_{t,k}^{\mathrm{R}}$ to denote the convex conjugate functions.

We provide a possible implementation for noninitial stage SSSO in Algorithm 7 based on Theorem 4.5. Its correctness is verified by the following corollary.

Algorithm 7 Single Stage Subproblem Oracle Implementation Under Assumption 4.3

Require: function h_t, over- and under-approximations $\overline{\mathcal{Q}}_t^i$ and $\underline{\mathcal{Q}}_t^i$, and a state $x_{t-1} \in \mathcal{X}_{t-1}$
Ensure: a linear cut $\mathcal{V}_{t-1}$, an overestimate v_{t-1}, a state x_t, and a gap value γ_t

1: Solve the minimization (4.26) with $\mathcal{Q}_t$ replaced by $\underline{\mathcal{Q}}_t^i$ and store the optimal value v_{t-1}^*, optimal solutions λ_t^* and $(x_{t,k}^*, \zeta_{t,k}^*)_{k=1}^{n_t}$ and the dual solutions u_t associated with the constraints $z_t = x_{t-1}$
2: **for** $k = 1, \ldots, n_t$ **do**
3: Compute the gap value $\gamma_{t,k} := \overline{\mathcal{Q}}_t^i(x_{t,k}^*) - \underline{\mathcal{Q}}_t^i(x_{t,k}^*)$
4: **end for**
5: Set $\mathcal{V}_{t-1}(\cdot) \leftarrow v_{t-1}^* + u_t^{\mathsf{T}}(\cdot)$
6: Set $v_{t-1} \leftarrow v_{t-1}^* + \frac{1}{n_t}\sum_{k=1}^{n_t} \gamma_{t,k}$
7: Take any $k^* \in \arg\max\{\gamma_{t,k} : k = 1, \ldots, n_t\}$ and set $x_t \leftarrow x_{t,k^*}^*$, $\gamma_t \leftarrow \gamma_{t,k^*}$

Corollary 4.4. *Under the same assumptions of Theorem 4.5, the outputs* $(\mathcal{V}_{t-1}, v_{t-1}, x_t; \gamma_t)$ *of Algorithm 7 satisfy the conditions in Definition 3.2.*

Proof. To check the validness of v_{t-1}, let $(z_t^*, x_{t,k}^*, \zeta_{t,k}^*, \lambda_t^*)$ denote an optimal solution in the minimization (4.26) with $\mathcal{Q}_t$ replaced by $\underline{\mathcal{Q}}_t^i$. Then we have

$$\begin{aligned} v_{t-1} &= v_{t-1}^* + \frac{1}{n_t}\sum_{k=1}^{n_t} \gamma_{t,k} \\ &= \sum_{j=0}^{m_t} \rho_{t,j}\lambda_{t,j}^* + \frac{1}{n_t}\sum_{k=1}^{n_t}[h_{t,k}(z_t^*, x_{t,k}^*, \zeta_{t,k}^*, \lambda_t^*) + \underline{\mathcal{Q}}_t^i(x_{t,k}^*) + \gamma_{t,k}] \\ &\geq \sum_{j=0}^{m_t} \rho_{t,j}\lambda_{t,j}^* + \frac{1}{n_t}\sum_{k=1}^{n_t}[h_{t,k}(z_t^*, x_{t,k}^*, \zeta_{t,k}^*, \lambda_t^*) + \mathcal{Q}_t(x_{t,k}^*)] \geq \mathcal{Q}_{t-1}(x_{t-1}), \end{aligned}$$

the last inequality is due to the feasibility of $(\mathcal{V}_{t-1}, v_{t-1}, x_t; \gamma_t)$ in the minimization (4.26). For the validness of $\mathcal{V}_{t-1}(\cdot)$, note that the value v_{t-1}^* and the dual solution u_t define a valid linear under-approximation for the function $\mathcal{Q}'_{t-1}(\cdot)$ defined by replacing $\mathcal{Q}_t$ with $\underline{\mathcal{Q}}_t^i$ in the minimization (4.26). Since clearly $\mathcal{Q}'_{t-1}(x_{t-1}) \leq \mathcal{Q}_{t-1}(x_{t-1})$ for all $x_{t-1} \in \mathcal{X}_{t-1}$, we see that $\mathcal{V}_{t-1}(\cdot)$ is a valid under-approximation for $\mathcal{Q}_{t-1}(\cdot)$. Finally the gap $v_{t-1} - \mathcal{V}_{t-1}(x_{t-1}) = \frac{1}{n_t}\sum_{k=1}^{n_t} \gamma_{t,k} \leq \gamma_t$ is controlled. □

Theorem 4.5 and Algorithm 7 would be most useful when the functions $h_{t,k}$ can be written explicitly as minimization problems. We thus spend the rest of this section to derive the form of $h_{t,k}$ in a special yet practically important case, where the local cost function f_t can be written as

$$\begin{aligned} f_t(x_{t-1}, x_t; \xi_t) = \min \quad & (A_t\xi_t + a_t)^\mathsf{T} y_t \\ \text{s.t.} \quad & (x_{t-1}, y_t, x_t) \in \mathcal{F}_t, \end{aligned} \tag{4.27}$$

for some compact convex set $\mathcal{F}_t \subseteq \mathcal{X}_{t-1} \times \mathbb{R}^{d'_t} \times \mathcal{X}_t$ in each stage $t \in \mathcal{T}$. It is straightforward to check that f_t in (4.27) is lower semicontinuous and convex in (x_{t-1}, x_t) for any $\xi_t \in \Xi_t$. To simplify our discussion, we assume that $f_t(\cdot, x_t; \xi_t)$ is M_t-Lipschitz continuous, so by Lemma 4.4 we have $f_t^{\mathrm{R}} = f_t$ and consequently $\mathcal{Q}_t = \mathcal{Q}_t^{\mathrm{R}}$ for all $t \in \mathcal{T}$. The problem (4.27)

is a common formulation in the usual MSCO literature, such as [40], where $\mathcal{F}_t$ is supposed to be a polytope.

Proposition 4.3. *Suppose the local cost function $f_t(x_{t-1}, x_t; \xi_t)$ is given in the form* (4.27) *and uniformly M_t-Lipschitz continuous in the variable x_{t-1}. Fix any point $\bar{\xi}_t \in \mathrm{int}\Xi_t$ and let $\sigma_t(\zeta) := \sup_{\xi\in\Xi_t} \zeta^\mathsf{T}(\xi - \bar{\xi}_t)$ denote the support function of the set $\Xi_t - \bar{\xi}_t$. If the functions $g_{t,j}(\xi_t) = \xi_t^\mathsf{T} B_{t,j}\xi_t + b_{t,j}^\mathsf{T}\xi_t$ are quadratic with coefficients $B_{t,j} \in \mathcal{S}_{\succeq 0}^{\delta_t}$ and $b_{t,j} \in \mathbb{R}^{\delta_t}$ for $j = 1, \ldots, m_t$, then we can write*

$$
\begin{aligned}
h_{t,k}(x_{t-1}, x_{t,k}, \zeta_{t,k}, \lambda_t) = \\
\min_{y_{t,k}, w_{t,j}, w'_{t,j}, \kappa_j} \quad & (A_t\bar{\xi}_t + a_t)^\mathsf{T} y_{t,k} - \sum_{j=1}^{m_t} \lambda_{t,j}[\bar{\xi}_t^\mathsf{T} B_{t,j}\bar{\xi}_t + b_{t,j}^\mathsf{T}\bar{\xi}_t] \\
& + \zeta_{t,k}^\mathsf{T}(\bar{\xi}_t - \hat{\xi}_{t,k}) + \sum_{j=1}^{m_t} \kappa_{t,j} + \sigma_t(w_{t,0}) \\
\text{s.t.} \quad & \sum_{j=0}^{m_t} w_{t,j} = \zeta_{t,k} + A_t^\mathsf{T} y_{t,k} - \sum_{j=1}^{m_t} \lambda_{t,j}\big[2B_{t,j}\bar{\xi}_t + b_{t,j}\big], \\
& \kappa_{t,j} \geq 0, \quad j = 1, \ldots, m, \\
& w_{t,j} \in \mathbb{R}^{\delta_t}, \quad j = 0, \ldots, m, \\
& \kappa_{t,j} + \lambda_{t,j} \geq \|(\kappa_{t,j} - \lambda_{t,j}, U_{t,j} w_{t,j})\|_2, \quad j = 1, \ldots, m_t, \\
& w_{t,j} = B_{t,j} w'_{t,j},\ w'_{t,j} \in \mathbb{R}^{\delta_t}, \quad j = 1, \ldots, m, \\
& (x_{t-1}, y_{t,k}, x_{t,k}) \in \mathcal{F}_t.
\end{aligned}
\tag{4.28}
$$

Here, $U_{t,j}$ is a $\delta_t \times \delta_t$ real matrix such that $U_{t,j}^\mathsf{T} U_{t,j} = B_{t,j}^\dagger$ is the pseudoinverse of $B_{t,j}$.

Proof. Under the assumptions, we can write the function $h_{t,k}$ as

$$
\begin{aligned}
h_{t,k}(x_{t-1}, x_{t,k}, \zeta_{t,k}, \lambda_t) = \sup_{\xi_t\in\Xi_t} \min_{y_{t,k}} \quad & (A_t\xi_t + a_t)^\mathsf{T} y_{t,k} - \sum_{j=1}^{m_t} \lambda_{t,j}(\xi_t^\mathsf{T} B_{t,j}\xi_t + b_{t,j}^\mathsf{T}\xi_t) + \zeta_{t,k}^\mathsf{T}(\xi_t - \hat{\xi}_{t,k}) \\
\text{s.t.} \quad & (x_{t-1}, y_{t,k}, x_{t,k}) \in \mathcal{F}_t.
\end{aligned}
$$

Note that the objective function in (4.28) is continuous in both y_t and $\xi_{t,k}$, and the projec-

tion of $\mathcal{F}_t$ onto the variables $y_{t,k}$ is compact. Thus by the minimax theorem [84], we can exchange the supremum and minimum operations

$$
\begin{aligned}
h_{t,k}(x_{t-1}, x_{t,k}, \zeta_{t,k}, \lambda_t) = \qquad & (4.29)\\
\min_{y_{t,k}} \quad & (A_t\bar{\xi}_t + a_t)^{\mathsf{T}} y_{t,k} - \sum_{j=1}^{m_t} \lambda_{t,j}[\bar{\xi}_t^{\mathsf{T}} B_{t,j}\bar{\xi}_t + b_{t,j}^{\mathsf{T}}\bar{\xi}_t] + \zeta_{t,k}^{\mathsf{T}}(\bar{\xi}_t - \hat{\xi}_{t,k})\\
+ \sup_{\xi_t \in \mathbb{R}^{\delta_t}} & \left\{ \zeta_{t,k}^{\mathsf{T}}\xi_t - \iota_t(\xi_t) + y_{t,k}^{\mathsf{T}} A_t \xi_t - \sum_{j=1}^{m_t} \lambda_{t,j}(\xi_t^{\mathsf{T}} B_{t,j}\xi_t + 2\bar{\xi}_t B_{t,j}\xi_t + b_{t,j}^{\mathsf{T}}\xi_t) \right\}\\
\text{s.t.} \quad & (x_{t-1}, y_{t,k}, x_{t,k}) \in \mathcal{F}_t,
\end{aligned}
$$

where ι_t is the convex indicator function of the set $\Xi_t - \bar{\xi}_t$, the convex conjugate of which is the support function σ_t by definition. If we further denote $\varphi_{t,j}(\xi_t; \lambda_{t,j}) := \lambda_{t,j}(\xi_t^{\mathsf{T}} B_{t,j}\xi_t)$, the supremum can be written using convex conjugacy as

$$
\left(\iota_t + \sum_{j=1}^{m_t} \varphi_{t,j}(\cdot; \lambda_{t,j}) \right)^* \left(\zeta_{t,k} + A_t^{\mathsf{T}} y_{t,k} - \sum_{j=1}^{m_t} \lambda_{t,j}\left[2B_{t,j}\bar{\xi}_t + b_{t,j}\right] \right).
$$

Note that for each $j = 1, \ldots, m_t$, the parametrized conjugate function $\varphi_{t,j}^*(\cdot; \lambda_{t,j})$ can be written as [72, Example 11.10]

$$
\begin{aligned}
\varphi_{t,j}^*(w; \lambda_{t,j}) &= \begin{cases} \dfrac{w^{\mathsf{T}} B_{t,j}^{\dagger} w}{4\lambda_{t,j}}, \text{ if } w \in \mathrm{range} B_{t,j}, \\ +\infty, \quad \text{otherwise}, \end{cases}\\
&= \min\left\{ \kappa_{t,j} \geq 0 : 4\kappa_{t,j}\lambda_{t,j} \geq (U_{t,j}w)^{\mathsf{T}}(U_{t,j}w), w = B_{t,j}w' \right\}\\
&= \min\left\{ \kappa_{t,j} \geq 0 : \kappa_{t,j} + \lambda_{t,j} \geq \|(\kappa_{t,j} - \lambda_{t,j}, U_{t,j}w)\|_2, w = B_{t,j}w' \right\},
\end{aligned}
$$

which is nonnegative and second-order conic representable. Here the convention for $\lambda_{t,j} = 0$ is consistent: we have $\varphi_{t,j}^*(0; 0) = 0$ and $\varphi_{t,j}^*(w; 0) = +\infty$ for any $w \neq 0$ because $(U_{t,j}B_{t,j})^{\mathsf{T}}(U_{t,j}B_{t,j}) = B_{t,j}B_{t,j}^{\dagger}B_{t,j} = B_{t,j}$, which implies that $U_{t,j}w = U_{t,j}B_{t,j}w' \neq 0$.

Now using the formula for convex conjugate of sum of convex functions, we have

$$\left(\iota_t + \sum_{j=1}^{m_t} \varphi_{t,j}(\cdot;\lambda_{t,j})\right)^* = \mathrm{cl}\left(\sigma_t \square \varphi_{t,1}^*(\cdot;\lambda_{t,1}) \square \cdots \square \varphi_{t,m_t}(\cdot;\lambda_{t,m_t})\right), \tag{4.30}$$

where $\square$ denotes the infimal convolution (a.k.a. epi-addition) of two convex functions and cl denotes the lower semicontinuous hull of a proper function. Since $\bar{\xi}_t \in \mathrm{int}\Xi_t$, the support function is coercive, i.e., $\lim_{\|w\|\to\infty} \sigma_t(w) = +\infty$. Moreover, each $\varphi_{t,j}^*$ is bounded below as it is nonnegative. Therefore, the closure operation is superficial and the convex conjugate of the sum is indeed lower semicontinuous [74, Proposition 12.14]. The rest of the proof follows from substitution of this convex conjugate expression (4.30) into the supremum in (4.29). □

4.2.3 Subproblem Oracles: Convex Uncertain Cost Functions

We provide another useful reformulation of the recursion (4.16) in this section, based on the following assumption.

Assumption 4.4. *The local cost function $f_t(x_{t-1}, x_t; \xi_t)$ is jointly convex in the state variable x_t and the uncertainty ξ_t, for any $x_{t-1} \in \mathcal{X}_{t-1}$. Moreover, the uncertainty set Ξ_t is a pointed polyhedron and the distance function $d_{t,k}(\cdot)$ is polyhedrally representable.*

We first mention some direct consequences of Assumption 4.4. First, the value function $Q_t(x_{t-1};\xi_t)$ would be a convex function in the uncertainty ξ_t for each state $x_{t-1} \in \mathcal{X}_{t-1}$, although it may not be a jointly convex function. Second, recall that a polyhedron is pointed if it does not contain any lines. Any point in a pointed polyhedron can be written as a convex combination of its extreme points and rays [85, Theorem 3.37]. Now under Assumption 4.4, we may define a lifted uncertainty set as $\tilde{\Xi}_{t,k} := \{(\zeta,\xi) : \xi \in \Xi_t,\, \zeta \geq d_{t,k}(\xi)\}$. It is easy to see that $\tilde{\Xi}_{t,k}$ is also a pointed polyhedron. We denote the finite set of extreme points of it as $\mathrm{ext}\,\tilde{\Xi}_{t,k} = \{(\tilde{\zeta}_l, \tilde{\xi}_l)\}_{l\in E_{t,k}}$ where $E_{t,k}$ is the set of indices, and want to

show that the maximization in (4.16) can be taken over the finite set $\{(\tilde{\zeta}_l, \tilde{\xi}_l)\}_{l \in E_{t,k}}$ in two important cases. First, we consider problems with bounded uncertainty sets Ξ_t.

Proposition 4.4. *Under Assumption 4.4, if we further assume that Ξ_t is bounded and all functions $g_{t,j}$ are concave for $j = 1, \ldots, m_t$, then the problem* (4.16) *can be equivalently reformulated as*

$$\begin{aligned} \mathcal{Q}_{t-1}(x_{t-1}) = \min_{\lambda_t, \tau_t} \quad & \sum_{j=0}^{m_t} \rho_{t,j} \lambda_{t,j} + \frac{1}{n_t} \sum_{k=1}^{n_t} \tau_{t,k} \\ \text{s.t.} \quad & \lambda_t \geq 0, \\ & \tau_{t,k} \geq Q_t(x_{t-1}; \tilde{\xi}_l) - \lambda_{t,0} \tilde{\zeta}_l - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\tilde{\xi}_l),\ l \in E_{t,k},\ k = 1, \ldots, n_t. \end{aligned} \tag{4.31}$$

Proof. From the definition of lifted uncertainty set $\tilde{\Xi}_t$, we have

$$\begin{aligned} & \sup_{\xi_k \in \Xi_t} \left\{ Q_t(x_{t-1}; \xi_k) - \lambda_{t,0} d_{t,k}(\xi_k) - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_k) \right\} \\ & = \max_{\xi_k \in \Xi_t, \zeta_k \in \mathbb{R}} \left\{ Q_t(x_{t-1}; \xi_k) - \lambda_{t,0} \zeta_k - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_k) : \zeta_k \geq d_{t,k}(\xi_k) \right\} \\ & = \max_{(\zeta_k, \xi_k) \in \tilde{\Xi}_{t,k}} \left\{ Q_t(x_{t-1}; \xi_k) - \lambda_{t,0} \zeta_k - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_k) \right\} \\ & = \max_{l \in E_{t,k}} \left\{ Q_t(x_{t-1}; \tilde{\xi}_l) - \lambda_{t,0} \tilde{\zeta}_l - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\tilde{\xi}_l) \right\}. \end{aligned}$$

To see the last equality, note that if Ξ_t is bounded, then the only recession direction of the lifted uncertainty set $\tilde{\Xi}_{t,k}$ is $(1, 0)$. Since $\lambda_{t,0} \geq 0$, any maximum solution (ζ_k^*, ξ_k^*) lies in the convex hull of $\operatorname{ext} \tilde{\Xi}_{t,k}$. Now the last equality follows from the convexity of the function $Q_t(x_{t-1}; \xi_k) - \lambda_{t,0} \zeta_k - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_k)$ in terms of ξ_k and ζ_k. Finally, the reformulation is done by replacing the maximum of finitely many functions by its epigraphical representation $\tau_{t,k} \geq Q_t(x_{t-1}; \xi_l) - \lambda_{t,0} \zeta_l - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_l)$ for all $l \in E_{t,k}$ and $k = 1, \ldots, n_t$. $\square$

If the uncertainty sets Ξ_t are unbounded, then in general the supremum in (4.16) can take $+\infty$ in some unbounded directions of Ξ_t, even when the value function $Q_t(x_{t-1};\cdot)$ has finite values everywhere. To avoid such situation, we consider the growth rate of the value function $Q_t(x_{t-1};\cdot)$ defined as

$$r_t(x_{t-1}) := \limsup_{\substack{d_{t,k}(\xi_t)\to\infty,\\ \xi_t\in\Xi_t}} \frac{Q_t(x_{t-1};\xi_t) - Q_t(x_{t-1};\hat{\xi}_{t,k})}{d_{t,k}(\xi_t)} \geq 0, \tag{4.32}$$

for any real-valued $Q_t(x_{t-1},\cdot)$, where the limit superior is in fact independent of the choice of $k = 1,\ldots,n_t$, and the inequality is due to that $Q_t(x_{t-1};\cdot)$ is assumed to be lower bounded by 0. Our convention is to set $r_t(x_{t-1}) \equiv 0$ when Ξ_t is bounded. We now consider problems with unbounded uncertainty sets Ξ_t.

Proposition 4.5. *Under Assumption 4.4, if $Q_t(x_{t-1};\cdot)$ has finite growth rate $r_t(x_{t-1})$ and all functions $g_{t,j}$ are bounded and concave for $j = 1,\ldots,m_t$, then the problem* (4.16) *with any $x_{t-1} \in \mathcal{X}_{t-1}$ such that $\mathcal{Q}_{t-1}(x_{t-1}) < +\infty$ can be equivalently reformulated as*

$$\begin{aligned} \mathcal{Q}_{t-1}(x_{t-1}) = \min_{\lambda_t,\tau_t} \quad & \sum_{j=0}^{m_t} \rho_{t,j}\lambda_{t,j} + \frac{1}{n_t}\sum_{k=1}^{n_t} \tau_{t,k} \\ \text{s.t.} \quad & \lambda_t \geq 0, \\ & \lambda_{t,0} \geq r_t(x_{t-1}), \\ & \tau_{t,k} \geq Q_t(x_{t-1};\tilde{\xi}_l) - \lambda_{t,0}\tilde{\zeta}_l - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\tilde{\xi}_l),\ l \in E_{t,k},\ k = 1,\ldots,n_t. \end{aligned} \tag{4.33}$$

Proof. We claim that the supremum

$$\sup_{\xi_k\in\Xi_t} \left\{ Q_t(x_{t-1};\xi_k) - \lambda_{t,0} d_{t,k}(\xi_k) - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_k) \right\} < +\infty$$

if and only if $\lambda_{t,0} \geq r_t(x_{t-1})$, for each $k = 1,\ldots,n_t$. Suppose $\lambda_{t,0} < r_t(x_{t-1})$. By definition (4.32), there exists a sequence $\{\xi_k^{(i)}\}_{i\in\mathbb{N}} \subseteq \Xi_t$ and a constant $\varepsilon > 0$ such that

$d_{t,k}(\xi_k^{(i)}) \to \infty$ as $i \to \infty$ and $Q_t(x_{t-1}; \xi_k^{(i)}) \geq Q_t(x_{t-1}; \hat{\xi}_{t,k}) + (\lambda_{t,0} + \varepsilon) d_{t,k}(\xi_k^{(i)})$. Thus $\sup_{i \in \mathbb{N}} \{Q_t(x_{t-1}; \xi_k^{(i)}) - \lambda_{t,0} d_{t,k}(\xi_k^{(i)}) - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_k^{(i)})\} \geq \sup_{i \in \mathbb{N}} \{\varepsilon d_{t,k}(\xi_k^{(i)}) - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_k^{(i)})\} = +\infty$ as $g_{t,j}(\xi_k^{(i)})$ for $j = 1, \ldots, m_t$ are bounded.

Conversely, by definition (4.32), there exists a constant $\bar{d} > d_{t,k}(\hat{\xi}_{t,k})$ such that

$$Q_t(x_{t-1}; \xi_k) \leq Q_t(x_{t-1}; \hat{\xi}_{t,k}) + \lambda_{t,0} d_{t,k}(\xi_k)$$

for all $\xi_k \in \Xi_t$ with $d_{t,k}(\xi_k) \geq \bar{d}$. Thus we have

$$\begin{aligned}
&\sup_{\xi_k \in \Xi_t} \left\{ Q_t(x_{t-1}; \xi_k) - \lambda_{t,0} d_{t,k}(\xi_k) - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_k) \right\} \\
&\leq \sup_{d_{t,k}(\xi_k) \leq \bar{d}} \left\{ Q_t(x_{t-1}; \xi_k) - \lambda_{t,0} d_{t,k}(\xi_k) \right\} + \sup_{\xi_k \in \Xi_t} \sum_{j=1}^{m_t} \left(-\lambda_{t,j} g_{t,j}(\xi_k) \right) \\
&= \max_{(\zeta_k, \xi_k) \in \tilde{\Xi}_t(\bar{d})} \left\{ Q_t(x_{t-1}; \xi_k) - \lambda_{t,0} \zeta_k \right\} + \sup_{\xi_k \in \Xi_t} \sum_{j=1}^{m_t} \left(-\lambda_{t,j} g_{t,j}(\xi_k) \right) < +\infty,
\end{aligned}$$

where $\tilde{\Xi}_t(\bar{d}) := \{(\zeta, \xi) : \xi \in \Xi_t, d_{t,k}(\xi) \leq \bar{d}, \zeta \geq d_{t,k}(\xi)\}$, and the maximum is finite because it is attained on some extreme point $(\bar{\zeta}_k, \bar{\xi}_k) \in \tilde{\Xi}_t(\bar{d})$ by convexity, so $Q_t(x_{t-1}; \bar{\xi}_k) - \lambda_{t,0} \bar{\zeta}_k < +\infty$.

Now from this claim, we see that for any $x_{t-1} \in \mathcal{X}_{t-1}$ such that $\mathcal{Q}_{t-1}(x_{t-1}) < +\infty$, the problem (4.16) can be formulated equivalently as

$$\begin{aligned}
\mathcal{Q}_{t-1}(x_{t-1}) = \min_{\lambda_t \geq 0} \quad & \sum_{j=0}^{m_t} \rho_{t,j} \lambda_{t,j} + \frac{1}{n_t} \sum_{k=1}^{n_t} \sup_{(\zeta_k, \xi_k) \in \Xi_t} \left\{ Q_t(x_{t-1}; \xi_k) - \lambda_{t,0} \zeta_k - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\xi_k) \right\} \\
\text{s.t.} \quad & \lambda_{t,0} \geq r_t(x_{t-1}).
\end{aligned}$$

The supremum can be attained in $\tilde{\Xi}'_{t,k} := \operatorname{conv}(\operatorname{ext} \tilde{\Xi}_{t,k})$: otherwise there exists a point $(\check{\zeta}_k, \check{\xi}_k) \in \tilde{\Xi}_{t,k} \setminus \tilde{\Xi}'_{t,k}$ and $(\bar{\zeta}_k, \bar{\xi}_k) \in \tilde{\Xi}'_{t,k}$ such that

$$Q_t(x_{t-1}; \check{\xi}_k) - \lambda_{t,0} \check{\zeta}_k - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\check{\xi}_k) > Q_t(x_{t-1}; \bar{\xi}_k) - \lambda_{t,0} \bar{\zeta}_k - \sum_{j=1}^{m_t} \lambda_{t,j} g_{t,j}(\bar{\xi}_k).$$

In other words, $(\check{\zeta}_k, \check{\xi}_k) - (\bar{\zeta}_k, \bar{\xi}_k)$ defines a strictly increasing ray of $\tilde{\Xi}_t$, which by convexity implies that the supremum is $+\infty$, a contradiction. Using the convexity again as in the proof of Proposition 4.4, we conclude that the supremum is indeed attained in $\operatorname{ext} \tilde{\Xi}_{t,k}$, and this completes the proof. □

Proposition 4.5 reduces to Proposition 4.4 since the growth rate $r_t(x_{t-1}) = 0$ and any continuous function $g_{t,j}$ over a bounded polyhedron is bounded. The finite growth rate condition is often satisfied, especially when the value function $Q_t(x_{t-1}; \cdot)$ is Lipschitz continuous. However, it is in general difficult to estimate the growth rate (4.32).

Note that the problems (4.31) and (4.33) are standard linear optimization problems in the variables λ_t and τ_t. Thus by strong duality, we can write the dual problem as

$$\begin{aligned} \mathcal{Q}_{t-1}(x_{t-1}) = \max_{\theta_t, \kappa_{t,k,l} \geq 0} \quad & \theta_t r_t(x_{t-1}) + \sum_{k=1}^{n_t} \sum_{l \in E_{t,k}} \kappa_{t,k,l} Q_t(x_{t-1}; \tilde{\xi}_l) \\ \text{s.t.} \quad & \sum_{l \in E_{t,k}} \kappa_{t,k,l} = \frac{1}{n_t}, \quad k = 1, \ldots, n_t, \\ & \theta_t + \sum_{k=1}^{n_t} \sum_{l \in E_{t,k}} \tilde{\zeta}_l \kappa_{t,k,l} \leq \rho_{t,0}, \\ & \sum_{k=1}^{n_t} \sum_{l \in E_{t,k}} g_{t,j}(\tilde{\xi}_l) \kappa_{t,k,l} \leq \rho_{t,j}, \quad j = 1, \ldots, m_t. \end{aligned} \tag{4.34}$$

Consequently, any feasible dual solutions θ_t and $\kappa_{t,k,l}$ to the dual (4.34) define a valid under-approximation

$$\mathcal{Q}_{t-1}(x_{t-1}) \geq \theta_t r_t(x_{t-1}) + \sum_{k=1}^{n_t} \sum_{l \in E_{t,k}} \kappa_{t,k,l} Q_t(x_{t-1}; \tilde{\xi}_l), \quad \forall\, x_{t-1} \in \mathcal{X}_{t-1}. \tag{4.35}$$

We now describe an SSSO implementation in Algorithm 8. Its correctness is verified in the following corollary.

Corollary 4.5. *Suppose that the growth rate function $r_t(\cdot)$ is convex. Under the assumptions of of Proposition 4.5, the outputs $(\mathcal{V}_{t-1}, v_{t-1}, x_t; \gamma_t)$ of Algorithm 8 satisfy the condi-*

Algorithm 8 Single Stage Subproblem Oracle Implementation Under Assumption 4.4

Require: over- and under-approximations $\overline{\mathcal{Q}}_t^i$ and $\underline{\mathcal{Q}}_t^i$, a state $x_{t-1} \in \mathcal{X}_{t-1}$, growth rate $r_t(x_{t-1})$, and extreme point sets $\operatorname{ext} \tilde{\Xi}_{t,k}$ for $k = 1, \ldots, n_t$

Ensure: a linear cut $\mathcal{V}_{t-1}$, an overestimate v_{t-1}, a state x_t, and a gap value γ_t

for $k = 1, \ldots, n_t$ **do**
  for $l \in E_{t,k}$ **do**
    Evaluate the approximate value function

$$\underline{Q}_t(x_{t-1}; \tilde{\xi}_l) := \min_{x_t \in \mathcal{X}_t} f_t(x_{t-1}, x_t; \tilde{\xi}_l) + \underline{\mathcal{Q}}_t^i(x_t)$$

    with a minimizer stored as $x_{t,k,l}$ and a subgradient vector as $u_{t,k,l} \in \partial \underline{Q}_t(\cdot; \tilde{\xi}_l)$ at x_{t-1}
    Calculate $\gamma_{t,k,l} := \overline{\mathcal{Q}}_t^i(x_{t,k,l}) - \underline{\mathcal{Q}}_t^i(x_{t,k,l})$
  end for
end for
Solve the problem (4.33) (or (4.31) if Ξ_t is bounded) with $Q_t(x_{t-1}; \tilde{\xi}_l)$ replaced by $\underline{Q}_t(x_{t-1}; \tilde{\xi}_l)$ and store the optimal value v_{t-1}^* and dual solutions θ_t^*, $\kappa_{t,k,l}^*$ to (4.34)
for $k = 1, \ldots, n_t$ **do**
  Take any $l^* \in \arg\max \left\{ \overline{\mathcal{Q}}_t^i(x_{t,k,l}) - \underline{\mathcal{Q}}_t^i(x_{t,k,l}) : l \in E_{t,k} \right\}$
  Set $\gamma_{t,k} \leftarrow \gamma_{t,k,l^*}$ and $x_{t,k} \leftarrow x_{t,k,l^*}$
end for
Take a subgradient $w_t \in \partial r_t(\cdot)$ at x_{t-1}
Set $\mathcal{V}_{t-1}(\cdot) \leftarrow v_{t-1}^* + \theta_t^* w_t^\mathsf{T}(\cdot - x_{t-1}) + \sum_{k=1}^{n_t} \sum_{l \in E_t} \kappa_{t,k,l}^* u_{t,k,l}^\mathsf{T}(\cdot - x_{t-1})$
Set $v_{t-1} \leftarrow v_{t-1}^* + \frac{1}{n_t} \sum_{k=1}^{n_t} \gamma_{t,k}$
Take any $k^* \in \arg\max\{\gamma_{t,k} : k = 1, \ldots, n_t\}$ and set $x_t \leftarrow x_{t,k^*}$, $\gamma_t \leftarrow \gamma_{t,k^*}$

tions in Definition 3.2.

Proof. The validness of $\mathcal{V}_{t-1}(\cdot)$ follows directly from the inequality (4.35) and the fact that $\underline{Q}_t(x_{t-1}; \xi_k) \le Q_t(x_{t-1}; \xi_k)$ for any $x_{t-1} \in \mathcal{X}_{t-1}$ and $\xi_k \in \Xi_t$ by definition. To see the validness of v_{t-1}, note that for any $k = 1, \ldots, n_t$ and $l \in E_{t,k}$, we have

$$\begin{aligned} Q_t(x_{t-1}; \tilde{\xi}_l) &\le \min_{x_t \in \mathcal{X}_t} \left[f_t(x_{t-1}, x_t; \tilde{\xi}_l) + \overline{\mathcal{Q}}_t^i(x_t) \right] \\ &\le f_t(x_{t-1}, x_{t,k,l}; \tilde{\xi}_l) + \overline{\mathcal{Q}}_t^i(x_{t,k,l}) \\ &\le f_t(x_{t-1}, x_{t,k,l}; \tilde{\xi}_l) + \underline{\mathcal{Q}}_t^i(x_{t,k,l}) + \gamma_{t,k,l} \\ &\le \underline{Q}_t(x_{t-1}; \tilde{\xi}_l) + \gamma_{t,k} \end{aligned}$$

by the definition of $\gamma_{t,k}$ in Algorithm 8. Thus for any optimal solution λ_t^* to the prob-

lem (4.33) with $Q_t(x_{t-1};\tilde{\xi}_l)$ replaced by $\underline{Q}_t(x_{t-1};\tilde{\xi}_l)$, we have

$$\max_{l\in E_{t,k}} Q_t(x_{t-1};\tilde{\xi}_l)-\lambda^*_{t,0}\tilde{\zeta}_l-\sum_{j=1}^{m_t}\lambda^*_{t,j}g_{t,j}(\tilde{\xi}_l) \le \max_{l\in E_{t,k}} \underline{Q}_t(x_{t-1};\tilde{\xi}_l)-\lambda^*_{t,0}\tilde{\zeta}_l-\sum_{j=1}^{m_t}\lambda^*_{t,j}g_{t,j}(\tilde{\xi}_l)+\gamma_{t,k},$$

and consequently $v_{t-1} \ge \mathcal{Q}_{t-1}(x_{t-1})$ since λ^*_t is also a feasible solution to the minimization in (4.33). Finally, the gap is controlled since $v_{t-1} - \mathcal{V}_{t-1}(x_{t-1}) = \frac{1}{n_t}\sum_{k=1}^{n_t}\gamma_{t,k} \le \gamma_{t,k^*} = \overline{\mathcal{Q}}^i_t(x_t) - \underline{\mathcal{Q}}^i_t(x_t)$. □

4.3 Numerical Experiments

In this section, we first introduce baseline models used for comparison against the MDRO model (4.1). Then we present comprehensive numerical studies of two application problems: the multi-commodity inventory problem with either uncertain demands or uncertain prices, and the hydro-thermal power system planning problem with uncertain water inflows.

4.3.1 Baseline Models and Experiment Settings

For performance comparison, we introduce two types of baseline models in addition to the MDRO with Wasserstein ambiguity sets (4.7). The first baseline model is the simple multistage robust convex optimization (MRCO) model, where we simply consider the worst-case outcome out of the uncertainty set Ξ_t in each stage t. Namely, the cost-to-go functions of the MRCO can be defined recursively as

$$\mathcal{Q}^{\text{Robust}}_{t-1}(x_{t-1}) := \sup_{\xi_t\in\Xi_t}\min_{x_t\in\mathcal{X}_t} f_t(x_{t-1},x_t;\xi_t) + \mathcal{Q}^{\text{Robust}}_t(x_t), \quad t=T,T-1,\ldots,2. \tag{4.36}$$

When the sum $f_t(x_{t-1},x_t;\xi_t) + \mathcal{Q}^{\text{Robust}}_t(x_t)$ is jointly convex in the state x_t and the uncertainty ξ_t for any given x_{t-1}, then the supremum can be attained at some extreme point of the convex hull of Ξ_t if it is finite. In particular, if we have relatively complete recourse, (i.e., the sum is always finite for any given x_{t-1}), and if Ξ_t is a polytope, (i.e., it is a convex

hull of finitely many points), then we can enumerate over the extreme points of Ξ_t to find the supremum, which allows us to solve the simple MRCO by Algorithm 5. In general, if the uncertainty set Ξ_t is unbounded, then the cost-to-go functions of the MRCO model can take $+\infty$ everywhere, so we will only use the baseline MRCO model when we have polytope uncertainty sets Ξ_t.

The second type of baseline models consists of risk-neutral and risk-averse multistage stochastic convex optimization (MSCO) models constructed from the empirical probability measures $\hat{\nu}_t$. That is, in each stage $t = 2, \ldots, T$, we only consider the outcomes $\hat{\xi}_{t,1}, \ldots, \hat{\xi}_{t,n_t}$ that have appeared in the empirical probability measure. The risk measure we use is called conditional value-at-risk (CVaR, a.k.a. average value-at-risk or expected shortfall). Its coherence leads to a dual representation [52], that allows the risk-averse MSCO models solved by Algorithm 5 with a straightforward implementation of SSSO. For simplicity, we only introduce the CVaR risk-averse MSCO based on this dual representation, and any interested reader is referred to [12] for the primal definition and the proof of duality.

Given parameters $\alpha \in (0, 1)$ and $\beta \in [0, 1]$, we define the cost-to-go functions associated with the (α, β)-CVaR risk measures recursively for $t = T, T-1, \ldots, 2$ as

$$\mathcal{Q}_{t-1}^{\mathrm{CVaR}}(x_{t-1}) := \max_{p_t \in \mathcal{P}_t^{\mathrm{CVaR}}} \sum_{k=1}^{n_t} p_{t,k} \left\{ \min_{x_t \in \mathcal{X}_t} f_t(x_{t-1}, x_t; \hat{\xi}_{t,k}) + \mathcal{Q}_t^{\mathrm{CVaR}}(x_t) \right\}, \tag{4.37}$$

where the ambiguity set is defined as

$$\mathcal{P}_t^{\mathrm{CVaR}} := \left\{ p_t = (p_{t,1}, \ldots, p_{t,n_t}) \in \mathbb{R}_{\geq 0}^{n_t} : \sum_{k=1}^{n_t} p_{t,k} = 1,\ p_{t,k} \leq \frac{\beta}{n_t} + \frac{1-\beta}{\alpha n_t}, k = 1, \ldots, n_t \right\}. \tag{4.38}$$

Note that when $\beta = 1$, the ambiguity set $\mathcal{P}_t^{\mathrm{CVaR}} = \{(\frac{1}{n_t}, \ldots, \frac{1}{n_t})\}$ has only one element corresponding to the empirical probability measure. Thus the CVaR risk-averse MSCO model (4.37) reduces to the risk-neutral nominal MSCO in this case. Alternatively, if $\beta = 0$ and $\alpha \leq \frac{1}{n_t}$, then the CVaR risk-averse MSCO model 4.37 considers only the worst

outcome of the empirical probability measure in each stage. We remark that both the simple MRCO model (4.36) and the CVaR risk-averse MSCO model (4.37) can be solved by Algorithm 5 since only finitely many outcomes need to be considered in each stage t. More details on the SSSO for these two baseline models can be found in [69].

Our numerical experiments aim to demonstrate two attractive aspects of the MDRO models on some application problems: better out-of-sample performance compared to the baseline models, and ability to achieve out-of-sample performance guarantee with reasonable conservatism. For ease of evaluation, we assume that we have the knowledge of the true underlying probability measures ν_t for each stage $t = 2, \ldots, T$ on infinite support sets Ξ_t. The experiments are then carried out in the following procedures.

1. Draw n_t iid samples from ν_t to form the empirical probability measures $\hat{\nu}_t$;

2. Construct the baseline models and MDRO models using $\hat{\nu}_t$;

3. Solve these models using our DDP algorithm (Algorithm 5) to a desired accuracy or within the maximum number of iterations or computation time;

4. Draw N iid sample paths from $(\nu_2, \ldots, \nu_T)$ and evaluate the performance profiles (mean, variance, and quantiles) of the models on these sample paths.

In particular, we focus on limited or moderate training sample sizes $n_t \in \{5, 10, 20, 40\}$, while keeping our sizes of evaluation sample paths to be large ($N = 100,000$). In each independent test run of our numerical experiment, the training samples used in a smaller-sized test are kept in larger-sized tests, and the evaluation sample paths are held unchanged for all models and sample sizes.

Our algorithms and numerical examples are implemented using Julia 1.6 [86], with Gurobi 9.0 [87] interfaced through the JuMP package (version 0.23) [78]. We use 25 single-core 2.1GHz Intel Xeon processors (24 for the worker processes and 1 for the manager process) with 50 GByte of RAM to allow parallelization of the SSSO (Algorithm 8).

4.3.2 Multi-commodity Inventory Problems

We consider a multi-commodity inventory problem which is adapted from the ones studied in [39, 69]. Let $\mathcal{J} \coloneqq \{1, 2, \ldots, J\}$ denote the set of product indices. We first describe the variables in each stage $t \in \mathcal{T}$. We use $x^l_{t,j}$ to denote the variable of inventory level, $y^a_{t,j}$ (resp. $x^b_{t,j}$) to denote the amount of express (resp. standard) order fulfilled in the current (resp. subsequent) stage, and $y^r_{t,j}$ to denote the amount of rejected order of each product $j \in \mathcal{J}$. Let $x_t \coloneqq (x^l_{t,1}, \ldots, x^l_{t,K}, x^b_{t,1}, \ldots, x^b_{t,K})$ be the state variable and $y_t \coloneqq (y^a_{t,1}, \ldots, y^a_{t,K}, y^r_{t,1}, \ldots, y^r_{t,K})$ be the internal variable for each stage $t \in \mathcal{T}$. The stage t subproblem can be defined through the local cost functions f_t as

$$f_t(x_{t-1}, x_t; \xi_t) \coloneqq \min_{y_t} \quad C^F + \sum_{j \in \mathcal{J}} \left(C^a_{t,j} y^a_{t,j} + C^b_{t,j} x^b_{t,j} + C^r_j y^r_{t,j} + C^H_j [x^l_{t,j}]_+ + C^B_j [x^l_{t,j}]_- \right) \tag{4.39}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{j \in \mathcal{J}} y^a_{t,j} \leq B^c, \\
& x^l_{t,j} \leq x^l_{t-1,j} + y^a_{t,j} + x^b_{t-1,j} + y^r_{t,j} - D_{t,j}, \quad \forall\, j \in \mathcal{J}, \\
& y^a_{t,j} \in [0, B^a_j], \quad \forall\, j \in \mathcal{J}, \\
& x^b_{t,j} \in [0, B^b_j], \quad \forall\, j \in \mathcal{J}, \\
& y^r_{t,j} \in [0, D_{t,j}], \quad \forall\, j \in \mathcal{J}, \\
& x^l_{t,j} \in [B^{l,-}_j, B^{l,+}_j], \quad \forall\, j \in \mathcal{J}.
\end{aligned}$$

In the definition (4.39), we use $C^a_{t,j} = C^a_{t,j}(\xi_t)$ (resp. $C^b_{t,j} = C^b_{t,j}(\xi_t)$) to denote the uncertain express (resp. standard) order unit cost, C^H_j (resp. C^B_j) the inventory holding (resp. backlogging) unit cost, C^r_j the penalty on order rejections, $C^F \equiv 1$ a positive fixed cost, B^a_j (resp. B^b_j) the bound for the express (resp. standard) order, and $B^{l,-}_j$, $B^{l,+}_j$ the bounds on the backlogging and inventory levels, $D_{t,j} = D_{t,j}(\xi_t)$ the uncertain demand for the product j, respectively. The first constraint in (4.39) is a cumulative bound B^c on the express orders,

the second constraint characterizes the change in the inventory level, and the rest are bounds on the decision variables with respect to each product. The notations $[x]_+ := \max\{x, 0\}$ and $[x]_- := -\min\{0, x\}$ are used to denote the positive and negative parts of a real number x. The initial state is given by $x^b_{0,j} = x^l_{0,j} = 0$ for all $j \in \mathcal{J}$. Before we discuss the details of the uncertain parameters $C^a_{t,j}, C^b_{t,j}$ or $D_{t,j}$, we make the following remarks on the definition (4.39).

First, it is easy to check that if we $C^a_{t,j}, C^b_{t,j}$ (resp. $D_{t,j}$) are deterministic, then Assumption 4.4 (resp. Assumption 4.3) is satisfied so we are able to apply the SSSO implementations discussed in Sections 4.2.3 and 4.2.2. Second, as the bounds $B^{l,-}_j, B^{l,+}_j$ do not change with t and all orders can be rejected (i.e., $y^r_{t,j} = D_{t,j}$ is feasible for all $j \in \mathcal{J}$), we see that the problem (4.39) has relatively complete recourse. Third, the Lipschitz constant of the value functions $Q_t(\cdot; \xi_t)$ is uniformly bounded by $\sum_{j\in\mathcal{J}} C^r_j$, so Proposition 4.2 can be applied here if we set the regularization factors to be sufficiently large $M_t \geq \sum_{j\in\mathcal{J}} C^r_j$. Besides, the Lipschitz continuity guarantees the in-sample adjustable conservatism by Theorem 4.4. Last, since all state variables are bounded, together with the above observation, we know by Theorem 3.1 that Algorithm 5 would always converge on our inventory problem (4.39).

4.3.3 Inventory Problems with Uncertain Demands

First, we consider the inventory problems with uncertain demands, where the goal is to seek a policy with minimum mean inventory cost plus the penalty on order rejections. The uncertain demands are modeled by the following expression:

$$D_{t,j}(\xi_t) := D_0 \left[1 + \cos\left(\frac{2\pi(t+j)}{\tau}\right)\right] + \bar{D} \cdot \xi_{t,j}, \quad j \in \mathcal{J}, t \in \mathcal{T}. \tag{4.40}$$

Here, D_0 is a factor and τ is the period for the base demands, and $\bar{D}$ is the bound on the uncertain demands. The uncertain vector $\xi_t \in [0, 1]^J$ has its components described as

follows: $\xi_{t,1} \sim \text{Uniform}(0,1)$, and for $j = 2, \ldots, J$, we have

$$\xi_{t,j} \mid \xi_{t,j-1} \sim \begin{cases} \text{Uniform}(0, (1+\xi_{t,j-1})/2), & \text{if } \xi_{t,j-1} \leq \dfrac{1}{2}, \\ \text{Uniform}(\xi_{t,j-1}/2, 1), & \text{otherwise.} \end{cases} \tag{4.41}$$

For the experiments, we consider $J = 3$ products and $T = \tau = 5$ stages. The unit prices of each product are deterministically set to $C^a_{t,j} = 5$ and $C^b_{t,j} = 1$ for all $t \in \mathcal{T}$; the inventory and holding costs are $C^H_j = 2$ and $C^B_j = 10$, and the rejection costs are $C^r_j = 100$, for each $j \in \mathcal{J}$. The bounds are set to $B^c = 15$, $B^a_j = 10$, $B^b_j = 20$, $B^{l,-}_j = 10$, and $B^{l,+}_j = 100$ for each $j \in \mathcal{J}$. We pick the uncertainty parameters $D_0 = 5$ and $\bar{D} = 50$. We terminate the DDP algorithm if it reaches 1% relative optimality or 2000 iterations. For Wasserstein ambiguity sets, we only consider the radius constraint (i.e., $m_t = 0$) with radius set to be relative to the following estimation of the distance among data points:

$$\hat{d}_t := \max_{k=1,\ldots,n_t} \frac{1}{n_t} \sum_{k' \neq k} \|\hat{\xi}_{t,k} - \hat{\xi}_{t,k'}\|. \tag{4.42}$$

For the CVaR baseline models, we consider parameters $\alpha \in \{0.01, 0.05, 0.10\}$ and $\beta \in \{0.0, 0.25, 0.50, 0.75\}$.

Using the experiment procedure described in Section 4.3.1, we present the results of three independent test runs of our data-driven MDRO model with Wasserstein ambiguity sets and the baseline models.

Figures 4.1, 4.2, and 4.3 display the out-of-sample performance quantiles of the nominal stochastic model and the MDRO models with different Wasserstein radii, constructed from the empirical probability measures $\hat{\nu}_t$. Here we use the log radius $-\infty$ to denote the nominal stochastic model, i.e., $\rho_{t,0} = 0$. From the figures, we see that in small-sample case ($n_t = 5$), the Wasserstein MDRO model significantly reduces the top 10% out-of-sample evaluation costs when the radius is set to be $10^{-1.4} - 10^{-1.0}$ of the estimation $\hat{d}_t$. Moreover, the difference between top 10% and bottom 10% of the out-of-sample evalua-

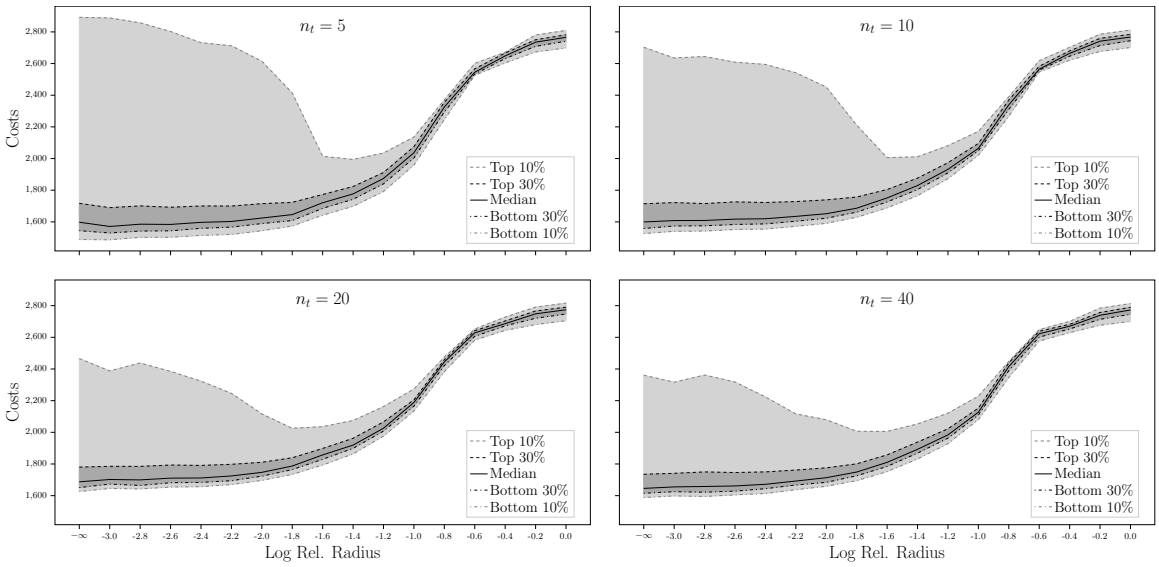


Figure 4.1: Out-of-sample Performance of (4.1) with Different Wasserstein Radii, Test Run No. 1

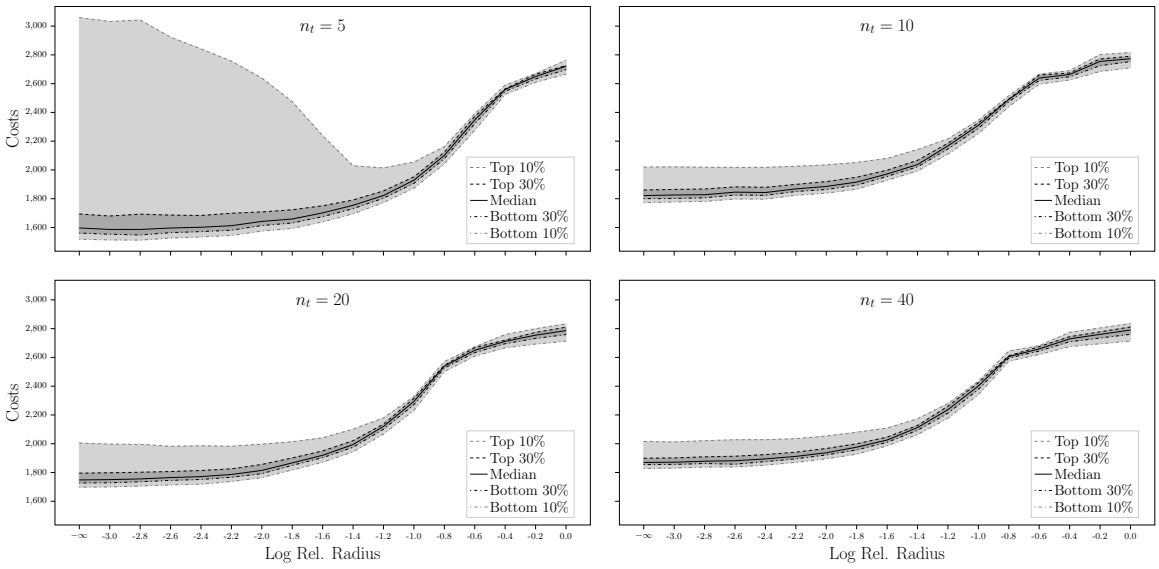


Figure 4.2: Out-of-sample Performance of (4.1) with Different Wasserstein Radii, Test Run No. 2

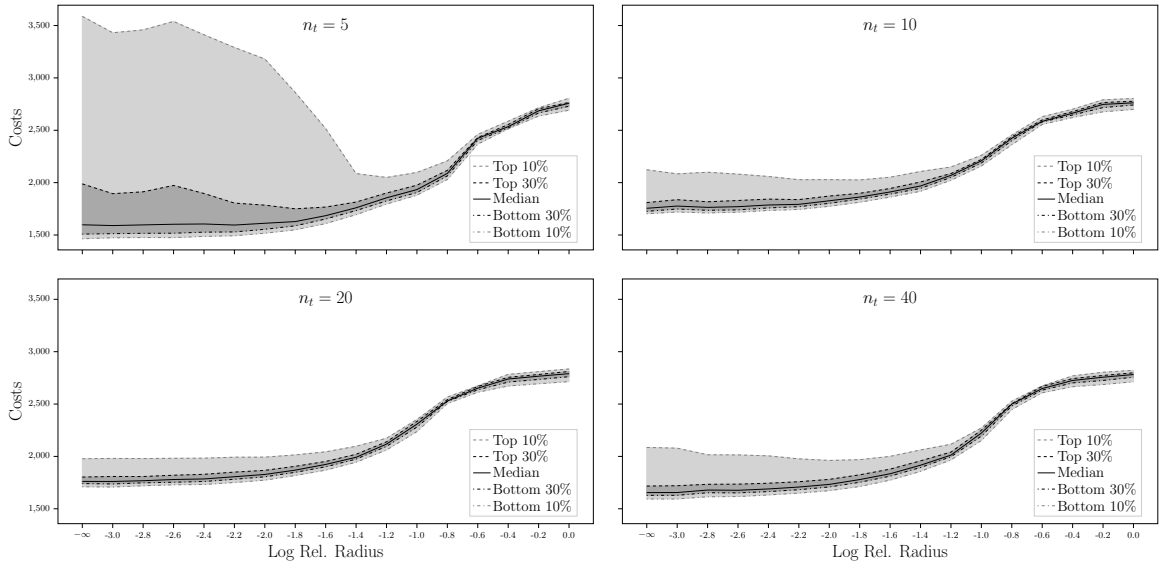


Figure 4.3: Out-of-sample Performance of (4.1) with Different Wasserstein Radii, Test Run No. 3

tion costs becomes smaller around the $10^{-1.0} \cdot \hat{d}_t$ even for larger sample sizes. However, the median out-of-sample cost increases with the Wasserstein radius, suggesting that larger Wasserstein radii in the MDRO model could lead to overly conservative policies.

To better quantify the trade-off between mean and variance (or equivalently standard deviation) of the out-of-sample evaluation costs on the policies from different models, we present Figures 4.4, 4.5, and 4.6. Here, the lines connect the points representing Wasserstein MDRO models from the smallest radius to the largest one. We say one policy dominates another policy if the former has smaller mean and standard deviation than the latter does, and have the following observations. First, in all cases, the policy from the robust model is dominated by some policy from the Wasserstein MDRO model. Second, the policy from the nominal stochastic model is dominated by some policy from the MDRO model with a small Wasserstein radius, when the sample size is small ($n_t = 5$ or sometimes 10). Last, policies from the CVaR model are dominated by policies from the MDRO model, until the sample sizes increase to $n_t = 20$ or 40.

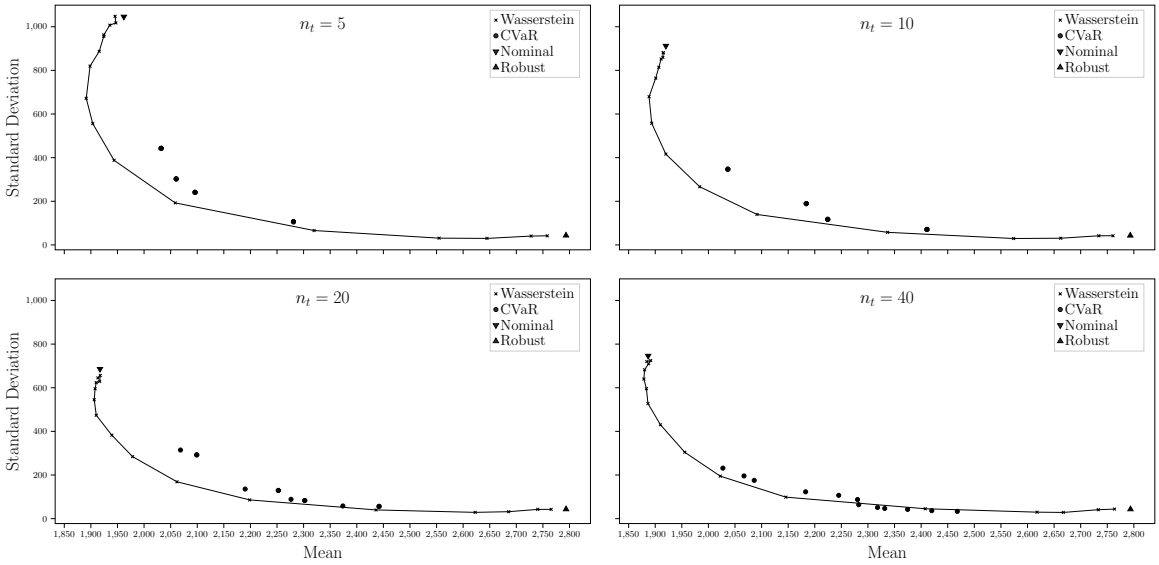


Figure 4.4: Out-of-sample Performance Comparison against Baseline Models, Test Run No. 1

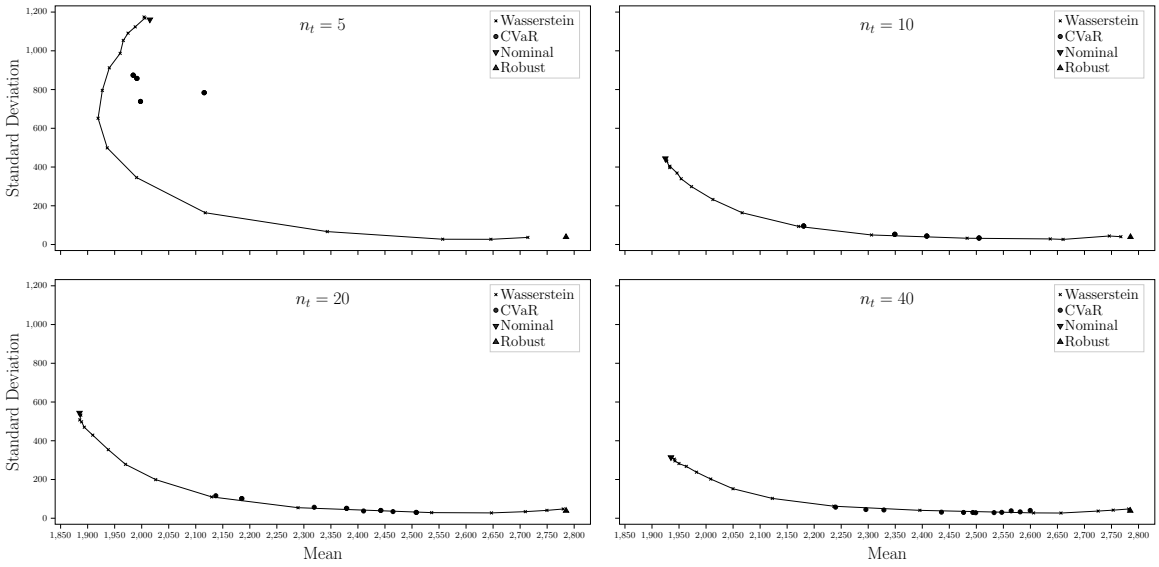


Figure 4.5: Out-of-sample Performance Comparison against Baseline Models, Test Run No. 2

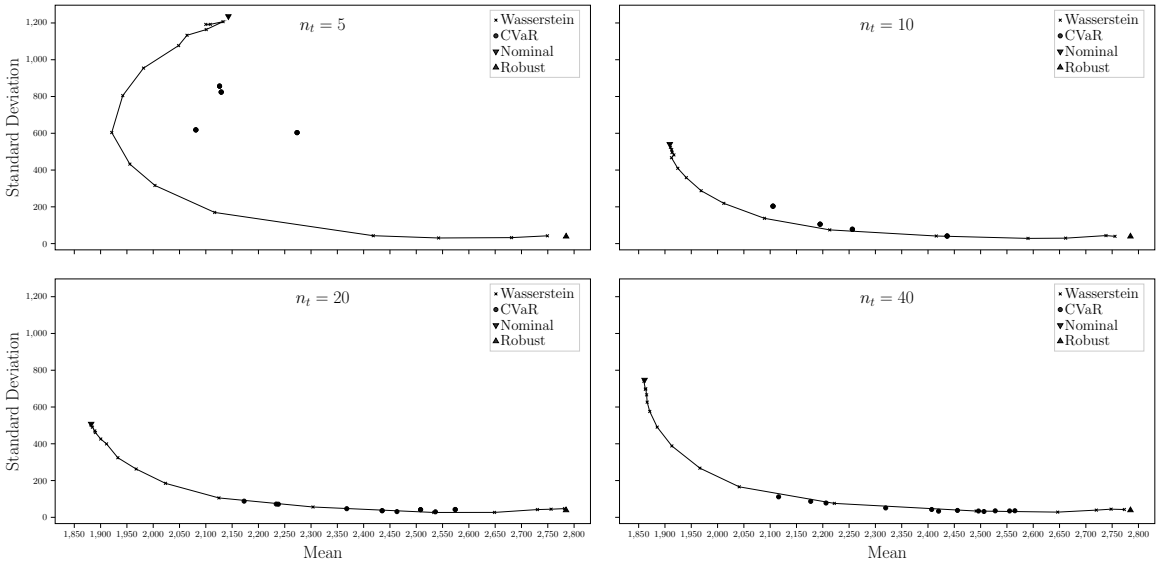


Figure 4.6: Out-of-sample Performance Comparison against Baseline Models, Test Run No. 3

4.3.4 Inventory Problems with Uncertain Prices

Now we discuss the inventory problems with uncertain prices and fixed demands. These problems can be viewed as a simplified model for supply contract problems [88], where the goal is to estimate the total cost of such supply contract and under-estimation would be very undesirable. The uncertain prices are modeled by the following expression:

$$C^b_{t,j}(\xi_t) := \xi_{t,j}, \quad C^a_{t,j}(\xi_t) := C_1 \cdot \xi_{t,j}, \quad j \in \mathcal{J}, t \in \mathcal{T}. \tag{4.43}$$

Here, C_1 is a factor for express orders. The uncertain vector $\xi_t \in \mathbb{R}^J_{\geq 0}$ follows a truncated multivariate normal distribution:

$$\xi_t := \max\left\{\text{Normal}(\mu_t, \bar{C} \cdot \Sigma_t), \underline{C}\right\}, \quad \mu_t := C_0 \left[1 + \sin\left(\frac{2\pi(t+j)}{\tau}\right)\right], \tag{4.44}$$

where the maximum is taken componentwise, C_0 is a factor for base prices, τ is the period, $\bar{C}$ is the magnitude on the price variation, $\underline{C}$ is the lower bound on the prices, and the covariance matrix Σ_t is randomly generated (by multiplying a uniformly distributed random

matrix with its transpose) and normalized to have its maximum eigenvalue equal to 1. The demands are deterministically given by

$$D_{t,j} := D_0 \left[1 + \cos\left(\frac{2\pi(t+j)}{\tau} \right) \right] + \bar{D}, \quad j \in \mathcal{J}, t \in \mathcal{T}. \tag{4.45}$$

For the experiments, we consider $J = 5$ products, $T = 10$ stages, and the period $\tau = 5$. The price uncertainty has parameters $C_0 = 1$, $C_1 = 5$, $\bar{C} = 0.1$, and $\underline{C} = 0.001$. We choose the demand parameters $D_0 = 5$ and $\bar{D} = 10$. The inventory and holding costs are $C_j^H = 1$ and $C_j^B = 10$, and the rejection costs are $C_j^r = 100$, for each $j \in \mathcal{J}$. The bounds are set to $B^c = 15$, $B_j^a = 10$, $B_j^b = 20$, $B_j^{l,-} = 20$, and $B_j^{l,+} = 20$ for each $j \in \mathcal{J}$. The Wasserstein radii in the MDRO models are set relatively with respect to $\hat{d}_t$ defined in (4.42).

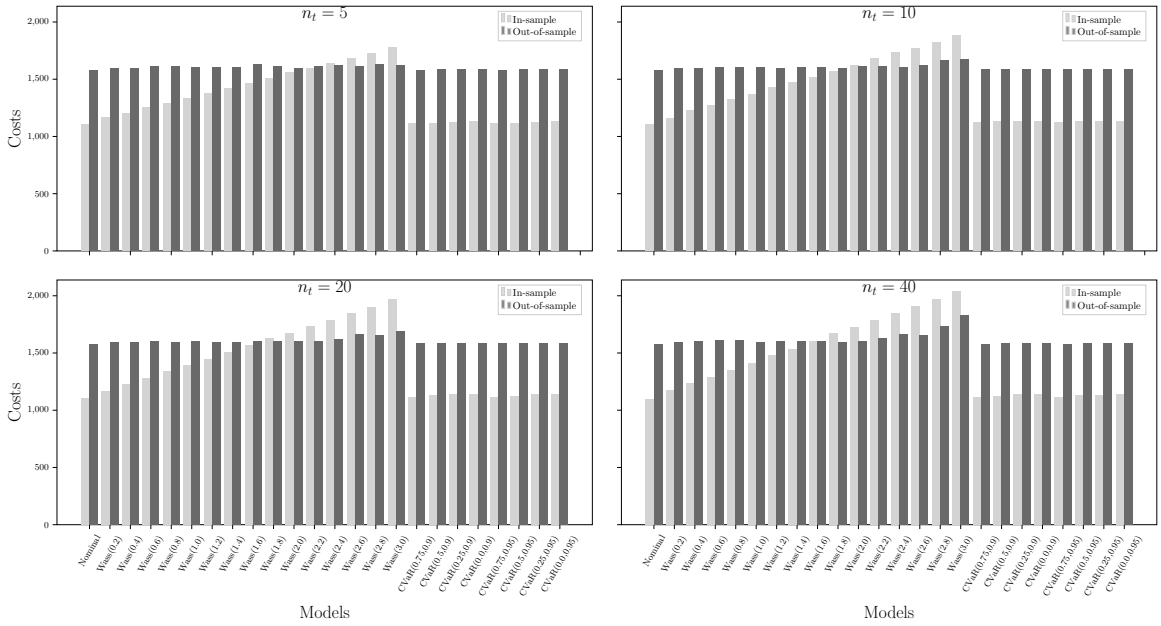


Figure 4.7: Comparison of In-sample and Out-of-sample Costs, Test Run No. 1

We plot the in-sample objective costs and out-of-sample mean evaluation costs in Figures 4.7, 4.8, and 4.9. The label $\mathrm{Nominal}$ refers to the nominal stochastic model; $\mathrm{Wass}(\gamma)$ refers to the Wasserstein MDRO model with radius $\rho_{t,0} = \gamma \cdot \hat{d}_t$ in each stage $t \geq 2$; and $\mathrm{CVaR}(\alpha, \beta)$ refers to the CVaR risk-averse model with parameters α and β. As the uncertainty vectors now have an unbounded support, the robust model is no longer appli-

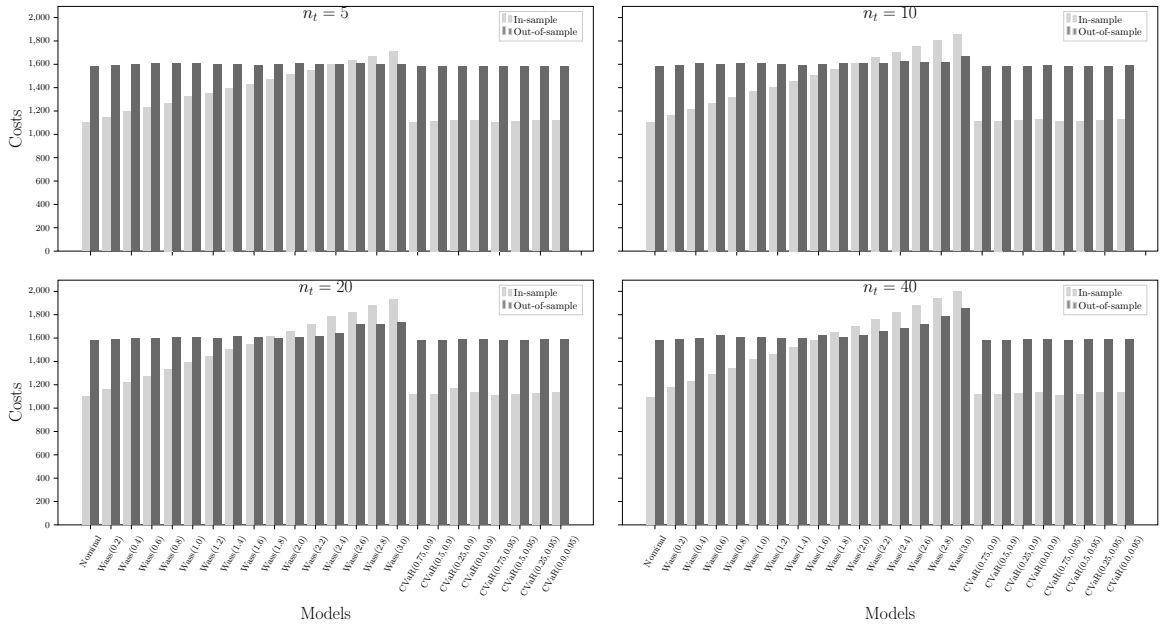


Figure 4.8: Comparison of In-sample and Out-of-sample Costs, Test Run No. 2

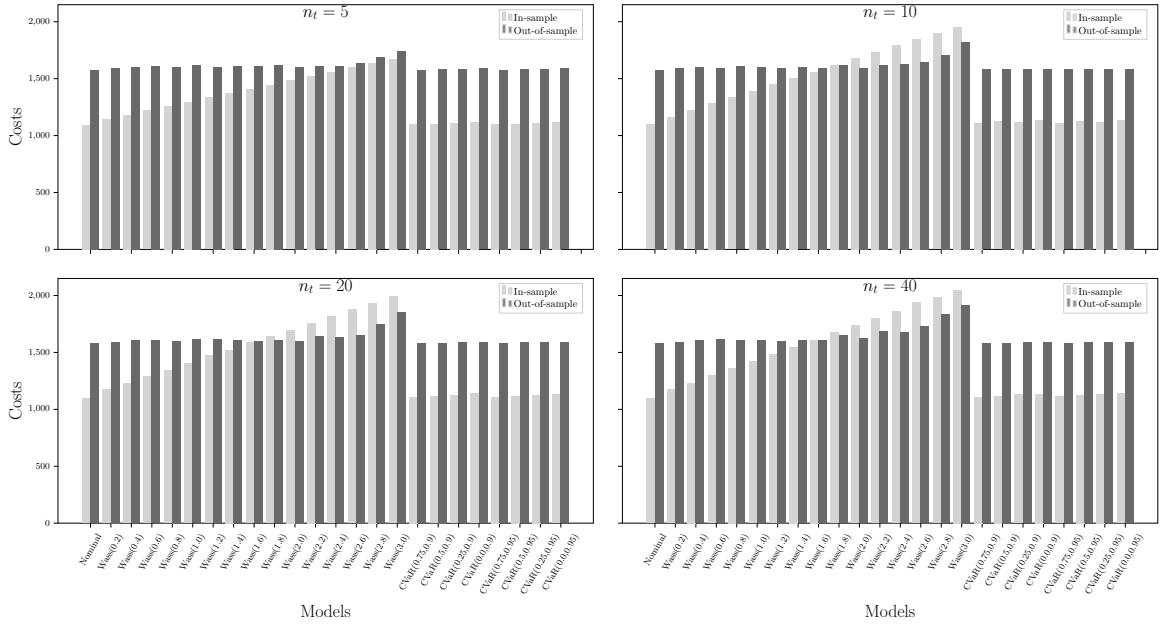


Figure 4.9: Comparison of In-sample and Out-of-sample Costs, Test Run No. 3

cable. We see that in all cases, the in-sample objective cost grows linearly with respect to the Wasserstein distance, as predicted by Theorem 4.4. As the nominal stochastic model inevitably under-estimates the mean evaluation costs, using Wasserstein MDRO models with a relative radius $\gamma \in [1.6, 2.4]$ depending on the sample size n_t could achieve the out-of-sample performance guarantee in almost all test cases. Moreover, none of the CVaR risk-averse model in the experiments could achieve similar effect. Thus we believe that the Wasserstein MDRO models are particularly more favorable in the context of supply contracts. It is however worth mentioning that we do not observe any improvement of the mean or the variance of evaluation costs from the Wasserstein MDRO model over the baseline models.

CHAPTER 5
CONCLUSION

In this dissertation, we first propose three algorithms in a unified framework of DDP for solving mixed-integer nonlinear MSO problems. The first algorithm is a generalization of the classic nested Benders decomposition algorithm, which deals with general scenario trees without the stagewise independence property. The second and third algorithms generalize DDP with sampling procedures on a stagewise independent scenario tree, where the second algorithm uses a deterministic sampling approach, and the third one uses a randomized sampling approach. The proposed algorithms are built on regularization of value functions, which enables them to handle problems with value functions that are non-Lipschitzian or discontinuous. We show that the regularized problem preserves the feasibility and optimality of the original multistage program, when the corresponding penalty reformulation satisfies exact penalization. The key ingredient of the proposed algorithms is a new class of cuts based on generalized conjugacy for approximating nonconvex cost-to-go functions of the regularized problems.

We obtain upper and lower bounds on the iteration complexity of the proposed algorithms on mixed-integer nonlinear MSO problem classes that allow exact Lipschitz regularization with predetermined penalty functions and parameters. The complexity analysis is new and deepens our understanding of the behavior of SDDP. For example, it is the first time to prove that the iteration complexity of the deterministic DDP depends polynomially on the number of stages, not exponentially, for both convex and nonconvex multistage stochastic programs, and this complexity dependence can be reduced to linear if the optimality gap is allowed to scale linearly with the number of stages, or if all the state spaces are finite sets. These findings resolve a conjecture on the scalability of DDP-type algorithms.

Second, we proposed a new class of algorithms that generalize and strengthen DDP

algorithms to solve a broad class of convex MDRO problems. The new algorithms use regularization to effectively control the growth of Lipschitz constants in the approximation and to handle problems without relatively complete recourse. We provide a thorough complexity analysis of the new algorithms given SSSO, proving both upper complexity bounds and a matching lower bound, which reveal, in a precise way, the dependence of the complexity of the DDP-type algorithms on the number of stages, the dimension of the decision space, and various regularity characteristics of MDRO. This is the first complexity analysis of DDP-type algorithms in such a general setting, and we believe it provides key insights for further developing efficient computational tools for the very many applications of sequential decision making under uncertainty. We also provide numerical examples to show the capability of the DDP-type algorithms method to solve problems without relatively complete recourse, and reduction in computation time and number of subproblem oracle evaluations, due to the regularization technique.

Then we apply the DDP algorithms to convex data-driven MDRO models with Wasserstein ambiguity sets. Using a verifiable generalized Slater condition, we derive a strong duality result that allows finite dimensional dual recursion of the MDRO models. We also review the out-of-sample performance guarantee in the multistage setting, and prove that the in-sample conservatism is adjustable when the value functions are Lipschitz continuous in the uncertainty vectors. By exploiting either the concavity or the convexity of uncertain local cost functions, we propose two new implementations of SSSO that ensures the convergence of DDP algorithms on these MDRO models. Extensive numerical experiments on inventory problems are conducted to measure the performance of policies obtained from the MDRO models against those from the risk-neutral and CVaR-based risk-averse empirical MSO models, and the standard MRO model. We observe that the MDRO models yield policies that reduce the out-of-sample mean cost by up to 3.24%-10.35% with 35.75%-51.13% reduction of the standard deviation, compared with the risk-neutral empirical MSO model, and dominate the risk-averse empirical MSO models for small sample sizes and the MRO

model in all sample sizes. Moreover, for inventory problems with uncertain prices, the policies from MDRO models are able to achieve the desired out-of-sample performance guarantee with no more than 2.59% increase of the out-of-sample mean cost in almost all cases.

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