

SYMMETRY, GEOMETRY, MODALITY.

by

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Abstract

This thesis consists of four studies into symmetry and geometry in modal homotopy type theory. First, we prove a higher analogue of Schreier’s classification of group extensions by means of non-abelian cohomology. Second, we put forward a definition of modal fibration suitable for synthetic algebraic topology, and characterize the modal fibrations for the homotopy type modality as those maps for which the homotopy types of their fibers form a local system on the homotopy type of the base. Third, we put forward a synthetic definition of orbifold, and show that all proper étale groupoids are orbifolds in this sense. And fourth, we construct the modal fracture hexagon of a higher group, and use this to derive the differential cohomology hexagon in synthetic differential geometry.

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Chapter 1

Introduction

Je n'ai fait celle-ci plus longue
que parce que je n'ai pas eu le
loisir de la faire plus courte.

Blaise Pascal

Henri Poincaré quipped that “mathematics is the art of calling different things by the same name”. Contemporary mathematicians are not satisfied with calling different things by the same name; we want to find explicit ways to identify them. This algebra of differences between things which are in some other sense the same is known as homotopical algebra, and it is steadily transforming the state of the art in differential geometry, algebraic geometry, and mathematical physics.

Homotopy type theory (HoTT) [Uni13] is a novel formal system for constructing mathematical objects and proving theorems which takes the notion of *identification* of mathematical objects as primitive, and therefore bakes homotopical algebra into the basics of logic. As a consequence, all constructions performed in HoTT are already *derived*; they are functorial with respect to

identifications. For simple types, this functoriality may just involve substituting equal elements; but in types with higher structure, it implies a general homotopy invariance.

Constructions and theorems in HoTT may be interpreted in *models*, which are universes of mathematics constructed in the usual foundations. As an example of modeling formal systems, ZF set theory without the law of excluded middle and the axiom of choice has models in all (1-)toposes of sheaves of sets as defined in ZFC (Zermelo-Frankel with choice). That is, when proving things about “sets” in a suitably constructive set theory, one gets theorems about sheaves of sets over any site. This process of interpreting theorems in constructive set theory into theorems about sheaves is known as the Kripke-Joyal semantics [MM92], and it can be used to prove tricky theorems about sheaves using arguments which would feel at home in an undergraduate algebra course [Ble18].

HoTT extends this interpretation dramatically. HoTT has models in all ∞ -toposes, a difficult theorem whose proof was initiated by Voevodsky, extended by Kapulkin and Lumsdaine [KL18], and finally completed by Shulman [Shu17; Shu19a]. As a result of this theorem, proofs written in HoTT prove general theorems about higher stacks, valid in any ∞ -topos. As an example, the Blakers-Massey theorem has an elegant proof in HoTT [HF+16] which not only applies in all ∞ -toposes but shows that the core of the argument can be given in *modal* terms [Ane+20].

In other words, homotopy type theory is a logical framework for working directly with higher stacks as if they were defined by what elements they have

— just like sets. Theorems proven in modal HoTT apply to wide classes of ∞ -toposes, specializing to results in differential, analytic, and algebraic geometry, as well as algebraic topology and homotopy theory.

An important benefit of working in ∞ -toposes over 1-toposes is that cohomology theories become representable. That is, for any cohomology theory E^\bullet defined over the site of the topos, there is a sequence of objects E_n (a *spectrum*) so that that (identification classes of) maps $X \rightarrow E_n$ correspond to cohomology classes in $E^n(X)$. Often, cohomology theories will be invariant over a certain class of objects: discrete cohomology theories (which classify bundle gerbes with *flat* connection) are invariant over contractible spaces; motivic cohomology theories are invariant over deformations by \mathbb{A}^1 ; crystalline cohomology is invariant over infinitesimal deformation; equivariant cohomology theories may be invariant over some subgroups. When the cohomology theory is represented by an object E in an ∞ -topos, its invariance in deformation by an object A can be expressed by saying that E is *A-local*: every map $A \rightarrow E$ factors uniquely (up to identification) through the point. For any object E , we may construct its *A-localization* $E \rightarrow L_A E$, the universal map from E to an *A-local* type $L_A E$. This operation L_A of localization by A is an example of a *modality*.

In higher topos theory, a modality is an orthogonal factorization system in which the left class is stable under pullback [Ane+20; Ane+18a]. The corresponding theory of localizations and modalities was developed in HoTT by [RSS20]. Modalities in HoTT resemble the $S4$ possibility operator from classical modal logic — whence the name — but may be applied to any type and not just to a proposition [Rey91; RZ91].

Modalities abound in mathematics. The Postnikov sections of a space are

given by localizing at the homotopy n -spheres [Uni13]; the homotopy type of a manifold is given by localizing it at the real line \mathbb{R} [Shu18a]; the motivic homotopy type of a simplicial scheme is given by localizing it at the affine line \mathbb{A}^1 [MF99]; the de Rham stack of a stack is its localization at infinitesimal varieties [Sch13a]; the strict quotient of an equivariant homotopy type is given by localizing it at an equivariant delooping of the equivariant group itself [Rez14]; the n -excisive approximations of a finitary functor also arise as modalities [Ane+18a]. Modalities can also be used to give a synthetic theory of spectra in HoTT [RFL21], giving modalities a place not just in describing the invariance properties of cohomology theories, but also in providing a synthetic theory of the cohomology theories themselves.

When HoTT is interpreted in an ∞ -topos, the types have both a *homotopical* structure, in the form of identifications between their points, and a spatial *cohesion* amongst their points, coming from the site of the ∞ -topos. *Cohesion* is Lawvere’s term [Law07] for a general notion of continuity encompassing not only continuity in the ordinary sense, but also more particularly smoothness, analyticity, holomorphicity, or algebraicness, depending on the site in question. Lawvere axiomatizes this general notion of cohesion for 1-toposes using adjoint functors; Shulman and Schreiber extended this theory to ∞ -toposes using modalities [SS14; Sch13a].

In *real cohesive* HoTT [Shu18a], the site is the category of Euclidean spaces \mathbb{R}^n and the continuous maps between them; this gives the types a cohesion resembling that of a continuous manifold. Localizing a type at the real line \mathbb{R} then collapses this cohesion into the homotopical structure of types by identifying points connected by paths. The resulting modality is known as the

shape modality, and it can be thought of as sending a type to its *fundamental* ∞ -groupoid.

My work in HoTT concerns the *modal geometry of higher groups*. Higher groups are homotopical generalizations of groups which give rise to *non-abelian cohomology theories*. Modalities give a way to study invariance properties of these non-abelian cohomology theories. This thesis consists of four studies into symmetry and geometry in modal homotopy type theory:

1. In Chapter 2, we will review the theory of higher groups in homotopy type theory and prove a higher analogue of Schreier's classification of group extensions.
2. In Chapter 3, I put forward a definition of *modal fibrations*. For the homotopy type modality on topological stacks, a modal fibration is a map $\pi : E \rightarrow B$ for which the homotopy types of the fibers $\mathbf{fib}_\pi(b)$ form a local system on the base B .
3. In Chapter 4, I put forward a synthetic definition of orbifold in the setting of synthetic differential geometry, and show that proper étale groupoids are orbifolds in this new sense.
4. In Chapter 5, I study differential cohomology from the modal point of view, constructing the modal fracture hexagon of a higher group. Furthermore, I construct the classifying types for circle gerbes with connection which represent ordinary differential cohomology, and derive the differential cohomology hexagon from the modal fracture hexagon of these classifying types.

1.1 Chapter 2: Higher groups and the higher Schreier theorem

As an example of the change in perspective which HoTT brings, we can consider the case of group theory. Traditionally, a group G is presented as a set with a binary operation satisfying some axioms, but in HoTT, we work directly with a type BG — known as a *delooping*, a term borrowed from homotopy theory — of mathematical objects that the group is the symmetries of. For example, instead of working with the group $GL_n(\mathbb{R})$, we would work with the type $BGL_n(\mathbb{R})$ of n -dimensional real vector spaces, since the symmetries of \mathbb{R}^n considered as an n -dimensional real vector space is $GL_n(\mathbb{R})$. When modeled in an ∞ -topos of sheaves on suitable spaces, the type $BGL_n(\mathbb{R})$ becomes the classifying stack for n -dimensional vector bundles. This shows a powerful feature of HoTT: to give the definition of some structure is to construct the classifying stack for bundles whose fibers have that structure.

When working with deloopings, there is no reason to assume that the symmetries form a set — they might form a type with higher dimensional identifications. This means that group theory in HoTT is natively higher group theory [BDR18a]. This change of perspective lets us prove powerful theorems about higher groups with ease. As an example, we can consider the case of Schreier theory. In 1926, Schreier [Sch26] classified all extensions of a group G by a group K using explicit cocycle conditions. Schreier's theorem was then extended by Eilenberg and Mac Lane [EM42], who showed that central extensions by an abelian group may be classified by group cohomology, Giraud [Gir71] who interpreted the theorem in terms of the non-abelian cohomology of stacks, and

Breen [Bre92], who extended the classification result to stacks of 2-groups. I have extended Schreier’s classification to all higher groups.¹

Theorem 2.5.7. *Let G and K be higher groups. Then the type of extensions of G by K is equivalent to the type of actions of G on a delooping \mathbf{BK} of K .*

Since HoTT has models in all ∞ -toposes, we get as a corollary the Schreier theorem for all stacks of ∞ -groups. From this, I recover the classical formulation of Schreier’s theorem in terms of group cohomology, and its extension to stacks of groups by Breen. Despite its generality, the argument in HoTT is both simple and concrete, and gives an explicit construction of the equivalence — a feature which typifies synthetic arguments given in HoTT.

1.2 Chapter 3: Modal fibrations

In Chapter 3, I develop a notion of *modal fibration* suitable for doing algebraic topology in modal HoTT. A map is a *shape fibration* when the canonical map from any of its fibers to the fibers of its action on shapes is a shape-equivalence. Though the definition looks very similar to the classical notion of *quasi-fibration* [DT58a] which says that the canonical map from the strict fiber to the homotopy fiber is a homotopy equivalence, the class of modal fibrations is much better behaved. Not only does a shape fibration induce a long exact sequence on homotopy groups (as a quasi-fibration does), but it induces a full monodromy action of the shape of the base on the shape of the fibers. In other words, the

¹My proof of this theorem does not use any cocycle conditions at all. The cocycle conditions can be derived by applying this theorem to an explicit presentation of n -groups and their homomorphisms as higher algebraic structures.

homotopy types of the fibers of a shape fibration form local system on the base, and moreover:

Theorem 3.1.2. *A map $\pi : E \rightarrow B$ is a shape fibration if and only if the shapes of its fibers $\int \text{fib}_\pi(b)$ form a local system on the base B .*

Since I prove this characterization for any modality — not just the shape modality — it also applies in many other contexts. For example, it characterizes the \mathbb{A}^1 -local fibrations as those maps whose \mathbb{A}^1 -localization of their fibers form an \mathbb{A}^1 -local system on the base. Or, in a simple example, I prove that the n -truncation fibrations are precisely those maps which are surjective on π_{n+1} ; this theorem then shows that a map of (sheaves of) homotopy types is (locally) surjective on π_{n+1} if and only if the n -truncation of its fibers are acted upon trivially by n^{th} stage of the Whitehead tower of the codomain. Examples of shape fibrations are produced by:

Theorem 3.6.1. *If there is a (crisply) discrete type F such that all the shapes $\int \text{fib}_\pi(b)$ of the fibers of a map $\pi : E \rightarrow B$ are identifiable with F , then π is a shape fibration.*

This theorem justifies a motto that if a map has a generic fiber up to homotopy, then it is a fibration, and is valid in any sort of cohesion — smooth, analytic, algebraic — not only real cohesion. This let me produce a litany of examples of shape fibrations, including shape fibrations over orbifolds and Lie groupoids.

1.3 Chapter 4: The synthetic geometry and homotopy theory of orbifolds

Using the theory of shape fibrations, I developed the theory of covering spaces in cohesive HoTT (see Chapter 3) in a way which extends seamlessly to orbifolds and Lie groupoids.

I have developed some of the theory of orbifolds in cohesive HoTT using both the modalities of cohesion and the de Rham stack modality [Che17]. In Chapter 4, I give constructions of orbifolds as homotopy quotients of higher group actions which in HoTT may be presented directly in terms of elements; for example $\mathcal{M}_{1,1}$ may be constructed as the type of pairs consisting of a 1-dimensional complex vector space and a lattice within it. Orbifolds so constructed are *micro-linear*, a theorem which would connect the synthetic homotopy theory of modal HoTT with *synthetic differential geometry* [Law79; Law80; Koc06; MBL18]. A benefit to working in HoTT is that, unlike in the traditional theory, suitably weak maps between orbifolds can be defined pointwise, as opposed to the traditional theory where maps of orbifolds are defined in terms of covering theory [Yam90] or bicategories of fractions [Cou+14]. For example, the universal cover of $\mathcal{M}_{1,1}$ by the upper half plane \mathcal{H} is the map sending $\tau \in \mathcal{H}$ to the pair $(\mathbb{C}, \mathbb{Z} \oplus \tau \mathbb{Z}) \in \mathcal{M}_{1,1}$.

It is not just the so-called “good orbifolds” — the quotients of smooth spaces by the actions of discrete groups — which are microlinear. Any orbifold which may be presented as a proper étale groupoid ([MP97]) is microlinear. Intuitively, however, an orbifold is simply a smooth space whose points have finite groups of internal symmetries. For this reason, I put forward a synthetic definition of

orbifold.

Definition 4.1.2. An orbifold is a *microlinear* type for which the type of identifications between any two points is *properly finite*.

A set is properly finite when it is discrete and a subquotient of a finite set. The usual definition of microlinearity from synthetic differential geometry, when simply imported into homotopy type theory, continues to give a suitable notion of “smooth space”. The main theorem of Chapter 4 is that any proper étale groupoid is an orbifold in this sense.

Theorem 4.6.37. *A (crisp, ordinary) proper étale groupoid is an orbifold in the sense of Definition 4.1.2.*

1.4 Chapter 5: Differential Cohomology and Modal Fracture

There are many situations where cohomology is useful but we need more than just the information of cohomology classes and their relations in cohomology — we need the information of specific cocycles which give rise to those classes and cochains which witness these relations. A striking example of this situation is *differential cohomology* [CS85]. The differential cohomology of a manifold X is characterized by its relationship to the integral cohomology of X and the differential forms on X by *character diagram* or the *differential cohomology hexagon* [SS08]. In his book [Sch13a], Schreiber notes that the differential cohomology hexagon arises from the adjointness between the shape modality and its right adjoint flat co-modality in cohesive ∞ -toposes.

In Chapter 5, I show that every higher group in cohesive HoTT sits within a modal fracture hexagon which renders it into its discrete, infinitesimal, and contractible components. This gives an internal and unstable version of Schreiber’s differential cohomology hexagon. As an example of this modal fracture hexagon, I recover the character diagram characterizing ordinary differential cohomology by its relation to its underlying integral cohomology and differential form data.

$$\begin{array}{ccccc}
 & & \mathbf{B}_{\nabla}^k \mathbb{R} & \xrightarrow{\quad} & \Lambda_{\text{cl}}^{k+1} \\
 & \nearrow^{(-)_b} & \searrow^{\pi} & \nearrow^{F(-)} & \searrow^{\text{dR}} \\
 \flat \mathbf{B}^k \mathbb{R} & & \mathbf{B}_{\nabla}^k U(1) & & \flat \mathbf{B}^{k+1} \mathbb{R} \\
 & \searrow & \nearrow^{(-)_b} & \searrow^c & \nearrow \\
 & \flat \mathbf{B}^k U(1) & \xrightarrow{\beta} & \mathbf{B}^{k+1} \mathbb{Z} &
 \end{array}$$

Theorem 5.2.31. *Any (crisp) higher group sits in the middle of a modal fracture hexagon. In particular, the modal fracture hexagon of the classifying stacks $\mathbf{B}_{\nabla}^k U(1)$ of circle k -gerbes with connection is as above. Both squares are pull-backs and the top, bottom, and diagonal sequences are fiber sequences. Here, c is the underlying topological class, F is the curvature $(k + 1)$ -form, dR takes the de Rham class of a closed $(k + 1)$ -form in discrete real cohomology, $\mathbf{B}_{\nabla}^k \mathbb{R}$ is the classifying stack for affine k -gerbes with connection and $\flat \mathbf{B}^k U(1)$ classifies circle k -gerbes with flat connection.*

Chapter 2

Higher Schreier Theory

2.1 Introduction

Homotopy type theory takes a novel perspective on the theory of groups which allows for the construction of quotients by group actions without using any colimits. This approach takes very seriously the idea that a group is to be considered as the type of symmetries of a given mathematical object. Instead of working with a group G itself, we work instead with a type $\mathbf{B}G$ of *exemplars* of G — mathematical objects whose group of symmetries is G , at least up to conjugation — together with a canonical exemplar $\mathbf{pt}_{\mathbf{B}G} : \mathbf{B}G$ whose self-identifications ($\mathbf{pt}_{\mathbf{B}G} =_{\mathbf{B}G} \mathbf{pt}_{\mathbf{B}G}$) we identify with G .

For example, we may take $\mathbf{BGL}_n(\mathbb{R})$ to be the type of n -dimensional real vector spaces. That is, we take an exemplar of the group $\mathbf{GL}_n(\mathbb{R})$ to be an n -dimensional real vector space. We have a canonical exemplar of $\mathbf{GL}_n(\mathbb{R})$: the canonical n -dimensional real vector space $\mathbb{R}^n : \mathbf{BGL}_n(\mathbb{R})$. We have a canonical identification of $\mathbf{GL}_n(\mathbb{R})$ with the type of identifications ($\mathbb{R}^n =_{\mathbf{BGL}_n(\mathbb{R})} \mathbb{R}^n$), which can be proven to be equivalent to the type of automorphisms of \mathbb{R}^n considered as an n -dimensional real vector space.

As another example, we may take $\mathbf{BAut}(\underline{n})$ to be the type of n -element sets. That is, we take an exemplar of the symmetric group $\mathbf{Aut}(\underline{n})$ to be an n -element set. We again have a canonical exemplar, namely the canonical n -element set $\underline{n} \equiv \{0, \dots, n-1\}$, and again the group $\mathbf{Aut}(\underline{n})$ is canonically identified with the automorphisms of \underline{n} as an n -element set.

The HoTT approach to groups, reviewed in Section 2.2, works with the types \mathbf{BG} of exemplars—known as *deloopings* of the group G —rather than the group G itself with its algebraic structure. This translation is lossless—everything we might want to do with a group G can be done in terms of a delooping \mathbf{BG} . In particular, as explained in Section 2.3, an action of G on a type X may be equivalently given by a function $X^{\circ-} : \mathbf{BG} \rightarrow \mathbf{Type}$ which sends any exemplar $e : \mathbf{BG}$ of G to a type $X^{\circ e}$ which we call “ X twisted by e ”, together with an identification $\mathbf{pt}_{X^{\circ-}} : X^{\circ \mathbf{pt}_{\mathbf{BG}}} \simeq X$ of X twisted by the canonical exemplar $\mathbf{pt}_{\mathbf{BG}}$ with X itself.

For example, the action of $\mathbf{GL}_n(\mathbb{R})$ on the set $\mathbb{R}^n - \{0\}$ of non-zero vectors in \mathbb{R}^n may be given by the function $V \mapsto V - \{0\} : \mathbf{BGL}_n(\mathbb{R}) \rightarrow \mathbf{Type}$, noting that this function sends the canonical exemplar \mathbb{R}^n to the set $\mathbb{R}^n - \{0\}$ we were trying to act on. Or, the action of the symmetric group $\mathbf{Aut}(\underline{n})$ on the vector space \mathbb{R}^n given by permuting the coordinates may be equivalently given by the function $X \mapsto \mathbb{R}^X : \mathbf{BAut}(\underline{n}) \rightarrow \mathbf{BGL}_n(\mathbb{R})$ sending a finite set X to the vector space of real-valued functions on X , noting that we may canonically identify $\mathbb{R}^{\underline{n}}$ with \mathbb{R}^n .

This approach to groups is as radical as it is elementary. It begins to pay major dividends in the construction of quotients by group actions. In Section 2.4, we describe a construction of the quotient $X // G$ of the type X

by the action of a group G as the type of a pairs (e, x) where $e : \mathbf{B}G$ is an exemplar of G , and $x : X^{\circ e}$ is an element of X twisted by the exemplar e . For example, the configuration space $\mathbb{R}^n // \mathbf{Aut}(\underline{n})$ of n unlabelled points in \mathbb{R}^n may be defined as the type of pairs (X, v) where X is an n -element set and $v : \mathbb{R}^X$ is a real-valued function on X . The quotient map itself is given by sending $x : X$ to $(\mathbf{pt}_{\mathbf{B}G}, x)$, remembering that we identify $X^{\circ \mathbf{pt}_{\mathbf{B}G}}$ with X . In our example, the quotient $\mathbb{R}^n \rightarrow \mathbb{R}^n // \mathbf{Aut}(\underline{n})$ is given by sending $v : \mathbb{R}^n$ to the pair (\underline{n}, v) .

To see that this type of pairs constructs the quotient, let's consider an identification $p : (\underline{n}, v) = (\underline{n}, w)$ between the images of two vectors v and $w : \mathbb{R}^n$ under the quotient map. By a few elementary HoTT lemmas concerning identifications, an identification p is equivalently given by a pair of identifications (σ, q) where $\sigma : \underline{n} = \underline{n}$ is a self-identification of the canonical n -element set with itself — a permutation — and where $q : v \circ \sigma = w$ is an identification of v with w *relative* to the identification σ . That is, the type $(\underline{n}, v) = (\underline{n}, w)$ is equivalent to the set of all permutations $\sigma : \mathbf{Aut}(\underline{n})$ which send v to w under the action of permuting coordinates: $v \circ \sigma = w$.

As you can see, we have gotten something more out of this simple process of taking pairs than just the usual set-theoretic quotient. Yes, if we have a σ so that $v \circ \sigma = w$, then we will get an identification $(\underline{n}, v) = (\underline{n}, w)$ of their images in the quotient, so that two vectors in the same orbit of this action are identified in the quotient. But furthermore, we remember exactly which permutations σ send v to w : the type $(\underline{n}, v) = (\underline{n}, w)$ is the *set* of all such permutations. The quotients $X // G$ constructed by taking pairs $(e : \mathbf{B}G) \times X^{\circ e}$ are often called *weak quotients* or *homotopy quotients*, though they are stronger than the usual set theoretic quotient in that they contain more information, and they have

nothing in particular to do with continuous deformation.

In Section 2.5, we will prove a higher generalization of Schreier’s classification of group extensions. This higher Schreier theorem shows that the extensions of a higher group G by a higher group K correspond to actions of G on a delooping BK of K . We can see this as a generalization of the elementary characterization of split extensions — semi-direct products — by homomorphic actions of G on K .

We will begin this chapter by reviewing the theory of (higher) groups in homotopy type theory in Section 2.2. In homotopy type theory, one works with a *delooping* of a group G , rather than the group itself. A delooping of G is a type BG with a fixed element $\mathbf{pt}_{BG} : BG$ whose group of symmetries is G — that is we have an isomorphism $G \simeq (\mathbf{pt}_{BG} = \mathbf{pt}_{BG})$ — and where every other element $e : BG$ is somehow identifiable with \mathbf{pt}_{BG} , though not canonically. As an example, we may take $BGL_n(\mathbb{R})$ to be the type of n -dimensional real vector spaces with $\mathbf{pt}_{BGL_n(\mathbb{R})}$ defined to be \mathbb{R}^n ; by definition, $GL_n(\mathbb{R})$ is the group of linear automorphisms of \mathbb{R}^n , and every n -dimensional vector space is isomorphic to \mathbb{R}^n , though of course not canonically since such isomorphisms are equivalent to a choice of basis. In Definition 2.2.4, we will introduce terminology for the elements of deloopings of groups: we will call $e : BG$ an *exemplar* of G , and we will call \mathbf{pt}_{BG} the *canonical exemplar*.¹ We’ll give a number of examples of exemplars of common groups to help this concept settle.

Next, in Section 2.3, we will review how actions of a group can be described in terms of a delooping BG . In particular, an action of a group G on a type

¹Far be it from the author to tell you how to pronounce your own words, but in my opinion, pronouncing the final syllable of “exemplar” as in the word “exemplary” makes it sound much less pompous than pronouncing it as the final syllable of “templar”.

X is equivalently a function which assigns to any exemplar $e : \mathbf{B}G$ of G a type $X^{\circ e}$ — called “ X twisted by e ” — in such a way that $X^{\circ \mathbf{pt}_{\mathbf{B}G}}$ is identified with X . Like with deloopings, this is best understood in terms of examples, and so we provide them.

Then, in Section 2.4, we receive a delightful payout for this reformulation of group theory. We will see in Definition 2.4.1 that the quotient of a type X by the action of a group G may be constructed as the type of pairs (e, x) with $e : \mathbf{B}G$ is an exemplar of G , and $x : X^{\circ e}$ is an element of the type X twisted by e . The quotient map itself sends $x : X$ to the pair $(\mathbf{pt}_{\mathbf{B}G}, x)$, remembering that we identify X with $X^{\circ \mathbf{pt}_{\mathbf{B}G}}$. We’ll note that this quotient is the “weak quotient” or “homotopy quotient” of the action: an identification $p : (\mathbf{pt}_{\mathbf{B}G}, x) = (\mathbf{pt}_{\mathbf{B}G}, y)$ is equivalently given by an element $g : G$ such that $gx = y$ (see Lemma 3.7.5 and Remark 2.4.4). In particular, the automorphisms of the point $(\mathbf{pt}_{\mathbf{B}G}, x)$ may be identified with the stabilizer of x . In this way, the elements of quotients so constructed pick up non-trivial internal symmetries.

2.2 (Higher) Groups in homotopy type theory

In homotopy type theory, we take the maxim that “a group is the group of symmetries of some mathematical object” as a definition. A symmetry is a self-identification of this object, considered as an object of a given type. We might therefore think of defining a group as a pair (X, x) of a type X of objects and an object $x : X$ of this type. The group G itself would then be the type of symmetries of this object (as an element of the type X):

$$G \equiv (x =_X x).$$

However, this definition keeps around too much baggage. For the pair (X, x) to be uniquely determined by the group $G \equiv (x = x)$ that it represents, we would need to show that two such pairs (X, x) and (Y, y) are equivalent if and only if their associated groups of symmetries $(x =_X x)$ and $(y =_Y y)$ are equivalent. However, if X has other elements x' which are not somehow identifiable with x , then there is no hope for this. Conversely, though, if every element of X is somehow identifiable with the chosen object x — if X is 0-connected — then we can prove the following *fundamental theorem of higher groups*.

Theorem 2.2.1 (Folklore). *Let X and Y be pointed, 0-connected types. That is, suppose that $\mathbf{pt}_X : X$ and $\mathbf{pt}_Y : Y$, and that for any $x : X$ there is merely an identification of x with x_0 , and similarly for $y : Y$. That is, suppose we have $(x : X) \rightarrow \|x = \mathbf{pt}_X\|$ and similarly $(y : Y) \rightarrow \|y = \mathbf{pt}_Y\|$. Then any function $f : X \rightarrow Y$ with $\mathbf{pt}_f : \mathbf{pt}_Y = f(\mathbf{pt}_X)$ is an equivalence if and only if the induced function*

$$\Omega f := p \mapsto \mathbf{pt}_f \bullet f_* p \bullet \mathbf{pt}_f^{-1} : (\mathbf{pt}_X = \mathbf{pt}_X) \rightarrow (\mathbf{pt}_Y = \mathbf{pt}_Y)$$

is an equivalence.

Proof. If f is an equivalence, it is straightforward to show that Ωf is as well. So, we prove the converse.

Suppose that Ωf is an equivalence. We will show that f is by showing that its fiber over any $y : Y$ is contractible. Since contractibility is a proposition and Y is 0-connected, we may assume a $p : \mathbf{pt}_Y = y$, which gives us an equivalence $\mathbf{fib}_f(\mathbf{pt}_Y) \simeq \mathbf{fib}_f(y)$. So, it will suffice to show that $\mathbf{fib}_f(\mathbf{pt}_Y)$ is contractible.

Now,

$$\begin{aligned} \text{fib}_f(\text{pt}_Y) &::= (x : X) \times (\text{pt}_Y = f(x)) \\ &\simeq (x : X) \times (f(\text{pt}_X) = f(x)) \end{aligned}$$

by $\text{pt}_f : \text{pt}_Y = f(\text{pt}_X)$. Now, $\Omega f = \text{pt}_f \cdot f_* \cdot \text{pt}_f^{-1}$ is an equivalence, so its conjugate $f_* : (\text{pt}_X = \text{pt}_X) \rightarrow (f(\text{pt}_X) = f(\text{pt}_X))$ is an equivalence. But we would like for $f_* : (\text{pt}_X = x) \rightarrow (f(\text{pt}_X) = f(x))$ to be an equivalence, because if it is, then

$$\begin{aligned} \text{fib}_f(\text{pt}_Y) &\simeq (x : X) \times (f(\text{pt}_X) = f(x)) \\ &\simeq (x : X) \times (\text{pt}_X = x) \\ &\simeq *. \end{aligned}$$

Luckily, $f_* : (\text{pt}_X = x) \rightarrow (f(\text{pt}_X) = f(x))$ being an equivalence is also a proposition, and since X is 0-connected we may assume a $q : \text{pt}_X = x$. Then we have a commuting square

$$\begin{array}{ccc} (\text{pt}_X = \text{pt}_X) & \xrightarrow{f_*} & (f(\text{pt}_X) = f(\text{pt}_X)) \\ \bullet q \downarrow & & \downarrow \bullet f_* q \\ (\text{pt}_X = x) & \xrightarrow{f_*} & (f(\text{pt}_X) = f(x)) \end{array}$$

by the functoriality of f_* . In this square, the top map and vertical maps are equivalences, Therefore, the bottom map is an equivalence, which proves the theorem. \square

With this theorem in hand, we can make the following definition of (higher)

group in homotopy type theory.

Definition 2.2.2 ([BDR18b]). A higher group is a type G identified with the type

$$\Omega\mathbf{B}G := (\mathbf{pt}_{\mathbf{B}G} = \mathbf{pt}_{\mathbf{B}G})$$

of self-identifications of the base point of a pointed, 0-connected type $\mathbf{B}G$. We refer to $\mathbf{B}G$ as a *delooping* of G . We say that G is an n -group if $\mathbf{B}G$ is n -truncated, or equivalently if G itself is $(n - 1)$ -truncated.

The basic theory of higher groups is developed in [BDR18b], with a further development forthcoming in the textbook [Bez+22]. By way of summary, in homotopy type theory we work with deloopings as pointed, 0-connected types, rather than with groups as algebraic structures. This change of perspective has deep ramifications for concrete calculations, which we will try to describe now.

If G is an ordinary (1-)group, then we can always deloop it by taking $\mathbf{B}G$ to be the type of G -torsors.

Proposition 2.2.3 ([Bez+22]). Let G be a 1-group. A G -torsor is a free, transitive, and inhabited (left) action of G on a set. The type \mathbf{Tors}_G of G -torsors, pointed at G acting on itself on the left, is a delooping of G .

In order to make this definition a bit more concrete, here is the full definition

of the type of G torsors:

$$\begin{aligned}
& (X : \mathbf{Type}) \times ((x, y : X) \rightarrow (p, q : x = y) \rightarrow (p = q)) \\
& \quad \times (\alpha : G \times X \rightarrow X) \\
& \quad \times ((x : X) \rightarrow (\alpha(1, x) = x)) \\
& \quad \times ((x : X) \rightarrow (g, h : G) \rightarrow (\alpha(gh, x) = \alpha(g, \alpha(h, x)))) \\
& \quad \times \left((x, y : X) \rightarrow \left\{ \begin{array}{l} ((g, p) : (g : G) \times (\alpha(g, x) = y)) \\ \times (((h, q) : (h : G) \times (\alpha(h, x) = y)) \rightarrow ((g, p) = (h, q))) \end{array} \right. \right) \\
& \quad \times \|X\|
\end{aligned}$$

The point of writing this type out in full is to show how a definition of an algebraic structure satisfying certain axioms can be written out using a few basic type constructors. Note that this is a tuple (\times) consisting of many functions (\rightarrow), some of which land in types of identifications ($=$).

The first pair $(X : \mathbf{Type}) \times ((x, y : X) \rightarrow (p, q : x = y) \rightarrow (p = q))$ has elements the *sets*, as defined in homotopy type theory. A set is a type where the type $x = y$ of identifications between two elements is a proposition — namely, the proposition that x and y are *equal*. A proposition is a type where any two elements may be identified; any witness to the truth of a proposition is as good as any other.

The next element $\alpha : G \times X \rightarrow X$ is the action map itself, and it is followed by the two axioms which define a group action. The second to last element witnesses that this action is a torsor. It says that for any x and y in X , there

is a unique g in G for which $\alpha(g, x) = y$. Since we assumed that X was a set, the rest of this data after the action map α is a proposition. The last element says that X is inhabited.

Using common type-theoretical shorthand, we could write this type more succinctly and clearly as:

$$\text{Tors}_G \equiv \left\{ \begin{array}{l} (X : \mathbf{Set}) \times (\alpha : G \times X \rightarrow X) \\ \quad \times (\forall x : X, \alpha(1, x) = x) \times (\forall x : X, \forall g, h : G, \alpha(gh, x) = \alpha(g, \alpha(h, x))) \\ \quad \times (\forall x, y : X, \exists! g : G, \alpha(g, x) = y) \\ \quad \times \|X\|. \end{array} \right.$$

In homotopy type theory, once you know the definition of a type of object, you also have constructed the stack which classifies bundles whose fibers are that type of object: the classifying stack is just the type itself. In particular, the type Tors_G of G -torsors classifies G -principal bundles.

Depending on what we are trying to do with our group, it might be useful to have different constructions of its delooping. In any case, we will need some special terminology for the elements of a particular delooping $\mathbf{B}G$, since we will be using these elements to do all our work with the group G .

Definition 2.2.4. Let G be a higher group and let $\mathbf{B}G$ be a delooping of G . We refer to the elements of $\mathbf{B}G$ as *exemplars* of G . We refer to the base point $\text{pt}_{\mathbf{B}G} : \mathbf{B}G$ as the *canonical exemplar* of G in $\mathbf{B}G$. We may refer to $\mathbf{B}G$ itself as a *type of exemplars* for G , and we note that there may be many different (though equivalent) types of exemplars for a given group G .

This terminology is best explained through examples. If our (higher) group of interest is the group of symmetries $\mathbf{Aut}_X(x) \equiv (x = x)$ of an object $x : X$, then we can always take x to be a canonical exemplar and define an exemplar to be an element of X which is *identifiable* with x .

Definition 2.2.5 (Standard). Let $x : X$ be an element. Then

$$\mathbf{B}\mathbf{Aut}_X(x) \equiv (y : X) \times \|y = x\|$$

is the type of all elements $y : X$ which are identifiable with x . Pointed at $(x, |\mathbf{refl}|)$, this type deloops the automorphism group $\mathbf{Aut}_X(x) \equiv (x = x)$.

We might also deloop a group G by giving a *categorical* definition of an object whose group of symmetries is G — a definition such that any two instances are identifiable.

Example 1. Let Σ_n denote the symmetric group on n elements. We can deloop Σ_n with the type

$$\begin{aligned} \mathbf{B}\Sigma_n &\equiv \mathbf{B}\mathbf{Aut}_{\mathbf{Set}}(\underline{n}) \\ &\equiv (F : \mathbf{Set}) \times \|F = \underline{n}\|, \end{aligned}$$

since Σ_n is the group of automorphisms of the standard n -element set $\underline{n} \equiv \{0, \dots, n-1\}$. Note that we can also see $\mathbf{B}\Sigma_n$ as the type of n -element sets — those sets which admit some bijection with the standard n -element set. In other words, we define an exemplar of Σ_n to be an n -element set, and take the canonical exemplar to be \underline{n} .

Example 2. Let $U(1) := \{z : \mathbb{C} \mid z\bar{z} = 1\}$ be the unit circle in the complex plane, considered as a group under multiplication. We can deloop $U(1)$ with the type $BU(1)$ of 1-dimensional Hermitian vector spaces, pointed at \mathbb{C} . Explicitly, a Hermitian vector space is a vector space V over \mathbb{C} equipped with a Hermitian inner product $\langle -, - \rangle : V \times V \rightarrow \mathbb{C}$ which is linear in the first component, conjugate symmetric, and for which $\langle x, x \rangle > 0$ for non-zero x .² Any 1-dimensional Hermitian vector space \mathcal{L} is identifiable with \mathbb{C} by some unitary isomorphism — if $\ell : \mathcal{L}$ gives a basis for \mathcal{L} , then the map $1 \mapsto \frac{\ell}{\langle \ell, \ell \rangle}$ gives a unitary isomorphism of \mathbb{C} with \mathcal{L} .

In general, $BU(n)$ may be defined to be the type of Hermitian vector spaces identifiable with \mathbb{C}^n with its standard inner product. In other words, we define an exemplar of $U(n)$ to be an n -dimensional Hermitian vector space with positive definite inner product and take the canonical exemplar to be \mathbb{C}^n with its standard inner product.

Example 3. Of course, other matrix groups work in a similar way. We can deloop $GL_n(\mathbb{R})$ with the type $BGL_n(\mathbb{R})$ of n -dimensional real vector spaces, pointed at \mathbb{R}^n . That is, we define an exemplar of $GL_n(\mathbb{R})$ to be an n -dimensional real vector space. Note that $BGL_n(\mathbb{R})$ classifies real vector bundles of rank n in a truly immediate way: the vector bundle $\pi : E \rightarrow B$ is classified by the map $b \mapsto \text{fib}_\pi(b) : B \rightarrow BGL_n(\mathbb{R})$ sending every point to the vector space sitting over it in the bundle.

We could take an exemplar of $SL_n(\mathbb{R})$ to be an n -dimensional real vector

²In the setting of this paper — specifically the axioms of synthetic differential geometry found in Section 4.4.1 — it is appropriate to ask that x be non-zero. However, in pure homotopy type theory with no classical assumptions, we should ask instead that x be *apart* from 0, meaning that there is some positive rational ε with either $x < \varepsilon$ or $x > \varepsilon$.

space V equipped with a non-zero³ element of the exterior power $\Lambda^n V$ — or equivalently a non-trivial alternating n -form on V . The canonical exemplar is \mathbb{R}^n equipped with the element $e_1 \wedge \cdots \wedge e_n$.

We could take an exemplar of $\mathbf{O}(n)$ to be an n -dimensional real vector space equipped with an inner product. The canonical exemplar is \mathbb{R}^n with its standard inner product.

We could take an exemplar of the symplectic group $\mathbf{Sp}(2n, \mathbb{R})$ to be a $2n$ -dimensional real vector space equipped with a non-degenerate alternating 2-form. The canonical exemplar is \mathbb{R}^{2n} equipped with its standard symplectic form

$$\omega(v, w) := \sum_{i=1}^n x^i(v)y^i(w) - y^i(v)x^i(w)$$

where $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ is the standard basis of \mathbb{R}^{2n} and x^i and y^i are the associated conjugate basis of $(\mathbb{R}^{2n})^*$.

Example 4. If V is a real vector space considered as an additive group, then we can take \mathbf{BV} to be the type of affine spaces whose difference vectors land in V . This is not so different than defining \mathbf{BV} to be the type of V -torsors. In other words, we define an exemplar of V to be an affine space over V , with the canonical exemplar being V itself.

If we want to deloop the full affine group of \mathbb{R}^n , we can take an exemplar to be a pair consisting of an n -dimensional real vector space V and an affine space over it. The canonical exemplar is \mathbb{R}^n paired with itself. That is,

$$\mathbf{BAffine}(\mathbb{R}^n) := (V : \mathbf{BGL}_n(\mathbb{R}^n)) \times \mathbf{BV}.$$

³Again, without classical assumptions this must instead mean “apart from zero”.

No matter what we use to deloop our group G , there is always an equivalence $\mathbf{B}G \xrightarrow{\sim} \mathbf{Tors}_G$ associating a G -torsor to any exemplar of G .

Proposition 2.2.6 ([Bez+22]). Let $\mathbf{B}G$ be a delooping of a 1-group G . Then for any exemplar $t : \mathbf{B}G$, the type $(t = \mathbf{pt}_{\mathbf{B}G})$ of identifications of t with the canonical exemplar is a G -torsor, and the function

$$t \mapsto (t = \mathbf{pt}_{\mathbf{B}G}) : \mathbf{B}G \rightarrow \mathbf{Tors}_G$$

sending the exemplar t to its *associated torsor* $(t = \mathbf{pt}_{\mathbf{B}G})$ is an equivalence.

Remark 2.2.7. A similar theorem would work for n -groups for $n > 1$ so long as an appropriate notion of torsor could be defined. For example, a 2-group is equivalently a monoidal groupoid where for every $g : G$, $g \otimes -$ is an equivalence. We could then define a torsor for a 2-group as a action $\alpha : G \times X \rightarrow X$ of this monoidal groupoid (an “actegory”) on an inhabited groupoid X for which the map $\alpha(-, x) : G \rightarrow X$ is an equivalence for every $x : X$. However, a careful proof of this would require a good deal of work.

Until a suitable theory of simplicial types can be developed in HoTT (a famous open problem), we will likely not be able to give a general theorem along the lines of Proposition 2.2.6 for general higher groups.

Remark 2.2.8. In the case that $G \equiv \mathbf{GL}_n(\mathbb{R})$, then the associated torsor of an n -dimensional vector space $V : \mathbf{BGL}_n(\mathbb{R})$ is the type $V = \mathbb{R}^n$ of linear isomorphisms of V with \mathbb{R}^n . A linear isomorphism with \mathbb{R}^n is the same thing as a basis of V — a *frame* — since V was assumed to be n -dimensional. Therefore, we see that the torsor associated to V is its space $\mathbf{Frame}(V)$ of frames.

If $E : B \rightarrow \mathbf{BGL}_n(\mathbb{R})$ classifies a vector bundle, then the composite $B \xrightarrow{E} \mathbf{BGL}_n(\mathbb{R}) \xrightarrow{\sim} \mathbf{Tors}_G$ classifies the frame bundle of that vector bundle.

As a sanity check, note that the associated torsor of a torsor is itself.

Lemma 2.2.9. The associated torsor of a G -torsor T is T itself. Explicitly, T is equivalent as a G -torsor (and therefore also as a type) to the torsor $(G =_{\mathbf{Tors}_G} T)$ of its identifications with G as a G -torsor.

Proof. An identification of G with T is determined by the image of $1 : G$ by equivariance. □

2.3 Homomorphisms and actions

We can define homomorphisms between higher groups using just their deloopings.

Definition 2.3.1 ([BDR18b]). A homomorphism $\varphi : G \rightarrow H$ between higher groups is a pointed map $\mathbf{B}\varphi : \mathbf{B}G \rightarrow \mathbf{B}H$ between their deloopings.

In other words, a homomorphism $G \rightarrow H$ is a function assigning exemplars of G to exemplars of H , together with an identification of the image of the canonical exemplar of G with that of H . The map $\varphi : G \rightarrow H$ itself is defined to be

$$\Omega \mathbf{B}\varphi(g) := \mathbf{pt}_{\mathbf{B}\varphi}^{-1} \cdot (B\varphi)_* g \cdot \mathbf{pt}_{\mathbf{B}\varphi}.$$

We can always deloop a homomorphism between ordinary groups into a map between their types of torsors by tensoring up along the homomorphism.

Proposition 2.3.2 ([Bez+22]). Let G and H be 1-groups, and let $\varphi : G \rightarrow H$ be a homomorphism. Then the map

$$\mathbf{B}\varphi := T \mapsto H \otimes_G T : \mathbf{Tors}_G \rightarrow \mathbf{Tors}_H$$

given by sending a G -torsor T to the H -torsor $H \otimes_G T$ defined by

$$H \otimes_G T := \frac{H \times T}{(h\varphi(g), t) \sim (h, gt)}$$

deloops the homomorphism φ when it is pointed at the equivalence

$$\mathbf{pt}_{\mathbf{B}\varphi} : H \otimes_G G = H$$

given by $h \otimes g \mapsto h\varphi(g)$.

Example 5. If $\varphi : G \rightarrow H$ is a *surjective* homomorphism, then we may deloop its kernel by defining $\mathbf{Bker}\varphi := \mathbf{fib}_{\mathbf{B}\varphi}(\mathbf{pt}_{\mathbf{B}H})$ to be the fiber of any delooping $\mathbf{B}\varphi : \mathbf{B}G \rightarrow \mathbf{B}H$. We need surjectivity for the fiber of $\mathbf{B}\varphi$ to be 0-connected; this is an if-and-only-if, since $\mathbf{B}\varphi$ is 0-connected if and only if its fiber is and if and only if its delooping φ is surjective (which is by definition means -1 -connected).

For example, we can reconstruct the delooping of $\mathbf{SL}_n(\mathbb{R})$ given in Example 3 by seeing $\mathbf{SL}_n(\mathbb{R})$ as the kernel of the determinant $\det : \mathbf{GL}_n(\mathbb{R}) \rightarrow \mathbf{GL}_1(\mathbb{R})$. The determinant may be delooped by the function $V \mapsto \Lambda^n V : \mathbf{BGL}_n(\mathbb{R}) \rightarrow \mathbf{BGL}_1(\mathbb{R})$, pointed at the identification $\Lambda^n \mathbb{R}^n = \mathbb{R}$ induced by the standard basis element $e_1 \wedge \cdots \wedge e_n : \Lambda^n \mathbb{R}^n$ (where e_i are the standard basis vectors of \mathbb{R}^n). Therefore, the kernel of \det may be delooped by the fiber of the function $\Lambda^n : \mathbf{BGL}_n(\mathbb{R}) \rightarrow \mathbf{BGL}_1(\mathbb{R})$, which is the type of n -dimensional vector spaces V equipped with a

linear isomorphism $\Lambda^n V = \mathbb{R}$.

We may describe an action of a group G on an object $x : X$ as a homomorphism $\alpha : G \rightarrow \mathbf{Aut}_X(x)$, which is the same as a pointed map $\mathbf{B}\alpha : \mathbf{B}G \rightarrow \mathbf{B}\mathbf{Aut}_X(x)$. In other words, we can see an action of the group G on an object x as a way of taking exemplars $t : \mathbf{B}G$ of G to objects $\mathbf{B}\alpha(t)$ identifiable with x , together with an identification $\mathbf{pt}_{\mathbf{B}\alpha} : \mathbf{B}\alpha(\mathbf{pt}_{\mathbf{B}G}) = x$ of the image of the canonical exemplar with x itself.

Definition 2.3.3. An action of a higher group G on a object $x : X$ is a pointed map $x^{\circ(-)} : \mathbf{B}G \rightarrow \mathbf{B}\mathbf{Aut}_X(x)$. An action $X^{\circ(-)} : \mathbf{B}G \rightarrow \mathbf{B}\mathbf{Aut}_X(x)$ takes an exemplar $t : \mathbf{B}G$ of G to the object $x^{\circ t} : X$ which is identifiable with x ; we say that $x^{\circ t}$ is x *twisted* by t .

The action of a higher group on a type is itself given by transport in the type family.

Lemma 2.3.4. Let $X^{\circ(-)} : \mathbf{B}G \rightarrow \mathbf{Type}$ be an action of a higher group G on a type $X := X^{\circ \mathbf{pt}_{\mathbf{B}G}}$. Then for $g : G$ and $x : X$, we have

$$gx = \mathbf{tr}(X^{\circ(-)}, g)(x).$$

Proof. The action gx is by definition given by $(X^{\circ(-)})_*g(x)$, so the desired identification follows from the fact that transporting over an identification in a type family is the same as applying the type family to the identification. \square

We can always deloop an action of a group G on a set X by twisting the action with a torsor.

Proposition 2.3.5 ([Bez+22]). Let $\alpha : G \rightarrow \text{Aut}(X)$ be an action of a group G on a set X . Then the map

$$T \mapsto T \otimes_G X : \text{Tors}_G \rightarrow \text{Set}$$

sending a G -torsor T to the tensor product $T \otimes_G X$ defined by

$$T \otimes_G X \equiv \frac{T \times X}{(t, gx) \sim (g^{-1}t, x)}$$

deloops the action α when pointed at the identification

$$\text{pt}_{\mathbf{B}\alpha} : G \otimes_G X = X$$

given by $g \otimes x \mapsto g^{-1}x$.

Remark 2.3.6. The appearance of the inverses in Proposition 2.3.5 is due to our choice to use *left* G -torsors and *left* actions. If we used *right* G -torsors and left actions then the tensor product formulas would not need any inverses.

A *representation* of a group G is therefore a pointed map $\mathbf{B}G \cdot \rightarrow \mathbf{B}\text{GL}_n(\mathbb{R})$; that is, it is a way of turning exemplars of G into n -dimensional vectors spaces in such a way that the canonical exemplar gets turned into \mathbb{R}^n .

Example 6. A representation of the symmetric group Σ_k is a function $\mathbf{B}\Sigma_k \rightarrow \mathbf{Vect}_{\mathbb{R}}$ which sends a k -element set to a vector space.

For example, we have the canonical representation of Σ_k on \mathbb{R}^k which is given by the function $\mathbf{B}\rho \equiv X \mapsto \mathbb{R}^X$ sending a k -element set X to the vector space \mathbb{R}^X which is free on it, together with the identification $\text{pt}_{\mathbf{B}\rho} : \mathbb{R}^k = \mathbb{R}^k$ which we may as well take as definitional.

Example 7. Since actions and representations of groups are given as functions of their exemplars, it is sometimes useful to come up with a new type of exemplars for the group in order to easily define an action. For example, suppose we are trying to define the action of the cyclic group C_n on the plane by rotation (for $n \geq 1$). Which delooping should we use?

First, we can think of C_n as the group μ_n of n^{th} roots of unity acting on the complex plane \mathbb{C} by multiplication. We can imagine an exemplar of μ_n as a set of n points equi-distantly arranged on a circle in a 1-dimensional complex vector space. So, we can start with a 1-dimensional Hermitian vector space \mathcal{L} , and equip this with n -element subset of its unit circle $\mathbb{S}^{\mathcal{L}}$ of equidistantly placed points. It is somewhat difficult to say that the points are equidistantly placed without an ordering on them; instead, we can say that the angle between any two of them evenly divides the circle, and that rotating by an n -th root of unity keeps us within C .

Definition 2.3.7. Let \mathcal{L} be a 1-dimensional Hermitian vector space. A *cycle of n elements* in \mathcal{L} is a subset $C \subseteq \mathbb{S}^{\mathcal{L}}$ of its unit circle such that for any $x, y \in C$, we have $\langle x, y \rangle^n = 1$ and for any n^{th} -root of unity $\zeta \in \mu_n$ and $x \in C$, the rotation $\zeta x \in C$ is also in C .

We will take a pair (\mathcal{L}, C) of a 1-dimensional Hermitian vector space and a cycle of n elements in it as an exemplar of μ_n . That is, we define

$$\mathbf{B}\mu_n := (\mathcal{L} : \mathbf{B}U(1)) \times \text{Cycle}_n(\mathcal{L}).$$

As a canonical exemplar, we take the subset $\mu_n \subseteq U(1) \subseteq \mathbb{C}$ of n^{th} roots of unity. It remains to show that $\mathbf{B}\mu_n$ is 0-connected, and that it deloops μ_n .

Suppose that $C \subseteq \mathcal{L}$ is a cycle of n -elements, seeking to show that this is identifiable with $\mu_n \subseteq \mathbb{C}$. We can identify \mathcal{L} with \mathbb{C} via a unitary transformation $U : \mathcal{L} = \mathbb{C}$, which gives us a cycle U_*C of n -elements in \mathbb{C} (since U is unitary). Now since $n \geq 1$, there is some element in U_*C , say $Ux \in U_*C$. Then the unitary transformation $(Ux)^{-1}U : \mathcal{L} = \mathbb{C}$ sends C to μ_n . Consider $(Ux)^{-1}Uy \in (Ux)^{-1}U_*C$; then $((Ux)^{-1}Uy)^n = \langle Uy, Ux \rangle^n = \langle x, y \rangle^n = 1$, so that $(Ux)^{-1}Uy$ is in μ_n . Conversely, if $\zeta \in \mu_n$, then we have $\zeta x \in C$ and so $\zeta = (Ux)^{-1}U(\zeta x) \in (Ux)^{-1}U_*C$.

Finally, $\mathbf{B}\mu_n$ does actually deloop μ_n since an unitary automorphism of \mathbb{C} which fixes μ_n setwise is given by multiplication by an element of μ_n .

Now, to define the representation of C_n on the plane is easy; we can send an exemplar $(\mathcal{L}, X) : \mathbf{B}\mu_n$ to \mathcal{L} considered as a 2-dimensional real vector space. We point this operation at the canonical identification $\mathbb{C} = \mathbb{R}^2$ given by $1 \mapsto e_1$ and $i \mapsto e_2$.

Note that in this example, we tailored the delooping of C_n to the task of constructing its action on the plane by considering a delooping of $U(1)$ which easily described its action on the plane, and then restricting this delooping to C_n by equipping the exemplars of $U(1)$ — the complex lines \mathcal{L} — with cycles of n -elements. Another way to describe a cycle of n elements in a complex line \mathcal{L} is as a μ_n -torsor which is a subaction of the μ_n action on \mathcal{L} by scalar multiplication. This recipe gives us a general way to restrict a delooping of a group to a subgroup.

Definition 2.3.8. Let G be a group, X a G -action, and Γ a subgroup of G . A Γ -subtorsor of X is a Γ -subaction of the G -action of X restricted to Γ which is

a Γ -torsor in its own right — that is, it is free, transitive, and inhabited as a Γ -action. We denote the type of Γ -subtorsors of X by $\mathbf{Subtors}_\Gamma(X)$.

Lemma 2.3.9. Let $g : G$ and let X be a G -action. Then $\mathrm{tr}(\mathbf{Subtors}_\Gamma(X^{\circ-}), g) : \mathbf{Subtors}_\Gamma(X) \rightarrow \mathbf{Subtors}_\Gamma(X)$ sends T to $g^{-1}T := \{x : X \mid \exists t : T. x = g^{-1}t\}$.

Proof. We note that a Γ -subtorsor T of X is in particular a subset $T \subseteq X$ of X which satisfies a property. For this reason, we only need to think about how transporting subsets works: $\mathrm{tr}(X^{\circ-} \rightarrow \mathbf{Prop}, g) : \mathbf{Subset}(X) \rightarrow \mathbf{Subset}(X)$ is given by taking the inverse image under $\mathrm{tr}(X^{\circ}, g) : X \rightarrow X$, which is the action by g . This is because subsets are equivalently described by the property of being in that subset, $\mathbf{Subset}(X) = (X \rightarrow \mathbf{Prop})$, and transport in the latter is given by precomposition. Therefore, $\mathrm{tr}(X^{\circ-} \rightarrow \mathbf{Prop}, g)(T) = \{x : X \mid gx \in T\} = g^{-1}T$. \square

Proposition 2.3.10. Let G be a group and Γ a subgroup. Suppose that \mathbf{BG} is a delooping of G giving us a type of exemplars for G . We may then define an exemplar of Γ to be an exemplar of G equipped with a Γ -subtorsor of its associated torsor:

$$\mathbf{B}\Gamma := (t : \mathbf{B}G) \times \mathbf{Subtors}_\Gamma(t = \mathbf{pt}_{\mathbf{B}G}).$$

We take the canonical exemplar to be the canonical exemplar $\mathbf{pt}_{\mathbf{B}G}$ of G paired with Γ considered as a Γ -subtorsor of the associated G -torsor ($\mathbf{pt}_{\mathbf{B}G} = \mathbf{pt}_{\mathbf{B}G}$), identified with G acting on itself.

Furthermore, the inclusion $\Gamma \hookrightarrow G$ is delooped by the first projection $\mathbf{fst} : \mathbf{B}\Gamma \rightarrow \mathbf{B}G$.

Proof. An identification $(\mathbf{pt}_{\mathbf{BG}}, \Gamma) = (\mathbf{pt}_{\mathbf{BG}}, \Gamma)$ is an identification $g : \mathbf{pt}_{\mathbf{BG}} = \mathbf{pt}_{\mathbf{BG}}$ so that $g^{-1}\Gamma = \Gamma$ by Lemma 2.3.9. However, Γ contains 1, so we may conclude that $g^{-1} \in \Gamma$ and therefore $g \in \Gamma$. On the other hand, any $\gamma : \Gamma$ clearly sends elements in Γ to elements in Γ , so we have a bijection between self identifications of the canonical exemplar of \mathbf{BG} with Γ . \square

2.4 Quotients as types of pairs

Now we are ready to define the quotient of a type by the action of a higher group. A beautiful feature of homotopy type theory is that the quotient has a *mapping-in* property, in addition to its usual mapping out property defining it as a quotient. What this means in practice is that we can define quotients by their elements, without forcing any equivalence relations or freely generating any structure. In fact, the construction couldn't be simpler: the quotient $X // G$ of the action X of a higher group G is the type of pairs (t, x) of an exemplar $t : \mathbf{BG}$ and an element $x : X^{\circ t}$ of X twisted by t .

Definition 2.4.1 ([Bez+22]). Given an action $X^{\circ(-)} : \mathbf{BG} \rightarrow \mathbf{Type}$ of the higher group G on the type $X := X^{\circ \mathbf{pt}_{\mathbf{BG}}}$, define the quotient

$$X // G := (t : \mathbf{BG}) \times X^{\circ t}$$

to be the type of pairs of an exemplar $t : \mathbf{BG}$ and an element of $X^{\circ t}$. The quotient map

$$[-] : X \rightarrow X // G$$

is given by pairing with the canonical exemplar: $x \mapsto (\mathbf{pt}_{\mathbf{BG}}, x)$.

This definition is justified by an elementary lemma about identifications in

types of pairs. As a corollary, we may deduce that the symmetries of an element in the quotient are precisely its stabilizer.

Lemma 2.4.2. Let $X^{\circ(-)} : \mathbf{BG} \rightarrow \mathbf{Type}$ be an action of a higher group G on a type X . For $x, y : X$, we have an equivalence

$$([x] = [y]) \simeq (g : G) \times (gx = y)$$

Proof. This follows immediately from Lemma 2.3.4 and Theorem 2.7.2 of the HoTT Book [Uni13] which characterizes identifications in pair types:

$$((\mathbf{pt}_{\mathbf{BG}}, x) = (\mathbf{pt}_{\mathbf{BG}}, y)) \simeq (g : \mathbf{pt}_{\mathbf{BG}} = \mathbf{pt}_{\mathbf{BG}}) \times (\mathbf{tr}(X^{\circ(-)}, g)(x) = y).$$

□

Corollary 2.4.3. Let $X^{\circ(-)} : \mathbf{BG} \rightarrow \mathbf{Type}$ be an action of a higher group on a type X . For $x : X$, the self-identifications of x in the quotient is the stabilizer of x :

$$\mathbf{Aut}_{X//G}([x]) \simeq (g : G) \times (gx = x) \equiv \mathbf{Stab}(x).$$

Remark 2.4.4. It is worth emphasizing that this construction of $X // G$ given in ?? (which is entirely standard in homotopy type theory) constructs what is usually known as the “homotopy” quotient, but it constructs it *on the nose*, and not “up to homotopy”. The terminology here really gets in the way, but the point is that it is *up to identification* — which is the only sort of equality in HoTT — and not up to continuous deformation that the quotient $X // G$ behaves like it is supposed to.

We can also characterize the homotopy quotient maps $q : X \rightarrow X // G$ quite simply, at least when G is a 1-group: they are precisely those maps whose fibers

are G -torsors. Or, in other words, the homotopy quotients by G are precisely the G -principal bundles.

Theorem 2.4.5. *Let G be a group $q : X \rightarrow Y$ be a map. Suppose that for any $y : Y$ we have a G -torsor structure on $\text{fib}_q(y)$. Then G acts on X and q is equivalent to the homotopy quotient $X \rightarrow X // G$ by this action.*

Proof. We will construct an action α of G on X by constructing its delooping $\mathbf{B}\alpha : \mathbf{Tors}_G \rightarrow \mathbf{BAut}(X)$. We define

$$\mathbf{B}\alpha(T) := (y : Y) \times (T = \text{fib}_q(y))$$

where the identification is taken as G -torsors. Then

$$\begin{aligned} \mathbf{B}\alpha(G) &\equiv (y : Y) \times (G = \text{fib}_q(y)) \\ &\simeq (y : Y) \times \text{fib}_q(Y) \\ &\simeq X. \end{aligned}$$

gives us a pointing of $\mathbf{B}\alpha$. Furthermore, since \mathbf{Tors}_G is 0-connected, this shows that $\mathbf{B}\alpha(T)$ is identifiable with X for all $T : \mathbf{Tors}_G$, so that $\mathbf{B}\alpha$ really does land in $\mathbf{BAut}(X) \equiv (Z : \mathbf{Type}) \times \|Z = X\|$. The middle equivalence follows from Lemma 2.2.9, and the last equivalence is a general fact about any map — it is always the sum of its fibers. Therefore, this equivalence identifies $x : X$ with $(q(x), (1 \mapsto (x, \text{refl})))$ where $(1 \mapsto (x, \text{refl})) : G = \text{fib}_q(q(x))$ is the identification of G -torsors determined by sending 1 to (x, refl) and the rest by G -equivariance.

Now, we will give an equivalence $Y = X // G$ which commutes with the

quotient maps. This follows quickly by the substitution lemma:

$$\begin{aligned} X // G &\equiv (T : \mathbf{Tors}_G) \times (y : Y) \times (T = \mathbf{fib}_q(y)) \\ &\simeq Y. \end{aligned}$$

Explicitly, this equivalence $Y = X // G$ is given by $y \mapsto (\mathbf{fib}_q(y), y, \mathbf{refl})$. It remains to show that for $x : X$, we have $(\mathbf{fib}_q(q(x)), q(x), \mathbf{refl}) = (G, q(x), (1 \mapsto (x, \mathbf{refl})))$. Since we may take $(1 \mapsto (x, \mathbf{refl}))^{-1} : \mathbf{fib}_q(q(x)) = G$ and transporting by this sends $(1 \mapsto (x, \mathbf{refl}))$ to $\mathbf{refl} : G = G$, we have our desired identification. \square

2.5 Schreier Theory for Higher Groups

The aim of this section is to prove the fundamental result of ‘‘Schreier Theory’’, namely to give for any groups F and G an equivalence

$$\{\text{Extensions of } G \text{ by } F\} \simeq (\mathbf{B}G \cdot \rightarrow \mathbf{BAut}(\mathbf{B}F))$$

between extensions E of G by F and actions of G on $\mathbf{B}F$. The classical theory can be found on the *nlab* page ‘‘Group Extensions’’.

Definition 2.5.1. An *extension* of the ∞ -group G by the ∞ -group F is an ∞ -group E together with homomorphisms $i : F \rightarrow E$ and $p : E \rightarrow G$ whose deloopings

$$\mathbf{B}F \cdot \rightarrow \mathbf{B}E \cdot \rightarrow \mathbf{B}G$$

form a fiber sequence. We denote the type of these extensions by $\mathbf{Ext}(G; F)$

This definition is justified by the following lemma.

Lemma 2.5.2. If F and G are (1-)groups, then the type of ∞ -group extensions of G by F is equivalent to the type of group extensions of G by F .

We will in fact prove something stronger than the traditional Schreier theorem, namely we will prove the theorem for any two ∞ -groups F and G . What's remarkable is that the proof is basically a trivial rearrangement of terms.

We need two lemmas which extend fundamental lemmas of type theory.

Definition 2.5.3. Given a pointed type X , a *pointed type family* E on X is a type family $E : X \rightarrow \mathbf{Type}$ together with $\mathbf{pt}_E : E(\mathbf{pt}_X)$.

Given a pointed type family E on X , the type of *dependent pointed functions*

$$(x : X) \cdot \rightarrow E(x) \equiv (f : (x : X) \rightarrow E(x)) \times (f(\mathbf{pt}_X) = \mathbf{pt}_E)$$

First, we classify pointed maps into a fixed pointed type B .

Lemma 2.5.4. For any pointed type B , we have an equivalence

$$(E : \mathbf{Type}_*) \times (\pi : E \cdot \rightarrow B) \simeq (c : B \rightarrow \mathbf{Type}) \times c(\mathbf{pt}_B)$$

between pointed maps over B and pointed type families varying over B , extending the usual equivalence of maps into B with type families varying in B .

Proof.

$$\begin{aligned}
(E : \mathbf{Type}_*) \times (\pi : E \cdot \rightarrow B) &\simeq \left\{ \begin{array}{l} (E : \mathbf{Type}) \times (\mathbf{pt}_E : E) \times \\ (\pi : E \rightarrow B) \times \\ (\mathbf{pt}_\pi : \pi(\mathbf{pt}_E) = \mathbf{pt}_B) \end{array} \right. \\
&\simeq \left\{ \begin{array}{l} (c : B \rightarrow \mathbf{Type}) \times (\mathbf{pt}_E : (t : B) \times c(t)) \times \\ (\mathbf{fst}(\mathbf{pt}_E) = \mathbf{pt}_B) \end{array} \right. \\
&\simeq (c : B \rightarrow \mathbf{Type}) \times c(\mathbf{pt}_B)
\end{aligned}$$

□

And then we classify solutions to pointed lifting problems.

Lemma 2.5.5. Let $f : X \cdot \rightarrow B$ and $\pi : E \cdot \rightarrow B$ be pointed functions. Then we have an equivalence

$$\left\{ \begin{array}{ccc} & & E \\ & \nearrow & \downarrow \pi \\ X & \xrightarrow{f} & B \end{array} \right\} \simeq (x : X) \cdot \rightarrow \mathbf{fib}_\pi(fx)$$

Now, we set out to prove the theorem. First, we need to simplify the definition of an ∞ -group extension.

Lemma 2.5.6. For any ∞ -groups F and G , we have an equivalence

$$\mathbf{Ext}(G; F) \simeq (\mathbf{BE} : \mathbf{Type}_*) \times (\mathbf{B}p : \mathbf{BE} \cdot \rightarrow \mathbf{B}G) \times (\mathbf{fib}_{\mathbf{B}p} \doteq \mathbf{B}F)$$

Proof. This follows from the universal property of the fiber.

$$\begin{aligned}
\text{Ext}(G; F) &\equiv \left\{ \begin{array}{l} (\mathbf{B}E : \mathbf{Type}_*^{>0}) \times (\mathbf{B}p : \mathbf{B}E \cdot \rightarrow \mathbf{B}G) \times \\ (\mathbf{B}i : \mathbf{B}F \cdot \rightarrow \mathbf{B}E) \times \\ (\text{null} : \mathbf{B}p \circ \mathbf{B}i = \text{pt}_{\mathbf{B}F \rightarrow \mathbf{B}G}) \times \\ ((\mathbf{B}i, \text{null}) : \mathbf{B}F \cdot \rightarrow \text{fib}_{\mathbf{B}p}) \text{ is an equiv} \end{array} \right. \\
&\simeq \left\{ \begin{array}{l} (\mathbf{B}E : \mathbf{Type}_*^{>0}) \times (\mathbf{B}p : \mathbf{B}E \cdot \rightarrow \mathbf{B}G) \times \\ (e : \mathbf{B}F \cdot \rightarrow \text{fib}_{\mathbf{B}p}) \times \\ e \text{ is an equiv} \end{array} \right. \\
&\simeq \left\{ \begin{array}{l} (\mathbf{B}E : \mathbf{Type}_*^{>0}) \times (\mathbf{B}p : \mathbf{B}E \cdot \rightarrow \mathbf{B}G) \times \\ (\mathbf{B}F \doteq \text{fib}_{\mathbf{B}p}) \end{array} \right.
\end{aligned}$$

Finally, we don't need to assume that $\mathbf{B}E$ is 0-connected. Since $\mathbf{B}G$ and $\mathbf{B}F$ are 0-connected, the fiber of $\mathbf{B}p$ over the point of $\mathbf{B}G$ is and so all the fibers of $\mathbf{B}p$ are. Therefore, $\mathbf{B}E$ is always the sum of 0-connected types indexed by a 0-connected type, and so is 0-connected. \square

Theorem 2.5.7. *Let F and G be ∞ -groups. Then the type of extensions of G by F is equivalent to the type of actions of G on a delooping $\mathbf{B}F$ of F :*

$$\text{Ext}(G; F) \simeq (\mathbf{B}G \cdot \rightarrow \mathbf{BAut}(\mathbf{B}F)).$$

Proof. We put Lemma 2.5.6 together with Lemma 2.5.4.

$$\begin{aligned}
\text{Ext}(G; F) &\simeq (c : \mathbf{B}G \rightarrow \mathbf{Type}) \times (\text{pt}_c : c(\text{pt}_{\mathbf{B}G})) \times (c(\text{pt}_{\mathbf{B}G}) \doteq \mathbf{B}F) \\
&\simeq (c : \mathbf{B}G \rightarrow \mathbf{Type}) \times (\text{pt}_c : c(\text{pt}_{\mathbf{B}G})) \times (e : \mathbf{B}F = c(\text{pt}_{\mathbf{B}G})) \times (\text{pt}_c = e\text{pt}_{\mathbf{B}F}) \\
&\simeq (c : \mathbf{B}G \rightarrow \mathbf{Type}) \times (c(\text{pt}_{\mathbf{B}G}) = \mathbf{B}F)
\end{aligned}$$

Since $\mathbf{B}G$ is connected, for any $t : \mathbf{B}G$ we have $\|c(t) = \mathbf{B}F\|$, so that we finally have

$$\simeq (\mathbf{B}G \cdot \rightarrow \mathbf{BAut}(\mathbf{B}F))$$

□

2.6 Centers and Central Extensions

In this section, we will define the *center* of a higher group, and classify *central* extensions of higher groups. Traditionally, central extensions are classified by the second delooping B^2A of an abelian group A . But the notion of *central* extension does not generalize uniquely to higher groups. We will show that B^2ZA , the second delooping of the *center* of A , classifies what we would naively call central extensions. For A an abelian group, $B^2ZA = B^2A$, but this is not true in general for higher braided groups A .

A central element is an element which conjugates the identity to itself. For a group G this is the same as an unpointed automorphism of the identity $\text{id}_G : \mathbf{B}G = \mathbf{B}G$. Such an automorphism $p : \text{id}_G = \text{id}_G$ can be equivalently given the type $p : (t : \mathbf{B}G) \rightarrow (t = t)$, which shows that such a p is a fixed point of the action of G on itself by conjugation, which is delooped by sending an exemplar

$t : \mathbf{B}G$ to $(t = t)$.

Definition 2.6.1. The *center* ZG of an ∞ -group G is the type of *unpointed* automorphisms if the identity $\mathrm{id}_G : \mathbf{B}G = \mathbf{B}G$.

$$ZG := ((\mathrm{id}_G) = (\mathrm{id}_G)) = ((t : \mathbf{B}G)) \rightarrow (t = t).$$

So far, we have defined ZG as a type; it should probably be an ∞ -group as well. To show that it is an ∞ -group, we need to give a delooping of it. Delooping is more of an art than a science – it’s not always possible. However, in this case, we are in luck because ZG is already type of loops. Namely,

$$ZG := ((t : \mathbf{B}G) \rightarrow t = t) \simeq (\mathrm{id}_{\mathbf{B}G} = \mathrm{id}_{\mathbf{B}G}) \equiv: \mathrm{Aut}(\mathrm{id}_{\mathbf{B}G})$$

where $\mathrm{id}_{\mathbf{B}G} : \mathbf{B}G \rightarrow \mathbf{B}G$ is the *unpointed* identity function of $\mathbf{B}G$. So ZG is an automorphism ∞ -group, and therefore we can define

$$\mathbf{B}ZG := \mathbf{B}\mathrm{Aut}(\mathrm{id}_{\mathbf{B}G}) \equiv (p : \mathbf{B}G = \mathbf{B}G) \times \|\mathrm{id}_{\mathbf{B}G} = p\|.$$

In 1-group theory, the usual group theory, we would also know that the center is abelian. We might expect the same thing here. We could go about proving it by hand, but we can jump start ourselves a bit by remembering the Eckmann-Hilton argument.

Theorem 2.6.2 (Eckmann-Hilton). *Let X be a pointed type and $p, q : \Omega^2 X$. Then $p \cdot q = q \cdot p$.*

In other words, elements in double loop spaces commute. So, instead of directly proving that elements in ZG commute, we can just deloop it one more

time. Saying ‘just’ here is only really OK because I know how to do it – usually it is a pretty tough challenge to deloop something.

In this case, we have some help again since $\mathbf{B}ZG$ consists of (*unpointed*) automorphisms of $\mathbf{B}G$. This suggests that we might be able to pull the same trick again and define \mathbf{B}^2ZG to be $\mathbf{BAut}_{\mathbf{Type}}(\mathbf{B}G)$. But, it isn’t the type of *all* automorphisms of $\mathbf{B}G$, so we’ll have to get a little clever. We will take

$$\mathbf{B}^2ZG := (X : \mathbf{Type}) \times \|\mathbf{B}G = X\|_0$$

to be the type of types together with a connected component of their identification with $\mathbf{B}G$. We take $(\mathbf{B}G, |\mathbf{refl}|_0)$ for the point. This definition is due to Ulrik Buchholtz, and is justified by the following calculation:

$$\begin{aligned} \Omega\mathbf{B}^2ZG &:= (\mathbf{B}G, |\mathbf{refl}|_0) = (\mathbf{B}G, |\mathbf{refl}|_0) \\ &\simeq (p : \mathbf{B}G = \mathbf{B}G) \times |p|_0 = |\mathbf{refl}|_0 \\ &\simeq (p : \mathbf{B}G = \mathbf{B}G) \times \|p = \mathbf{refl}\| \\ &\equiv: \mathbf{B}ZG. \end{aligned}$$

Of course, we have a map $ZG \rightarrow G$ given by sending $z : (t : \mathbf{B}G) \rightarrow (t = t)$ to $z(\mathbf{pt}_{\mathbf{B}G})$. To show that this is a homomorphism, we need to deloop it by evaluating at $\mathbf{pt}_{\mathbf{B}G}$ again, sending $e : \mathbf{B}ZG$ to $e(\mathbf{pt}_{\mathbf{B}G})$, pointed at \mathbf{refl} .

Now, classically there is an exact sequence $ZG \rightarrow G \rightarrow \mathbf{Inn}(G)$ witnessing G as an extension of its center by its group of inner automorphisms. An inner automorphism e is a group automorphism – and so given by a pointed equivalence $\mathbf{B}e : \mathbf{B}G \xrightarrow{=} \mathbf{B}G$ – which is conjugate to the identity. Since an *unpointed*

identification between pointed maps is given (at least in the case of a 1-group) by a conjugating element, to say that there is some conjugating element relating e and id_G means that there is some way to identify $\mathbf{B}e$ and $\text{id}_{\mathbf{B}G}$ as unpointed equivalences, or $\|\mathbf{B}e_- = \text{id}_{\mathbf{B}G_-}\|$. This motivates the following definition.

Definition 2.6.3. Let G be an ∞ -group. Then

$$\text{Inn}(G) := (\mathbf{B}e : \mathbf{B}G \doteq \mathbf{B}G) \times \|\mathbf{B}e_- = \text{id}_{\mathbf{B}G_-}\|$$

is its ∞ -group of *inner automorphisms*, with delooping

$$\text{BInn}(G) := (X : \mathbf{Type}_*) \times \|X_- = \mathbf{B}G_-\|_0$$

the type of *pointed* types with a connected component of *unpointed* equivalences with $\mathbf{B}G$.

The function $\text{BInn}(G) \cdot \rightarrow \mathbf{B}^2ZG$ which forgets the extra point has fiber $\mathbf{B}G$ (since it is a projection from a sum type), and so we get a long fiber sequence:

$$\begin{array}{ccccc}
 ZG & \longrightarrow & G & \longrightarrow & \text{Inn}(G) \\
 & & & & \downarrow \\
 \text{BZG} & \longrightarrow & \mathbf{B}G & \longrightarrow & \text{BInn}(G) \\
 & & & & \downarrow \\
 & & & & \mathbf{B}^2ZG
 \end{array}$$

This is a paradigmatic example of a *central extension* (of $\text{Inn}(G)$ by ZG).

It is in particular an extension, and so is classified by a map $\text{BInn}(G) \cdot \rightarrow \mathbf{B}\text{Aut}(\mathbf{B}G)$. Which one?

Note that \mathbf{B}^2ZG gave rise to this extension by taking a fiber, whereas maps into $\mathbf{B}\text{Aut}(\mathbf{B}ZG)$ give rise to extensions by taking the dependent sum. But the

fiber is constructed as a certain kind of dependent sum, so we can mediate these two ways of classifying extensions through the map

$$T \mapsto T = \mathbf{pt}_{\mathbf{B}^2 ZG} : \mathbf{B}^2 ZG \cdot \rightarrow \mathbf{BAut}(\mathbf{B}ZG),$$

pointed at the calculation of $\Omega \mathbf{B}^2 ZG = \mathbf{B}ZG$ we did above. Now, $\mathbf{B}^2 ZG$ is 1-connected, so this map factors through $t : \mathbf{BAut}(\mathbf{B}ZG) \parallel t = \mathbf{pt}_{\mathbf{BAut}(\mathbf{B}ZG)} \parallel_0$. By inspecting definitions, we see that this is $\mathbf{B}^2 Z^2 G$!

$$\begin{array}{ccc} & & \mathbf{B}^2 Z^2 G \\ & \nearrow \text{dashed} & \downarrow \\ \mathbf{B} \text{Inn}(G) & \longrightarrow & \mathbf{BAut}(\mathbf{B}ZG) \end{array}$$

This leads us to make the following definition.

Definition 2.6.4. An extension of G by A classified by $c : \mathbf{B}G \cdot \rightarrow \mathbf{BAut}(\mathbf{B}A)$ is *central* if c factors through the forgetful map $\mathbf{B}^2 ZA \cdot \rightarrow \mathbf{BAut}(\mathbf{B}A)$.

$$\begin{array}{ccc} & & \mathbf{B}^2 ZA \\ & \nearrow \text{dashed} & \downarrow \\ \mathbf{B}G & \xrightarrow{c} & \mathbf{BAut}(\mathbf{B}A) \end{array}$$

It may seem odd that we did not instead define a “centrality” of an extension to be the data of such a lift. It turns out that the type of such lifts is a proposition, and so being central is a property of an extension and not extra structure.

Remark 2.6.5. Just as for extensions, we note that we have *not* assumed that A is commutative in any way in the above definition. The traditional definition of a central extension – that $A \hookrightarrow E$ lies in the center of E – implies that A is abelian. As we will see, our notion of central extension makes sense even

for *non-abelian* A , and will correspond with the traditional notion of central extension by ZA .

A close look at our definition of \mathbf{B}^2ZA as $(X : \mathbf{Type}) \times \|\mathbf{B}A = X\|_0$ reveals that it is the fiber over $|\mathbf{B}A|_1$ of the 1-truncation map

$$\mathbf{B} \mathbf{Aut}(\mathbf{B}A) \xrightarrow{|\cdot|_1} \|\mathbf{B} \mathbf{Aut}(\mathbf{B}A)\|_1,$$

since $(|\mathbf{B}A|_1 = |X|_1) \simeq \|\mathbf{B}A = X\|_0$. We can understand the 1-truncation $\|\mathbf{B} \mathbf{Aut}(\mathbf{B}A)\|_1$ in more elementary terms as well: it is a delooping $\mathbf{B} \mathbf{Out}(A)$ of the group of *outer* automorphisms of A .

Definition 2.6.6. For a higher group A , we define its group $\mathbf{Out}(A)$ of outer automorphisms to be $\|\mathbf{Aut}(\mathbf{B}A)\|_0$, delooped by

$$\mathbf{B} \mathbf{Out}(A) := \|\mathbf{B} \mathbf{Aut}(\mathbf{B}A)\|_1.$$

To justify this definition, note that $\mathbf{B} \mathbf{Aut}(A)$ is delooped by $\mathbf{B} \mathbf{Aut}_*(\mathbf{B}A)$, the type of pointed types which are identifiable with $\mathbf{B}A$ as a *pointed type*. We have a map $\mathbf{B} \mathbf{Aut}(A) \rightarrow \mathbf{B} \mathbf{Out}(A)$ given by $(X, \mathbf{pt}_X) \mapsto |X|_1$, and the fiber of this map is precisely $\mathbf{B} \mathbf{Inn}(A)$. That is, we have a fiber sequence

$$\mathbf{B} \mathbf{Inn}(A) \rightarrow \mathbf{B} \mathbf{Aut}(A) \rightarrow \mathbf{B} \mathbf{Out}(A),$$

and this shows us that $\mathbf{Out}(A) := \|\mathbf{Aut}(\mathbf{B}A)\|_0$ is the quotient $\mathbf{Aut}(A) // \mathbf{Inn}(A)$.

The fiber sequence

$$\mathbf{B}^2ZA \rightarrow \mathbf{B} \mathbf{Aut}(\mathbf{B}A) \rightarrow \mathbf{B} \mathbf{Out}(A)$$

gives us a useful and classical condition for an extension to be central.

Definition 2.6.7. Let $c : \mathbf{B}G \rightarrow \mathbf{B}\text{Aut}(\mathbf{B}A)$ classify an extension $p : E \rightarrow G$ of G by A . The *abstract kernel* of $p : E \rightarrow A$ is the homomorphism $G \rightarrow \text{Out}(A)$ delooped by the composite

$$\mathbf{B}G \xrightarrow{c} \mathbf{B}\text{Aut}(\mathbf{B}A) \xrightarrow{|\cdot|_1} \mathbf{B}\text{Out}(A).$$

Theorem 2.6.8. *An extension of higher groups classified by $c : \mathbf{B}G \rightarrow \mathbf{B}\text{Aut}(\mathbf{B}A)$ is central if and only if its abstract kernel vanishes. As a corollary, being central is a proposition.*

Proof. This is just the universal property of the fiber sequence

$$\mathbf{B}^2ZA \rightarrow \mathbf{B}\text{Aut}(\mathbf{B}A) \rightarrow \mathbf{B}\text{Out}(A).$$

Namely, the type of lifts of $c : \mathbf{B}G \rightarrow \mathbf{B}\text{Aut}(\mathbf{B}A)$ to \mathbf{B}^2ZA is equivalent to the type of trivializations of the abstract kernel $|\cdot|_1 \circ c : \mathbf{B}G \rightarrow \mathbf{B}\text{Out}(A)$.

It remains only to show that it is a proposition that the abstract kernel vanishes, which is to say that the type of trivializations of $\mathbf{B}\kappa : \mathbf{B}G \rightarrow \mathbf{B}\text{Out}(A)$ as a pointed map is a proposition. But by construction, $\mathbf{B}\text{Out}(A)$ is a 1-type, so the abstract kernel factors uniquely through $|\cdot|_1 : \mathbf{B}G \rightarrow \|\mathbf{B}G\|_1$ as a pointed map $\mathbf{B}\bar{\kappa} : \|\mathbf{B}G\|_1 \rightarrow \mathbf{B}\text{Out}(A)$. Therefore, the type of trivializations of $\mathbf{B}\kappa$ is equivalent to the type of trivializations of $\mathbf{B}\bar{\kappa}$. The type $\|\mathbf{B}G\|_1 \rightarrow \mathbf{B}\text{Out}(A)$ of pointed maps is equivalent to the set of group homomorphisms $\|G\|_0 \rightarrow \text{Out}(A)$. In particular, a trivialization of $\mathbf{B}\bar{\kappa}$ is equivalent to an equality $\overline{\kappa} = 0$ in the set of such homomorphisms. This shows that the vanishing of the abstract kernel is a proposition, and as a corollary that being central is a proposition. \square

2.6.1 B^2A versus B^2ZA

Now, some extensions are clearly classified by maps $\mathbf{B}G \cdot \rightarrow \mathbf{B}^2F$ by taking the fiber to get a sequence

$$\begin{array}{ccccc}
 F & \longrightarrow & E & \longrightarrow & G \\
 & \searrow & & \searrow & \\
 \mathbf{B}F & \longrightarrow & \mathbf{B}E & \longrightarrow & \mathbf{B}G \\
 & \searrow & & \searrow & \\
 & & \mathbf{B}^2F & &
 \end{array}$$

The question is, which extensions are classified in this way? Note that a map $c : \mathbf{B}G \cdot \rightarrow \mathbf{B}^2F$ classifies an extension by taking the fiber $\mathbf{fib}_c := (t : \mathbf{B}G) \times t = \mathbf{pt}_{\mathbf{B}^2F}$, while a map $\tilde{c} : \mathbf{B}G \cdot \rightarrow \mathbf{BAut}(\mathbf{B}F)$ classifies an extension by taking the total space $(t : \mathbf{B}G) \times \tilde{c}(t)$. We can reconcile these two different ways of classifying extensions via the map

$$\varphi : \mathbf{B}^2F \cdot \rightarrow \mathbf{BAut}(\mathbf{B}F)$$

$$T \mapsto T = \mathbf{pt}_{\mathbf{B}^2F}$$

pointed by the fact that $\mathbf{B}F = (\mathbf{pt}_{\mathbf{B}^2F} = \mathbf{pt}_{\mathbf{B}^2F})$. Conceptually, φ is the left action of $\mathbf{B}F$ on itself. Given a $c : \mathbf{B}G \cdot \rightarrow \mathbf{B}^2F$, if we define $\tilde{c} : \mathbf{B}G \cdot \rightarrow \mathbf{BAut}(\mathbf{B}F)$ by $\tilde{c} := \varphi \circ c$ then c and \tilde{c} classify the same extension with delooping $(t : \mathbf{B}G) \times c(t) = \mathbf{pt}_{\mathbf{B}^2F}$. The question of whether an extension classified by $\tilde{c} : \mathbf{B}G \cdot \rightarrow \mathbf{BAut}(\mathbf{B}F)$ is classified by \mathbf{B}^2F then becomes a question of whether it factors through φ , and conceptually of whether the fibers of $\mathbf{B}p : \mathbf{B}E \cdot \rightarrow \mathbf{B}G$ are $\mathbf{B}F$ -torsors, or just merely equivalent to $\mathbf{B}F$.

Now, given a map $c : \mathbf{B}G \cdot \rightarrow \mathbf{B}^2F$ classifying $\mathbf{B}E := \mathbf{fib}_c$, we can get a map $\tilde{c} : \mathbf{B}^2F \cdot \rightarrow \mathbf{B}^2ZE$ by $t \mapsto \mathbf{fib}_c(t)$ and satisfying $\|\mathbf{fib}_c(t) = \mathbf{B}E\|_0$ by appealing to

the 1-connectedness of \mathbf{B}^2F . We take the point of \tilde{c} to be **refl**. If we take the total space of \tilde{c} we get an equivalence

$$\begin{aligned} (t : \mathbf{B}^2F) \times \tilde{c}(t) &\equiv (t : \mathbf{B}^2F) \times \mathbf{fib}_c(t) \\ &\simeq (t : \mathbf{B}^2F) \times (s : \mathbf{B}G) \times c(s) = t \\ &\simeq \mathbf{B}G \end{aligned}$$

whose inverse is given by sending $s : \mathbf{B}G$ to $(c(s), s, \mathbf{refl})$, so that the projection back to \mathbf{B}^2F recovers c . If we take the induced map $\Omega\tilde{c} : \mathbf{B}F \rightarrow \mathbf{B}ZE$ and push forward to $\mathbf{B}E$ by evaluating at $\mathbf{pt}_{\mathbf{B}E} \equiv (\mathbf{pt}_{\mathbf{B}G}, \mathbf{pt}_c)$, we get for $t : \mathbf{B}F$

$$\begin{aligned} \Omega\tilde{c}(t)(\mathbf{pt}_{\mathbf{B}E}) &\equiv (\mathbf{ap} \tilde{c} t)(\mathbf{pt}_{\mathbf{B}E}) \\ &= (\mathbf{tr}(\lambda T. \mathbf{fib}_c(T)) t)(\mathbf{pt}_{\mathbf{B}E}) \\ &= (\mathbf{pt}_{\mathbf{B}G}, \mathbf{pt}_c \cdot t) \end{aligned}$$

which is the map $\mathbf{B}F \rightarrow \mathbf{B}E$ given by extending the fiber sequence $\mathbf{B}E \rightarrow \mathbf{B}G \xrightarrow{c} \mathbf{B}^2F$.

This suggests that \mathbf{B}^2F classifies *braided central extensions* by F , namely extensions $\mathbf{B}F \xrightarrow{\mathbf{B}i} \mathbf{B}E \xrightarrow{\mathbf{B}p} \mathbf{B}G$ such that $\mathbf{B}i$ factors through $\mathbf{B}ZE$ via a *braided* ∞ -group homomorphism, namely via a pointed map $\mathbf{B}^2F \rightarrow \mathbf{B}^2ZE$.

Chapter 3

Modal fibrations

3.1 Introduction

While homotopy theory — the study of identifications — has been well developed in homotopy type theory, algebraic topology — the study of the connectivity of space — has been somewhat lacking. This is because Book HoTT (the homotopy type theory of the HoTT Book [Uni13]) has no way of saying that a type is *the homotopy type* of another type. While we can define both the homotopy circle S^1 as a higher inductive type and the topological circle

$$\mathbb{S}^1 := \{(x, y) : \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

in Book HoTT alone we do not have the tools to say that S^1 is the homotopy type of \mathbb{S}^1 .

In his Real Cohesive Homotopy Type Theory [Shu18b], Shulman solves this issue by adding a system of modalities which includes the *shape* modality \int that takes a type X to its homotopy type $\int X$.¹ In Real Cohesive HoTT, every type

¹The symbol “ \int ” is an *esh*, the IPA symbol for the voiceless palato-alveolar fricative phoneme /*sh*/ that begins the word “shape”. It is not an integral sign.

has a spatial structure and every map is continuous with respect to this spatial structure. This spatial structure is distinct from the *homotopical* structure of identifications that every type has in homotopy type theory. But these two structures are brought together by the \mathbb{J} modality, which allows us to identify points by giving spatial paths between them. Formally, the \mathbb{J} modality is given by localizing at the type of Dedekind real numbers \mathbb{R} — in other words, by identifying points which are connected by paths $\gamma : \mathbb{R} \rightarrow X$.²

As with any modality, there is a *modal unit* $(-)^{\mathbb{J}} : X \rightarrow \mathbb{J}X$, a quotient map of sorts, which is the universal map from X to a *discrete* type — one with only homotopical and no spatial structure.³ For any map $f : X \rightarrow Y$, we have a naturality square which induces a map from the fiber of f over $y : Y$ to its *homotopy fiber*, the fiber of $\mathbb{J}f$:

$$\begin{array}{ccc}
 \mathbf{fib}_f(y) & \overset{\delta}{\dashrightarrow} & \mathbf{fib}_{\mathbb{J}f}(y^{\mathbb{J}}) \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{(-)^{\mathbb{J}}} & \mathbb{J}X \\
 f \downarrow & & \downarrow \mathbb{J}f \\
 Y & \xrightarrow{(-)^{\mathbb{J}}} & \mathbb{J}Y
 \end{array}$$

The fibers of maps between discrete types are themselves discrete, so the map $\delta : \mathbf{fib}_f(y) \rightarrow \mathbf{fib}_{\mathbb{J}f}(y^{\mathbb{J}})$ factors uniquely through $(-)^{\mathbb{J}} : \mathbf{fib}_f(y) \rightarrow \mathbb{J}\mathbf{fib}_f(y)$ by the universal property of the unit. This gives us a useful diagram (Figure 3.1) which

²In this chapter, we reserve the term *path* (in X) for function $\gamma : \mathbb{R} \rightarrow X$, while we use the term *identification* for points of the type $x = y$ (for $x, y : X$). This conflicts with the terminology of the HoTT Book, in which “path” is used for what we call identifications. But, in our setting, the shape modality \mathbb{J} takes a path $\gamma : \mathbb{R} \rightarrow X$ and gives an identification $\gamma(0)^{\mathbb{J}} = \gamma(1)^{\mathbb{J}}$ in the homotopy type $\mathbb{J}X$. So, when one is working with homotopy types $\mathbb{J}X$, the difference between our terminology and the terminology of the HoTT Book is blurred.

³That is, every path is constant in a discrete type, but there may still be non-trivial identifications between its points.

I like to call *the modal prism*.

$$\begin{array}{ccc}
 \mathbf{fib}_f(y) & \xrightarrow{\delta} & \mathbf{fib}_{\int f}(y^{\int}) \\
 \searrow^{(-)^{\int}} & & \nearrow^{\gamma} \\
 & \int \mathbf{fib}_f(y) &
 \end{array}$$

Figure 3.1: The Modal Prism.

Looking through the modal prism, we see a rainbow of different possibilities for a function $f : X \rightarrow Y$.

Definition 3.1.1. Let $f : X \rightarrow Y$ and consider the modal prism as in Figure 1. Then f is

- \int -*modal* if its fibers are discrete, that is, if $(-)^{\int}$ is an equivalence for all $y : Y$,
- \int -*connected* if its fibers are homotopically contractible, that is, if $\int \mathbf{fib}_f(y)$ is contractible for all $y : Y$,
- \int -*étale* if its fibers are its homotopy fibers, that is, if δ is an equivalence for all $y : Y$.
- a \int -*equivalence* if its homotopy fibers are contractible, that is, if $\mathbf{fib}_{\int f}(y^{\int})$ is contractible for all $y : Y$,
- a \int -*fibration* if the homotopy type of its fibers are its homotopy fibers, that is, if γ is an equivalence for all $y : Y$.

For the shape modality, a map is modal when it has discrete fibers, and is a modal equivalence, or (weak) homotopy equivalence, when it induces an

equivalence on homotopy types. It is modally connected when it has the stronger property that its fibers are homotopically contractible; for comparison, consider the inclusion $x : \mathbb{R} \rightarrow \mathbb{R}^2$ of the x -axis, which is clearly a homotopy equivalence but is not \mathcal{J} -connected since some of its fibers are empty. Finally, a \mathcal{J} -étale map is a weak relative of a covering map; it has a *unique* lifting against any homotopy equivalence.

The notions of modal maps, connected maps, and modal equivalences appear in the HoTT Book ([Uni13]). For the n -truncation modality, these are n -truncated and n -connected maps respectively, with modal equivalences not given a specific name. The notion of modal étale map is due to Wellen as a “formally étale map” in [Wel17], building on work of Schreiber in the setting of higher topos theory [Sch13b]. In the case of \mathcal{J} , it appears as a “modal covering” in [Wel18a].

The notion of modality has also made its way into the ∞ -categorical literature through the work of Anel, Biederman, Finster, and Joyal (see [Ane+17] and [Ane+18b]). In these papers, they define a modality as a *stable orthogonal factorization system* (one of the equivalent ways of defining a modality in HoTT), and translate a homotopy type theoretic generalized Blakers-Massey Theorem into the language of ∞ -categories and apply it to the Goodwillie calculus of functors. As Shulman has proven that every ∞ -topos models HoTT ([Shu19b]), the results in this chapter concerning modal fibrations (in Section 3.3) apply in any ∞ -topos as well.

The notion of modal fibration is, as far as I know, novel to this chapter. It gives a good notion of fibration in real cohesion which works not just for set level spaces (e.g. manifolds) but also spaces with both topological and homotopical

content (e.g. orbifolds and Lie groupoids). A map is a \mathcal{J} -fibration when the homotopy type of its fibers are the fibers of its action on homotopy types; this gives us the long fiber sequence on homotopy groups we expect from a fibration in real cohesion. This definition closely resembles the classical notion of *quasi-fibration* due to Dold and Thom [DT58b], though it is much better behaved (see Remark 3.3.1).

In Section 3.2, we will refresh ourselves on modalities and look through the modal prism to see the different kinds of functions associated with a modality. Then, in Section 3.3 we will develop the basic theory of \diamond -fibrations for an arbitrary modality \diamond , and justify the name. In summary, the \diamond -fibrations are closed under composition and pullback and may be characterized in any one of the following ways.

Theorem 3.1.2. *For a map $f : X \rightarrow Y$, the following are equivalent:*

1. f is a \diamond -fibration.
2. \diamond preserves all fibers of f .
3. \diamond preserves all pullbacks along f .
4. The \diamond -connected/ \diamond -modal and \diamond -equivalence/ \diamond -étale factorizations of f agree.
5. The \diamond -modal factor of f is \diamond -étale.
6. The \diamond -equivalence factor of f is \diamond -connected.
7. The \diamond -naturality square of f is \diamond -cartesian.

8. The connecting map $\mathbf{tot}(\gamma)$ between the two factorizations of f is a \diamond -fibration.
9. f has \diamond -locally constant \diamond -fibers in the sense that $\diamond \mathbf{fib}_f : Y \rightarrow \mathbf{Type}_\diamond$ factors through $\diamond Y$.
10. (If \diamond -units are surjective:) For every $x : X$, the induced map $\mathbf{fib}_{(-)^\diamond}(x^\diamond) \rightarrow \mathbf{fib}_{(-)^\diamond}((fx)^\diamond)$ is \diamond -connected.

In particular, we will prove in Theorem 3.3.14 that a map $f : X \rightarrow Y$ is an \diamond -fibration if and only if the type family $\diamond \mathbf{fib}_f : Y \rightarrow \mathbf{Type}$ factors through the modal unit $(-)^\diamond : Y \rightarrow \diamond Y$. For the modality \mathfrak{J} , this means that a map is a \mathfrak{J} -fibration if and only if the homotopy type of its fiber over $y : Y$ is locally constant in y ; that is, a map is a \mathfrak{J} -fibration if and only if its fibers form a *local system* on its codomain.

We will also characterize the $\|-\|_n$ -fibrations as those maps which are surjective on π_{n+1} in Corollary 3.3.19.

In Section 3.4, we give a brief review of Shulman's Real Cohesive HoTT. We then prove in Section 3.5 that the classifying types of bundles of discrete structures are themselves discrete (see Theorem 3.5.9 for the precise statement). As a corollary, we find in Theorem 3.6.1 that maps whose fibers have a merely constant homotopy type are \mathfrak{J} -fibrations. Morally, this result says that if all the fibers of a map have the same homotopy type so that one can comfortably write

$$F \rightarrow E \xrightarrow{p} B$$

with F well defined up to homotopy, then p is a \mathfrak{J} -fibration.

In the remaining sections, we will show how this theory can be applied to synthetic algebraic topology. Because the homotopy type of the fibers of a \mathfrak{f} -fibration are its homotopy fibers, whenever

$$F \rightarrow E \xrightarrow{p} B$$

is a fiber sequence with p a \mathfrak{f} -fibration, $\mathfrak{f}F \rightarrow \mathfrak{f}E \xrightarrow{\mathfrak{f}p} \mathfrak{f}B$ is also a fiber sequence. Using the fact that the fibers of the map $(\cos, \sin) : \mathbb{R} \rightarrow \mathbb{S}^1$ are merely equivalent to \mathbb{Z} , Theorem 3.6.1 implies that this map is a \mathfrak{f} -fibration, and that therefore,

$$\mathbb{Z} \rightarrow \mathfrak{f}\mathbb{R} \rightarrow \mathfrak{f}\mathbb{S}^1$$

is a fiber sequence. Since $\mathfrak{f}\mathbb{R} \simeq *$ is contractible, this calculates the loop space of the topological circle \mathbb{S}^1 without passing through the higher inductive circle S^1 . We consider this and other examples of \mathfrak{f} -fibrations, including:

- The map $(\cos, \sin) : \mathbb{R} \rightarrow \mathbb{S}^1$ (in Section 3.6.1).
- The homogeneous coordinates $\mathbb{S}^n \rightarrow \mathbb{R}P^n$, $\mathbb{S}^{2n+1} \rightarrow \mathbb{C}P^n$, and $\mathbb{S}^{4n+3} \rightarrow \mathbb{H}P^n$, including as special cases the Hopf fibration $\mathbb{S}^3 \rightarrow \mathbb{C}P^1$ and the quaternionic Hopf fibration $\mathbb{S}^7 \rightarrow \mathbb{H}P^1$ (in Section 3.6.2).
- The rotation map $\mathbf{SO}(n+1) \rightarrow \mathbb{S}^n$ (in Section 3.7.1).
- The homotopy quotient $\mathbb{R} \vee \mathbb{R} \rightarrow (\mathbb{R} \vee \mathbb{R}) // C_2$, and many other homotopy quotients (in Section 3.7.2).

After this, we prove some corollaries for the theory of higher groups in Sections 3.7 and 3.8. We begin by reviewing the definition of higher groups, and then show that the homotopy quotient $X \rightarrow X // G$ of a type by the action of

a crisp higher group is always a \mathcal{J} -fibration. We then prove that \mathcal{J} preserves the connectedness of crisp types, and conclude that the homotopy type of a higher group is itself a higher group.

Finally, in Section 3.9, we turn to the theory of covering spaces. We define the notion of covering following Wellen [Wel18a], and show that the type of coverings on a type is equivalent to the type of actions of its fundamental groupoid on discrete sets. We then show that every pointed type has a universal cover, and prove that this universal cover has the expected universal property. We end by showing that the universal cover of a higher group is a higher group.

3.2 Modalities and the Modal Prism

A modality is a way of changing what it means for two elements of a type to be identified. To each type X , we associate a new type $\diamond X$ and a function $(-)^{\diamond} : X \rightarrow \diamond X$. For two points $x, y : X$ to be identified by the modality then means that $x^{\diamond} = y^{\diamond}$ as elements of $\diamond X$. Here are a few examples of modalities, with emphasis on those we will focus on in this paper.

- With the trivial modality $\diamond X = *$, any two points are uniquely identified.
- With the n -truncation modality $\|-\|_n$, two points are identified by giving an $(n - 1)$ -truncated identification between them. The base case is $\|X\|_{-2} = *$, the trivial modality.
- With the *shape* modality \mathcal{J} , two points may be identified by giving a path between them (that is, a map from the real line \mathbb{R} which sends 0 to one point and 1 to the other). We call $\mathcal{J}X$ the *homotopy type* of a type X .⁴

⁴The modality \mathcal{J} appears as Definition 9.6 of [Shu18b], and we review it in Section 3.4.

- With the *crystalline* modality \mathfrak{I} , two points may be identified by giving an *infinitesimal* path between them. We call $\mathfrak{I}X$ the *de Rham stack* of a type X .⁵

While the elementary theory of modalities appeared in the HoTT Book [Uni13], the notion was developed more fully by Rijke, Shulman, and Spitters in [RSS17a]. In that paper, they give equivalences between four different notions of modality and prove a number of useful lemmas along the way. We will take our modalities to be “higher modalities”, one of the many equivalent notions of modality.

Definition 3.2.1. A *higher modality* consists of a *modal operator* $\diamond : \mathbf{Type} \rightarrow \mathbf{Type}$ together with:

- For each type X , a *modal unit*

$$(-)^\diamond : X \rightarrow \diamond X$$

- For every $A : \mathbf{Type}$ and $P : \diamond A \rightarrow \mathbf{Type}$, an *induction principle*

$$\mathbf{ind}_A^\diamond : ((a : A) \rightarrow \diamond P(a^\diamond)) \rightarrow ((u : \diamond A) \rightarrow \diamond P(u)),$$

- For every $A : \mathbf{Type}$, $P : \diamond A \rightarrow \mathbf{Type}$, $f : (a : A) \rightarrow \diamond P(a^\diamond)$ and $x : A$, a *computation rule*

$$\mathbf{comp}_A^\diamond : \mathbf{ind}_A^\diamond(f)(x^\diamond) = f(x),$$

⁵The crystalline modality appears formally as Axiom 3.4.1 in [Wel17], and in the higher categorical setting in Definition 4.2.1 of [Sch13b], where it is called the *infinitesimal shape* modality

- For any $u, v : \diamond A$, a witness that the modal unit $(-)^{\diamond} : u = v \rightarrow \diamond(u = v)$ is an equivalence.

We say a type X is \diamond -*modal* if $(-)^{\diamond} : X \rightarrow \diamond X$ is an equivalence, and we define

$$\mathbf{Type}_{\diamond} \equiv (X : \mathbf{Type}) \times \text{is}\diamond\text{Modal}(X)$$

to be the universe of \diamond -modal types. A type X is \diamond -*separated* if for all $x, y : X$, the type of identifications $x = y$ is \diamond -modal.

A modality is in particular a *reflective subuniverse*: pre-composition by $(-)^{\diamond}$ gives an equivalence

$$(\diamond X \rightarrow Z) \xrightarrow{\sim} (X \rightarrow Z)$$

whenever Z is \diamond -modal (see Theorem 1.13 of [RSS17a]). Any map $\eta : X \rightarrow K$ from X to a modal type K which satisfies the same property is called a \diamond -*unit*, since from this property it can be show that $K \simeq \diamond X$ and $\eta = (-)^{\diamond}$ under this equivalence.

Modal types are closed under the basic operations of dependent type theory in the following way.

Lemma 3.2.2. Let X be a type and $P : X \rightarrow \mathbf{Type}$ a family of types.

- If X is modal and for all $x : X$, Px is modal, then $(x : X) \times Px$ is modal.
- If for all $x : X$, Px is modal, then $(x : X) \rightarrow Px$ is modal.

Proof. See Theorem 1.32 and Lemma 1.26 of [RSS17a]. □

As a corollary, a number of useful properties of modal types are also modal.

Corollary 3.2.3. Let A be a modal type. Then

$$\text{isContractible}(A) := (a : A) \times ((a' : A) \rightarrow (a = a'))$$

is modal. If B is also a modal type and $f : A \rightarrow B$, then

$$\text{isEquiv}(f) := (b : B) \rightarrow \text{isContractible}(\text{fib}_f(b))$$

is modal.

When we use the induction principle of a modality, it often makes sense to think of it “backwards”. That is, we think of the induction principle as saying that in order to map out of $\diamond A$ into a modal type, it suffices to map out of A . Or, with variables, in order to define $T(u) : \diamond P(u)$ for $u : \diamond A$, it suffices to assume that $u \equiv a^\diamond$ for $a : A$. In prose, we will just say that \diamond -induction lets us assume u is of the form a^\diamond .

We can extend the operation of \diamond to a functor using the induction principle. If $f : X \rightarrow Y$, then define $\diamond f : \diamond X \rightarrow \diamond Y$ by $\diamond f(x^\diamond) := f(x)^\diamond$, or explicitly by

$$\diamond f := \mathbf{ind}_X^\diamond((-)^\diamond \circ f).$$

Using the computation rule, we get a *naturality square*

$$\begin{array}{ccc} X & \xrightarrow{(-)^\diamond} & \diamond X \\ f \downarrow & & \downarrow \diamond f \\ Y & \xrightarrow{(-)^\diamond} & \diamond Y \end{array}$$

Any commuting square induces a map from the fiber of the left map to the fiber of the right. Therefore, we get the map $\delta : \text{fib}_f(y) \rightarrow \text{fib}_{\diamond f}(y^\diamond)$ for any

$y : Y$ given by

$$\delta((x : X), (p : fx = y)) := (x^\diamond, \mathbf{comp}^\diamond \cdot (\mathbf{ap}(-)^\diamond p)).$$

As the sum of modal types is modal, $\mathbf{fib}_{\diamond f}(y^\diamond) \equiv (u : \diamond X) \times (\diamond f(u) = y^\diamond)$ is modal. Therefore, this map factors through $\diamond \mathbf{fib}_f(y)$ uniquely, giving us the *modal prism*.

$$\begin{array}{ccc} \mathbf{fib}_f(y) & \xrightarrow{\delta} & \mathbf{fib}_{\diamond f}(y^\diamond) \\ & \searrow^{(-)^\diamond} & \nearrow^{\gamma} \\ & \diamond \mathbf{fib}_f(y) & \end{array}$$

The modal prism divides functions in 5 possible kinds. Four of these possibilities arrange themselves into orthogonal factorization systems; the other gives a mediating notion which is the focus of this paper.

Definition 3.2.4. Let $f : X \rightarrow Y$ and consider the modal prism as in Figure

1. Then f is

- \diamond -*modal* if $(-)^{\diamond}$ is an equivalence for all $y : Y$,
- \diamond -*connected* if $\diamond \mathbf{fib}_f(y)$ is contractible for all $y : Y$,
- \diamond -*étale* if δ is an equivalence for all $y : Y$.
- a \diamond -*equivalence* if $\mathbf{fib}_{\diamond f}(y^\diamond)$ is contractible for all $y : Y$,
- a \diamond -*fibration* if γ is an equivalence for all $y : Y$.

Remark 3.2.5. By a quick application of \diamond -induction, we see that f is a \diamond -equivalence if and only if $\diamond f$ is an equivalence. And, by the lemma that a square

is a pullback if and only if the induced map on fibers is an equivalence, f is \diamond -étale if and only if its naturality square is a pullback.

We can see relations between these definitions right off the bat.

Lemma 3.2.6. Let $f : X \rightarrow Y$. Then:

- f is \diamond -étale if and only if it is \diamond -modal and a \diamond -fibration.
- f is \diamond -connected if and only if it is a \diamond -equivalence and a \diamond -fibration.

Proof. Since the modal prism commutes, if f is \diamond -modal and a \diamond -fibration, then it is \diamond -étale. On the other hand, since $\mathbf{fib}_{\diamond f}(y^\diamond)$ is modal, if f is \diamond -étale then $\mathbf{fib}_f(y)$ is \diamond -modal and so $(-)^{\diamond}$ is an equivalence and hence so is γ .

If f is a \diamond -equivalence and a \diamond -fibration, then $\diamond \mathbf{fib}_f(y)$ is contractible as it is equivalent to the contractible $\mathbf{fib}_{\diamond f}(y^\diamond)$. On the other hand, if f is \diamond -connected, then it is a \diamond -equivalence by Lemma 1.35 of [RSS17a], and so γ is a map between contractible types and is therefore an equivalence. \square

Recall that any function $f : X \rightarrow Y$ gives an equivalence $X \simeq (y : Y) \times \mathbf{fib}_f(y)$ over Y . Therefore, by totalizing the modal prism, we can find two factorizations of any map f , connected in the middle by $\mathbf{tot}(\gamma)$:

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow^{\mathbf{tot}((-)^{\diamond})} & \downarrow f & \searrow^{\mathbf{tot}(\delta)} & \\
 (y : Y) \times \diamond \mathbf{fib}_f(y) & & & & (y : Y) \times \mathbf{fib}_{\diamond f}(y^\diamond) \\
 & \swarrow^{\mathbf{fst}} & \downarrow f & \searrow^{\mathbf{fst}} & \\
 & & Y & &
 \end{array}$$

In [RSS17a], Rijke, Shulman, and Spitters prove that the left factorization is a *stable orthogonal factorization system*. In particular, $\mathbf{tot}((-)^{\diamond})$ is \diamond -connected,

and $\text{fst} : (y : Y) \times \diamond \text{fib}_f(y) \rightarrow Y$ is \diamond -modal, and these give the unique \diamond -connected/ \diamond -modal factorization of f . The connected/modal factorization of a map f is also preserved under pullback; if $y : A \rightarrow Y$ is any map, then the factorization of the pullback y^*f is the pullback of the factorization of f along y .

This can be seen most clearly by viewing the factorization system from the point of view of type families. A map $f : X \rightarrow Y$ corresponds to the type family $\text{fib}_f : Y \rightarrow \mathbf{Type}$, and its modal factor corresponds to the type family $\diamond \text{fib}_f : Y \rightarrow \mathbf{Type}$. On type families, pullback along $y : A \rightarrow Y$ corresponds to composition, so y^*f corresponds to $\lambda a : A. \text{fib}_f(ya) : A \rightarrow \mathbf{Type}$. The modal factorization of the pullback y^* is then $\lambda a : A. \diamond \text{fib}_f(ya)$, which is precisely the pullback of the modal factorization of f .

In his thesis [Rij18a], Rijke proves that the right factorization is an *orthogonal factorization system*. In particular, $\text{tot}(\delta)$ is a \diamond -equivalence and $\text{fst} : (y : Y) \times \text{fib}_{\diamond f}(y^\diamond) \rightarrow Y$ is \diamond -étale, and this is the unique \diamond -equivalence/ \diamond -étale factorization of f . This is, however, not a stable factorization system because the \diamond -equivalences are not in general preserved under pullback (see Remark 3.3.8 for an example).

Another important concept in the theory of modalities is that of a \diamond -cartesian square (see, for example, Definition 3.7.1 of [Ane+17]). We will make use of \diamond -cartesian squares in developing the theory of modal fibrations, so we will establish a few lemmas here.

Definition 3.2.7. A commuting square

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ f \downarrow & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

is \diamond -cartesian if the cartesian gap map $A \rightarrow B \times_D C$ is \diamond -connected.

Note that a **id**-cartesian square for the identity modality **id** is simply a pull-back. Before proving our lemmas concerning \diamond -cartesian squares,

Lemma 3.2.8. Consider a square

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ f \downarrow & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

commuting via $S : (x : A) \rightarrow (k(f(x)) = h(g(x)))$. Let $c : C$, and define the map $G : \mathbf{fib}_f(c) \rightarrow \mathbf{fib}_h(kc)$ by

$$G(x : A, w : fx = c) \equiv (gx, S(x)^{-1} \cdot k_* w).$$

Then for any $(b, p) : \mathbf{fib}_h(kc)$, we have an equivalence $\mathbf{fib}_G((b, p)) = \mathbf{fib}_{\text{gap}}((c, bp))$ with the fiber of the gap map $A \rightarrow B \times_D C$.

Proof. We find the equivalence as the following composite:

$$\begin{aligned}
\mathbf{fib}_G((b, p)) &: \equiv ((x, w) : \mathbf{fib}_g(c)) \times (G(x, w) = (b, p)) \\
&= (x : A) \times (w : fx = c) \times ((gx, S(x)^{-1} \cdot k_* w) = (b, p)) \\
&= (x : A) \times ((gx, fx, S(x)^{-1}) = (b, c, p)) \\
&= \mathbf{fib}_{\text{gap}}((b, c, p)).
\end{aligned}$$

□

Using this, we can give a characterization of \diamond -cartesian maps which resembles the usual characterization of pullbacks as fiberwise equivalences.

Lemma 3.2.9. A commuting square

$$\begin{array}{ccc}
A & \xrightarrow{g} & B \\
f \downarrow & & \downarrow h \\
C & \xrightarrow{k} & D
\end{array}$$

is \diamond -cartesian if and only if for every $c : C$, the induced map

$$G : \mathbf{fib}_f(c) \rightarrow \mathbf{fib}_h(kc)$$

induced on fibers is \diamond -connected.

Proof. By Lemma 3.2.8, the fibers of the gap map are the fibers of G ; so, the fibers of the gap map are \diamond -connected if and only if the fibers of G are. □

The following lemmas may be found in [Ane+17] as Lemmas 3.7.4 and 3.7.3 respectively. We will prove them in HoTT.

Lemma 3.2.10. Consider a pair of commuting squares:

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{k} & D & \longrightarrow & F \end{array}$$

Then

1. If the left square and the right square are \diamond -cartesian, then so is the composite square.
2. If the left square and the composite square are \diamond -cartesian, and k is surjective, then the right square is \diamond -cartesian.
3. If the right square is a pullback and the composite square is \diamond -cartesian, then the left square is \diamond -cartesian.

Proof. We will appeal to Lemma 3.2.9 a number of times. To prove the first fact, let $c : C$ and consider the following diagram:

$$\begin{array}{ccccc} \mathbf{fib}_f(c) & \longrightarrow & \mathbf{fib}_h(kc) & \longrightarrow & \mathbf{fib}_\ell(jkc) \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & E \\ f \downarrow & & \downarrow h & & \downarrow \ell \\ C & \xrightarrow{k} & D & \xrightarrow{j} & F \end{array}$$

The squares are \diamond -cartesian when the maps on fibers are \diamond -connected, and \diamond -connected maps are closed under composition, so the outer square is also \diamond -cartesian.

With a modification of the above argument, we can prove the third fact. Suppose instead that the right square is a pullback, so that $\mathbf{fib}_h(kc) \rightarrow \mathbf{fib}_\ell(jkc)$

is an equivalence. Then since the composite map $\mathbf{fib}_f(c) \rightarrow \mathbf{fib}_\ell(jkc)$ is \diamond -connected, so is $\mathbf{fib}_f(c) \rightarrow \mathbf{fib}_h(kc)$.

To prove the second fact, suppose that $d : D$; then, since k is assumed to be surjective and we are trying to prove a proposition, we may suppose we have a $c : C$ with $kc = d$. Then we can consider the above diagram again with $\mathbf{fib}_f(c) \rightarrow \mathbf{fib}_h(d)$ and $\mathbf{fib}_f(c) \rightarrow \mathbf{fib}_\ell(jd)$ modally connected. By right cancellability of modally connected maps (Lemma 1.33 of [RSS17a]), we see that therefore $\mathbf{fib}_h(d) \rightarrow \mathbf{fib}_\ell(jd)$ is \diamond -connected.

□

Lemma 3.2.11. Suppose that

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ f \downarrow & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

is a \diamond -cartesian square. In its modal factorization

$$\begin{array}{ccccc} A & \longrightarrow & (b : B) \times \diamond \mathbf{fib}_g(b) & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & (d : D) \times \diamond \mathbf{fib}_k(d) & \longrightarrow & D \end{array} \quad (3.1)$$

the right square is a pullback.

Proof. Here we will use the proof of this fact from Lemma 3.7.3 of [Ane+17].

Consider the following diagram:

$$\begin{array}{ccccccc} A & \longrightarrow & B \times_D C & \xrightarrow{x} & B \times_D ((d : D) \times \diamond \mathbf{fib}_k(d)) & \xrightarrow{y} & B \\ & \searrow f & \downarrow & & \downarrow & & \downarrow h \\ & & C & \xrightarrow{\ell} & (d : D) \times \diamond \mathbf{fib}_k(d) & \xrightarrow{r} & D \end{array}$$

where we have taken two pullbacks. By construction, ℓ is \diamond -connected and r is \diamond -modal. By stability of the \diamond -connected / \diamond -modal factorization system, x is also \diamond -connected and y is \diamond -modal. Since by hypothesis the gap map $A \rightarrow B \times_D C$ is \diamond -connected, the composite $A \rightarrow B \times_D ((d : D) \times \diamond \mathbf{fib}_k(d))$ is \diamond -connected, so by the uniqueness of \diamond -connected / \diamond -modal factorizations, we see that $B \times_D ((d : D) \times \diamond \mathbf{fib}_k(d))$ must be equivalent to \diamond -factorization $(b : B) \times \diamond \mathbf{fib}_g(b)$. Therefore, the right hand pullback square in the above diagram is equivalent to the right hand square in Diagram 3.1, showing that it is a pullback. \square

Using these lemmas, we can prove a slight improvement of the Proposition 5.1 of [CR20], using essentially the same proof.

Theorem 3.2.12. *Suppose that*

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ f \downarrow & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

is a \diamond -cartesian square, and that B and D are \diamond -modal. Then the square

$$\begin{array}{ccc} \diamond A & \xrightarrow{\tilde{g}} & B \\ \diamond f \downarrow & & \downarrow h \\ \diamond C & \xrightarrow{\tilde{k}} & D \end{array}$$

is a pullback, where the maps $\tilde{g} : \diamond A \rightarrow B$ and $\tilde{k} : \diamond C \rightarrow D$ are the unique factorizations of g and k respectively.

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
 A & \xrightarrow{r} & B \times_D \diamond C & \xrightarrow{x} & B \\
 f \downarrow & & \downarrow & & \downarrow h \\
 C & \xrightarrow{(-)^\diamond} & \diamond C & \xrightarrow{\bar{k}} & D
 \end{array} \tag{3.2}$$

We will start by showing that the map $r : A \rightarrow B \times_D \diamond C$ is \diamond -connected. Let $c : C$, and extend the diagram as follows:

$$\begin{array}{ccccc}
 \mathbf{fib}_f(c) & \xrightarrow{z} & \mathbf{fib}_{\text{snd}}(c^\diamond) & \xrightarrow{\sim} & \mathbf{fib}_h(kc) \\
 \downarrow & & \downarrow & & \downarrow \\
 A & \xrightarrow{r} & B \times_D \diamond C & \xrightarrow{x} & B \\
 f \downarrow & & \downarrow & & \downarrow h \\
 C & \xrightarrow{(-)^\diamond} & \diamond C & \xrightarrow{\bar{k}} & D
 \end{array}$$

Since the square on the bottom right is a pullback, we get an equivalence between the map $z : \mathbf{fib}_f(c) \rightarrow \mathbf{fib}_{\text{snd}}(c^\diamond)$ and the composite $G : \mathbf{fib}_f(c) \rightarrow \mathbf{fib}_h(kc)$. Since, by Lemma 3.2.9, G is \diamond -connected, we see for all $c : C$ the map $z : \mathbf{fib}_f(c) \rightarrow \mathbf{fib}_{\text{snd}}(c^\diamond)$ is \diamond -connected. Since $(-)^\diamond$ is always \diamond -connected, we may conclude by Lemma 1.39 of [RSS17a] that the map $r : A \rightarrow B \times_D \diamond C$ is \diamond -connected.

Now, as the pullback of maps between modal types, $B \times_D \diamond C$ is modal. Therefore, r is a \diamond -connected map into a \diamond -modal type, which makes it a \diamond -unit. Therefore, the square on the right in Diagram 3.2 is the square we are trying to show is a pullback. \square

Remark 3.2.13. We can also see Theorem 3.2.12 as a corollary of Lemma 3.2.11 by noting that the right square in that lemma will be the square in the conclusion of Theorem 3.2.12 when B and D are modal.

3.3 Modal Fibrations

Recall that a map $f : X \rightarrow Y$ is a \diamond -fibration if and only if the induced map $\gamma : \diamond \text{fib}_f(y) \rightarrow \text{fib}_{\diamond f}(y^\diamond)$ is an equivalence for all $y : Y$. In other words, $f : X \rightarrow Y$ is a \diamond -fibration if \diamond preserves its fibers in the sense that whenever

$$F \rightarrow X \xrightarrow{f} Y$$

is a fiber sequence (for any pointing of Y), so is

$$\diamond F \rightarrow \diamond X \xrightarrow{\diamond f} \diamond Y.$$

In other words, a \diamond -fibration is a map f whose fibers “correctly represent” the fibers of $\diamond f$.

For example, consider the shape modality \int . A \int -fibration is a map $f : X \rightarrow Y$ whose fibers have the same homotopy type as its homotopy fibers, the fibers of its induced map $\int f : \int X \rightarrow \int Y$ on homotopy types. A simple example of a \int -fibrations is the projection $\pi_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$; all the fibers of this map are identifiable with \mathbb{R} whose shape is contractible, and the fibers of its induced map on homotopy types are contractible. An example of a map which isn’t a fibration is the inclusion $i : * \rightarrow \mathbb{R}^2$ of the origin into the real plane. Over the point $(1, 1) : \mathbb{R}^2$, the fiber of i is empty, and so its homotopy type is empty. But the induced map $\int i : \int * \rightarrow \int \mathbb{R}^2$ is an equivalence since $\int \mathbb{R}^2$ is contractible, and so all the fibers of $\int i$ are equivalent to $*$ which is not empty.

Remark 3.3.1. This is the sense in which a \diamond -fibration is a “fibration”. It most closely resembles the notion of *quasi-fibration* of topological spaces introduced by Dold and Thom in [DT58b], which is a continuous map $f : X \rightarrow Y$ such

that for all $y \in Y$, the canonical map from the inverse image $f^{-1}(y)$ to the homotopy fiber $\mathbf{fib}_f(y)$ is a weak equivalence. If, seeking analogy, we take “weak equivalence” to be \diamond -equivalence (which, for \mathcal{J} , means that a map is a weak equivalence if it induces an equivalence on homotopy types), then a \diamond -fibration is map f whose fibers are weakly equivalent to its “modal fibers”, the fibers of $\diamond f$.

However, the notion of \diamond -fibration is somewhat more robust than the notion of quasi-fibration, even in the case of \mathcal{J} . As we will see, \diamond -fibrations are closed under pullback, while quasi-fibrations are not. In this sense, \diamond -fibrations more closely resemble the *universal quasi-fibrations* introduced by Goodwillie in an email to the ALGTOP mailing list [Goo01]. Intuitively, this is because universal quantification in type theory says more than it does in set theory — it implies a sort of continuity. We will come back to this subtle point in the next section when we introduce the notion of a crisp variable from Shulman’s real hoheision [Shu18b] in order to give a trick for showing a map is a \mathcal{J} -fibration.

Before we get there, let’s develop the basic theory of \diamond -fibrations for a general modality. First, we will characterize \diamond -fibrations as those maps on which the two factorization systems of \diamond agree.

Lemma 3.3.2. For $f : X \rightarrow Y$, the following are equivalent:

1. f is a \diamond -fibration.
2. The \diamond -modal factor of f is \diamond -étale.
3. The \diamond -equivalence factor of f is \diamond -connected.

4. The \diamond -connected/ \diamond -modal and \diamond -equivalence/ \diamond -étale factorizations of f are equal as factorizations of f .
5. The \diamond -naturality square for f is \diamond -cartesian.

Proof. We will first show that the first two conditions are equivalent; then we will argue that the next three are all equivalent by the uniqueness of each factorization. Finally, we note that the last condition is immediately equivalent to the third, since the \diamond -equivalence factor of f is the gap map of the \diamond -naturality square.

By Lemma 1.24 of [RSS17a], the unique factorization of the map

$$\lambda(y, x). (y, x^\diamond)^\diamond : (y : Y) \times \mathbf{fib}_f(y) \rightarrow \diamond((y : Y) \times \diamond \mathbf{fib}_f(y))$$

through $\diamond((y : Y) \times \mathbf{fib}_f(y))$ is an equivalence. Therefore, the composite

$$(y : Y) \times \diamond \mathbf{fib}_f(y) \xrightarrow{(-)^\diamond} \diamond((y : Y) \times \diamond \mathbf{fib}_f(y)) \xrightarrow{\sim} \diamond((y : Y) \times \mathbf{fib}_f(y))$$

is a \diamond -unit. So, for any $y : Y$, we get a diagram

$$\begin{array}{ccccc}
\mathbf{fib}_f(y) & \longrightarrow & \diamond \mathbf{fib}_f(y) & \xrightarrow{\gamma} & \mathbf{fib}_{\diamond f}(y^\diamond) \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & (y : Y) \times \diamond \mathbf{fib}_f(y) & \longrightarrow & \diamond X \\
f \downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow{\text{id}} & Y & \longrightarrow & \diamond Y
\end{array}$$

in which the bottom right square is a \diamond -naturality square. The map f is a \diamond -fibration if and only if the connecting map γ is an equivalence for all $y : Y$, and this happens if and only if the bottom right square is a pullback. But the bottom right square is a pullback precisely when $\mathbf{fst} : (y : Y) \times \diamond \mathbf{fib}_f(y) \rightarrow Y$ is

\diamond -étale.

On the other hand, the fourth condition implies the second and third by simply transporting the properties. Each of the second and third also imply the fourth by the uniqueness of each factorization. Without loss of generality, consider the second condition. The \diamond -connected factor of f is always a \diamond -equivalence, so if the modal factor of f is \diamond -étale then the \diamond -connected/ \diamond -modal factorization is a \diamond -equivalence/ \diamond -étale factorization and so is equal to the canonical one by the uniqueness of such factorizations. \square

As a corollary, we can prove that \diamond -fibrations are closed under pullback, and give a descent theorem for \diamond -fibrations.

Corollary 3.3.3. Let

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ g \downarrow & & \downarrow f \\ B & \xrightarrow{y} & Y \end{array}$$

be a \diamond -cartesian square. If f is a fibration, then so is g . In particular, \diamond -fibrations are closed under pullback.

Proof. Consider the following cube:

$$\begin{array}{ccccc} & & \diamond A & \longrightarrow & \diamond X \\ & \nearrow & \downarrow & \nearrow & \downarrow \diamond f \\ A & \longrightarrow & X & \longrightarrow & Y \\ g \downarrow & & \downarrow & & \downarrow \\ B & \nearrow & \diamond B & \longrightarrow & \diamond Y \\ & \searrow & \downarrow & \searrow & \\ & & Y & & \end{array} \quad (3.3)$$

By hypothesis, the front face is \diamond -cartesian and, since f is a \diamond -fibration, so is the

rightmost face. Therefore, by Lemma 3.2.10, the diagonal square is \diamond -cartesian. Then, by Theorem 3.2.12, the back face is a pullback. Then, by Lemma 3.2.10 again, the leftmost face is \diamond -cartesian, which shows that g is a \diamond -fibration. \square

Remark 3.3.4. It is at this point that we require a full modality, rather than just a *reflective subuniverse*. The proof of Theorem 3.2.12 uses the fact that \diamond -units are \diamond -connected, a fact which characterizes modalities amongst localizations (also known as reflective subuniverses). However, if one could prove Theorem 3.2.12 without using this fact, or prove that the pullback of a \diamond -étale map is \diamond -étale for \diamond a reflective subuniverse, then we could prove the pullback stability of \diamond -fibrations and so the rest of the theory of \diamond -fibrations would go through as well.

Using Lemma 3.2.10 and the characterization of \diamond -fibrations as those maps whose naturality squares are \diamond -cartesian, we can show that \diamond -fibrations have the same closure properties as \diamond -cartesian squares.

Theorem 3.3.5. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be maps.*

1. *If f and g are \diamond -fibrations, then $g \circ f$ is a \diamond -fibration.*
2. *If f and $g \circ f$ are \diamond -fibrations, and $\diamond f$ is surjective, then g is a \diamond -fibration.*
3. *If g is \diamond -étale and $g \circ f$ is a \diamond -fibration, then f is a \diamond -fibration.*

Proof. We apply Lemma 3.2.10 to the squares

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
 \downarrow & & \downarrow & & \downarrow \\
 \diamond X & \xrightarrow{\diamond f} & \diamond Y & \xrightarrow{\diamond g} & \diamond Z
 \end{array}$$

For the third part, remember that g is \diamond -étale precisely when its naturality square is a pullback. \square

We now have the tools to characterize \diamond -fibrations in another way. A modality is called *lex* if it preserves all pullbacks. Not all modalities are lex; for example, the truncation modalities are not, and nor is \mathbb{J} . The \diamond -fibrations are precisely the maps along which \diamond is lex. That is, \diamond preserves all pullbacks of a map f if and only if that map is a \diamond -fibration.

Theorem 3.3.6. *A map $f : X \rightarrow Y$ is a \diamond -fibration if and only if \diamond preserves every pullback of it in the sense that whenever the square on the left is a pullback, so is the square on the right.*

$$\begin{array}{ccc}
 A & \xrightarrow{x} & X \\
 g \downarrow & & \downarrow f \\
 B & \xrightarrow{y} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 \diamond A & \xrightarrow{\diamond x} & \diamond X \\
 \diamond g \downarrow & & \downarrow \diamond f \\
 \diamond B & \xrightarrow{\diamond y} & \diamond Y
 \end{array}$$

Remark 3.3.7. For the case of \mathbb{J} , Theorem 3.3.6 gives us a sufficient condition for a pullback to be a *homotopy pullback* (that is, a pullback on homotopy types): if one of the legs is a \mathbb{J} -fibration, then the pullback is a homotopy pullback.

Proof. If \diamond preserves all pullbacks of f , then by taking $B \equiv *$, we see that \diamond preserves all fibers of f which by definition makes it a \diamond -fibration.

On the other hand, suppose that f is a \diamond -fibration and that the square on the left above is a pullback. Then the connecting map $\alpha : \mathbf{fib}_g(a) \rightarrow \mathbf{fib}_f(ya)$ is an equivalence for all $a : A$. Furthermore, g is also a \diamond -fibration by Corollary 3.3.3 and therefore the maps $\gamma_f : \diamond \mathbf{fib}_f(ya) \rightarrow \mathbf{fib}_{\diamond f}((ya)^\diamond)$ and $\gamma_g : \diamond \mathbf{fib}_g(a) \rightarrow \mathbf{fib}_{\diamond g}(a^\diamond)$ are equivalences for all $a : A$. These maps fit together into a commuting

square:

$$\begin{array}{ccc}
\diamond \mathbf{fib}_g(a) & \xrightarrow{\diamond \alpha} & \diamond \mathbf{fib}_f(ya) \\
\gamma_g \downarrow & & \downarrow \gamma_f \\
\mathbf{fib}_{\diamond g}(a^\diamond) & \longrightarrow & \mathbf{fib}_{\diamond f}((ya)^\diamond)
\end{array}$$

Since the sides and top are equivalences, the bottom is also an equivalence.

Now, in order to show that the square on the right is a pullback, we need for the induced map $\zeta: \mathbf{fib}_{\diamond g}(u) \rightarrow \mathbf{fib}_{\diamond f}(\diamond y(u))$ to be an equivalence for all $u: \diamond B$. But we have only shown it for $u \equiv a^\diamond$, since $\diamond y(a^\diamond) = (ya)^\diamond$ by naturality. Luckily, as both $\mathbf{fib}_{\diamond g}(u)$ and $\mathbf{fib}_{\diamond f}(\diamond y(u))$ are \diamond -modal, $\mathbf{isEquiv}(\zeta)$ is also \diamond -modal for all $u: \diamond B$. We may therefore assume that $u \equiv a^\diamond$ by \diamond -induction. \square

As a corollary of this, we can prove a partial stability of the \diamond -equivalence/ \diamond -étale factorization system. A factorization system is stable if the left class is stable under pullback.

Remark 3.3.8. The class of \diamond -equivalences is not stable under pullback in general. For example, consider the following pullback

$$\begin{array}{ccc}
\emptyset & \longrightarrow & * \\
\downarrow & & \downarrow 1 \\
* & \xrightarrow{0} & \mathbb{R}
\end{array}$$

Though the bottom map is a \mathfrak{f} -equivalence since \mathbb{R} is homotopically contractible, the top map is not a \mathfrak{f} -equivalence.

On the other hand, \diamond -equivalences are preserved by pullback along \diamond -fibrations.

Corollary 3.3.9. Suppose that the following square is a pullback. If f is a \diamond -fibration and y a \diamond -equivalence, then x is a \diamond -equivalence.

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ g \downarrow & & \downarrow f \\ B & \xrightarrow{y} & Y \end{array}$$

Proof. Since f is a \diamond -fibration, the square

$$\begin{array}{ccc} \diamond A & \xrightarrow{\diamond x} & \diamond X \\ \diamond g \downarrow & & \downarrow \diamond f \\ \diamond B & \xrightarrow{\diamond y} & \diamond Y \end{array}$$

is also a pullback. But $\diamond y$ is an equivalence by hypothesis, and therefore so is $\diamond x$. □

All of this pullback preserving lets us add a few more conditions to the long list of equivalent conditions for lexness in Theorem 3.1 of [RSS17a].

Proposition 3.3.10. The following are equivalent:

1. The modality \diamond is lex.
2. Every map is a \diamond -fibration.
3. If every map $f_i : A_i \rightarrow B_i$ is a \diamond -fibration in a family of maps f , then the total map $\text{tot}(f) : (i : I) \times A_i \rightarrow (i : I) \times B_i$ is a \diamond -fibration.
4. For any map $f : X \rightarrow Y$, the connecting map $\text{tot}(\gamma) : (y : Y) \times \diamond \text{fib}_f(y) \rightarrow (y : Y) \times \text{fib}_{\diamond f}(y^\diamond)$ between factorizations of f is a \diamond -fibration.
5. The universal map $\mathbf{Type}_* \rightarrow \mathbf{Type}$ is a \diamond -fibration.

Proof. Conditions 1 and 2 are equivalent by the characterization of \diamond -fibrations in terms of pullback preservation, and condition 2 trivially implies conditions 3, 4, and 5. Every map between \diamond -modal types is \diamond -étale since for \diamond -modal types the modal units are equivalences. Therefore, the connecting map $\gamma : \diamond \mathbf{fib}_f(y) \rightarrow \mathbf{fib}_{\diamond f}(y^\diamond)$ is \diamond -étale and in particular a \diamond -fibration for any map $f : X \rightarrow Y$ and $y : Y$. This means that condition 3 implies condition 4. On the other hand, since \diamond -fibrations are closed under composition, if $\mathbf{tot}(\gamma)$ is a \diamond -fibration then the \diamond -modal factor of any map $f : X \rightarrow Y$ is a \diamond -fibration, as it is the composite of $\mathbf{tot}(\gamma)$ and the \diamond -étale factor of f . Therefore, by Lemma 3.3.2, f is a \diamond -fibration, so that condition 4 implies condition 2.

Finally, the last condition implies the second since \diamond -fibrations are closed under pullback. \square

All objects are “fibrant” with respect to \diamond -fibrations in the sense that the terminal map is always a \diamond -fibration. We can say something more — every projection map $\mathbf{fst} : A \times B \rightarrow A$ is a \diamond -fibration.

Lemma 3.3.11. For any types A and B , the projection map $\mathbf{fst} : A \times B \rightarrow A$ is a \diamond -fibration.

Proof. This follows directly from the fact that \diamond preserves products. The map $(-)^{\diamond} \times (-)^{\diamond} : A \times B \rightarrow \diamond A \times \diamond B$ is a \diamond -unit by Lemma 1.27 of [RSS17a], and

so for any $a : A$ we get a map of fiber sequences:

$$\begin{array}{ccc}
 B & \xrightarrow{(-)^\diamond} & \diamond B \\
 \downarrow & & \downarrow \\
 A \times B & \xrightarrow{(-)^\diamond \times (-)^\diamond} & \diamond A \times \diamond B \\
 \text{fst} \downarrow & & \downarrow \text{fst} \\
 A & \xrightarrow{(-)^\diamond} & \diamond A
 \end{array}$$

where the bottom square is a \diamond -naturality square. The induced map $\gamma : \diamond \text{fib}_{\text{fst}}(a) \rightarrow \text{fib}_{\diamond \text{fst}}(a^\diamond)$ is therefore equal to the identity map of $\diamond B$, and so is an equivalence. \square

A map $f : X \rightarrow Y$ is equal to a projection $\text{fst} : Y \times Z \rightarrow Y$ if and only if $\text{fib}_f : Y \rightarrow \mathbf{Type}$ is constant, that is, if it factors through the point.

$$\begin{array}{ccc}
 Y & \xrightarrow{\text{fib}_f} & \mathbf{Type} \\
 \downarrow & \nearrow Z & \\
 * & &
 \end{array}$$

We have just shown that such maps are \diamond -fibrations, but we can do better. We can show that a map is a \diamond -fibration if and only if it has \diamond -locally constant \diamond -fibers in the sense made precise in the upcoming Theorem 3.3.14. First, we prove a similar characterization of \diamond -étale maps. This is the *modal descent* theorem of [CR20].

Lemma 3.3.12. Let $E : Y \rightarrow \mathbf{Type}_\diamond$ be a family of modal types. Then E factors through the modal unit of Y if and only if $\text{fst} : (y : Y) \times Ey \rightarrow Y$ is \diamond -étale. In particular, the type of such factorizations is a proposition.

Proof. If fst is \diamond -étale, then $\gamma : Ey \rightarrow \text{fib}_{\diamond \text{fst}}(y^\diamond)$ is an equivalence; therefore, $\text{fib}_{\diamond \text{fst}} : \diamond Y \rightarrow \mathbf{Type}_\diamond$ is such a factorization.

On the other hand, suppose that $\tilde{E} : \diamond Y \rightarrow \mathbf{Type}_\diamond$ with $w : (y : Y) \rightarrow (Ey \simeq \tilde{E}y^\diamond)$ is a factorization. Then the square

$$\begin{array}{ccc} (y : Y) \times Ey & \xrightarrow{\text{tot}(w)} & (u : \diamond Y) \times \tilde{E}u \\ \text{fst} \downarrow & & \downarrow \text{fst} \\ Y & \longrightarrow & \diamond Y \end{array}$$

is a pullback. Since the unit $Y \rightarrow \diamond Y$ is \diamond -connected and \diamond -connected maps are closed under pullback, $\text{tot}(w)$ is \diamond -connected. As $(u : \diamond Y) \times \tilde{E}u$ is a sum of modal types over a modal type, it is modal, and therefore $\text{tot}(w)$ is a \diamond -unit and this square is a \diamond -naturality square. But then $\text{fst} : (y : Y) \times Ey \rightarrow Y$ is \diamond -étale since its \diamond -naturality square is a pullback.

To show that the type of such factorizations is a proposition, we just need to show that any factorization equals $(\text{fib}_{\diamond \text{fst}}, \gamma)$. This follows immediately from the uniqueness of \diamond -units. \square

As a corollary, we can characterize the \diamond -étale maps into a type Y .

Corollary 3.3.13. For any type Y , the type

$$\text{Ét}_\diamond(Y) := (X : \mathbf{Type}) \times (f : X \rightarrow Y) \times \text{is}\diamond\text{étale}(f)$$

is equivalent to the type $\diamond Y \rightarrow \mathbf{Type}_\diamond$ of families of modal types varying over $\diamond Y$.

Proof. Consider the following equivalence:

$$\begin{aligned}
\text{Ét}_\diamond(Y) &::= (X : \mathbf{Type}) \times (f : X \rightarrow Y) \times \text{is}\diamond\text{étale}(f) \\
&\simeq (X : \mathbf{Type}) \times (f : X \rightarrow Y) \times (\tilde{E} : \diamond Y \rightarrow \mathbf{Type}_\diamond) \times \text{fib}_f = \tilde{E} \circ (-)^\diamond \\
&\simeq (E : Y \rightarrow \mathbf{Type}_\diamond) \times (\tilde{E} : \diamond Y \rightarrow \mathbf{Type}_\diamond) \times (E = \tilde{E} \circ (-)^\diamond) \\
&\simeq \diamond Y \rightarrow \mathbf{Type}_\diamond \qquad \square
\end{aligned}$$

We may now prove the main theorem of this section, characterizing \diamond -fibrations as those maps with \diamond -locally constant \diamond -fibers.

Theorem 3.3.14. *Let $E : Y \rightarrow \mathbf{Type}$ be a family of types. Then $\text{fst} : (y : Y) \times Ey \rightarrow Y$ is a \diamond -fibration if and only if there is a type family $\tilde{E} : \diamond Y \rightarrow \mathbf{Type}_\diamond$ making the following square commute:*

$$\begin{array}{ccc}
Y & \xrightarrow{E} & \mathbf{Type} \\
\downarrow & & \downarrow \diamond \\
\diamond Y & \dashrightarrow_{\tilde{E}} & \mathbf{Type}_\diamond
\end{array}$$

Remark 3.3.15. In the case of the \int modality, Theorem 3.3.14 can be understood as characterizing the \int -fibrations as those maps whose fibers form a *local system* on their codomain. The factorization $\tilde{E} : \int Y \rightarrow \mathbf{Type}_\int$ of $\int E : Y \rightarrow \mathbf{Type}_\int$ shows that the homotopy types of the fibers Ey are *locally constant* in y . Moreover, the usual transport of identifications in $\int Y$ gives rise to a *monodromy* action of the homotopy type $\int Y$ on the homotopy types $\int Ey$ of the fibers Ey .

Proof. By Lemma 3.3.2, fst is a fibration if and only if its modal factor $\mathcal{R}(\text{fst}) :$

$(y : Y) \times \diamond(Ey) \rightarrow Y$ is \diamond -étale. By Lemma 3.3.12, $\mathcal{R}(\mathbf{fst})$ is \diamond -étale if and only if $\diamond E : Y \rightarrow \mathbf{Type}_\diamond$ factors through $\diamond Y$. But this is exactly what we are asking for! \square

What is a $\|-\|_n$ -fibration? A map is a $\|-\|_n$ -equivalence exactly when it induces an equivalence on the homotopy groups π_k for $0 \leq k \leq n$ (see Theorem 8.8.3 of [Uni13]), and is $\|-\|_n$ -connected when it furthermore induces a surjection on π_{n+1} (see Corollary 8.8.6 of [Uni13]). Since a map is a $\|-\|_n$ -fibration if and only if its $\|-\|_n$ -equivalence factor is $\|-\|_n$ -connected, we might expect that a map is a $\|-\|_n$ -fibration if it induces a surjection on π_{n+1} . We can prove this naive conjecture by giving one more equivalent characterization of \diamond -fibrations — this time with a small caveat.

We first need an elementary lemma concerning fibers.

Lemma 3.3.16. Consider a square

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ f \downarrow & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

commuting via $S : (x : A) \rightarrow (k(f(x)) = h(g(x)))$. Let $a : A$, and define $F : \mathbf{fib}_g(ga) \rightarrow \mathbf{fib}_k(kfa)$ by

$$F(x : A, p : gx = ga) := (fx, S(x) \cdot h_* p \cdot S(a)^{-1}).$$

For $(c, q) : \mathbf{fib}_k(kfa)$, define $G : \mathbf{fib}_f(c) \rightarrow \mathbf{fib}_h(kfa)$ by

$$G(x : A, w : fx = c) := (gx, S(x)^{-1} \cdot k_* w \cdot q).$$

Then we have an equivalence $\mathbf{fib}_F(c, q) = \mathbf{fib}_G(ga, S(a)^{-1})$ giving a (judgementally) commuting square

$$\begin{array}{ccc} \mathbf{fib}_F(c, q) (= \mathbf{fib}_G(ga, S(a)^{-1})) & \longrightarrow & \mathbf{fib}_f(c) \\ \downarrow & & \downarrow \\ \mathbf{fib}_g(ga) & \longrightarrow & A \end{array}$$

Proof. We find the equivalence as the following composite:

$$\begin{aligned} \mathbf{fib}_F(c, q) &::= ((x, p) : \mathbf{fib}_g(ga)) \times (F(x, p) = (c, q)) \\ &= (x : A) \times (p : gx = ga) \times ((fx, S(x) \cdot h_*p \cdot S(a)^{-1}) = (c, q)) \\ &= (x : A) \times (p : gx = ga) \times (w : fx = c) \times (k_*w^{-1} \cdot S(x) \cdot h_*p \cdot S(a)^{-1} = q) \\ &= (x : A) \times (w : fx = c) \times (p : gx = ga) \times (h_*p^{-1} \cdot S(x) \cdot k_*w \cdot q = S(a)^{-1}) \\ &= (x : A) \times (w : fx = c) \times (G(x, w) = (ga, S(a)^{-1})) \\ &= \mathbf{fib}_G(ga, S(a)^{-1}). \end{aligned}$$

Note that throughout this equivalence, $x : A$ is not affected by the equivalences.

Therefore, we end up with the judgementally commuting square as desired. \square

Theorem 3.3.17. *Let $f : X \rightarrow Y$.*

1. *If f is a \diamond -fibration, then for all $x : X$ the induced map $\mathbf{fib}_{(-)\diamond}(x^\diamond) \rightarrow \mathbf{fib}_{(-)\diamond}((fx)^\diamond)$ is \diamond -connected.*
2. *If the modal unit $(-)\diamond : X \rightarrow \diamond X$ is surjective, and for all $x : X$ the induced map $\mathbf{fib}_{(-)\diamond}(x^\diamond) \rightarrow \mathbf{fib}_{(-)\diamond}((fx)^\diamond)$ is \diamond -connected, then f is a \diamond -fibration.*

Proof. First, suppose that f a \diamond -fibration, and let $x : X$ seeking to show that the induced map $\mathbf{fib}_{(-)\diamond}(x^\diamond) \rightarrow \mathbf{fib}_{(-)\diamond}((fx)^\diamond)$ is \diamond -connected. By Lemma 3.3.16, the fiber of the induced map over $(y, p) : \mathbf{fib}_{(-)\diamond}((fx)^\diamond)$ is equivalent to the fiber of $\delta : \mathbf{fib}_f(y) \rightarrow \mathbf{fib}_{\diamond f}(y^\diamond)$ over $(x^\diamond, S(x)^{-1})$ where $S : (x : X) \rightarrow (fx)^\diamond = \diamond f(x^\diamond)$ is witness to the commutativity of the naturality square. Since f is a \diamond -fibration, this δ is a \diamond -equivalence; but it is a \diamond -equivalence landing in a modal type, and is therefore a \diamond -unit, which is to say it is \diamond -connected.

Conversely, suppose that the modal unit $(-)\diamond : X \rightarrow \diamond X$ is surjective. We aim to show that $f : X \rightarrow Y$ is a \diamond -fibration, so it suffices to prove that the maps $\delta : \mathbf{fib}_f(y) \rightarrow \mathbf{fib}_{\diamond f}(y^\diamond)$ are \diamond -connected for all $y : Y$. So, suppose we have $(u, p) : \mathbf{fib}_{\diamond f}(y^\diamond)$, seeking to show that $\mathbf{fib}_\delta(u, p)$ is \diamond -connected. By the surjectivity of $(-)\diamond : X \rightarrow \diamond X$, we may assume u is of the form x^\diamond . Then Lemma 3.3.16 tells us that $\mathbf{fib}_\delta(x^\diamond, p)$ is equivalent to the fiber of the induced map $\mathbf{fib}_{(-)\diamond}(x^\diamond) \rightarrow \mathbf{fib}_{(-)\diamond}((fx)^\diamond)$ over $(fx, S(x))$. But by hypothesis, this fiber was \diamond -connected. \square

Remark 3.3.18. The condition that $(-)\diamond : X \rightarrow \diamond X$ be surjective is often trivially satisfied. For many modalities — the n -truncation modalities and the shape modality included — *all* modal units are surjective. In this case, Theorem 3.3.17 characterizes the \diamond -fibrations with no caveats. We might refer to modalities whose units are surjective as *global* modalities; they are counterposed to topological modalities, which are given by a nullification at a family of propositions, since any global topological modality is trivial. More specifically, any global modality is cotopological in the sense of Theorem 3.22 of [RSS17a].

Corollary 3.3.19. A map $f : X \rightarrow Y$ is a $\|-\|_n$ -fibration if and only if for all

$y : Y$ and $(x, p) : \mathbf{fib}_f(y)$, the induced map $\pi_{n+1}(X, x) \rightarrow \pi_{n+1}(Y, y)$ is surjective.

Proof. By Theorem 3.3.17, f is a $\|- \|_n$ -fibration if and only if the induced map $\mathbf{fib}_{|-|_n}(x) \rightarrow \mathbf{fib}_{|-|_n}(y)$ is $\|- \|_n$ -connected. As the fibers of $\|- \|_n$ -units, $\mathbf{fib}_{|-|_n}(x)$ and $\mathbf{fib}_{|-|_n}(y)$ are $\|- \|_n$ -connected, so the induced map is $\|- \|_n$ -connected if and only if the induced map

$$pi_{n+1}(\mathbf{fib}_{|-|_n}(x), (x, \mathbf{refl})) \rightarrow \pi_{n+1}(\mathbf{fib}_{|-|_n}(y), (y, \mathbf{refl}))$$

is a surjection. But this map is equivalent to the induced map $\pi_{n+1}(X, x) \rightarrow \pi_{n+1}(Y, y)$. \square

Before moving on, let's briefly consider a pair of modalities $\diamond \leq \blacklozenge$, where every \diamond -modal type is \blacklozenge -modal. For example, $\|- \|_n \leq \|- \|_{n+1}$. In particular, $\diamond X$ is \blacklozenge -modal, and so the unit $(-)^{\diamond} : X \rightarrow \diamond X$ factors uniquely through $(-)^{\blacklozenge} : X \rightarrow \blacklozenge X$, giving us a commuting diagram:

$$\begin{array}{ccc} X & \xrightarrow{(-)^{\blacklozenge}} & \blacklozenge X \\ & \searrow^{(-)^{\diamond}} & \downarrow c \\ & & \diamond X \end{array}$$

Lemma 3.3.20. Suppose that every \diamond -modal type is \blacklozenge -modal. Then the connecting map $c : \blacklozenge X \rightarrow \diamond X$ is a \diamond -unit. As a corollary, for any $f : X \rightarrow Y$, we get a \diamond -naturality square

$$\begin{array}{ccc} \blacklozenge X & \longrightarrow & \diamond X \\ \blacklozenge f \downarrow & & \downarrow \diamond f \\ \blacklozenge Y & \longrightarrow & \diamond Y \end{array}$$

Proof. Let Z be a \diamond -modal type. It is therefore also \blacklozenge -modal. Precomposing

by the above commutative triangle gives us a commutative diagram:

$$\begin{array}{ccc}
 (X \rightarrow Z) & \xleftarrow{\sim} & (\blacklozenge X \rightarrow Z) \\
 & \swarrow \sim & \uparrow \\
 & & (\lozenge X \rightarrow Z)
 \end{array}$$

Because Z is both \lozenge -modal and \blacklozenge -modal, the two horizontal maps are equivalences, and therefore the vertical map is an equivalence, as desired. \square

We aim to demonstrate the following relations between the different kinds of maps associated to these modalities.

Theorem 3.3.21. *Suppose that every \lozenge -modal type is \blacklozenge -modal, and that $f : X \rightarrow Y$. Then:*

1. *If f is \lozenge -modal, then it is \blacklozenge -modal.*
2. *If f is \lozenge -étale, then it is \blacklozenge -étale.*
3. *If f is a \blacklozenge -equivalence, then it is a \lozenge -equivalence.*
4. *If f is \blacklozenge -connected, then it is \lozenge -connected.*
5. *If f is a \blacklozenge -fibration and $\blacklozenge f$ is a \lozenge -fibration, then f is a \lozenge -fibration.*

Proof of Theorem 3.3.21.

1. If f is \lozenge -modal, then its fibers are \lozenge -modal and so by hypothesis \blacklozenge -modal, so that f is \blacklozenge -modal.
2. If f is \lozenge -étale, then by Lemma 3.3.12, \mathbf{fib}_f factors through $\lozenge X$ as $E : \lozenge X \rightarrow \mathbf{Type}$. But then $E \circ c : \blacklozenge X \rightarrow \mathbf{Type}$ is a factorization of \mathbf{fib}_f through $\blacklozenge X$, so that f is \blacklozenge -étale.

3. If f is a \blacklozenge -equivalence, then $\blacklozenge f$ is an equivalence. But then since $\blacklozenge\blacklozenge f$ is equivalent to $\blacklozenge f$ by Lemma 3.3.20, $\blacklozenge f$ is an equivalence.
4. If f is \blacklozenge -connected, then $\blacklozenge \text{fib}_f(y)$ is contractible for all $y : Y$. But then $\blacklozenge \text{fib}_f(y) = \blacklozenge\blacklozenge \text{fib}_f(y)$ is contractible for all $y : Y$, so f is \blacklozenge -connected.
5. Consider the following diagram.

$$\begin{array}{ccc}
Y & \xrightarrow{\text{fib}_f} & \mathbf{Type} \\
\downarrow & & \downarrow \blacklozenge \\
\blacklozenge Y & \xrightarrow{\text{fib}_{\blacklozenge f}} & \mathbf{Type}_{\blacklozenge} \\
\downarrow & & \downarrow \blacklozenge \\
\blacklozenge\blacklozenge Y & \xrightarrow{\text{fib}_{\blacklozenge\blacklozenge f}} & \mathbf{Type}_{\blacklozenge\blacklozenge}
\end{array}$$

If f is a \blacklozenge -fibration then the upper square commutes, and if $\blacklozenge f$ is a \blacklozenge -fibration then the lower square commutes. If the outer square commutes, then fib_f factors through $\blacklozenge Y$, and so is a \blacklozenge -fibration. \square

3.4 A Brief Review of Cohesive HoTT

In this section, we review Mike Shulman’s Real Cohesive Homotopy Type Theory (as found in [Shu18b]). The *shape* modality \mathfrak{J} which sends a type to its homotopy type is defined in the context of Real Cohesive HoTT. It is the interplay of this modality with the *comodality* \mathfrak{b} that defines real cohesion, and that we will exploit to give a trick for showing that a map is a \mathfrak{J} -fibration.

For the reader who isn’t too familiar with real cohesion and doesn’t feel like getting too familiar with it, worry not. The details in this section revolve around the notion of *crisp* objects, which will be explained below. But every object (type or element) which appears in the empty context — that is to say, with no

free variables in its definition — is crisp. Therefore, if you need a heuristic for understanding what it means to, say, have a crisp type $Z :: \mathbf{Type}$, just imagine that this means that Z has no free variables in its definition. For example, \mathbb{N} , \mathbb{Z} , \mathbb{R} , and \mathbf{Type} are all crisp types, while $0 : \mathbb{N}$, $\pi : \mathbb{R}$, and $\lambda x. x^2 + 2 : \mathbb{R} \rightarrow \mathbb{R}$ are all crisp elements since they have no free variables. Furthermore, any natural number may be assumed to be crisp, so that types like \mathbb{R}^n may be taken as crisp even though they involve a free variable $n : \mathbb{N}$.

In type theory, if you can argue that for all $x : X$, there is an $f(x) : Y$, then you have given a function $f : X \rightarrow Y$ in the process. In Shulman’s Real Cohesive HoTT, all functions will be *continuous* in a topological sense. So, saying that for $x : X$ we have a $f(x) : Y$ means that $f(x)$ must depend *continuously* on x . But not all dependencies are continuous. What if we want to express a discontinuous dependence?

To address this concern, Shulman introduces the notion of a “crisp variable”

$$a :: A$$

to express a discontinuous dependence. Hypothesizing $a :: A$ means that we can use a in a discontinuous manner; one way this is realized is in the crisp Law of Excluded middle.

Axiom 1 (Crisp excluded middle). For any crisp $P :: \mathbf{Prop}$, we have $P \vee \neg P$.

This axiom lets us use case analysis when assuming a crisp element of a set, even if the set has a native topology that wouldn’t admit case analysis constructively (such as the Dedekind real numbers \mathbb{R} , which cannot constructively be separated into two disjoint parts).

Any variable appearing in the type of a crisp variable must also be crisp, and a crisp variable may only be substituted by expressions that *only* involve crisp variables. When all the variables in an expression are crisp, we say that that expression is crisp; so, we may only substitute crisp expressions in for crisp variables. Constants — like $0 : \mathbb{N}$ or $\mathbb{N} : \mathbf{Type}$ — appearing in an empty context are therefore always crisp. This means that one cannot give a closed form example of a term which is *not* crisp; all terms with no free variables are crisp. For emphasis, we will say that a term which is not crisp is *cohesive*. The rules for crisp type theory can be found in Section 2 of [Shu18b].

One way to think of the difference between a cohesive dependence — for all $x : X, f(x) : Y$ — and a crisp dependence — for all $x :: X, f(x) : Y$ — is that the former expresses that $f(x)$ depends on a generic $x : X$, whereas in the latter we are saying that for *each* individual x , there is an $f(x)$.⁶

Given a *crisp* type X , we can remove its spatial structure to get a type $\flat X$. If X is a set, $\flat X$ can be thought of as its set of points.⁷ The rules for \flat can be found in Section 4 of [Shu18b]. They may be summed up by saying that $\flat X$ is inductively generated by elements of the form x^\flat for *crisp* $x :: X$. In particular, whenever we have a type family $C : \flat X \rightarrow \mathbf{Type}$, an $x : \flat X$, and an element $f(u) : C(u^\flat)$ depending on a *crisp* $u :: X$, we get an element

$$(\text{let } u^\flat := x \text{ in } f(u)) : C(x)$$

and if $x \equiv v^\flat$, then $(\text{let } u^\flat := x \text{ in } f(u)) \equiv f(v)$. This allows us to think of $\flat X$

⁶In particular, by the crisp excluded middle axiom, we may deal with each $x :: X$ on a case by case basis.

⁷This intuition really only works for sets, since if G is a group then $\flat BG$ behaves like the moduli stack of principal G -bundles with flat connection, and not “the type of points of BG ”.

as “the type of *crisp* points of X ”.

We have an inclusion $(-)_b : \flat X \rightarrow X$ given by $x_b := x$ in u . Since we are thinking of a dependence on a crisp variable as a *discontinuous* dependence, if this map $(-)_b : \flat X \rightarrow X$ is an equivalence then every *discontinuous* dependence on $x :: X$ underlies a *continuous* dependence on x . This leads us to the following definition:

Definition 3.4.1. A crisp type $X :: \mathbf{Type}$ is *crisply discrete* if the counit $(-)_b : \flat X \rightarrow X$ is an equivalence.⁸

We would like our formal notion of continuity coming from crisp types to match our topological notion of continuity as measured by continuous paths. We have a notion of discreteness coming from crisp variables — *crisply discrete* — but we also need a topological notion of discreteness.

Definition 3.4.2. A type X is *discrete* if every path in it is constant in the sense that the inclusion of constant paths $X \rightarrow (\mathbb{R} \rightarrow X)$ is an equivalence.

Remark 3.4.3. The real numbers \mathbb{R} in Definition 3.4.2 — and throughout this paper — are the *Dedekind* real numbers and not the Cauchy real numbers. It can be proven in real cohesion (with a form of the axiom of choice) that the Cauchy real numbers are discrete, and that indeed they are equivalent to $\flat \mathbb{R}$ — see Corollary 8.28 of [Shu18b].

Note that we can form the proposition “is discrete” for any type, while we can only form the proposition “is crisply discrete” for crisp types, since to form $\flat X$, X must be crisp. The main axiom of real cohesion, which ties the liminal

⁸See Remark 6.13 of [Shu18b] for a discussion on some of the subtleties in the notion of crisp discreteness.

sort of topology implied by the use of crisp variables to the concrete topology of the real numbers, is that for crisp types being discrete and being crisply discrete coincide.

Axiom 2 ($\mathbb{R} \flat$). A crisp type $X :: \mathbf{Type}$ is crisply discrete if and only if it is discrete.

We can now define the shape modality as a localization.

Definition 3.4.4. The *shape* or *homotopy type* $\int X$ of a type X is defined to be the localization of X at the type of Dedekind real numbers \mathbb{R} (see Definition 9.6 of [Shu18b]). By construction, a type is \int -modal if and only if it is discrete.

Since \int is given by localization at a small type,⁹ it is accessible in the sense of [RSS17a]. Therefore, by Lemma 2.24 of [RSS17a], it may be extended canonically to any larger universe. For this reason, and because \flat is universe polymorphic, we will elide the size issues in the use of \int and, for example, consider the type of discrete types \mathbf{Type}_{\int} to be \int -separated.

In the upcoming sections, we will need not only the shape modality \int , but the *n-truncated* shape modality \int_n .

Definition 3.4.5. Let \int_n be the modality whose modal types are discrete, n -truncated types. It can be constructed by localizing at the real line \mathbb{R} and the homotopy n -sphere S^n .

It may be tempting to define $\int_n X$ as $\|\int X\|_n$, but it is not currently known whether $\|D\|_n$ of a discrete type D is discrete; the author suspects that it is not true in general. However, for crisp types, this is true.

⁹Assuming propositional resizing, \mathbb{R} is as small as \mathbb{N} ; without propositional resizing, \mathbb{R} has the size of the universe of \mathbb{N} . We will assume propositional resizing here, as is common in homotopy type theory and valid in any ∞ -topos.

Proposition 3.4.6. Let $X :: \mathbf{Type}$ be a crisp type. Then $\int_n X = \|\int X\|_n$.

Proof. Since X is crisp, so is $\int X$. Since $\int X$ is crisp, $\|\int X\|_n$ is crisply an n -type. Then, by Corollary 6.7 of [Shu18b], $\flat \|\int X\|_n = \|\flat \int X\|_n$. But $\int X$ is discrete, so by Axiom $\mathbb{R} \flat$, $\flat \int X = \int X$. Therefore, $\|\int X\|_n$ is a discrete n -type and so the canonical map $\|\int X\|_n \rightarrow \int_n X$ is an equivalence. \square

We can think of $\int_n X$ as the “fundamental n -groupoid” of X . In particular,

- $\int_0 X$ is the set of connected components of X .
- $\int_1 X$ is the fundamental groupoid of X .

We can prove that $\int_0 X$ is the set of connected components of X in a naive sense.

Definition 3.4.7. Let X be a type. A *connected component* of X is a subtype $C : X \rightarrow \mathbf{Prop}$ of X which is

1. Inhabited: there is merely an $x : X$ such that $C(x)$.
2. Connected: If $C \subseteq P \cup \neg P$, then $C \subseteq P$ or $C \subseteq \neg P$.¹⁰
3. Detachable: For any $x : X$, either $C(x)$ or $\neg C(x)$.¹¹

We denote the set of connected components of X by $\pi_0 X$.

Connected components are quite rigid; if two connected components have non-empty intersection, then they are equal.

¹⁰This expresses the connectivity of C because it says that if C is contained in a *disjoint* union, it is contained wholly in one part.

¹¹This says that C is a *component* of X in the sense that X is the disjoint union of C and its complement.

Lemma 3.4.8. Suppose that C and D are connected components of X . Then $C = D$ if and only if $C \cap D$ is non-empty.

Proof. If $C = D$, then $C \cap D$ is C and so is inhabited.

Since D is detachable, we have that $X \subseteq D \cup \neg D$, and therefore $C \subseteq D \cup \neg D$. Now, C is connected, so $C \subseteq D$ or $C \subseteq \neg D$; but it can't be the latter because then their intersection would be empty. So, $C \subseteq D$ and symmetrically $D \subseteq C$. \square

Intuitively, $\int_0 X$ should be the set of connected components of X and $(-)^{\int_0} : X \rightarrow \int_0 X$ should send $x : X$ to the connected component x^{\int_0} it is contained in. We can justify this intuition with the following theorem.

Lemma 3.4.9. Let $u : \int_0 X$, and let $C_u : X \rightarrow \mathbf{Prop}$ be defined by

$$C_u(x) :\equiv u = x^{\int_0}$$

Then C_u is a connected component of X , giving us a map $C : \int_0 X \rightarrow \pi_0 X$.

Proof. We need to prove that C_u is inhabited, connected, and detachable.

1. C_u is inhabited because $(-)^{\int_0}$ is surjective (by the same proof as that of Corollary 9.12 of [Shu18b]).
2. Suppose that $C_u \subseteq P \cup \neg P$. Consider the map $\chi : (x : X) \times C_u(x) \rightarrow \{0, 1\}$ sending x to 0 if $P(x)$ and x to 1 if $\neg P(x)$. As $\{0, 1\}$ is a discrete set (by Theorems 6.19 and 6.21 of [Shu18b], noting that $\{0, 1\} = \{0\} + \{1\}$), χ factors uniquely through $\int_0((x : X) \times C_u(x))$. But $(x : X) \times C_u(x) \equiv \mathbf{fib}_{(-)^{\int_0}}$ is a fiber of a \int_0 -unit, and so is \int_0 -connected. Therefore χ is constant, and so either all x in C_u satisfy P , or they all satisfy $\neg P$.

3. Since $\int_0 X$ is a discrete set, it has decidable equality by Lemma 8.15 of [Shu18b]. So, for any $x : X$, either $u = x^{f_0}$ or not. But that exactly means that $C_u(x)$ or not. \square

Theorem 3.4.10. *Let X be a type. Then the map $C : \int_0 X \rightarrow \pi_0 X$ of Lemma 3.4.9 is an equivalence.*

Proof. We will show that the map C is surjective and injective.

1. To show that C is surjective, suppose that U is a connected component of X , seeking to witness $\|\mathbf{fib}_C(U)\|$. Since we are seeking a proposition and U is inhabited, we may assume that $x : X$ is in U . Then x is in $C_{x^{f_0}} \cap U$, so that $C_{x^{f_0}} = U$ by Lemma 3.4.8.
2. To show that C is injective, suppose that $C_u = C_v$ seeking to show that $u = v$. If $C_u = C_v$, then $C_u \cap C_v = C_u$ is merely inhabited. Since we are seeking a proposition, let x be an element in the intersection. But then $u = x^{f_0}$ and $v = x^{f_0}$, so $u = v$. \square

Remark 3.4.11. Though we have framed this paper as taking place in the setting of Real Cohesion, it will in fact mostly use the “locally contractible” part of the theory — namely, crisp variables, the comodality \flat , the modality \int , and the axiom relating them for crisp types. The only extra condition is that \flat commute with propositional truncation, which, as proven in [Shu18b], uses the codiscrete modality $\#$. It also follows from the fact (Proposition 8.8 of [Shu18b]) that propositions are discrete which only uses that \int is given by localization at a family of pointed types.

In particular, Theorem 3.5.9 relies only on crisp type theory, while Theorem 3.6.1 relies on the adjoint relationship of \mathfrak{f} and \mathfrak{b} (namely, that crisp types are \mathfrak{f} -modal if and only if they are \mathfrak{b} -comodal). Theorems 3.7.7 and 3.8.6 relies only on Theorem 3.6.1, and are therefore also valid in general cohesion. On the other hand, the specific examples in Sections 3.6, 3.7, 3.8 and 3.9 take place in real cohesion.

Therefore, the theory of \mathfrak{f} -fibrations and coverings in the coming sections should work equally well in other settings that have an adjoint $\diamond \dashv \square$ modality/comodality pair implemented using crisp variables in which \square preserves propositional truncation. A likely example of such a situation would be the adjoint pair $\mathfrak{J} \dashv \&$ between the crystalline modality \mathfrak{J} which is given by localizing at a family of infinitesimal types, and the infinitesimal flat modality $\&$ which appears (in the language of ∞ -toposes, rather than type theory) in Schreiber’s [Sch13b]. Since \mathfrak{J} is the localization at a family of pointed types, propositions are crystalline and so $\&$ commutes with propositional truncation. In this setting, Theorem 3.6.1 would be used with Lemma 3.3.12 to show that the projections of certain bundles are \mathfrak{J} -étale (that is, formally étale or locally diffeomorphic).

The modality \mathfrak{J} is left exact, and so every map is an \mathfrak{J} -fibration. However, \mathfrak{J} -étale maps include the *formally étale* maps, or *local diffeomorphisms*. So the applications to covering theory of Section 3.9 can be interpreted in this setting as well.

3.5 Classifying Types of Discrete Structures are Discrete

In this section, we will show that the classifying types of bundles of crisply discrete structures are themselves discrete. As a corollary, the fibers of such a bundle depend only on the homotopy type of the base space. We will use this fact to show that maps whose fibers have a merely constant homotopy type — merely equivalent to some crisply discrete type — are \mathfrak{f} -fibrations.

First, we need a good notion of “type of discrete objects”. We will call these types *locally discrete*.

Definition 3.5.1. A type X is *locally discrete* if it is \mathfrak{f} -separated, that is, for all $x, y : X$, $x = y$ is discrete. A crisp type X is *locally crisply discrete* if for all crisp $x, y :: X$, $x = y$ is crisply discrete; more explicitly, for all $x, y : \flat X$, $x_{\flat} = y_{\flat}$ is crisply discrete.

Remark 3.5.2. We can’t explicitly quantify over crisp elements $x, y :: X$ in Shulman’s crisp type theory, but we can quantify over cohesive elements $x, y : \flat X$. These amount to the same thing, since if x and y are crisp elements of X , then $x_{\flat} = y_{\flat}$ is the same type as $x = y$.

In Agda, which has incorporated the \flat modality since version 2.6, we can quantify over crisp variables.

That we can think of locally discrete types as being types *of* discrete objects is justified by the following lemma.

Lemma 3.5.3. The type $\mathbf{Type}_{\mathfrak{f}}$ of discrete types is locally discrete.

Proof. For any modality, the types of identifications between modal types are equivalent to modal types. In particular, \mathbf{Type}_j is separated relative to the canonical extension of j to any universe containing \mathbf{Type} . \square

In [Chr+18], Christensen, Opie, Rijke, and Scoccola show that if a modality \diamond is given by localization at a type X , then the \diamond -separated types also form a modality whose operator is given by localization at the suspension ΣX (see Lemma 2.15 and Remark 2.16 of [Chr+18]). As a corollary, by Lemma 3.2.2 we get that locally discrete types are closed under dependent sums.

Lemma 3.5.4. If X is locally discrete and $P : X \rightarrow \mathbf{Type}$ is a family of locally discrete types, then $(x : X) \times Px$ is locally discrete.

We can package this result into a useful extension of the idea that a locally discrete type is a type *of* discrete objects. Many structured objects are captured by the notion of a *standard notion of structure*, which appears in the HoTT Book [Uni13] in Section 9.8 as a tool to prove the structure identity principle. A standard notion of structure on a category \mathcal{C} is a pair (P, H) where $P : \mathcal{C}_0 \rightarrow \mathbf{Type}$ assigns to each object of \mathcal{C} its type of (P, H) -structures (and H gives a notion of homomorphism between such structures). For example, a group is a standard notion of structure on the category of sets by letting P take each set to the set of group structures on it. We can read the previous lemma as saying that discretely structured discrete objects are also discrete, in the following way.

Corollary 3.5.5. Let \mathcal{C} be a category whose type of objects \mathcal{C}_0 is locally discrete type, and (P, H) be a standard notion of structure on \mathcal{C} such that for all $x : \mathcal{C}_0$, Px is discrete. Then the type of (P, H) structures is locally discrete.

Proof. The type of structures is just the dependant sum $(x : \mathcal{C}_0) \times Px$, which is locally discrete by the above corollary. \square

There are two ways to say a crisp type $X :: \mathbf{Type}$ is discrete: either $(-)_b : \flat X \rightarrow X$ is an equivalence or $(-)^f : X \rightarrow \flat X$ is an equivalence. Correspondingly, there are two ways to say that a crisp type is locally discrete, which we have given the names of locally discrete and locally crisply discrete. Though a crisp type which is locally discrete will always be locally crisply discrete, these two notions are likely *not* equivalent in general since the latter only quantifies over *crisp* elements of X . We can, however, give another characterization of locally crisply discrete types.

Lemma 3.5.6. A crisp type X is locally crisply discrete if and only if $(-)_b : \flat X \rightarrow X$ is an embedding.

Proof. Recall the left exactness of \flat (Theorem 6.1 of [Shu18b]); we have an equivalence $\flat(x = y) \simeq (x^b = y^b)$ for all crisp $x, y :: X$ making the following diagram commute:

$$\begin{array}{ccc}
 \flat(x = y) & \xrightarrow{\simeq} & x^b = y^b \\
 \searrow^{(-)_b} & & \swarrow^{\text{ap}(-)_b} \\
 & x = y &
 \end{array}$$

Now, X is locally crisply discrete if and only if the downwards map on the left is an equivalence, and $(-)_b$ is an embedding if and only if the downwards map on the right is an equivalence. \square

Let's turn our attention to classifying types. In general, any type X can be seen as “classifying” the maps into it. This rather abstract way of thinking is more useful the more readily the objects of X can be turned into types, since

maps into **Type** correspond to arbitrary bundles of types. For an $x : X$, the following general definition gives a classifying type for “bundles of x s”.

Definition 3.5.7. For a type X and a term $x : X$, we define

$$\mathbf{BAut}_X(x) := (y : X) \times \|x = y\|$$

This notation is inspired by the notation for the classifying space \mathbf{BG} of principal G -bundles for a topological group G . If $G \simeq \mathbf{Aut}_X(x)$ is the group of automorphisms of some object (as, for example, $\mathbf{GL}_n(\mathbb{R}) \simeq \mathbf{Aut}_{\mathbf{Vect}_{\mathbb{R}}}(\mathbb{R}^n)$), then $\mathbf{BAut}_X(x)$ as defined above does classify principal G -bundles. If $\mathbf{Aut}_X(x)$ has a recognizable name G , we will write \mathbf{BG} for $\mathbf{BAut}_X(x)$.

We will now show that if X is crisply locally discrete, and $x :: X$ is a crisp element, then $\mathbf{BAut}_X(x)$ is discrete.

Lemma 3.5.8. For any crisp type X and crisp $x :: X$, we have an equivalence $\flat \mathbf{BAut}_X(x) \simeq \mathbf{BAut}_{\flat X}(x^\flat)$ making the following triangle commute:

$$\begin{array}{ccc} \flat \mathbf{BAut}_X(x) & \xrightarrow{\simeq} & \mathbf{BAut}_{\flat X}(x^\flat) \\ & \searrow^{(-)_\flat} & \swarrow_{(y:p) \mapsto y_\flat, \dots} \\ & & \mathbf{BAut}_X(x) \end{array}$$

Proof. Consider the following equivalence:

$$\begin{aligned}
\flat \mathbf{BAut}_X(x) &:\equiv \flat((y : X) \times \|x = y\|) \\
&\simeq (u : \flat X) \times \text{let } y^\flat := u \text{ in } \flat \|x = y\| \\
&\simeq (u : \flat X) \times \text{let } y^\flat := u \text{ in } \|\flat(x = y)\| \\
&\simeq (u : \flat X) \times \text{let } y^\flat := u \text{ in } \|x^\flat = y^\flat\| \\
&\simeq \mathbf{BAut}_{\flat X}(x^\flat).
\end{aligned}$$

The first equivalence follows from Lemma 6.8, the second from Corollary 6.7, and the third from Theorem 6.1 of [Shu18b]. The final equivalence follows from Lemma 4.4 of [Shu18b], which says that $(\text{let } y^\flat := u \text{ in } f(y^\flat)) = f(u)$.

On $(y, p)^\flat : \flat \mathbf{BAut}_X(x)$, this equivalence yields $(y^\flat, \dots) : \mathbf{BAut}_{\flat X}(x^\flat)$, and so when applying $(-)_\flat$ to either side, we find that the result is the same. \square

Theorem 3.5.9. *Suppose X is locally crisply discrete and $x :: X$. Then $\mathbf{BAut}_X(x)$ is (crisply) discrete.*

Proof. By the above lemma, it suffices to prove that $(y, \cdot) \mapsto (y_\flat, \cdot) : \mathbf{BAut}_{\flat X}(x^\flat) \rightarrow \mathbf{BAut}_X(x)$ is an equivalence. Now, $(-)_\flat : \flat X \rightarrow X$ is an embedding because X is locally crisply discrete, so the map in question is an embedding as well. We just need to show it is surjective.

Suppose $y : \mathbf{BAut}_X(x)$. To prove surjectivity, we need to inhabit $\|\mathbf{fib}(y)\|$. Because we are trying to prove a proposition, we may assume that $p : x = y$; but then $(x^\flat, p) : \mathbf{fib}(y)$. \square

3.6 Examples of \mathfrak{f} -Fibrations

By using Theorem 3.5.9 together with Theorem 3.3.14, we get a nice trick for showing that a map $f : X \rightarrow Y$ is a \mathfrak{f} -fibration. We just need give a crisply discrete type $F :: \mathbf{Type}_{\mathfrak{f}}$ such that $\mathfrak{f} \mathbf{fib}_f(y)$ is merely equivalent to F for all $y : Y$.

Theorem 3.6.1. *Let $f : X \rightarrow Y$. If there is a crisp type $F :: \mathbf{Type}_{\mathfrak{f}}$ such that for all $y : Y$, $\|F = \mathfrak{f} \mathbf{fib}_f(y)\|$, then f is a \mathfrak{f} -fibration. If furthermore we have that $\|F = \mathbf{fib}_f(y)\|$ for all $y : Y$, then f is \mathfrak{f} -étale. If F is an n -type, then f is a \mathfrak{f}_{n+1} -fibration (resp. \mathfrak{f}_{n+1} -étale).*

Proof. By hypothesis, $\mathfrak{f} \mathbf{fib}_f$ factors through $\mathbf{BAut}(F)$. Since F is a crisp element of a locally discrete type, $\mathbf{BAut}(F)$ is discrete by Theorem 3.5.9 and therefore $\mathfrak{f} \mathbf{fib}_f$ factors through $\mathfrak{f}Y$. But then, by Theorem 3.3.14, f is a \mathfrak{f} -fibration. The second claim follows in the same way from Lemma 3.3.12. If F is an n -type, then $\mathbf{BAut}(F)$ is an $(n + 1)$ -type, and so the maps factor further through $\mathfrak{f}_{n+1}X$. \square

With a little effort, we can extend this trick to classify fibrations over disconnected spaces whose fibers over each part are different. A little care must be taken around crispness.

Corollary 3.6.2. Let $X, Y :: \mathbf{Type}$ and $f :: X \rightarrow Y$. Assuming the crisp axiom of choice, f is a \mathfrak{f} -fibration if and only if there is a $F :: \|\mathfrak{f}Y\|_0 \rightarrow \mathbf{Type}$ such that for all $y : Y$, $\|F(|y^{\mathfrak{f}0}|) = \mathfrak{f} \mathbf{fib}_f(y)\|$.

Proof. First, if there is an $F :: \|\mathfrak{f}Y\|_0 \rightarrow \mathbf{Type}$ such that for all $y : Y$, $\|F(|y^{\mathfrak{f}0}|) = \mathfrak{f} \mathbf{fib}_f(y)\|$, then $\mathfrak{f} \mathbf{fib}_f : Y \rightarrow \mathbf{Type}$ factors through $(u : \|\mathfrak{f}Y\|_0) \times$

$\mathbf{BAut}(F(u))$. Since $\|\mathbb{f}Y\|_0$ is crisply discrete (by Proposition 3.4.6) and for all $z : \flat\|\mathbb{f}Y\|_0$ we have that (let $v^b := z$ in $\mathbf{isdiscrete}(\mathbf{BAut}(F(v)))$) by Theorem 3.5.9, we find that $(u : \|\mathbb{f}Y\|_0) \times \mathbf{BAut}(F(u))$ is crisply discrete by Theorem 6.20 of [Shu18b]. Therefore, $\mathbb{f}\mathbf{fib}_f$ factors through $(-)^{\mathbb{f}}$, proving that f is an \mathbb{f} -fibration.

On the other hand, suppose that f is a fibration. Assuming the crisp axiom of choice (Theorem 6.30 of [Shu18b]), there is a crisp section $s :: \|\mathbb{f}Y\|_0 \rightarrow Y$ of $|(-)^{\mathbb{f}}|_0 : Y \rightarrow \|\mathbb{f}Y\|_0$; that is, we may choose an element in every fiber. Define $F(u) := \mathbb{f}\mathbf{fib}_f(su)$. It remains to show that $\|F(|y^{\mathbb{f}}|_0) = \mathbb{f}\mathbf{fib}_f(y)\|$ for all $y : Y$. Since f is a fibration, we have that $\mathbb{f}\mathbf{fib}_f = \mathbf{fib}_{\mathbb{f}f} \circ (-)^{\mathbb{f}}$ and so

$$\|F(|y^{\mathbb{f}}|_0) = \mathbb{f}\mathbf{fib}_f(y)\| \simeq \|\mathbf{fib}_{\mathbb{f}f}((s|y^{\mathbb{f}}|_0)^{\mathbb{f}}) = \mathbf{fib}_{\mathbb{f}f}(y^{\mathbb{f}})\|$$

It will suffice to show that $\|s|y^{\mathbb{f}}|_0^{\mathbb{f}} = y^{\mathbb{f}}\|$. But this is equivalent to $|s|y^{\mathbb{f}}|_0^{\mathbb{f}}|_0 = |y^{\mathbb{f}}|_0$, which holds since s is a section. \square

We can now use Theorem 3.6.1 to give a number of examples of \mathbb{f} -fibrations. In this section, we will be working in real cohesion, assuming that \mathbb{f} is given by localization at the type \mathbb{R} of Dedekind real numbers. We will add two more examples later, in Section 3.7.

3.6.1 The Universal Cover of the Circle

We will now show that the map $(\cos, \sin) : \mathbb{R} \rightarrow \mathbb{S}^1$ is a \mathbb{f} -fibration, where \mathbb{S}^1 is the unit circle in \mathbb{R}^2 . In Section 3.9, we will show that it is indeed the universal cover of the circle \mathbb{S}^1 .

Lemma 3.6.3. The map $(\cos, \sin) : \mathbb{R} \rightarrow \mathbb{S}^1$ is \mathbb{f}_1 -étale, and so in particular is

a \int -fibration.

Proof. Let $r \equiv (\cos, \sin)$. Over $(x, y) : \mathbb{S}^1$, the fiber of r is $r^*(x, y) := \{\theta : \mathbb{R} \mid \cos \theta = x, \sin \theta = y\}$. We will show that $r^*(x, y)$ is merely equivalent to \mathbb{Z} .

For any $\theta : r^*(x, y)$ and $k : \mathbb{Z}$, we have that $\theta + 2\pi k$ is in $r^*(x, y)$. This gives map $\lambda k. \theta + 2\pi k : \mathbb{Z} \rightarrow r^*(x, y)$. Moreover, given any other $\varphi : r^*(x, y)$, the difference $\varphi - \theta$ is an integral multiple of 2π , which gives us a map $\lambda \varphi. \frac{\varphi - \theta}{2\pi} : r^*(x, y) \rightarrow \mathbb{Z}$. These maps are clearly inverse, and since r is merely surjective there is always some θ we may choose to make this equivalence.

We have therefore shown that $r^* : \mathbb{S}^1 \rightarrow \mathbf{Type}$ factors through $\mathbf{BAut}(\mathbb{Z})$.¹² But \mathbb{Z} is a crisply discrete set, so by Theorem 3.6.1, r is a fibration. \square

We can now use the fact that (\cos, \sin) is a fibration to calculate the fundamental group of the circle.

Theorem 3.6.4. *Let \mathbb{S}^1 be the unit circle in \mathbb{R}^2 . Then $\Omega \int \mathbb{S}^1 \simeq \mathbb{Z}$.*

Proof. Since

$$\mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{S}^1$$

is a fiber sequence and (\cos, \sin) is a \int -fibration,

$$\mathbb{Z} \rightarrow * \rightarrow \int \mathbb{S}^1$$

is a fiber sequence, showing that $\Omega \int \mathbb{S}^1 \simeq \mathbb{Z}$. \square

¹²In fact, since the fibers are actually \mathbb{Z} -torsors, r^* factors through $\mathbf{B}\mathbb{Z}$, which would work just as well.

3.6.2 Hopf Fibrations

In the following, let \mathbb{K} be the real numbers \mathbb{R} , the complex numbers \mathbb{C} , or the quaternions \mathbb{H} . We will denote the apartness relation on any of these number systems by $x\#y$; for real numbers this means $|x - y| > 0$, and for the other two number systems this means $\|x - y\| > 0$. If X is a set with an apartness relation and $x : X$, we will denote by $X\#\{x\}$ the set of elements $y : X$ with $x\#y$.

Remark 3.6.5. In the presence of Shulman’s Axiom T of [Shu18b], the notions of apartness and non-equality in \mathbb{R} , \mathbb{C} , and \mathbb{H} coincide (see Theorem 8.32 of that paper). In this case, we could replace all instances of apartness by non-equality. Otherwise, we make no use of Axiom T.

Definition 3.6.6. A *line* in \mathbb{K}^{n+1} is a proposition $\mathcal{L} : \mathbb{K}^{n+1} \rightarrow \mathbf{Prop}$ satisfying:

1. There is (merely) an $x\#0$ element in \mathcal{L} which is apart from 0.
2. For any element x in \mathcal{L} and $c : \mathbb{K}$, the scaled element cx is in \mathcal{L} .
3. For any elements x and y in \mathcal{L} , there is a unique $c : \mathbb{K}$ such that $cx = y$.

For a line \mathcal{L} , we define $\{\mathcal{L}\} := (x : \mathbb{K}^{n+1}) \times \mathcal{L}(x)$ to be its extent. We denote the type of lines in \mathbb{K}^{n+1} by $\mathbb{K}P^n$.

Quite obviously, every line is somehow identifiable with \mathbb{K} .

Lemma 3.6.7. Let $\mathcal{L} : \mathbb{K}P^n$ be a line. Then

$$\|\{\mathcal{L}\} = \mathbb{K}\|.$$

Proof. Since we are proving a proposition and since there exists a element apart from zero on \mathcal{L} , we may assume we have such an element x . Then the map $y \mapsto c$ where c is the unique element of \mathbb{K} such that $cx = y$ determines a map $\{\mathcal{L}\} \rightarrow \mathbb{K}$. Since for any $c \in \mathbb{K}$, cx is on \mathcal{L} , this map is surjective. It is injective by the uniqueness condition (3). \square

For any $x \in \mathbb{K}^{n+1} \setminus \{0\}$, we get the line $\mathbb{K}x$ in the direction of x defined as

$$\mathbb{K}x(y) := \exists c \in \mathbb{K}, cx = y.$$

We have a function $\tilde{h} : \mathbb{K}^{n+1} \setminus \{0\} \rightarrow \mathbb{K}P^n$, sending x to $\mathbb{K}x$. We refer to its restriction $h : \mathbb{S}_{\mathbb{K}^{n+1}} \rightarrow \mathbb{K}P^n$ to the unit sphere of \mathbb{K}^{n+1} as the *generalized Hopf map*.

Suppose that $\mathcal{L} \subset \mathbb{K}P^n$ is a line and consider the fiber $\text{fib}_{\tilde{h}}(\mathcal{L})$. By definition, this is the type of all elements $x \in \mathbb{K}^{n+1} \setminus \{0\}$ such that $\mathbb{K}x = \mathcal{L}$.

Lemma 3.6.8. For any line $\mathcal{L} \subset \mathbb{K}P^n$,

$$\text{fib}_{\tilde{h}}(\mathcal{L}) = \{\mathcal{L}\} \setminus \{0\}$$

And, as a corollary,

$$\text{fib}_h(\mathcal{L}) = (x \in \{\mathcal{L}\}) \times (\|x\| = 1)$$

consists of the elements on the line \mathcal{L} of unit length.

Proof. Suppose that x is in \mathcal{L} . By property 2, cx is in \mathcal{L} for any $c \in \mathbb{K}$, and by property 3, every element of \mathcal{L} may be so expressed in a unique way. Therefore, $\mathbb{K}x = \mathcal{L}$.

On the other hand, if $\mathbb{K}x = \mathcal{L}$, then in particular $1 \cdot x = x$ is in \mathcal{L} . \square

Putting together these two lemmas, we conclude that for all $\mathcal{L} : \mathbb{K} P^n$, the fiber of h over \mathcal{L} is merely equivalent to the unit sphere of \mathbb{K} :

$$\|\text{fib}_h(\mathcal{L}) = \mathbb{S}_{\mathbb{K}}\|.$$

In particular, their homotopy types are merely equivalent, and so by Theorem 3.6.1,

$$\mathbb{S}_{\mathbb{K}} \rightarrow \mathbb{S}_{\mathbb{K}^{n+1}} \rightarrow \mathbb{K} P^n$$

is a \mathfrak{f} -fibration.

Substituting \mathbb{R} , \mathbb{C} , and \mathbb{H} back in for \mathbb{K} , we see that:

Theorem 3.6.9.

- $\mathbb{S}^0 \rightarrow \mathbb{S}^n \rightarrow \mathbb{R} P^n$ is a \mathfrak{f} -fibration.¹³
- $\mathbb{S}^1 \rightarrow \mathbb{S}^{2n+1} \rightarrow \mathbb{C} P^n$ is a \mathfrak{f} -fibration. This includes the original Hopf fibration $\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{C} P^1$.
- $\mathbb{S}^3 \rightarrow \mathbb{S}^{4n+3} \rightarrow \mathbb{H} P^n$ is a \mathfrak{f} -fibration. This includes the quaternionic Hopf fibration $\mathbb{S}^3 \rightarrow \mathbb{S}^7 \rightarrow \mathbb{H} P^1$.

3.6.3 A \mathfrak{f} -Fibration which is not a Hurewicz Fibration

In this example we will prove that the projection of the x and y -axes onto the x -axis is a \mathfrak{f} -fibration. This is a classic example of a quasi-fibration which is not a Hurewicz fibration, since the x -axis cannot be lifted to a path going through a point $y \neq 0$ in the fiber over $x = 0$.

First, we need a useful and straightforward lemma.

¹³We will see in the next section that it is a covering map.

Lemma 3.6.10. Let X be a type with a point $x_0 : X$ and suppose that for every $x : X$, we have a path $\gamma_x : \mathbb{R} \rightarrow X$ with $\gamma_x(0) = x$ and $\gamma_x(1) = x_0$. Then $\mathfrak{f}X$ is contractible.

Proof. Define the map $\tilde{\gamma} : \mathbb{R} \rightarrow (X \rightarrow X)$ by $\tilde{\gamma}(t)(x) = \gamma_x(t)$ and note that $\tilde{\gamma}(0) = \text{id}_X$ and $\tilde{\gamma}(1) = \text{const}_{x_0}$, the constant map at x_0 . This gives us an identification $\text{id}_x^{\mathfrak{f}} = \text{const}_{x_0}^{\mathfrak{f}}$ in $\mathfrak{f}(X \rightarrow X)$. It remains to show that such an identification implies that $\mathfrak{f}X$ is contractible.

The functorial action of \mathfrak{f} gives a map $(X \rightarrow X) \rightarrow (\mathfrak{f}X \rightarrow \mathfrak{f}X)$, and since the latter is \mathfrak{f} -modal this factors uniquely through $\mathfrak{f}(X \rightarrow X)$. By construction, the map $\mathfrak{f}(X \rightarrow X) \rightarrow (\mathfrak{f}X \rightarrow \mathfrak{f}X)$ sends $\text{id}_X^{\mathfrak{f}}$ to $\mathfrak{f}\text{id}_X$, which equals $\text{id}_{\mathfrak{f}X}$ by functoriality. Furthermore, $\text{const}_{x_0}^{\mathfrak{f}}$ gets sent to $\mathfrak{f}(\text{const}_{x_0}) = \mathfrak{f}(x_0 \circ !)$ where $! : X \rightarrow *$ is the terminal morphism. By functoriality, this equals the composite $\mathfrak{f}X \xrightarrow{\mathfrak{f}!} \mathfrak{f}* \xrightarrow{\mathfrak{f}x_0} \mathfrak{f}X$, which is the constant map at $x_0^{\mathfrak{f}}$. Therefore, the identity of $\mathfrak{f}X$ factors through a constant map, and so $\mathfrak{f}X$ is contractible. \square

Remark 3.6.11. We can think of the function $\gamma_{(-)}(-) : X \rightarrow (\mathbb{R} \rightarrow X)$ of Lemma 3.6.10 as a weak form of multiplicative action of \mathbb{R} on X . If we write $t \cdot x \equiv \gamma_x(t)$, then the assumptions $\gamma_x(0) = x_0$ and $\gamma_x(1) = x$ read as $0 \cdot x = x_0$ and $1 \cdot x = x$. Seen this way, Lemma 3.6.10 shows us that any type with such a multiplicative action of \mathbb{R} — say, a vector space — is \mathfrak{f} -connected.

As a corollary, we find that the projection

$$\{(x, y) : \mathbb{R}^2 \mid xy = 0\} \rightarrow \{x : \mathbb{R}\}$$

is \mathfrak{f} -connected (and is therefore in particular a \mathfrak{f} -fibration). The fiber of this projection over $x : \mathbb{R}$ is $\{y : Y \mid xy = 0\}$, and for every y in the fiber we have

the path $t \mapsto ty$ from 0 to y .

Remark 3.6.12. We shouldn't expect all quasi-fibrations to be \mathfrak{f} -fibrations. The closest analogue of a quasi-fibration in real hoheision would be a map $f : X \rightarrow Y$ such that for every *crisp* $y :: Y$, $\gamma : \mathfrak{f} \text{fib}_f(y) \rightarrow \text{fib}_{\mathfrak{f}f}(y^{\mathfrak{f}})$ is an equivalence. This is strictly weaker than our definition of \mathfrak{f} -fibration; it amounts to the claim that the pullback of f along $(-)_b : \flat Y \rightarrow Y$ is a \mathfrak{f} -fibration.

3.7 Homotopy Quotients are \mathfrak{f} -Fibrations.

In this section, we show that the quotient map $X \rightarrow X // G$ from a type X to the homotopy quotient $X // G$ of X by an action of the ∞ -group G is a fibration whenever G is crisp. If the action is crisp and transitive, then for any crisp point $x :: X$, the map $G \rightarrow X$ given by acting on x is a fibration as well. We will then give two more examples of \mathfrak{f} -fibrations.

Before we prove these things, we should review the definition of ∞ -group and ∞ -group action. These notions can be found in [BDR18c], which develops the basic theory of ∞ -groups and proves a stabilization theorem about them.

Definition 3.7.1. An ∞ -group is a type G identified with the loop space ΩBG of a pointed, 0-connected type BG (called the *delooping* of G). Since singleton types are contractible, the type of ∞ -groups is equivalent to the type of pointed, 0-connected types.

$$\begin{aligned} \infty\text{-Grp} &: \equiv (G : \mathbf{Type}) \times (BG : \mathbf{Type}_*^{>0}) \times (G = \Omega BG) \\ &\simeq \mathbf{Type}_*^{>0}. \end{aligned}$$

For this reason, we will often identify G with $\Omega\mathbf{B}G$.

We may think of the elements of $\mathbf{B}G$ as G -torsors, and the point $\mathbf{pt}_{\mathbf{B}G} : \mathbf{B}G$ as G acting on itself. Indeed, for any group G in the axiomatic sense (a set equipped with operations satisfying laws), we may construct its delooping $\mathbf{B}G$ as the type of G -torsors, pointed at G .

Definition 3.7.2. An *action* of the ∞ -group G on types is a map $X^{(-)} : \mathbf{B}G \rightarrow \mathbf{Type}$. We write $X := X^{\mathbf{pt}_{\mathbf{B}G}}$ for the image of the point $\mathbf{pt}_{\mathbf{B}G} : \mathbf{B}G$.

Given an element $g : G$, we get an automorphism of X by applying $X^{(-)}$ to g . That is, given $x : X$, define

$$gx := \mathbf{ap}(X^{(-)}, g) \text{ at } x.^{14}$$

We can think of an action $X^{(-)} : \mathbf{B}G \rightarrow \mathbf{Type}$ as an action of G on $X := X^{\mathbf{pt}_{\mathbf{B}G}}$, and we can think of the image X^t of $t : \mathbf{B}G$ as the action of G on X twisted by the torsor t .

Definition 3.7.3. Given an action $X^{(-)} : \mathbf{B}G \rightarrow \mathbf{Type}$, and $x, y : X$, define

$$x \xrightarrow{G} y := (g : G) \times (gx = y)$$

$$\mathbf{Orbit}(x) := (y : X) \times (x \xrightarrow{G} y)$$

$$\mathbf{Stab}(x) := x \xrightarrow{G} x$$

We say that the action is *free* if for all $x, y : X$, $x \xrightarrow{G} y$ is a proposition and *transitive* if $\left\| x \xrightarrow{G} y \right\|$.

¹⁴where $\mathbf{at} : (f = g) \rightarrow (x : X) \rightarrow fx = gx$ is the function that applies an equality of functions at a point.

With this terminology in hand, we can easily define the homotopy quotient of a type by the action of an ∞ -group.

Definition 3.7.4. If $X^{(-)} : \mathbf{BG} \rightarrow \mathbf{Type}$ is an action of the ∞ -group G , then

$$X // G := (t : \mathbf{BG}) \times X^t$$

is the *homotopy quotient* of X by G . The quotient map $[-] : X \rightarrow X // G$ is defined by

$$[x] := (\mathbf{pt}_{\mathbf{BG}}, x).$$

This definition is justified by the computation of identity types in dependent pair types.

Lemma 3.7.5. Let $X^{(-)} : \mathbf{BG} \rightarrow \mathbf{Type}$ be an action of the ∞ -group G and $x, y : X$. Then

$$([x] = [y]) \simeq (x \xrightarrow{G} y)$$

Proof. This follows immediately from Theorem 2.7.2 of [Uni13] after expanding the definition of each side. \square

Following through the definitions, we get the following long fiber sequence associated to any ∞ -group action.

Proposition 3.7.6. For any ∞ -group G , action $X^{(-)} : \mathbf{BG} \rightarrow \mathbf{Type}$, and point $x : X^{\mathbf{pt}}$, there is a long fiber sequence ending

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathbf{Stab}(x) & \longrightarrow & \mathbf{Orbit}(x) & & \\ & & & & \text{fst} & \searrow & \\ & & & & & \curvearrowright & \\ & & & & & \swarrow & \\ & & & & & & \\ X^{\mathbf{pt}} & \longrightarrow & X // G & \longrightarrow & \mathbf{BG} & & \end{array}$$

In particular, for all $x : X$, $\mathbf{Orbit}(x) \simeq G$.

Now we can prove our main theorem for this section.

Theorem 3.7.7. *Let G be a crisp ∞ -group, and $X^{(-)} : \mathbf{BG} \rightarrow \mathbf{Type}$ an action of G . Then the quotient map $[-] : X \rightarrow X // G$ is a \mathfrak{f} -fibration.*

If furthermore $X^{(-)}$ is crisp, then the classifying map $\mathbf{fst} : X // G \rightarrow \mathbf{BG}$ is a \mathfrak{f} -fibration, and if the action is transitive and $x :: X$, then the map $g \mapsto gx : G \rightarrow X$ is a \mathfrak{f} -fibration.

Proof. Each fact follows quickly from Proposition 3.7.6 and Theorem 3.6.1.

Since \mathbf{BG} is 0-connected, the map $x \mapsto [x] :\equiv (\mathbf{pt}_{\mathbf{BG}}, x)$ is surjective. Since by Proposition 3.7.6 the fiber $\mathbf{fib}_{[-]}([x]) \simeq G$ for all $x : X$; in particular for all $(t, y) : X // G$ we have a term of $\|\mathbf{fib}_{[-]}((t, y)) = G\|$. Since G is crisp, we may take the homotopy type of each side to discover (by Theorem 3.6.1) that $[-] : X \rightarrow X // G$ is a \mathfrak{f} -fibration.

If $X^{(-)}$ is crisp, then so is $X :\equiv X^{\mathbf{pt}_{\mathbf{BG}}}$ (since the ∞ -group G , and hence its delooping \mathbf{BG} and its basepoint $\mathbf{pt}_{\mathbf{BG}}$ are assumed crisp). Since \mathbf{BG} is 0-connected, all the fibers of $\mathbf{fst} : X // G \rightarrow \mathbf{BG}$ are merely equivalent to X , and therefore their homotopy types are merely equivalent to its homotopy type. So, by Theorem 3.6.1, the classifying map $\mathbf{fst} : X // G \rightarrow \mathbf{BG}$ is a \mathfrak{f} -fibration.

Suppose that $x :: X$. If the action is transitive, then for any $y : X$, $\|\mathbf{Stab}(y) = \mathbf{Stab}(x)\|$. Since x is crisp, so is $\mathbf{Stab}(x)$, so by Theorem 3.6.1 this proves that the map $g \mapsto gx : G \rightarrow X$ (whose fiber over $y : X$ is $\mathbf{Stab}(y)$ by Proposition 3.7.6) is a \mathfrak{f} -fibration. \square

We can use Theorem 3.7.7 to give two more examples of \mathfrak{f} -fibrations.

3.7.1 $\mathbf{SO}(n) \rightarrow \mathbf{SO}(n+1) \rightarrow \mathbb{S}^n$

We will first construct a delooping $\mathbf{BSO}(n)$ of the special orthogonal group, and then define the action of $\mathbf{SO}(n+1)$ on the n -sphere as a map $\mathbf{BSO}(n+1) \rightarrow \mathbf{Type}$ (with $n \geq 1$). We will prove that the fiber of the map $\mathbf{SO}(n+1) \rightarrow \mathbb{S}^n$ given by acting on the base point has fiber $\mathbf{SO}(n)$. Finally, by Theorem 3.7.7, we will conclude that the map $\mathbf{SO}(n+1) \rightarrow \mathbb{S}^n$ is a \mathfrak{f} -fibration.

Definition 3.7.8. An *orientation* on a normed real n -dimensional vector space V is a unit length element of its exterior power $\Lambda^n V$, equipped with the norm

$$\langle v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_n \rangle := \det[\langle v_i, w_j \rangle_V]$$

We define $\mathbf{BSO}(n)$ to be the type of normed real n -dimensional vector spaces V equipped with an orientation that are merely isomorphic to \mathbb{R}^n with its standard norm and orientation. We point $\mathbf{BSO}(n)$ at \mathbb{R}^n with its standard norm and orientation.

We need to justify this definition of $\mathbf{BSO}(n)$.

Lemma 3.7.9. $\Omega\mathbf{BSO}(n) = \mathbf{SO}(n)$.

Proof. A linear automorphism of \mathbb{R}^n which preserves the norm is given by an orthogonal matrix. If this furthermore preserves the standard orientation on \mathbb{R}^n , that means its n^{th} -exterior power is the identity; but this is given by multiplying by its determinant, so its determinant must be 1. \square

We can now define the action of $\mathbf{SO}(n+1)$ on the n -sphere \mathbb{S}^n .

Definition 3.7.10. For $(V, \langle -, - \rangle)$ a normed vector space, let $\mathbb{S}_V \equiv \{v : V \mid \|v\| = 1\}$ be its unit sphere. Note that $\mathbb{S}_{\mathbb{R}^n} \equiv \mathbb{S}^{n-1}$ by definition.

The map $(V, \langle -, - \rangle, \omega) \mapsto \mathbb{S}_V : \mathbf{BSO}(n+1) \rightarrow \mathbf{Type}$ induces the action of $\mathbf{SO}(n+1)$ on \mathbb{S}^n .

Lemma 3.7.11. The action of $\mathbf{SO}(n+1)$ on \mathbb{S}^n is transitive, and the stabilizer of the basepoint $1 : \mathbb{S}^n$ may be identified with $\mathbf{SO}(n)$.

Proof. For $v : \mathbb{S}^n$, consider v as a unit vector in \mathbb{R}^{n+1} . Then v may be merely extended to a orthonormal basis of \mathbb{R}^{n+1} by the Gram-Schmidt process. The resulting matrix will have determinant either 1 or -1 , but since $\{-1, 1\}$ has decidable equality, we can choose to swap two of these basis vectors to get a special orthogonal matrix that sends $(1, 0, \dots, 0) : \mathbb{S}^n$ to v .

The stabilizer of the basepoint $1 : \mathbb{S}^n$ may be identified with the special orthogonal matrices whose first column has its first entry 1 and all other entries 0. Since the matrix is orthogonal, there can be nothing but 0s in the first row as well. Therefore, the bottom minor given by removing the first row and first column is also special orthogonal, and this gives an identification of the stabilizer with $\mathbf{SO}(n)$. \square

Finally, by Theorem 3.7.7, we may conclude that

$$\mathbf{SO}(n) \rightarrow \mathbf{SO}(n+1) \rightarrow \mathbb{S}^n$$

is a \mathfrak{f} -fibration.

3.7.2 A \mathfrak{f} -fibration over a 1-type

So far we have only seen \mathfrak{f} -fibrations over sets. But with Cohesive HoTT, we can work directly with topological stacks as well. In this example, we will see an example of a \mathfrak{f} -fibration over a 1-type — a stacky version of the real numbers.

Often, a map will fail to be a fibration at a few points because it is ramified there. For example, the map $\mathbb{R} \vee \mathbb{R} \rightarrow \mathbb{R}$ induced by the identity maps

$$\begin{array}{ccc}
 * & \xrightarrow{0} & \mathbb{R} \\
 0 \downarrow & & \downarrow \\
 \mathbb{R} & \longrightarrow & \mathbb{R} \vee \mathbb{R} \\
 & \searrow \text{id} & \downarrow \text{id} \\
 & & \mathbb{R}
 \end{array}$$

is almost a \mathfrak{f} -fibration (indeed, almost a covering), but it is ramified over 0. However, when such a “ramified fibration” appears as the quotient of a group action, it can be rectified into a \mathfrak{f} -fibration by replacing the base by the homotopy quotient.

In the above example, note that we can also see this map as the quotient

$$\mathbb{R} \vee \mathbb{R} \rightarrow \mathbb{R} \vee \mathbb{R} / C_2$$

of the action of the cyclic group C_2 of order 2 on $\mathbb{R} \vee \mathbb{R}$ given by permuting the factors. The homotopy quotient $\mathbb{R} \vee \mathbb{R} // C_2$ will be a stacky version of the reals where 0 has automorphism group C_2 . Now the fiber over 0 consists of both a point over 0 (of which there is just one), together with an identification of its image with 0, of which there are now two. So the fibers have become locally constant; they are in fact merely equivalent to the group C_2 .

This can be made formal by appealing to the upcoming Theorem 3.7.7. We will construct the example above.

Definition 3.7.12. Let \mathbf{BC}_2 be the type of 2-element sets pointed at $\{0, 1\}$, noting that $C_2 = \Omega \mathbf{BC}_2$.

For $T : \mathbf{BC}_2$, let X^T be the cofiber of $(\text{id}, 0) : T \rightarrow T \times \mathbb{R}$. Note that

$X \equiv X^{\text{pt}_{\mathbf{B}C_2}}$ may be identified with $\mathbb{R} \vee \mathbb{R}$. This gives the action of C_2 on $\mathbb{R} \vee \mathbb{R}$ by permuting the factors.

Theorem 3.7.7 then tells us that

$$C_2 \rightarrow \mathbb{R} \vee \mathbb{R} \rightarrow \mathbb{R} \vee \mathbb{R} // C_2$$

is a \mathfrak{f} -fibration. Explicitly $\mathbb{R} \vee \mathbb{R} // C_2$ is the type of pairs $(T : \mathbf{B}C_2) \times X^T$ of 2-element sets T and elements of the cofiber of the inclusion $(\text{id}, 0) : T \rightarrow T \times \mathbb{R}$.

A map can be a “ramified fibration” even if each fiber¹⁵ is the same. An example of this is the Mobius band given by rotating $[-1, 1]$ around a circle with a half turn mapping down onto $[-1, 1]/\text{sgn}$ sending each longitudinal circle to the set of points it intersects in a fixed copy of $[-1, 1]$ in the Mobius band.

Each fiber of this map is a circle, but as one travels from $[1]$ to $[0]$ in $[-1, 1]/\text{sgn}$, the fibers double over. So while each fiber is the same, they do not have a well defined transport along paths as a \mathfrak{f} -fibration would. The trick here is the word “each”; it is true that every fiber is a circle over each *crisp* point of $[-1, 1]/\text{sgn}$, but not over a generic point as Theorem 3.6.1 requires.

This ramification can be fixed by considering the map to $[-1, 1] // \text{sgn}$, a stacky version of $[0, 1]$ in which 0 has an automorphism group C_2 .

3.8 The Shape of a Crisp n -Connected Type is n -Connected

One might expect that if X is $\|-\|_n$ -connected, then its homotopy type $\mathfrak{f}X$ would also be $\|-\|_n$ -connected. While we do not know whether this is true in

¹⁵That is, over each *crisp* point.

general, we can prove it for crisp types $X :: \mathbf{Type}$. To do this, we need to recall a bit of the theory of *separated* types for a modality from [Chr+18].

Definition 3.8.1. A type X is \diamond -separated if for all $x, y : X$, the type of identifications $x = y$ is \diamond -modal. By Theorem 2.26 of [Chr+18], the \diamond -separated types form a modality \diamond' , and we may inductively define

$$\diamond^{(0)} := \diamond$$

$$\diamond^{(n+1)} := \diamond^{(n)'}$$

We now need to import a few lemmas from [Chr+18].

Lemma 3.8.2. Any \diamond -modal type is $\diamond^{(n)}$ -modal, and the canonical factorization $\diamond^{(n)}X \rightarrow \diamond X$ of the \diamond -unit through the $\diamond^{(n)}$ -unit is a \diamond -unit.

Proof. By hypothesis, the identification types in $\diamond X$ are \diamond -modal, so that $\diamond X$ is \diamond' -modal, and so on. This proves the first statement.

The second statement now follows by Lemma 3.3.20. □

Lemma 3.8.3. For any modality \diamond and any pointed type X , there is an equivalence

$$\Omega^n \diamond^{(n)} X \simeq \diamond \Omega^n X$$

Proof. This follows immediately from Proposition 2.27 of [Chr+18] by induction. □

Lemma 3.8.4. Suppose that \diamond is given by localization at a map $A \rightarrow *$. Then $\diamond^{(n)}$ is given by localization at $\Sigma^n A \rightarrow *$.

Proof. This follows immediately from Lemma 2.15 of [Chr+18] by induction. \square

As a corollary, we find that the n -fold locally discrete modalities $\mathfrak{f}^{(n)}$ are given by localization at $\Sigma^n \mathbb{R} \rightarrow *$. Since \mathbb{R} is inhabited, as a corollary we find that $\mathfrak{f}^{(n)}$ preserves n -connected types.

Lemma 3.8.5. Suppose that $-1 \leq k \leq n$. If X is k -connected, then $\mathfrak{f}^{(n)}X$ is k -connected.

Proof. This follows immediately from Corollary 3.13 of [Chr+18] by induction. In particular, since \mathbb{R} is (-1) -connected, by Theorem 8.2.1 of [Uni13] $\Sigma^n \mathbb{R}$ is $(n-1)$ -connected and so $(k-1)$ -connected. Corollary 3.13 of [Chr+18] then applies to the map $\Sigma^n \mathbb{R} \rightarrow *$. \square

We are now ready to prove that \mathfrak{f} preserves n -connected crisp types.

Theorem 3.8.6. Let $X :: \mathbf{Type}$ be a crisp, n -connected type for $n \geq -1$. Then the canonical map $\mathfrak{f}^{(n+1)}X \rightarrow \mathfrak{f}X$ induced by factoring the \mathfrak{f} -unit through the $\mathfrak{f}^{(n+1)}$ -unit is an equivalence, and so in particular $\mathfrak{f}X$ is n -connected.

Proof. For $n \equiv -1$, the statement follows tautologically. It remains to show that assuming the statement for n implies $n+1$. We note here that since \mathbb{N} is crisply discrete, we may assume all natural numbers are crisp.

First, we argue that we may assume that X is crisply pointed. Since X is $(n+1)$ -connected and $n \geq -1$, in particular $\|X\|$ is contractible and so also $\flat\|X\|$ is contractible. By Corollary 6.7 of [Shu18b], $\flat\|X\| \simeq \|\flat X\|$ so that $\|\flat X\|$ is also contractible. Since we are trying to prove that a map is an equivalence,

which is a proposition, we may assume that we have a $u : \flat X$, and therefore assume that we have $u \equiv x^\flat$ for a crisp $x :: X$.

Now, assume that $x :: X$ is a crisp point of X and that X is $(n + 1)$ -connected. Then ΩX is a crisp, n -connected type and therefore $\int^{(n+1)} \Omega X \rightarrow \int \Omega X$ is an equivalence by hypothesis; in particular $\int^{(n+1)} \Omega X$ is discrete. Therefore, $\int^{(n+1)} \Omega X \simeq \Omega \int^{(n+2)} X$ is discrete. By Lemma 3.8.5, $\int^{(n+2)} X$ is $(n + 1)$ -connected and therefore in particular 0-connected; therefore, it is locally crisply discrete. Since it is pointed and 0-connected, it is also equivalent to $\mathbf{BAut}_{\int^{(n+2)} X}(x^{\int^{(n+1)}})$ and so by Theorem 3.5.9, it is discrete. But then the canonical map $\int^{(n+2)} X \rightarrow \int X$ is an equivalence by Lemma 3.8.2. \square

Using Theorem 3.8.6, we can show that the homotopy type of a higher group is a higher group.

Definition 3.8.7. A k -commutative ∞ -group is a type G identified with $\Omega^{k+1} \mathbf{B}^{k+1} G$ for a pointed, k -connected type $\mathbf{B}^{k+1} G$.¹⁶ A homomorphism of k -commutative ∞ -groups is a pointed map $\mathbf{B}^{k+1} G \rightarrow \mathbf{B}^{k+1} H$.

Lemma 3.8.8. The equivalence $\diamond \Omega^{(n)} = \Omega^{(n)} \diamond^{(n)}$ of Lemma 3.8.3 is natural. Let $f : X \rightarrow Y$ be a pointed map between pointed types. Then the following square commutes:

$$\begin{array}{ccc} \diamond \Omega^n X & \xrightarrow{\diamond \Omega^n f} & \diamond \Omega^n Y \\ \sim \downarrow & & \downarrow \sim \\ \Omega^n \diamond^{(n)} X & \xrightarrow{\Omega \diamond^{(n)} f} & \Omega^n \diamond^{(n)} Y \end{array}$$

Proof. Since $\Omega^n \diamond^{(n)} Y$ is modal, we may check that this commutes on $p : \Omega^n X$.

¹⁶In [BDR18c], k -commutative ∞ -groups are called $(k + 1)$ -tuply groupal, but I couldn't bear to subject the reader to such terminology.

When restricted to $\Omega^n X$, the square becomes Ω^n applied to the $\diamond^{(n)}$ -naturality square, which commutes. \square

Theorem 3.8.9. *Suppose that G is a crisp, k -commutative ∞ -group with $(k+1)$ -fold delooping $\mathbf{B}^{k+1}G$. Then $\int G$ is a k -commutative ∞ -group with delooping $\int \mathbf{B}^{k+1}G$ and the unit $(-)^{\int} : G \rightarrow \int G$ is a homomorphism.*

Proof. By Theorem 3.8.6, $\int \mathbf{B}^{k+1}G$ is k -connected and may be pointed at $\mathbf{pt}_{\mathbf{B}^{k+1}G}^{\int}$. By the same theorem,

$$\begin{aligned} \Omega^{k+1} \int \mathbf{B}^{k+1}G &\simeq \Omega^{k+1} \int^{(k+1)} \mathbf{B}^{k+1}G \\ &\simeq \int \Omega^{k+1} \mathbf{B}^{k+1}G \\ &\simeq \int G. \end{aligned}$$

By Lemma 3.8.8 and the fact that the composite $\mathbf{B}^{k+1}G \rightarrow \int^{(k+1)} \mathbf{B}^{k+1}G \xrightarrow{\simeq} \int \mathbf{B}^{k+1}G$ is equal to the unit $\mathbf{B}^{k+1}G \rightarrow \int \mathbf{B}^{k+1}G$, this unit deloops the unit $G \rightarrow \int G$, showing that the latter is a k -commutative homomorphism. \square

As a corollary, we can understand the homotopy type of some classifying types.

- Let $\mathbf{BGL}_1(\mathbb{R})$ be the type of 1-dimensional real vector spaces. Since $\int \mathbf{GL}_1(\mathbb{R}) = \{-1, 1\}$ may be identified with the group of signs, we get find that $\int \mathbf{BGL}_1(\mathbb{R}) = \mathbf{B}\mathbb{Z}/2$. We can call the \int -unit $w_1 : \mathbf{BGL}_1(\mathbb{R}) \rightarrow \mathbf{B}\mathbb{Z}/2$ the *first Stiefel-Whitney class*, since pushing forward by it sends a real line bundle to a first degree cocycle in $\mathbb{Z}/2$ cohomology. Since this is a \int -unit, we see that the first Stiefel-Whitney class is the universal discrete cohomological invariant of a real line bundle.

- Let $\mathbf{BU}(1)$ be the type of 1-dimensional normed complex vector spaces. Since $\int \mathbf{U}(1) = \mathbf{B}\mathbb{Z}$ is a pointed, connected type whose loop space is \mathbb{Z} , we find that $\int \mathbf{BU}(1) = \mathbf{B}^2\mathbb{Z}$. We can call the \int -unit $c_1 : \mathbf{BU}(1) \rightarrow \mathbf{B}^2\mathbb{Z}$ the *first Chern class*, since pushing forward by it sends a Hermitian line bundle to a second degree cocycle in integral cohomology. Since this is a \int -unit, we see that the first Chern class is the universal discrete cohomological invariant of a complex line bundle.

We can now show, with a quick modal argument, that the first Chern class of the Hopf fibration generates $H^2(\mathbb{S}^2; \mathbb{Z})$.

Proposition 3.8.10. The first Chern class $c_1(h)$ of the Hopf fibration $h : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ generates $H^2(\mathbb{S}^2; \mathbb{Z})$.

Proof. For the purpose of this proof, we make an identification of \mathbb{S}^2 with $\mathbb{C}P^1$ and so take the points of \mathbb{S}^2 to be complex lines in \mathbb{C}^2 . We will show that the \int_2 -unit $\mathbb{S}^2 \rightarrow \int_2 \mathbb{S}^2$ generates $H^2(\mathbb{S}^2; \mathbb{Z})$, and then that $c_1(h)$ factors uniquely through this unit.

Consider the long exact sequence of homotopy groups associated to the Hopf fibration. Since we have calculated (in Lemma 3.6.3) that $\Omega \int \mathbb{S}^1 \simeq \mathbb{Z}$, we see that $\pi_2(\int \mathbb{S}^2) \simeq \pi_1(\int \mathbb{S}^1) = \mathbb{Z}$. Therefore, $\int_2 \mathbb{S}^2$ is a $\mathbf{B}^2\mathbb{Z}$, and the \int_2 -unit $(-)^{\int_2} : \mathbb{S}^2 \rightarrow \int_2 \mathbb{S}^2$ induces the identity on π_2 and so generates $H^2(\mathbb{S}^2; \mathbb{Z})$.

It remains to show that $c_1(h) : \mathbb{S}^2 \rightarrow \mathbf{B}^2\mathbb{Z}$ is an \int_2 -unit. Let $\chi : \mathbb{S}^2 \rightarrow \mathbf{BU}(1)$ send a line $\mathcal{L} : \mathbb{S}^2$ in \mathbb{C}^2 to $\{\mathcal{L}\}$, the normed 1-dimensional complex vector space that it is as a subspace of \mathbb{C}^2 . This classifies the Hopf fibration by Lemma 3.6.8 and because a unitary isomorphism with \mathbb{C} is determined by an element of unit

norm:

$$\mathrm{fib}_\chi(\mathbb{C}) \equiv (\mathcal{L} : \mathbb{S}^2) \times (\{\mathcal{L}\} = \mathbb{C}) \simeq (\mathcal{L} : \mathbb{S}^2) \times (\ell : \{\mathcal{L}\}) \times (\|\ell\| = 1) \simeq (\mathcal{L} : \mathbb{S}^2) \times \mathrm{fib}_h(\mathcal{L})$$

In other words, $c_1(h) \equiv c_1 \circ \chi$. Now, the fibers of χ are merely equivalent to \mathbb{S}^3 , and $\int_2 \mathbb{S}^3 = *$, so it is \int_2 -connected. But c_1 is an \int_2 -unit and so also \int_2 -connected. Therefore, $c_1 \circ \chi$ is a \int_2 -connected map into a \int_2 -modal type; by Lemma 1.38 of [RSS17a], it is therefore a \int_2 -unit. □

3.9 A Bit of Covering Space Theory

In this section, we'll see a bit of modal covering theory and get a sense of how working with coverings using modalities feels. In his *Cohesive Covering Theory* extended abstract [Wel18a], Wellen defines a *modal covering* map $\pi : E \rightarrow B$ for a modality \diamond to be a \diamond -étale map. He then specializes to the modality \int_1 to recover the usual covering theory. Here, in light of further conversation with Wellen, we will make a slightly less general definition of covering map which relates more closely to the traditional theory.

Definition 3.9.1. A map $\pi : E \rightarrow B$ is a *cover* if it is \int_1 -étale and its fibers are sets.

Recall from Section 3.2 that \diamond -equivalences lift uniquely against \diamond -étale maps. In particular, in any square

$$\begin{array}{ccc} * & \longrightarrow & E \\ 0 \downarrow & & \downarrow \pi \\ \mathbb{R} & \longrightarrow & B \end{array}$$

there is a unique filler since \mathbb{R} is \int_1 -connected. Therefore, covers satisfy the *unique* path lifting property.

We can quickly prove the classical theorem that coverings of a space X correspond to actions of the fundamental groupoid of X on discrete sets.

Theorem 3.9.2. *Let X be a type and let $\text{Cov}(X)$ denote the type of covers of X . Then*

$$\text{Cov}(X) \simeq (\int_1 X \rightarrow \mathbf{Type}_{\int_0}).$$

Proof. This follows immediately from Corollary 3.3.13, applied to the modality \int_1 . This corollary says that \int_1 -étale maps into X correspond to maps from $\int_1 X$ to \mathbf{Type}_{\int_1} . If furthermore the fibers are sets, then the maps go from $\int_1 X$ to \mathbf{Type}_{\int_0} . \square

Classically, the universal cover is just any simply connected cover. We can let this characterization lead us to a definition of the universal cover of a pointed, homotopically connected space. Let X be a space and $\pi : \tilde{X} \rightarrow X$ a covering with \tilde{X} simply connected in the sense that $\int_1 \tilde{X} = *$. Since π is a covering, and hence \int_1 -étale, the \int_1 -naturality square

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \int_1 \tilde{X} \\ \pi \downarrow & & \downarrow \int_1 \pi \\ X & \longrightarrow & \int_1 X \end{array}$$

is a pullback. But $\int_1 X = *$, so this shows us that $\tilde{X} = \text{fib}_{(-)\int_1}(u)$ for some $u : \int_1 X$. This leads us to the following definition.

Definition 3.9.3. Let X be a type and $\text{pt}_X : X$ a base point. Suppose further that X is homotopically connected in the sense that $\|\int_1 X\|_0 = *$. Then the

universal cover $\pi : \tilde{X} \cdot \rightarrow X$ is defined to be $\text{fst} : \text{fib}_{(-)_{f_1}}(\text{pt}_X^{f_1}) \rightarrow X$, with $\text{pt}_{\tilde{X}} \equiv (\text{pt}_X, \text{refl})$ and $\text{pt}_\pi \equiv \text{refl}$:

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & * \\ \pi \downarrow & & \downarrow \text{pt}_X^{f_1} \\ X & \longrightarrow & \int_1 X \end{array}$$

Theorem 3.9.4. *The universal cover $\pi : \tilde{X} \rightarrow X$ is the initial pointed cover of X . That is, for any pointed cover $c : C \cdot \rightarrow X$, there is a unique pointed cover $\chi_c : \tilde{X} \cdot \rightarrow C$ such that $c \circ \chi_c = \pi$ as pointed maps.*

Proof. We need to show that the universal cover is a cover with the correct universal property.

First, note that as the fiber of a \int_1 -unit, \tilde{X} is \int_1 -connected (that is, simply connected). Therefore, the naturality square

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \int_1 \tilde{X} \\ \pi \downarrow & & \downarrow \int_1 \pi \\ X & \longrightarrow & \int_1 X \end{array}$$

is equal to the square

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & * \\ \pi \downarrow & & \downarrow \text{pt}_X^{f_1} \\ X & \longrightarrow & \int_1 X \end{array}$$

which is a pullback. As the \int_1 -naturality square of π is a pullback, π is \int_1 -étale. The fiber of π over any point $x : X$ is equivalent to $x^{f_1} = \text{pt}_X^{f_1}$, which is a type of identifications in the 1-type $\int_1 X$ and is therefore a set. This proves that π is a cover.

Now for the universal property. Note that since $\pi(\text{pt}_{\tilde{X}}) \equiv \text{pt}_X$, the data of

a pointed cover $c : C \rightarrow X$ can be expressed as a square

$$\begin{array}{ccc} * & \xrightarrow{\text{pt}_C} & C \\ \text{pt}_{\tilde{X}} \downarrow & & \downarrow c \\ \tilde{X} & \xrightarrow{\pi} & X \end{array}$$

in which the map c is a cover. A filler of that square is precisely a pointed map $\tilde{X} \rightarrow C$ over X . But \tilde{X} is \mathcal{J}_1 -connected and therefore the map $\text{pt}_{\tilde{X}} : * \rightarrow \tilde{X}$ is an \mathcal{J}_1 -equivalence. And since c is a \mathcal{J}_1 -étale map and \mathcal{J}_1 -equivalences are orthogonal to \mathcal{J}_1 -étale maps by Lemma 6.1.23 of [Rij18a], the type of fillers of this square is contractible.

It remains to show that the unique filler of the square is a cover. Since c and π are \mathcal{J}_1 -étale, it is \mathcal{J}_1 -étale. And since c and π have set fibers, it does as well. Therefore, it is a cover. \square

As promised, Lemma 3.6.3 does prove that $(\cos, \sin) : \mathbb{R} \rightarrow \mathbb{S}^1$ is the universal cover of the circle. This map is \mathcal{J}_1 -étale, its fibers are sets, and \mathbb{R} is simply connected.

Theorem 3.6.1 provides us with a simple trick for showing that a map is a cover.

Corollary 3.9.5. Let $\pi : E \rightarrow B$. If there is a crisply discrete set F such that $\|\text{fib}_\pi(b) = F\|$ for all $b : B$, then π is a cover.

Remark 3.9.6. As promised in Section 3.6.2, the map $\mathbb{S}^{n+1} \rightarrow \mathbb{R}P^n$ is a covering map, and since \mathbb{S}^{n+1} is simply connected for $n \geq 0$, this is the universal cover of $\mathbb{R}P^n$.

We can prove a seemingly suspect proposition with this trick: any map with finite fibers is a cover. To do this, we need to prove a bit of folklore.

Lemma 3.9.7. Let $\mathbf{Fin} := (X : \mathbf{Type}) \times \|(n : \mathbb{N}) \times X = \{1, \dots, n\}\|$ be the type of finite types (types X for which there exists an n such that $X = \{1, \dots, n\}$). There is an equivalence

$$\mathbf{Fin} \simeq (n : \mathbb{N}) \times \mathbf{BAut}(n)$$

between the type of finite types and the sum over $n : \mathbb{N}$ of the classifying types $\mathbf{BAut}(n) := (X : \mathbf{Type}) \times \|X = \{1, \dots, n\}\|$ of the symmetric group $\mathbf{Aut}(n)$.

Proof. Note that

$$\begin{aligned} (n : \mathbb{N}) \times \mathbf{BAut}(n) &\equiv (n : \mathbb{N}) \times (X : \mathbf{Type}) \times \|X = \{1, \dots, n\}\| \\ &\simeq (X : \mathbf{Type}) \times (n : \mathbb{N}) \times \|X = \{1, \dots, n\}\|. \end{aligned}$$

Therefore, it will suffice to show that

$$(n : \mathbb{N}) \times \|X = \{1, \dots, n\}\| \simeq \|(n : \mathbb{N}) \times X = \{1, \dots, n\}\|$$

assuming that $X : \mathbf{Type}$. But the obvious map $(n, |p|) \mapsto |(n, p)|$ is a $\|-\|$ -unit by Lemma 1.24 of [RSS17a], so it will suffice to show that $(n : \mathbb{N}) \times \|X = \{1, \dots, n\}\|$ is a proposition.

Suppose that (n, p) and (m, q) are of type $(n : \mathbb{N}) \times \|X = \{1, \dots, n\}\|$, seeking $(n, p) = (m, q)$. From p and q , we get $\|\{1, \dots, n\} = \{1, \dots, m\}\|$. A simple induction shows that this occurs if and only if $n = m$. \square

Proposition 3.9.8. Let $\pi : E \rightarrow B$ be a map whose fibers are finite in the sense that for every $b : B$, there exists an $n : \mathbb{N}$ such that $\|\mathbf{fib}_\pi(b) = \{1, \dots, n\}\|$. Then π is a cover.

Proof. Note that this condition says that the map $\text{fib}_\pi : B \rightarrow \mathbf{Type}$ factors through $\mathbf{Fin} \hookrightarrow \mathbf{Type}$. But by Lemma 3.9.7, $\mathbf{Fin} \simeq (n : \mathbb{N}) \times \mathbf{BAut}(n)$, and since \mathbb{N} is crisply discrete, we have an equivalence

$$(n : \mathbb{N}) \times \mathbf{BAut}(n) \simeq (n : \flat \mathbb{N}) \times \text{let } n := m^{\flat} \text{ in } \mathbf{BAut}(m).$$

Now, in the inner expression, $m :: \mathbb{N}$ is *crisp*, and so Theorem 3.5.9 applies and $\mathbf{BAut}(m)$ is discrete. Therefore, \mathbf{Fin} is a discretely indexed sum of discrete types, and so it is also discrete. It is, furthermore, a 1-type since it is a set indexed sum of 1-types.

Therefore, fib_b factors through $\int_1 B$ and so by Lemma 3.3.12, is \int_1 -étale. By hypothesis, its fibers are finite and therefore sets, so it is a cover. \square

Remark 3.9.9. What is strange about this theorem is that there appear to be counterexamples. Consider the map $\mathbb{R} \vee \mathbb{R} \rightarrow \mathbb{R}$ we looked at in Example 3.7.2. It seems like its fibers are finite. By a quick application of descent, we can see that its fiber over $r : \mathbb{R}$ is equivalent to the suspension $\Sigma(r = 0)$ of the proposition that $r = 0$. The inclusion of the endpoints of the suspension are always jointly surjective, so there is a *surjection* $\{0, 1\} \rightarrow \Sigma(r = 0)$. But we cannot prove this is a bijection, or that there is a bijection from $\Sigma(r = 0)$ to $\{0\}$ without deciding the proposition $r = 0$. We can't decide whether a real number is 0 (since the reals are connected), so we can't find a precise cardinality for the fiber. This example emphasizes the difference between cardinal finiteness (being equivalent to some $\{1, \dots, n\}$) and Kuratowski finiteness (admitting a surjection from some $\{1, \dots, n\}$) in real cohesion.

Remark 3.9.10. While the map $\mathbb{R} \vee \mathbb{R} \rightarrow \mathbb{R}$ we considered in Example 3.7.2 is

not a covering, the homotopy quotient $\mathbb{R} \vee \mathbb{R} \rightarrow \mathbb{R} \vee \mathbb{R} // C_2$ is a cover, and is in fact the universal cover of $\mathbb{R} \vee \mathbb{R} // C_2$. To see this, note that $\mathbb{R} \vee \mathbb{R}$ is contractible since it is given as a crisp pushout and \int preserves crisp pushouts. The fibers of the homotopy quotient are merely equivalent to C_2 , which is a discrete set, so the map is a covering. This gives an example of the universal cover of a space which is not a set.

For a particular example of these results, consider an n -fold cover of the circle \mathbb{S}^1 .

Definition 3.9.11. An n -fold cover $\pi : E \rightarrow B$ is a map whose fibers have n elements. By Corollary 3.9.5, an n -fold cover is indeed a cover.

Theorem 3.9.12. Let $n : \mathbb{N}$. The type of n -fold covers of \mathbb{S}^1 whose fiber over $(1, 0)$ is identified with a fixed n -element set $\{1, \dots, n\}$ is equivalent to the type $\mathbf{Aut}(n)$ of permutations of n elements.

Proof. First, we note that since \mathbb{N} is crisply discrete, we may assume without loss of generality that n is crisp and that the fixed n -element set $\{1, \dots, n\}$ is also crisp. The type in question is

$$(f : \mathbb{S}^1 \rightarrow \mathbf{BAut}(n)) \times (f(1, 0) = \{1, \dots, n\})$$

the type of *pointed* maps from the circle to $\mathbf{BAut}(n)$. But Theorem 3.5.9, $\mathbf{BAut}(n)$ is discrete and so this is equivalent to the type

$$(f : \int \mathbb{S}^1 \rightarrow \mathbf{BAut}(n)) \times (f(1, 0)^{\int} = \{1, \dots, n\}).$$

By Theorem 9.5 of [Shu18b], $(\mathbb{S}^1 \rightarrow X) \simeq (S^1 \rightarrow X)$ for any discrete X , and so

the above type is equivalent to

$$(f : S^1 \rightarrow \mathbf{BAut}(n)) \times (f(\mathbf{pt}) = \{1, \dots, n\})$$

which, by the universal property of S^1 , is equivalent to $\Omega \mathbf{BAut}(n) \simeq \mathbf{Aut}(n)$. \square

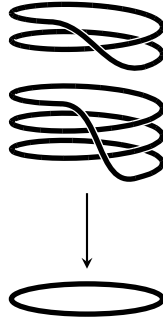


Figure 3.2: A 5-fold cover of the circle corresponding to the permutation $(12)(354)$. It has cycle type $(2, 3)$, corresponding to the 2 elements of the fiber in the top connected component, and the 3 elements in the bottom.

Looking at some examples of n -fold coverings (such as Figure 3.2), we might get the idea that the set of connected components of the total space corresponds to the cycle type of its induced permutation. Somewhat more objectively, we might expect that the set of connected components of the total space should correspond to the set of orbits of the action of the induced permutation on the elements of a fiber. We can prove this using a nice modal argument.

Theorem 3.9.13. *Let $\pi : E \rightarrow B$ be a cover over a pointed base B with fiber F which is connected in the sense that $\int_1 B$ is 0-connected. Then*

$$\int_1 E = F // \pi_1(B)$$

where $\pi_1(B) := \Omega(\int_1 B, \mathbf{pt}_B^1)$ is the fundamental group of B .

Proof. Since $\pi : E \rightarrow B$ is a cover, $\mathbf{fib}_\pi : B \rightarrow \mathbf{Type}$ factors through $\int_1 B$ as $\mathbf{fib}_{\int_1 \pi}$:

$$\begin{array}{ccc} B & \xrightarrow{\mathbf{fib}_\pi} & \mathbf{Type} \\ (-)^{\int_1} \downarrow & \nearrow \mathbf{fib}_{\int_1 \pi} & \\ \int_1 B & & \end{array}$$

witnessed by $\delta : \mathbf{fib}_\pi(b) \xrightarrow{\sim} \mathbf{fib}_{\int_1 \pi}(b^{\int_1})$. Taking total spaces, we find that the following square is a pullback:

$$\begin{array}{ccc} E & \xrightarrow{\mathbf{tot}(\delta)} & (t : \int_1 B) \times \mathbf{fib}_{\int_1 \pi}(t) \\ \pi \downarrow & & \downarrow \mathbf{fst} \\ B & \xrightarrow{(-)^{\int_1}} & \int_1 B \end{array}$$

Since $(-)^{\int_1} : B \rightarrow \int_1 B$ is \int_1 -connected (by Theorem 1.32 of [RSS17a]) and \int_1 -connected maps are preserved under pullback (by Theorem 1.34 of [RSS17a]), the top map $\mathbf{tot}(\delta)$ is also \int_1 -connected.

Now, since $\int_1 B$ is 0-connected, when pointed at $\mathbf{pt}_B^{\int_1}$ it can be considered as the delooping $\mathbf{B}\pi_1(B)$ of the fundamental group of B . Then, the homotopy quotient $\mathbf{fib}_\pi(\mathbf{pt}_B) \parallel \pi_1(B)$ can be constructed as the pair type

$$F \parallel \pi_1(B) \equiv (t : \int_1 B) \times \mathbf{fib}_{\int_1 \pi}(t).$$

See Section 3.7 for a brief introduction to the theory of higher groups and Lemma 3.7.5 for a justification of this construction.

So, the canonical map $E \rightarrow F \parallel \pi_1(B)$ is \int_1 -connected and therefore in particular a \int_1 -equivalence. But as a \int_1 -modally indexed sum of \int_1 -modal types, $\mathbf{fib}_\pi(\mathbf{pt}_B) \parallel \pi_1(B)$ is \int_1 -modal, so we find that $\int_1 E = F \parallel \pi_1(B)$. \square

Corollary 3.9.14. Let $\pi : E \rightarrow \mathbb{S}^1$ be an n -fold covering of the circle whose fiber

over $(1, 0)$ is identified with $\{1, \dots, n\}$, and let $\varphi : \text{Aut}(n)$ be the corresponding permutation. Then the set of connected components of the total space E is equivalent to the set of orbits of the action of φ on $\{1, \dots, n\}$.

Proof. The set of connected components of the total space may be constructed as $\|\int_1 E\|_0$, which by Theorem 3.9.13 is equivalent to $\|\text{fib}_\pi((1, 0)) // \pi_1(\mathbb{S}^1)\|_0$. As we calculated in Theorem 3.6.4, $\pi_1(\mathbb{S}^1) = \mathbb{Z}$, and by hypothesis $\text{fib}_\pi((1, 0)) = \{1, \dots, n\}$. So the connected components of E is equivalent to $\|\{1, \dots, n\} // \mathbb{Z}\|_0$ with the action given by $1 \mapsto \varphi$. By Lemma 3.7.5, two elements of $\|\{1, \dots, n\} // \mathbb{Z}\|_0$ are equal if and only if there is an integer that sends one to the other; in other words, this is the set of orbits of the action of φ , as desired. \square

We can extend the definition of a cover naturally to an “ n -cover” using the modality \int_n .

Definition 3.9.15. A map $\pi : E \rightarrow B$ is an n -cover if it is \int_n -étale and its fibers are $(n - 1)$ -types.

The theory of n -covers follows just as smoothly as the theory of covers. For every fact above about covers, there is an analogous fact about n -covers proved in the same way. In particular, a universal n -cover is just a \int_n -connected n -cover. We can describe the universal 2-cover of the 2-sphere.

Theorem 3.9.16. *Let $h : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ be the Hopf fibration. Then the \int -modal factor $\text{fst} : (s : \mathbb{S}^2) \times \int \text{fib}_h(s) \rightarrow \mathbb{S}^2$ of the Hopf fibration is the universal 2-cover of the 2-sphere.*

Proof. Let $\pi : E \rightarrow \mathbb{S}^2$ denote the \int -modal factor of the Hopf fibration. Note that $\text{fib}_\pi(s) = \int \text{fib}_h(s)$ is merely equivalent to the crisply discrete 1-type $\int \mathbb{S}^1$

for all $s : \mathbb{S}^2$, and is therefore by Theorem 3.6.1 is \int_2 -étale and so a 2-cover. Furthermore, $\int E \simeq \int \mathbb{S}^3$, so it is \int_2 -connected (since $\int \mathbb{S}^3 = S^3$ is 2-connected), and therefore the universal 2-cover. \square

The theory of n -covers seems related to the theory of Whitehead towers, but the precise relationship between these notions in Cohesive HoTT is not yet clear to the author.

We can show that the universal cover of a crisp ∞ -group is also an ∞ -group. If G is a crisp ∞ -group, then so is $\int_1 G \simeq \|\int G\|_1$ by Theorem 3.8.9 and so we get a long fiber sequence:

$$\begin{array}{ccccccc}
 & & & \cdots & \longrightarrow & \pi_1(G) & \\
 & & & & & \searrow & \\
 & & & & & \curvearrowright & \\
 & & & \tilde{G} & \longrightarrow & G & \longrightarrow \int_1 G \\
 & & & \searrow & & \searrow & \\
 & & & & & \curvearrowright & \\
 & & & \mathbf{B}\tilde{G} & \longrightarrow & \mathbf{B}G & \longrightarrow \int_2 \mathbf{B}G
 \end{array}$$

The delooping of \tilde{G} is defined to be the fiber of $(-)^{\int_2} : \mathbf{B}G \rightarrow \int_2 \mathbf{B}G$, and it is 0-connected since the unit $(-)^{\int_1} : G \rightarrow \int_1 G$ is surjective. Note that $\mathbf{B}\tilde{G}$ is the universal 2-cover of $\mathbf{B}G$.

We can continue this fiber sequence on as long as G can be delooped, taking $\int_{k+1} \mathbf{B}^k G$ as the delooping of $\int_k \mathbf{B}^{k-1} G$ and taking $\mathbf{B}^k \tilde{G}$ to be the universal $(k+1)$ -cover of $\mathbf{B}^k G$. In particular, we get a long fiber sequence:

$$\begin{array}{ccccccc}
& & & & \dots & \longrightarrow & \mathbb{Z} \\
& & & & & & \downarrow \\
& & & & \mathbb{R} & \longrightarrow & \mathbb{U}(1) \longrightarrow \mathbb{B}\mathbb{Z} \\
& & & & & & \downarrow \\
& & & & \mathbb{B}\mathbb{R} & \longrightarrow & \mathbb{B}\mathbb{U}(1) \longrightarrow \mathbb{B}^2\mathbb{Z} \\
& & & & & & \downarrow \\
& & & & \dots & &
\end{array}$$

This gives us a long exact sequence $H^*(-; \mathbb{Z}) \rightarrow H^*(-; \mathbb{R}) \rightarrow H^*(-; \mathbb{U}(1)) \rightarrow H^{*+1}(-; \mathbb{Z})$ in *continuous* cohomology.

In this paper, we have defined a notion of *modal fibration* and explored the fibrations for the *shape* modality of Real Cohesive HoTT. We have seen that it is often quite easy to prove a map is a \mathfrak{f} -fibration — indeed, if you know what the fiber is ahead of time, it is often trivial. After a fibration is found, many simple calculations can be done with purely modal arguments.

Chapter 4

Orbifolds

4.1 Introduction

4.1.1 What are orbifolds, and what could they be?

Informally, an orbifold is a smooth space whose points may have finitely many internal symmetries.

Informal Definition 4.1.1. An *orbifold* is a smooth space X whose points x have finite groups $\text{Aut}_X(x)$ of internal symmetries, known as their *isotropy groups*.

A paradigmatic example of an orbifold is the quotient of a manifold by the action of a finite group; the smooth structure comes from the manifold, and we may think of the stabilizer group of a point as its internal symmetries.

A common way to model the notion of orbifold uses *proper étale groupoids*, which are groupoids internal to the category of smooth manifolds (Lie groupoids) satisfying certain properties [MP97]. However, the correct notion of sameness for these sorts of orbifolds is not equivalence of groupoids, but a separate notion of *Morita equivalence*. Worse, to get all the morphisms between two such

orbifolds we may need to replace one of them by a Morita equivalent orbifold first. While this sort of situation is standard fair for categorical homotopy theory, it does not directly capture the intuitive idea of an orbifold as a smooth space whose points have internal symmetries. The issue is that in the usual set theoretic foundations, the elements of sets *cannot* have internal symmetries, and therefore we must carry the data of these symmetries around and account for them at every step of our theory.

In this chapter, we will investigate the notion of orbifold in the setting of homotopy type theory, where points can be non-trivially self identified on the foundational level. Homotopy type theory is a novel foundation of mathematics which is based on the intuition that each mathematical object is a certain *type* of mathematical object. For example, 3 is an integer, π is a real number, and $\mathbf{GL}_n(\mathbb{R})$ is a Lie group. For any two objects x and y of the same type — say, any two vector spaces — we may consider what it means to *identify* x with y — for vector spaces, we would identify x with y by giving a linear isomorphism between them. Because an identification between mathematical objects is another mathematical object, we also get a type of all identifications between x and y which we write as $(x = y)$.

Identification between mathematical objects is the only form of equality available in homotopy type theory — the more traditional *proposition* of equality just occurs in the case that there is at most one way to identify two objects of a given type. For example, there is at most one way to identify two natural numbers: when they're equal, we may trivially identify them, and otherwise we can't. We call types where identification is the proposition of equality — types where there is at most one way to identify two elements — *sets*. Just as in

set theory, the elements of sets cannot have any non-trivial self-identifications. However, there are types in which there can be multiple ways to identify two different objects, such as the type of vector spaces (since there are many linear isomorphisms between two vector spaces).

In these higher types, objects $x : X$ may have nontrivial self-identifications in the type $\mathbf{Aut}_X(x) := (x = x)$.¹ That is, homotopy type theory gracefully generalizes from set-level mathematics to groupoid-level mathematics and higher. Because the elements of types can have non-trivial self identifications — the type $(x = x)$ can have more than one element — we can quite directly formalize Informal Definition 4.1.1.

Definition 4.1.2. An *orbifold* is a *microlinear type* (Definition 4.4.22) X whose types of identifications $(x = y)$ are *properly finite* for all points $x, y : X$.

The notion of microlinearity is a good formalization of “smooth space”. The technical notion of “properly finite” is needed due to a quirk of constructive mathematics — saying that the identification types were simply finite would be much too strong. We will discuss these notions further in this introduction.

Our main goal in this chapter will be to justify Definition 4.1.2. We will do this in two main ways. First, in Section 4.2, we will construct explicit examples of orbifolds by saying what their points are. For example, in Definition 4.2.2 we will construct the moduli space $\mathcal{M}_{1,1}$ of complex elliptic curves as the type of pairs (\mathcal{L}, Λ) where \mathcal{L} is a 1-dimensional complex vector space and $\Lambda \subseteq \mathcal{L}$ is a lattice in it — the elliptic curve itself is the torus \mathcal{L}/Λ . The symmetries of

¹To distinguish between identifications ($=$) and definitional equalities, we use the symbol \equiv for a definitional equality. We will also put a colon next to the symbol — as in $:=$ — to show that the left hand side is defined to be the right hand side. For more on notation, see ??.

a point $(\mathcal{L}, \Lambda) : \mathcal{M}_{1,1}$ may be identified with \mathbb{C} -linear automorphisms of \mathcal{L} that fix the lattice Λ . This will let us show in Proposition 4.2.4 that $\mathcal{M}_{1,1}$ may be equivalently defined as the quotient $\mathfrak{h} // \mathrm{SL}_2(\mathbb{Z})$ of the upper half plane \mathfrak{h} by the action of $\mathrm{SL}_2(\mathbb{Z})$ via Mobius transformations.

Second, in Theorem 4.6.37, we will prove that any crisp, ordinary proper étale groupoid is an orbifold in the sense of Definition 4.1.2. The extra adjectives “crisp” and “ordinary” are just there to say that we mean proper étale groupoids in the ordinary, external sense. This theorem shows that our type theoretic definition of orbifold subsumes the standard definition.

This introduction is structured as an outline of the chapter. The remaining introductions in this introduction introduce Sections 2-6 of this chapter. We conclude in section 7 with a brief summary of our results.

4.1.2 Good Orbifolds.

Using the elementary construction of the quotients of types by the actions of groups described in Section 2.4, we will explicitly construct a number of examples of so-called *good orbifolds* — the orbifolds arising as quotients of smooth spaces by discrete groups — in Section 4.2. In general, to construct an orbifold knowing that it may be expressed as the quotient $X // \Gamma$ involves choosing a good notion of exemplar for Γ — that is, judiciously choosing a $\mathbf{B}\Gamma$ — so that the action of Γ on X takes a particularly nice form as a function $X^{\circ-} : \mathbf{B}\Gamma \rightarrow \mathbf{Type}$. In the end, the construction gives us an explicit definition of the orbifold in terms of its points: the points of $X // \Gamma$ are pairs (e, x) of an exemplar $e : \mathbf{B}\Gamma$ of Γ together with a point $x : X^{\circ e}$ of X twisted by e .

4.1.3 The homotopy theory of orbifolds via cohesion.

In his open letter to the homotopy theory community [Bar17], Clark Barwick makes the bold claim that

Homotopy theory is not a branch of topology.

If homotopy theory is not the study of *homotopies* — that is, continuous deformations of objects — what is it? Homotopy type theory offers a striking formal answer to this question: homotopy theory is the study of the way mathematical objects may be *identified*. For example, the homotopy circle S^1 may be freely generated as a type with a single point \mathbf{pt} a self-identification $\mathbf{loop} : \mathbf{pt} = \mathbf{pt}$.

However, if we intend to do algebraic topology — that is, to use homotopy theoretic methods to study topological spaces and their higher cousins such as orbifolds — then we will need to distinguish the actual circle

$$\mathbb{S}^1 := \{x : \mathbb{R}^2 \mid |x| = 1\}$$

from the homotopy circle S^1 . Even more, we should be able to prove that the homotopy circle S^1 is the type we end up with if we start with the actual circle \mathbb{S}^1 and identify points according to how they may be continuously deformed into each other. That is to say, S^1 should be the *homotopy type* of \mathbb{S}^1 .

In his paper “Brouwer’s fixed point theorem in real cohesive homotopy type theory” [Shu18a], Shulman gives us the tools to do synthetic algebraic topology in homotopy type theory by adding a system of *modalities* to HoTT which include the *shape* modality \int that sends a type X to its homotopy type $\int X$. The shape $\int X$ of a type X may be defined as the localization of X at the type \mathbb{R} of real numbers, so that any path $\gamma : \mathbb{R} \rightarrow X$ gives us an identification

$p(\gamma) : \gamma(0)^f = \gamma(1)^f$ in $\int X$. However, for this construction to behave right, we need the rest of Shulman’s cohesion, which we will review in Section 4.3.1.

In Section 4.3.2, we will use the theory of modal fibrations and coverings developed in Chapter 3 to compute the homotopy types of some orbifolds and hint at their general covering theory. For example, in Theorem 4.3.9 we will compute that

$$\int \mathcal{M}_{1,1} \simeq \mathrm{BSL}_2(\mathbb{Z})$$

the homotopy type of the moduli stack of elliptic curves $\mathcal{M}_{1,1}$ is a delooping of the group $\mathrm{SL}_2(\mathbb{Z})$. We’ll use this modal covering theory in Section 4.3.4 to briefly investigate maps between orbifolds.

Remark 4.1.3. It is precisely the fluency of covering theory in cohesive homotopy type theory which forces us to use a technical notion of “properly finite”, rather than “finite”, in our definition of orbifolds (Definition 4.1.2). A map $f : X \rightarrow Y$ whose fibers are finite is necessarily a covering map by the “good fibrations” trick of Chapter 3 (see also Remark 9.9 of *ibid.*). If for every $x : X$, the type of automorphisms $(x = x)$ of x were finite, then the projection from the inertia orbifold $X^{S^1} \rightarrow X$ (whose fiber over x is $(x = x)$) would be a finite covering; but this would in particular imply that the cardinality of the isotropy group $(x = x)$ is constant on any connected component of X . This is problematic for most orbifolds seen in practice.

4.1.4 Smooth spaces and synthetic differential geometry.

Orbifolds are *smooth spaces* whose points have internal symmetries, and while moving to homotopy type theory has given us direct access to types whose points

have internal symmetries, we have not yet talked about the smooth structure. The formal system of homotopy type theory admits models in all ∞ -toposes ([**Shulman:Models.of.HoTT**]), so that a type gets interpreted as a stack of homotopy types, and an element of a type gets interpreted as a map between these stacks. Identifications between elements get interpreted as homotopies between the corresponding maps. We can therefore get the smooth structure we need on our types by interpreting our homotopy type theory in an ∞ -topos of stacks on a suitable site — say a site consisting of smooth manifolds.

But we would like to be able to work with this smooth structure from within homotopy type theory itself. To give our types smooth structure, we will use the axioms of *synthetic differential geometry*, which we review in Section 4.4.1. Synthetic differential geometry is an axiom system for doing differential geometry with nilpotent infinitesimals, first put forward by Lawvere and developed further by Dubuc, Kock, Bunge, Moerdijk, Reyes, and many others. While this axiom system is usually interpreted in 1-toposes, it can be interpreted in ∞ -toposes just as well. The Dubuc ∞ -topos of stacks on a site of infinitesimally extended Euclidean spaces (with smooth maps between them) and the very similar ∞ -topos which Schreiber calls the topos of formal smooth ∞ -groupoids [Sch13c] are models of both cohesion and synthetic differential geometry. These will be our intended models for this chapter.²

²The site for the Dubuc ∞ -topos consists of (the opposite category of) \mathcal{C}^∞ -rings of the form $\mathcal{C}^\infty(\mathbb{R}^n)/I$ where I is a *germ-determined ideal*: $f \in I$ if and only if for all $x \in \mathbb{R}^n$, the germ f_x is in the ideal I_x generated by the germs at x of functions in I . The site for the topos of formal smooth ∞ -groupoids has as its objects the \mathcal{C}^∞ -rings of the form $\mathcal{C}^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} W$ where W is a *Weil algebra* — a finitely presented augmented \mathbb{R} -algebra with finitely generated and nilpotent augmentation ideal. Crucially, the category of euclidean spaces \mathbb{R}^n and smooth maps embeds into both of these sites by $\mathbb{R}^n \mapsto \mathcal{C}^\infty(\mathbb{R}^n)$. But these sites also have *infinitesimal spaces*, such as the dual numbers $\mathbb{R}[x]/(x^2)$, which enable us to work with infinitesimals in the homotopy type theory of these toposes. For a definition of these sites, see the standard

In synthetic differential geometry, we axiomatize the *smooth reals* \mathbb{R} as an ordered field. Crucially, since we are working constructively, just because a number is *not* non-zero does not imply that it is zero. The numbers which are not non-zero are known as *infinitesimals* (after Penon’s *Infinitesimaux et intuitionisme* [Pen81]). The most crucial axiom of synthetic differential geometry which make these infinitesimals behave as we would like them to is the Kock-Lawvere axiom. As a special case of this axiom, we see that every function $f : \mathbb{D} \equiv \{\varepsilon : \mathbb{R} \mid \varepsilon^2 = 0\} \rightarrow \mathbb{R}$ from the set \mathbb{D} of nilsquare elements of \mathbb{R} (the “first-order infinitesimals”) to \mathbb{R} is linear: that is, there is a unique $b : \mathbb{R}$ so that $f(\varepsilon) = f(0) + b\varepsilon$ for all $\varepsilon^2 = 0$.

As a corollary of this axiom, we may define the derivative f' of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ to be the unique function satisfying

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon$$

for all $x : \mathbb{R}$ and $\varepsilon^2 = 0$. This justifies calling \mathbb{R} the “smooth reals” — every function $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth.

In general, we may think of the type of functions $X^{\mathbb{D}} \equiv (\mathbb{D} \rightarrow X)$ as the tangent bundle of X , with the projection $\pi : X^{\mathbb{D}} \rightarrow X$ given by evaluation at 0. The tangent space $T_x X$ of $x : X$ is therefore the type of functions $v : \mathbb{D} \rightarrow X$ with $v(0) = x$. Because we can make this definition, there is a sense in which every type in synthetic differential geometry has a sort of differentiable structure. However, this structure isn’t very much like a manifold’s in general. For any type X , the tangent spaces $T_x X$ admit a scalar multiplication by \mathbb{R} defined by $(rv)(\varepsilon) := v(r\varepsilon)$, but $T_x X$ is not generally an \mathbb{R} -module.

reference [MR90] where Dubuc’s site is known as \mathbb{G} , and Section 6.5 of [Sch13c].

The natural question is then: what are the smooth spaces in synthetic differential geometry? There are a number of answers. We could of course repeat the usual definition of smooth manifold. Or, we could look at spaces that are only *infinitesimally* (and not necessarily locally) isomorphic to Euclidean space; this gives us Penon’s notion of manifold. Or, we could look at types which are infinitesimally isomorphic to Euclidean space, but this time in the sense of being related to Euclidean spaces by a zig-zag of étale maps; this gives us Schreiber’s notion of manifold. For each of these possible definitions of manifold, the tangent spaces $T_x X$ will be \mathbb{R} -modules.

But there is a wider class of spaces that includes all of the above and which the synthetic differential geometry community has settled into as the “right” notion of smooth space suitable for proving theorems: *microlinear* spaces. Microlinear spaces have all the infinitesimal linear properties that \mathbb{R} does, in a sense which we will make precise in Section 4.4.2. And, since microlinear spaces may be defined by lifting uniquely on the right against a given class of maps (Lemma 4.4.39), they have good closure properties.

As a further advantage, the definition of microlinearity applies just as well to higher types as to sets. In particular, a standard theorem in synthetic differential geometry proves that the tangent spaces of microlinear sets are \mathbb{R} -modules; this is likely the reason these spaces are called “microlinear”. In Theorem 4.4.29, we will prove this fact in such a way that it applies not only to sets but also groupoids and general higher types. That is, if X is a microlinear type, not necessarily a set, then its tangent spaces admit fully coherent \mathbb{R} -module structures.

Of the three definitions of manifold given above, only Schreiber’s generalizes

to higher types; but this definition relies on a choice of atlas, whereas microlinearity is “coordinate-free”.

Just as we introduced the shape modality \mathbb{J} to study the topology of orbifolds by trivializing it through the nullification of \mathbb{R} , we will introduce the *crystalline* modality \mathfrak{S} in Section 4.4.3 to study the diffeology of orbifolds by trivializing it through the nullification of the set \mathfrak{D} of infinitesimal real numbers. The modality \mathfrak{S} was called the “infinitesimal shape modality” by Schreiber in [Sch13c], and was studied in homotopy type theory by Cherubini in [Che17].

We will mainly use the \mathfrak{S} modality for its étale maps. A map is \mathfrak{S} -étale when its \mathfrak{S} -naturality square is a pullback:

$$\begin{array}{ccc} X & \xrightarrow{(-)^{\mathfrak{S}}} & \mathfrak{S}X \\ f \downarrow & \lrcorner & \downarrow \mathfrak{S}f \\ Y & \xrightarrow{(-)^{\mathfrak{S}}} & \mathfrak{S}Y \end{array}$$

In Theorem 4.4.42, we will show that microlinearity descends along surjective \mathfrak{S} -étale maps. That is, if X is microlinear and $f : X \rightarrow Y$ is surjective and \mathfrak{S} -étale, then Y is also microlinear. We will use this theorem together with the “good fibrations” trick of Chapter 3 to show that quotients of microlinear spaces by discrete groups are themselves microlinear (Theorem 4.5.26). This proves in particular that the good orbifolds constructed in Section 4.2 are microlinear.

4.1.5 Smooth spaces are microlinear

With Theorem 4.4.42 in hand, we will show that all sorts of smooth spaces are microlinear. In particular, we will show that ordinary smooth manifolds are microlinear (Section 4.5.1), as are the synthetic manifolds of Penon (Section 4.5.2)

and Schreiber (Section 4.5.3). We will also compare the modal notion of \mathfrak{S} -étale map with the usual notion of local diffeomorphism, showing in Corollary 4.5.20 that these notions coincide between crisp, ordinary manifolds.

Most importantly, in Section 4.5.4 we will prove in Theorem 4.5.34 that *étale groupoids* are microlinear. An étale groupoid is — roughly speaking — a groupoid \mathcal{G} for which the source map $s : \mathcal{G}_1 \rightarrow \mathcal{G}_0$ sending a morphism of \mathcal{G} to its source is \mathfrak{S} -étale. Classically, these are a class of locally discrete Lie groupoids which contain the proper étale Lie groupoids that present orbifolds. For this reason, Theorem 4.5.34 is a major step on the way to proving that all proper étale groupoids are orbifolds in the sense of Definition 4.1.2.

The proof of Theorem 4.5.34 involves a number of colimit-preserving properties of the modality \mathfrak{S} — properties which \mathfrak{S} shares with \mathfrak{J} . That \mathfrak{S} commutes with crisp pushouts and colimits of sequences follows from the assumption that the type \mathfrak{D} of infinitesimal real numbers is *tiny*. This is a crucial assumption of synthetic differential geometry which we do not fully explore in this chapter. Rather, we push the definition of tiny type and the requisite lemmas to Section 4.7.

The main lemma in the proof of Theorem 4.5.34 is an étale descent theorem, Theorem 4.5.32, which states that if the pullback of a crisp map f along itself is \mathfrak{S} -étale, then f is itself \mathfrak{S} -étale. We prove this descent theorem for any modality that commutes with crisp colimits, which also includes \mathfrak{J} .

In Section 4.5.5, we will investigate the microlinearity of deloopings $\mathbf{B}G$ of microlinear groups G , which include the Lie groups. While I was not able to prove that $\mathbf{B}G$ is microlinear, we can prove in Theorem 4.5.40 that $\mathbf{B}G$ is *infinitesimally linear* — a weaker condition than microlinearity — so that at

least its tangent spaces are (higher) \mathbb{R} -modules. The tangent space $T_{\mathbf{pt}_{BG}}BG$ of BG at its canonical exemplar \mathbf{pt}_{BG} is a delooping of the Lie algebra $\mathfrak{g} \equiv T_1G$, and the map $e \mapsto T_eBG : BG \rightarrow \mathbf{Type}$ is a delooping of the adjoint action of G on $B\mathfrak{g} \equiv T_{\mathbf{pt}_{BG}}BG$.

4.1.6 Finiteness and compactness

Finally, we turn our attention to proving Theorem 4.6.37. This theorem states that crisp ordinary proper étale groupoids are orbifolds in the sense of Definition 4.1.2. An ordinary proper étale groupoid is a groupoid \mathcal{G} for which the spaces \mathcal{G}_0 of objects and \mathcal{G}_1 of morphisms are both ordinary smooth manifolds, where the source map $s : \mathcal{G}_1 \rightarrow \mathcal{G}_0$ is \mathfrak{S} -étale (which by Corollary 4.5.20 means that s is a local diffeomorphism in the ordinary sense), and where the map $(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is *proper*.

The usual definition of a proper map is that the inverse image of any compact set is compact. If we used the usual definition of compact — that any cover admits a finitely enumerable subcover — then we could prove that the fibers of any proper map are in fact finite sets. This is of course to be expected, but remember: being finite is a strong condition in cohesive homotopy type theory. Namely, if a map has finite fibers, then it is a covering map. This won't do, because that would imply that $(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is a finite cover, which would mean that the cardinality of the isotropy groups $\mathcal{G}(x, x) = (s, t)^{-1}(x, x)$ would be constant over any connected component of \mathcal{G}_0 . This is almost never true of orbifolds in practice; for example, the quotient $\mathbb{R}^2 // C_k$ of the plane by rotation by $\frac{2\pi}{k}$ has non-trivial isotropy group C_k only at the origin.

For this reason, we will need a different definition of proper map, which

means a different definition of compact set, which ultimately relies on a different notion of “finite”. In Section 4.6.1, we will introduce *properly finite* sets: discrete subquotients of finite sets. We then relate this new notion of finiteness to an appropriate notion of compactness.

Luckily, Dubuc and Penon have already explored a beautifully creative definition of “compact set” in the setting of synthetic differential geometry. A set K is Dubuc-Penon compact if universal quantification over K commutes with logical or: that is, if for any proposition A and predicate $B : K \rightarrow \mathbf{Prop}$, if for all k it is the case that A holds or $B(k)$ holds, then either A holds or for all k , $B(k)$ holds:

$$(\forall k : K. A \vee B(k)) \Rightarrow (A \vee \forall k : K. B(k)).$$

This is an intrinsic property of the set K . In [DP86], Dubuc and Penon prove that in the various toposes of interest, a sheaf represented by a smooth manifold is Dubuc-Penon compact if and only if that manifold is compact in the ordinary sense.

In Section 4.6.2, we will prove an internal version of Dubuc and Penon’s theorem in Proposition 4.6.24: for any crisp, Dubuc-Penon compact subset K of an ordinary manifold, any crisp open cover of K admits a finite subcover. This result follows as a corollary of Theorem 4.6.22, which states that any Dubuc-Penon compact set K is countably compact: any countably enumerable Penon open cover of K admits a finitely enumerable subcover. This theorem is proven with a key lemma, Theorem 4.6.17, which states that for any Dubuc-Penon compact set K and any relation $r \subseteq K \times \mathbb{R}$, if $r(k, x)$ for all $k : K$, then there exists an $\varepsilon > 0$ such that $r(k, y)$ for all $k : K$ and $y \in B(x, \varepsilon)$ in the ε -ball

around x . This key lemma was extracted from the proof that Gago gives in his thesis [GC89] that any positive valued function $f : K \rightarrow (0, \infty)$ is bounded away from 0 (Corollary 4.6.19). All of this relies crucially on the Covering Property, originally due to Bunge and Dubuc [BD87], which is assumed of the smooth reals: if $A \cup B = \mathbb{R}$, then for any $x : \mathbb{R}$ there is an $\varepsilon > 0$ so that $B(x, \varepsilon) \subseteq A$ or $B(x, \varepsilon) \subseteq B$.

With this analysis of Dubuc-Penon compact subsets in hand, we begin Section 4.6.3. We will prove in Lemma 4.6.30 that discrete Dubuc-Penon compact subsets of ordinary manifolds are properly finite. To finish the proof of our main Theorem 4.6.37, then, it remains to show that if $(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is Dubuc-Penon proper and that $s : \mathcal{G}_1 \rightarrow \mathcal{G}_0$ is \mathfrak{S} -étale, then the hom sets $\mathcal{G}(x, y)$ of this proper étale groupoid \mathcal{G} are discrete. We accomplish this final lemma in Lemma 4.6.34, showing that crystalline subsets of ordinary manifolds are discrete.

We may then conclude that all (crisp, ordinary) proper étale groupoids are orbifolds in the sense of Definition 4.1.2, justifying that definition. In Section 4.6.4, we show that the quotient of a microlinear set by the action of a finite group is an orbifold, and quickly prove that orbifolds are closed under pullback.

4.2 Good orbifolds

We are ready to construct some orbifolds. In this section, we will focus on *good* orbifolds — those orbifolds which are the quotients of discrete groups acting on manifolds. The easy and concrete construction of quotients in homotopy type

theory makes constructing good orbifolds a breeze.

Remark 4.2.1. We will eventually be able to define *étale groupoids* (Definition 4.5.31), a notion which includes the presentations of orbifolds as proper étale groupoids due to Moerdijk and Pronk [MP97]. We will, in Theorem 4.6.37, prove that (crisp, ordinary) proper étale groupoids are orbifolds in the sense of Definition 4.1.2. But first we will need to develop the language of synthetic differential geometry.

Example 8. We can define an *elliptic point* of order n as $\mathbb{R}^2 // C_n$, with the action of C_n on \mathbb{R}^2 constructed as in Example 7. Explicitly, this means

$$\mathbb{R}^2 // C_n := (\mathcal{L} : BU(1)) \times (C : \text{Cycle}_n(\mathcal{L})) \times \mathcal{L}$$

consists of triples (\mathcal{L}, C, ℓ) where \mathcal{L} is a 1-dimensional Hermitian vector space, $C \subseteq \mathbb{S}^{\mathcal{L}}$ is a cycle of n -elements in \mathcal{L} , and $\ell : \mathcal{L}$ is a point of \mathcal{L} .

Example 9. Satake [Satake:Orbifold] defined orbifolds as spaces locally modelled on a quotient of \mathbb{R}^n by the action of a finite subgroup of $O(n)$. For any finite subgroup $\Gamma \subseteq O(n)$, we can construct the coordinate patch $\mathbb{R}^n // \Gamma$ by

$$\mathbb{R}^n // \Gamma := (V : BO(n)) \times \text{Subtors}_{\Gamma}(\text{Frame}(V)) \times V.$$

We're making use of Proposition 2.3.10 and Remark 2.2.8 to deloop Γ as $B\Gamma := (V : BO(n)) \times \text{Subtors}_{\Gamma}(\text{Frame}(V))$.

Example 10. We can describe the configuration space $X // n!$ of n unlabeled points in a given type X quite simply as a homotopy quotient. We may take n -element sets as our exemplars of the symmetric group $\text{Aut}(\underline{n})$, with the standard

finite cardinal $\underline{n} := \{0, \dots, n-1\}$ as our canonical exemplar. That is, we define

$$\mathbf{BAut}(n) := (F : \mathbf{Type}) \times \|F \simeq \underline{n}\|$$

to be the type of n -element sets, which are equivalently types F which are somehow identifiable with n . We may then act on the cartesian power X^n by sending an n -element set F to the type of functions X^F . This gives us the following construction of the configuration space of n unlabelled points as the type of pairs of an n -element set F and an F -tuple of elements of X :

$$X^n // n! := (F : \mathbf{BAut}(\underline{n})) \times X^F.$$

Example 11. We can describe the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves over \mathbb{C} as the homotopy quotient of the type $\mathbf{Lattice}(\mathbb{C})$ of lattices in \mathbb{C} by the action of \mathbb{C}^* .

To this end, we will define the notion of a lattice in a complex line. In fact, we might as well define the notion of lattice in n -dimensional (real) space.

Definition 4.2.2. Let $V : \mathbf{BGL}_n(\mathbb{R})$ be an n -dimensional real vector space. A *lattice* in V is a subset $\Lambda \subseteq V$ which is

1. an additive subgroup of V ,
2. metrically discrete, in that for any norm $\langle -, - \rangle$ on V there exists a (rational) $\varepsilon > 0$ so that if $x \in \Lambda$ has norm $\langle x, x \rangle$ less than ε , then $x = 0$.
3. non-degenerate, in that it has rank n as an abelian group.

We denote by $\mathbf{Lattice}(V)$ the type of lattices in V . To consider a lattice in a complex vector space, we first consider that vector space as a real vector space.

We can then define $\mathcal{M}_{1,1}$ as the type of pairs consisting of a 1-dimensional complex vector space and a lattice in it.

Definition 4.2.3. We define $\mathcal{M}_{1,1}$ to be the type of pairs consisting of a 1-dimensional complex vector space L , and a lattice Λ within it.

$$\mathcal{M}_{1,1} := \text{Lattice}(\mathbb{C}) // \text{GL}_1(\mathbb{C}) \equiv (\mathcal{L} : \text{BGL}_1(\mathbb{C})) \times \text{Lattice}(\mathcal{L}).$$

It is not clear from this description that $\mathcal{M}_{1,1}$ is an orbifold, however. In order to do that, we will use the recognition theorem, Theorem 2.4.5, to show that $\mathcal{M}_{1,1}$ is the homotopy quotient of the upper half plane $\mathfrak{h} := \{a + bi : \mathbb{C} \mid b > 0\}$ by the action of $\text{SL}_2(\mathbb{Z})$ via Möbius transformations.

We may define a map $q : \mathfrak{h} \rightarrow \mathcal{M}_{1,1}$ by $q(\tau) \equiv (\mathbb{C}, \mathbb{Z} \oplus \tau \mathbb{Z})$. We will show that the fibers of q may be equipped with the structure of a $\text{SL}_2(\mathbb{Z})$ -torsor; by Theorem 2.4.5, this will show that q is a homotopy quotient and that $\mathcal{M}_{1,1} = \mathfrak{h} // \text{SL}_2(\mathbb{Z})$.

Proposition 4.2.4. Let $q : \mathfrak{h} \rightarrow \mathcal{M}_{1,1}$ be the map

$$q(\tau) \equiv (\mathbb{C}, \mathbb{Z} \tau \oplus \mathbb{Z}).$$

Then every fiber $\text{fib}_q(\mathcal{L}, \Lambda)$ may be equipped with the structure of a $\text{SL}_2(\mathbb{Z})$ -torsor with the action given by Möbius transformations. Consequently,

$$\mathcal{M}_{1,1} \simeq \mathfrak{h} // \text{SL}_2(\mathbb{Z}).$$

Proof. Let \mathcal{L} be a 1-dimensional complex vector space and $\Lambda \subseteq \mathcal{L}$ a lattice in it. By definition,

$$\text{fib}_q(\mathcal{L}, \Lambda) := (\tau : \mathfrak{h}) \times ((\mathbb{C}, \mathbb{Z} \tau \oplus \mathbb{Z}) = (\mathcal{L}, \Lambda)).$$

By the calculation of identifications in pair types, this is equivalently

$$(\tau : \mathfrak{h}) \times (p : \mathbb{C} = \mathcal{L}) \times (\mathbb{Z}\tau \oplus \mathbb{Z} = p^{-1}(\Lambda)).$$

Note that since lattices are subsets, equality between lattices is a proposition. Therefore, we are free to consider the fiber as a type of pairs (τ, p) which satisfy a proposition. We will describe a $\mathrm{SL}_2(\mathbb{Z})$ action on this type and then prove that it is free and transitive.

Given a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} : \mathrm{SL}_2(\mathbb{Z})$, its associated Möbius transform is the function $f(z) = \frac{az+b}{cz+d}$ which acts on the upper half plane \mathfrak{h} . We define the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{fib}_q(\mathcal{L}, \Lambda)$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau, p) := \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)p \right).$$

We check that

$$\begin{aligned} \mathbb{Z} \left(\frac{a\tau + b}{c\tau + d} \right) \oplus \mathbb{Z} &= \frac{1}{c\tau + d} (\mathbb{Z}(c\tau + d) \oplus \mathbb{Z}(a\tau + b)) \\ &= \frac{1}{c\tau + d} (\mathbb{Z}\tau \oplus \mathbb{Z}) \\ &= \frac{1}{c\tau + d} p^{-1}(\Lambda) \\ &= ((c\tau + d)p)^{-1}(\Lambda). \end{aligned}$$

We also check that this is an action, which is to say that

$$\begin{bmatrix} x & y \\ u & v \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau, p) = \begin{bmatrix} xa + yc & xb + yd \\ ua + vc & ub + vd \end{bmatrix} (\tau, p).$$

That is, we need that

$$\begin{aligned} & \left(\frac{x \left(\frac{a\tau+b}{c\tau+d} \right) + y}{u \left(\frac{a\tau+b}{c\tau+d} \right) + v}, \left(u \left(\frac{a\tau+b}{c\tau+d} \right) + v \right) (c\tau + d)p \right) = \\ & \left(\frac{(xa + yc)\tau + (xb + yd)}{(ua + vc)\tau + (ub + vd)}, ((ua + vc)\tau + (ub + vd))p \right) \end{aligned}$$

which amounts to a bit of algebra. Next, we show that this action is free and transitive. If (τ, p) and (σ, q) are in the fiber, then we have that

$$\mathbb{Z}\tau \oplus \mathbb{Z} = p^{-1}(\Lambda) = p^{-1}(q(\mathbb{Z}\sigma \oplus \mathbb{Z})) = z(\mathbb{Z}\sigma \oplus \mathbb{Z})$$

where $z \equiv p^{-1}q(1)$ is a non-zero complex scalar. This tells us that z and $z\sigma$ are in $\mathbb{Z}\tau \oplus \mathbb{Z}$, or, in other words, we have that

$$z = a\tau + b \quad \text{for } a, b : \mathbb{Z}, \text{ and}$$

$$z\sigma = c\tau + d \quad \text{for } c, d : \mathbb{Z}, \text{ so that}$$

$$\sigma = \frac{a\tau + b}{c\tau + d}.$$

We also know that $z\sigma$ and z generate $\mathbb{Z}\tau \oplus \mathbb{Z}$ (as an ordered basis), since multiplication by z is an isomorphism of abelian groups. Therefore, the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ must have determinant 1, since it transforms one ordered basis of this rank 2 free abelian group into another. \square

Example 12. Let Λ be a lattice in a 1-dimensional complex vector space V — that is, let $(V, \Lambda) : \mathcal{M}_{1,1}$. The pillowcase orbifold $\mathfrak{P}(\Lambda)$ associated to Λ is $(V/\Lambda) // \mathbf{O}(1)$ with the action of $\mathbf{O}(1) = \{-1, 1\}$ on the torus V/Λ given by $[v] \mapsto [-v]$. We can describe the action of $\mathbf{O}(1)$ on V/Λ by acting on V via

the map $L \mapsto L \otimes_{\mathbb{R}} V : \mathbf{BO}(1) \rightarrow \mathbf{BAut}_{\mathbf{BGL}_1(\mathbb{C})}(V)$, where the complex action on $L \otimes_{\mathbb{R}} V$ is induced by $i(\ell \otimes v) := \ell \otimes iv$. Therefore, we can construct the pillowcase orbifold as type of pairs consisting of a 1-dimensional real inner product space L and an element of the torus $((L \otimes_{\mathbb{R}} V)/(L \otimes_{\mathbb{R}} \Lambda)$:

$$\mathfrak{P}(\Lambda) := (L : \mathbf{BO}(1)) \times ((L \otimes_{\mathbb{R}} V)/(L \otimes_{\mathbb{R}} \Lambda)).$$

The next few examples are described as quotients in Section 1.6 of [ALR07].

Example 13. The Kummer surface K is the quotient $\mathbb{T}^4 // \mathbf{Gal}(\mathbb{C} : \mathbb{R})$ of a 4-torus $\mathbb{T}^4 := (U(1))^4$ with the action of $\mathbf{Gal}(\mathbb{C} : \mathbb{R}) := \{1, \sigma\}$ given by complex conjugation: $\sigma(z_1, z_2, z_3, z_4) := (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$.

To describe the action of $\mathbf{Gal}(\mathbb{C} : \mathbb{R})$ on \mathbb{T}^4 , we should first describe its action on $U(1)$, since the action on \mathbb{T}^4 is diagonal. To do this smoothly, we should choose a judicious delooping of $\mathbf{Gal}(\mathbb{C} : \mathbb{R})$. We can define an exemplar of $\mathbf{Gal}(\mathbb{C} : \mathbb{R})$ to be an algebraic closure of \mathbb{R} — or, to be a bit safer, a degree 2 algebraic extension of \mathbb{R} . The canonical exemplar is of course taken to be \mathbb{C} . If \mathbb{K} is any degree 2 algebraic extension of \mathbb{R} , then its Galois group $\mathbf{Gal}(\mathbb{K} : \mathbb{R})$ has at most two elements, one of which must be the identity. Call the other element σ ; for $\mathbb{K} \equiv \mathbb{C}$ this is of course complex conjugation. We may then define

$$\mathbb{S}^1(\mathbb{K}) := \{z : \mathbb{K} \mid z\sigma(z) = 1\}$$

and then $\mathbb{T}^4(\mathbb{K}) := \mathbb{S}^1(\mathbb{K})^4$. This gives us the desired action of $\mathbf{Gal}(\mathbb{C} : \mathbb{R})$ on $\mathbb{T}^4 \equiv \mathbb{T}^4(\mathbb{C})$. We may therefore define the Kummer surface as

$$K := (\mathbb{K} : \mathbf{BGal}(\mathbb{C} : \mathbb{R})) \times \mathbb{T}^4(\mathbb{K}).$$

Example 14. The teardrop orbifold $\mathbb{S}^2(n)$ for $n : \mathbb{N}$ ($n > 1$) may be constructed as $\mathbb{S}^3 // U(1)$ where $U(1)$ acts on $\mathbb{S}^3 := \{z : \mathbb{C}^2 \mid |z| = 1\}$ via $(z_1, z_2) \mapsto (\lambda z_1, \lambda^n z_2)$. We can describe this action as the map which sends $L : \mathbf{BU}(1)$ to $\mathbb{S}(L \oplus L^{\otimes n})$, where we consider $L \oplus L^{\otimes n} : \mathbf{BU}(2)$ as a 2-dimensional Hermitian vector space and define the unit sphere $\mathbb{S}(V)$ for $V : \mathbf{BU}(n)$ by $\mathbb{S}(V) := \{v : V \mid \langle v, v \rangle = 1\}$. Therefore, we may construct the teardrop as the type of pairs of a 1-dimensional Hermitian vector space L and a unit length element of $L \oplus L^{\otimes n}$:

$$\mathbb{S}^2(n) := (L : \mathbf{BU}(1)) \times \mathbb{S}(L \oplus L^{\otimes n}).$$

We can generalize this definition to the weighted projective spaces $W \mathbb{P}(n_1, \dots, n_k)$ (for a natural numbers n_1, \dots, n_k all coprime).

$$W \mathbb{P}(n_1, \dots, n_k) := (L : \mathbf{BU}(1)) \times \mathbb{S}(L^{\otimes n_1} \oplus \dots \oplus L^{\otimes n_k}).$$

Example 15. A large class of orbifolds which appear in practice may be constructed as quotients $\mathbb{T}^n // \Gamma$ where $\Gamma \subseteq \mathbf{GL}_n(\mathbb{Z})$ is a finite subgroup of $\mathbf{GL}_n(\mathbb{Z})$ acting on $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ via the action of $\mathbf{GL}_n(\mathbb{Z})$ on \mathbb{R}^n by matrix multiplication. The Kummer surface of Example 13 is one example of this sort of orbifold, as are the pillowcases.

Here is one general construction of this sort of orbifold. We may deloop $\mathbf{GL}_n(\mathbb{Z})$ by noting that this is the type of symmetries of the vector space \mathbb{R}^n which preserve the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$. This suggests

$$\mathbf{BGL}_n(\mathbb{Z}) := (V : \mathbf{BGL}_n(\mathbb{R})) \times \mathbf{Lattice}(V)$$

pointed at $(\mathbb{R}^n, \mathbb{Z}^n)$. We need to check that this is 0-connected, so let (V, Λ)

be a lattice in an n -dimensional real vector space. There is some isomorphism $p : V = \mathbb{R}^n$, and this identifies Λ with the lattice $p(\Lambda)$. Now choose generators for $p(\Lambda)$; this gives us an isomorphism $q : p(\Lambda) = \mathbb{Z}^n$ considered as abstract groups; however, we may extend q to an automorphism $\bar{q} : \mathbb{R}^n = \mathbb{R}^n$ of \mathbb{R}^n by noting that the standard generators of \mathbb{Z}^n form a basis for \mathbb{R}^n . The composite $\bar{q}^{-1} \circ p : V = \mathbb{R}^n$ is an identification of V with \mathbb{R}^n which sends Λ to \mathbb{Z}^n .

For a finite subgroup Γ of $\mathrm{GL}_n(\mathbb{Z})$, we can now define $\mathbf{B}\Gamma$ out of $\mathbf{BGL}_n(\mathbb{Z})$ using Proposition 2.3.10. We can then define

$$\mathbb{T}^n // \Gamma \equiv ((V, \Lambda, T) : \mathbf{B}\Gamma) \times (V/\Lambda).$$

Explicitly, the points of $\mathbb{T}^n // \Gamma$ consist of an n -dimensional vector space V , a lattice Λ in V , a Γ -subtorsor T of the space of frames of V , and an point on the torus V/Λ .

4.3 Cohesion and the Homotopy Theory of Orbifolds

We will now move beyond bare homotopy type theory and into *modal* homotopy type theory. In this section, we will briefly survey the homotopy theory of orbifolds, and in particular their covering theory. This means, in particular, defining the *homotopy type* of an orbifold.

Defining the homotopy type of a type is luckily quite straightforward. We would like to be able to identify points by giving continuous deformations between them. In other words, we should have a sort of quotient map $(-)^{\mathcal{J}} : X \rightarrow \int X$ sending any point in X to its homotopy class in the homotopy type $\int X$, and

given a path $\gamma : \mathbb{R} \rightarrow X$ we should get an identification $\gamma(0)^f = \gamma(1)^f$ in $\mathfrak{f}X$. In other words, we want to nullify maps out of \mathbb{R} , or more precisely, we want to localize our type X at the terminal map $\mathbb{R} \rightarrow *$. The theory of localizations in HoTT is developed in [RSS:Modalities.in.HoTT] and [Lop], and we may use this theory to define the \mathfrak{f} modality.

Definition 4.3.1 (Definition 9.6 [Shu18a]). We define the *shape* modality \mathfrak{f} to be localization at the type of real numbers \mathbb{R} .³ The *n-shape* \mathfrak{f}_n modality is the localization at \mathbb{R} and the homotopy $(n + 1)$ -sphere S^{n+1} , and its modal types are those types which are both \mathfrak{f} -modal and n -truncated. A definition of this localization can be as Definition 9.6 of [Shu18a] or (for a general localization) in Section 2.2 of [RSS:Modalities.in.HoTT].

This definition is short and sweet, but without a supporting apparatus it is unfortunately underspecified. That supporting apparatus is the *cohesive homotopy type theory* which adds a *comodality* \flat that strips types of their spatial structure. For this reason, we begin this section with a review of cohesive homotopy type theory in Section 4.3.1.

In ??, we will then review the cohesive covering theory developed in Chapter 3. We will use this covering theory to quickly compute the homotopy type of $\mathcal{M}_{1,1}$ in Theorem 4.3.7: it is a $\mathrm{BSL}_2(\mathbb{Z})$.

Then, in ??, we will take a brief look at maps between good orbifolds. With the modal approach to covering theory and the HoTT approach to group theory, we will see that maps into a configuration space $X^n // n!$ correspond to maps out

³For now, I will leave ambiguous which type of real numbers we are localizing at. In Section 4.4.1, we will see axioms for the type of *smooth reals* which will play the role of the real numbers in synthetic differential geometry.

of n -fold covers (Proposition 4.3.13), and that maps into a quotient $X // \Gamma$ correspond to Γ -equivariant maps out of Γ -principal bundles (Proposition 4.3.14).

4.3.1 A review of cohesive homotopy type theory.

To understand what cohesion adds to type theory, let's think in terms of models for a bit. We intend to interpret our type theory in a topos of smooth stacks, such as the Dubuc ∞ -topos. As with any ∞ -topos, this topos of smooth stacks lives over the topos of homotopy types (stacks on the point) by its global sections functor.

$$\begin{array}{ccc} & \text{smooth stacks} & \\ & \uparrow & \\ \text{shape} \left(& \text{locally constant stacks} & \right) \text{global sections} \\ & \downarrow & \\ & \text{homotopy types} & \end{array}$$

Left adjoint to the global sections functor is the functor sending a homotopy type X the stack of locally constant sections valued in X . In our case, this functor really is the inclusion of the constant stacks — it is fully faithful. Then there is a further left adjoint: this sends a stack to its *shape* (in the sense of Lurie). If that stack is represented by a manifold, then its shape will be the homotopy type of that manifold. In the terminology of higher toposes, the topos of smooth stacks is ∞ -connected and locally ∞ -connected.⁴

Cohesive HoTT [Shu18a] formalizes this relationship between smooth stacks and homotopy types by adding *crisp* variables to homotopy type theory. Every expression in HoTT occurs in a *context*, which is a list of the free variables in

⁴It is furthermore *local*, in that the global sections functor also admits a *right* adjoint. This right adjoint is the inclusion of codiscretes. We will only need this other adjoint modality briefly, for Lemma 4.5.17, and so we will not dwell on it here.

the expression, together with the type these variables have. To define a function $f : X \rightarrow Y$, we construct $f(x) : Y$ in the context of a free variable $x : X$. Since any such function $f : X \rightarrow Y$ will be interpreted as a map of stacks — and if these stacks are represented by manifolds, therefore potentially as a smooth map of manifolds — the dependence of an expression $f(x)$ on its free variable $x : X$ implies a sort *smoothness*. But not all dependencies in mathematics should be smooth; sometimes an expression $f(x)$ should vary *discontinuously* in x .

Shulman allows for discontinuous dependency with a new type of free variable declaration: *crisp* variables $x :: X$. To say that $f(x) : Y$ for a *crisp variable* $x :: X$ is to say that $f(x)$ depends on x in a (possibly) discontinuous way. If all of the free variables in an expression f are crisp, we say that f is crisp. Importantly, crisp variables must also have crisp type. In particular, every expression with no free variables is *crisp*. For example, \mathbb{Z} and \mathbb{R} are crisp types, and $0 : \mathbb{Z}$ and $\pi : \mathbb{R}$ are crisp elements. While the expression $x^2 + 1 : \mathbb{R}$ with $x : \mathbb{R}$ is not crisp, the function $(x \mapsto x^2 + 1) : \mathbb{R} \rightarrow \mathbb{R}$ is crisp since the variable x has been bound.

One way to ensure that crisp variables behave discontinuously is the crisp law of excluded middle.

Axiom 3 ([Shu18a]). For any crisp proposition $P :: \mathbf{Prop}$, either P holds or $\neg P$ holds.

The crisp law of excluded middle lets us define functions of crisp variables by cases. For example, if $x :: \mathbb{R}$ is a crisp variable, then the proposition $(x > 0) : \mathbf{Prop}$ is also crisp (since every free variable in it is crisp). Therefore, either

$(x > 0)$ or $\neg(x > 0)$. We can therefore define the real number

$$f(x) := \begin{cases} -1 & \text{if } x > 0 \\ 1 & \text{otherwise} \end{cases}$$

by cases. Clearly, $f(x)$ is a discontinuous function of x which we were able to define using law of excluded middle. Without the law of excluded middle, it is impossible to define discontinuous functions $\mathbb{R} \rightarrow \mathbb{R}$.

An expression such as $f(x) : Y$ above depending on a crisp variable $x :: X$ can't give a function $X \rightarrow Y$, because $X \rightarrow Y$ is supposed to be the type of smooth functions (or, really, the mapping stack). To internalize the crisp variables, we can add a type $\flat X$ which is “freely generated by the crisp variables of X ” in the sense that $f(x) : Y$ depending on $x :: X$ gives rise to a function $\flat X \rightarrow Y$. We can think of $\flat X$ as X stripped of its smooth structure; in terms of stacks, $\flat X$ is the stack constant at the global sections of X .

More formally, for any *crisp* type X we have a type $\flat X$ and for every crisp $x :: X$ we have $x^\flat : \flat X$. We then have the following induction principle: if $C : \flat X \rightarrow \mathbf{Type}$ is any type family, and if for $x :: X$ we have $c(x) : C(x^\flat)$, then for any $u : \flat X$ we have an element

$$(\text{let } x^\flat := u \text{ in } c(x)) : C(u).$$

Furthermore, if $u \equiv y^\flat$ for $y :: X$, we have

$$(\text{let } x^\flat := y^\flat \text{ in } c(x)) \equiv c(y).$$

We can define a counit $(-)_\flat : \flat X \rightarrow X$ by $u_\flat := (\text{let } x^\flat := u \text{ in } x)$. Less formally, we might say that $(-)_\flat$ is defined by $(x^\flat)_\flat := x$. Given any smooth function

$f : X \rightarrow Y$, we can precompose by $(-)_b : \flat X \rightarrow X$ to get its underlying discontinuous function $f \circ (-)_b : \flat X \rightarrow Y$.

A type X should be discrete when any discontinuous function out of it is already smooth. That is, a type X should be discrete precisely when precomposition by $(-)_b$ gives an equivalence of $X \rightarrow Y$ with $\flat X \rightarrow Y$ for any type Y . This will happen precisely when $(-)_b$ is an equivalence.

Definition 4.3.2 ([Shu18a]). A crisp type $X :: \mathbf{Type}$ is (*crisply*) *discrete* if $(-)_b : \flat X \rightarrow X$ is an equivalence.

In terms of stacks, a crisply discrete type is a constant stack — that is, one for which the canonical map from the constant stack at its global sections into it is an equivalence.

In order to relate the liminal spatiality of crisp variables to the concrete topology of the reals, we will relate the discreteness of \flat with a discreteness measured by \mathbb{R} .

Axiom 4 ($\mathbb{R} \flat$: [Shulman:RealCohesion]). A crisp type $X :: \mathbf{Type}$ is discrete if and only if the inclusion $\mathbf{const} : X \rightarrow (\mathbb{R} \rightarrow X)$ of constant paths is an equivalence. That is, X is discrete if and only if every path $\gamma : \mathbb{R} \rightarrow X$ is constant.

$$(\flat X \simeq X) \iff (X \simeq \int X).$$

This axiom justifies extending the definition of discreteness to types which aren't crisp. We say a type X is discrete just when $\mathbf{const} : X \rightarrow (\mathbb{R} \rightarrow X)$ is an equivalence, or when it is \mathbb{R} -null. By construction, this is precisely when $(-)^{\int} : X \rightarrow \int X$ is an equivalence, so we see that a type is discrete if and only if it is \int -modal.

Let's end this review of cohesive homotopy type theory by quoting Theorem 9.15 of [Shu18a].

Theorem 4.3.3 (Theorem 9.15 [Shu18a]). *For any crisp types X and Y , we have an equivalence*

$$\flat(X \rightarrow \flat Y) \simeq \flat(\int X \rightarrow Y)$$

exhibiting the adjointness of \int and \flat .

4.3.2 Modal covering theory.

Let's take a minute to recall the modal covering theory developed in Section 9 of Chapter 3. A covering map $\pi : C \rightarrow X$ satisfies the *unique path lifting* property:

$$\begin{array}{ccc} * & \xrightarrow{\forall} & C \\ 0 \downarrow & \exists! \nearrow & \downarrow \pi \\ \mathbb{R} & \xrightarrow{\forall} & X \end{array}$$

For any path $\gamma : \mathbb{R} \rightarrow X$ and $c : C$ over $\gamma(0)$, there is a unique lift $\tilde{\gamma} : \mathbb{R} \rightarrow C$ of γ . Furthermore, if $f : A \rightarrow B$ is any map which induces an equivalence on fundamental groupoids, then $\pi : X \rightarrow Y$ lifts uniquely on the right against f . We can use this property to define the notion of covering using the fundamental groupoid modality \int_1 , which is given by localization both at \mathbb{R} and the homotopy 2-sphere S^2 and whose modal types are discrete groupoids.

We will use the notion of a *modal étale* map, studied in [CR21].

Definition 4.3.4 ([CR21]). For a modality \diamond , a map $f : X \rightarrow Y$ is \diamond -étale

when the modal naturality square

$$\begin{array}{ccc} X & \xrightarrow{(-)^\diamond} & \diamond X \\ f \downarrow & & \downarrow \diamond f \\ Y & \xrightarrow{(-)^\diamond} & \diamond Y \end{array}$$

is a pullback.

Definition 4.3.5 (Definition 9.1 Chapter 3). A map $\pi : C \rightarrow X$ is a *covering* if it is \int_1 -étale and its fibers are sets.

The justification of these definitions comes from Theorem 7.2 of [CR21] which shows that \diamond -equivalences and \diamond -étale maps form an orthogonal factorization system.

Theorem 4.3.6 (Theorem 7.2 [CR21]). *A map $f : X \rightarrow Y$ is \diamond -étale if and only if it lifts uniquely on the right against all \diamond -equivalences — maps $g : A \rightarrow B$ for which $\diamond g$ is an equivalence. Furthermore, the \diamond -étale maps and the \diamond -equivalences form the right and left classes respectively of an orthogonal factorization system.*

Finally, we need a theorem from Chapter 3. We can characterize coverings of X by the monodromy action of fundamental groupoid $\int_1 X$ of X .

Theorem 4.3.7 (Theorem 9.2 Chapter 3). *For a type X , let $\mathbf{Cov}(X)$ denote the type of coverings of X . Then we have an equivalence*

$$\mathbf{Cov}(X) \simeq (\int_1 X \rightarrow \mathbf{Type}_{\int_0})$$

between coverings of X and discrete set valued functions on the fundamental groupoid of X . Given such a map $E : \int_1 X \rightarrow \mathbf{Type}_{\int_0}$, the associated covering is

the first projection from $C := (x : X) \times E_{x^{f_1}}$.

Remark 4.3.8. Note that if X is connected in the sense that its fundamental groupoid $\int_1 X$ is 0-connected, and $x : X$ is any point, then $\int_1 X$ pointed at x^{f_1} is a delooping $\mathbb{B}\pi_1 X$ of the fundamental group $\pi_1 X$ (based at x). In this case, Theorem 4.3.7 specializes to the usual theorem that coverings of X are equivalent to actions $\mathbb{B}\pi_1 X$ on discrete sets, with the action given by monodromy.

4.3.3 Coverings of orbifolds.

With all this review out of the way, we can discuss the homotopy theory of orbifolds. We can begin by calculating the homotopy type of $\mathcal{M}_{1,1}$.

Theorem 4.3.9. *The homotopy type of $\mathcal{M}_{1,1}$ (Definition 4.2.2) is a $\mathbb{B}\mathrm{SL}_2(\mathbb{Z})$, and $q : \mathfrak{h} \rightarrow \mathcal{M}_{1,1}$ is the universal cover of $\mathcal{M}_{1,1}$.*

Proof. By Proposition 4.2.4, $q : \mathfrak{h} \rightarrow \mathcal{M}_{1,1}$ has fibers which are $\mathrm{SL}_2(\mathbb{Z})$ torsors and so is the fiber of a map $\mathrm{fib}_q : \mathcal{M}_{1,1} \rightarrow \mathrm{Tors}_{\mathrm{SL}_2(\mathbb{Z})}$. Since $\mathrm{SL}_2(\mathbb{Z})$ is a crisply discrete group, $\mathrm{Tors}_{\mathrm{SL}_2(\mathbb{Z})}$ is also discrete by Theorem 5.9 of Chapter 3. Since \mathfrak{h} is \mathfrak{f} -connected, we see that the map $\mathrm{fib}_q : \mathcal{M}_{1,1} \rightarrow \mathrm{Tors}_{\mathrm{SL}_2(\mathbb{Z})}$ is a \mathfrak{f} -connected map into a \mathfrak{f} -modal type, making it a \mathfrak{f} -unit.

To see that $q : \mathfrak{h} \rightarrow \mathcal{M}_{1,1}$ is the universal cover, note that it is the fiber of $\mathrm{fib}_q : \mathcal{M}_{1,1} \rightarrow \mathrm{Tors}_{\mathrm{SL}_2(\mathbb{Z})}$ over the canonical exemplar. Since fib_q is a \mathfrak{f}_1 -unit, this exhibits q as the universal cover (see Definition 9.3 and Theorem 9.4 of Chapter 3). □

The argument in Theorem 4.3.9 is completely general.

Theorem 4.3.10. *Let $X^{\circ-} : \mathbf{B}\Gamma \rightarrow \mathbf{Type}$ be an action of a discrete higher group Γ (in the sense that $\mathbf{B}\Gamma$ is \mathfrak{f} -modal) on a \mathfrak{f} -connected type X . Then the first projection*

$$\text{fst} : X // \Gamma \rightarrow \mathbf{B}\Gamma$$

is a \mathfrak{f} -unit.

Proof. The map $\text{fst} : X // \Gamma \rightarrow \mathbf{B}\Gamma$ is a map whose fibers are identifiable with the \mathfrak{f} -connected type X , and it is therefore a \mathfrak{f} -connected into the \mathfrak{f} -modal type $\mathbf{B}\Gamma$. Therefore, it is a \mathfrak{f} -unit. \square

As a corollary, we see that

$$\mathfrak{f}(\mathbb{R}^n // \Gamma) \simeq \mathbf{B}\Gamma$$

when $\mathbb{R}^n // \Gamma$ is a coordinate patch for a finite subgroup $\Gamma \subseteq \mathbf{O}(n)$ as in Example 9. We can be even more general so long as our higher group is crisp.

Theorem 4.3.11. *Let $X^{\circ-} : \mathbf{B}G \rightarrow \mathbf{Type}$ be an action of a crisp higher group G on a type X . Then there is a unique action of $\mathfrak{f}G$ on $\mathfrak{f}X$ so that*

$$\mathfrak{f}(X // G) \simeq \mathfrak{f}X // \mathfrak{f}G.$$

Proof. Consider the map $t \mapsto \mathfrak{f}X^{\circ t} : \mathbf{B}G \rightarrow \mathbf{Type}_{\mathfrak{f}}$ sending an exemplar t of G to the homotopy type of X twisted by t . This lands in the \mathfrak{f} -separated type of discrete types $\mathbf{Type}_{\mathfrak{f}}$ and so factors uniquely through the \mathfrak{f} -separated unit $(-)^{\mathfrak{f}(1)} : \mathbf{B}G \rightarrow \mathfrak{f}^{(1)}\mathbf{B}G$. But by the proof of Theorem 8.9 of Chapter 3, we see that the factorization $\mathfrak{f}^{(1)}\mathbf{B}G \rightarrow \mathfrak{f}\mathbf{B}G$ of the \mathfrak{f} -unit of $\mathbf{B}G$ is an equivalence, so

that $\mathfrak{J}X^{\circ-} : \mathbf{BG} \rightarrow \mathbf{Type}_f$ factors through $\mathfrak{J}\mathbf{BG}$; we take this as our action of $\mathfrak{J}G$ on $\mathfrak{J}X$, since $\mathfrak{J}\mathbf{BG}$ deloops $\mathfrak{J}G$ by Theorem 8.9 of Chapter 3.

It remains to show that $\mathfrak{J}X // \mathfrak{J}G$ is $\mathfrak{J}(X // G)$. For $t : \mathbf{BG}$, we have a \mathfrak{J} -unit $(-)^{\mathfrak{J}} : X^{\circ t} \rightarrow \mathfrak{J}X^{\circ t}$. We can assemble these into a map

$$(t, x) \mapsto (t^{\mathfrak{J}}, x^{\mathfrak{J}}) : X // G \rightarrow \mathfrak{J}X // \mathfrak{J}G.$$

Since pair types of modal types are modal, $\mathfrak{J}X // \mathfrak{J}G$ is \mathfrak{J} -modal; so it suffices to show that this map is \mathfrak{J} -connected. But it is the pairing of \mathfrak{J} -connected maps, so by Lemma 1.39 of [RSS17b], it is \mathfrak{J} -connected. \square

As a corollary, we can compute the homotopy types of a few more of our examples.

Corollary 4.3.12. Let Γ be a crisp finite subgroup of $\mathbf{GL}_n(\mathbb{Z})$. Then

$$\mathfrak{J}(\mathbb{T}^n // \Gamma) \simeq \mathbf{B}\tilde{\Gamma}$$

where $\tilde{\Gamma}$ is a *crystallographic group* extending Γ :

$$0 \rightarrow \mathbb{Z}^n \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 0.$$

Proof. Consider the fiber sequence

$$\mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{T}^n // \Gamma \xrightarrow{\text{fst}} \mathbf{B}\Gamma$$

where we recall from Example 15 that $\mathbb{T}^n // \Gamma := ((V, \Lambda, T) : \mathbf{B}\Gamma) \times (V/\Lambda)$ and that the canonical exemplar of Γ is $(\mathbb{R}^n, \mathbb{Z}^n, \Gamma)$. By Theorem 7.7 of Chapter 3,

the projection $\text{fst} : \mathbb{T}^n // \Gamma \rightarrow \mathbf{B}\Gamma$ is a \mathcal{J} -fibration and therefore

$$\mathcal{J}(\mathbb{R}^n / \mathbb{Z}^n) \rightarrow \mathcal{J}(\mathbb{T}^n // \Gamma) \rightarrow \mathcal{J}\mathbf{B}\Gamma$$

is a fiber sequence. Since Γ is a crisply discrete group, $\mathbf{B}\Gamma$ is crisply discrete by Theorem 5.9 of Chapter 3, and $\mathcal{J}(\mathbb{R}^n / \mathbb{Z}^n) = \mathcal{J}((\mathbb{S}^1)^n) = \mathbf{B}\mathbb{Z}^n$, so we have a fiber sequence

$$\mathbf{B}\mathbb{Z}^n \rightarrow \mathcal{J}(\mathbb{T}^n // \Gamma) \rightarrow \mathbf{B}\Gamma.$$

Now, we may point $\mathcal{J}(\mathbb{T}^n // \Gamma)$ at $\mathbf{pt} := (\mathbb{R}^n, \mathbb{Z}^n, \Gamma, [0])^{\mathcal{J}}$; it remains to show that $\mathcal{J}(\mathbb{T}^n // \Gamma)$ is 0-connected. Since $\mathbb{T}^n // \Gamma$ is crisp, $\|\mathcal{J}(\mathbb{T}^n // \Gamma)\|_0 = \int_0(\mathbb{T}^n // \Gamma)$ by Proposition 4.5 of Chapter 3. But $\mathbf{B}\Gamma$ is discrete and 0-connected, so it is also \int_0 -connected; likewise, the torus V/Λ is \int_0 -connected for any vector space V and lattice Λ in it since it is surjected by the \int_0 -connected type V . Therefore, $\mathbb{T}^n // \Gamma$ is \int_0 -connected as the sum of \int_0 -connected types.

Defining $\tilde{\Gamma} := \pi_1(\mathbb{T}^n // \Gamma)$, which in this case is equivalent to $\Omega(\mathcal{J}(\mathbb{T}^n // \Gamma), \mathbf{pt})$, we see that $\mathcal{J}(\mathbb{T}^n // \Gamma)$ is a $\mathbf{B}\tilde{\Gamma}$ and we have an extension

$$0 \rightarrow \mathbb{Z}^n \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 0. \quad \square$$

4.3.4 Maps between orbifolds

Another upside of working in homotopy type theory is that correct notion of map between orbifolds is simply a function, which can be defined by its action on points in the usual way. That is, if \mathcal{X} and \mathcal{Y} are orbifolds, then the mapping space between them is the space of functions $\mathcal{X} \rightarrow \mathcal{Y}$. In particular, since we have defined our orbifolds in terms of their points, it is fairly straightforward to understand what it means to map into them.

Proposition 4.3.13. Let X be a type, and consider the configuration space $X^n // n!$. A function $f : A \rightarrow X^n // n!$ is equivalently an n -fold cover $\pi : C_f \rightarrow A$ together with a map $\tilde{f} : C_f \rightarrow X$.

Proof. We may calculate directly:

$$\begin{aligned}
(A \rightarrow X^n // n!) &\simeq (A \rightarrow (F : \mathbf{B} \mathbf{Aut}(\underline{n})) \times X^F) \\
&\simeq (C : A \rightarrow \mathbf{B} \mathbf{Aut}(\underline{n})) \times ((a : A) \rightarrow X^{C_a}) \\
&\simeq ((C, \pi) : \mathbf{Cov}(A)) \times ((a : A) \rightarrow \|\mathbf{fib}_\pi(a) \simeq \underline{n}\|) \times ((a : A) \rightarrow X^{\mathbf{fib}_\pi(a)}) \\
&\simeq ((C, \pi) : \mathbf{Cov}(A)) \times ((a : A) \rightarrow \|\mathbf{fib}_\pi(a) \simeq \underline{n}\|) \times ((a : A) \times \mathbf{fib}_\pi(a) \rightarrow X) \\
&\simeq ((C, \pi) : \mathbf{Cov}(A)) \times ((a : A) \rightarrow \|\mathbf{fib}_\pi(a) \simeq \underline{n}\|) \times (C \rightarrow X).
\end{aligned}$$

We made use of Theorem 4.3.7 and the fact that finite sets are discrete. \square

We can prove something slightly more general but along the same lines.

Proposition 4.3.14. Suppose that a (higher) group Γ acts on a type X . Then maps $f : A \rightarrow X // \Gamma$ correspond to Γ -principal bundles $\pi : P \rightarrow A$ together with a Γ -equivariant map $P \rightarrow X$.

Proof. We may calculate directly:

$$\begin{aligned}
(A \rightarrow X // \Gamma) &\equiv (A \rightarrow (T : \mathbf{Tors}_\Gamma) \times X^{\circ T}) \\
&\simeq ((P : \mathbf{Type}) \times (\pi : P \rightarrow A) \times ((a : A) \rightarrow \mathbf{Tors}_\Gamma(\mathbf{fib}_\pi(a))) \times ((a : A) \rightarrow X^{\circ \mathbf{fib}_\pi(a)})) \\
&\simeq \left\{ \begin{array}{l} (P : \mathbf{Type}) \times (\pi : P \rightarrow A) \times ((a : A) \rightarrow \mathbf{Tors}_\Gamma(\mathbf{fib}_\pi(a))) \\ \times (((T, a, p) : (T : \mathbf{Tors}_\Gamma) \times (a : A) \times (\mathbf{fib}_\pi(a) = T)) \rightarrow X^{\circ \mathbf{fib}_\pi(a)}) \end{array} \right. \\
&\simeq \left\{ \begin{array}{l} (P : \mathbf{Type}) \times (\pi : P \rightarrow A) \times ((a : A) \rightarrow \mathbf{Tors}_\Gamma(\mathbf{fib}_\pi(a))) \\ \times ((T : \mathbf{Tors}_\Gamma) \rightarrow ((a, p) : (a : A) \times (\mathbf{fib}_\pi(a) = T)) \rightarrow X^{\circ \mathbf{fib}_\pi(a)}) \end{array} \right. \\
&\simeq \left\{ \begin{array}{l} (P : \mathbf{Type}) \times (\pi : P \rightarrow A) \times ((a : A) \rightarrow \mathbf{Tors}_\Gamma(\mathbf{fib}_\pi(a))) \\ \times \mathbf{Hom}_\Gamma(P, X) \end{array} \right.
\end{aligned}$$

The equivalence $\mathbf{Hom}_\Gamma(P, X) \simeq ((T : \mathbf{Tors}_\Gamma) \rightarrow ((a, p) : (a : A) \times (\mathbf{fib}_\pi(a) = T)) \rightarrow X^{\circ \mathbf{fib}_\pi(a)})$ is either a definition (for general higher groups Γ and types X) or a theorem (for ordinary groups Γ and sets X). To understand this last step, first note that the action of Γ on the total space P of a Γ -bundle $\pi : P \rightarrow A$ may be described by the map

$$(T : \mathbf{Tors}_\Gamma) \mapsto P^{\circ T} := (a : A) \times (\mathbf{fib}_\pi(a) = T).$$

When we apply this function to $\Gamma : \mathbf{Tors}_\Gamma$, we get may calculate that $P^{\circ\Gamma}$ is

$$\begin{aligned} ((a : A) \times (\mathbf{fib}_\pi(a) = \Gamma)) &\simeq ((a : A) \times \mathbf{fib}_\pi(a)) \\ &\simeq P, \end{aligned}$$

since a Γ -equivariant identification with Γ is determined by an element. Finally, a Γ -equivariant map $P \rightarrow X$ is equivalently a map $(T : \mathbf{Tors}_\Gamma) \rightarrow (P^{\circ T} \rightarrow X^{\circ T})$, which is what appears at the end of the calculation above. \square

4.4 Synthetic Differential Geometry

Synthetic differential geometry began in 1967 with a series of lectures by Lawvere in which he attempted to give a topos-theoretic foundation (in the sense of a distillation of established practices) for the sorts of differential geometry used in engineering and physics that made explicit use of infinitesimals [Law; Law80]. Lawvere was influenced by the use of nilpotent infinitesimal elements appearing in non-reduced schemes in Grothendieck's reformulated algebraic geometry. The field of synthetic differential geometry was further developed by Kock, Dubuc, Bunge, Penon, Lavendhomme, Reyes, Moerdijk, and others. An introductory text is [Bel08]; reference texts are [Koc06] and [Lav96]. See also [BGSL18]. For topos theoretic models, see [MR90].

The main idea of synthetic differential geometry is to formalize the common arguments using numbers ε which are so small that their square ε^2 is negligible. If we say that $\varepsilon^2 = 0$ is actually 0, then such numbers are *nilsquare infinitesimals*. The most crucial axiom of SDG, known as the Kock-Lawvere axiom, implies that for any function $f : \mathbb{R} \rightarrow \mathbb{R}$, there is a unique function $f' : \mathbb{R} \rightarrow \mathbb{R}$

so that for all $x : \mathbb{R}$ and $\varepsilon^2 = 0$, we have:

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon.$$

This axiom implies that every function $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth, and for that reason the real numbers \mathbb{R} of SDG are known as the *smooth reals*.

Of course, there cannot be any non-zero infinitesimals, at least if \mathbb{R} is to be a field where non-zero elements are invertible. If ε was non-zero, then it would be invertible, and we could conclude that

$$1 = \varepsilon^2 \cdot \frac{1}{\varepsilon^2} = 0 \cdot \frac{1}{\varepsilon^2} = 0.$$

Since 1 does not equal 0, we may conclude that if $\varepsilon^2 = 0$, then ε is *not non-zero*.

Classically, we could conclude from this that ε must be 0; but this follows from a separate law of logic, *double negation elimination*, which states that if a proposition is not false, then it is true. We do not have to take this law of logic as an axiom — it does not follow from the rules of type theory. With our extra logical wiggle room, we can have a non-trivial theory of infinitesimal calculus. In fact, we will follow Penon in defining an *infinitesimal* to be a number $\varepsilon : \mathbb{R}$ which is not non-zero in Definition 4.4.13.

So far, synthetic differential geometry has only been studied in 1-topos theory. We will see that the same axioms (introduced in Section 4.4.1), and the same definitions (such as that of microlinearity, Definition 4.4.22) give us access to the differential structure of orbifolds and other higher types when interpreted in cohesive homotopy type theory.

The SDG literature has settled on the notion of *microlinearity* (Definition 4.4.22) as a good notion of “smooth space” for the purposes of proving theorems in SDG. Beautifully, the definition of microlinearity generalizes smoothly from sets to higher types. A type is microlinear when, roughly speaking, it has the same infinitesimal lifting properties as \mathbb{R} . In Section 4.4.2, we review the definition of microlinearity, and prove in Theorem 4.4.29 that the tangent spaces of microlinear types have the structure of \mathbb{R} -modules. While this theorem is standard for microlinear sets, we prove it in such a way that it generalizes to higher types. Specifically, we show that tangent spaces of these higher types — which themselves may be higher types and not sets — are models of the Lawvere theory of \mathbb{R} -modules and so have a \mathbb{R} -module structure which is coherent up to higher identifications.

In Section 4.4.3, we will prove our main theorem of this section concerning the descent of microlinearity along \mathfrak{S} -étale maps. Theorem 4.4.42 states that if X is microlinear and $f : X \rightarrow Y$ is surjective and \mathfrak{S} -étale, then Y is also microlinear. This will allow us to give examples in Section 4.5 of higher microlinear types, such as étale groupoids (Theorem 4.5.34) and the quotients of microlinear types by higher groups (Theorem 4.5.26). Here, \mathfrak{S} is the *crystalline* modality given by localizing at the type of infinitesimals in \mathbb{R} (Definition 4.4.30). The type $\mathfrak{S}X$ is sometimes known as the *de Rham stack* of X , and a map $f : X \rightarrow Y$ is \mathfrak{S} -étale when the it’s \mathfrak{S} -naturality square is a pullback:

$$\begin{array}{ccc} X & \xrightarrow{(-)^{\mathfrak{S}}} & \mathfrak{S}X \\ f \downarrow & \lrcorner & \downarrow \mathfrak{S}f \\ Y & \xrightarrow{(-)^{\mathfrak{S}}} & \mathfrak{S}Y \end{array}$$

This is a useful and entirely modal notion of local diffeomorphism. In Proposition 4.4.34, we will see that any \mathfrak{S} -étale map $f : X \rightarrow Y$ induces an isomorphism $f_* : T_x X \rightarrow T_{fx} Y$ on tangent spaces, and later, in Corollary 4.5.19, we will see that this is an equivalent condition for f to be \mathfrak{S} -étale so long as X and Y are manifolds.

4.4.1 Axioms of synthetic differential geometry

Synthetic differential geometry proceeds by axiomatizing the *smooth real line*, which we will denote by \mathbb{R} . We will use the naming convention of [BGSL18] (except for Postulates E and S, which do not appear there, and the Covering Property, which is due to Bunge and Dubuc [BD87]).

Axiom 5. The *smooth real line* \mathbb{R} is a ring satisfying the following axioms:

- (Postulate K) \mathbb{R} is a field in the sense of Kock: $0 \neq 1$ and for any $n : \mathbb{N}$ and $x : \mathbb{R}^n$, we have

$$\neg \left(\bigwedge_{i=1}^n (x_i = 0) \right) \rightarrow \bigvee_{i=1}^n (x_i \text{ is invertible}).$$

Taking the case $n \equiv 1$ tells us that if $x \neq 0$ then x is invertible (and therefore the invertible elements coincide with the non-zero elements of \mathbb{R}). Taking the case $n \equiv 2$ tells us that \mathbb{R} is a local ring in the sense that if $x + y$ is invertible, then one of x or y is invertible.

- (Postulate O) \mathbb{R} is strictly ordered: there is a binary relation $<$ on \mathbb{R} satisfying the following axioms:

1. $1 > 0$, and if $x > 0$ and $y > 0$, then $x + y > 0$ and $xy > 0$.

2. It is never the case that $x > x$.
 3. If $x > y$, then either $x > z$ or $z > y$ for any z .
 4. If $x \neq 0$, then $x < 0$ or $x > 0$.⁵
 5. (Archimedean law) For any $x : \mathbb{R}$, there is an $n : \mathbb{N}$ with $x < n$.
- (Postulate E) There is an isomorphism of ordered groups $\exp : \mathbb{R} \simeq (0, \infty) : \log$ between the additive group of real numbers and the multiplicative group of positive real numbers.
 - (Covering Property) Let $A, B \subseteq \mathbb{R}$ be subsets of \mathbb{R} . If $A \cup B = \mathbb{R}$, then for every $x : \mathbb{R}$, either there is an $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq A$, or there is an $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq B$.
 - (Principle of Constancy) Let $f : \mathbb{R} \rightarrow \mathbb{R}$. If for all $x : \mathbb{R}$ and $\varepsilon : \mathbb{R}$ with $\varepsilon^2 = 0$, $f(x + \varepsilon) = f(x)$, then f is constant.
 - (Postulate W) The crisp infinitesimal varieties (Definition 4.4.11) and the type \mathfrak{D} of infinitesimals in \mathbb{R} (Definition 4.4.13) are *tiny* (Definition 4.7.1).
 - (Postulate J) The Kock-Lawvere axiom: For every Weil algebra W over \mathbb{R} , the evaluation map

$$w \mapsto \varphi \mapsto \varphi(w) : W \rightarrow (\mathrm{Spec}_{\mathbb{R}} W \rightarrow \mathbb{R})$$

is an isomorphism. We will explain these terms and the consequences of this axiom shortly.

⁵At this point in [MBL18], the authors have the axiom $\neg(\bigwedge_{i=1}^n x_i = 0) \rightarrow \bigvee_{i=1}^n (x_i < 0 \vee x_i > 0)$, but in light of Postulate K the axiom we are using here is equivalent.

Let's explain Postulate J, which is the axiom which underlies the differential geometric aspects of synthetic differential geometry. To do this, we need to understand the notion of a *Weil algebra*.

Definition 4.4.1 (Standard, see [Lav96]). Let R be a ring. A *Weil algebra* over R is an augmented finitely presented R -algebra $\pi : W \rightarrow R$ whose augmentation ideal $\ker \pi$ is finitely generated and nilpotent. The category of Weil algebras \mathbf{Weil} is the full subcategory of the augmented R -algebras spanned by the Weil algebras.

Remark 4.4.2. Note that a Weil algebra is something entirely different from a Weyl algebra.

Any Weil algebra may be put in a standard form as the quotient of a polynomial algebra where the augmentation is given by evaluating at 0.

Lemma 4.4.3 (Standard). Any augmented algebra with W finitely presented $\pi : W \rightarrow \mathbb{R}$ is merely equivalent to a augmented algebra of the form $\mathbf{ev}_0 : \mathbb{R}[x_1, \dots, x_n]/(f_1, \dots, f_m) \rightarrow \mathbb{R}$ with augmentation given by sending each x_i to 0.

Proof. This is a quick change of variables. By hypothesis, W is finitely presented as a \mathbb{R} -algebra, so it is of the form $\mathbb{R}[y_1, \dots, y_n]/(g_1, \dots, g_m)$. Let $\varphi : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[y_1, \dots, y_n]$ be the map given by $\varphi(x_i) := y_i - \pi(y_i)$, which we note is an equivalence. Define $f_i := \varphi^{-1}(g_i)$, which is to say that $f_i(x_1, \dots, x_n) = g_i(x_1 + \pi(y_1), \dots, x_n + \pi(y_n))$. By construction, φ descends to an equivalence $\mathbb{R}[x_1, \dots, x_n]/(f_1, \dots, f_n) = \mathbb{R}[y_1, \dots, y_n]/(g_1, \dots, g_n)$, and since $\pi \varphi(x_i) = \pi(y_i - \pi(y_i)) = 0$, this equivalence commutes with the augmentation. \square

Remark 4.4.4. The canonical example of a Weil algebra is $R[x]/(x^2)$ equipped with the augmentation $x \mapsto 0 : R[x]/(x^2) \rightarrow R$.

Definition 4.4.5 (Standard, see [Lav96]). Let A be an R -algebra. The *synthetic spectrum* $\text{Spec}_R(A)$ of A relative to R is the set of R -algebra homomorphisms A to R .

$$\text{Spec}_R(A) := \text{Hom}_R(A, R).$$

Remark 4.4.6. Note that if $A \equiv R[x_1, \dots, x_n](f_1, \dots, f_m)$ is finitely presented R -algebra, then by the universal properties of quotients of polynomial algebras, the synthetic spectrum of A over R is the set of solutions to the equations f_1, \dots, f_n :

$$\text{Spec}_R(A) = \{(r_1, \dots, r_n) : R^n \mid \forall i. f_i(\vec{r}) = 0\}$$

In particular, note that

$$\text{Spec}_{\mathbb{R}}(\mathbb{R}[x]/(x^2)) \simeq \{\varepsilon : \mathbb{R} \mid \varepsilon^2 = 0\} \equiv \mathbb{D}.$$

is the set of nilsquare infinitesimals \mathbb{D} . The evaluation map $\mathbb{R}[x]/(x^2) \rightarrow (\text{Spec}_{\mathbb{R}}(\mathbb{R}[x]/(x^2)) \rightarrow \mathbb{R})$ sends $a + bx$ to the function $\varepsilon \mapsto a + b\varepsilon$. The Kock-Lawvere axiom (Postulate J) says that this map is an equivalence. In other words, every function $f : \mathbb{D} \rightarrow \mathbb{R}$ is of the form $f(\varepsilon) = a + b\varepsilon$ for unique a and b in \mathbb{R} . Of course, plugging in 0 for ε shows us that $a = f(0)$, so we see that there is a unique $b : \mathbb{R}$ for which $f(\varepsilon) = f(0) + b\varepsilon$ for all $\varepsilon^2 = 0$.

This axiom is valid in every context; that is, we can make use of it even when there are other free variables floating around. In particular, it gives us the following lemma.

Lemma 4.4.7 (Standard, see [Lav96]). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Then there is a unique function $f' : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x : \mathbb{R}$ and $\varepsilon^2 = 0$, we have

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon.$$

We refer to f' as the *derivative* of f .

Proof. Given $x : \mathbb{R}$, define $g_x(\varepsilon) := f(x + \varepsilon)$ and note that $g_x : \mathbb{D} \rightarrow \mathbb{R}$. Therefore, there is a unique $b_x : \mathbb{R}$ for which $g_x(\varepsilon) = g_x(0) + b_x\varepsilon$. We may therefore define $f'(x) := b_x$. \square

In general, a function $v : \mathbb{D} \rightarrow X$ plays the role of a tangent vector in X , based at $v(0) : X$. In particular, a function $v : \mathbb{D} \rightarrow \mathbb{R}$ can be considered as a tangent vector based at $v(0)$, and we see that the type of all such vectors is equivalent to \mathbb{R} by the association of b with $w(\varepsilon) := v(0) + b\varepsilon$.

Definition 4.4.8 (Standard, see [Lav96]). Let X be a type, and $x : X$ an element. We define the tangent space $T_x X$ of X based at x to be the type of pointed function $v : \mathbb{D} \rightarrow X$ sending 0 to x .

$$T_x X := (v : \mathbb{D} \rightarrow X) \times (v(0) = x).$$

The tangent bundle is the projection $\text{fst} : TX := (x : X) \times T_x X \rightarrow X$.

Here's an example of how we might compute a tangent space. Specifically, we will show that the Lie algebra of $U(1)$ is \mathbb{R} .

Lemma 4.4.9 (Standard). The tangent space $T_1 U(1)$ of $U(1) := \{z : \mathbb{C} \mid z\bar{z} = 1\}$ at 1 is identifiable with the set $1 + i\mathbb{R}$ of numbers of the form $1 + bi$ in \mathbb{C} — which is itself identifiable with \mathbb{R} .

Proof. Let $v : \mathbb{D} \rightarrow U(1)$ be a tangent vector at 1, so that $v(0) = 1$. We can write $v(\varepsilon) = a(\varepsilon) + b(\varepsilon)i$, and then note that $1 = v(0) = a(0) + b(0)i$ so that $a(0) = 1$ and $b(0) = 0$. We can further expand $v(\varepsilon)$ as

$$v(\varepsilon) = a(\varepsilon) + b(\varepsilon)i = (1 + a'(0)\varepsilon) + (b'(0)\varepsilon)i$$

However, we still know that $v(\varepsilon)\overline{v(\varepsilon)} = 1$, so in particular

$$1 = (1 + a'(0)\varepsilon)^2 + (b'(0)\varepsilon)^2 = 1 + 2a'(0)\varepsilon$$

from which we may conclude that $a'(0) = 0$. Therefore, we see that $v(\varepsilon) = 1 + b'(0)\varepsilon i$ for a unique element $b'(0) : \mathbb{R}$, which proves the lemma. \square

Remark 4.4.10. Note that if X is a higher type (such as an orbifold), then the isotropy group $\mathbf{Aut}_X(x)$ of $x : X$ acts on $T_x X$. This action is easy to define as a function $T_x X^{\circ-} : \mathbf{B} \mathbf{Aut}_X(x) \rightarrow \mathbf{Type}$, namely:

$$T_x X^{\circ y} := T_y X.$$

Just because every type has a tangent bundle doesn't mean that every type is smooth. While we can always define a scalar action of \mathbb{R} on $T_x X$ by $rv := \varepsilon \mapsto v(r\varepsilon)$, this action does not in general extend to the structure of an \mathbb{R} -module on $T_x X$. That is, we can't necessarily add tangent vectors. One pass at a definition of "smooth type" would be a type for which the tangent spaces are \mathbb{R} -modules. But we don't just want the first order algebraic structure of infinitesimals from \mathbb{R} , we want the higher order structure as well: we want all the algebraic structure of higher order infinitesimals in \mathbb{R} to be present in "smooth types". This leads us to the notion of *microlinear* types.

4.4.2 Microlinear types

In this section, we will review the notion of *microlinear* types, and characterize them in terms of the algebraic theory of *finite order algebras*. The synthetic differential geometry community has settled on the notion of microlinearity as the correct notion of “smooth space” in the context of SDG. The main theme of this half of the paper will be that all reasonable notions of manifold and orbifold give rise to microlinear types. This means that by naively extending this notion of smoothness to higher types, we correctly pick up the intuitively smooth higher types such as orbifolds.

Intuitively, a microlinear type is one which shares all of the infinitesimal algebraic structure that \mathbb{R} has. However, the definition of microlinear types does not seem to capture this intuition immediately. We will formalize this intuition, and it will be our main theorem of this section: ??.

In order to define microlinear types, we will need the notion of *infinitesimal variety*.

Definition 4.4.11. An *infinitesimal variety* V is the spectrum of a Weil algebra. More formally, a pointed type V is an infinitesimal variety if there merely exists a Weil algebra W for which $V \simeq \mathbf{Spec}_{\mathbb{R}}(W)$ as pointed types, where $\mathbf{Spec}_{\mathbb{R}}(W)$ is pointed by the augmentation of W .

The category \mathbf{InfVar} of infinitesimal varieties is the full subcategory of pointed sets spanned by the infinitesimal varieties.

The walking tangent vector $\mathbb{D} = \mathbf{Spec}_{\mathbb{R}}(\mathbb{R}[x]/(x^2))$ is an example of an infinitesimal variety. In fact, the category of infinitesimal varieties is dual to the category of Weil algebras by the Kock-Lawvere axiom.

Lemma 4.4.12. We have an equivalence of categories:

$$\begin{array}{ccc} & \mathbb{R}^{(-)} & \\ & \curvearrowright & \\ \text{InfVar}^{\text{op}} & & \text{WeilAlg.} \\ & \curvearrowleft & \\ & \text{Spec} & \end{array}$$

Proof. First, let's note that this is a contravariant adjunction. That is, we have the unit $\eta : \equiv w \mapsto [\varphi \mapsto \varphi(w)] : W \rightarrow \mathbb{R}^{\text{Spec}(W)}$ and counit $\varepsilon : \equiv v \mapsto [f \mapsto f(v)] : V \rightarrow \text{Spec}(\mathbb{R}^V)$, both given by evaluation. The Kock-Lawvere axiom (Postulate J) says that the unit η is an isomorphism for Weil algebras W . Therefore, the left adjoint $\text{Spec} : \text{WeilAlg} \rightarrow \text{InfVar}^{\text{op}}$ is fully faithful. Since it is by definition essentially surjective, this concludes our proof. \square

We can justify the name “infinitesimal variety” if we have a good definition of “infinitesimal” (due to Penon [Pen81]).

Definition 4.4.13 ([Pen81]). A real number $x : \mathbb{R}$ is *infinitesimal* if it is not non-zero. More generally, for $x, y : X$ of any set X , define the *neighbor* relation $x \approx y$ by

$$(x \approx y) : \equiv \neg\neg(x = y).$$

An infinitesimal is x such that $x \approx 0$. We denote the set of infinitesimals by

$$\mathfrak{D} : \equiv \{x : \mathbb{R} \mid x \approx 0\}.$$

If $x : X$, then we may define $\mathfrak{D}_x X : \equiv \{y : X \mid y \approx x\}$.

Remark 4.4.14. In the Dubuc topos, the type \mathfrak{D} of infinitesimals is representable by the \mathcal{C}^∞ -algebra $\mathcal{C}_0^\infty(\mathbb{R})$ of germs of smooth functions on \mathbb{R} at 0. This is Proposition 11.5 of [BGSL18].

Remark 4.4.15. Note that by the Archimedean property (Postulate O.5), a number x is infinitesimal if and only if $x < \frac{1}{n}$ for all $n : \mathbb{N}$.

Lemma 4.4.16 ([Pen81]). Any function $f : X \rightarrow Y$ preserves the neighbor relation, in that we have a map

$$f_* : (x \approx y) \rightarrow (fx \approx fy).$$

As a corollary, for any point $x : X$ there is a pushforward

$$f_* : \mathfrak{D}_x X \rightarrow \mathfrak{D}_{fx} Y.$$

Proof. We apply $\neg\neg$ functorially to $\mathbf{ap} f$. □

The infinitesimal neighborhoods of 0 in \mathbb{R}^n consist of the points with infinitesimal coordinates.

Lemma 4.4.17. For any $n : \mathbb{N}$, we have an equality of subsets of \mathbb{R}^n :

$$\mathfrak{D}_0(\mathbb{R}^n) = \mathfrak{D}^n.$$

Proof. We prove both inclusions. Going from the left hand side to the right hand side is straightforward. Suppose that $\vec{x} \approx 0$, and let x_i be its i^{th} coefficient. If $x_i \neq 0$, then $\vec{x} \neq 0$ since $\vec{x} = 0$ if and only if all its coefficients are 0; therefore, we conclude that x_i is not non-zero, which is to say that $x_i \approx 0$.

It's the other direction that requires Postulate K. Suppose that each coefficient of \vec{x} is not non-zero. Suppose that \vec{x} were non-zero; since $\vec{x} = 0$ means precisely that all of its coefficients are 0, we are assuming

$$\neg \left(\bigwedge_{i=1}^n (x_i = 0) \right).$$

By Postulate K, we may therefore conclude that one of the x_i is invertible, which in particular means that it is non-zero. But this contradicts our assumption that each of the x_i are not non-zero, so we conclude that \vec{x} is not non-zero, which is to say that $\vec{x} \approx 0$. \square

Remark 4.4.18. If we assume that the smooth reals \mathbb{R} are a field in the sense that every non-zero element is invertible, we can see Lemma 4.4.17 as a reformulation of Postulate K.

Infinitesimal varieties are the zero locuses of functions on infinitesimals.

Lemma 4.4.19. Let V be an infinitesimal variety. Then there is merely a (polynomial) function $f : \mathfrak{D}^n \rightarrow \mathfrak{D}^m$ which send 0 to 0 and for which $V \simeq \{x : \mathfrak{D}^n \mid f(x) = 0\}$, identifying the base point of V with 0.

Proof. Let $V = \mathbf{Spec}(W)$ be the spectrum of a Weil algebra in standard form $W = \mathbb{R}[x_1, \dots, x_n]/(f_1, \dots, f_m) \xrightarrow{\mathbf{ev}_0} \mathbb{R}$, which we may assume by Lemma 4.4.3. The polynomials f_1, \dots, f_m assemble into a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. We note that since $\mathbf{ev}_0(f_j) = 0$, we have that $f(0) = 0$. We may restrict these functions to $\{a : \mathbb{R}^n \mid a \approx 0\} \rightarrow \{b : \mathbb{R}^m \mid b \approx 0\}$. We will show that V is equivalent to the fiber over 0 of this restricted function.

To show that $V \simeq \{a : \mathfrak{D}^n \mid f(a) = f(0)\}$, we'll construct an explicit equivalence. Let $a : \mathfrak{D}^n$ so that $f(a) = 0$, and define $\varphi_a : W \rightarrow \mathbb{R}$ by $\varphi_a(x_i) = a_i$. For this to be well defined, we need to know that $f_j(a) = 0$, but this was presumed. This gives a map $\{a : \mathfrak{D}^n \mid f(a) = 0\} \rightarrow \mathbf{Spec}(W)$ which is evidently injective. To show surjectivity, suppose $\psi : W \rightarrow \mathbb{R}$ is a \mathbb{R} -homomorphism; we will show that $(\psi(x_1), \dots, \psi(x_n)) \in \{a : \mathfrak{D}^n \mid f(a) = 0\}$, splitting the map

$a \mapsto \varphi_a$. First, we note that $f(\psi(x_1), \dots, \psi(x_n)) = \psi(f)(x_1, \dots, x_n) = 0$. It remains to show that $\psi(x_i) \approx 0$. Note that since $\mathbf{ev}_0(x_i) = 0$, x_i is in the augmentation $\ker(\mathbf{ev}_0)$ which was assumed to be nilpotent. Therefore, x_i is nilpotent, and so $\psi(x_i)$ is nilpotent, so it cannot be non-zero. \square

We can now define microlinear types, though it will take a bit more work to explain why they are a useful class of types.

Definition 4.4.20. A square as on the left is said to be an X -pushout (for a given type X) if the square on the right given by precomposition is a pullback:

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array} \qquad \begin{array}{ccc} X^A & \longleftarrow & X^C \\ \uparrow & & \uparrow \\ X^B & \longleftarrow & X^D \end{array}$$

Definition 4.4.21. An *infinitesimal \mathbb{R} -pushout* is a *crisp* commuting square of pointed maps between infinitesimal varieties which is an \mathbb{R} -pushout.

Definition 4.4.22 (Standard, see [Lav96]). A type X is *microlinear* if every infinitesimal \mathbb{R} -pushout is also an X -pushout.

We will show that the tangent spaces of microlinear types have a canonical \mathbb{R} -module structure. We will do this by showing that for any $x : X$, the functor $V \mapsto (v : V \rightarrow X) \times (v(0) = x) : \mathbf{InfVar}^{\text{op}} \rightarrow \mathbf{Type}$ restricts to a product preserving functor $\mathbf{fgFreeMod}_{\mathbb{R}}^{\text{op}} \rightarrow \mathbf{Type}$ sending \mathbb{R} to $T_x X$. This shows that $T_x X$ is a model of the algebraic theory of \mathbb{R} -modules.

First, we begin by defining the first order infinitesimal patches of the origin in \mathbb{R}^n .

Definition 4.4.23 (Standard, see [Lav96]). The first order infinitesimal patch of the origin $\mathbb{D}(n)$ in \mathbb{R}^n is

$$\mathbb{D}(n) := \{x : \mathbb{R}^n \mid \forall i, j, x_i x_j = 0\} = \{x : \mathbb{R}^n \mid xx^T = 0\}.$$

Note that $\mathbb{D}(1) \equiv \mathbb{D}$ is the set of first order infinitesimals.

Lemma 4.4.24. There is a fully faithful functor $\mathbb{D} : \mathbf{fgFreeMod}_{\mathbb{R}} \rightarrow \mathbf{InfVar}$ sending \mathbb{R}^n to $\mathbb{D}(n)$.

Proof. First, we will show that the object assignment $\mathbb{R}^n \mapsto \mathbb{D}(n)$ is functorial simply by restricting a linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to $\mathbb{D}(n)$. If $x \in \mathbb{D}(n)$, we will show that $f(x) \in \mathbb{R}^m$. By the Kock-Lawvere axiom, we have that

$$f(x) = f(0) + \sum_i x_i A_i$$

for unique vectors $A_i : \mathbb{R}^m$. By linearity, $f(0) = 0$, so we see that $f(x) = Ax$ where A is the $m \times n$ matrix with columns A_i . Finally,

$$(Ax)(Ax)^T = Axx^T A^T = 0.$$

because by hypothesis $xx^T = 0$.

To show that this assignment is fully faithful, we need to show that any pointed function $f : \mathbb{D}(n) \rightarrow \mathbb{D}(m)$ extends to a unique linear map $\mathbb{R}^n \rightarrow \mathbb{R}^m$. The key is again to use the Kock-Lawvere axiom to see that $f(x) = f(0) + Ax$ for a unique matrix A , and since we are only looking at pointed functions we know that $f(0) = 0$. Therefore we have a matrix A , which gives us a linear map $\mathbb{R}^n \rightarrow \mathbb{R}^m$. \square

Lemma 4.4.25 (Proposition 2.2.6 of [Lav96]). The square

$$\begin{array}{ccc}
 * & \xrightarrow{0} & \mathbb{D}(m) \\
 0 \downarrow & & \downarrow y \mapsto (0,y) \\
 \mathbb{D}(n) & \xrightarrow{x \mapsto (x,0)} & \mathbb{D}(n+m)
 \end{array} \tag{4.1}$$

is an infinitesimal \mathbb{R} -pushout. As a corollary, the functor $\mathbb{D} : \mathbf{fgFreeMod}_{\mathbb{R}} \rightarrow \mathbf{InfVar}$ sends coproducts to infinitesimal \mathbb{R} -pushouts.

Proof. Suppose $f : \mathbb{D}(n) \rightarrow \mathbb{R}$ and $g : \mathbb{D}(m) \rightarrow \mathbb{R}$ are such that $f(0) = g(0)$. By the Kock-Lawvere axiom, we have that $f(x) = f(0) + a \cdot x$ and $g(y) = g(0) + b \cdot y$ for unique vectors $a : \mathbb{R}^n$ and $b : \mathbb{R}^m$.

Define $\langle f, g \rangle : \mathbb{D}(n+m) \rightarrow \mathbb{R}$ by $\langle f, g \rangle(x, y) := f(0) + a \cdot x + b \cdot y$. By definition, $\langle f, g \rangle(x, 0) = f(x)$ and $\langle f, g \rangle(0, y) = g(y)$. The uniqueness part of the Kock-Lawvere axiom shows that this extension is unique.

Finally, we note that every coproduct diagram in $\mathbf{fgFreeMod}_{\mathbb{R}}$ is given by the inclusion of axis as in the above square, at least up to isomorphism. \square

Lemma 4.4.25 shows that for any microlinear type X , the square

$$\begin{array}{ccc}
 X^{\mathbb{D}(n+m)} & \longrightarrow & X^{\mathbb{D}(n)} \\
 \downarrow & \lrcorner & \downarrow \\
 X^{\mathbb{D}(m)} & \longrightarrow & X
 \end{array}$$

is a pullback. This condition is weaker than microlinearity, but useful in its own right. It appears as Definition 6.3 in [Koc06].

Definition 4.4.26 ([Koc06]). A type X is *infinitesimally linear* if the squares

$$\begin{array}{ccc}
 X^{\mathbb{D}(n+m)} & \longrightarrow & X^{\mathbb{D}(n)} \\
 \downarrow & \lrcorner & \downarrow \\
 X^{\mathbb{D}(m)} & \longrightarrow & X
 \end{array}$$

give by precomposing by the \mathbb{R} -pushout squares in Lemma 4.4.25 are pullbacks for all n and m .

Remark 4.4.27. As an immediate corollary of Lemma 4.4.25, we see that microlinear types are infinitesimally linear. It is somewhat unfortunate that these two notions have such similar names; it is really “microlinearity” which should be called something else, since it also involves higher order infinitesimals.

Remark 4.4.28. While Kock [Koc06] and Lavendhomme [Lav96] use “microlinear” and “infinitesimally linear” in the same sense that we are using here, Bunge, Gago, and San Luis [**Bunge-Gago-SanLuis:SDT**] use the term “infinitesimally linear” to refer to what we here call “microlinear”.

Theorem 4.4.29. *If X is infinitesimally linear (in particular if X is microlinear), then for all $x : X$, the tangent space $T_x X$ has the (coherent) structure of an \mathbb{R} -module — even if X is not a set.*

Proof. If X is microlinear, then the functor

$$V \mapsto (\text{ev}_{\text{pt}} : X^V \rightarrow X) : \mathbf{InfVar}^{\text{op}} \rightarrow \mathbf{Type}_{/X}$$

sends infinitesimal \mathbb{R} -pushouts to pullbacks. The functor $\text{fib}_{(-)}(x) : \mathbf{Type}_{/X} \rightarrow \mathbf{Type}$ taking the fiber over x preserves pullbacks, and so the composite functor $V \mapsto (v : V \rightarrow X) \times (v(0) = x)$ sends infinitesimal \mathbb{R} -pushouts to pullbacks. Since the functor $\mathbb{D} : \mathbf{fgFreeMod}_{\mathbb{R}} \rightarrow \mathbf{InfVar}$ sends coproducts to infinitesimal \mathbb{R} -pushouts, the composite functor $\mathbb{R}^n \mapsto (v : \mathbb{D}(n) \rightarrow X) \times (v(0) = x)$ preserves products. This makes $T_x X := (v : \mathbb{D}(1) \rightarrow X) \times (v(0) = x)$ into a model of the Lawvere theory of \mathbb{R} -algebras, which was to be shown. \square

4.4.3 The crystalline modality \mathfrak{S}

Just as we used the shape modality \mathfrak{J} , defined by nullifying the reals \mathbb{R} , to study the homotopy theory of orbifolds, we can use the *crystalline* modality \mathfrak{S} , defined by nullifying the infinitesimals \mathfrak{D} , to study the differential structure of orbifolds.

In this section, we will introduce the \mathfrak{S} modality (and in particular the \mathfrak{S} -étale maps). We will then prove a crucial result: microlinearity descends along surjective \mathfrak{S} -étale maps. In the next section, we will use this theorem to show that all reasonable notions of “smooth space” are microlinear.

Definition 4.4.30. We define the *crystalline* modality \mathfrak{S} to be nullification at $\mathfrak{D} := \{x : \mathbb{R} \mid x \approx 0\}$. We refer to the \mathfrak{S} -modal types as *crystalline*.

Remark 4.4.31. The crystalline modality \mathfrak{S} appears in [Sch13a] as the *infinitesimal shape* modality. The type $\mathfrak{S}X$ is sometimes called the “de Rham stack” of X . It was studied in homotopy type theory in [Cherubini:Cartan.Geometry] (see also the appendix to [CR21]). I have decided to call it the *crystalline* modality because a map $E : \mathfrak{S}X \rightarrow \mathbf{Vect}$ is a *crystal* on X , in the sense of [Lur].

In the setting of [Sch13a], \mathfrak{S} may be equivalently defined as the nullification of all crisp infinitesimal varieties. But in the Dubuc ∞ -topos, it is not clear that these two modalities coincide; we have gone for the definition which works best in the Dubuc ∞ -topos.

Arguing externally, we could use the tinyness of \mathfrak{D} to prove that \mathfrak{S} has an external right adjoint. However, in our intended models, \mathfrak{S} also has an external left adjoint, and so in particular preserves crisp pullbacks. We take this preservation property as an axiom.

Axiom 6. The \mathfrak{S} modality preserves crisp pullbacks.

Remark 4.4.32. In the intended models, \mathfrak{S} is in fact part of an (external) adjoint triple

$$\mathbf{R} \dashv \mathfrak{S} \dashv \&$$

which Schreiber refers to as *infinitesimal cohesion*. Here, both \mathbf{R} and $\&$ are *comodalities*, or comonadic modalities. To add nontrivial comodalities requires modifying the underlying type theory, so we refrain from that here. The comodality $\&$ behaves very much like \flat , and adjunction $\mathfrak{S} \dashv \&$ mirrors that of $\int \dashv \flat$. For this reason, a modification of Shulman’s crisp type theory to accommodate $\&$ should be rather straightforward. But the comodality \mathbf{R} is not lex (even externally), and so a different method would be necessary to incorporate it into the type theory. We make use of neither here.

The \mathfrak{S} -connected types — those types X for which $\mathfrak{S}X$ is contractible — are a good class of *infinitesimal types*. We can show that the crisp infinitesimal varieties are \mathfrak{S} -connected, using the crisp lexness of \mathfrak{S} .

Lemma 4.4.33. Every crisp infinitesimal variety is \mathfrak{S} -connected.

Proof. By Lemma 4.4.19, every infinitesimal variety is the zero-locus of a map between powers of \mathfrak{D} . Since \mathfrak{S} is lex for crisp diagrams and powers of \mathfrak{D} are \mathfrak{S} -connected, the fiber of a crisp map $f :: \mathfrak{D}^n \rightarrow \mathfrak{D}^m$ over the crisp element $0 :: \mathfrak{D}^m$ is \mathfrak{S} -connected. □

We will mainly use the modality \mathfrak{S} for its étale maps. Recall from Definition 4.3.4 that a map $f : X \rightarrow Y$ is \mathfrak{S} -étale if and only if the \mathfrak{S} -naturality

square is a pullback:

$$\begin{array}{ccc} X & \xrightarrow{(-)^{\mathfrak{S}}} & \mathfrak{S}X \\ f \downarrow & \lrcorner & \downarrow \mathfrak{S}f \\ Y & \xrightarrow{(-)^{\mathfrak{S}}} & \mathfrak{S}Y \end{array}$$

The \mathfrak{S} -étale maps $f : X \rightarrow Y$ (see) are a reasonable class of “local diffeomorphisms” because they lift uniquely against the base points $\mathbf{pt} : * \rightarrow V$ for any crisp infinitesimal variety V :

$$\begin{array}{ccc} * & \xrightarrow{\forall} & C \\ 0 \downarrow & \exists! \nearrow & \downarrow \pi \\ V & \xrightarrow{\forall} & X \end{array}$$

In particular, an \mathfrak{S} -étale map $f : X \rightarrow Y$ induces an equivalence $T_x X \xrightarrow{\sim} T_{fx} Y$ on tangent spaces for all $x : X$.

Proposition 4.4.34. Let $f : X \rightarrow Y$ be \mathfrak{S} -étale. Then the pushforward $f_* : T_x^V X \rightarrow T_{fx}^V Y$ on the V -tangent space for any crisp infinitesimal variety V and $x : X$ is an equivalence.

Proof. Let $x : X$ and $(v, w) : T_{fx}^V Y$. Consider the following diagram:

$$\begin{array}{ccc} * & \xrightarrow{x} & C \\ 0 \downarrow & \exists! \nearrow & \downarrow \pi \\ V & \xrightarrow{v} & X \end{array} \tag{4.2}$$

The square commutes by the witness $w : v(0) = fx$ that v sends 0 to fx .

The type of fillers to this square is

$$(\tilde{v} : V \rightarrow X) \times (\tilde{w} : \tilde{v}(0) = x) \times (q : f \circ \tilde{v} = v) \times ((q^{-1} \text{ at } 0) \bullet f_*(\tilde{w}) = w).$$

This is equivalent to $\mathbf{fib}_{f_*}(v, w)$ of $f_* : T_x^V X \rightarrow T_{fx}^V Y$ over (v, w) :

$$\begin{aligned} \mathbf{fib}_{f_*}(v, w) &\equiv ((\tilde{v}, \tilde{w}) : T_x^V X) \times ((f \circ \tilde{v}, f_* \tilde{w}) = (v, w)). \\ &= ((\tilde{v}, \tilde{w}) : T_x^V) \times (q : f \circ \tilde{v} = v) \times (\mathbf{tr}(\lambda z.z(0) = fx, q)(f_* \tilde{w}) = w). \end{aligned}$$

By Lemma 4.4.33, the base point inclusion $\mathbf{pt} : * \rightarrow V$ is a \mathfrak{S} -equivalence. Therefore, the type of fillers to (4.2) is contractible since f is \mathfrak{S} -étale. By the equivalence between $\mathbf{fib}_{f_*}(v, w)$ and the type of fillers, we see that $\mathbf{fib}_{f_*}(v, w)$ is contractible, showing that $f_* : T_x^V X \rightarrow T_{fx}^V Y$ is an equivalence. \square

Remark 4.4.35. It would be really useful to know that the \mathfrak{S} -étale maps are exactly those that lift on the right against the base point inclusion $0 : * \rightarrow \mathfrak{D}$. But I do not know under which conditions the étale maps of a modality given by nullifying at a pointed type are defined by lifting against the inclusion of the base point. In Theorem 3.10 of [CR21], Cherubini and Rijke show that this holds for the n -truncation modality, and ask the question in general.

Let's find some ways to construct \mathfrak{S} -étale maps. Firstly, any covering map is \mathfrak{S} -étale. Even more generally, every \mathfrak{f} -modal map is \mathfrak{S} -étale.

Lemma 4.4.36. Any discrete (\mathfrak{f} -modal) type is crystalline (\mathfrak{S} -modal). As a corollary:

1. Any \mathfrak{f} -modal map is \mathfrak{S} -modal.
2. Any \mathfrak{f} -étale map is \mathfrak{S} -étale.
3. Any \mathfrak{S} -connected map is \mathfrak{f} -connected.
4. Any \mathfrak{S} -equivalence is a \mathfrak{f} -equivalence.

Proof. This follows from the fact that \mathfrak{D} is \mathfrak{f} -connected, since it admits a multiplicative action by \mathbb{R} giving us an explicit contraction $x \mapsto tx$ onto 0. Therefore, if X is discrete — which is to say, \mathfrak{f} -modal — then any map $\mathfrak{D} \rightarrow X$ factors uniquely through the shape of \mathfrak{D} , which is the point.

The corollaries follow from Theorem 3.17 of Chapter 3. □

Remark 4.4.37. Combining Lemma 4.4.36 with Proposition 4.4.34 shows us that $q : \mathfrak{h} \rightarrow \mathcal{M}_{1,1}$ is \mathfrak{S} -étale and that therefore it induces an isomorphism on tangent spaces. In particular, we can say that $\mathcal{M}_{1,1}$ is a 2-dimensional orbifold.

Warning 4.4.38. Thanks to Lemma 4.4.36, we can prove that \mathfrak{S} is *not* lex. Since all propositions are discrete (Lemma 8.8 of [Shu18a]), all propositions are crystalline. This means that every embedding is \mathfrak{S} -modal. If \mathfrak{S} were lex, this would mean that every embedding would be \mathfrak{S} -étale. But then the inclusion $\{0\} \hookrightarrow \mathbb{R}$ would be \mathfrak{S} -étale, which would imply that the inclusion $\mathfrak{D} \hookrightarrow \mathbb{R}$ is constant at 0 — that is, every infinitesimal would be 0. This obviously trivializes the theory.

Let's now turn our attention to the relationship between microlinearity and \mathfrak{S} -étale maps. Since \mathfrak{S} -étale maps are intuitively local diffeomorphisms and microlinearity is a local — or even infinitesimal — property, it stands to reason that if $f : X \rightarrow Y$ is \mathfrak{S} -étale, then it should be the case that X is microlinear if and only if Y is. We will show that this is the case, as long as f is surjective as well.

First we will show that microlinearity ascends along \mathfrak{S} -étale maps. In order to prove this, we will need to re-express the notion of microlinearity as a lifting condition.

Lemma 4.4.39. A type X is microlinear if and only if it lifts uniquely on the right against all codiagonal maps $\nabla_V : V_2 +_{V_1} V_3 \rightarrow V_4$ where

$$\begin{array}{ccc} V_1 & \longrightarrow & V_3 \\ \downarrow & & \downarrow \\ V_2 & \longrightarrow & V_4 \end{array} \quad V$$

is an infinitesimal \mathbb{R} -pushout.

Proof. The unique lifting condition says that pre-composition by ∇_V gives an equivalence between X^{V_4} and $(V_2 +_{V_1} V_3 \rightarrow X)$. But by the universal property of the pushout, this latter type is equivalent to the pullback $X^{V_2} \times_{X^{V_1}} X^{V_3}$, and so V is an X -pushout if and only if X lifts uniquely against ∇_V . \square

Lemma 4.4.40. If Y is microlinear and $f : X \rightarrow Y$ is \mathfrak{S} -étale, then X is microlinear.

Proof. Let V be an infinitesimal \mathbb{R} -pushout. Then the codiagonal $\nabla_V : V_2 +_{V_1} V_3 \rightarrow V_4$ is a map between \mathfrak{S} -connected types, and is therefore an \mathfrak{S} -equivalence. Since f is \mathfrak{S} -étale, it therefore lifts against ∇_V , making f microlinear. Since the terminal map $X \rightarrow *$ is the composite $X \xrightarrow{f} Y \rightarrow *$ and by hypothesis $Y \rightarrow *$ was microlinear, $X \rightarrow *$ is microlinear as well (by closure under composition of maps defined by lifting on the right against a given class of maps). \square

Next, we will show that microlinearity descends along surjective \mathfrak{S} -étale maps. To do this we will need a general lemma.

Lemma 4.4.41. For any modality \diamond , the \diamond -connected types are closed under pushout.

Proof. Consider a pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ h \downarrow & & \downarrow k \\ C & \xrightarrow{g} & D \end{array}$$

in which A , B , and C are \diamond -connected. Then for any \diamond -modal X , consider the following cube:

$$\begin{array}{ccccc} & & X & \xlongequal{\quad} & X \\ & \swarrow \cong & \downarrow & & \downarrow \\ X & \xlongequal{\quad} & X & \xlongequal{\quad} & X \\ \downarrow & & \downarrow & \rightarrow & \downarrow \\ & & X^D & \longrightarrow & X^C \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ X^B & \longrightarrow & X^A & & \end{array}$$

Since the square we began with was a pushout, the bottom face is a pullback. The top face is also trivially a pullback. Since A , B , and C are \diamond -connected, the front three vertical edges are equivalences. Therefore, the back edge is also an equivalence, which proves that D is also \diamond -connected. \square

Theorem 4.4.42. *Let $f : X \rightarrow Y$ be a surjective, \mathfrak{S} -étale map. If X is microlinear, then so is Y .*

Proof. We will show that Y lifts on the right against the gap maps $\nabla_V : V_2 +_{V_1} V_3 \rightarrow V_4$ of infinitesimal \mathbb{R} -pushouts. This means showing that for any $k : V_2 +_{V_1} V_3 \rightarrow Y$, the type $(g : V_4 \rightarrow Y) \times (g \circ \nabla_V = k)$ is contractible. As we are seeking to prove a proposition and f is surjective, we may assume we have $(x, p) : \text{fib}_f(k(0))$. Then, since by Lemma 4.4.41 the inclusion $0 : * \rightarrow V_2 +_{V_1} V_3$

is a \mathfrak{S} -equivalence, we have a unique dashed filler in the following square:

$$\begin{array}{ccc} * & \xrightarrow{x} & X \\ 0 \downarrow & \nearrow \tilde{k} & \downarrow f \\ V_2 +_{V_1} V_3 & \xrightarrow{k} & Y \end{array}$$

Call the bottom commutative triangle $\beta : f \circ \tilde{k} = k$. We will show that $(g : V_4 \rightarrow Y) \times (g \circ \nabla_V = k)$ is equivalent to the type $(\tilde{g} : V_4 \rightarrow X) \times (\tilde{g} \circ \nabla_V = \tilde{k})$. Since this latter type is contractible by the assumption that X is microlinear, this will complete our proof.

Define $\varphi : (\tilde{g} : V_4 \rightarrow X) \times (\tilde{g} \circ \nabla_V = \tilde{k}) \rightarrow (g : V_4 \rightarrow Y) \times (g \circ \nabla_V = k)$ by

$$(\tilde{g}, \tilde{w}) \mapsto (f \circ \tilde{g}, (f \circ)_* \tilde{w} \bullet \beta).$$

We will show that the fibers of this map are contractible. We begin with a calculation:

$$\begin{aligned} \text{fib}_\varphi(g, w) &= (\tilde{g} : V_4 \rightarrow X) \times (\tilde{w} : \tilde{g} \circ \nabla_V = \tilde{k}) \times ((f \circ \tilde{g}, (f \circ)_* \tilde{w} \bullet \beta) = (g, w)) \\ &= (\tilde{g} : V_4 \rightarrow X) \times \left\{ \begin{array}{l} (\tilde{w} : \tilde{g} \circ \nabla_V = \tilde{k}) \times (z : f \circ \tilde{g} = g) \times \\ (\text{tr}(\lambda h. h \circ \nabla_V = k) z ((f \circ)_* \tilde{w} \bullet \beta) = w) \end{array} \right. \\ &= (\tilde{g} : V_4 \rightarrow X) \times (\tilde{w} : \tilde{g} \circ \nabla_V = \tilde{k}) \times (z : f \circ \tilde{g} = g) \times ((\circ \nabla_V)_* z \bullet (f \circ)_* \tilde{w} \bullet \beta = w) \\ &= (\tilde{g} : V_4 \rightarrow X) \times (\tilde{w} : \tilde{g} \circ \nabla_V = \tilde{k}) \times (z : f \circ \tilde{g} = g) \times ((\circ \nabla_V)_* z \bullet (f \circ)_* \tilde{w} = w \bullet \beta^{-1}) \end{aligned}$$

This final type is the type of fillers to the square:

$$\begin{array}{ccc}
 V_2 +_{V_1} V_3 & \xrightarrow{\tilde{k}} & X \\
 \nabla_V \downarrow & \nearrow \tilde{g} & \downarrow f \\
 V_4 & \xrightarrow{g} & Y
 \end{array}$$

where the underlying square commutes by $w \bullet \beta^{-1} : g \circ \nabla_V = f \circ \tilde{k}$. But since ∇_V is an \mathfrak{S} -equivalence and f is \mathfrak{S} -étale, there is a unique filler to this square. Therefore, $\text{fib}_\varphi(g, w)$ is contractible. \square

4.5 Smooth Spaces are Microlinear

In this section, we will use Theorem 4.4.42 to show that most reasonable notions of smooth space in synthetic differential geometry are microlinear. Our main theorem will be Theorem 4.5.34, which proves that étale groupoids are microlinear.

We will begin in Section 4.5.1 by recalling the definition of a manifold in the ordinary sense, which we will call ordinary manifolds for emphasis. We will make use of ordinary manifolds in our final theorem, Theorem 4.6.37, which proves that crisp ordinary proper étale groupoids — which are proper étale groupoids in the ordinary sense — are orbifolds in the sense of Definition 4.1.2.

In Theorem 4.5.12, we will show that ordinary manifolds are microlinear. We will do this by showing that any ordinary manifold is a *Penon manifold* (??), and then by showing that all Penon manifolds are microlinear — a result which is likely folklore. We will also take the opportunity in Theorem 4.5.18 to show that Penon’s notion of étale map coincides with that of \mathfrak{S} -étale maps, at least between crisp Penon manifolds.

In Section 4.5.3, we will recall Schreiber’s notion of V -manifold and show in Theorem 4.5.25 that any Schreiber V -manifold for microlinear V is itself microlinear. Unlike ordinary manifolds and Penon manifolds, which must be sets, Schreiber manifolds can be higher types. This is therefore our first taste of higher microlinear types.

We will fully turn our attention to higher microlinear types in Section 4.5.4. In Theorem 4.5.26, we will use Theorem 4.4.42 to quickly prove that the quotient $X // \Gamma$ of a microlinear type X by the action of a crisply discrete higher group Γ is microlinear. This is enough to show that all of our examples of good orbifolds from Section 4.2 are microlinear. But not all orbifolds are good orbifolds. In order to justify Definition 4.1.2 and show that any orbifold in the ordinary sense — that is, any proper étale groupoid — is microlinear, we prove in Theorem 4.5.34 that crisp étale groupoids are microlinear. This provides one pillar of the upcoming Theorem 4.6.37 which shows that all crisp, ordinary proper étale groupoids are orbifolds in the sense of Definition 4.1.2.

The proof of Theorem 4.5.34 is not trivial. It relies on a form of étale descent, Theorem 4.5.32, which proves that if $f :: X \rightarrow Y$ is crisp and surjective and its pullback along itself is \mathfrak{S} -étale, then it is \mathfrak{S} -étale. This makes essential use of the commutation of \mathfrak{S} with crisp pushouts and colimits of crisp sequences, a fact which appears in this section as Theorem 4.5.33 but whose proof is deferred to Section 4.7.

Finally, we show in Section 4.5.5 that a delooping of an infinitesimally linear group (such as a Lie group) is itself infinitesimally linear. This gives examples of infinitesimally linear higher types which are not locally discrete. However, the method we use to prove Theorem 4.5.40 does not enable us to show that a

delooping of a microlinear group is microlinear. I have not been able to prove this, nor have I come up with a counterexample.

Before even starting, let's observe a number of well known closure properties of microlinear types. This will suffice to show that a wide number of spaces encountered in practice are microlinear, without relying on any other notions of smooth space.

Proposition 4.5.1. Microlinear types are closed under limits, and if X is microlinear then X^A is microlinear for any A .

Proof. By Lemma 4.4.39, microlinearity is characterized by lifting uniquely on the right against a class of maps. These closure properties are the closure properties of the class of maps defined by such a lifting property. \square

As a corollary, we see that the zero loci of (arbitrarily indexed) families of real valued functions are microlinear.

Proposition 4.5.2. Let $f : \mathbb{R}^N \rightarrow \mathbb{R}^M$ be any map where N and M are any types. Then the zero-locus $Z_f := \{x : \mathbb{R}^N \mid f(x) = 0\}$ is microlinear.

Proof. As \mathbb{R} is microlinear, \mathbb{R}^N and \mathbb{R}^M are microlinear. The result then follows by the closure of microlinear types under pullback. \square

4.5.1 Ordinary manifolds

First we begin by giving a formulation of the standard notion of manifold in homotopy type theory. By the standard notion of manifold, I mean a second countable Hausdorff topological space which is locally homeomorphic to \mathbb{R}^n . For emphasis, we will always call these “ordinary manifolds” in this chapter.

Classically, this would only define a continuous manifold; but we are using the *smooth reals*, so any transition function between charts will already be smooth.

Since we are working in a constructive setting, we should take a bit of care about the topological points. Namely, instead of asking that an ordinary manifold be Hausdorff (T_2), we will ask that it be regular Hausdorff (T_3) — that is, regular and T_0 , which together imply Hausdorff.

Definition 4.5.3. We recall the following topological definitions:

1. A topological space X is *regular* if for any open set U and $x \in U$, there is an open neighborhood V of U and an open set G disjoint from V (that is, $G \cap V = \emptyset$) but complementary with U (that is, $G \cup U = X$).
2. A topological space is T_0 if for any pair of *distinct* points x and y (that is, $x \neq y$), there is either an open set containing x but not y , or an open set containing y but not x .
3. A topological space is regular Hausdorff if it is regular and T_0 .
4. A topological space X is Hausdorff if for any pair of *distinct* points x and y , there are disjoint open sets $U_x \cap U_y = \emptyset$ with $x \in U_x$ and $y \in U_y$.

Let's recall the simple proof that regular Hausdorff spaces as defined above are also Hausdorff.

Lemma 4.5.4. A regular Hausdorff topological space is Hausdorff.

Proof. Let x and y be distinct points of a regular Hausdorff space X . Since X is T_0 , there is (without loss of generality) an open set U containing x but not y . By regularity, there is then an open set V containing x and an open set G

disjoint from V and with $G \cup U = X$. Therefore y is either in G or U ; but we already know it is not in U , so it must be in G . Therefore, there are disjoint open neighborhoods separating x and y . \square

There is a good reason for using regular Hausdorff instead of just Hausdorff: open sets in regular spaces are *infinitesimally stable* in the sense that if x is in an open set U and $x \approx y$, then y is also in U . We will use this property in Proposition 4.5.10 to show that manifolds have the correct *infinitesimal* structure in addition to the local structure which we assume.

Lemma 4.5.5. Let X be a regular topological space. Then open sets U of X are infinitesimally stable in the sense that if $x \in U$ and $y \approx x$, then $y \in U$.

Proof. Let $x \in U$ and suppose that it is not the case that $x \neq y$. Since X is regular, for $x \in U$ there is an open V containing x and an open G disjoint from V and complementary to U . Therefore, $y \in G$ or U . But G is disjoint from V and $x \in V$; if y were in G , then y would be distinct from x (we would have $x \neq y$). Therefore, $y \in U$. \square

Before going forward to define manifold, we should equip our Euclidean spaces with an appropriate topology and check that it is regular Hausdorff. We will use the metric topology.

Definition 4.5.6. A subset $U \subseteq \mathbb{R}^n$ is *metrically open* if for every $x \in U$ there is a $\varepsilon > 0$ (which may be chosen to be rational, by the Archimedean property) so that for all $y : X$, if

$$\sum_{i=1}^n (x_i - y_i)^2 < \varepsilon^2$$

then $y \in U$.

Defining $B(x, \varepsilon) := \{y : \mathbb{R}^n \mid \sum_{i=1}^n (x_i - y_i)^2 < \varepsilon^2\}$, this means that a set U is metrically open if for all $x \in U$ there is an $\varepsilon > 0$ for which $B(x, \varepsilon) \subseteq U$.

Theorem 4.5.7. *The metrically open subsets of \mathbb{R}^n form a regular Hausdorff topology.*

Proof. We explain the proof of the 1-dimensional case to more clearly communicate the main ideas. First, we will show that \mathbb{R} is T_0 . Suppose that $x \neq y$ are real numbers; then either $x < y$ or $y < x$, so suppose the latter without loss of generality. Then $x - y \neq 0$ and is therefore invertible. Let n be a natural number so that $\frac{1}{x-y} < n$; then $x - y < \frac{1}{n}$. Then the ball $B(x, \frac{1}{n})$ is a metrically open subset containing x but not y .

Next, we will show that \mathbb{R} is regular. Let U be metrically open and let $x \in U$. Then there is an $\varepsilon \in \mathbb{Q}$ for which $B(x, \varepsilon) \subseteq U$. Define $V := B(x, \frac{\varepsilon}{2})$. Define $G := \{y : \mathbb{R} \mid (x - y)^2 > (\frac{\varepsilon}{2})^2\}$, and note that $G \cap V = \emptyset$. It remains to show that $G \cup U = \mathbb{R}$; it suffices to show that $G \cup B(x, \varepsilon) = \mathbb{R}$. But this follows from Postulate O.3: since $\varepsilon^2 > (\frac{\varepsilon}{2})^2$, either $\varepsilon^2 > (x - y)^2$ or $(x - y)^2 > (\frac{\varepsilon}{2})^2$. \square

Now we can define the notion of “ordinary manifold”.

Definition 4.5.8. *A ordinary n -dimensional manifold M is a regular Hausdorff topological space which is locally isomorphic to \mathbb{R}^n in that for any point $p \in M$, there is merely a *chart* around p , an open subset $U \subseteq \mathbb{R}^n$ of the origin and an open embedding $\phi : U \hookrightarrow M$ so that $\phi(0) = p$. We will assume that ordinary manifolds are second countable in that they have a countable base of charts.*

In the next section, we will show that ordinary manifolds are Penon manifolds, and that Penon manifolds are microlinear.

4.5.2 Penon manifolds

In his paper *Infinitesimaux et Intuisionisme* [Pen81], Jacques Penon emphasizes that infinitesimal neighbors of a point x of a space (for us, a set or a 0-type) X are the points y which are *not distinct* from x in the sense that

$$\neg\neg(y = x).$$

This is the relation $x \approx y$ of Definition 4.4.13.

Accordingly, Penon suggests that a “manifold” should be a set which is infinitesimally isomorphic to \mathbb{R}^n in the following sense.

Definition 4.5.9. A *Penon manifold* of dimension n is a set M so that for all $p : M$, the infinitesimal neighborhood $\mathfrak{D}_p M \equiv \{x : M \mid x \approx p\}$ is identifiable with the infinitesimal neighborhood $\mathfrak{D}^n \subseteq \mathbb{R}^n$ of the origin of real n -space as a pointed set. That is, for all p , we have

$$\|(f : \mathfrak{D}_p M = \mathfrak{D}^n) \times (f(p) = 0)\|.$$

Ordinary manifolds are Penon manifolds.

Proposition 4.5.10. Every ordinary manifold is a Penon manifold.

Proof. Let M be an ordinary manifold and let $p : M$ be a point, seeking to prove that $\mathfrak{D}_p M$ is identifiable with $\mathfrak{D}^n \subseteq \mathbb{R}^n$. Since we are seeking to prove a proposition, we may take as given a chart $\phi : U \hookrightarrow M$ around p — that is, with $\phi(0) = p$. Since ϕ is an open embedding, its image $\phi(U)$ is open. Since M is regular, $\mathfrak{D}_p M \subseteq \phi(U)$ by Lemma 4.5.5. Since ϕ is an embedding, it restricts to an equivalence on its image; therefore, it also restricts to an equivalence

$\phi : \phi^{-1}(\mathfrak{D}_p M) \simeq \mathfrak{D}_p$. It remains to show that $\phi^{-1}(\mathfrak{D}_p M) = \mathfrak{D}^n$. First, we note that $0 \in \phi^{-1}(\mathfrak{D}_p M)$ since by assumption $\phi(0) = p$. Then, by Lemma 4.4.16, the equivalence ϕ restricts to an equivalence \mathfrak{D}^n with $\mathfrak{D}_p M$. \square

We will now show that Penon manifolds are microlinear by proving that microlinearity is a local property.

Lemma 4.5.11. Let X be a type. Then X is microlinear if and only if for each $x : X$, the infinitesimal neighborhood $\mathfrak{D}_x X := (y : X) \times (y \approx x)$ of x is microlinear.

Proof. For any $x : X$, the projection $i : \mathfrak{D}_x X \rightarrow X$ is an embedding since $(y \approx x)$ is a proposition. Furthermore, if V_4 is any infinitesimal variety, then since every $\varepsilon : V_4$ is near 0 — $\varepsilon \approx 0$ — we have that $v(\varepsilon) \approx v(0)$ for every $v : V_4 \rightarrow X$. Therefore, we have a lift of ∇_V into X for any infinitesimal \mathbb{R} -pushout if and only if we have a lift into $\mathfrak{D}_x X$ where x is the image of the base point 0 . \square

Theorem 4.5.12. *Every Penon manifold, and hence every ordinary manifold, is microlinear.*

Proof. Since for every point $p : M$ in a Penon manifold, the infinitesimal neighborhood \mathfrak{D}_p is identifiable with \mathfrak{D}^n , it will suffice to show that \mathfrak{D}^n is microlinear by Lemma 4.5.11. But $\mathfrak{D}^n = \mathfrak{D}_0(\mathbb{R}^n)$ and \mathbb{R}^n is microlinear, so this also follows by Lemma 4.5.11. \square

In [Pen81], Penon gives a definition of étale map between his manifolds.

Definition 4.5.13. A map $f : X \rightarrow Y$ between sets is Penon étale if for all $x : X$, the induced map $f_* : \mathfrak{D}_x \rightarrow \mathfrak{D}_{fx}$ is an equivalence.

We already have a notion of étale map: the \mathfrak{S} -étale maps. This is not to mention the ordinary notion of local diffeomorphism between manifolds: a map $f : X \rightarrow Y$ for which the pushforward $f_* : T_x X \rightarrow T_{fx} X$ is an isomorphism for all $x : X$. Luckily, all of these notions coincide where they are jointly defined, at least for crisp manifolds.

To prove this, we take a definition from Cherubini's [**Cherubini:Cartan.Geometry**].

Definition 4.5.14 ([**Cherubini:Cartan.Geometry**]). Let X be a type and $x : X$ be an element. The \mathfrak{S} -disk around x is defined to be the fiber of the unit $(-)^{\mathfrak{S}} : X \rightarrow \mathfrak{S}X$ over $x^{\mathfrak{S}}$:

$$D_x^{\mathfrak{S}} X \equiv \text{fib}_{(-)^{\mathfrak{S}}}(x^{\mathfrak{S}}) \equiv ((y : X) \times (x^{\mathfrak{S}} = y^{\mathfrak{S}})).$$

Cherubini and Rijke show in Proposition 3.7 of [CR21] that if the modal unit $(-)^{\mathfrak{S}} : X \rightarrow \mathfrak{S}X$ is surjective, then a map $f : X \rightarrow Y$ is \mathfrak{S} -étale if and only if the induced map $f_* : D_x^{\mathfrak{S}} X \rightarrow D_{fx}^{\mathfrak{S}} Y$ is an equivalence. But as \mathfrak{S} is given by localizing at a pointed type, all modal units are surjective.

We will show that in a crisp set X , the \mathfrak{S} -disk $D_x^{\mathfrak{S}} X$ around x coincides with the infinitesimal disk $\mathfrak{D}_x X$ around x . Together, this will show that a map between crisp Penon manifolds is Penon étale if and only if it is \mathfrak{S} -étale. However, we will need a few lemmas to do this. First, we need to know that for a crisp set X , $\mathfrak{S}X$ is also a set.

Lemma 4.5.15. Let X be a crisp set. Then $\mathfrak{S}X$ is a set.

Proof. A type X is a set if and only if the square

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \parallel & & \downarrow \Delta \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

is a pullback. If X is a crisp set, then that square is a pullback, and since \mathfrak{S} preserves crisp pullbacks (and binary products) so is the square

$$\begin{array}{ccc} \mathfrak{S}X & \xlongequal{\quad} & \mathfrak{S}X \\ \parallel & & \downarrow \Delta \\ \mathfrak{S}X & \xrightarrow{\Delta} & \mathfrak{S}X \times \mathfrak{S}X \end{array}$$

which shows that $\mathfrak{S}X$ is a set. \square

Next, we need to investigate the relationship between \mathfrak{S} and the other modality of cohesion: the codiscrete modality \sharp (see Section 3 of [Shu18a]). We have not talked about \sharp very much in this chapter, and we won't need it for anything but this.

Lemma 4.5.16. Every codiscrete type is crystalline. As a corollary, the unit $(-)^{\sharp} : X \rightarrow \sharp X$ factors uniquely through the unit $(-)^{\mathfrak{S}} : X \rightarrow \mathfrak{S}X$, and the factor $\mathfrak{S}X \rightarrow \sharp X$ is itself a \sharp -counit.

Proof. It suffices to show that \mathfrak{D} is \sharp -connected: $\sharp \mathfrak{D} \simeq *$. For this, it suffices to show that $\flat \mathfrak{D} \simeq *$ by Theorem 6.22 of [Shu18a]. We note that $\flat \mathfrak{D}$ is a set, and so by the crisp law of excluded middle, for any $u : \flat Dc$, either $u = 0^{\flat}$ or not. Suppose that $u \neq 0^{\flat}$; then, since $(-)^{\flat} : \flat \mathfrak{D} \rightarrow \mathfrak{D}$ is an embedding since \mathfrak{D} is a set by Theorem 8.21 of [Shu18a], it follows that $u_{\flat} \neq 0$. But every element of \mathfrak{D} is not distinct from 0, so this is a contradiction and we may conclude that $u = 0^{\flat}$, so that $\flat \mathfrak{D}$ is contractible. \square

Finally, we can prove that infinitesimal disks and \mathfrak{S} -disks coincide for Penon manifolds.

Lemma 4.5.17. Let X be a crisp Penon manifold. Then for $x : X$, we have

$$\mathfrak{D}_x X \simeq D_x^{\mathfrak{S}} X$$

over X .

Proof. By Lemma 4.5.15, $\mathfrak{S}X$ is a set as well and so for any $x, y : X$, the type $(x^{\mathfrak{S}} = y^{\mathfrak{S}})$ is a proposition. Therefore, $D_x^{\mathfrak{S}}$ is a subset of X for any $x : X$, so it suffices to show that for $y : X$ we have that $(x \approx y)$ if and only if $(x^{\mathfrak{S}} = y^{\mathfrak{S}})$.

First, suppose that $x \approx y$, seeking to show the proposition $(x^{\mathfrak{S}} = y^{\mathfrak{S}})$. Since X is a Penon manifold, choose a coordinate chart $\phi : \mathfrak{D}^n \simeq \mathfrak{D}_x X$, and note that the composite $\mathfrak{D}^n \xrightarrow{\phi} \mathfrak{D}_x X \hookrightarrow X \xrightarrow{(-)^{\mathfrak{S}}} \mathfrak{S}X$ is constant at $x^{\mathfrak{S}}$, which shows in particular that $(x^{\mathfrak{S}} = y^{\mathfrak{S}})$.

On the other hand, suppose that $(x^{\mathfrak{S}} = y^{\mathfrak{S}})$. By Lemma 4.5.16, we have a map $\mathfrak{S}X \rightarrow \sharp X$, and this gives us a map $(x^{\mathfrak{S}} = y^{\mathfrak{S}}) \rightarrow (x^{\sharp} = y^{\sharp})$. By Theorem 3.7 of [Shu18a], we have an equivalence $(x^{\sharp} = y^{\sharp}) \simeq \sharp(x = y)$, and by Theorem 3.15 of [Shu18a] we have an equivalence $\sharp(x = y) \simeq \neg\neg(x = y)$, which by definition was $(x \approx y)$. In total, we see that $(x^{\mathfrak{S}} = y^{\mathfrak{S}})$ implies $(x \approx y)$. \square

As a corollary, we can finally deduce that Penon étale maps between crisp Penon manifolds are the same as \mathfrak{S} -étale maps.

Theorem 4.5.18. *Let X and Y be crisp Penon manifolds and let $f : X \rightarrow Y$.⁶ Then f is \mathfrak{S} -étale if and only if it is Penon étale.*

⁶Note, f does not need to be crisp here, just the manifolds do.

Proof. By Lemma 4.5.17, for any $f : X \rightarrow Y$ between crisp Penon manifolds and $x : X$, we have a commuting square

$$\begin{array}{ccc} \mathfrak{D}_x X & \xrightarrow{\sim} & D_x^{\mathfrak{S}} X \\ f_* \downarrow & & \downarrow f_* \\ \mathfrak{D}_{fx} Y & \xrightarrow{\sim} & D_{fx}^{\mathfrak{S}} X \end{array}$$

Therefore, the left vertical map is an equivalence if and only if the right vertical map is. The map f is Penon étale if and only if the left vertical map is an equivalence, and the right vertical map is an equivalence if and only if f is \mathfrak{S} -étale by Proposition 3.7 of [CR20] (noting that the modal units of \mathfrak{S} are surjective, since it is given by nullifying a pointed type). \square

Corollary 4.5.19. Let $S \subseteq X$ be a crisp subset of a (necessarily crisp) Penon manifold X . If S is infinitesimally stable (meaning that if $x \in S$ and $x \approx y$, then $y \in S$) the inclusion $S \hookrightarrow X$ is \mathfrak{S} -étale.

Proof. Infinitesimal stability implies that for any $x \in S$, the infinitesimal neighborhood $\mathfrak{D}_x S$ of x in S is equivalent to the infinitesimal neighborhood $\mathfrak{D}_x X$ of x in X . Therefore, the result follows by Theorem 4.5.18. \square

Using the final proposition of [Pen81], we can prove as a corollary that the \mathfrak{S} -étale maps between the crisp ordinary manifolds are precisely the local diffeomorphisms in the usual sense. This does require an extra axiom which is considered by Penon (and proved to hold in the intended models by the same): the “infinitesimal inverse function theorem”.

Corollary 4.5.20. Suppose that the “infinitesimal inverse function theorem” holds: for every $f : \mathfrak{D}^n \rightarrow \mathfrak{D}^n$ with $f(0) = 0$, if f is invertible when restricted

to the first order infinitesimals $\mathbb{D}(n)$, then f is invertible.

Let X and Y be crisp Penon manifolds (or ordinary manifolds). Then $f : X \rightarrow Y$ is \mathfrak{S} -étale if and only if the square

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback.

Proof. Penon shows in the final proposition of [Pen81] that such a map is Penon étale if and only if the square of tangent bundles is a pullback, and so the result follows by Theorem 4.5.18. \square

4.5.3 Schreiber manifolds

Schreiber describes his notion of manifold in Definition 5.3.88 of [Sch13c]. We will use a slightly more general definition allowing for multiple different sorts of coordinate spaces.

Definition 4.5.21 (Definition 5.3.88 [Sch13c]). Let $V : I \rightarrow \mathbf{Type}$ be a fixed family of *coordinate spaces* indexed by a discrete type I . A *Schreiber V -manifold* is a type M for which there merely exists a V -atlas, which consists of:

1. A family of types $U : A \rightarrow \mathbf{Type}$ indexed by a discrete type A .
2. For every $a : A$, an \mathfrak{S} -étale map $i_a : U_a \hookrightarrow M$. We assume that these are jointly surjective: for every $p : M$ there is merely some $a : A$ and $u : U_a$ with $p = i_a(u)$.
3. For every $a : A$, an index $ka : I$ and an \mathfrak{S} -étale embedding $c_a : U_a \hookrightarrow V_{ka}$.

A variant of Schreiber manifolds were studied in homotopy type theory by Cherubini [**Cherubini:Cartan.Geometry**]. As a special case, we can consider Satake’s notion of orbifold as a space locally modelled on $\mathbb{R}^n // \Gamma$ where Γ is a finite subgroup of $O(n)$.

Definition 4.5.22. Let $I := \mathfrak{b}\{\text{finite subgroup of } O(n)\}$ be the type of crisp, finite subgroups of $O(n)$, and let $V : I \rightarrow \mathbf{Type}$ be defined by $V_{\Gamma^{\flat}} := \mathbb{R}^n // \Gamma$. A *Schreiber-Satake orbifold* is a Schreiber V -manifold for this choice of V .

As a quick corollary of Theorem 4.4.42, we can prove that any Schreiber V -manifold where V is a family of microlinear types is also microlinear. We need two quick lemmas first: crystalline sum of \mathfrak{S} -étale maps are \mathfrak{S} -étale and crystalline sum of microlinear types are microlinear.

Lemma 4.5.23. Let $U : A \rightarrow \mathbf{Type}$ and $X : I \rightarrow \mathbf{Type}$ be type families indexed by \mathfrak{S} -crystalline types A and I . Let $k : A \rightarrow I$ be a map and let $i_a : U_a \rightarrow X_{ka}$ be \mathfrak{S} -étale for all $a : A$. Then the map $\sum_k i : (a : A) \times U_a \rightarrow (i : I) \times X_i$ defined by $\sum_k i(a, u) := i_a(u)$ is \mathfrak{S} -étale.

Proof. Consider the square

$$\begin{array}{ccc} (a : A) \times U_a & \longrightarrow & (a : A) \times \mathfrak{S}U_a \\ \sum_k i \downarrow & & \downarrow \sum_k \mathfrak{S}i \\ (i : I) \times X_i & \longrightarrow & (i : I) \times \mathfrak{S}X_i \end{array}$$

where $\sum_k \mathfrak{S}i(a, z) := (ka, \mathfrak{S}i_a(z))$. By Lemma 1.24 of [**RSS:Modalities.in.HoTT**] and the fact that A and I are \mathfrak{S} -modal, the horizontal maps are \mathfrak{S} -units and this square is an \mathfrak{S} -naturality square. Therefore, to show that $i \equiv \sum_a i_a$ is \mathfrak{S} -étale, we just need to show that this square is a pullback. Consider a point

$(i, x) : (i : I) \times X_i$ and the induced map

$$(((a, p) : \mathbf{fib}_k(i)) \times \mathbf{fib}_{i_a}(x)) \rightarrow (((a, p) : \mathbf{fib}_k(i)) \times \mathbf{fib}_{\mathfrak{S}i_a}(x^{\mathfrak{S}}))$$

on fibers of the vertical maps. Note that this map is the sum of the maps $\mathbf{fib}_{i_a}(x) \rightarrow \mathbf{fib}_{\mathfrak{S}i_a}(x^{\mathfrak{S}})$ induced by the \mathfrak{S} -naturality squares of $i_a : U_a \rightarrow X_{ka}$, which by hypothesis were equivalences. Therefore, this map is an equivalence, and $\sum_k i$ is \mathfrak{S} -étale. \square

Lemma 4.5.24. Let $X : I \rightarrow \mathbf{Type}$ be a family of microlinear types indexed by a crystalline type I . Then the sum $(i : I) \times X_i$ is microlinear.

Proof. Let V be an infinitesimal \mathbb{R} -pushout and consider a map $v : V_2 +_{V_1} V_3 \rightarrow (i : I) \times X_i$, seeking to show that the type of lifts $(\tilde{v} : V_4 \rightarrow (i : I) \times X_i) \times (\tilde{v} \circ \nabla_V = v)$ is contractible. Now, since $V_2 +_{V_1} V_3$ and V_4 are \mathfrak{S} -connected, any map from them into a crystalline type is constant and so

$$\begin{aligned} (V_2 +_{V_1} V_3 \rightarrow (i : I) \times X_i) &\simeq ((v^1 : V_2 +_{V_1} V_3 \rightarrow I) \times ((\varepsilon : V_2 +_{V_1} V_3) \rightarrow X_{v^1\varepsilon})) \\ &\simeq (i : I) \times (V_2 +_{V_1} V_3 \rightarrow X_i) \end{aligned}$$

and similarly $V_4 \rightarrow (i : I) \times X_i$ is equivalent to $(i : I) \times (V_4 \rightarrow X_i)$. Let v correspond to (i_0, v') under this equivalence; the type of lifts is therefore equivalent to $(i : I) \times (\tilde{v}' : V_4 \rightarrow X_i) \times (\tilde{v}' \circ \nabla_V = v') \times (i = i_0)$. We can contract i into i_0 so that this type is equivalent to $(\tilde{v}' : V_4 \rightarrow X_{i_0}) \times (\tilde{v}' \circ \nabla_V = v')$ which was contractible by hypothesis. \square

Theorem 4.5.25. Any Schreiber V -manifold for a family of microlinear types $V : I \rightarrow \mathbf{Type}$ is itself microlinear.

Proof. Consider an atlas of X and the associated span

$$\begin{array}{ccc}
 & (a : A) \times U_a & \\
 \Sigma_k c \swarrow & & \searrow \Sigma i \\
 (i : I) \times V_i & & X
 \end{array}$$

given by summing up the maps $i_a : U_a \rightarrow X$ and $c_a : U_a \rightarrow V_{ka}$. By Lemma 4.5.23, these maps are \mathfrak{S} -étale, and by hypothesis the right leg i is surjective. Therefore, by combining Lemma 4.4.40 and Theorem 4.4.42, we see that X is microlinear when $(i : I) \times V_i$ is. And $(i : I) \times V_i$ is microlinear when each of the V_i is by Lemma 4.5.24. \square

4.5.4 Étale groupoids

Given Theorem 4.4.42, it is easy to show that the quotient $X // \Gamma$ of the action of a crisply discrete higher group Γ on a microlinear type X is microlinear.

Theorem 4.5.26. *Let Γ be a crisply discrete group, and let $X^{\circ-} : \mathbf{B}\Gamma \rightarrow \mathbf{Type}$ be an action of Γ on a microlinear type $X := X^{\circ\text{pt}_{\mathbf{B}\Gamma}}$. Then the homotopy quotient $X // \Gamma$ is microlinear.*

Proof. By Theorem 7.7 of Chapter 3, the quotient map $q : X \rightarrow X // \Gamma$ is \mathfrak{f} -étale and therefore by Lemma 4.4.36 is \mathfrak{S} -étale. Since q is surjective and \mathfrak{S} -étale, by Theorem 4.4.42, we see that if X is microlinear then $X // \Gamma$ is as well. \square

As a corollary, we can give a satisfying condition for the microlinearity of a crisp type.

Corollary 4.5.27. *Suppose that X is a crisp, pointed type, and that X is path connected in the sense that $\int_1 X$ is 0-connected. If the universal cover \tilde{X} of X is microlinear, then X is microlinear.*

Proof. Since $\int_1 X$ was presumed to be 0-connected, it is a $\mathbf{B}\pi_1(X)$ (where we define $\pi_1(X) \equiv \|\Omega \int_1 X\|_0$ to be the fundamental group of X). By definition, the universal cover \tilde{X} is the fiber of the unit $(-)^{\int_1} : X \rightarrow \int_1 X$. Therefore, we can see the map $t \mapsto \mathbf{fib}_{(-)^{\int_1}}(t) : \mathbf{B}\pi_1 X \rightarrow \mathbf{Type}$ as giving the monodromy action of $\pi_1(X)$ on the universal cover \tilde{X} . The homotopy quotient is therefore $(t : \mathbf{B}\pi_1(X)) \times \mathbf{fib}_{(-)^{\int_1}}(t)$, which is equivalent to X ; in other words, we have $X \simeq \tilde{X} // \pi_1(X)$. Since $\pi_1(X)$ is a crisp group, Theorem 4.5.26 then shows that if \tilde{X} is microlinear, so is X . \square

However, not every orbifold may be presented as the quotient of a smooth space by the action of a discrete group. A general way to present orbifolds is with *proper étale groupoids*, as first defined by Moerdijk and Pronk [MP97]. In this section, we will show that the wider class of étale groupoids are microlinear; in the next section, we'll discuss the notion of compactness appropriate for synthetic differential geometry, define proper étale groupoids, and prove that they are orbifolds in the sense of Definition 4.1.2.

First, let's recall the notion of *pregroupoid*. In HoTT, a groupoid is best understood as a type which is 1-truncated. However, the traditional definition of a groupoid as having a set of objects and a set of isomorphism between these objects can still be performed in HoTT; the resulting notion is that of a *pregroupoid*. The terminology is by analogy with the relation between preorders and ordered sets.

Definition 4.5.28. A *precategory* \mathcal{C} consists of a type of objects \mathcal{C}_0 , and for each two objects $x, y : \mathcal{C}_1$ a set $\mathcal{C}(x, y)$ of morphisms, together with an associative, unital composition of morphisms. A *pregroupoid* is a precategory where every

morphism is an isomorphism.

A precategory is a *category* (or a *univalent category*, for emphasis) if the map $\text{idtoiso} : (x = y) \rightarrow \text{Iso}_{\mathcal{C}}(x, y)$ defined by $\text{refl}_x \mapsto \text{id}_x$ is an equivalence for all objects $x, y : \mathcal{C}_0$.

Remark 4.5.29. A univalent pregroupoid \mathcal{G} — one for which the map $\text{idtoiso} : (x = y) \rightarrow \text{Iso}_{\mathcal{G}}(x, y)$ is an equivalence for all $x, y : \mathcal{G}_0$ — carries no more information than its type \mathcal{G}_0 of objects, since $\text{Iso}_{\mathcal{G}}(x, y) = \mathcal{G}(x, y)$. Furthermore, since by hypothesis there is a set of morphisms $\mathcal{G}(x, y)$, we find that there is a set of identifications $(x = y)$, making \mathcal{G}_0 into a groupoid in the sense of being a 1-type. Therefore, we are free to identify univalent pregroupoids with their groupoids (1-types) of objects. We will therefore drop the subscript on \mathcal{G}_0 when talking about groupoids.

There is a universal groupoid generated by any pregroupoid: the *Rezk completion*. For more, see Section 9.9 of the HoTT Book [HoTT.Book].

Definition 4.5.30. The *Rezk completion* $r\mathcal{C}$ of a precategory \mathcal{C} is the essential image of the Yoneda embedding — the full subcategory of the category $\hat{\mathcal{C}}$ of presheaves on \mathcal{C} spanned by the representable functors. Explicitly $r\mathcal{C}$ has objects those presheaves F for which there merely exists an $x : \mathcal{C}$ and a natural isomorphism $\mathcal{C}(-, x) \cong F$.

With these definitions, we can now define the notion of étale groupoid.

Definition 4.5.31. An *étale pregroupoid* is a pregroupoid \mathcal{G} where

- The type \mathcal{G}_0 of objects is microlinear.

- The source map $((x, y, p) \mapsto x) : \mathcal{G}_1 \rightarrow \mathcal{G}_0$ from the type of morphisms $\mathcal{G}_1 \equiv (x, y : \mathcal{G}_0) \times \mathcal{G}(x, y)$ to the type of objects is \mathfrak{S} -étale.

An *étale groupoid* is a groupoid which is equivalent to the Rezk completion of an étale pregroupoid.

In order to prove that étale groupoids are microlinear, we will show that the Rezk completion of étale pregroupoids are microlinear. To do this, we will show that the Yoneda embedding $y : \mathcal{G}_0 \rightarrow r\mathcal{G}$ is \mathfrak{S} -étale. Since y is by definition surjective, and since \mathcal{G}_0 is by hypothesis microlinear, it will follow by Theorem 4.4.42 that $r\mathcal{G}$ is étale.

The proof that $y : \mathcal{G}_0 \rightarrow r\mathcal{G}$ is \mathfrak{S} -étale for an étale pregroupoid \mathcal{G} is not trivial. It will follow from the following theorem, and only in the case that \mathcal{G} is crisp.

Theorem 4.5.32. *Suppose that \diamond is a modality with surjective units which preserves \emptyset , crisp pushouts, and colimits of crisp sequences. Let $f :: A \rightarrow B$ be a crisp, surjective map with inhabited domain A , and suppose that the pullback $\text{fst} : A \times_B A \rightarrow A$ of f along itself is \diamond -étale:*

$$\begin{array}{ccc}
 A \times_B A & \xrightarrow{\text{snd}} & A \\
 \text{fst} \downarrow & \lrcorner & \downarrow f \\
 A & \xrightarrow{f} & B
 \end{array}$$

Then f is also \diamond -étale

We state Theorem 4.5.32 in this abstract way so that it applies not only to \mathfrak{S} , but also to \mathfrak{f} . That Theorem 4.5.32 applies to the modality \mathfrak{S} at all is due to Postulate W : the type \mathfrak{D} is presumed to be *tiny*. Results of Section 4.7 allow us to show that \mathfrak{S} preserves crisp pushouts and colimits of sequences.

Theorem 4.5.33. *The modality \mathfrak{S} preserves crisp pushouts and colimits of crisp sequences.*

Proof. This is a special case of Theorem 4.7.9, since \mathfrak{D} is tiny. \square

Before proving Theorem 4.5.32, we will use it to show that crisp étale groupoids are microlinear.

Theorem 4.5.34. *Crisp étale groupoids are microlinear.*

Proof. If \mathcal{H} is a crisp étale groupoid, then by hypothesis there is a crisp étale pregroupoid \mathcal{G} with $r\mathcal{G} \simeq \mathcal{H}$. It will therefore suffice to show that $r\mathcal{G}$ is microlinear. Since \mathcal{G}_0 is by hypothesis microlinear, it suffices to show that $y : \mathcal{G}_0 \rightarrow r\mathcal{G}$ is \mathfrak{S} -étale.

Now, for every pair of objects $x, y : \mathcal{G}_0$, we have that $(y(x) = y(y)) \simeq \mathcal{G}(x, y)$ by the Yoneda lemma. Therefore, we have a pullback square:

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{t} & \mathcal{G}_0 \\ s \downarrow & & \downarrow y \\ \mathcal{G}_0 & \xrightarrow{y} & r\mathcal{G} \end{array}$$

Since y is crisp and surjective, and s is \mathfrak{S} -étale, it follows by Theorem 4.5.32 that y is \mathfrak{S} -étale. \square

In order to prove Theorem 4.5.32, we will need a few useful lemmas. The first two lemmas that we need show that if all the maps in a crisp diagram \mathcal{D} are \mathfrak{S} -étale, then so are all the maps in the cocone $\mathcal{D} \rightarrow \text{colim } \mathcal{D}$. In all of the following lemmas, we will assume that \diamond is a modality with surjective units which preserves \emptyset , crisp pushouts, and crisp colimits of sequences.

Lemma 4.5.35. Suppose that

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ f \downarrow & & \downarrow k \\ C & \xrightarrow{h} & D \end{array}$$

is a crisp pushout. If f and g are \diamond -étale, then so are h and k .

Proof. This follows by the fact that \diamond preserves crisp pushouts and by Mather's cube theorem, which is proven in HoTT in Theorem 2.2.11 of [Rij18b]. Consider the cube

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & | & \searrow & \\ B & & & & C \\ & \swarrow & \downarrow & \searrow & \\ & & \diamond A & & \\ \diamond B & & & & \diamond C \\ & \swarrow & \downarrow & \searrow & \\ & & \diamond D & & \end{array}$$

The top face is a pushout by hypothesis, and since \diamond was presumed to preserve crisp pushouts, the bottom face is as well. If f and g are \diamond -étale, then the back two faces are pullbacks. This implies that the front two faces are pullbacks, which shows that h and k are \diamond -étale. \square

Lemma 4.5.36. Suppose that

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} \dots$$

is a crisp sequence of types with colimit A_∞ . If i_j is \diamond -étale for all j , then so are the maps $i_{j,\infty} : A_j \rightarrow A_\infty$.

Proof. This follows from the analogous descent property for sequences as Mather's cube theorem is for pushouts, by essentially the same argument. \square

Next, we need a useful closure property of étale maps.

Lemma 4.5.37. Suppose that $f : A \rightarrow B$ is \diamond -étale and that $g : B \rightarrow C$. If $g \circ f$ is \diamond -étale and $\diamond f$ is surjective, then g is \diamond -étale.

Proof. This follows from the analogous property for pullbacks. In the given situation, we have a diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ \diamond A & \xrightarrow{\diamond f} & \diamond B & \xrightarrow{\diamond g} & \diamond C \end{array}$$

in which the left and composite squares are pullbacks, and where $\diamond f$ is surjective. It follows that the right square is also a pullback, which means that g is \diamond -étale. \square

Finally, we are ready to prove Theorem 4.5.32.

Proof of Theorem 4.5.32. By Rijke's join construction [Rij17], B is the sequential colimit of the sequence

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \dots$$

which is inductively defined by defining A_0 to be \emptyset , $i_0 : A_0 \rightarrow A_1$ and $f_0 : A_0 \rightarrow B$ to be the unique maps, and then defining $A_{n+1} \equiv A_n *_B A$ and $f_{n+1} : A_{n+1} \rightarrow B$, and $i_{n+1} : A_n \rightarrow A_{n+1}$ by the universal property:

$$\begin{array}{ccc} A_n \times_B A & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \\ A_n & \xrightarrow{i_{n+1}} & A_{n+1} \\ & \searrow f_n & \downarrow f_{n+1} \\ & & B \end{array}$$

Here, the outer square is a pullback, and the inner square is a pushout defining A_{n+1} as the join of A_n and A over B . Note that this sequence is crisp, and that $A_1 \simeq A$. By Lemma 4.5.36, it will suffice to show that each i_n is \diamond -étale. We can argue this by induction.

First, we note that $i_0 : A_0 \rightarrow A_1$ is \diamond -étale since \diamond preserves \emptyset and any square

$$\begin{array}{ccc} \emptyset & \xrightarrow{i_0} & A_1 \\ \parallel & & \downarrow \\ \diamond\emptyset & \xrightarrow{\diamond i_0} & \diamond A_1 \end{array}$$

is a pullback. Now, suppose that all maps i_j for $j < n$ are \diamond -étale, seeking to show that i_n is \diamond -étale. Since i_n is constructed as a pushout inclusion

$$\begin{array}{ccc} A_n \times_B A & \xrightarrow{\text{snd}} & A \\ \text{fst} \downarrow & \lrcorner & \downarrow \\ A_n & \xrightarrow{i_{n+1}} & A_{n+1} \end{array}$$

it will suffice to prove that both projections $\text{fst} : A_n \times_B A \rightarrow A_n$ and $\text{snd} : A_n \times_B A \rightarrow A$ are \diamond -étale by Lemma 4.5.35. Consider the following diagram:

$$\begin{array}{ccccc} & & \text{snd} & & \\ & & \curvearrowright & & \\ A \times_B A & \xrightarrow{i \times_B A} & A_n \times_B A & \xrightarrow{\text{snd}} & A \\ \text{fst} \downarrow & \lrcorner & \text{fst} \downarrow & \lrcorner & \downarrow f \\ A & \xrightarrow{i} & A_n & \xrightarrow{f_n} & B \\ & & \text{fst} & & \\ & & \curvearrowleft & & \\ & & f & & \end{array}$$

Here the map $i : A \rightarrow A_n$ is the composite of all maps $A \xrightarrow{\sim} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \cdots \xrightarrow{i_{n-1}} A_n$. Note that since A is inhabited, each $i_j : A_j \rightarrow A_{j+1}$ is surjective

and therefore $i : A \rightarrow A_n$ is surjective. The right square is a pullback by definition, and so is the outer square; therefore, the left square is also a pullback. We know by assumption that $\text{fst} : A \times_B A \rightarrow A$ is \diamond -étale, and by inductive hypothesis that $i : A \rightarrow A_n$ is \diamond -étale. By symmetry, we also know that $\text{snd} : A \times_B A \rightarrow A$ is \diamond -étale.

To show that both $\text{fst} : A_n \times_B A \rightarrow A_n$ and $\text{snd} : A_n \times_B A \rightarrow A$ are \diamond -étale, it therefore suffices by Lemma 4.5.37 to show that $i \times_B A : A \times_B A \rightarrow A_n \times_B A$ is surjective and \diamond -étale. But this is the pullback of $i : A \rightarrow A_n$, which as we noted above is surjective and \diamond -étale, so this concludes our proof. \square

4.5.5 Deloopings of infinitesimally linear groups are infinitesimally linear

So far, we have been showing that various notions of smooth spaces are micro-linear. This in particular implies that their tangent spaces are \mathbb{R} -modules by Theorem 4.4.29. But there is a weaker condition which implies the same thing: infinitesimal linearity (??). Infinitesimal linearity says essentially nothing but the fact that the tangent spaces are \mathbb{R} -modules.

Theorem 4.4.29 was proven with no truncation conditions on the infinitesimally linear type X . We have seen some higher types which are infinitesimally linear since they are microlinear — for example $X // \Gamma$ where Γ is a discrete higher group (Theorem 4.5.26). But these examples have all had discrete types of identifications. If we restrict ourselves to infinitesimal linearity, we can find examples of higher types whose spaces of identifications are not discrete. In particular, we can show that if G is a crisp infinitesimally linear higher group (for example, a Lie group), then any delooping BG will be infinitesimally linear.

To prove this, we first need a general lemma about higher groups.

Lemma 4.5.38. Consider a square of homomorphisms of higher groups:

$$\begin{array}{ccc}
 \mathbf{B}G_1 & \xrightarrow{\mathbf{B}f} & \mathbf{B}G_3 \\
 \mathbf{B}h \downarrow & & \downarrow \mathbf{B}k \\
 \mathbf{B}G_2 & \xrightarrow{\mathbf{B}g} & \mathbf{B}G_4
 \end{array}
 \qquad
 \begin{array}{ccc}
 G_1 & \xrightarrow{f} & G_3 \\
 h \downarrow & \lrcorner & \downarrow k \\
 G_2 & \xrightarrow{g} & G_4
 \end{array}$$

That is, consider a square of pointed maps between pointed, 0-connected types on the left. If $k \equiv \Omega \mathbf{B}k$ is surjective and the looped square on the right is a pullback, then the square on the left is a pullback.

Proof. Because $\mathbf{B}G_4$ is 0-connected and k is surjective, $\mathbf{B}k$ is 0-connected. Since the square on the right is a pullback, h is also surjective and so $\mathbf{B}h$ is 0-connected. To show that the square on the left is a pullback, it suffices to show that the map $\mathbf{B}f_* : \mathbf{fib}_{\mathbf{B}h}(\mathbf{pt}_{\mathbf{B}G_2}) \rightarrow \mathbf{fib}_{\mathbf{B}k}(\mathbf{pt}_{\mathbf{B}G_4})$ is an equivalence. But these types are both pointed and 0-connected, so by the “fundamental theorem of higher groups”, it suffices to prove that $\Omega \mathbf{B}f_*$ is an equivalence, or equivalently that $f_* : \mathbf{fib}_h(1_{G_2}) \rightarrow \mathbf{fib}_k(1_{G_4})$ is an equivalence. But the square on the right is a pullback, so f_* is an equivalence. \square

Warning 4.5.39. The surjectivity of k is necessary. Let $k : G_3 \rightarrow G_4$ be the doubling map $2 : \mathbb{Z} \rightarrow \mathbb{Z}$, and let $G_2 \equiv *$ be trivial. Then $G_1 = \ker k = *$ is trivial, but $\mathbf{fib}_{\mathbf{B}k}(\mathbf{pt}_{\mathbf{B}G_4})$ is the quotient $\mathbb{Z}/2$ of the action $2 : \mathbb{Z} \rightarrow \mathbb{Z}$. This is not 0-connected and so does not deloop G_1 .

Theorem 4.5.40. *Let G be a crisp, infinitesimally linear higher group. Then $\mathbf{B}G$ is infinitesimally linear.*

Proof. We need to show that the squares

$$\begin{array}{ccc}
(\mathbf{B}G)^{\mathbb{D}(n+m)} & \longrightarrow & (\mathbf{B}G)^{\mathbb{D}(n)} \\
\downarrow & & \downarrow \\
(\mathbf{B}G)^{\mathbb{D}(m)} & \longrightarrow & \mathbf{B}G
\end{array}$$

are pullbacks. By Postulate W, the crisp infinitesimal varieties $\mathbb{D}(n)$ are tiny in the sense of Definition 4.7.1. Therefore, by Corollary 4.7.8, the types $(\mathbf{B}G)^{\mathbb{D}(n)}$ are 0-connected. Furthermore, the projection $G^{\mathbb{D}(n)} \rightarrow G$ is surjective since it splits (by precomposition along the map $\mathbb{D}(n) \rightarrow *$). By hypothesis, the looped square is a pullback since G was assumed to be infinitesimally linear. Therefore, Lemma 4.5.38 applies and we may conclude that this square is a pullback. \square

Remark 4.5.41. Note that the proof of Theorem 4.5.40 made essential use of the surjectivity of the projection $G^{\mathbb{D}(n)} \rightarrow G$. This is why the same argument does not show that the delooping of a crisp microlinear group is microlinear; in the general case, precomposition $G^{V_2} \rightarrow G^{V_1}$ along a map $V_1 \rightarrow V_2$ of crisp infinitesimal varieties need not be surjective. I do not know whether $\mathbf{B}G$ of a crisp microlinear (higher) group G is necessarily microlinear.

Corollary 4.5.42. Let G be a crisp, infinitesimally linear higher group, and define $\mathfrak{g} := T_1G$ to be its “higher Lie algebra”. Then \mathfrak{g} is a higher group with delooping $\mathbf{B}\mathfrak{g} := T_{\mathbf{pt}_{\mathbf{B}G}}\mathbf{B}G$, and its delooping $\mathbf{B}\mathfrak{g}$ is a (higher) \mathbb{R} -module.

Proof. By definition, $T_{\mathbf{pt}_{\mathbf{B}G}}\mathbf{B}G := (v : \mathbb{D} \rightarrow \mathbf{B}G) \times (v(0) = \mathbf{pt}_{\mathbf{B}G})$ is the fiber of the tangent bundle projection $\mathbf{ev}_0 : (\mathbf{B}G)^{\mathbb{D}} \rightarrow \mathbf{B}G$. By Corollary 4.7.8, $(\mathbf{B}G)^{\mathbb{D}} = \mathbf{B}(G^{\mathbb{D}})$ is 0-connected and deloops the tangent space of G , and the projection $(\mathbf{B}G)^{\mathbb{D}} \rightarrow \mathbf{B}G$ deloops the projection $G^{\mathbb{D}} \rightarrow G$. Therefore, the fiber $T_{\mathbf{pt}_{\mathbf{B}G}}\mathbf{B}G$ of $(\mathbf{B}G)^{\mathbb{D}} \rightarrow \mathbf{B}G$ deloops the fiber \mathfrak{g} of $G^{\mathbb{D}} \rightarrow G$ over $1 : G$. \square

4.6 Finiteness, compactness, and proper étale groupoids

We have seen that étale groupoids are microlinear. But not all étale groupoids are orbifolds — there is a finiteness condition in the definition of an orbifold. In an étale pregroupoid, this is traditionally handled by asking that the map $(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$ be proper (see [MP97]). Using the usual definition of proper (inverse images of compact sets are compact) with the usual definition of compact (every open cover admits a finitely enumerable subcover), we can actually show that the isotropy groups of such an étale pregroupoid are *finite*.

But we don't actually want for the isotropy groups to be finite, because this would imply that they have a constant cardinality over any connected component of our orbifold. This is because there is a function $\text{card} : \text{Fin} \rightarrow \mathbb{N}$ which sends a finite set F to its cardinality $\text{card}(F)$. If for all $x, y : X$, the type $(x = y)$ were finite, then we would have a function $(x, y) \mapsto \text{card}(x = y) : X \times X \rightarrow \mathbb{N}$. Because \mathbb{N} is discrete, this map would factor through the set $\int_0(X \times X)$ of connected components of $X \times X$. In other words, the cardinality of $(x = y)$ would be constant on each connected component.

This discussion reveals that being finite is a very strong condition on a set in constructive mathematics. Luckily, and sometimes frustratingly, there are many different, weaker notions of finiteness in constructive mathematics, which we survey in Definition 4.6.1. In Section 4.6.1, we will introduce a new notion of finiteness for our own purposes: *properly finite* (see Definition 4.6.3). A set is properly finite when it is discrete and a subquotient of a finite set.

Changing our notion of finiteness means that we will also need to change

our notion of compactness to match. Dubuc and Penon have investigated a useful notion of compactness which, in the intended models, corresponds on crisp ordinary manifolds with external compactness ([DP86]). We recall Dubuc and Penon's notion of compactness in Definition 4.6.10.

Using Dubuc-Penon compactness, in Definition 4.6.36 we will define an ordinary proper étale pregroupoid to be an étale pregroupoid \mathcal{G} where \mathcal{G}_0 and \mathcal{G}_1 are ordinary manifolds, and where the map $(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is Dubuc-Penon proper in the sense that the inverse image of any Dubuc-Penon compact subset is Dubuc-Penon compact. With this definition of ordinary proper étale groupoid, we may then state Theorem 4.6.37: the Rezk completion of a crisp, ordinary proper étale pregroupoid is an orbifold in the sense of Definition 4.1.2.

Since we have already shown that the Rezk completion $\mathbf{r}\mathcal{G}$ of a crisp étale pregroupoid is microlinear, it only remains to show that the types of identifications in $\mathbf{r}\mathcal{G}$ are properly finite. The types of identifications in $\mathbf{r}\mathcal{G}$ are identifiable with the sets of maps $\mathcal{G}(x, y)$ in the pregroupoid \mathcal{G} . It follows from the assumption that \mathcal{G} is proper étale that $\mathcal{G}(x, y)$ is crystalline and Dubuc-Penon compact. We can therefore argue that $\mathcal{G}(x, y)$ is properly finite in two stages: first, in Lemma 4.6.34, that crystalline subsets of ordinary manifolds are discrete, and second, in Lemma 4.6.30, that discrete Dubuc-Penon compact subsets of ordinary manifolds are properly finite.

In Section 4.6.2, we will explore the notion of Dubuc-Penon compactness (Definition 4.6.10). In particular, we will show that Dubuc-Penon compact sets are compact in ways more closely resembling ordinary compactness: in Theorem 4.6.22 we show that any Dubuc-Penon compact set is countably compact and *subcountably subcompact*, and in Corollary 4.6.23 that any Dubuc-Penon

compact subset of an ordinary manifold is *refinement subcompact*. See Definition 4.6.20 for the definitions of these subtly differing notions of compactness. These results depend in an essential way on the Covering Property appearing in Axiom 5. At the end, we will prove Lemma 4.6.30: discrete Dubuc-Penon compact subsets of ordinary manifolds are properly finite.

In Section 4.6.3, we will prove Lemma 4.6.34: crystalline subsets of ordinary manifolds are discrete. This completes the proof of Theorem 4.6.37.

Finally, in Section 4.6.4, we note that the quotient of a microlinear set by the action of a finite group is an orbifold, and that orbifolds are closed under pullback. As a corollary, the inertia orbifold X^{S^1} of any orbifold is itself an orbifold.

4.6.1 Notions of finiteness

In constructive mathematics, the notion of “finiteness” fractures into a number of inequivalent notions.

Definition 4.6.1 (Standard). Let X be a set. Then:

1. X is *finite* if there is an $n : \mathbb{N}$ and an equivalence $X \simeq \underline{n}$ with the standard finite ordinal $\underline{n} := \{0, \dots, n - 1\}$.
2. X is *subfinite* if there is an $n : \mathbb{N}$ and an embedding $X \hookrightarrow \underline{n}$.
3. X is *finitely enumerable* (also known as Kurotowski finite) if there is an $n : \mathbb{N}$ and a surjection $\underline{n} \twoheadrightarrow X$.
4. X is *subfinitely enumerable* if there is a subfinite set \tilde{X} and a surjection $\tilde{X} \twoheadrightarrow X$.

These notions of finiteness are liminal in the sense that they are only distinct for non-crisp sets (sets which, in some sense, vary continuously in a free variable). Crisply, there is only one notion of finiteness.

Proposition 4.6.2. Any crisp subfinitely enumerable set is finite.

Proof. Let X be a crisp subfinitely enumerable set. Then there is a crisp subset $\tilde{X} \subseteq \underline{n}$ and a crisp surjection $f : \tilde{X} \rightarrow X$. Since \tilde{X} is crisp and \underline{n} is crisply discrete, the predicate $i \in \tilde{X}$ for $i :: \underline{n}$ is crisp, and therefore decidable. So, as a decidable subset of a finite set, \tilde{X} is finite.

This leaves X as the crisp quotient of a finite set. The relation $x \sim y$ defined by $f(x) = f(y)$ on \tilde{X} may therefore also be taken as crisp and therefore decidable. So, as the quotient of a finite set by a decidable relation, X is finite. \square

We will add an additional notion of finiteness to this list, one which has already appeared in our definition of orbifold (Definition 4.1.2).

Definition 4.6.3. A set X is said to be *properly finite* if it is subfinitely enumerable and has decidable equality — for all $x, y : X$, $(x = y) \vee (x \neq y)$. By Lemma 8.15 of [Shu18a], a set is properly finite if and only if it is discrete and subfinitely enumerable.

Remark 4.6.4. It is folklore that a discrete finitely enumerable set is necessarily finite. This is because such a set is the quotient of a finite set \underline{n} by a decidable equivalence relation $\sim \subseteq \underline{n} \times \underline{n}$, which is a decidable subset of a finite set and therefore itself finite. Therefore, the quotient can also be shown to be finite.

We might therefore expect that a properly finite subset must necessarily be subfinite. This would be very nice, but I have neither managed to prove it nor construct a counterexample.

Lemma 4.6.5. Subfinite sets (and so also finite sets) are properly finite.

Proof. As subsets of discrete sets, subfinite sets are discrete. They are also of course subfinitely enumerable. \square

Warning 4.6.6. Though subfinite sets are properly finite, it is not necessarily the case that any subfinitely enumerable set is properly finite. Let $z : \mathbb{R}$ and let $Q_z = \{0, 1\} / \sim$ where $0 \sim 1$ if and only if $z = 0$ (type theoretically, this is the suspension $\Sigma(z = 0)$ of the proposition $z = 0$). This set is finitely enumerable, and therefore also subfinitely enumerable. But it is not discrete, since the proposition $[0] = [1]$ in Q_z is equivalent to $z = 0$, which is not decidable.

The following proposition gives us another useful definition of a subfinitely enumerable set, and therefore also of a properly finite set.

Proposition 4.6.7 (Standard). A set X is subfinitely enumerable when there exists an $n : \mathbb{N}$ and a relation $r : X \times \underline{n} \rightarrow \mathbf{Prop}$ such that:

1. For all $x : X$, there is some $i : \underline{n}$ with $r(x, i)$.
2. If $r(x, i)$ and $r(x, j)$, then $i = j$.

We will refer to such a relation as an *association* of X with \underline{n} . If $r \subseteq X \times Y$ is an association of X with Y and Y is subfinitely enumerable, then X is also subfinitely enumerable

Proof. The relation itself, considered as a subset of the product $X \times \underline{n}$, defines the \tilde{X} which is a subset of \underline{n} and which surjects onto X . If $r \subseteq X \times Y$ and $s \subseteq Y \times \underline{n}$ are associations, then the composite relation $s \circ r \subseteq X \times \underline{n}$ defined by

$$(s \circ r)(x, i) := \exists y : Y. r(x, y) \wedge s(y, i)$$

is also an association. For all $x : X$, there is a $y : Y$ with $r(x, y)$, and for this y and $i : \underline{n}$ with $s(y, i)$, so that $(s \circ r)(x, i)$. If there is a y such that $r(x, y)$ and $s(y, i)$ and a y' such that $r(x, y')$ and $s(y', j)$, then $y = y'$ and therefore $i = j$. \square

We can also show that (subfinitely) enumerable sets may be equally defined as sub(finitely enumerable) sets.

Lemma 4.6.8 (Standard). X is subfinitely enumerable if and only if there exists a finitely enumerable \hat{X} and an embedding $X \hookrightarrow \hat{X}$.

Proof. Consider the following square:

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & \underline{n} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \hat{X} \end{array}$$

If X is subfinitely enumerable, then \tilde{X} exists and we may define \hat{X} by pushout. Conversely, if \hat{X} exists then we may define \tilde{X} by pullback. \square

Let's now prove some closure properties of properly finite sets.

Proposition 4.6.9. The following closure properties hold of properly finite sets.

1. Subsets of properly finite sets are properly finite.
2. The product of two properly finite sets is properly finite.
3. The pullback of properly finite sets is properly finite.
4. A finite disjoint union of properly finite sets is properly finite.
5. If X is subfinitely enumerable and S is discrete and $f : X \rightarrow S$, then $\text{im } f \subseteq S$ is properly finite.
6. Properly finite subsets of a discrete set are closed under finite union.

Proof. We will make use of Proposition 4.6.7.

1. Suppose that $S \subseteq X$ and X is properly finite as witnessed by the association $r \subseteq X \times \underline{n}$. Then r restricted to S remains an association, so that S is subfinitely enumerable. Furthermore, as the subset of a discrete set, it is also discrete; it is therefore properly finite.
2. Suppose that X and Y are properly finite as witnessed by the associations $r \subseteq X \times \underline{n}$ and $s \subseteq Y \times \underline{m}$. Then the relation $(r \times s)((x, y), (i, j)) := r(x, i) \wedge s(y, j)$ gives an association $r \times s \subseteq (X \times Y) \times (\underline{n} \times \underline{m})$. The product of discrete types is discrete as well, since f is a modality.
3. The pullback $X \times_Z Y$ of sets is a subset of the product $X \times Y$, so this follows by the previous two properties.
4. Let X_i be properly finite sets for $i : \underline{k}$; we will show that $(i : \underline{k}) \times X_i$ is properly finite. Suppose that $r_i \subseteq X_i \times \underline{n}_i$ is an association witnessing the proper finiteness of X_i . Define $r((i, x), (i', j)) := (i = i') \wedge r_i(x, j)$

to be a relation $r \subseteq ((i : \mathbf{k}) \times X_i) \times ((i : \mathbf{k}) \times \mathbf{n}_i)$; we will show that this is an association. First, for any (i, x) , there is a j with $r_i(x, j)$, so that $r((i, x), (i, j))$. Second, if $r((i, x), (i_1, j_1))$ and $r((i, x), (i_2, j_2))$, then $i_1 = i = i_2$ and so $r_i(x, j_1)$ and $r_i(x, j_2)$, from which we conclude that $j_1 = j_2$.

5. If X is subfinitely enumerable and $f : X \rightarrow S$, then $\text{im } f$ is also subfinitely enumerable. But if S is discrete, then $\text{im } f$ is also discrete as the subset of a discrete set; therefore, it is properly finite.
6. We combine the previous two closure properties. Let X_i be properly finite subsets of a discrete set S for $i : \mathbf{k}$. Then $(i : \mathbf{k}) \times X_i$ is properly finite and if we define $f : (i : \mathbf{k}) \times X_i \rightarrow S$ by $f(i, x) = x$, we see that $\text{im } f = \bigcup_{i:\mathbf{k}} X_i$.

□

4.6.2 Dubuc-Penon compactness

In their paper [DP86], Dubuc and Penon introduce a very creative notion of compactness suitable for synthetic differential geometry.

Definition 4.6.10 (Dubuc-Penon, [DP86]). A type K is *Dubuc-Penon compact* if for every $A : \mathbf{Prop}$ and $B : K \rightarrow \mathbf{Prop}$,

$$(\forall x : K. A \vee B(x)) \rightarrow (A \vee \forall x : K. B(x)).$$

A map $f : X \rightarrow Y$ is *Dubuc-Penon proper* if the inverse image of any Dubuc-Penon compact subtype of Y is Dubuc-Penon compact.

Dubuc and Penon prove in [DP86] that the sheaves represented by compact ordinary manifolds in our intended models are Dubuc-Penon compact. This will allow us to assume at least that the unit interval is Dubuc-Penon compact.

Axiom 7. The unit interval $[0, 1]$ is Dubuc-Penon compact.

Lemma 4.6.11. We prove the following basic fact about Dubuc-Penon compact sets.

1. Finitely enumerable sets are Dubuc-Penon compact.
2. If K is Dubuc-Penon compact, and $f : K \rightarrow X$ is surjective, then X is Dubuc-Penon compact.
3. If $X + Y$ is Dubuc-Penon compact, then so are X and Y .

Proof. First, finite sets are Dubuc-Penon compact because universal quantification over a finite set is a finite conjunction, and disjunction commutes with finite conjunction. As the images of maps from finite sets, finitely enumerable sets will be compact by the second property.

Suppose that K is Dubuc-Penon compact and $f : K \rightarrow X$ is surjective. Let $A : \mathbf{Prop}$ and $B : X \rightarrow \mathbf{Prop}$, and suppose that for all $x : X$, A holds or $B(x)$ holds. Then also for all $k : K$, A holds or $B(f(k))$ holds, so by the compactness of K , A holds or for all $k : K$, $B(f(k))$ holds. But since for all $x : X$, there is a k such that $f(k) = x$, this suffices to show that A holds or for all $x : X$, $B(x)$ holds.

Suppose that $X + Y$ is Dubuc-Penon compact, and let $A : \mathbf{Prop}$ and $B : X \rightarrow \mathbf{Prop}$ such that for all $x : X$, A holds or $B(x)$ holds. Define $C : X + Y \rightarrow \mathbf{Prop}$ by $C(z) \equiv (z \in X) \rightarrow B(x)$. For all $z : X + Y$, either z is in X and so A holds or $B(z)$ holds, or z is in Y and so $C(z)$ holds trivially; in either case, A holds or $C(z)$ holds, so by compactness of $X + Y$, conclude that either A holds or for all $z : X + Y$, $C(z)$ holds. But in that latter case, we see that $x : X$,

$B(x)$ holds, and so X is Dubuc-Penon compact. Of course, the argument for Y is symmetric. \square

Warning 4.6.12. Although finitely enumerable sets are Dubuc-Penon compact, subfinite sets are not in general Dubuc-Penon compact. Every proposition is subfinite, but a proposition P is Dubuc-Penon compact if and only if it is decidable: $P \vee \neg P$.

Proposition 4.6.13. A map $f : X \rightarrow Y$ is Dubuc-Penon proper if and only if its fibers are Dubuc-Penon compact. As a corollary, Dubuc-Penon proper maps are closed under pullback.

Proof. Since singletons are clearly compact, if $f : X \rightarrow Y$ is proper, then its fibers are compact. Conversely, suppose that the fibers of f are compact, and let $K \subseteq Y$ be a compact subset; we aim to show that $f^*K := \{x : X \mid f(x) \in K\}$ is compact. So, let $A : \mathbf{Prop}$ and $B : f^*K \rightarrow \mathbf{Prop}$ and suppose that for all $x \in f^*K$, A holds or $B(x)$ holds. Then also for all $y \in K$, and for all $x \in f^*\{y\}$, A holds or $B(x)$ holds. By hypothesis, $f^*\{y\}$ is compact, so this means that for all $y \in K$, A holds or for all $x \in f^*\{y\}$ $B(x)$ holds. But then we may appeal to the compactness of K and see that either A holds or for all $y \in K$ and $x \in f^*\{y\}$, $B(x)$ holds. But this means that either A holds or for all $x \in f^*K$, $B(x)$ holds.

If $f : X \rightarrow Y$ is proper, and $g : A \rightarrow X$, then the pullback $g^*f : A \times_Y X \rightarrow A$ has equivalent fibers to those of f , and so is also proper. \square

Proposition 4.6.14. Let K be a Dubuc-Penon compact type and let $F : K \rightarrow \mathbf{Type}$ be a family of Dubuc-Penon compact types. Then the type of pairs

$(k : K) \times F(k)$ is Dubuc-Penon compact. In particular, the product of two Dubuc-Penon compact types is Dubuc-Penon compact.

Proof. Let $A : \mathbf{Prop}$ and $B : (k : K) \times F(k) \rightarrow \mathbf{Prop}$ and suppose that for all $(k, x) : (k : K) \times F(k)$, A holds or $B(k, x)$ holds. We may compute:

$$\begin{aligned}
\forall (k, x) : (k : K) \times F(k). (A \vee B(k, x)) &\iff \forall k : K \forall x : F(k). (A \vee B(k, x)) \\
&\Rightarrow \forall k : K. (A \vee \forall x : F(k). B(k, x)) \\
&\Rightarrow A \vee (\forall k : K. \forall x : F(k). B(k, x)) \\
&\iff A \vee (\forall (k, x) : (k : K) \times F(k). B(k, x)) \square
\end{aligned}$$

In his thesis, [GC89], Gago proved that any positive, real valued function on a Dubuc-Penon compact set valued in \mathbb{R} was bounded away from 0 (Corollary 4.6.19). We will extract the method which Gago uses in his proof as Theorem 4.6.17. This method makes use of Penon's logical definition of open set, and relies crucially on the Covering Property.

Definition 4.6.15. A subtype $U \subseteq X$ is *Penon open* if for all $x \in U$ and $y : X$, either $x \neq y$ or $y \in U$. A subtype $C \subseteq X$ is *Penon closed* if its complement $X - C$ is Penon open.

Penon opens form a topology on any type, and any function is continuous for the Penon topology. Any regular topology is finer than the Penon topology on its set of points; in particular, every open set in an ordinary manifold is Penon open.

Lemma 4.6.16. Let X be a regular topological space. Then any open set in X is Penon open.

Proof. Let U be open in X , and let $x \in U$ and $y \in X$. By the regularity of X , there is an open $V \subseteq U$ and G with $x \in V$, $V \cap G = \emptyset$, and $U \cup G = X$. Therefore, $y \in U$ or $y \in G$; but if $y \in G$, y cannot equal x , so we conclude that either $x \neq y$ or $y \in U$. \square

Theorem 4.6.17. Let K be Dubuc-Penon compact and let $r : K \times \mathbb{R} \rightarrow \mathbf{Prop}$ be a relation which is Penon open as a subset of the product. If for all k , we have $r(k, x)$, then there is an $\varepsilon > 0$ so that $r(k, y)$ for all $y \in B(x, \varepsilon)$ and $k : K$.

Proof. That r is Penon open means that for any k and x so that $r(k, x)$ and any other q and y , we have $((k, x) \neq (q, y) \text{ or } r(q, y))$. So, supposing that $r(k, x)$ for all k , let $y \in \mathbb{R}$ and note that for any $k : K$,

$$((k, x) \neq (k, y)) \vee r(k, y).$$

Now, $(k, x) \neq (k, y)$ if and only if $x \neq y$, so we conclude that for any $k : K$, $(x \neq y) \text{ or } r(k, y)$. Therefore, by the compactness of K , either $(x \neq y)$ or for all $k : K$, $r(k, y)$. By the Covering Principle, then, either there is an $\varepsilon > 0$ so that $B(x, \varepsilon) \subseteq \{y \mid x \neq y\}$ or there is an $\varepsilon > 0$ so that $B(x, \varepsilon) \subseteq \{y \mid \forall k : K. r(k, y)\}$; since the former can't possibly be true, we conclude the latter. \square

Let's give a careful proof that cartesian products of Penon open sets are open before deriving some corollaries of Theorem 4.6.17.

Lemma 4.6.18. Let $U \subseteq X$ and $V \subseteq Y$ be Penon open subsets. Then $U \times V \subseteq X \times Y$ is Penon open.

Proof. Suppose that $(u, v) \in U \times V$ and let $(x, y) : X \times Y$, seeking

$$((u, v) \neq (x, y)) \vee ((x, y) \in U \times V).$$

Expanding these assumptions out a bit, we see that $u \in U$ and $v \in V$, and it will suffice to prove

$$(u \neq x) \vee (v \neq y) \vee (x \in U \wedge y \in V),$$

which is equivalently

$$((u \neq x) \vee (v \neq y) \vee (x \in U)) \wedge ((u \neq x) \vee (v \neq y) \vee (y \in V)).$$

By hypothesis we have that $(u \neq x) \vee (x \in U)$ and $(v \neq y) \vee (y \in V)$ by the openness of U and V . This clearly suffices. \square

Corollary 4.6.19 (Gago, [GC89]). Let K be a Dubuc-Penon compact set.

1. For every $f : K \rightarrow (0, \infty)$, there is an $\varepsilon > 0$ so that $\varepsilon < f(k)$ for all $k : K$.
2. For every $f : K \rightarrow \mathbb{R}$, there is a $B > 0$ so that $-B < f(x) < B$ for all $x : K$.

Proof. To prove the first statement, let $r(k, x) \equiv (x < f(k))$ in Theorem 4.6.17, and conclude that there is some $\delta > 0$ so that for all $x \in B(0, \delta)$, we have $r(k, x)$ for all $k : K$. Define $\varepsilon \equiv \frac{\delta}{2}$, and conclude that for all $k : K$, we have $\varepsilon < f(k)$.

To prove the second statement, we use Postulate E to transform f into a positive valued function. By the first statement, we have $\varepsilon > 0$ so that $\varepsilon < \exp f(k)$ for all k , and we also have $\delta > 0$ so that $\delta < \frac{1}{\exp f(k)}$. By using a smooth approximation to the minimum function, we can ensure that both ε

and δ are less than 1. As a result, we see that $\log \varepsilon < f(k) < -\log \delta$ for all $k : K$, and both $\log \varepsilon$ and $\log \delta$ are positive. Then define B to be any number bigger than both $\log \varepsilon$ and $\log \delta$. \square

We can use Theorem 4.6.17 to show that Dubuc-Penon compact subsets of \mathbb{R}^n are “compact” in suitably weak senses. Let’s give names to a few of these senses now.

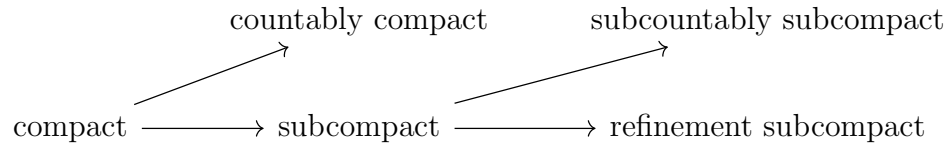
Definition 4.6.20. Consider the following notions of compactness. Let K be a topological space.

1. K is (open-cover) *compact* if every open cover admits a finitely enumerable subcover. Explicitly, K is compact if for any open cover $\mathcal{U} \subseteq \mathcal{O}(K)$, there is a finitely enumerable subset $\mathcal{V} \subseteq \mathcal{U}$ for which $\bigcup_{V \in \mathcal{V}} V = K$.⁷
2. K is *countably compact* if every countably enumerable open cover admits a finitely enumerable subcover.
3. K is *subcompact* if every open cover admits a subfinitely enumerable subcover.
4. K is *subcountably subcompact* if every subcountably enumerable cover admits a subfinitely enumerable subcover.
5. K is *refinement subcompact* if every open cover admits a subfinitely enumerable refinement. Explicitly, K is refinement subcompact if for any open cover $\mathcal{U} \subseteq \mathcal{O}(K)$, there is a subfinitely enumerable cover $\mathcal{V} \subseteq \mathcal{O}(K)$, so that for any $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ such that $V \subseteq U$.

⁷We will not be using this notion in this chapter; it is included only for comparison with the subsequent notions. In particular, when I use the term “compact” in a proof, it usually means Dubuc-Penon compact. This should be clear from context.

The definitions of “compact” and “countably compact” are standard in constructive mathematics. On the other hand, the definitions of “subcompact”, “subcountably subcompact”, and “refinement subcompact” are, as far as I know, novel.

Remark 4.6.21. Clearly, any compact set is countably compact, subcompact, subcountably subcompact, and refinement subcompact. Any subcompact set is subcountably subcompact, and refinement subcompact. However, countable compactness and subcountable subcompactness are generally incomparable; the latter applies to more covers but gives a weaker condition on the resulting subcover. This is summarized in the following diagram:



We can prove that any Dubuc-Penon compact set is both countably compact and subcountably subcompact.

Theorem 4.6.22. *Let K be Dubuc-Penon compact. Then K is subcountably subcompact and countably compact with regard to the Penon topology. That is, for any $I \subseteq \mathbb{N}$, any I -indexed Penon open cover $U : I \rightarrow \mathcal{O}(K)$ admits a subfinitely enumerable subcover. More explicitly, there is an $n : \mathbb{N}$ so that if $I_{<n} := \{i : I \mid i < n\}$, we have $K \subseteq \bigcup_{i \in I_{<n}} U_i$. In the case that $I = \mathbb{N}$, we see that $I_{<n} = \underline{n}$ is actually finite, so that we have a finitely enumerable subcover.*

Proof. Let $r(k, x)$ be the relation

$$r(k, x) := \exists i : I. (k \in U_i) \wedge \left(x < \frac{1}{i} \right).$$

This relation is Penon open, since it may be described as the union of a cartesian product of opens (Lemma 4.6.18).

$$r = \bigcup_{i:I} \left((K \cap U_i) \times \left(-\infty, \frac{1}{i} \right) \right).$$

For all $k : K$, we have $r(k, 0)$; therefore, by Theorem 4.6.17 we may conclude that there is an $\varepsilon > 0$ so that $B(0, \varepsilon) \subseteq \{x : \mathbb{R} \mid \forall k : K. r(k, x)\}$. In particular, letting $n : \mathbb{N}$ being any number greater than $\frac{1}{\varepsilon}$, we see that for all $k : K$ there is an $i : \mathbb{N}$ with $k \in U_i$ and $\frac{1}{n} < \frac{1}{i}$. This shows that $K = \bigcup_{i=0}^n K \cap U_i$, which is a finite union. \square

Using Theorem 4.6.22, we can prove that Dubuc-Penon compact subsets of ordinary manifolds are refinement subcompact.

Corollary 4.6.23. Let K be a Dubuc-Penon compact subset of a second countable topological space X whose opens are Penon open. Then K is refinement subcompact.

Proof. Let \mathcal{U} be an open cover of K , and let $B : \mathbb{N} \rightarrow \mathcal{O}(X)$ be a countable base for X . Define $I := \{i : \mathbb{N} \mid \exists U \in \mathcal{U}. B_i \subseteq U\}$ be the the set of indices of the base opens which are contained in opens of \mathcal{U} . For any $x \in K$, there is some $U \in \mathcal{U}$ with $x \in U$; since B is a base, there is also some $i : \mathbb{N}$ so that $x \in B_i \subseteq U$. Therefore, $\mathcal{B} := \{B_i \mid i \in I\}$ remains an open cover and is a subcountable refinement of \mathcal{U} . But then, by Theorem 4.6.22, \mathcal{B} admits a subfinitely enumerable subcover by the subcountable subcompactness of K . \square

Clearly, in the prescence of the law of excluded middle, a space would be refinement subcompact if and only if it were compact — LEM implies that

subfinitely enumerable sets are finite, and we could then choose a subcover out of our refinement since the product of finitely many inhabited types is always inhabited. In fact, we can internalize this observation into a theorem about crisp refinement subcompact sets.

Proposition 4.6.24. Let K be a crisp topological space which is crisply refinement subcompact. Then any crisp open cover $\mathcal{U} \subseteq \mathcal{O}(X)$ admits a finite subcover.

Proof. By hypothesis, \mathcal{U} admits a subfinitely enumerable refinement \mathcal{V} , and we may take \mathcal{V} to be crisp. Since \mathcal{V} is crisply subfinitely enumerable, by Proposition 4.6.2, \mathcal{V} is finite. But then we may choose for every $v \in \mathcal{V}$ a $u_v \in \mathcal{U}$ with $v \subseteq u_v$, which gives us a crisp, finitely enumerable subcover of \mathcal{U} , which is therefore also finite by Proposition 4.6.2. \square

Remark 4.6.25. Putting together Proposition 4.6.24 with Corollary 4.6.23 proves that any crisp open cover of any crisp Dubuc-Penon compact ordinary manifold admits a finite subcover. This gives an internal proof of the external theorem of Dubuc and Penon [**Dubuc-Penon:compact**] that if the object of the Dubuc topos represented by a manifold M is Dubuc-Penon compact, then M is compact.

While we're here, we might as well prove the Heine-Borel property for Dubuc-Penon compact sets: they are both closed and bounded. The converse fails, however, see Warning 4.6.29.

Proposition 4.6.26. Let X be separated in the sense of Dubuc-Penon [DP86], namely if $x \neq y$, then for all z , either $x \neq z$ or $z \neq y$. Then if $K \subseteq X$ is Dubuc-Penon compact, it is Penon closed.

Proof. We will show that $X - K$ is Penon open. Suppose that $x \notin K$, and let $z : X$, seeking $x \neq z$ or $z \notin K$. Now, for all $k \in K$, either $x \neq z$ or $z \neq k$ by the separatedness of X . Therefore, by the compactness of K , either $x \neq z$ or for all $k \in K$, $z \neq k$. But in the latter case, z cannot be in K . \square

Lemma 4.6.27. The smooth reals \mathbb{R} are separated in the sense of Dubuc-Penon.

Proof. It will suffice to show that if $x \neq 0$, then for any $y : \mathbb{R}$, either $x \neq y$ or $y \neq 0$. This is equivalently asking whether $x - y \neq 0$ or $y \neq 0$. The result then follows since \mathbb{R} is a local ring and a field (Postulate K): $(x - y) + y = x$ is invertible, and therefore either $(x - y)$ or y is nonzero. \square

Theorem 4.6.28. Any Dubuc-Penon compact subset of \mathbb{R} is closed and bounded.

Proof. By Corollary 4.6.19, a Dubuc-Penon compact subset K of \mathbb{R} is bounded. Furthermore, as a compact subset of a separated space, K is closed. \square

Warning 4.6.29. While Dubuc-Penon compact subsets of \mathbb{R} are closed and bounded, the full Heine-Borel theorem does not hold. We can give an example of a closed and bounded subset which is not Dubuc-Penon compact. Let $z : \mathbb{R}$ be a real number, and let $S = \{x : \mathbb{R} \mid (x = 0) \wedge (z = 0)\}$. If $z = 0$, then $S = \{0\}$, and if $z \neq 0$, then $S = \emptyset$ — in fact the set S is equivalent to the proposition $(z = 0)$. If S were Dubuc-Penon compact, then $(z = 0)$ or $(z \neq 0)$; this is because

$$(S \rightarrow ((z = 0) \vee \emptyset)) \simeq ((z = 0) \vee (S \rightarrow \emptyset)) \simeq ((z = 0) \vee (z \neq 0)),$$

where we use the compactness of S and the fact that $S \simeq (z = 0)$. We can conclude that S is *not* Dubuc-Penon compact, since equality of the reals is

not decidable. However, S is clearly bounded, and using Postulate K, we can also show that it is closed. Let $x \notin S$ and $y : \mathbb{R}$, seeking $y \neq x$ or $y \notin S$. The statement $x \notin S$ means $\neg((x = 0) \wedge (z = 0))$, which by Postulate K is equivalent to $(x \neq 0) \vee (z \neq 0)$; similarly, we are seeking to show that $(y \neq x) \vee (y \neq 0) \vee (z \neq 0)$. Of course, in the case that $(z \neq 0)$, we're done. On the other hand, if $(x \neq 0)$, then by the separatedness of \mathbb{R} (Lemma 4.6.27), either $(x \neq y)$ or $(y \neq 0)$.

The usual proof of the Heine-Borel theorem relies on the fact that closed subsets of compact sets are compact, in order to conclude the compactness of closed bounded subsets from the compactness of closed intervals. This is also false in general. Suppose that closed subsets of compact sets were compact; then, since $\underline{1}$ is compact, every proposition is compact. But a proposition P is Dubuc-Penon compact if and only if $P \vee \neg P$, so this would imply the law of excluded middle.

Both of these counterexamples make essentially use of non-crisp propositions. This is because, by Axiom 3, all crisp propositions P are decidable: $P \vee \neg P$. This leaves room for a theorem proving that crisp, closed and bounded subsets of \mathbb{R} are Dubuc-Penon compact, but I do not know how to prove this.

We can now set about proving the technical lemmas used in Theorem 4.6.37. First, we will show in Lemma 4.6.30 that discrete, Dubuc-Penon compact subsets of ordinary manifolds are properly finite. Then, in the next section, we will show in Lemma 4.6.34 that crystalline subsets of crisp ordinary manifolds are discrete.

Lemma 4.6.30. Let K be a discrete, Dubuc-Penon compact subset of an ordinary manifold M . Then K is properly finite.

Proof. By Corollary 4.6.23, K is refinement subcompact. Since it is discrete, every singleton is Penon open: for any x , either $y \neq x$ or $y = x$ (and so $y \in \{x\}$). Therefore, there is a subfinitely enumerable refinement of the cover of K by its singletons. That is, there is a subfinitely enumerable set $\mathcal{S} \subset \mathcal{O}(K)$ consisting of subsingletons, such that $\bigcup_{S \in \mathcal{S}} S = K$.

The relation $r(k, S) := (k \in S)$ gives an association of K with \mathcal{S} . For $k \in K$, there is some $S \in \mathcal{S}$ with $k \in S$ since this is a cover. If $k \in S$ and $k \in S'$, then $S = S'$ because they were assumed to be subsingletons and they both contain k and are therefore both the singleton $\{k\}$. It follows by Proposition 4.6.7 that K is itself subfinitely enumerable. Since it was assumed to be discrete, this makes K properly finite. \square

4.6.3 Crisp, ordinary proper étale groupoids are orbifolds

Lemma 4.6.31. If a subset $C \subseteq \mathbb{R}^n$ is crystalline, then it is discrete.

Proof. We will show that every path $\gamma : \mathbb{R} \rightarrow C$ is constant. Since C is crystalline, for any $t \in \mathbb{R}$, the composite

$$\mathcal{D}_t \hookrightarrow \mathbb{R} \xrightarrow{\gamma} C$$

is constant at $\gamma(t)$. In particular, we have that $\gamma(t + \varepsilon) = \gamma(t)$ for each $\varepsilon^2 = 0$, since $t + \varepsilon \approx t$. But then the coordinate functions $\gamma_i : \mathbb{R} \rightarrow \mathbb{R}$ are similarly unmoved by first order displacement, and so by the Principle of Constancy, the coordinate functions γ_i are constant and so γ is constant. \square

Lemma 4.6.32. Let $f : X \rightarrow Y$ be \mathfrak{S} -étale. If C is crystalline and $c : C \rightarrow Y$ is any map, then the pullback $f^*c : X \times_Y C \rightarrow X$ is crystalline.

Proof. By Theorem 3.20 of [Che17], the pullback of a modally étale map is modally étale; therefore, the map $c^*f : X \times_Y C \rightarrow C$ is \mathfrak{S} -étale, and so the naturality square

$$\begin{array}{ccc} X \times_Y C & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow \\ \mathfrak{S}(X \times_Y C) & \longrightarrow & \mathfrak{S}C \end{array}$$

is a pullback. By hypothesis, the unit $C \rightarrow \mathfrak{S}C$ is an equivalence, so we conclude that the unit $X \times_Y C \rightarrow \mathfrak{S}(X \times_Y C)$ is an equivalence. \square

Proposition 4.6.33 (Shulman, Theorem 11.1 [Shu18a]). The reals \mathbb{R} are connected in the sense that if $X \cup Y = \mathbb{R}$ and both X and Y are nonempty, then $X \cap Y$ is nonempty.

Proof. Suppose that $X \cap Y = \emptyset$. Then $\mathbb{R} \simeq X + Y$ is a disjoint union, and can define a function $f : \mathbb{R} \rightarrow \underline{2}$ defined by $f(x) = 0$ if $x \in X$ and $f(x) = 1$ if $x \in Y$. But $\underline{2}$ is crisply discrete, and so is in particular discrete, so f factors through the shape $\int \mathbb{R}$, which is the point. In other words, f is constant, and so for all $x : \mathbb{R}$, $x \in X$, or for all $x : \mathbb{R}$, $x \in Y$. That is, $X = \mathbb{R}$ or $Y = \mathbb{R}$. Suppose that $X = \mathbb{R}$; then since $X \cap Y = \emptyset$, we can conclude that $Y = \emptyset$. But this contradicts our assumption that Y was nonempty. Similarly, if $Y = \mathbb{R}$, then $X = \emptyset$, a contradiction. In either case, $X \cap Y$ cannot be empty. \square

Lemma 4.6.34. If a subset $C \subseteq M$ of a crisp ordinary manifold M is crystalline, then it is discrete.

Proof. Let $\gamma : \mathbb{R} \rightarrow C$; we will show that γ is constant by showing that $\gamma(t) = \gamma(0)$ for all $t : \mathbb{R}$. First, let's show that it suffices to prove that $\gamma(t) \approx \gamma(0)$. Consider the following pullback:

$$\begin{array}{ccc} \mathfrak{D}_{\gamma(0)} \cap C & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow \\ \mathfrak{D}_{\gamma(0)} & \hookrightarrow & M \end{array}$$

By Lemma 4.5.17, the inclusion $\mathfrak{D}_{\gamma(0)} \hookrightarrow M$ is \mathfrak{S} -étale, and so by Lemma 4.6.32 the pullback $\mathfrak{D}_{\gamma(0)} \cap C$ is crystalline. But $\mathfrak{D}_{\gamma(0)}$ embeds into \mathbb{R}^n since ordinary manifolds are Penon manifolds, so $\mathfrak{D}_{\gamma(0)} \cap C$ is a crystalline subset of \mathbb{R}^n and therefore by Lemma 4.6.31 is discrete. Therefore, equality in $\mathfrak{D}_{\gamma(0)} \cap C$ is decidable; but since for any $x \in \mathfrak{D}_{\gamma(0)}$, $x \approx \gamma(0)$, we can conclude that if x is also in C then $x = \gamma(0)$. That is, $\mathfrak{D}_{\gamma(0)} \cap C = \{\gamma(0)\}$, and so if we prove that $\gamma(t) \approx \gamma(0)$, this will imply that $\gamma(t) = \gamma(0)$.

Since M was assumed to be crisp, we can take a crisp, countable open cover $M = \bigcup_{i \in \mathbb{N}} U_i$. Any chart is infinitesimally stable, because M is regular. Therefore, for any chart $\phi_i : \mathbb{R}^n \rightarrow M$ (with $U_i = \phi_i(\mathbb{R}^n)$) in this cover, $\phi_i^{-1} C \subseteq \mathbb{R}^n$ is crystalline by Corollary 4.5.19 and Lemma 4.6.32. The restriction $\gamma|_{U_i} : \gamma^{-1}(U_i) \rightarrow \phi_i^{-1} C$ is therefore constant.

Now, let $t : \mathbb{R}$, seeking to prove that $\gamma(t) \approx \gamma(0)$. Since we are trying to prove a negative statement (namely, $\neg\neg(\gamma(t) = \gamma(0))$), we are free to use the law of excluded middle and double negation elimination. Let N be a number so that t and 0 are both in $(-N, N)$, and therefore also $[-N, N]$. By ??, $[-N, N]$ is Dubuc-Penon compact and so by Theorem 4.6.22 there is an $n : \mathbb{N}$ such that $[-N, N] \subseteq V_1, \dots, V_n$. Therefore, there is some i and j so that $0 \in V_i$ and

$t \in V_j$; let $W = \bigcup_{j \neq k \neq i} V_k$ be the rest of this finite cover. Now, either W is empty or it isn't. If W is empty, then $(-N, N) \subseteq V_i \cup V_j$ and so $V_i \cap V_j$ is nonempty by Proposition 4.6.33. But then we can assume that $x \in V_i \cap V_j$, so that $\gamma(t) = \gamma(x) = \gamma(0)$. On the other hand, if W is nonempty, then either $V_i \cap V_j$ is nonempty or both of $W \cap V_i$ and $W \cap V_j$ are nonempty. In either case, we can assume we have inhabitants, and conclude that $\gamma(t) = \gamma(0)$. \square

Warning 4.6.35. We scraped through Lemma 4.6.34 by the skin of our teeth. It feels like it should be easier to prove. The real line is connected, and so we would expect that a locally constant function $f : \mathbb{R} \rightarrow X$ (valued in a set) should be constant. One way we might try to prove this general theorem is by showing that every open cover of \mathbb{R} admits a *chain* from x to y : opens U_1, \dots, U_n in the cover with $x \in U_1$ and $y \in U_n$ and $U_i \cap U_{i+1}$ inhabited. We could then give $f(x) = f(y)$ by $f(x) = f(x_1) = \dots = f(x_{n-2}) = f(y)$ with $x_i \in U_i \cap U_{i+1}$, appealing to the constancy of f on each U_i . There are a number of issues, however. First, the usual proof of this relies on showing that the set of all points y for which there is a chain from x is clopen and inhabited. However, I do not know how to prove that a clopen, inhabited subset U of \mathbb{R} is all of \mathbb{R} — the usual argument makes use of the classical fact that $U \cup (\mathbb{R} - U) = \mathbb{R}$, but this does not hold constructively. Second, even in the binary case it is not clear that if $U \cup V = \mathbb{R}$ with U and V open and inhabited, then $U \cap V$ must be inhabited (as opposed to ?? which shows that $U \cap V$ is nonempty). This property is called *overt connectedness* by Taylor [Tay10], and is proven for crisp subsets of \mathbb{R} as Theorem 11.3 of [Shu18a].

Another approach to proving Lemma 4.6.34 would be to prove that the union

of discrete subsets is discrete, or more narrowly that the union of countably many discrete subsets is discrete. However, discrete subsets are not even closed under binary union. Let $z : \mathbb{R}$ and consider the quotient $Q_z = \{0, 1\} / \sim$ where $0 \sim 1$ iff $z = 0$ (in more type theoretic language, this is the suspension $\Sigma(z = 0)$ of the proposition $(z = 0)$). By definition, $[0] = [1]$ in Q_z if and only if $z = 0$, so Q_z is not discrete; if it were, then it would have decidable equality and so we could decide whether or not $(z = 0)$. But $\{[0]\} \cup \{[1]\} = Q_z$ since the quotient map $[-] : \{0, 1\} \rightarrow Q_z$ is surjective, and singletons are discrete. There is a nice topological interpretation of this counterexample: the projection $\text{fst} : (z : \mathbb{R}) \times Q_z \rightarrow \mathbb{R}$ is the codiagonal map $\mathbb{R} \vee \mathbb{R} \rightarrow \mathbb{R}$ from the wedge of \mathbb{R} with itself, pointed at 0. The failure of the discreteness of Q_z reflects the failure of this map to be a covering map; see also Remark 9.9 of Chapter 3.

However, the use of compactness — and therefore of Axiom 7 which asserts the Dubuc-Penon compactness of the unit interval — is likely inessential.

Finally, we can introduce the definition of an ordinary proper étale pregroupoid, and prove that the Rezk completions of crisp ordinary proper étale pregroupoids are orbifolds.

Definition 4.6.36. An *ordinary proper étale pregroupoid* is a pregroupoid \mathcal{G} satisfying the following conditions:

1. The type \mathcal{G}_0 of objects and \mathcal{G}_1 of morphisms are ordinary manifolds.
2. The source map $s : \mathcal{G}_1 \rightarrow \mathcal{G}_0$ is \mathfrak{S} -étale (which, by Corollary 4.5.20, means that it is a local diffeomorphism in the usual sense).
3. The map $(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$ sending a morphism to its source and target

is Dubuc-Penon proper.

Theorem 4.6.37. *The Rezk completion of a crisp ordinary proper étale pregroupoid is an orbifold in the sense of Definition 4.1.2.*

Proof. Since a crisp ordinary proper étale pregroupoid \mathcal{G} is in particular a crisp étale pregroupoid, its Rezk completion $\mathbf{r}\mathcal{G}$ is microlinear by Theorem 4.5.34. Therefore, it remains to show that the types of identifications in $\mathbf{r}\mathcal{G}$ are properly finite. By the Yoneda lemma, $(\mathbf{y}(x) = \mathbf{y}(y))$ is equivalent to $\mathcal{G}(x, y) \subseteq \mathcal{G}_1$, so it will suffice to show that $\mathcal{G}(x, y)$ is properly finite. Since $(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is Dubuc-Penon proper and singletons are Dubuc-Penon compact, the inverse image $\mathcal{G}(x, y) \simeq \mathbf{fib}_{(s,t)}(x, y)$ is Dubuc-Penon compact. As a subset of the crystalline set $(z : \mathcal{G}_0) \times \mathcal{G}(x, z)$ — which is the fiber of $s : \mathcal{G}_1 \rightarrow \mathcal{G}_0$ over x — the set $\mathcal{G}(x, y)$ is crystalline. Therefore, it is discrete by Lemma 4.6.34. Then, by Lemma 4.6.30, we may conclude that $\mathcal{G}(x, y)$ is properly finite.

□

4.6.4 Global quotient orbifolds, and pullback of orbifolds

We finally have the definitions of microlinear types and properly finite sets so that Definition 4.1.2 is a fully precise definition of orbifold. Let's quickly show that the good orbifolds — the quotients of the actions of finite groups on microlinear types — are orbifolds in this sense. Then, we will show that orbifolds are closed under pullback, and use this fact to conclude that the *inertia orbifold* of an orbifold is, in fact, an orbifold.

Theorem 4.6.38. *The quotient $X // \Gamma$ of a microlinear set X by the action of a crisp, finite group Γ is an orbifold in the sense of Definition 4.1.2. These are*

the global quotient *orbifolds*.

Proof. By Theorem 4.5.26, $X // \Gamma$ is microlinear; it remains to show that its types of identifications are properly finite. Since $q : X \rightarrow X // \Gamma$ is surjective, we may consider the types of identifications $q(x) = q(y)$ for $x, y : X$. By Lemma 3.7.5, this type is equivalent to the type $(\gamma : \Gamma) \times (\gamma x = y)$. Since X is a set, the type $(\gamma x = y)$ is a proposition, and so $(\gamma : \Gamma) \times (\gamma x = y)$ is a subset of Γ . Therefore, $q(x) = q(y)$ is subfinite, and so properly finite. \square

Proposition 4.6.39. Orbifolds are closed under pullback.

Proof. Suppose that

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ k \downarrow & \lrcorner & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

is a pullback square commuting via $S : (a : A) \rightarrow fha = gka$, and that B, C and D are orbifolds. Since microlinear types are closed under pullback, A is also microlinear; it remains to show that it has properly finite identification types. We have an equivalence

$$(a = a') \simeq ((p : ha = ha') \times (q : ka = ka') \times (f_*p \bullet S_{a'} = S_a \bullet g_*q))$$

given by the computation of identification types of pair types. The first two factors are properly finite by assumption; the third is a proposition. Therefore, $(a = a')$ is a subset of a product of properly finite sets and is therefore properly finite by Proposition 4.6.9. \square

We can use Proposition 4.6.39 to show that the *inertia orbifolds* are orbifolds.

Definition 4.6.40. If X is an orbifold, its *inertia orbifold* is the type $(x : X) \times (x = x)$, or equivalently the type X^{S^1} , the free loop type of X .

Corollary 4.6.41. The inertia orbifold X^{S^1} of an orbifold is an orbifold in the sense of Definition 4.1.2.

Proof. We have a pullback

$$\begin{array}{ccc} X^{S^1} & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \Delta \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

So, by Proposition 4.6.39, X^{S^1} is an orbifold. □

4.7 Tiny Types

A remarkable feature of synthetic differential geometry is the *tinyness* of the infinitesimals and infinitesimal varieties. This is Postulate W of Axiom 5. The importance of tiny objects for SDG was first realized by Lawvere [**Lawvere:SDG.bodies**], and their elementary theory was worked out by Yetter [Yet87].

In this subsection, we will develop just enough of the theory of tiny types for our purposes in this chapter — in particular, enough to prove that localization at the type \mathfrak{D} of infinitesimals preserves crisp colimits (Theorem 4.5.33). A great deal more can be done with tiny types in synthetic differential geometry; for example, the construction of differential form classifiers, and from them the construction of classifying types for principal bundles with connection. We leave the further development of the theory of tiny types to future work.

A type T is tiny if the functor $X \mapsto X^T$ has a *right* adjoint, in addition to its usual left adjoint $X \mapsto T \times X$. This formulation is not quite correct: the

adjunction can only exist *externally* (see ??). To refer to the external world internally, we will use crisp types and the \flat comodality.

Tiny types in a type theory with \flat have been defined before by Licata, Orton, Pitts, and Spitters in Figure 1 of [Lic+18]. However, their definition is only coherent for set level objects, which suits their purposes because they interpret the type theory in a 1-topos. Their axioms are also not propositional when applied to higher types. We will use a different definition which is coherent for higher types as well, and where being tiny is a proposition.

Definition 4.7.1. A crisp type T is *tiny* when the following structure exists crisply:

1. For any crisp type X , a type $X^{\frac{1}{T}}$ and a map $\xi : (X^{\frac{1}{T}})^T \rightarrow X$.
2. For any crisp types X and Y , the map

$$\Xi := \omega \mapsto [v \mapsto \xi(\omega \circ v)] : (X \rightarrow Y^{\frac{1}{T}}) \rightarrow (X^T \rightarrow Y)$$

is a \flat -equivalence: $\flat\Xi$ is an equivalence. That is,

$$\flat\Xi : \flat(X \rightarrow Y^{\frac{1}{T}}) \simeq \flat(X^T \rightarrow Y)$$

This definition is coherent because the assignment $X \mapsto X^T$ is ∞ -functorial, and an ∞ -functor $L : \mathcal{C} \rightarrow \mathcal{D}$ has a right adjoint if and only if the slice category L/d has a terminal object for each object $d \in \mathcal{D}$ — with no functoriality in d assumed. Definition 4.7.1 gives, roughly speaking, a terminal object $\xi : (Y^{\frac{1}{T}})^T \rightarrow Y$ in $(-)^T/Y$; that $\flat\Xi$ is an equivalence says that for every object $X^T \rightarrow Y$ of $(-)^T/Y$, there is a unique map $X \rightarrow Y^{\frac{1}{T}}$ so that the triangle

commutes. Since we want all of these statements to be external, we put them under a \flat .

Warning 4.7.2. By Theorem 1.4 of [Yetter:Tiny] (or, a suitable adaptation to ∞ -toposes), T should be tiny in *any* context, even one which is not crisp. However, as mentioned in the warning immediately following that theorem, it is not generally the case that the adjoint $X^{\frac{1}{T}}$ is stable under base-change. For this reason, we restrict X to be crisp, since $X^{\frac{1}{T}}$ is stable under crisp base change.

Remark 4.7.3. In [Law04], Lawvere says that “this possibility [of the existence of tiny types] does not seem to have been contemplated by combinatory logic; the formalism should be extended to enable treatment of so basic a situation.”. Definition 4.7.1 does not constitute such an extension of the formalism of type theory. Rather, it is more of a hack, using the Shulman’s modalities to internalize the external. Working with tiny objects as in Definition 4.7.1 is not very different then working with them externally — which is how they must be worked with in an internal logic without externalizing modalities, since the defining adjunctions only exist externally. A real solution to Lawvere’s challenge could be a novel type theory for tiny objects.

Lemma 4.7.4. Let $f :: X' \rightarrow X$ be a crisp map. Then the square

$$\begin{array}{ccc} (X \rightarrow Y^{\frac{1}{T}}) & \xrightarrow{\Xi} & (X^T \rightarrow Y) \\ \text{-}\circ f \downarrow & & \downarrow \text{-}\circ f^T \\ (X' \rightarrow Y^{\frac{1}{T}}) & \xrightarrow{\Xi} & (X'^T \rightarrow Y) \end{array}$$

commutes.

Proof. Let $\omega : X \rightarrow Y^{\frac{1}{T}}$ and $v : T \rightarrow X'$, and compute:

$$\begin{aligned}\Xi(\omega) \circ f^T(v) &\equiv \xi(\omega \circ f \circ v) \\ &\equiv \Xi(\omega \circ f)(v).\end{aligned}$$

□

Since mapping out of a tiny type has a right adjoint, it commutes with all colimits. This is the sense in which tiny types are “tiny”: that $X \mapsto (T \rightarrow X)$ commutes with all colimits is a very strong compactness property. Of course, the adjoint only exists for crisp types and the adjunction only holds for crisp maps, so we can only hope to commute with crisp colimits.

Proposition 4.7.5. Let T be a tiny type. Then the functor $X \mapsto X^T$ preserves all crisp colimits, but in particular the following:

1. If $f, g :: A \rightarrow B$, then

$$\text{coeq}(f, g)^T \simeq \text{coeq}(f^T, g^T)$$

where $f^T, g^T :: A^T \rightarrow B^T$ are given by post-composition.

2. If the square on the left is a crisp pushout, then so is the square on the right:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \quad \begin{array}{ccc} A^T & \longrightarrow & B^T \\ \downarrow & & \downarrow \\ C^T & \longrightarrow & D^T \end{array}$$

3. If $A_0 \rightarrow A_1 \rightarrow \dots$ is a crisp sequence with colimit A_∞ , then A_∞^T is the colimit of the sequence $A_0^T \rightarrow A_1^T \rightarrow \dots$.

Proof. In general, the arguments will go by showing that both $\text{colim } D_i^T$ and $(\text{colim } D_i)^T$ have the same universal property in the ∞ -category of crisp types with mapping types from X to Y being $\mathfrak{b}(X \rightarrow Y)$. This will give an equivalence $\mathfrak{b}(\text{colim } D_i^T \simeq (\text{colim } D_i)^T)$, from which we may conclude that $\text{colim } D_i^T \simeq (\text{colim } D_i)^T$ as types (and therefore that they have the same universal property using the full mapping types $X \rightarrow Y$).

We will prove the case of coequalizers; the other cases are similar. To that end, let $f, g :: A \rightarrow B$ be crisp maps. For any crisp type Y , we have equivalences:

$$\begin{aligned}
\mathfrak{b}(\text{coeq}(f, g)^T \rightarrow Y) &\simeq \mathfrak{b}(\text{coeq}(f, g) \rightarrow Y^{\frac{1}{T}}) \\
&\simeq (z : \mathfrak{b}(B \rightarrow Y^{\frac{1}{T}})) \times (\text{let } h^{\flat} : \equiv z \text{ in } \mathfrak{b}(h \circ f = h \circ g)) \\
&\simeq (z : \mathfrak{b}(B^T \rightarrow Y)) \times (\text{let } h^{\flat} : \equiv \Xi^{-1}(z) \text{ in } \mathfrak{b}(h \circ f = h \circ g)) \\
&\simeq (z : \mathfrak{b}(B^T \rightarrow Y)) \times (\text{let } h^{\flat} : \equiv \Xi^{-1}(z) \text{ in } (h^{\flat} \circ f^{\flat} = h^{\flat} \circ g^{\flat})) \\
&\equiv (z : \mathfrak{b}(B^T \rightarrow Y)) \times (\Xi^{-1}(z) \circ f^{\flat} = \Xi^{-1}(z) \circ g^{\flat}) \\
&\simeq (z : \mathfrak{b}(B^T \rightarrow Y)) \times (\Xi^{-1}(z \circ (f^T)^{\flat}) = \Xi^{-1}(z \circ (g^T)^{\flat})) \\
&\simeq (z : \mathfrak{b}(B^T \rightarrow Y)) \times (z \circ (f^T)^{\flat} = z \circ (g^T)^{\flat}) \\
&\simeq \mathfrak{b}(\text{coeq}(f^T, g^T) \rightarrow Y)
\end{aligned}$$

By substituting in $\text{coeq}(f, g)^T$ and $\text{coeq}(f^T, g^T)$ in for Y , respectively, we can give an equivalence between them. \square

Lemma 4.7.6. Let T be any type. If A is T -null (the inclusion $A \rightarrow A^T$ of

constants is and equivalence), then for any family $B : A \rightarrow \mathbf{Type}$, we have

$$((a : A) \times B(a))^T \simeq (a : A) \times B(a)^T.$$

Proof. If A is T -null, then the fibers $(a : A) \times (f = \mathbf{const}_a)$ of the inclusion of constants $\mathbf{const} : A \rightarrow A^T$ are contractible. Therefore, we have the following equivalences:

$$\begin{aligned} (T \rightarrow (a : A) \times B(a)) &\simeq (f : T \rightarrow A) \times ((t : T) \rightarrow B(ft)) \\ &\simeq (a : A) \times (f : T \rightarrow A) \times (f = \mathbf{const}_a) \times ((t : T) \rightarrow B(ft)) \\ &\simeq (a : A) \times ((t : T) \rightarrow B(\mathbf{const}_a t)) \\ &\equiv (a : A) \times B(a)^T. \end{aligned}$$

□

Theorem 4.7.7. *Suppose that T is a tiny type. Let I be a crisply discrete, T -null type, and let $f_i :: P_i \rightarrow Q_i$ be a crisp family of T -null, sequentially compact types indexed by $i :: I$. (More formally, we have a crisp function $f :: I \rightarrow (X, Y : \mathbf{Type}) \times (X \rightarrow Y)$, and since I is crisply discrete, we may assume any element of I to be crisp) If X is any crisp type, then*

$$(L_f X)^T \simeq L_f(X^T)$$

where L_f is the localization at the family f . As a corollary, we have

$$\|X\|_n^T \simeq \|X^T\|_n$$

for any $n : \mathbb{N}$.

Proof. We will use the construction of $L_f X$ as in Section 7.2 of Rijke's thesis [Rij18b]. This construction proceeds as follows: for any type A , define $\mathbf{QL}_f A$ to be the pushout

$$\begin{array}{ccc} \sum_{i:I} ((P_i \times A^{P_i}) +_{(P_i \times A^{Q_i})} (Q_i \times A^{Q_i})) & \longrightarrow & \sum_{i:I} Q_i \times A^{P_i} \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathbf{QL}_f A \end{array}$$

where the top horizontal map is given by pushout-product and the left vertical map is given by evaluation (see Definition 7.2.6 of [Rij18b] for details). We then define $L_f X$ to be the colimit of the sequence $X \rightarrow \mathbf{QL}_f X \rightarrow \mathbf{QL}_f \mathbf{QL}_f X \rightarrow \dots$. By Proposition 4.7.5, $(L_f X)^T$ is equivalently the colimit of the sequence $X^T \rightarrow (\mathbf{QL}_f X)^T \rightarrow (\mathbf{QL}_f \mathbf{QL}_f X)^T \rightarrow \dots$. Therefore, it will suffice to show that for any crisp type A , we have $(\mathbf{QL}_f A)^T \simeq \mathbf{QL}_f(A^T)$, natural for crisp maps in A .

For this, we will appeal to Lemma 4.7.6 and Proposition 4.7.5 again to see that we have equivalences:

$$\begin{array}{ccc} (\sum_{i:I} ((P_i \times A^{P_i}) +_{(P_i \times A^{Q_i})} (Q_i \times A^{Q_i})))^T & \longrightarrow & (\sum_{i:I} Q_i \times A^{P_i})^T \\ \parallel & & \parallel \\ \sum_{i:I} ((P_i \times (A^T)^{P_i}) +_{(P_i \times (A^T)^{Q_i})} (Q_i \times (A^T)^{Q_i})) & \longrightarrow & \sum_{i:I} Q_i \times (A^T)^{P_i} \end{array}$$

We are making use of the fact that I , P_i and Q_i are all T -null, and that $(-)^T$ commutes with crisp pushouts.

□

Corollary 4.7.8. Let T be a tiny type. If X is crisp and k -connected, then

X^T is k -connected. In particular, for k -commutative higher groups G ,

$$\mathbf{B}^{k+1}(G^T) \simeq (\mathbf{B}^{k+1}G)^T.$$

Proof. By Theorem 4.7.7, we have $\|X^T\|_k \simeq \|X\|_k^T \simeq *^T \simeq *$. □

Theorem 4.7.9. *Let I be a crisply discrete type and let $T :: I \rightarrow \mathbf{Type}$ be a family of tiny types. Let L_T be the modality given by nullifying the T_i . Then L_T commutes with crisp colimits: in particular, with pushouts and colimits of sequences.*

$$L_T(\operatorname{colim} D_j) \simeq \operatorname{colim} L_T D_j.$$

Proof. The argument is the same for any expressible colimit (coequalizers, pushouts, colimits of sequences, etc.). Both $L_T(\operatorname{colim} D_j)$ and $\operatorname{colim} L_T D_j$ are universal for cones under D mapping into L_T -modal types. Since $L_T(\operatorname{colim} D_j)$ is an L_T -modal type admitting a cone under D , we have a map $\operatorname{colim} L_T D_j \rightarrow L_T(\operatorname{colim} D_j)$. To show that this map is an equivalence, it will suffice to show that $\operatorname{colim} L_T D_j$ is L_T -modal; then we can discharge the universal property of $L_T(\operatorname{colim} D_j)$ to construct an inverse.

Being L_T -modal means being T_i -null for all $i : I$. That is, we need to show that $\operatorname{const} : \operatorname{colim} L_T D_j \rightarrow (\operatorname{colim} L_T D_j)^T$ is an equivalence. But by Proposition 4.7.5, we have $(\operatorname{colim} L_T D_j)^T \simeq \operatorname{colim}((L_T D_j)^T)$, and $L_T D_j$ is T -null by construction. □

4.8 Conclusion

We began this chapter by seeing that examples of orbifolds can be constructed explicitly and intuitively in homotopy type theory using the yoga of higher

groups. Orbifolds constructed in this way have meaningful points, and we can study these orbifolds in terms of their points — just as we might study a particular set by understanding its elements.

We then showed that the axioms and notions of synthetic differential geometry generalize smoothly to higher types when interpreted in homotopy type theory. In particular, we saw that a variety of locally discrete higher types — types with discrete types of identifications — are microlinear, using the exact definition which has become standard in SDG.

In particular, we saw a definition of orbifold which closely reflects the intuitive idea of an orbifold as a smooth space whose points have finite automorphism groups. Since finiteness has many constructive incarnations, we had to take a detour to understand finiteness and compactness in SDG. But, in the end, we saw that any proper étale groupoid in the ordinary, external sense (which internally is a crisp ordinary proper étale pregroupoid) presents an orbifold in the sense of the new definition.

Chapter 5

Modal fracture of Higher Groups

5.1 Introduction

There are many situations where cohomology is useful but we need more than just the information of cohomology classes and their relations in cohomology — we need the information of specific cocycles which give rise to those classes and cochains which witness these relations. A striking example of this situation is *ordinary differential cohomology*. To give a home for calculations done in [CS74], Cheeger and Simons [CS85] gave a series of lectures in 1973 defining and studying *differential characters*, which equip classes in ordinary integral cohomology with explicit differential form representatives. Slightly earlier, Deligne [Del71] had put forward a cohomology theory in the complex analytic setting which would go on to be called Deligne cohomology. It was later realized that when put in the differential geometric setting, Deligne cohomology gave a presentation of the theory of differential characters. This combined theory has become known as ordinary differential cohomology.

The ordinary differential cohomology $D_k(X)$ of a manifold X is characterized by its relationship to the ordinary cohomology of X and the differential forms on

X by a diagram known as the *differential cohomology hexagon* or the *character diagram* [SS08]:

$$\begin{array}{ccccc}
 & & \Lambda^k(X)/\text{im}(d) & \xrightarrow{d} & \Lambda_{\text{cl}}^{k+1}(X) & & \\
 & \nearrow & & \searrow & & \nearrow & \\
 H^k(X; \mathbb{R}) & & & & D_k(X) & & H^{k+1}(X; \mathbb{R}) \\
 & \searrow & & \nearrow & & \searrow & \\
 & & H^k(X; U(1)) & \xrightarrow{\beta} & H^{k+1}(X; \mathbb{Z}) & &
 \end{array} \tag{5.1}$$

In this diagram, the top and bottom sequences are long exact, and the diagonal sequences are exact in the middle. The bottom sequence is the Bockstein sequence associated to the universal cover short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1) \rightarrow 0$$

while the top sequence is given by de Rham's theorem representing real cohomology classes by differential forms.

This sort of diagram is characteristic of differential cohomology theories in general. Bunke, Nikolaus, and Vokel [BNV14] interpret differential cohomology theories as sheaves on the site of smooth manifolds and construct differential cohomology hexagons very generally in this setting:

$$\begin{array}{ccccc}
 & & \mathcal{A}(\hat{E})^k(X) & \xrightarrow{\quad} & \mathcal{Z}(\hat{E})^k(X) & & \\
 & \nearrow & & \searrow & & \nearrow & \\
 H^{k-1}(X; Z(\hat{E})) & & & & \hat{E}^k(X) & & H^k(X; Z(\hat{E})) \\
 & \searrow & & \nearrow & & \searrow & \\
 & & H^k(X; S(\hat{E})) & \xrightarrow{\quad} & H^k(X; U(E)) & &
 \end{array}$$

Here, \hat{E} is the differential cohomology theory, $U(\hat{E})$ is its underlying topological

cohomology, $\mathcal{Z}(\hat{E})$ are the differential cycles, \mathcal{S} is the secondary cohomology theory given by flat classes, and \mathcal{A} classifies differential deformations (this summary discussion is lifted from [BNV14]). Here, as with ordinary differential cohomology, the top and bottom sequences are exact, and the diagonal sequences are exact in the middle.

The arguments of Bunke, Nikolaus, and Vokel are abstract and *modal* in character. This is emphasized by Schreiber in his book [Sch13a], where he constructs similar diagrams in the setting of an adjoint triple

$$\Pi_\infty \left(\begin{array}{c} \mathcal{E} \\ \uparrow \\ + \Delta + \\ \downarrow \\ \mathcal{H} \end{array} \right) \Gamma \quad (5.2)$$

in which the middle functor is fully faithful and the leftmost adjoint Π_∞ preserves products. In the case that Γ is the global sections functor of an ∞ -topos \mathcal{E} landing in the ∞ -topos of homotopy types \mathcal{H} , this structure makes \mathcal{E} into a *strongly ∞ -connected* ∞ -topos. In the case that \mathcal{E} is the ∞ -topos of sheaves of homotopy types on manifolds, the leftmost adjoint Π_∞ is given by localizing at the sheaf of real-valued functions; for a representable, this recovers the homotopy type or fundamental ∞ -groupoid of the manifold. Schreiber shows in Proposition 4.1.17 of [Sch13a] that any such adjoint triple gives rise to differential cohomology hexagons, specializing to those of Bunke, Nikolaus, and Vokel in the case that \mathcal{E} is the ∞ -topos of sheaves of homotopy types on smooth manifolds.

This abstract re-reading of the differential cohomology hexagons shows that there is nothing specifically “differential” about them, and that they arise in

situations where there is no differential calculus to be found. Schreiber emphasizes this point in an *nLab* article [Sch21] where he refigures these hexagons as *modal fracture squares*. To recover more traditional fracture theorems, Schreiber considers the case where $\mathcal{E} = A\text{-Mod}$ is the ∞ -category of modules over an \mathbb{E}^2 -ring A ; $\Gamma = \Gamma_I$ is the reflection in to I -nilpotent modules (with $I \subseteq \pi_0 A$ a finitely generated ideal) constructed by Lurie in [Lur11, Notation 4.1.13], and $\Pi_\infty M = M_I^\wedge$ is the I -completion constructed in [Lur11, Notation 4.2.3]. The subcategories of I -nilpotent and I -complete modules are distinct but equivalent, allowing us to see them as a single ∞ -category \mathcal{H} .

In this chapter, we will construct the *modal fracture hexagon* associated to a higher group — a homotopy type which may be delooped — synthetically, working in an appropriately modal homotopy type theory. We will work in Shulman’s *flat homotopy type theory* [Shu18a], a variant of homotopy type theory that adds a *comodality* \flat which may be thought of as the comonad $\Delta\Gamma$ from Diagram 5.2. We will equip this type theory equipped with a modality \flat (which may be thought of as the monad $\Delta\Pi_\infty$) which satisfies a *unity of opposites* axiom (Axiom 8) implying that \flat is left adjoint to \flat in the sense that

$$\flat(\flat X \rightarrow Y) = \flat(X \rightarrow \flat Y).$$

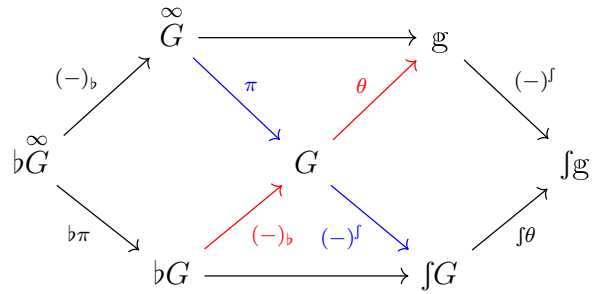
We can think of this axiom as the internalization of the adjunction $\Delta\Pi_\infty \dashv \Delta\Gamma$ induced by Diagram 5.2.

This type theory should have models in all *strongly ∞ -connected geometric morphisms* between ∞ -toposes. A geometric morphism $f : \mathcal{E} \rightarrow \mathcal{S}$ is strongly ∞ -connected when its inverse image $f^* : \mathcal{S} \rightarrow \mathcal{E}$ is fully faithful and has a left

adjoint $f_! : \mathcal{E} \rightarrow \mathcal{S}$ which preserves finite products. We then have $\int = f^* f_!$ and $\flat = f^* f_*$. An ∞ -topos \mathcal{E} is strongly ∞ -connected when its terminal geometric morphism $\Gamma : \mathcal{E} \rightarrow \infty\mathbf{Grpd}$ is strongly ∞ -connected.

Our main theorem (an unstable and synthetic version of Proposition 4.1.17 in [Sch13a]) is as follows:

Theorem 5.2.31. *For a crisp ∞ -group G , there is a modal fracture hexagon:*



where

- $\theta : G \rightarrow \mathfrak{g}$ is the infinitesimal remainder of G , the quotient $G // \flat G$, and
- $\pi : \tilde{G} \rightarrow G$ is the universal (contractible) ∞ -cover of G .

Moreover,

1. The middle diagonal sequences are fiber sequences.
2. The top and bottom sequences are fiber sequences.
3. Both squares are pullbacks.

Furthermore, the homotopy type of \mathfrak{g} is a delooping of $\flat \tilde{G}$:

$$\int \mathfrak{g} = \flat \mathbf{B}\tilde{G}.$$

Therefore, if G is k -commutative for $k \geq 1$ (that is, admits further deloopings $\mathbb{B}^{k+1}G$), then we may continue the modal fracture hexagon on to $\mathbb{B}^k G$.

We will define the notions of *universal ∞ -cover* and of *infinitesimal remainder* in Sections 5.2.2 and 5.2.3 respectively. Our proof of this theorem will make extensive use of the theory of modalities in homotopy type theory developed by Rijke, Shulman, and Spitters [RSS20], as well as the theory of modal étale maps developed in [CR21] and the theory of modal fibrations developed in Chapter 3.

Having proven this theorem, we will turn our attention to providing interesting examples of it. To that end, in Section 5.3 we will construct ordinary differential cohomology (in the guise of the classifying bundles $\mathbb{B}_{\nabla}^k U(1)$ of connections on k -gerbes with band $U(1)$, see Definition 5.3.5) in smooth real cohesive homotopy type theory. For this, we assume the existence of a long exact sequence

$$0 \rightarrow \flat \mathbb{R} \rightarrow \mathbb{R} \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \rightarrow \dots$$

where the Λ^k classify *differential k -forms*. It should be possible to construct the Λ^k from the axioms of synthetic differential geometry with tiny infinitesimals, but we do not do so here for reasons of space and self-containment. See Remark 5.3.1 for a full discussion.

Our construction of ordinary differential cohomology is clean, conceptual, and modal. We do not, however, recover exactly the character diagram 5.1, because de Rham's theorem does not hold for all types (see Proposition 5.3.20). In Section 5.3.3, we do recover a very similar diagram and find that the obstruction to these two diagrams being the same lies in a shifted version of ordinary differential cohomology.

Because the arguments given in Section 5.3 are abstract and modal in character, they are applicable in other settings. In Section 5.3.4, we express our construction of ordinary differential cohomology in the abstract setting of a *contractible and infinitesimal resolution* of a crisp abelian group. In Section 5.3.5, we briefly describe how to construct combinatorial analogues of ordinary differential cohomology in symmetric simplicial homotopy types, making use of an observation of Lawvere that cocycle classifiers may be constructed using the tinyness of the simplices.

5.2 The Modal Fracture Hexagon

In this section, we will construct the modal fracture hexagon of a higher group.

A higher group G is a type equipped with a 0-connected delooping $\mathbf{B}G$. An ordinary group G may be considered as a higher group by taking $\mathbf{B}G$ to be the type of G -torsors and equating G with the group of automorphisms of G considered as a G -torsor.

The theory of higher groups is expressed in terms of their deloopings: for example a homomorphism $G \rightarrow H$ is equivalently a pointed map $\mathbf{B}G \rightarrow \mathbf{B}H$. See [BDR18a] for a development of the elementary theory of higher groups in homotopy type theory.

The modal fracture hexagon associated to a (crisp) higher group G will factor G into its *universal ∞ -cover* $\overset{\infty}{G}$ and its *infinitesimal remainder* \mathfrak{g} . We will therefore introduce $\overset{\infty}{G}$ and \mathfrak{g} and prove some lemmas about them which will set the stage for the modal fracture hexagon.

Notation. We will use the Agda-inspired notation for dependent pair types

(also known as dependent sum types) and dependent function types (also known as dependent product types):

$$(a : A) \times B(a) \equiv \sum_{a:A} B(a)$$

$$(a : A) \rightarrow B(a) \equiv \prod_{a:A} B(a).$$

If X is a pointed type, we refer to its base point as $\mathbf{pt}_X : X$. If X and Y are pointed types, then we define $X \cdot \rightarrow Y$ to be the type of pointed functions between them:

$$(X \cdot \rightarrow Y) \equiv (f : X \rightarrow Y) \times (f(\mathbf{pt}_X) = \mathbf{pt}_Y).$$

5.2.1 Preliminaries

In this section, we will review Shulman’s flat type theory [Shu18a] and the necessary lemmas.

In constructive mathematics, the proposition that all functions $\mathbb{R} \rightarrow \mathbb{R}$ are continuous is undecided — there are models of constructive set theory (and homotopy type theory) in which every function $\mathbb{R} \rightarrow \mathbb{R}$ is continuous (and, of course, familiar models where there are discontinuous functions $\mathbb{R} \rightarrow \mathbb{R}$). Since, in type theory, such a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by giving the image $f(x)$ of a free variable $x : \mathbb{R}$, we see that in a pure constructive setting, the dependence of terms on their free variables confers a liminal sort of continuity. This is a very powerful observation which, in its various guises, lets us avoid the menial checking of continuity, smoothness, regularity, and so on for various sorts of functions in various models of homotopy type theory. It extends far beyond real

valued functions; for example, the assignment of a vector space V_p to a point p in a manifold M , constructively, gives a vector bundle $(p : M) \times V_p \rightarrow M$ over M with all its requirements of continuity or smoothness (depending on the model). Since all sorts of continuity (continuity, smoothness, regularity, analyticity) can be captured in various models, Lawvere named the general notion “cohesion” in his paper [Law07], whose generalization to ∞ -categories in [Sch13a] inspired the type theory of [Shu18a].

However, not every dependency is cohesive (continuous, smooth, etc.). To enable discontinuous dependencies, then, we must mark our free variables as varying cohesively or not. For this reason, Shulman introduces *crisp* variables, which are free variables in which terms depend discontinuously:

$$a :: A.$$

Any variable appearing in the type of a crisp variable must also be crisp, and a crisp variable may only be substituted by expressions that *only* involve crisp variables. When all the variables in an expression are crisp, we say that that expression is crisp; so, we may only substitute crisp expressions in for crisp variables. Constants — like $0 : \mathbb{N}$ or $\mathbb{N} : \mathbf{Type}$ — appearing in an empty context are therefore always crisp.¹ This means that one cannot give a closed form example of a term which is *not* crisp; all terms with no free variables are crisp. For emphasis, we will say that a term which is not crisp is *cohesive*. The rules for crisp type theory can be found in Section 2 of [Shu18a].

¹Note that as these are terms and not free variables, we don’t need to use the special syntax $a :: A$. The double colon introduces a crisp *free* variable.

Given this notion of discontinuous dependence of terms on their free variables, we can now define an operation on types which removes the cohesion amongst their points. Given a *crisp* type X , we have a type $\flat X$ whose points are, in a sense, the *crisp* points of X . Since it is free variables that may be crisp, we express this idea by allowing ourselves to assume that a (cohesive) variable $x : \flat X$ is of the form u^\flat for a *crisp* $u :: X$. More precisely, whenever we have type family $C : \flat X \rightarrow \mathbf{Type}$, an $x : \flat X$, and an element $f(u) : C(u^\flat)$ depending on a *crisp* $u :: X$, we get an element

$$(\text{let } u^\flat := x \text{ in } f(u)) : C(x)$$

and if $x \equiv v^\flat$, then $(\text{let } u^\flat := x \text{ in } f(u)) \equiv f(v)$. We refer to this method of proof as “ \flat -induction”. The full rules for \flat can be found in Section 4 of [Shu18a].

We have an inclusion $(-)_\flat : \flat X \rightarrow X$ given by $x_\flat := \text{let } u^\flat := x \text{ in } u$. Since we are thinking of a dependence on a crisp variable as a *discontinuous* dependence, if this map $(-)_\flat : \flat X \rightarrow X$ is an equivalence then every *discontinuous* dependence on $x :: X$ underlies a *continuous* dependence on x . This leads us to the following definition:

Definition 5.2.1. A crisp type $X :: \mathbf{Type}$ is *crisply discrete* if the counit $(-)_\flat : \flat X \rightarrow X$ is an equivalence.²

Note that this definition is only sensible for crisp types, since we may only form $\flat X$ for crisp $X :: \mathbf{Type}$. We would also like a notion of discreteness which applies to any type, and a reflection $X \rightarrow \flat X$ of a type into a discrete type. For that reason, we will also presume that there is a modality \flat called the *shape*

²See Remark 6.13 of [Shu18a] for a discussion on some of the subtleties in the notion of crisp discreteness.

(and which we think of as sending a type X to its homotopy type or shape $\int X$). We refer to the \int -modal types as discrete. To make sure that these two notions of discreteness coincide, we assume the following axiom:

Axiom 8 (Unity of Opposites). For any *crisp* type X , the counit $(-)^{\flat} : \flat X \rightarrow X$ is an equivalence if and only if the unit $(-)^{\sharp} : X \rightarrow \int X$ is an equivalence.

This axiom implies that \int is left adjoint to \flat , at least for crisp maps. In [Shu18a], Shulman assumes an axiom C0 which lets him define the \int modality as a localization and prove our Unity of Opposites axiom. The two axioms have roughly the same strength, though C0 is slightly stronger since it assumes that \int is an *accessible* modality.

Theorem 5.2.2. *Let X and Y be crisp types. Then*

$$\flat(X \rightarrow \flat Y) = \flat(\int X \rightarrow Y).$$

Proof. This is Theorem 9.15 of [Shu18a]. Note that Axiom C0 is only used via our Unity of Opposites axiom. □

Remark 5.2.3. Theorem 5.2.2 justifies the use of the symbol “ \flat ” in flat type theory. If we think of $\int X$ as the homotopy type of X , then the adjointness of \int with \flat tells us that $\flat \mathbf{B}G$ modulates principal G -bundles with a *homotopy invariant parallel transport* — that is, bundles with *flat* connection. This terminology is due to Schreiber in [Sch13a].

We may also define the *truncated shape modalities* \int_n to have as modal types the types which are both n -truncated and \int -modal. It is not known whether

$\int_n X = \|\int X\|_n$ for general X , but it is true for *crisp* X (see Proposition 4.5 of Chapter 3).

We will now prepare ourselves by proving a few preservation properties of the \flat comodality and the \int modality. A first time reader may content themselves with the the statements of the lemmas, as the proofs are mere technicalities.

Lemma 5.2.4. The comodality \flat preserves fiber sequences. Let $f :: X \rightarrow Y$ be a crisp map and $y :: Y$ a crisp point. Then we have an equivalence $\flat \mathbf{fib}_f(y) = \mathbf{fib}_{\flat f}(y^\flat)$ such that

$$\begin{array}{ccc}
 \flat \mathbf{fib}_f(y) & \xrightarrow{(-)_\flat} & \mathbf{fib}_f(y) \\
 \parallel & & \nearrow \delta \\
 \mathbf{fib}_{\flat f}(y^\flat) & &
 \end{array}$$

commutes. In particular, $\flat \mathbf{fib}_f(y) \rightarrow \flat X \rightarrow \flat Y$ is a fiber sequence and that the naturality squares give a map of fiber sequences:

$$\begin{array}{ccc}
 \flat \mathbf{fib}_f(y) & \xrightarrow{\delta} & \mathbf{fib}_f(y) \\
 \downarrow & & \downarrow \\
 \flat X & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 \flat Y & \longrightarrow & Y
 \end{array}$$

Proof. We begin by constructing the equivalence:

$$\begin{aligned}
\flat \mathbf{fib}_f(y) &\equiv \flat ((x : X) \times (f(x) = y)) \\
&= (u : \flat X) \times (\text{let } x^\flat := u \text{ in } \flat(f(x) = y)) && \text{[Shu18a, Lemma. 6.8]} \\
&= (u : \flat X) \times (\text{let } x^\flat := u \text{ in } f(x)^\flat = y^\flat) && \text{[Shu18a, Theorem. 6.1]} \\
&\equiv (u : \flat X) \times (\text{let } x^\flat := u \text{ in } \flat f(x^\flat) = y^\flat) \\
&= (u : \flat X) \times (\flat f(u) = y^\flat) && \text{[Shu18a, Lemma. 4.4]} \\
&\equiv \mathbf{fib}_{\flat f}(y^\flat).
\end{aligned}$$

We will need to understand what this equivalence does on elements $(x, p)^\flat$ for $(x, p) :: \mathbf{fib}_f(y)$. The first equivalence in the composite sends $(x, p)^\flat$ to (x^\flat, p^\flat) , and no other equivalence affects the first component, so the first component of the result will be x^\flat . The second equivalence will send p^\flat to $\mathbf{ap}_\flat(-)^\flat p$, where \mathbf{ap}_\flat is the crisp application function. The next equivalence is given by reflexivity, since $\flat f(x^\flat) \equiv f(x)^\flat$. In total, then, this equivalence acts as

$$(x, p)^\flat \mapsto (x^\flat, \mathbf{ap}_\flat(-)^\flat p).$$

Now, to show the triangle commutes, it will suffice to show that it commutes for $(x, p)^\flat$ where $(x, p) :: \mathbf{fib}_f(y)$. This is to say, we need to show that sending $(x, p)^\flat$ through the above equivalence and then into $\mathbf{fib}_f(y)$ yields (x, p) . The map δ from $\mathbf{fib}_{\flat f}(y^\flat)$ to $\mathbf{fib}_f(y)$ sends (u, q) to $(u_\flat, \square_\flat(u) \bullet \mathbf{ap}(-)_\flat q)$, where $\square_\flat(u) : f(u_\flat) = \flat f(u)_\flat$ is the naturality square. So, the round trip

$\flat \mathbf{fib}_f(y) \rightarrow \mathbf{fib}_{\flat f}(y^\flat) \rightarrow \mathbf{fib}_f(y)$ acts as

$$(x, p)^\flat \mapsto (x^\flat, \mathbf{ap}_\flat(-)^\flat p) \mapsto (x^\flat_\flat, \square_\flat(x^\flat) \bullet \mathbf{ap}(-)_\flat(\mathbf{ap}_\flat(-)^\flat p)).$$

Now, $x^\flat_\flat \equiv x$, so it remains to show that $\square_\flat(x^\flat) \bullet \mathbf{ap}(-)_\flat(\mathbf{ap}_\flat(-)^\flat p) = p$. However, the naturality square is defined by $\square_\flat(x^\flat) \equiv \mathbf{refl}_{f(x)} : f(x^\flat_\flat) = \flat f(x^\flat)_\flat$, so it only remains to show that the two applications cancel. This can easily be shown by a crisp path induction. \square

Lemma 5.2.5. Let $f :: X \rightarrow Y$ be a crisp map between crisp types. The following are equivalent:

1. For every crisp $y :: Y$, $\mathbf{fib}_f(y)$ is discrete.
2. The naturality square

$$\begin{array}{ccc} \flat X & \xrightarrow{(-)_\flat} & X \\ \flat f \downarrow & & \downarrow \pi \\ \flat Y & \xrightarrow{(-)_\flat} & Y \end{array}$$

is a pullback.

Proof. We note that the naturality square being a pullback is equivalent to the induced map

$$\mathbf{fib}_{\flat f}(u) \rightarrow \mathbf{fib}_f(u_\flat)$$

begin an equivalence for all $u : \flat Y$. By the universal property of $\flat Y$, we may assume that u is of the form y^\flat for a crisp $y :: Y$. By Lemma 5.2.4, we have

that

$$\begin{array}{ccc}
 \mathfrak{b} \mathbf{fib}_f(y) & \xrightarrow{(-)_\mathfrak{b}} & \mathbf{fib}_f(y) \\
 \parallel & \nearrow & \\
 \mathbf{fib}_{\mathfrak{b}f}(y^\mathfrak{b}) & &
 \end{array}$$

commutes. Therefore, the naturality square is a pullback if and only if for all crisp $y :: Y$, we have that $(-)_\mathfrak{b} : \mathfrak{b} \mathbf{fib}_f(y) \rightarrow \mathbf{fib}_f(y)$ is an equivalence; but this is precisely what it means for $\mathbf{fib}_f(y)$ to be discrete. \square

Lemma 5.2.6. Let X be a crisp type, and let $a, b :: X$ be crisp elements. Then there is an equivalence $e : (a^\mathfrak{b} = b^\mathfrak{b}) \simeq \mathfrak{b}(a = b)$ together with a commutation of the following triangle:

$$\begin{array}{ccc}
 (a^\mathfrak{b} = b^\mathfrak{b}) & \xrightarrow{\mathbf{ap}(-)_\mathfrak{b}} & (a = b) \\
 \parallel & \nearrow & \\
 e & & \\
 \parallel & \searrow & \\
 \mathfrak{b}(a = b) & \xrightarrow{(-)_\mathfrak{b}} &
 \end{array}$$

Proof. For the construction of the equivalence e we refer to [Shu18a, Theorem. 6.1]. For the commutativity, we use function extensionality to work from $u : \mathfrak{b}(a = b)$ seeking $e^{-1}(u) = u_\mathfrak{b}$ and proceed by \mathfrak{b} -induction and then identity induction in which case both sides reduce to \mathbf{refl} . \square

Lemma 5.2.7. Let G be a crisp higher group; that is, suppose that $\mathbf{B}G$ is a crisp, 0-connected type and its base point $\mathbf{pt} :: \mathbf{B}G$ is also crisp. Then $\mathfrak{b}G$ is also a higher group and we may take

$$\mathbf{B}\mathfrak{b}G \equiv \mathfrak{b}\mathbf{B}G$$

pointed at $\mathbf{pt}^\mathfrak{b}$. Furthermore, the counit $(-)_\mathfrak{b} : \mathfrak{b}G \rightarrow G$ is a homomorphism

delooped by the counit $(-)_b : \mathfrak{b}BG \rightarrow BG$.

Proof. We need to show that $\mathfrak{b}BG$ deloops $\mathfrak{b}G$ via an equivalence $e : \Omega\mathfrak{b}BG = \mathfrak{b}G$, that it is 0-connected, and that looping the counit $(-)_b : \mathfrak{b}BG \rightarrow BG$ corresponds to the counit $(-)_b : \mathfrak{b}G \rightarrow G$ along the equivalence e .

For the equivalence $e : \Omega\mathfrak{b}BG = \mathfrak{b}G$, we may take the equivalence $(\mathfrak{p}\mathfrak{t}^{\mathfrak{b}} = \mathfrak{p}\mathfrak{t}^{\mathfrak{b}}) = \mathfrak{b}(\mathfrak{p}\mathfrak{t} = \mathfrak{p}\mathfrak{t})$ of Lemma 5.2.6. The commutation of the triangle

$$\begin{array}{ccc}
 (\mathfrak{p}\mathfrak{t}^{\mathfrak{b}} = \mathfrak{p}\mathfrak{t}^{\mathfrak{b}}) & \xrightarrow{\text{ap}(-)_b} & (\mathfrak{p}\mathfrak{t} = \mathfrak{p}\mathfrak{t}) \\
 \parallel \scriptstyle e & & \nearrow \scriptstyle (-)_b \\
 \mathfrak{b}(\mathfrak{p}\mathfrak{t} = \mathfrak{p}\mathfrak{t}) & &
 \end{array}$$

shows that $(-)_b : \mathfrak{b}BG \rightarrow BG$ deloops $(-)_b : \mathfrak{b}G \rightarrow G$.

To show that $\mathfrak{b}BG$ is connected, we rely on [Shu18a, Corollary. 6.7] which says that $\|\mathfrak{b}BG\|_0 = \mathfrak{b}\|BG\|_0$, which is $*$ by the hypothesis that BG is 0-connected.

□

We end with a useful lemma: \mathfrak{b} preserves long exact sequences of groups.

Lemma 5.2.8. The comodality \mathfrak{b} preserves crisp short and long exact sequences of groups.

Proof. A sequence

$$0 \rightarrow K \rightarrow G \rightarrow H \rightarrow 0$$

of groups is short exact if and only if its delooping

$$BK \cdot \rightarrow BG \cdot \rightarrow BH$$

is a fiber sequence. But \flat preserves crisp fiber sequences by Lemma 5.2.4 and by Lemma 5.2.7 we have that the fiber sequence

$$\flat \mathbf{B}K \cdot \rightarrow \flat \mathbf{B}G \cdot \rightarrow \flat \mathbf{B}H$$

deloops the sequence $\flat K \rightarrow \flat G \rightarrow \flat H$, so this sequence is also short exact.

Now, a complex of groups

$$\cdots \rightarrow A_{n-1} \xrightarrow{d} A_n \xrightarrow{d} A_{n+1} \rightarrow \cdots$$

satisfying $d \circ d = 0$ is long exact if and only if the sequences

$$0 \rightarrow K_n \rightarrow A_n \rightarrow K_{n+1} \rightarrow 0$$

are short exact, where $K_n := \ker(A_n \rightarrow A_{n+1})$. Now, we have a complex

$$\cdots \rightarrow \flat A_{n-1} \xrightarrow{\flat d} \flat A_n \xrightarrow{\flat d} \flat A_{n+1} \rightarrow \cdots$$

by the functoriality of \flat . Since \flat preserves short exact sequences, the sequences

$$0 \rightarrow \flat K_n \rightarrow \flat A_n \rightarrow \flat A_{n+1} \rightarrow 0$$

are short exact. Now, since \flat preserves fibers we have that

$$\flat K_n = \ker(\flat A_n \rightarrow \flat A_{n+1}),$$

so that the \flat -ed complex is long exact. □

5.2.2 The Universal ∞ -Cover of a Higher Group

An ∞ -cover of a type X is a generalization of the notion of cover from a theory concerning 1-types (the fundamental groupoid of X , with the universal cover being simply connected) to arbitrary types (the homotopy type of X , with the universal ∞ -cover being contractible).

Recall that, classically, a covering map $\pi : \tilde{X} \rightarrow X$ satisfies a *unique path lifting property*; that is, every square of the following form admits a unique filler:

$$\begin{array}{ccc} * & \xrightarrow{\tilde{x}} & \tilde{X} \\ 0 \downarrow & \exists! \nearrow & \downarrow \pi \\ \mathbb{R} & \xrightarrow{\gamma} & X \end{array}$$

This property can be extended to a unique lifting property against any map which induces an equivalence fundamental groupoids. That is, whenever $f : A \rightarrow B$ induces an equivalence $\int_1 f : \int_1 A \rightarrow \int_1 B$, every square of the following form admits a unique filler:

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\gamma}} & \tilde{X} \\ f \downarrow & \exists! \nearrow & \downarrow \pi \\ B & \xrightarrow{\gamma} & X \end{array}$$

For any modality $!$, there is an orthogonal factorization system where $!$ -equivalences (those maps f such that $!f$ is an equivalence) lift uniquely against *! $\text{-}\acute{e}$ tale maps* ([Rij18b; CR21]).

Definition 5.2.9. A map $f : A \rightarrow B$ is *! $\text{-}\acute{e}$ tale* for a modality $!$ if the naturality square

$$\begin{array}{ccc} A & \xrightarrow{(-)!} & !A \\ f \downarrow & & \downarrow !f \\ B & \xrightarrow{(-)!} & !B \end{array}$$

is a pullback.

We may single out the covering maps as the \int_1 -étale maps whose fibers are sets. For more on this point of view, see the last section of Chapter 3. Here, however, we will be more concerned with \int -étale maps, which we will call ∞ -covers. This notion was called a “modal covering” in [Wel18b], and was referred to as an ∞ -cover in the setting of ∞ -categories by Schreiber in [Sch13a].

Definition 5.2.10. A map $\pi : E \rightarrow B$ is an ∞ -cover if the naturality square

$$\begin{array}{ccc} E & \xrightarrow{(-)^{\int}} & \int E \\ \pi \downarrow & & \downarrow \int \pi \\ B & \xrightarrow{(-)^{\int}} & \int B \end{array}$$

is a pullback. That is, an ∞ -cover is precisely a \int -étale map.

A map $\pi : E \rightarrow B$ is an n -cover if it is \int_{n+1} -étale and its fibers are n -types. We call a 1-cover just a cover, or a covering map.

Theorem 6.1 of Chapter 3 gives a useful way for proving that a map is an ∞ -cover.

Proposition 5.2.11 (Theorem 6.1 of Chapter 3). Let $\pi : E \rightarrow B$ and suppose that there is a crisp, discrete type F so that for all $b : B$, $\|\mathbf{fib}_{\pi}(b) = F\|$. Then π is an ∞ -cover.

Example 16. As an example of an ∞ -cover, consider the exponential map $\mathbb{R} \rightarrow \mathbb{S}^1$ from the real line to the circle. The fibers of this map are all merely \mathbb{Z} , so by Theorem 5.2.11, this map is an ∞ -cover. Since \mathbb{R} is contractible, it is in fact the *universal* ∞ -cover of the circle.

Just as the universal cover of a space X is any simply connected cover \tilde{X} , the universal ∞ -cover $\overset{\infty}{X}$ of a type X is any *contractible* cover — contractible in the sense of being \mathfrak{f} -connected, meaning $\mathfrak{f}\overset{\infty}{X} = *$. Since units of a modality are modally connected, we may always construct a universal ∞ -cover by taking the fiber of the \mathfrak{f} -unit $(-)^{\mathfrak{f}} : X \rightarrow \mathfrak{f}X$.

Definition 5.2.12. The *universal* ∞ -cover of a pointed type X is defined to be the fiber of the \mathfrak{f} -unit:

$$\overset{\infty}{X} := \text{fib}((-\)^{\mathfrak{f}} : X \rightarrow \mathfrak{f}X).$$

Since the units of modalities are modally connected, $\overset{\infty}{X}$ is homotopically contractible:

$$\mathfrak{f}\overset{\infty}{X} = *.$$

Let’s take a bit to get an image of the universal ∞ -cover of a type. The universal ∞ -cover of a type only differs from its universal cover in the identifications between its points; in other words, it is a “stacky” version of the universal cover.

Proposition 5.2.13. Let X be a crisp type. Then the map $\overset{\infty}{X} \rightarrow \tilde{X}$ from the universal ∞ -cover of X to its universal cover induced by the square

$$\begin{array}{ccccc} \overset{\infty}{X} & \xrightarrow{\pi} & X & \xrightarrow{(-)^{\mathfrak{f}}} & \mathfrak{f}X \\ \downarrow \text{---} & & \parallel & & \downarrow \\ \tilde{X} & \xrightarrow{\pi} & X & \xrightarrow{(-)^{\mathfrak{f}_1}} & \mathfrak{f}_1 X \end{array}$$

is $\|_0$ -connected and \mathfrak{f} -modal. In particular, if X is a set then $\left\| \overset{\infty}{X} \right\|_0 = \tilde{X}$. Furthermore, its fibers may be identified with the loop space $\Omega((\mathfrak{f}X)\langle 1 \rangle)$ of the

first stage of the Whitehead tower of the shape of X .

Proof. We begin by noting that the unique factorization $f : \int X \rightarrow \int_1 X$ of the \int_1 unit $(-)^{\int_1} : X \rightarrow \int_1 X$ through the \int unit is a $\|-\|_1$ unit. We note that $(-)^{\int_1} : X \rightarrow \int_1 X$ and $\|(-)^{\int_1}\|_1 : X \rightarrow \|\int X\|_1$ have the same universal property: any map from X to a discrete 1-type factors uniquely through them. However, unless X is crisp, we do not know that $\|\int X\|_1$ is itself discrete; in general, we can only conclude that there is a map $\|\int X\|_1 \rightarrow \int_1 X$. This is why we must assume that X is crisp. By Proposition 4.5 of Chapter 3, $\|\int X\|_1$ is discrete and therefore the map $\|\int X\|_1 \rightarrow \int_1 X$ is an equivalence. Since f factors uniquely through this map (since \int_1 is a 1-type), we see that f is equal to the $\|-\|_1$ unit of $\int X$ and is therefore a $\|-\|_1$ unit. In particular, $f : \int X \rightarrow \int_1 X$ is 1-connected.

Now we will show that the fibers of the induced map $\tilde{X}^\infty \rightarrow \tilde{X}$ are 0-connected and discrete. Consider the following diagram:

$$\begin{array}{ccccc}
 \text{fib}(p) & \longrightarrow & * & \longrightarrow & \text{fib}((\pi p)^{\int_1}) \\
 \downarrow & & \downarrow & & \downarrow \\
 \tilde{X}^\infty & \xrightarrow{\pi} & X & \xrightarrow{(-)^{\int}} & \int X \\
 \downarrow & & \parallel & & \downarrow \\
 \tilde{X} & \xrightarrow{\pi} & X & \xrightarrow{(-)^{\int_1}} & \int_1 X
 \end{array}$$

All vertical sequences are fiber sequences, and the bottom two sequences are fiber sequences; therefore, the top sequence is a fiber sequence, which tells us that the fiber over any point $p : \tilde{X}$ is equivalent to $\Omega \text{fib}_f((\pi p)^{\int_1})$. As we have shown that the fibers of f are 1-connected, their loop spaces are 0-connected. And as f is a map between discrete types, its fibers are discrete and so their loop spaces are discrete. Finally, we note that the fiber of the 1-truncation

$|\cdot|_1 : \mathfrak{f}X \rightarrow \|\mathfrak{f}X\|_1$ is the first stage $(\mathfrak{f}X)\langle 1 \rangle$ of the Whitehead tower of $\mathfrak{f}X$. \square

We are now ready to prove a simple sort of fracture theorem for any crisp, pointed type. This square will make up the left square of our modal fracture hexagon.

Proposition 5.2.14. Let X be a crisp, pointed type. Then the \flat naturality square of the universal ∞ -cover $\pi : \overset{\infty}{X} \rightarrow X$ is a pullback:

$$\begin{array}{ccc} \flat \overset{\infty}{X} & \xrightarrow{(-)_\flat} & \overset{\infty}{X} \\ \flat \pi \downarrow & & \downarrow \pi \\ \flat X & \xrightarrow{(-)_\flat} & X \end{array}$$

Proof. By Lemma 5.2.5, it will suffice to show that over a crisp point $x :: X$, $\text{fib}_\pi(x)$ is discrete. But since π is, by definition, the fiber of $(-)^{\mathfrak{f}}$, we have that

$$\text{fib}_\pi(x) = \Omega(\mathfrak{f}X, x^{\mathfrak{f}}).$$

Since $\mathfrak{f}X$ is discrete by assumption, so is $\Omega(\mathfrak{f}X, x^{\mathfrak{f}})$. \square

Although \mathfrak{f} is *not* a left exact modality — it does not preserve all pullbacks — it does preserve pullbacks and fibers of \mathfrak{f} -fibrations. The theory of modal fibrations was developed in Chapter 3. Included amongst the \mathfrak{f} -fibrations are the \mathfrak{f} -étale maps, and so \mathfrak{f} preserves fiber sequences of \mathfrak{f} -étale maps.

Lemma 5.2.15. Let $f : X \rightarrow Y$ be an ∞ -cover. Then for any $y : Y$, the sequence

$$\mathfrak{f} \text{fib}_f(y) \rightarrow \mathfrak{f}X \xrightarrow{\mathfrak{f}f} \mathfrak{f}Y$$

is a fiber sequence.

Proof. Since a \mathfrak{J} -étale map f is modal, its étale and modal factors agree (they are equivalently f), so by Theorem 1.2 of Chapter 3, f is a \mathfrak{J} -fibration. The result then follows since \mathfrak{J} preserves all fibers of \mathfrak{J} -fibrations (see also Theorem 1.2 of Chapter 3). \square

Importantly, it is also true that the shape of a *crisp* n -connected type is also n -connected by Theorem 8.6 of Chapter 3. It follows that $\mathfrak{J}BG$ is a delooping of $\mathfrak{J}G$ for crisp higher groups G , and that this can continue for higher deloopings.

Proposition 5.2.16. Let G be a crisp higher group. Then its universal ∞ -cover $\overset{\infty}{G}$ is a higher group and $\pi : \overset{\infty}{G} \rightarrow G$ is a homomorphism. Furthermore, if G is k -commutative, then so is $\overset{\infty}{G}$.

Proof. We may define

$$\mathbf{B}^i \overset{\infty}{G} := \text{fib}((-)^{\mathfrak{J}} : \mathbf{B}^i G \rightarrow \mathfrak{J}\mathbf{B}^i G).$$

This lets us extend the fiber sequence:

$$\begin{array}{ccccc} \overset{\infty}{G} & \xrightarrow{\pi} & G & \xrightarrow{(-)^{\mathfrak{J}}} & \mathfrak{J}G \\ \downarrow & & & & \downarrow \\ \mathbf{B}\overset{\infty}{G} & \xrightarrow{\mathbf{B}\pi} & \mathbf{B}G & \xrightarrow{(-)^{\mathfrak{J}}} & \mathfrak{J}\mathbf{B}G \\ \downarrow & & & & \downarrow \\ \mathbf{B}^2\overset{\infty}{G} & \xrightarrow{\mathbf{B}^2\pi} & \mathbf{B}^2G & \xrightarrow{(-)^{\mathfrak{J}}} & \mathfrak{J}\mathbf{B}^2G \\ \downarrow & & & & \downarrow \\ \dots & & & & \dots \end{array}$$

\square

5.2.3 The Infinitesimal Remainder of a Higher Group

In this section, we will investigate the infinitesimal remainder $\theta : G \rightarrow \mathfrak{g}$ of higher group G . The infinitesimal remainder is what is left of a higher group when all of its crisp points have been made equal. Having trivialized all substantial difference between points, we are left with the infinitesimal differences that remain.

Definition 5.2.17. Let G be a higher group. Define its *infinitesimal remainder* to be

$$\mathfrak{g} := \text{fib}((-)_b : \flat BG \rightarrow BG).$$

Then, continuing the fiber sequence, we have

$$\begin{array}{ccccc} \flat G & \xrightarrow{(-)_b} & G & \xrightarrow{\theta} & \mathfrak{g} \\ & & & \searrow & \\ & & \flat BG & \longrightarrow & BG \end{array}$$

which defines the quotient map $\theta : G \rightarrow \mathfrak{g}$.

Remark 5.2.18. By its construction, we can see that \mathfrak{g} modulates flat connections on trivial principal G -bundles, with respect to the interpretation of $\flat BG$ given in Remark 5.2.3. In the setting of differential geometry, such flat connections on trivial principal G -bundles are given by closed \mathfrak{g} -valued 1-forms, where here \mathfrak{g} is the Lie algebra of the Lie group G . In this setting, θ is the Maurer-Cartan form on G . This is why we adopt the name $\theta : G \rightarrow \mathfrak{g}$ for the infinitesimal remainder in general. This can in fact be *proven* in the setting of synthetic differential geometry with tiny infinitesimals satisfying a principle of constancy using a purely modal argument. See Remark 5.3.1 for a further discussion.

Remark 5.2.19. The infinitesimal remainder \mathfrak{g} is defined as the *de Rham coefficient object* of $\mathbf{B}G$ in Definition 5.2.59 of [Sch13a]. Schreiber defined $\flat_{dR}X$ for any (crisp) pointed type X as the fiber of $(-)^{\flat} : \flat X \rightarrow X$, so that $\mathfrak{g} \equiv \flat_{dR}\mathbf{B}G$. We focus on the case that X is 0-connected — of the form $\mathbf{B}G$ — and so only consider the infinitesimal remainder of a higher group G .

While the infinitesimal remainder exists for any (crisp) higher group, it is not necessarily itself a higher group. However, if G is braided, then \mathfrak{g} will be a higher group.

Proposition 5.2.20. If G is a crisp k -commutative higher group, then \mathfrak{g} is a $(k - 1)$ -commutative higher group. In particular, if G is a braided higher group, then \mathfrak{g} is a higher group and the remainder map $\theta : G \rightarrow \mathfrak{g}$ is a homomorphism.

Proof. We may define

$$\mathbf{B}^i \mathfrak{g} := \text{fib}((-)^{\flat} : \flat \mathbf{B}^{i+1}G \rightarrow \mathbf{B}^{i+1}G),$$

which lets us continue the fiber sequence:

$$\begin{array}{ccccc}
 \flat G & \xrightarrow{(-)^{\flat}} & G & \xrightarrow{\theta} & \mathfrak{g} \\
 \downarrow & & \downarrow & & \downarrow \\
 \flat \mathbf{B}G & \longrightarrow & \mathbf{B}G & \xrightarrow{\mathbf{B}\theta} & \mathbf{B}\mathfrak{g} \\
 \downarrow & & \downarrow & & \downarrow \\
 \flat \mathbf{B}^2 G & \longrightarrow & \mathbf{B}^2 G & \xrightarrow{\mathbf{B}^2 \theta} & \mathbf{B}^2 \mathfrak{g} \\
 \downarrow & & \downarrow & & \downarrow \\
 \dots & & & &
 \end{array}$$

□

Remark 5.2.21. We can see the delooping $\mathbf{B}\theta : \mathbf{B}G \rightarrow \mathbf{B}\mathfrak{g}$ of the infinitesimal remainder $\theta : G \rightarrow \mathfrak{g}$ as taking the *curvature* of a principal G -bundle, in that

$\mathbf{B}\theta$ is an obstruction to the flatness of that bundle since

$$\mathfrak{b}\mathbf{B}G \rightarrow \mathbf{B}G \rightarrow \mathbf{B}\mathfrak{g}$$

is a fiber sequence.

As with any good construction, the infinitesimal remainder is functorial in its higher group. This is defined easily since the infinitesimal remainder is constructed as a fiber.

Definition 5.2.22. Let $f :: G \rightarrow H$ be a crisp homomorphism of higher groups with delooping $\mathbf{B}f : \mathbf{B}G \rightarrow \mathbf{B}H$. Then we have a pushforward $f_* : \mathfrak{g} \rightarrow \mathfrak{h}$ given by $(t, p) \mapsto (\mathfrak{b}\mathbf{B}f(t), (\mathbf{a}p \mathbf{B}f p) \bullet \mathbf{pt}_{\mathbf{B}f})$. This is the unique map fitting into the following diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{f_*} & \mathfrak{h} \\ \downarrow & & \downarrow \\ \mathfrak{b}\mathbf{B}G & \xrightarrow{\mathfrak{b}\mathbf{B}f} & \mathfrak{b}\mathbf{B}H \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\mathbf{B}f} & \mathbf{B}H \end{array}$$

If G and H are k -commutative and f is a k -commutative homomorphism, then f_* admits a unique structure of a $(k-1)$ -commutative homomorphism by defining $\mathbf{B}^{k-1}f_*$ to be the map induced by $\mathfrak{b}\mathbf{B}^k f$ on the fiber.

We record a useful lemma: the fibers of the quotient map $\theta : G \rightarrow \mathfrak{g}$ are all identifiable with $\mathfrak{b}G$.

Lemma 5.2.23. Let G be a crisp higher group. For $t : \mathfrak{g}$, we have $\|\mathbf{fib}_\theta(t) = \mathfrak{b}G\|$.

Proof. By definition, $t : \mathfrak{g}$ is of the form (T, p) for $T : \mathfrak{b}\mathbf{B}G$ and $p : T_b = \mathbf{pt}_{\mathbf{B}G}$. Since $\mathfrak{b}\mathbf{B}G$ is 0-connected and we are trying to prove a proposition, we may

suppose that $q : T = \mathbf{pt}_{\mathfrak{b}BG}$. We then have that $t = (\mathbf{pt}_{\mathfrak{b}BG}, (-)_{\mathfrak{b}*}(q) \bullet p)$, and therefore:

$$\begin{aligned}
\mathbf{fib}_\theta(t) &\equiv (g : G) \times ((\mathbf{pt}_{\mathfrak{b}BG}, g) = (\mathbf{pt}_{\mathfrak{b}BG}, (-)_{\mathfrak{b}*}(q) \bullet p)) \\
&= (g : G) \times (a : \mathfrak{b}G) \times (a_{\mathfrak{b}} \bullet g = (-)_{\mathfrak{b}*}(q) \bullet p) \\
&= (g : G) \times (a : \mathfrak{b}G) \times (g = a_{\mathfrak{b}}^{-1} \bullet (-)_{\mathfrak{b}*}(q) \bullet p) \\
&= \mathfrak{b}G. \qquad \square
\end{aligned}$$

The infinitesimal remainder is *infinitesimal* in the sense that it has a single crisp point.

Proposition 5.2.24. Let G be a higher group. Then its infinitesimal remainder \mathfrak{g} is infinitesimal in the sense that

$$\mathfrak{b}\mathfrak{g} = *.$$

Proof. By Lemma 5.2.4, \mathfrak{b} preserves the fiber sequence

$$\mathfrak{g} \rightarrow \mathfrak{b}BG \rightarrow BG.$$

But $\mathfrak{b}(-)_{\mathfrak{b}} : \mathfrak{b}\mathfrak{b}BG \rightarrow \mathfrak{b}BG$ is an equivalence by Theorem 6.18 of [Shu18a], so $\mathfrak{b}\mathfrak{g}$ is contractible. \square

Despite being infinitesimal, we will see that \mathfrak{g} has (in general) a highly non-trivial homotopy type.

Remark 5.2.25. The infinitesimal remainder \mathfrak{g} is of special interest when G is a Lie group, since in this case the vanishing of the cohomology groups $H^*(\mathfrak{g}; \mathbb{Z}/p)$

for all primes p is equivalent to the Friedlander-Milnor conjecture. The fact that this conjecture remains unproven is a testament to the intricacy of the homotopy type of the infinitesimal space \mathfrak{g} .

We note that \mathfrak{g} itself represents an obstruction to the discreteness of G .

Proposition 5.2.26. A crisp higher group G is discrete if and only if its infinitesimal remainder \mathfrak{g} is contractible.

Proof. If G is discrete, then $(-)_\flat : \flat G \rightarrow G$ is an equivalence and so $(-)_\flat : \flat \mathbf{B}G \rightarrow \mathbf{B}G$ is an equivalence: this implies that $\mathfrak{g} = *$. On the other hand, if $\mathfrak{g} = *$ then $(-)_\flat : \flat \mathbf{B}G \rightarrow \mathbf{B}G$ is an equivalence and so its action on loops is an equivalence. \square

Using Proposition 5.2.11, we can quickly show that $\theta : G \rightarrow \mathfrak{g}$ is an ∞ -cover. This gives us the right hand pullback square in our modal fracture hexagon.

Proposition 5.2.27. Let G be a crisp ∞ -group. Then the infinitesimal remainder $\theta : G \rightarrow \mathfrak{g}$ is an ∞ -cover. In particular, the \mathfrak{f} -naturality square:

$$\begin{array}{ccc} G & \xrightarrow{\theta} & \mathfrak{g} \\ (-)_\mathfrak{f} \downarrow & & \downarrow (-)_\mathfrak{f} \\ \mathfrak{f}G & \xrightarrow{\mathfrak{f}\theta} & \mathfrak{f}\mathfrak{g} \end{array}$$

is a pullback. If, furthermore, G is (crisply) an n -type, then θ is an $(n+1)$ -cover.

Proof. By Proposition 5.2.11, to show that $\theta : G \rightarrow \mathfrak{g}$ is an ∞ -cover (resp. an $(n+1)$ -cover) it suffices to show that the fibers are merely equivalent to a crisply discrete type (resp. a crisply discrete n -type). But by Lemma 5.2.23, the fibers of $\theta : G \rightarrow \mathfrak{g}$ are all merely equivalent to $\flat G$ which is crisply discrete (and, by Theorem 6.6 of [Shu18a], if G is (crisply) an n -type then so is $\flat G$). \square

There is a sense in which the infinitesimal remainder of a higher group behaves like its Lie algebra. Just as the Lie algebra of a Lie group is the same as the Lie algebra of its universal cover, we can show that the infinitesimal remainder of a higher group is the same as that of its universal ∞ -cover.

Proposition 5.2.28. Let $G \xrightarrow{\phi} H \xrightarrow{\psi} K$ be a crisp exact sequence of higher groups. Then

1. K is discrete if and only if $\phi_* : \mathfrak{g} \rightarrow \mathfrak{h}$ is an equivalence.
2. G is discrete if and only if $\psi_* : \mathfrak{h} \rightarrow \mathfrak{k}$ is an equivalence.

Proof. We consider the following diagram in which each horizontal and vertical sequence is a fiber sequence:

$$\begin{array}{ccccc}
 \mathfrak{g} & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{k} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathfrak{b}BG & \longrightarrow & \mathfrak{b}BH & \longrightarrow & \mathfrak{b}BK \\
 \downarrow & & \downarrow & & \downarrow \\
 BG & \longrightarrow & BH & \longrightarrow & BK
 \end{array}$$

If K is discrete, then $\mathfrak{k} = *$ and so $\mathfrak{g} = \mathfrak{h}$. On the other hand, if $\mathfrak{g} = \mathfrak{h}$, then the bottom left square of the above diagram is a pullback. Therefore, it induces an equivalence on the fibers of the horizontal maps:

$$\begin{array}{ccccc}
 \mathfrak{b}K & \longrightarrow & \mathfrak{b}BG & \longrightarrow & \mathfrak{b}BH \\
 \downarrow \sim & & \downarrow & & \downarrow \\
 K & \longrightarrow & BG & \longrightarrow & BH
 \end{array}$$

This shows that K is discrete.

If $\psi_* : \mathfrak{h} \rightarrow \mathfrak{k}$ is an equivalence, then its fiber \mathfrak{g} is contractible. Therefore, G is discrete. On the other hand, if G is discrete, then the bottom right square is

a pullback, and therefore the induced map on vertical fibers is an equivalence.

This map is $\psi_* : \mathfrak{h} \rightarrow \mathfrak{k}$. □

Corollary 5.2.29. The universal ∞ -cover $\pi : \tilde{G}^\infty \rightarrow G$ induces an equivalence $\mathfrak{g}^\infty = \mathfrak{g}$ fitting into the following commutative diagram:

$$\begin{array}{ccccc}
 \mathfrak{g}^\infty & \xlongequal{\quad} & \mathfrak{g} & \longrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathfrak{b}B\tilde{G}^\infty & \longrightarrow & \mathfrak{b}BG & \longrightarrow & \mathfrak{b}jBG \\
 \downarrow & & \downarrow & & \parallel \\
 B\tilde{G}^\infty & \longrightarrow & BG & \longrightarrow & jBG
 \end{array}$$

In particular, this gives us a long fiber sequence

$$\begin{array}{c}
 \mathfrak{b}\tilde{G}^\infty \xrightarrow{(-)_\mathfrak{b}} \tilde{G}^\infty \xrightarrow{\theta} \mathfrak{g} \\
 \curvearrowright \\
 \mathfrak{b}B\tilde{G}^\infty \xrightarrow{B\pi} B\tilde{G}^\infty
 \end{array}$$

which forms the top fiber sequence of the modal fracture hexagon.

5.2.4 The Modal Fracture Hexagon

We have seen the two main fiber sequences

$$\begin{array}{c}
 \tilde{G}^\infty \xrightarrow{\pi} G \xrightarrow{(-)^f} jG \\
 \curvearrowright \\
 B\tilde{G}^\infty \xrightarrow{B\pi} BG \xrightarrow{(-)^f} jBG
 \end{array}
 \quad \text{and} \quad
 \begin{array}{c}
 \mathfrak{b}G \xrightarrow{(-)_\mathfrak{b}} G \xrightarrow{\theta} \mathfrak{g} \\
 \curvearrowright \\
 \mathfrak{b}BG \longrightarrow BG
 \end{array}$$

associated to a higher group G . Now, when we apply \flat to the left sequence and \int to the right sequence, we find the sequences

$$\begin{array}{ccc}
 \flat G^\infty \xrightarrow{\flat \pi} \flat G \xrightarrow{(-)^{\int \circ (-)}_\flat} \int G & & \flat G \xrightarrow{(-)^{\int \circ (-)}_\flat} \int G \xrightarrow{\int \theta} \int \mathfrak{g} \\
 \curvearrowright & \text{and} & \curvearrowleft \\
 \flat \mathbf{B}G^\infty \xrightarrow{\flat \mathbf{B}\pi} \flat \mathbf{B}G \xrightarrow{(-)^{\int \circ (-)}_\flat} \int \mathbf{B}G & & \flat \mathbf{B}G \longrightarrow \int \mathbf{B}G
 \end{array}$$

which are *the same sequence, just shifted over*. This gives us the bottom exact sequence of our modal fracture hexagon, reading the sequence on the left. But it also proves that $\int \mathfrak{g} = \flat \mathbf{B}G^\infty$, which gives us the top exact sequence of our modal fracture hexagon by Corollary 5.2.29.

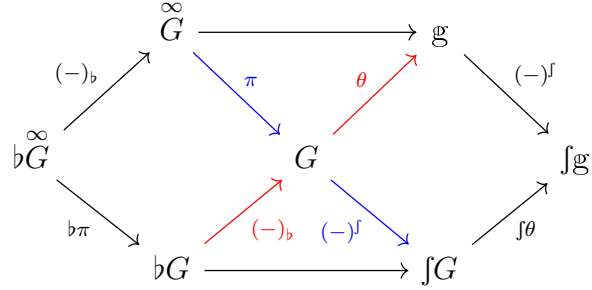
Of course, we need to be able to apply \int freely to fiber sequences to fulfil this argument. But \int is not left exact, and so does not preserve fiber sequences in general. Luckily, Theorem 6.1 of Chapter 3 gives us a trick for showing that a map is a \int -fibration which allows us to prove this general lemma.

Lemma 5.2.30. Let G be a crisp higher group (that is, $\mathbf{B}G$ is a crisply pointed 0-connected type). Then any crisp map $f :: X \rightarrow \mathbf{B}G$ is a \int -fibration.

Proof. Since $\mathbf{B}G$ is 0-connected, all the fibers of f are merely equivalent to the fiber $\mathbf{fib}_f(\mathbf{pt})$ over the basepoint. Therefore, their homotopy types are merely equivalent to $\int \mathbf{fib}_f(\mathbf{pt})$, which is a crisp, discrete type. It follows by Theorem 6.1 of Chapter 3 that f is a \int -fibration. \square

This means that we can freely apply \int to crisp fiber sequences of 0-connected types. This concludes our proof of the main theorem.

Theorem 5.2.31. *For a crisp ∞ -group G , there is a modal fracture hexagon:*



where

- $\theta : G \rightarrow \mathfrak{g}$ is the infinitesimal remainder of G , the quotient $G // \mathfrak{b}G$, and
- $\pi : \overset{\infty}{G} \rightarrow G$ is the universal (contractible) ∞ -cover of G .

Moreover,

1. The middle diagonal sequences are fiber sequences.
2. The top and bottom sequences are fiber sequences.
3. Both squares are pullbacks.

Furthermore, the homotopy type of \mathfrak{g} is a delooping of $\overset{\infty}{\mathfrak{b}G}$:

$$\mathfrak{j}\mathfrak{g} = \mathfrak{b}\overset{\infty}{\mathfrak{B}G}.$$

Therefore, if G is k -commutative for $k \geq 1$ (that is, admits further deloopings $\mathfrak{B}^{k+1}G$), then we may continue the modal fracture hexagon on to $\mathfrak{B}^k G$.

Proof. We assemble the various components of the proof here.

1. The middle diagonal sequences are fiber sequences by definition (see Definition 5.2.12 and Definition 5.2.22).

2. The top sequence was shown to be a fiber sequence in Corollary 5.2.29. We showed that the bottom sequence is a fiber sequence at the beginning of this subsection.
3. The left square was shown to be a pullback in Proposition 5.2.14, and the right sequence in Proposition 5.2.27.

Finally, we calculated the homotopy type of \mathfrak{g} at the beginning of this subsection. □

5.3 Ordinary Differential Cohomology

In this section, we will use modal fracture to construct ordinary differential cohomology in cohesive homotopy type theory. We will recover a differential hexagon for ordinary differential cohomology which very closely resembles the classical hexagon; however, as de Rham’s theorem does not hold for all types, we will not recover the classical hexagon exactly. For more discussion of these subtleties, see Section 5.3.3.

In [BNV14], Bunke, Nikolaus, and Vokel show that differential cohomology theories can be understood as spectra in the ∞ -topos of sheaves on a site of manifolds. Schreiber notes in Proposition 4.4.9 of [Sch13a] that the simpler site consisting of Euclidean spaces and smooth maps between them yields the same topos of sheaves, and proves in Proposition 4.4.8 that this ∞ -topos is cohesive. This topos, and the similar ∞ -Dubuc topos (called the ∞ -Cahiers topos in Remark 4.5.6 and **SynthDiff ∞ Grpd** in Definition 4.5.7 of *ibid.*), will be our intended model for cohesive homotopy type theory in this section.

The theme of this chapter is that the main feature of differential cohomology

— the differential cohomology hexagon — is not of a particularly differential character, but arises from the more basic opposition between an adjoint modality \mathbb{f} and comodality \mathbb{b} . As we saw in the previous section, in the presence of these (co)modalities, any higher group may be fractured in a manner resembling the differential cohomology hexagon.

We will take a similarly general view in constructing ordinary differential cohomology. The key idea in ordinary differential cohomology is the equipping of differential form data to integral cohomology. We will therefore focus on cohomology theories (in particular, ∞ -commutative higher groups or connective spectra) which arise by equipping an existing cohomology theory with extra data representing the cocycles. Our exposition will focus on ordinary differential cohomology, but this extra generality will enable us to define combinatorial analogues of ordinary differential cohomology as well (see Section 5.3.5).

5.3.1 Assumptions and Preliminaries

For this section, we make the following assumption.

Assumption 1. In cohesive homotopy type theory with the axioms of synthetic differential geometry, *tiny infinitesimal varieties*, and a *principle of constancy*, we have a contractible and infinitesimal resolution of $U(1)$

$$0 \rightarrow \mathbb{b}U(1) \rightarrow U(1) \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \dots \quad (5.3)$$

given by the *differential k -form classifiers* Λ^k . That is:

- The Λ^k are crisp \mathbb{R} -vector spaces.
- The maps $d :: \Lambda^k \rightarrow \Lambda^{k+1}$ are crisply \mathbb{b} - \mathbb{R} -linear (not \mathbb{R} -linear!), and the

sequence Eq. (5.3) is crisply long exact.

- The Λ^k are infinitesimal: $\flat\Lambda^k = *$. Therefore also the closed k -form classifiers $\Lambda_{\text{cl}}^k := \ker(d : \Lambda^k \rightarrow \Lambda^{k+1})$ are infinitesimal.

Here, $U(1) \equiv \{z : \mathbb{C} \mid z\bar{z} = 1\}$ is the abelian group of units in the *smooth* complex numbers, which are defined as $\mathbb{C} \equiv \mathbb{R}[i]/(i^2 + 1)$ where \mathbb{R} are the *smooth real numbers* presumed by synthetic differential geometry.

Remark 5.3.1. For reasons of space, we will not justify this assumption in this chapter. In forthcoming work, we will show how one can construct the form classifiers and their long exact sequence

$$0 \rightarrow \flat\mathbb{R} \rightarrow \mathbb{R} \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \rightarrow \dots \quad (5.4)$$

from the axioms of synthetic differential geometry with tiny infinitesimals and a principle of constancy. Synthetic differential geometry is an axiomatic system for working with nilpotent infinitesimals put forward first by Lawvere [Law79] and developed by Bunge, Dubuc, Kock, Wraith, and others. It admits a model in sheaves on infinitesimally extended Euclidean spaces, known as the Dubuc topos (or Cahiers topos) [Dub79]; for a review of models see [MR91]. The Dubuc topos is cohesive, and is our intended model for this section.

It was noted by Lawvere [Law80] that the exceptional projectivity enjoyed by the infinitesimal interval $\mathbb{D} = \{\epsilon : \mathbb{R} \mid \epsilon^2 = 0\}$ was equivalent to the existence of an (external) right adjoint to the exponential functor $X \mapsto X^{\mathbb{D}}$. We will follow Yetter [**Yetter:Tiny**] in calling objects T for which the functor $X \mapsto X^T$ admits a right adjoint *tiny objects*. Lawvere and Kock showed how one could

use this “amazing” right adjoint to construct the form classifiers Λ^k (see Section I.20 of [Koc06] for a construction of Λ^1).

However, working with the form classifiers was difficult in synthetic differential geometry since the adjoint which defines them only exists externally. This may be remedied by using Shulman’s Cohesive HoTT, where the \sharp modality allows for an internalization of the external. This allows us to give a fully internal theory of the form classifiers. We will, however, post-pone a discussion of this internal theory of tiny objects to future work.

The principle of constancy says that if the differential of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ vanishes uniformly, then f is constant. This extra principle has been long considered in synthetic differential geometry (see, for example, the second chapter of [MBL18]), but when combined with real cohesion it implies the exactness of the sequence

$$0 \rightarrow \flat \mathbb{R} \rightarrow \mathbb{R} \xrightarrow{d} \Lambda^1$$

and so begins the theory of differential cohomology we will see shortly. The interaction with cohesion is non-trivial in many ways for synthetic differential geometry. For example, the principle of constancy in the presence of real cohesion implies the existence of primitives, and the exponential functions $\exp(-) : \mathbb{R} \rightarrow \mathbb{R}^+$ and $\exp(2\pi i -) : \mathbb{R} \rightarrow U(1)$ (where $U(1) := \{z : \mathbb{C} \mid z\bar{z} = 1\}$).

Remark 5.3.2. Externally, the smooth reals \mathbb{R} correspond to the sheaf of smooth real valued functions, $U(1)$ corresponds to the sheaf of smooth $U(1)$ -valued functions, and Λ^k is the sheaf sending a manifold to its set of differential k -forms.

We have assumed the existence of a crisp long exact sequence of abelian

groups in which each of the Λ^i are real vector spaces (but $d : \Lambda^i \rightarrow \Lambda^{i+1}$ are not \mathbb{R} -linear). These are to be the *differential form classifiers*, which externally are the sheaves of \mathbb{R} -valued n -forms on manifolds (or infinitesimally extended manifolds).

If we define

$$\Lambda_{\text{cl}}^n := \ker(d : \Lambda^n \rightarrow \Lambda^{n+1})$$

to be the closed n -form classifier, then we can reorganize the long exact sequence (5.4) into a series of short exact sequences of abelian groups

$$0 \rightarrow \Lambda_{\text{cl}}^n \rightarrow \Lambda^n \xrightarrow{d} \Lambda_{\text{cl}}^{n+1} \rightarrow 0.$$

The first of these short exact sequences is

$$0 \rightarrow {}_b\mathbb{R} \rightarrow \mathbb{R} \xrightarrow{d} \Lambda_{\text{cl}}^1 \rightarrow 0$$

which we may extend into a long fiber sequence

$$\begin{array}{ccccc} {}_b\mathbb{R} & \xrightarrow{(-)_b} & \mathbb{R} & \xrightarrow{d} & \Lambda_{\text{cl}}^1 \\ & & & & \downarrow \\ & & {}_b\mathbf{B}\mathbb{R} & \longrightarrow & \mathbf{B}\mathbb{R} \longrightarrow \dots \end{array}$$

This shows that Λ_{cl}^1 is the infinitesimal remainder of the additive Lie group \mathbb{R} . Since \mathbb{R} has contractible shape by definition, we see that $\mathbb{R} \xrightarrow{d} \Lambda_{\text{cl}}^1$ is the universal ∞ -cover of Λ_{cl}^1 . This gives us the following theorem, a form of de Rham's theorem in smooth cohesion.

Lemma 5.3.3. The n -form classifiers Λ^n are contractible.

Proof. This follows immediately from the assumption that they are real vector spaces. By Lemma 6.9 of Chapter 3, to show that Λ^n is contractible it suffices

to give for every $\omega : \Lambda^n$ a path $\gamma : \mathbb{R} \rightarrow \Lambda^n$ from ω to 0. We can of course define

$$\gamma(t) := t\omega$$

which gives our desired contraction. \square

Theorem 5.3.4. *Let $\Lambda_{cl}^n := \ker(d : \Lambda^n \rightarrow \Lambda^{n+1})$ be the closed n -form classifier.*

Then

$$\int \Lambda_{cl}^n = \flat \mathbf{B}^n \mathbb{R}.$$

Proof. Since Λ_{cl}^1 is the infinitesimal remainder of \mathbb{R} , this follows from Theorem 5.2.31:

$$\int \Lambda_{cl}^1 = \flat \mathbf{B} \mathbb{R}.$$

We then proceed by induction. We have a short exact sequence of abelian groups

$$0 \rightarrow \Lambda_{cl}^n \rightarrow \Lambda^n \xrightarrow{d} \Lambda_{cl}^{n+1} \rightarrow 0.$$

We note that since $d : \Lambda^n \rightarrow \Lambda_{cl}^{n+1}$ is an abelian group homomorphism, all of its fibers are identifiable with the crisp type Λ_{cl}^n and therefore, by the “good fibrations” trick (Theorem 6.1 of Chapter 3), it is a \mathfrak{f} -fibration. Therefore, we get a fiber sequence

$$\int \Lambda_{cl}^n \rightarrow \int \Lambda^n \rightarrow \int \Lambda_{cl}^{n+1}.$$

Now, since Λ^n is contractible by Lemma 5.3.3, we see that

$$\Omega \int \Lambda_{cl}^{n+1} = \int \Lambda_{cl}^n$$

By inductive hypothesis, $\int \Lambda_{cl}^n = \flat \mathbf{B}^n \mathbb{R}$, so all that remains is to show that $\int \Lambda_{cl}^{n+1}$ is n -connected. We will do this by showing that for any $u : \int \Lambda_{cl}^{n+1}$, the loop

space $\Omega(\mathfrak{f}\Lambda_{\text{cl}}^{n+1}, u)$ is $(n-1)$ -connected. By Corollary 9.12 of [Shu18a], the \mathfrak{f} -unit $(-)^{\mathfrak{f}} : \Lambda_{\text{cl}}^{n+1} \rightarrow \mathfrak{f}\Lambda_{\text{cl}}^{n+1}$ is surjective, so there exists an $\omega : \Lambda_{\text{cl}}^{n+1}$ with $u = \omega^{\mathfrak{f}}$. We then have a fiber sequence

$$\mathbf{fib}_d(\omega) \rightarrow \Lambda^n \xrightarrow{d} \Lambda_{\text{cl}}^{n+1}$$

which, since $\Lambda^n \xrightarrow{d} \Lambda_{\text{cl}}^{n+1}$ is a \mathfrak{f} -fibration descends to a fiber sequence

$$\mathfrak{f}\mathbf{fib}_d(\omega) \rightarrow \mathfrak{f}\Lambda^n \xrightarrow{\mathfrak{f}} \mathfrak{f}\Lambda_{\text{cl}}^{n+1}. \quad (5.5)$$

Since d is surjective, there is a $\alpha : \Lambda^n$ with $d\alpha = \omega$, and we may therefore contract $\mathfrak{f}\Lambda^n$ onto $\alpha^{\mathfrak{f}}$. This lets us equate the sequence (5.5) with the sequence

$$\Omega(\mathfrak{f}\Lambda_{\text{cl}}^{n+1}, \omega^{\mathfrak{f}}) \rightarrow * \xrightarrow{\omega^{\mathfrak{f}}} \mathfrak{f}\Lambda_{\text{cl}}^{n+1}.$$

But as $d : \Lambda^n \rightarrow \Lambda_{\text{cl}}^{n+1}$ is an abelian group homomorphism, its fibers are all identifiable with its kernel Λ_{cl}^n ; this means that $\Omega(\mathfrak{f}\Lambda_{\text{cl}}^{n+1}, u)$ is identifiable with $\mathfrak{f}\Lambda_{\text{cl}}^n$, which by inductive hypothesis is $(n-1)$ -connected. \square

We may understand this theorem as a form of de Rham theorem in smooth cohesive homotopy type theory. We may think of the unit $(-)^{\mathfrak{f}} : \Lambda_{\text{cl}}^n \rightarrow \mathfrak{b}B^n \mathbb{R}$ as giving the de Rham class of a closed n -form. That this map is the \mathfrak{f} -unit says that this is the universal discrete cohomological invariant of closed n -forms. Explicitly, if E_{\bullet} is a loop spectrum, then $H^k(\Lambda_{\text{cl}}^n; E_{\bullet}) := \|\Lambda_{\text{cl}}^n \rightarrow E_k\|_0$. Therefore, if the E_k are discrete, then any cohomology class $c : \Lambda_{\text{cl}}^n \rightarrow E_k$ factors through the de Rham class $(-)^{\mathfrak{f}} : \Lambda_{\text{cl}}^n \rightarrow \mathfrak{b}B^n \mathbb{R}$. In this sense, every discrete cohomological invariant of closed n -forms is in fact an invariant of their de Rham class in discrete real cohomology.

5.3.2 Circle k -Gerbes with Connection

We can now go about defining ordinary differential cohomology. We understand ordinary differential cohomology as equipping integral cohomology with differential form data. Hopkins and Singer define (Definition 2.4 of [HS05]) a *differential cocycle* of degree $k + 1$ on X to be a triple (c, h, ω) consisting of an underlying cocycle $c \in Z^{k+1}(X, \mathbb{Z})$ in integral cohomology, a *curvature* form $\omega \in \Lambda_{\text{cl}}^{k+1}(X)$, and a *monodromy* term $h \in C^k(X, \mathbb{R})$ satisfying the equation $dh = \omega - c$.

We will follow their lead, at least in spirit. In true homotopy type theoretic fashion, we will define the classifying types first and then derive the cohomology theory by truncation.

Definition 5.3.5. We define the classifier $\mathbb{B}_{\nabla}^k U(1)$ of degree $(k + 1)$ classes in ordinary differential cohomology to be the pullback:

$$\begin{array}{ccc} \mathbb{B}_{\nabla}^k U(1) & \xrightarrow{F_-} & \Lambda_{\text{cl}}^{k+1} \\ \downarrow & \lrcorner & \downarrow (-)^{\flat} \\ \mathbb{B}^{k+1} \mathbb{Z} & \longrightarrow & \flat \mathbb{B}^{k+1} \mathbb{R} \end{array}$$

Therefore, a cocycle $\tilde{c} : X \rightarrow \mathbb{B}_{\nabla}^k U(1)$ in differential cohomology will consist of an underlying cocycle $c : X \rightarrow \mathbb{B}^{k+1} \mathbb{Z}$, a curvature form $\omega : X \rightarrow \Lambda_{\text{cl}}^{k+1}$, together with an identification $h : c = \omega$ in $X \rightarrow \flat \mathbb{B}^{k+1} \mathbb{R}$. Since h lands in types identifiable with $\Omega \flat \mathbb{B}^{k+1} \mathbb{R}$, which equals $\flat \mathbb{B}^k \mathbb{R}$, we may consider it as the monodromy term in discrete real cohomology. We will now set about justifying this terminology.

We may note immediately from this definition that the map $\mathbb{B}_{\nabla}^k U(1) \rightarrow$

$\mathbf{B}^{k+1} \mathbb{Z}$ (which we may think of as taking the underlying class in ordinary cohomology) is the \mathfrak{f} -unit. This means that the underlying cocycle is the universal discrete cohomological invariant of a differential cocycle.

Lemma 5.3.6. The pullback square

$$\begin{array}{ccc} \mathbf{B}_{\nabla}^k U(1) & \longrightarrow & \Lambda_{\text{cl}}^{k+1} \\ \downarrow & \lrcorner & \downarrow (-)^{\mathfrak{f}} \\ \mathbf{B}^{k+1} \mathbb{Z} & \longrightarrow & \mathfrak{b}\mathbf{B}^{k+1} \mathbb{R} \end{array}$$

is a \mathfrak{f} -naturality square. That is, $\mathbf{B}_{\nabla}^k U(1) \rightarrow \mathbf{B}^{k+1} \mathbb{Z}$ is a \mathfrak{f} -unit.

Proof. Since $\mathbf{B}_{\nabla}^k U(1) \rightarrow \mathbf{B}^{k+1} \mathbb{Z}$ is a map into a \mathfrak{f} -modal type, to show that it is a \mathfrak{f} unit it suffices to show that it is \mathfrak{f} -connected. Since we have a pullback square, the fibers of $\mathbf{B}_{\nabla}^k U(1) \rightarrow \mathbf{B}^{k+1} \mathbb{Z}$ are the same as those of $(-)^{\mathfrak{f}} : \Lambda_{\text{cl}}^{k+1} \rightarrow \mathfrak{b}\mathbf{B}^{k+1} \mathbb{R}$. But as this map is a \mathfrak{f} -unit, its fibers are \mathfrak{f} -connected. \square

The reason for our change of index — defining $\mathbf{B}_{\nabla}^k U(1)$ to represent degree $(k + 1)$ classes — is because we would like to think of $\mathbf{B}_{\nabla}^k U(1)$ as more directly classifying connections on k -gerbes with band $U(1)$. To reify this idea, let's give the map $\mathbf{B}_{\nabla}^k U(1) \rightarrow \mathbf{B}^k U(1)$ which we think of as taking the underlying k -gerbe.

Construction 5.3.7. We construct a map $\mathbf{B}_{\nabla}^k U(1) \rightarrow \mathbf{B}^k U(1)$ which makes the following triangle commute:

$$\begin{array}{ccc} \mathbf{B}_{\nabla}^k U(1) & \longrightarrow & \mathbf{B}^k U(1) \\ & \searrow (-)^{\mathfrak{f}} & \swarrow (-)^{\mathfrak{f}} \\ & & \mathbf{B}^{k+1} \mathbb{Z} \end{array}$$

Construction. Since $\mathbb{R} \rightarrow U(1)$ is the universal ∞ -cover of $U(1)$, by Corollary 5.2.29, $U(1)$ has the same infinitesimal remainder as \mathbb{R} , which is Λ_{cl}^1 . Therefore, by modal fracture Theorem 5.2.31, we have a pullback square

$$\begin{array}{ccc} \mathbf{B}^k U(1) & \longrightarrow & \mathbf{B}^k \Lambda_{\text{cl}}^1 \\ (-)^f \downarrow & \lrcorner & \downarrow \\ \mathbf{B}^{k+1} \mathbb{Z} & \longrightarrow & \mathbf{b}\mathbf{B}^{k+1} \mathbb{R} \end{array}$$

Now, since we have a series of short exact sequences

$$0 \rightarrow \Lambda_{\text{cl}}^n \rightarrow \Lambda^n \xrightarrow{d} \Lambda_{\text{cl}}^{n+1} \rightarrow 0$$

we have long fiber sequences

$$\begin{array}{ccccc} \Lambda_{\text{cl}}^n & \longrightarrow & \Lambda^n & \xrightarrow{d} & \Lambda_{\text{cl}}^{n+1} \\ \downarrow & & & & \downarrow \\ \mathbf{B}\Lambda_{\text{cl}}^n & \longrightarrow & \mathbf{B}\Lambda^n & \longrightarrow & \dots \end{array}$$

for each n . In particular, we have maps $\mathbf{B}^n \Lambda_{\text{cl}}^{m+1} \rightarrow \mathbf{B}^{n+1} \Lambda_{\text{cl}}^m$ for all n and m .

Taking repeated pullbacks along these maps gives us a diagram

$$\begin{array}{ccc}
 \mathbb{B}_{\nabla}^k U(1) & \longrightarrow & \Lambda_{\text{cl}}^{k+1} \\
 \downarrow \lrcorner & & \downarrow \\
 \bullet & \longrightarrow & \mathbb{B} \Lambda_{\text{cl}}^k \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 \bullet & \longrightarrow & \mathbb{B}^{k-1} \Lambda_{\text{cl}}^2 \\
 \downarrow \lrcorner & & \downarrow \\
 \mathbb{B}^k U(1) & \longrightarrow & \mathbb{B}^k \Lambda_{\text{cl}}^1 \\
 \downarrow \lrcorner & & \downarrow \\
 \mathbb{B}^{k+1} \mathbb{Z} & \longrightarrow & \flat \mathbb{B}^{k+1} \mathbb{R}
 \end{array} \tag{5.6}$$

The dashed composite in this diagram is what we were seeking to construct. \square

Remark 5.3.8. Diagram 5.6 shows us that the following square is a pullback:

$$\begin{array}{ccc}
 \mathbb{B}_{\nabla}^k U(1) & \longrightarrow & \Lambda_{\text{cl}}^{k+1} \\
 \downarrow \lrcorner & & \downarrow (-)^{\flat} \\
 \mathbb{B}^k U(1) & \longrightarrow & \mathbb{B}^k \Lambda_{\text{cl}}^1
 \end{array}$$

If we note that $\mathbb{B}^k \Lambda_{\text{cl}}^1$ is $\mathbb{B}^k \mathfrak{u}(1)$, we get an alternate definition of $\mathbb{B}_{\nabla}^k U(1)$ by this pullback. This shows that our definition agrees with Schreiber's Definition 4.4.93 in [Sch13a].

We can now see that the map $\mathbb{B}_{\nabla}^k U(1) \rightarrow \Lambda_{\text{cl}}^{k+1}$ takes the the *curvature* $(k+1)$ -*form*. We can justify this by showing that the fiber of this map is $\flat \mathbb{B}^k U(1)$; in other words, a circle k -gerbe with connection is flat if and only if its curvature vanishes.

Lemma 5.3.9. The map $F_{(-)} : \mathbf{B}_{\nabla}^k U(1) \rightarrow \Lambda_{\text{cl}}^{k+1}$ has fiber $\flat \mathbf{B}^k U(1)$. Since this map gives an obstruction to flatness, we refer to it as the *curvature* $(k+1)$ -form.

Proof. By considering the top part from the diagram (5.6), we find a pullback square

$$\begin{array}{ccc} \mathbf{B}_{\nabla}^k U(1) & \longrightarrow & \Lambda_{\text{cl}}^{k+1} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{B}^k U(1) & \longrightarrow & \mathbf{B}^k \Lambda_{\text{cl}}^1 \end{array}$$

For this reason, we get an equivalence on fibers:

$$\begin{array}{ccccc} \bullet & \longrightarrow & \mathbf{B}_{\nabla}^k U(1) & \longrightarrow & \Lambda_{\text{cl}}^{k+1} \\ \wr \downarrow & & \downarrow & \lrcorner & \downarrow \\ \flat \mathbf{B}^k U(1) & \longrightarrow & \mathbf{B}^k U(1) & \longrightarrow & \mathbf{B}^k \Lambda_{\text{cl}}^1 \end{array} \quad \square$$

As a corollary, we may characterize the curvature $F_{(-)} : \mathbf{B}_{\nabla}^k U(1) \rightarrow \Lambda_{\text{cl}}^{k+1}$ modally.

Corollary 5.3.10. The curvature $F_{(-)} : \mathbf{B}_{\nabla}^k U(1) \rightarrow \Lambda_{\text{cl}}^{k+1}$ is a unit for the $(k-1)$ -truncation modality. In particular,

$$\|\mathbf{B}_{\nabla}^k U(1)\|_j = \Lambda_{\text{cl}}^{k+1}$$

for any $0 \leq j < k$.

Proof. As $\Lambda_{\text{cl}}^{k+1}$ is 0-truncated and so $(k-1)$ truncated, it will suffice to show that $F_{(-)}$ is $(k-1)$ -connected. But by Lemma 5.3.9, the fiber of $F_{(-)}$ over any point $\omega : \Lambda_{\text{cl}}^{k+1}$ is identifiable with $\flat \mathbf{B}^k U(1)$, which is $(k-1)$ -connected. \square

Though our notation may have suggested that the $\mathbf{B}_{\nabla}^k U(1)$ form a loop spectrum, they do not. Indeed, $\Omega \mathbf{B}_{\nabla}^k U(1) = \mathfrak{b} \mathbf{B}^{k-1} U(1)$, as can be seen by taking loops of the pullback square defining $\mathbf{B}_{\nabla}^k U(1)$ and noting that $\Lambda_{\text{cl}}^{n+1}$ is a set (0-type). In total,

$$\pi_* \mathbf{B}_{\nabla}^k U(1) = \begin{cases} \Lambda_{\text{cl}}^{k+1} & \text{if } * = 0 \\ \mathfrak{b} U(1) & \text{if } * = k \\ 0 & \text{otherwise.} \end{cases}$$

Nevertheless, each $\mathbf{B}_{\nabla}^k U(1)$ is an infinite loop space in its own right.

Definition 5.3.11. For $n, k \geq 0$, define $\mathbf{B}^n \mathbf{B}_{\nabla}^k U(1)$ to be the following pullback:

$$\begin{array}{ccc} \mathbf{B}^n \mathbf{B}_{\nabla}^k U(1) & \longrightarrow & \mathbf{B}^n \Lambda_{\text{cl}}^{k+1} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{B}^{n+k+1} \mathbb{Z} & \longrightarrow & \mathfrak{b} \mathbf{B}^{n+k+1} \mathbb{R} \end{array}$$

It is immediate from this definition and the commutation of taking loops with taking pullbacks that $\Omega \mathbf{B}^{n+1} \mathbf{B}_{\nabla}^k U(1) = \mathbf{B}^n \mathbf{B}_{\nabla}^k U(1)$. We have already seen these deloopings before in Diagram (5.6):

$$\begin{array}{ccc} \mathbf{B}_{\nabla}^k U(1) & \longrightarrow & \Lambda_{\text{cl}}^{k+1} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{B} \mathbf{B}_{\nabla}^{k-1} U(1) & \longrightarrow & \mathbf{B} \Lambda_{\text{cl}}^k \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathbf{B}^{k-1} \mathbf{B}_{\nabla} U(1) & \longrightarrow & \mathbf{B}^{k-1} \Lambda_{\text{cl}}^2 \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{B}^k U(1) & \longrightarrow & \mathbf{B}^k \Lambda_{\text{cl}}^1 \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{B}^{k+1} \mathbb{Z} & \longrightarrow & \mathfrak{b} \mathbf{B}^{k+1} \mathbb{R} \end{array}$$

These maps along the left hand side give us maps of loop spectra

$$\mathbf{B}^\bullet \mathbf{B}_{\nabla}^k U(1) \rightarrow \Sigma \mathbf{B}^\bullet \mathbf{B}_{\nabla}^{k-1} U(1)$$

We will see in Section 5.3.3 that for $\bullet = 0$, these maps give obstructions to de Rham's theorem for general types.

Each $\mathbf{B}_{\nabla}^k U(1)$ is a higher group itself. We may therefore ask: what is it's infinitesimal remainder?

Lemma 5.3.12. The infinitesimal remainder of $\mathbf{B}_{\nabla}^k U(1)$ is the curvature $F_{(-)}$: $\mathbf{B}_{\nabla}^k U(1) \rightarrow \Lambda_{\text{cl}}^{k+1}$.

Proof. Consider the following diagram:

$$\begin{array}{ccccccc}
 \mathfrak{b}_{\text{dR}} \mathbf{B} \mathbf{B}_{\nabla}^k U(1) & \xrightarrow{\sim} & \Lambda_{\text{cl}}^{k+1} & & & & \\
 \downarrow & \lrcorner & \downarrow & \searrow & & & \\
 * & \longrightarrow & * & \longrightarrow & \mathfrak{b} \mathbf{B} \mathbf{B}_{\nabla}^k U(1) & \xrightarrow{\sim} & \mathfrak{b} \mathbf{B} \Lambda_{\text{cl}}^{k+1} \\
 & \searrow & & \downarrow & \lrcorner & \downarrow & \\
 & & & \mathbf{B}^{k+2} \mathbb{Z} & \xrightarrow{\sim} & \mathfrak{b} \mathbf{B}^{k+2} \mathbb{R} & \longrightarrow & \mathbf{B} \mathbf{B}_{\nabla}^k U(1) & \xrightarrow{\sim} & \mathbf{B} \Lambda_{\text{cl}}^{k+1} \\
 & & & \parallel & & \parallel & \downarrow & \lrcorner & \downarrow & \\
 & & & \mathbf{B}^{k+2} \mathbb{Z} & \longrightarrow & \mathfrak{b} \mathbf{B}^{k+2} \mathbb{R} & & & &
 \end{array}$$

The diagonal sequences in this diagram are fiber sequences of the \mathfrak{b} -counits which define the infinitesimal remainders. Now, since $\mathfrak{b} \mathbf{B} \Lambda_{\text{cl}}^{k+1} = *$ since the form classifiers are infinitesimal, we find that the fiber of the \mathfrak{b} -counit $(-)_\mathfrak{b} : \mathfrak{b} \mathbf{B} \Lambda_{\text{cl}}^{k+1} \rightarrow \mathbf{B} \Lambda_{\text{cl}}^{k+1}$ is $\Omega \mathbf{B} \Lambda_{\text{cl}}^{k+1}$, which is $\Lambda_{\text{cl}}^{k+1}$.

Now, the frontmost square is a crisp pullback and \mathfrak{b} is left exact, so the middle square is also a pullback. Then, since the diagonal sequences are fiber sequences, the back square is also a pullback. But this shows that the infinitesimal remainder of $\mathbf{B}_{\nabla}^k U(1)$ is $\Lambda_{\text{cl}}^{k+1}$.

If we take one more fiber, we can continue the diagram to give us the following diagram:

$$\begin{array}{ccccccc}
\mathbb{B}_{\nabla}^k U(1) & \xrightarrow{F_{(-)}} & \Lambda_{\text{cl}}^{k+1} & & & & \\
\downarrow & \lrcorner & \downarrow & \xrightarrow{\theta} & & & \\
\mathbb{B}^{k+1} \mathbb{Z} & \longrightarrow & \mathfrak{b}\mathbb{B}^{k+1} \mathbb{R} & \xrightarrow{\mathfrak{b}_{\text{dR}} \mathbb{B}\mathbb{B}_{\nabla}^k U(1)} & \Lambda_{\text{cl}}^{k+1} & & \\
& & \downarrow & \lrcorner & \downarrow & \longrightarrow & \mathfrak{b}\mathbb{B}\mathbb{B}_{\nabla}^k U(1) \longrightarrow \mathfrak{b}\mathbb{B}\Lambda_{\text{cl}}^{k+1} \\
& & * & \longrightarrow & * & & \downarrow \\
& & & & & \longrightarrow & \mathbb{B}^{k+2} \mathbb{Z} \longrightarrow \mathfrak{b}\mathbb{B}^{k+2} \mathbb{R}
\end{array}$$

This shows that the infinitesimal remainder θ is equal, modulo our constructed equivalence, to the curvature $F_{(-)}$. \square

Now that we know the infinitesimal remainder of $\mathbb{B}_{\nabla}^k U(1)$, we are almost ready to understand its modal fracture hexagon. But first, we must understand its universal ∞ -cover. We will show that the universal ∞ -cover of $\mathbb{B}_{\nabla}^k U(1)$ is an analogous type $\mathbb{B}_{\nabla}^k \mathbb{R}$.

Definition 5.3.13. For $n, k \geq 0$, define $\mathbb{B}^n \mathbb{B}_{\nabla}^k \mathbb{R}$ to be the universal ∞ -cover of $\mathbb{B}^n \Lambda_{\text{cl}}^{k+1}$:

$$\begin{array}{ccc}
\mathbb{B}^n \mathbb{B}_{\nabla}^k \mathbb{R} & \xrightarrow{F_{(-)}} & \mathbb{B}^n \Lambda_{\text{cl}}^{k+1} \\
\downarrow & \lrcorner & \downarrow \\
* & \longrightarrow & \mathfrak{b}\mathbb{B}^{n+k+1} \mathbb{R}
\end{array}$$

We refer to the cohomology theories $\mathbb{B}_{\nabla}^k \mathbb{R}$ as *pure differential cohomology*.

Just as we may think of $\mathbb{B}_{\nabla}^k U(1)$ as classifying circle k -gerbes with connection, we may think of $\mathbb{B}_{\nabla}^k \mathbb{R}$ as classifying affine k -gerbes with connection. We can now show that $\mathbb{B}_{\nabla}^k \mathbb{R}$ is the universal ∞ -cover of $\mathbb{B}_{\nabla}^k U(1)$.

Proposition 5.3.14. The map $(\omega, p) \mapsto (\text{pt}_{\mathbb{B}^{k+1}\mathbb{Z}}, \omega, \lambda_{\cdot} p) : \mathbb{B}_{\nabla}^k \mathbb{R} \rightarrow \mathbb{B}_{\nabla}^k U(1)$ is the universal ∞ -cover of $\mathbb{B}_{\nabla}^k U(1)$.

Proof. Consider the following cube:

$$\begin{array}{ccccc}
 \mathbb{B}_{\nabla}^k \mathbb{R} & \longrightarrow & \Lambda_{\text{cl}}^{k+1} & & \\
 \downarrow & \lrcorner & \downarrow & \searrow & \\
 * & \longrightarrow & \flat \mathbb{B}^{k+1} \mathbb{R} & \longrightarrow & \mathbb{B}_{\nabla}^k U(1) \longrightarrow \Lambda_{\text{cl}}^{k+1} \\
 & \searrow & \downarrow & \lrcorner & \downarrow \\
 & & \mathbb{B}^{k+1} \mathbb{Z} & \longrightarrow & \flat \mathbb{B}^{k+1} \mathbb{R}
 \end{array}$$

In this cube, the front and back spaces are pullbacks by definition, and the right face is a pullback because its top and bottom sides are identities. Therefore, the left face is a pullback. Since $\mathbb{B}_{\nabla}^k U(1) \rightarrow \mathbb{B}^{k+1} \mathbb{Z}$ is a \mathfrak{f} -unit by Lemma 5.3.6, this shows that the dashed map is the fiber of a \mathfrak{f} -unit, and therefore the universal ∞ -cover. \square

Remark 5.3.15. The fiber sequence

$$\mathbb{B}_{\nabla}^k \mathbb{R} \rightarrow \mathbb{B}_{\nabla}^k U(1) \rightarrow \mathbb{B}^{k+1} \mathbb{Z}$$

expresses the informal identity

ordinary differential cohomology = pure differential cohomology + ordinary cohomology.

We are now ready to assemble what we have learned into the modal facture

hexagon of $\mathbb{B}_{\nabla}^k U(1)$:

$$\begin{array}{ccccc}
 & & \mathbb{B}_{\nabla}^k \mathbb{R} & \xrightarrow{\quad} & \Lambda_{\text{cl}}^{k+1} \\
 & \nearrow^{(-)_b} & & \searrow^{\pi} & \nearrow^{F_{(-)}} \\
 \mathfrak{b}\mathbb{B}^k \mathbb{R} & & & & & & \mathfrak{b}\mathbb{B}^{k+1} \mathbb{R} \\
 & \searrow & & \nearrow^{(-)_b} & \searrow^{(-)_f} & & \\
 & & \mathfrak{b}\mathbb{B}^k U(1) & \xrightarrow{\quad} & \mathbb{B}^{k+1} \mathbb{Z} & & \\
 & & & \nearrow^{(-)_b} & \searrow^{(-)_f} & & \\
 & & \mathfrak{b}\mathbb{B}^k U(1) & \xrightarrow{\quad \beta \quad} & \mathbb{B}^{k+1} \mathbb{Z} & &
 \end{array} \tag{5.7}$$

5.3.3 Descending to Cohomology and the Character Diagram

In this section, we will discuss how the modal fracture hexagon (5.7) descends to cohomology. In general, if E_{\bullet} is a loop spectrum, then we may define the cohomology groups of a type valued in E_{\bullet} to be the 0-truncated types of maps:

$$H^k(X; E_{\bullet}) := \|X \rightarrow E_k\|_0.$$

However, these abelian groups are not discrete — externally, they are (possible non-constant) sheaves of abelian groups. We will want the discrete (externally, constant) invariants. With this in mind, we make the following definitions.

Definition 5.3.16. Let X be a crisp type. We then make the following definitions:

$$H^n(X; \mathbb{Z}) := \|\mathfrak{b}(X \rightarrow \mathbf{B}^n \mathbb{Z})\|_0$$

$$H^n(X; \mathfrak{b} \mathbb{R}) := \|\mathfrak{b}(X \rightarrow \mathfrak{b} \mathbf{B}^n \mathbb{R})\|_0$$

$$H^n(X; \mathfrak{b} U(1)) := \|\mathfrak{b}(X \rightarrow \mathfrak{b} \mathbf{B}^n U(1))\|_0$$

$$H_{\nabla}^{n,k}(X; U(1)) := \|\mathfrak{b}(X \rightarrow \mathbf{B}^n \mathbf{B}_{\nabla}^k U(1))\|_0$$

$$H_{\nabla}^{n,k}(X; \mathbb{R}) := \|\mathfrak{b}(X \rightarrow \mathbf{B}^n \mathbf{B}^k \mathbb{R})\|_0$$

$$\Lambda^k(X) := \mathfrak{b}(X \rightarrow \Lambda^k)$$

$$\Lambda_{\text{cl}}^k(X) := \mathfrak{b}(X \rightarrow \Lambda_{\text{cl}}^k).$$

Remark 5.3.17. In full cohesion, it would be better to work with codiscrete cohomology groups, rather than discrete cohomology groups. This way the definition could be given for all types and not just crisp ones. But we will continue to use discrete groups so that we do not need to work with the codiscrete modality \sharp in this chapter.

We note that with these definitions we may reduce the calculation of ordinary differential cohomology for discrete and homotopically contractible types.

Proposition 5.3.18. Let X be a crisp type and let $k \geq 1$.

1. If X is discrete (that is, $X = \int X$), then $H_{\nabla}^{n,k}(X; U(1)) = H^{n+k}(X; \mathfrak{b} U(1))$.
2. If X is homotopically contractible (that is, $\int X = *$), then $H_{\nabla}^{n,k}(X; U(1)) =$

$$H^n(X; \Lambda_{\text{cl}}^{k+1}).$$

We may make similar calculations for pure differential cohomology:

1. If X is discrete, then $H^{n,k}(X; \mathbb{R}) = H_{\nabla}^{n+k}(X; \flat \mathbb{R})$.
2. If X is homotopically contractible, then $H_{\nabla}^{n,k}(X; \mathbb{R}) = H^n(X; \Lambda_{\text{cl}}^{k+1})$.

Proof. We will only prove the identities for ordinary differential cohomology; the proofs for pure differential cohomology are identical. We take advantage of the adjointness between \int and \flat .

1. Suppose that X is discrete. Then

$$\begin{aligned} H_{\nabla}^{n,k}(X; U(1)) &= \|\flat(X \rightarrow \mathbf{B}^n \mathbf{B}_{\nabla}^k U(1))\|_0 \\ &= \|\flat(\int X \rightarrow \mathbf{B}^n \mathbf{B}_{\nabla}^k U(1))\|_0 \\ &= \|\flat(X \rightarrow \flat \mathbf{B}^n \mathbf{B}_{\nabla}^k U(1))\|_0 \\ &= \|\flat(X \rightarrow \flat \mathbf{B}^{n+k} U(1))\|_0 \\ &= H^{n+k}(X; \flat U(1)) \end{aligned}$$

2. Suppose that X is homotopically contractible, and let $i : \mathbf{B}^{n+k+1} \mathbb{Z} \rightarrow$

$\mathfrak{b}\mathbf{B}^{n+k+1} \mathbb{R}$ denote the (delooping of) the inclusion. Then

$$\begin{aligned}
H_{\nabla}^{n,k}(X; U(1)) &= \left\| \mathfrak{b}(X \rightarrow \mathbf{B}^n \mathbf{B}_{\nabla}^k U(1)) \right\| \\
&= \mathfrak{b} \left\| \left(\begin{array}{l} (\omega : X \rightarrow \mathbf{B}^n \Lambda_{\text{cl}}^{k+1}) \times (c : X \rightarrow \mathbf{B}^{n+k+1} \mathbb{Z}) \\ \times (h : ic = \omega^f) \end{array} \right) \right\|_0 \\
&= \mathfrak{b} \left\| \left(\begin{array}{l} (\omega : X \rightarrow \mathbf{B}^n \Lambda_{\text{cl}}^{k+1}) \times (c : \mathbf{B}^{n+k+1} \mathbb{Z}) \\ \times (h : (x : X) \rightarrow ic = \omega(x)^f) \end{array} \right) \right\|_0
\end{aligned}$$

Since X is homotopically contractible, we have an equivalence $e : (X \rightarrow \mathfrak{b}\mathbf{B}^{n+k+1} \mathbb{R}) \simeq \mathfrak{b}\mathbf{B}^{n+k+1}$. We therefore have $e((-)^f \circ \omega) : \mathfrak{b}\mathbf{B}^{n+k+1}$ and for all $x : X$ a witness $\omega(x)^f = e((-)^f \circ \omega)$. We may therefore continue:

$$\begin{aligned}
&= \mathfrak{b} \left\| \left(\begin{array}{l} (\omega : X \rightarrow \mathbf{B}^n \Lambda_{\text{cl}}^{k+1}) \times (c : \mathbf{B}^{n+k+1} \mathbb{Z}) \\ \times (h : (x : X) \rightarrow ic = e((-)^f \circ \omega)) \end{array} \right) \right\|_0 \\
&= \mathfrak{b} \left\| \left(\begin{array}{l} (\omega : X \rightarrow \mathbf{B}^n \Lambda_{\text{cl}}^{k+1}) \times (c : \mathbf{B}^{n+k+1} \mathbb{Z}) \\ \times (ic = e((-)^f \circ \omega)) \end{array} \right) \right\|_0
\end{aligned}$$

Now, both $\mathbf{B}^{n+k+1} \mathbb{Z}$ and $(ic = e((-)^f \circ \omega))$ are 0-connected. The latter because it is identifiable with $\Omega \mathfrak{b}\mathbf{B}^{n+k+1} \mathbb{R}$, which is $\mathfrak{b}\mathbf{B}^{n+k} \mathbb{R}$ and so 0-connected for $k \geq 1$ and any n . We may therefore continue:

$$\begin{aligned}
&= \mathfrak{b} \left\| X \rightarrow \mathbf{B}^n \Lambda_{\text{cl}}^{k+1} \right\| \\
&= H^n(X; \Lambda_{\text{cl}}^{k+1}). \quad \square
\end{aligned}$$

Remark 5.3.19. We note here that since every type X lives in the center of a

fiber sequence

$$\overset{\infty}{X} \rightarrow X \rightarrow \int X$$

between a homotopically contractible type and a discrete type, we get a Serre spectral sequence converging to the k^{th} ordinary differential cohomology of X with E_2 page depending on its $bU(1)$ cohomology and the $\Lambda_{\text{cl}}^{k+1}$ valued cohomology of its universal ∞ -cover.

Now, since the top, bottom, and diagonal sequences in the modal fracture hexagon (5.7) of $B_{\nabla}^k U(1)$ are fiber sequences, when we take b and 0-truncations we will get long exact sequences. With the above definitions, we get the following diagram:

$$\begin{array}{ccccc}
 & H_{\nabla}^{0,k}(X; \mathbb{R}) & \xrightarrow{\quad} & \Lambda_{\text{cl}}^{k+1}(X) & \\
 & \nearrow & & \nearrow & \searrow \\
 H^k(X; b\mathbb{R}) & & H_{\nabla}^{0,k}(X; U(1)) & & H^{k+1}(X; b\mathbb{R}) \\
 & \searrow & \nearrow & \searrow & \nearrow \\
 & H^k(X; bU(1)) & \xrightarrow{\quad \beta \quad} & H^{k+1}(X; \mathbb{Z}) &
 \end{array} \tag{5.8}$$

in which the top and bottom sequences are long exact, and the diagonal sequences are exact in the middle. This looks very much like the character diagram for ordinary differential cohomology [SS08] except for two differences:

1. Where we have the pure cohomology $H_{\nabla}^{0,k}(X; \mathbb{R})$, one would normally find $\Lambda^k(X)/\text{im}(d)$, the abelian group which fits into an exact sequence

$$\Lambda^{k-1}(X) \xrightarrow{d} \Lambda^k(X) \rightarrow \Lambda^k(X)/\text{im}(d) \rightarrow 0.$$

2. Where we have $H^{k+1}(X; b\mathbb{R})$, which is ordinary (discrete) cohomology with real coefficients, one would normally find the de Rham cohomology

$H_{\text{dR}}^{k+1}(X)$. The de Rham cohomology is defined as closed forms mod exact forms, and so $H_{\text{dR}}^{k+1}(X)$ is the abelian group fitting the following exact sequence:

$$\Lambda^k(X) \xrightarrow{d} \Lambda_{\text{cl}}^{k+1}(X) \rightarrow H_{\text{dR}}^{k+1}(X) \rightarrow 0.$$

Both of these discrepancies are instances of de Rham's theorem that the de Rham cohomology of forms is the (discrete) ordinary cohomology with real coefficients. Classically and externally, this holds for smooth manifolds. We note that de Rham's theorem cannot hold for all types for rather trivial reasons: the form classifiers are sets, and so $\Lambda^k(X)$ depends only on the set truncation of X whereas $H^k(X; \mathfrak{b}\mathbb{R})$ can depend on the k -truncation of X .

Proposition 5.3.20. The de Rham theorem does not hold for the delooping $\mathfrak{b}\mathbb{R}$ of the discrete additive group of real numbers. Explicitly,

$$H_{\text{dR}}^1(\mathfrak{b}\mathbb{R}) = 0$$

$$H^1(\mathfrak{b}\mathbb{R}; \mathfrak{b}\mathbb{R}) = \mathfrak{b}\text{Hom}(\mathfrak{b}\mathbb{R}, \mathfrak{b}\mathbb{R}) \neq 0$$

Proof. Since $\mathfrak{b}\mathbb{R}$ is 0-connected and the form classifiers are sets, every map $\mathfrak{b}\mathbb{R} \rightarrow \Lambda^k$ is constant for all k . Therefore,

$$H_{\text{dR}}^1(\mathfrak{b}\mathbb{R}) = \Lambda_{\text{cl}}^1(\mathfrak{b}\mathbb{R}) / \Lambda^0(\mathfrak{b}\mathbb{R}) = 0$$

On the other hand, $H^1(\mathfrak{b}\mathbb{R}; \mathfrak{b}\mathbb{R}) = \|\mathfrak{b}(\mathfrak{b}\mathbb{R} \rightarrow \mathfrak{b}\mathbb{R})\|_0$ is the set of group homomorphisms from $\mathfrak{b}\mathbb{R}$ to itself (modulo conjugacy, which makes no difference). The identity is not conjugate to 0, and so this group is not trivial. \square

We can, however, make explicit the obstruction to de Rham's theorem

lying in the first (cohomological) degree pure differential cohomology groups $H_{\nabla}^{1,k}(X; \mathbb{R})$. We begin first by trying to construct an exact sequence

$$\Lambda^k(X) \xrightarrow{d} \Lambda^{k+1}(X) \rightarrow H_{\nabla}^{0,k}(X; \mathbb{R}) \rightarrow 0.$$

Recall Diagram 5.6. There is a similar diagram for pure differential cohomology:

$$\begin{array}{ccc}
 \mathbf{B}_{\nabla}^k \mathbb{R} & \longrightarrow & \Lambda_{\text{cl}}^{k+1} \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbf{B}\mathbf{B}_{\nabla}^{k-1} \mathbb{R} & \longrightarrow & \mathbf{B}\Lambda_{\text{cl}}^k \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 \mathbf{B}^{k-1}\mathbf{B}_{\nabla} \mathbb{R} & \longrightarrow & \mathbf{B}^{k-1}\Lambda_{\text{cl}}^2 \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbf{B}^k \mathbb{R} & \longrightarrow & \mathbf{B}^k \Lambda_{\text{cl}}^1 \\
 \downarrow & \lrcorner & \downarrow \\
 * & \longrightarrow & \mathfrak{b}\mathbf{B}^{k+1} \mathbb{R}
 \end{array}$$

If we focus at the top, we see that we have a pullback square which induces an equivalence on fibers:

$$\begin{array}{ccc}
 \bullet & \xrightarrow{\sim} & \Lambda^k \\
 \downarrow & & \downarrow d \\
 \mathbf{B}_{\nabla}^k \mathbb{R} & \longrightarrow & \Lambda_{\text{cl}}^{k+1} \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbf{B}\mathbf{B}_{\nabla}^{k-1} \mathbb{R} & \longrightarrow & \mathbf{B}\Lambda_{\text{cl}}^k
 \end{array} \tag{5.9}$$

This gives us a fiber sequence

$$\Lambda^k \rightarrow \mathbb{B}_{\nabla}^k \mathbb{R} \rightarrow \mathbb{B}\mathbb{B}_{\nabla}^{k-1} \mathbb{R},$$

which we may deloop as much as we like. Noting that $\Omega\mathbb{B}_{\nabla}^k \mathbb{R} = \mathbb{b}\mathbb{B}^{k-1} \mathbb{R}$, we therefore have a long exact sequence:

$$0 \rightarrow H^{k-1}(X; \mathbb{b}\mathbb{R}) \rightarrow H_{\nabla}^{0,k-1}(X; \mathbb{R}) \rightarrow \Lambda^k(X) \rightarrow H_{\nabla}^{0,k}(X; \mathbb{R}) \rightarrow H^{1,k-1}(X; \mathbb{R}) \rightarrow \dots$$

From this, we see that the surjectivity of the map $\Lambda^k(X) \rightarrow H_{\nabla}^{0,k}(X; \mathbb{R})$ is determined by the vanishing of the map $H_{\nabla}^{0,k}(X; \mathbb{R}) \rightarrow H^{1,k-1}(X; \mathbb{R})$. Furthermore, the version of Diagram 5.9 for $k-1$ shows us that $d : \Lambda^{k-1}(X) \rightarrow \Lambda^k(X)$ factors through $H_{\nabla}^{0,k}(X; \mathbb{R})$. This means that for the kernel of $\Lambda^k(X) \rightarrow H_{\nabla}^{0,k}(X; \mathbb{R})$ to be the image of $d : \Lambda^{k-1}(X) \rightarrow \Lambda^k(X)$, we need for $\Lambda^{k-1}(X) \rightarrow H_{\nabla}^{0,k-1}(X; \mathbb{R})$ to be surjective; this is controlled by the vanishing of $H_{\nabla}^{0,k-1}(X; \mathbb{R}) \rightarrow H_{\nabla}^{1,k-2}(X; \mathbb{R})$. In general, we see the obstructions to having exact sequences

$$\Lambda^{k-1}(X) \rightarrow \Lambda^k(X) \rightarrow H_{\nabla}^{0,k}(X; \mathbb{R})$$

lie in $H_{\nabla}^{1,k-1}(X; \mathbb{R})$ and $H_{\nabla}^{1,k-2}(X; \mathbb{R})$.

First cohomological degree pure differential cohomology groups also control obstructions to de Rham's theorem for general types X . By definition we have a fiber sequence $\mathbb{B}_{\nabla}^k \mathbb{R} \rightarrow \Lambda_{\text{cl}}^{k+1} \rightarrow \mathbb{b}\mathbb{B}^{k+1} \mathbb{R}$ which may be delooped arbitrarily. We therefore get exact sequences

$$0 \rightarrow H^k(X; \mathbb{b}\mathbb{R}) \rightarrow H_{\nabla}^{0,k}(X; \mathbb{R}) \rightarrow \Lambda_{\text{cl}}^{k+1}(X) \rightarrow H^{k+1}(X; \mathbb{b}\mathbb{R}) \rightarrow H_{\nabla}^{1,k}(X; \mathbb{R}) \dots$$

This exact sequence shows us that the surjectivity of the map $\Lambda_{\text{cl}}^{k+1}(X) \rightarrow$

$H^{k+1}(X; \flat \mathbb{R})$ is controlled by the vanishing of the map $H^{k+1}(X; \flat \mathbb{R}) \rightarrow H_{\nabla}^{1,k}(X; \mathbb{R})$. Furthermore, in order for the kernel of $\Lambda_{\text{cl}}^{k+1}(X) \rightarrow H^{k+1}(X; \flat \mathbb{R})$ to be $d : \Lambda^k(X) \rightarrow \Lambda_{\text{cl}}^{k+1}$, we need for $\Lambda^k(X) \rightarrow H_{\nabla}^{0,k}(X; \mathbb{R})$ to be surjective. As we saw above, for the map $\Lambda^k(X) \rightarrow H_{\nabla}^{0,k}(X; \mathbb{R})$ to be surjective, we must have that $H_{\nabla}^{0,k}(X; \mathbb{R}) \rightarrow H_{\nabla}^{1,k-1}(X; \mathbb{R})$ vanishes.

Remembering the classical, external differential cohomology hexagon, we are led to the following conjecture:

Conjecture 5.3.21. Let X be a crisp smooth manifold. Then $H_{\nabla}^{1,k}(X; \mathbb{R})$ vanishes for all k .

5.3.4 Abstract Ordinary Differential Cohomology

In the above sections, we constructed ordinary differential cohomology from the assumption of a long exact sequence of form classifiers. Apart from the concrete differential geometric input of the form classifiers, the construction was entirely abstract. In this section, we will describe the abstract ordinary differential cohomology theory from an axiomatic perspective.

The role of the form classifiers will be played by a *contractible and infinitesimal resolution* of a crisp abelian group C .

Definition 5.3.22. Let C be a crisp abelian group. A *contractible and infinitesimal resolution* (CIR) of C is a crisp long exact sequence

$$0 \rightarrow \flat C \rightarrow C \xrightarrow{d} C_1 \xrightarrow{d} C_2 \xrightarrow{d} \dots$$

where the C_n are homotopically contractible — $\int C_n = *$ — and where the kernels $Z_n \equiv \ker(d : C_n \rightarrow C_{n+1})$ are infinitesimal — $\flat Z_n = *$. We may think of C_n as

the abelian group of n -cochains, and Z_n as the abelian group of n -cocycles.

Remark 5.3.23. In an ∞ -topos of sheaves of homotopy types, an abelian group C in the empty context (which would therefore be crisp) is a sheaf of abelian groups. In this setting, we can understand a contractible and infinitesimal resolution of C as presenting a cohomology theory on the site. The C_n are the sheaves of n -cochains, and the Z_n the sheaves of n -cocycles. To suppose that the chain complex $d : C_n \rightarrow C_{n+1}$ is exact is to say that representables have vanishing cohomology. To say that Z_n is infinitesimal for $n > 0$ is to say that there is a unique n -cocycle on the terminal sheaf, namely 0. To say that the C_n are contractible may be understood as saying that for any two objects of the site, there is a homotopically unique *concordance* between any n -cochains on them.

Remark 5.3.24. It's likely that the generality could be pushed even further by taking C to be a spectrum and giving the following definition of a contractible and infinitesimal resolution of C :

- Two sequences C_n and Z_n of spectra, $n \geq 0$, with $C_0 = C$ and $Z_0 = \mathfrak{b}C$. We may think of C_n as the spectrum of n -cochains, and Z_n as the spectrum of n -cocycles.
- Fiber sequences $Z_n \xrightarrow{i} C_n \xrightarrow{d} Z_{n+1}$ in which all maps d are \mathfrak{f} -fibrations, and where $i_0 : Z_0 \rightarrow C_0$ is $(-)_\mathfrak{b} : \mathfrak{b}C \rightarrow C$.
- The C_n are contractible, and the Z_n are infinitesimal.

This definition re-expresses the long exact sequence $C_{n-1} \xrightarrow{d} C_n \xrightarrow{d} C_{n+1}$ in

terms of the short exact sequences

$$0 \rightarrow Z_n \rightarrow C_n \xrightarrow{d} Z_{n+1} \rightarrow 0$$

where $Z_n \equiv \ker(d : C_n \rightarrow C_{n+1})$. As we have no concrete examples at this level of generality in mind, we leave the details of this generalization to future work.

For the rest of this section, we fix a crisp abelian group C and an contractible and infinitesimal resolution of it. We can then prove analogues of the lemmas in the above sections. We begin by an analogue of Theorem 5.3.4.

Lemma 5.3.25. Let C be a crisp abelian group and C_\bullet a contractible and infinitesimal resolution of C . Then $d : C \rightarrow Z_1$ is the infinitesimal remainder of C .

Proof. By hypothesis, we have a short exact sequence

$$0 \rightarrow \mathfrak{b}C \rightarrow C \xrightarrow{d} Z_1 \rightarrow 0.$$

We therefore have a long fiber sequence

$$C \xrightarrow{d} Z_1 \rightarrow \mathfrak{b}BC \rightarrow BC$$

which exhibits $d : C \rightarrow Z_1$ as the infinitesimal remainder of C . □

Theorem 5.3.26. For C a crisp abelian group and C_\bullet a contractible and infinitesimal resolution of C , we have

$$\int Z_n = \mathfrak{b}B^\infty C.$$

Proof. The same as the proof of Theorem 5.3.4. □

We now define analogues of the ordinary differential geometry classifiers $\mathbb{B}^n \mathbb{B}_{\nabla}^k U(1)$.

Definition 5.3.27. For $n, k \geq 0$, define $\mathbb{B}^n D_k$ to be the following pullback:

$$\begin{array}{ccc} \mathbb{B}^n D_k & \xrightarrow{\mathbb{B}^n F_-} & \mathbb{B}^n Z_{k+1} \\ \downarrow & \lrcorner & \downarrow (-)^{\mathfrak{J}} \\ \int \mathbb{B}^{n+k} C & \longrightarrow & \flat \mathbb{B}^{n+k+1} \overset{\infty}{C} \end{array}$$

We refer to $F_{(-)} : D_k \rightarrow Z_{k+1}$ as the *curvature*.

We begin by noting that D_0 is simply C . This went without saying before; we refrained from defining $\mathbb{B}_{\nabla}^0 U(1)$, but if we had it would have been $U(1)$.

Lemma 5.3.28. As abelian groups, $D_0 = C$.

Proof. The defining pullback of D_0 is

$$\begin{array}{ccc} D_0 & \xrightarrow{F_-} & Z_1 \\ \downarrow & \lrcorner & \downarrow (-)^{\mathfrak{J}} \\ \int C & \longrightarrow & \flat \mathbb{B}^1 \overset{\infty}{C} \end{array}$$

But Z_1 is the infinitesimal remainder of C , so the right square in the modal fracture hexagon of C shows that C is the pullback of the same diagram. \square

We can prove an analogue of Lemma 5.3.6.

Lemma 5.3.29. The defining diagram

$$\begin{array}{ccc} \mathbb{B}^n D_k & \xrightarrow{\mathbb{B}^n F_-} & \mathbb{B}^n Z_{k+1} \\ \downarrow & \lrcorner & \downarrow (-)^{\mathfrak{J}} \\ \int \mathbb{B}^{n+k} C & \longrightarrow & \flat \mathbb{B}^{n+k+1} \overset{\infty}{C} \end{array}$$

is a \mathfrak{f} -naturality square. In particular, $\mathfrak{f}D_k = \mathfrak{f}\mathbf{B}^k C$.

Proof. Since $\mathfrak{f}\mathbf{B}^{n+k}C$ is discrete, it suffices to show that the fibers of $\mathbf{B}^n D_k \rightarrow \mathfrak{f}\mathbf{B}^{n+k}C$ are \mathfrak{f} -connected. But they are equivalent to the fibers of $(-)^{\mathfrak{f}} : \mathbf{B}^n Z_{k+1} \rightarrow \mathfrak{b}\mathbf{B}^{n+k+1}\overset{\infty}{C}$, which are \mathfrak{f} -contractible. \square

As a corollary, we can deduce an analogue of Proposition 5.3.14.

Corollary 5.3.30. The curvature $F_{(-)} : D_k \rightarrow Z_{k+1}$ induces an equivalence on universal ∞ -covers: $\overset{\infty}{D}_k = \overset{\infty}{Z}_{k+1}$.

Proof. The defining pullback

$$\begin{array}{ccc} D_k & \xrightarrow{F_{(-)}} & Z_{k+1} \\ \downarrow & \lrcorner & \downarrow (-)^{\mathfrak{f}} \\ \mathfrak{f}\mathbf{B}^k C & \longrightarrow & \mathfrak{b}\mathbf{B}^{k+1}\overset{\infty}{C} \end{array}$$

induces an equivalence on the fibers of the vertical maps. Since these maps are \mathfrak{f} -units, the fibers are by definition the respective universal ∞ -covers. \square

As with $\mathbf{B}_{\nabla}^k U(1)$, we may see D_k as equipping k -gerbes with band C with

cocycle data coming from Z_{k+1} . We have the analogue of Diagram 5.6:

$$\begin{array}{ccc}
 D_k & \longrightarrow & Z_{k+1} \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbf{B}D_k & \longrightarrow & \mathbf{B}Z_k \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 \mathbf{B}^{k-1}D_1 & \longrightarrow & \mathbf{B}^{k-1}Z_2 \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbf{B}^k C & \longrightarrow & \mathbf{B}^k Z_1 \\
 \downarrow & \lrcorner & \downarrow \\
 \int \mathbf{B}^k C & \longrightarrow & \flat \mathbf{B}^{k+1} \mathbb{R}
 \end{array} \tag{5.10}$$

By applying \flat to this diagram and recalling that the Z_i (and therefore their deloopings) are infinitesimal, we see that $\flat D_k = \flat \mathbf{B}^k C$. We can use a composite square from this diagram to prove an analogue of Lemma 5.3.9.

Lemma 5.3.31. The fiber of the curvature $F_{(-)} : D_k \rightarrow Z_{k+1}$ is $\flat \mathbf{B}^k C$. We are therefore justified in seeing the curvature as an obstruction to the flatness of the underlying gerbe.

Proof. The defining pullback

$$\begin{array}{ccc}
 D_k & \xrightarrow{F_-} & Z_{k+1} \\
 \downarrow & \lrcorner & \downarrow (-)^{\flat} \\
 \int \mathbf{B}^k C & \longrightarrow & \flat \mathbf{B}^{k+1} \overset{\infty}{C}
 \end{array}$$

induces an equivalence on the fibers of the horizontal maps. But the fiber of $\int \mathbf{B}^k C \rightarrow \flat \mathbf{B}^{k+1} \overset{\infty}{C}$ is $\flat \mathbf{B}^k C$. \square

Corollary 5.3.32. The curvature $F_{(-)} : D_k \rightarrow Z_{k+1}$ is a unit for the $(k - 1)$ -truncation modality.

Finally, we record an analogue to Lemma 5.3.12.

Lemma 5.3.33. The infinitesimal remainder of D_k is the curvature $F_{(-)} : D_k \rightarrow Z_{k+1}$.

Proof. Exactly as Lemma 5.3.12. □

We may put these results together to find the modal fracture hexagon of D_k .

Theorem 5.3.34. *The modal fracture hexagon of D_k (Definition 5.3.27) is:*

$$\begin{array}{ccccc}
 & & Z_{k+1}^\infty & \xrightarrow{\quad} & Z_{k+1} \\
 & \nearrow^{(-)_b} & \searrow^\pi & \nearrow^{F_{(-)}} & \searrow^{(-)^f} \\
 \mathfrak{b}\mathbf{B}^k C^\infty & & D_k & & \mathfrak{b}\mathbf{B}^{k+1} C^\infty \\
 & \searrow & \nearrow^{(-)_b} & \searrow^{(-)^f} & \nearrow \\
 & \mathfrak{b}\mathbf{B}^k C & \xrightarrow{\quad} & \mathfrak{f}\mathbf{B}^k C &
 \end{array} \tag{5.11}$$

5.3.5 A combinatorial analogue of differential cohomology

Our arguments in the preceding sections have been abstract and modal in character. This abstraction means that we can apply these arguments in settings other than differential geometry. In this subsection, we will sketch a combinatorial analogue of differential cohomology taking place in the cohesive ∞ -topos of symmetric simplicial homotopy types. We will mix internal and external reasoning in sketching the setup. We will give a fuller — and properly internal — exploration of symmetric simplicial cohesion in future work.

A symmetric simplicial homotopy type S is an ∞ -functor $X : \mathbf{Fin}_{>0} \rightarrow \mathbf{H}$ from the category of non-empty finite sets into the ∞ -category of homotopy types. These are the unordered analogue of simplicial homotopy types.

The ∞ -topos of symmetric simplicial homotopy types is cohesive. The modalities operate on a symmetric simplicial homotopy type X in the following ways:

- $\flat X$ is the discrete (0-skeletal) inclusion of $X_0 \equiv X([0])$ the homotopy type of 0-simplices in X .
- $\int X$ is the discrete inclusion of the *geometric realization* (or colimit) of X .
- $\sharp X$ is the codiscrete (0-coskeletal) inclusion of X_0 .

We have thus far avoided using the codiscrete modality \sharp in this chapter, but it plays a crucial role in this section. This is because the n -simplex $\Delta[n]$ may be defined to be the codiscrete reflection of the $(n+1)$ -element set $[n] \equiv \{0, \dots, n\}$.

$$\Delta[n] := \sharp[n].$$

We may therefore axiomatize symmetric simplicial cohesion internally with the following axiom:

Axiom 9 (Symmetric Simplicial Cohesion). A crisp type X is crisply discrete if and only if it is $\sharp[n]$ -local for all n .

We may therefore define $\int = \text{Loc}_{\{\sharp[n] \mid n \in \mathbb{N}\}}$ to be the localization at the simplices, and the Symmetric Simplicial Cohesion axiom will ensure that \int is adjoint to \flat as required by the Unity of Opposites axiom.

In his paper [Law21], Lawvere points out that the simplices $\Delta[n]$ are *tiny*, much like the infinitesimal disks in synthetic differential geometry. $\Delta[n]$ being tiny means the functor $(-)^{\Delta[n]}$ admits an *external right* adjoint. We may refer to this adjoint as $(-)^{\frac{1}{\Delta[n]}}$, following Lawvere. If C is a crisp codiscrete abelian group, then the external adjointness shows that maps $X \rightarrow C^{\frac{1}{\Delta[n]}}$ correspond to maps $X^{\Delta[n]} \rightarrow C$, which, since C is codiscrete, correspond to maps $\flat X^{\Delta n} \rightarrow C$; but $\flat X^{\Delta n} = X^{\Delta n}([0]) = X([n])$, so such maps ultimately correspond to maps $X([n]) \rightarrow C$ — that is, to C -valued n -cochains on the symmetric simplicial homotopy type X ! In total, $C^{\frac{1}{\Delta[n]}}$ classifies n -cochains, much in the way that Λ^n classifies differential n -forms. We note that $C^{\frac{1}{\Delta[n]}}$ inherits the (crisp) algebraic structure of C since $(-)^{\frac{1}{\Delta[n]}}$ is a right adjoint.

If furthermore C is a ring, then the $C^{\frac{1}{\Delta[n]}}$ will be modules and since codiscretes are contractible in this topos (by Theorem 10.2 of [Shu18a], noting that it satisfies Shulman’s Axiom C2), we see that the $C^{\frac{1}{\Delta[n]}}$ are contractible. We may use the face inclusions $\Delta[n] \rightarrow \Delta[n+1]$ to give maps $C^{\frac{1}{\Delta[n]}} \rightarrow C^{\frac{1}{\Delta[n+1]}}$, and taking their alternating sum gives us a chain complex

$$C \xrightarrow{d} C^{\frac{1}{\Delta[1]}} \xrightarrow{d} C^{\frac{1}{\Delta[2]}} \xrightarrow{d} \dots$$

Reasoning externally, we can see that this sequence will be exact since the C -valued cohomology of the n -simplices is trivial. Furthermore, since $\flat C_k = \flat C$ by adjointness ($\flat(* \rightarrow C_k) = \flat(*^{\Delta k} \rightarrow C)$), we see that the Z_k are infinitesimal:

$$\flat Z_k = \ker(\flat d : \flat C_k \rightarrow \flat C_{k+1}) = *.$$

For this reason, we may make the following assumption in the setting of symmetric

simplicial cohesion, mirroring Assumption 1 of the existence of form classifiers in synthetic differential cohesion.

Assumption 2. Let C be a codiscrete ring, and define $C_n := C^{\frac{1}{\Delta[n]}}$. Then the sequence

$$0 \rightarrow \flat C \rightarrow C \xrightarrow{d} C_1 \xrightarrow{d} C_2 \xrightarrow{d} \dots$$

forms a contractible and infinitesimal resolution of C .

We can now interpret the abstract language of Section 5.3.4 into the more concrete language of Assumption 2:

- We have begun with a codiscrete abelian group C . We note that since C is codiscrete, it is homotopically contractible: $\int C = *$. Therefore, $\overset{\infty}{C} = C$.
- The abelian groups C_k are the k -cochain classifiers.
- The kernels $Z_k := \ker(d : C_k \rightarrow C_{k+1})$ classify k -cocycles. Applying Theorem 5.3.26 here shows us that

$$\int Z_k = \flat B^k C.$$

From this, we see that cohomology valued in the discrete group $\flat C$ is the universal discrete cohomological invariant of k -cocycles value in C . This justifies a remark of Lawvere in [Law21] that the Z_k have the homotopy type of the Eilenberg-MacLane space $K(\flat C, k)$.

- Since C is contractible, we have that D_k as defined in Definition 5.3.27 is the universal ∞ -cover $\overset{\infty}{Z}_{k+1}$ of Z_{k+1} . We see that D_k classifies $(k + 1)$ -cocycles together with witnesses that their induced cohomology class vanishes in $\flat \mathbf{B}^{k+1} C$.

The D_k in this setting have more in common with pure differential cohomology $B_{\nabla}^k \mathbb{R}$ than with ordinary differential cohomology $\mathbf{B}_{\nabla}^k U(1)$ on account of being contractible. We can remedy this by introducing some new data. Suppose that we have an exact sequence

$$0 \rightarrow K \rightarrow C \rightarrow G \rightarrow 0$$

of crisp codiscrete abelian groups. We may then redefine D_k to instead be the following pullback:

$$\begin{array}{ccc} D_k & \longrightarrow & Z_{k+1} \\ \downarrow & \lrcorner & \downarrow \\ \mathfrak{b}\mathbf{B}^{k+1}K & \longrightarrow & \mathfrak{b}\mathbf{B}^{k+1}C \end{array}$$

We will then have $\int D_k = \mathfrak{b}\mathbf{B}^{k+1}K$, $\overset{\infty}{D}_k = \overset{\infty}{Z}_{k+1}$, and $\mathfrak{b}D_k = \mathfrak{b}\mathbf{B}^k G$, giving us a modal fracture hexagon:

$$\begin{array}{ccccc} & \overset{\infty}{Z}_{k+1} & \xrightarrow{\quad} & Z_{k+1} & \\ & \swarrow^{(-)_b} & & \searrow^{(-)^f} & \\ \mathfrak{b}\mathbf{B}^k C & & \xrightarrow{\pi} & D_k & \xrightarrow{F(-)} & \mathfrak{b}\mathbf{B}^{k+1} C \\ & \searrow & \swarrow^{(-)_b} & \swarrow^{(-)^f} & & \\ & \mathfrak{b}\mathbf{B}^k G & \xrightarrow{\beta} & \mathfrak{b}\mathbf{B}^{k+1} K & & \end{array} \quad (5.12)$$

Taking the short exact sequence $0 \rightarrow K \rightarrow C \rightarrow G \rightarrow 0$ to be

$$0 \rightarrow \sharp \mathbb{Z} \rightarrow \sharp \mathbb{R} \rightarrow \sharp U(1) \rightarrow 0$$

gives us a bona-fide combinatorial analogue of ordinary differential cohomology, fitting within a similar hexagon. However, instead of equipping the integral cohomology of manifolds with differential form data, we are equipping the integral

cohomology of symmetric simplicial sets with real cocycle data.

We intend to give this combinatorial analogue of ordinary differential cohomology a fully internal treatment in future work.

Bibliography

- [ALR07] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and stringy topology (Cambridge tracts in mathematics, series number 171)*. Cambridge, England: Cambridge University Press, 2007.
- [Ane+17] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal. *A Generalized Blakers-Massey Theorem*. 2017. arXiv: 1703.09050 [math.AT].
- [Ane+18a] M. Anel, G. Biedermann, E. Finster, and A. Joyal. “Goodwillie’s calculus of functors and higher topos theory”. In: *Journal of Topology* 11.4 (2018), 1100–1132. ISSN: 1753-8416. DOI: 10.1112/topo.12082. URL: <http://dx.doi.org/10.1112/topo.12082>.
- [Ane+18b] M. Anel, G. Biedermann, E. Finster, and A. Joyal. “Goodwillie’s calculus of functors and higher topos theory”. In: *Journal of Topology* 11.4 (2018), 1100–1132. ISSN: 1753-8416. DOI: 10.1112/topo.12082. URL: <http://dx.doi.org/10.1112/topo.12082>.
- [Ane+20] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal. “A generalized Blakers–Massey theorem”. In: *Journal of Topology* 13.4 (2020), 1521–1553. ISSN: 1753-8424. DOI: 10.1112/topo.12163. URL: <http://dx.doi.org/10.1112/topo.12163>.
- [Bar17] Clark Barwick. *The future of homotopy theory*. 2017. URL: <https://www.maths.ed.ac.uk/~cbarwick/papers/future.pdf>.
- [BD87] Marta Bunge and Eduardo Dubuc. “Local concepts in synthetic differential geometry and germ representability”. In: 1987. Chap. 5.
- [BDR18a] Ulrik Buchholtz, Floris van Doorn, and Egbert Rijke. “Higher Groups in Homotopy Type Theory”. In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science* (2018).

- [BDR18b] Ulrik Buchholtz, Floris van Doorn, and Egbert Rijke. *Higher Groups in Homotopy Type Theory*. 2018. arXiv: 1802.04315 [cs.LO].
- [BDR18c] Ulrik Buchholtz, Floris van Doorn, and Egbert Rijke. *Higher Groups in Homotopy Type Theory*. 2018. arXiv: 1802.04315 [cs.LO].
- [Bel08] John L Bell. *A Primer of Infinitesimal Analysis*. 2nd ed. Cambridge, England: Cambridge University Press, 2008.
- [Bez+22] Marc Bezem, Ulrik Buchholtz, Pierre Cagne, Bjørn Ian Dundas, and Daniel R. Grayson. *Symmetry*. <https://github.com/UniMath/SymmetryBook>. 2, 2022.
- [BGSL18] Marta Bunge, Felipe Gago, and Ana María San Luis. *Synthetic Differential Topology*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2018. DOI: 10.1017/9781108553490.
- [Ble18] Ingo Blechschmidt. *An elementary and constructive proof of Grothendieck’s generic freeness lemma*. 2018. arXiv: 1807.01231 [math.AC].
- [BNV14] Ulrich Bunke, Thomas Nikolaus, and Michael Völkl. “Differential cohomology theories as sheaves of spectra”. In: *Journal of Homotopy and Related Structures* 11.1 (2014), pp. 1–66. DOI: 10.1007/s40062-014-0092-5. URL: <https://doi.org/10.1007/s40062-014-0092-5>.
- [Bre92] Lawrence Breen. “Théorie de Schreier supérieure”. fr. In: *Annales scientifiques de l’École Normale Supérieure* 4e série, 25.5 (1992), pp. 465–514. DOI: 10.24033/asens.1656.
- [Che17] Felix Cherubini. “Formalizing Cartan Geometry in Modal Homotopy Type Theory”. PhD thesis. Karlsruher Institut für Technologie, 2017.
- [Chr+18] J. Daniel Christensen, Morgan Opie, Egbert Rijke, and Luis Scoccola. “Localization in Homotopy Type Theory”. In: *arXiv e-prints*, arXiv:1807.04155 (2018), arXiv:1807.04155. arXiv: 1807.04155 [math.AT].
- [Cou+14] Vesta Coufal, Dorette Pronk, Carmen Rovi, Laura Scull, and Courtney Thatcher. *Orbispace and their Mapping Spaces via Groupoids: A Categorical Approach*. 2014. arXiv: 1401.4772 [math.CT].
- [CR20] Felix Cherubini and Egbert Rijke. “Modal descent”. In: *Mathematical Structures in Computer Science* (2020), 1–29. ISSN: 1469-8072. DOI: 10.1017/S0960129520000201. URL: <http://dx.doi.org/10.1017/S0960129520000201>.

- [CR21] Felix Cherubini and Egbert Rijke. “Modal descent”. 2021.
- [CS74] Shiing-Shen Chern and James Simons. “Characteristic Forms and Geometric Invariants”. In: *Annals of Mathematics* 99.1 (1974), pp. 48–69. ISSN: 0003486X. URL: <http://www.jstor.org/stable/1971013>.
- [CS85] Jeff Cheeger and James Simons. “Differential characters and geometric invariants”. In: *Geometry and Topology*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1985. ISBN: 978-3-540-39738-0.
- [Del71] Pierre Deligne. “Théorie de Hodge : II”. fr. In: *Publications Mathématiques de l’IHÉS* 40 (1971), pp. 5–57. URL: http://www.numdam.org/item/PMIHES_1971__40__5_0/.
- [DP86] Eduardo J. Dubuc and Jacques Penon. “Objets compacts dans les topos”. In: *Journal of The Australian Mathematical Society* 40 (1986), pp. 203–217.
- [DT58a] Albrecht Dold and Rene Thom. “Quasifaserungen und Unendliche Symmetrische Produkte”. In: *Annals of Mathematics* 67.2 (1958), pp. 239–281. ISSN: 0003486X. URL: <http://www.jstor.org/stable/1970005>.
- [DT58b] Albrecht Dold and Rene Thom. “Quasifaserungen und Unendliche Symmetrische Produkte”. In: *Annals of Mathematics* 67.2 (1958), pp. 239–281. ISSN: 0003486X. URL: <http://www.jstor.org/stable/1970005>.
- [Dub79] Eduardo J. Dubuc. “Sur les modèles de la géométrie différentielle synthétique”. fr. In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 20.3 (1979), pp. 231–279. URL: http://www.numdam.org/item/CTGDC_1979__20_3_231_0/.
- [EM42] Samuel Eilenberg and Saunders MacLane. “Group Extensions and Homology”. In: *Annals of Mathematics* 43.4 (1942), pp. 757–831. ISSN: 0003486X. URL: <http://www.jstor.org/stable/1968966>.
- [GC89] Felipe Gago-Couso. *Internal weak opens, stability and Morse theory for synthetic germs*. Universidade de Santiago de Compostela, 1989. ISBN: 84-600-7007-7.
- [Gir71] Jean Giraud. *Cohomologie non abélienne*. Springer Berlin Heidelberg, 1971. DOI: 10.1007/978-3-662-62103-5. URL: <https://doi.org/10.1007/978-3-662-62103-5>.

- [Goo01] Tom Goodwillie. *quasifibrations*. Email to ALGTOP mailing list. 2001.
- [HF+16] Kuen-Bang Hou Favonia, Eric Finster, Daniel R. Licata, and Peter LeFanu Lumsdaine. “A Mechanization of the Blakers–Massey Connectivity Theorem in Homotopy Type Theory”. In: *2016 31st Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. 2016, pp. 1–10.
- [HS05] M.J. Hopkins and I.M. Singer. “Quadratic functions in geometry, topology, and M-theory”. In: *Journal of Differential Geometry* 70.3 (2005), pp. 329–452. DOI: 10.4310/jdg/1143642908. URL: <https://doi.org/10.4310/jdg/1143642908>.
- [KL18] Chris Kapulkin and Peter LeFanu Lumsdaine. *The Simplicial Model of Univalent Foundations (after Voevodsky)*. 2018. arXiv: 1211.2851 [math.LO].
- [Koc06] Anders Kock. *Synthetic differential geometry (London mathematical society lecture note series, number 333)*. en. 2nd ed. London Mathematical Society lecture note series. Cambridge, England: Cambridge University Press, 2006.
- [Lav96] Rene Lavendhomme. *Basic concepts of synthetic differential geometry*. en. 1996th ed. Texts in the Mathematical Sciences. Dordrecht, Netherlands: Springer, 1996.
- [Law] “Categorical Dynamics”. In: vol. Proceedings of Aarhus. Open House on Topos Theoretic Methods in Geometry. 1979.
- [Law04] F.William Lawvere. “Left and right adjoint operations on spaces and data types”. In: *Theoretical Computer Science* 316.1 (2004), pp. 105–111. ISSN: 0304-3975. DOI: <https://doi.org/10.1016/j.tcs.2004.01.026>. URL: <https://www.sciencedirect.com/science/article/pii/S0304397504000799>.
- [Law07] F Lawvere. “Axiomatic Cohesion”. In: *Theory and Applications of Categories* 19 (2007), pp. 41–49.
- [Law21] F. William Lawvere. “Toposes generated by codiscrete objects in combinatorial topology and functional analysis”. In: *Reprints in Theory and Applications of Categories* 27 (2021), pp. 1–11. URL: <http://www.tac.mta.ca/tac/reprints/articles/27/tr27abs.html>.

- [Law79] *Categorical Dynamics*. Vol. 30. Topos Theoretic Methods in Geometry. Aarhus University, 1979, pp. 1–28.
- [Law80] F. William Lawvere. “Toward the description in a smooth topos of the dynamically possible motions and deformations of a continuous body”. en. In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 21.4 (1980), pp. 377–392. URL: http://www.numdam.org/item/CTGDC_1980__21_4_377_0/.
- [Lic+18] Daniel R. Licata, Ian Orton, Andrew M. Pitts, and Bas Spitters. “Internal Universes in Models of Homotopy Type Theory”. en. In: (2018). DOI: 10.4230/LIPICS.FSCD.2018.22. URL: <http://drops.dagstuhl.de/opus/volltexte/2018/9192/>.
- [Lop] “Localization in homotopy type theory”. In: *Higher Structures* 4 (1 2020).
- [Lur] Jacob Lurie. *Notes on crystals and algebraic D-modules*. URL: <https://ncatlab.org/nlab/files/LurieCrystals.pdf>.
- [Lur11] Jacob Lurie. *Derived Algebraic Geometry XII: Proper Morphisms, Completions, and the Grothendieck Existence Theorem*. <http://people.math.harvard.edu/~lurie/papers/DAG-XII.pdf>. 2011.
- [MBL18] Felipe Gago Marta Bunge and Ana María San Luis. *Synthetic Differential Topology*. Vol. 448. 2018. ISBN: 9781108447232.
- [MF99] Vladimir Voevodsky Morel Fabien. “ \mathbb{A}^1 -homotopy theory of schemes”. In: *Publications Mathématiques de l’Institut des Hautes Études Scientifiques* (1999), pp. 45–143. URL: <https://doi.org/10.1007/BF02698831>.
- [MM92] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic*. en. 1st ed. Universitext. New York, NY: Springer, 1992.
- [MP97] Ieke Moerdijk and Dorette Pronk. “Orbifolds, sheaves and groupoids”. In: *K-theory* 12 (1997), pp. 3–21.
- [MR90] Ieke Moerdijk and Gonzalo E Reyes. *Models for smooth infinitesimal analysis*. en. 1991st ed. New York, NY: Springer, 1990.
- [MR91] Ieke Moerdijk and Gonzalo E. Reyes. *Models for Smooth Infinitesimal Analysis*. Springer New York, 1991. DOI: 10.1007/978-1-4757-4143-8. URL: <https://doi.org/10.1007/978-1-4757-4143-8>.

- [Pen81] Jacques Penon. “Infinitésimaux et intuitionnisme”. In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 22 (1981), pp. 67–72.
- [Rey91] Gonzalo E. Reyes. “A topos-theoretic approach to reference and modality.” In: *Notre Dame Journal of Formal Logic* 32.3 (1991), pp. 359–391. DOI: 10.1305/ndjfl/1093635834. URL: <https://doi.org/10.1305/ndjfl/1093635834>.
- [Rez14] Charles Rezk. *Global Homotopy Theory and Cohesion*. <https://faculty.math.illinois.edu/cohesion.pdf>. 2014.
- [RFL21] Mitchell Riley, Eric Finster, and Daniel R. Licata. *Synthetic Spectra via a Monadic and Comonadic Modality*. 2021. arXiv: 2102.04099 [math.CT].
- [Rij17] Egbert Rijke. *The join construction*. 2017. DOI: 10.48550/ARXIV.1701.07538. URL: <https://arxiv.org/abs/1701.07538>.
- [Rij18a] Egbert Rijke. “Classifying Types: Topics in synthetic homotopy theory”. PhD thesis. Carnegie Mellon University, 2018.
- [Rij18b] Egbert Rijke. “Classifying Types: Topics in synthetic homotopy theory”. PhD thesis. Carnegie Mellon University, 2018.
- [RSS17a] Egbert Rijke, Michael Shulman, and Bas Spitters. “Modalities in homotopy type theory”. In: *arXiv e-prints* (2017). arXiv: 1706.07526 [math.CT].
- [RSS17b] Egbert Rijke, Michael Shulman, and Bas Spitters. “Modalities in homotopy type theory”. In: *arXiv e-prints* (2017). arXiv: 1706.07526 [math.CT].
- [RSS20] Egbert Rijke, Michael Shulman, and Bas Spitters. “Modalities in homotopy type theory”. In: *Logical Methods in Computer Science* Volume 16, Issue 1 (2020). DOI: 10.23638/LMCS-16(1:2)2020. URL: <https://lmcs.episciences.org/6015>.
- [RZ91] Gonzalo E. Reyes and Houman Zolfaghari. “Topos-theoretic approaches to modality”. In: *Category Theory*. Ed. by Aurelio Carboni, Maria Cristina Pedicchio, and Guisepppe Rosolini. Berlin, Heidelberg: Springer Berlin Heidelberg, 1991, pp. 359–378. ISBN: 978-3-540-46435-8.
- [Sch13a] Urs Schreiber. “Differential cohomology in a cohesive infinity-topos”. In: (2013). arXiv: 1310.7930 [math-ph].

- [Sch13b] Urs Schreiber. “Differential cohomology in a cohesive infinity-topos”. In: (2013). arXiv: 1310.7930 [math-ph].
- [Sch13c] Urs Schreiber. “Differential cohomology in a cohesive infinity-topos”. In: (2013). URL: <https://ncatlab.org/schreiber/files/dcct170811.pdf>.
- [Sch21] Schreiber, Urs. *Differential cohesion and idelic structure*. <http://ncatlab.org/nlab/show/differential%20cohesion%20and%20idelic%20structure>. 2021.
- [Sch26] Otto Schreier. “Über die Erweiterung von Gruppen I”. In: *Monatshefte für Mathematik und Physik* 34 (1 1926), pp. 165–180.
- [Shu17] Michael Shulman. *Univalence for inverse EI diagrams*. 2017. arXiv: 1508.02410 [math.AT].
- [Shu18a] Michael Shulman. “Brouwer’s fixed-point theorem in real-cohesive homotopy type theory”. In: *Mathematical Structures in Computer Science* 28.6 (2018), 856–941. DOI: 10.1017/S0960129517000147.
- [Shu18b] Michael Shulman. “Brouwer’s fixed-point theorem in real-cohesive homotopy type theory”. In: *Mathematical Structures in Computer Science* 28.6 (2018), 856–941. DOI: 10.1017/S0960129517000147.
- [Shu19a] Michael Shulman. *All $(\infty, 1)$ -toposes have strict univalent universes*. 2019. DOI: 10.48550/ARXIV.1904.07004. URL: <https://arxiv.org/abs/1904.07004>.
- [Shu19b] Michael Shulman. *All $(\infty, 1)$ -toposes have strict univalent universes*. 2019. arXiv: 1904.07004 [math.AT].
- [SS08] James Simons and Dennis Sullivan. “Axiomatic characterization of ordinary differential cohomology”. In: *Journal of Topology* 1.1 (2008), pp. 45–56. DOI: <https://doi.org/10.1112/jtopol/jtm006>. eprint: <https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/jtopol/jtm006>. URL: <https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/jtopol/jtm006>.
- [SS14] Urs Schreiber and Michael Shulman. “Quantum Gauge Field Theory in Cohesive Homotopy Type Theory”. In: *Electronic Proceedings in Theoretical Computer Science* 158 (2014), 109–126. ISSN: 2075-2180. DOI: 10.4204/eptcs.158.8. URL: <http://dx.doi.org/10.4204/EPTCS.158.8>.

- [Tay10] Paul Taylor. “A Lambda Calculus for Real Analysis”. In: *Journal of Logic and Analysis* 2.5 (2010), pp. 1–115. DOI: 10.4115/jla.2010.2.5. URL: PaulTaylor.EU/ASD/lamcra.
- [Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: <https://homotopytypetheory.org/book>, 2013.
- [Wel17] Felix Wellen. “Formalizing Cartan Geometry in Modal Homotopy Type Theory”. PhD thesis. Karlsruher Institut für Technologie, 2017.
- [Wel18a] Felix Wellen. “Cohesive Covering Theory”. In: *Proceedings of HoTT/UF 2018*. Ed. by Simon Huber Benedikt Ahrens and Anders Mörtberg. 2018. URL: https://hott-uf.github.io/2018/abstracts/HoTTUF18_paper_15.pdf.
- [Wel18b] Felix Wellen. “Cohesive Covering Theory”. In: *Proceedings of HoTT/UF 2018*. Ed. by Huber Ahrens and Mörtberg. 2018. URL: https://hott-uf.github.io/2018/abstracts/HoTTUF18_paper_15.pdf.
- [Yam90] Masayuki Yamasaki. “Maps between orbifolds”. In: *Proceedings of the American Mathematical Society* 109.1 (1990), pp. 223–223. DOI: 10.1090/s0002-9939-1990-1017853-2. URL: <https://doi.org/10.1090/s0002-9939-1990-1017853-2>.
- [Yet87] David Yetter. “On right adjoints to exponential functors”. In: *Journal of Pure and Applied Algebra* 45.3 (1987), pp. 287–304. ISSN: 0022-4049. DOI: [https://doi.org/10.1016/0022-4049\(87\)90077-6](https://doi.org/10.1016/0022-4049(87)90077-6). URL: <https://www.sciencedirect.com/science/article/pii/0022404987900776>.