An elastic flow for nonlinear spline interpolations in $\mathbb{R}^n$

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November 27, 2021

Keywords: fourth-order geometric flow, elastic spline, spline interpolation, curve fitting
MSC(2010): primary 35K55; secondary 41A15, 53C44

Abstract

In this paper we use the method of geometric flow on the problem of nonlinear spline interpolations for non-closed curves in n-dimensional Euclidean spaces. The method applies theory of fourth-order parabolic PDEs to each piece of the curve between two successive knot points at which certain dynamic boundary conditions are imposed. We show the existence of global solutions of the elastic flow in suitable Hölder spaces. In the asymptotic limit, as time approaches infinity, solutions subconverge to a stationary solution of the problem. The method of geometric flows provides a new approach for the problem of nonlinear spline interpolations.

1 Introduction

Let $\mathcal{P} = \{p_0, p_1, ..., p_K\}$ be an ordered set of points in $\mathbb{R}^n$. We want to ask the following question: Can one find a sufficiently smooth curve starting from p_0 to p_K, which passes through all intermediate points p_i, $i = 1, ..., K-1$ in the given order and in a prescribed smooth manner? Such a problems, either in linear or nonlinear settings, have been investigated in literature under the name of spline interpolations or curve-fitting problems, e.g., see [2, 3, 13, 14, 15, 16, 20, 24]. Almost all approaches in literature to such problems are variational methods.

In this paper we apply the elastic flow of — non-closed curves in n-dimensional Euclidean spaces to spline interpolations. We use the theory of fourth-order parabolic PDEs to each piece of the curves between two successive knot points, where certain dynamic boundary conditions are imposed at these knot points. Since each piece of the curve, from p_i to p_{i+1}, evolves by the elastic flow under specified boundary conditions, the evolution equation is set up as a coupled fourth-order parabolic system. In this article, we prove the existence of global solutions to the elastic flow in suitable Hölder spaces. In the asymptotic limit, on a subsequence of times approaching infinity, solutions converge to equilibrium configurations of the elastic energy among the class of curves with given knot points and clamped ends. It is worth to mention that the so-called minimal-energy splines in [15] correspond to our asymptotic curves in Theorem 1. However, the result in [15] is restricted to curves in $\mathbb{R}^2$. We provide a new approach via long-time solutions to parabolic PDEs for curve fitting and nonlinear spline interpolation problems, rather than variational approaches to the equilibrium problem found in literature. Furthermore, for the latter, most

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articles investigate the planar case only. To the best of the authors' knowledge, the new approach via parabolic PDEs has also been proposed by Barrett, Garcke and Nürnberg in [2], together with a numerical implementation. Here, the aim of our work is to give a rigorous proof on the analytical aspect including the higher dimensional case.

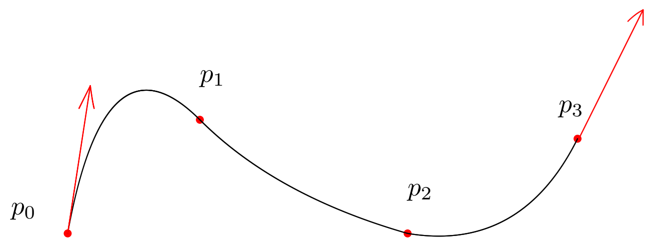

Figure 1: An example of an initial curve passing through $\mathcal{P} = \{p_0, p_1, p_2, p_3\}$ in the order with prescribed directions at p_0 and p_3.

Let $I = (0,1)$ and $f_i : \bar{I} \to \mathbb{R}^n$ represent a regular curve in $\mathbb{R}^n$, fulfilling $f_i \in C^{k,\alpha}\left(\bar{I}\right)$, $\forall\ i \in \{1, ..., K\}$. Let $f = (f_1, ..., f_K) \in \mathbb{R}^{n\times K}$, where f is understood as an $n \times K$ matrix valued function. Define $\Gamma_f : [0, K] \to \mathbb{R}^n$ by $\Gamma_f(t, x) = f_i(t, x-i+1)$, which represents the curve passing through the points $p_0, ..., p_K$ in the given order by imposing proper boundary conditions on each f_i.

Denote by $ds = |\partial_x f_i|\, dx$ the arclength element of f_i, and $\partial_s = |\partial_x f_i|^{-1} \partial_x$ the arclength differentiation on f_i. Further let $\tau_i = \partial_s f_i$ the unit tangent vector of f_i and $\vec{\kappa}_i = \partial_s^2 f_i$ the curvature vector of f_i. For convenience, as we reparametrize the curve f_i by its arclength parameter, i.e., $\bar{f}_i(s) = (f_i \circ x)(s)$, we still denote the curve by $f_i = f_i(s)$.

Define the bending energy of curves by

$$\mathcal{E}[f_i] := \int_I \frac{1}{2}|\vec{\kappa}_i|^2\, ds, \tag{1.1}$$

and the elastic energy (also called the penalized elastic energy) of f_i by

$$\mathcal{E}_\lambda[f_i] := \mathcal{E}[f_i] + \lambda \cdot \mathcal{L}[f_i], \tag{1.2}$$

where the constant $\lambda > 0$ is — called *tension modulus*, and $\mathcal{L}[f_i] := \int_I |\partial_x f_i|\, dx$ is the length of curve f_i. The bending energy corresponds in the literature to the so-called *Euler-Bernoulli* model of elastic rods. We define the total elastic energy of entire curve $f = (f_1, ..., f_K)$ by

$$\mathcal{E}_\lambda\left[f\right] := \sum_{i=1}^{K} \mathcal{E}_\lambda[f_i]. \tag{1.3}$$

To discuss the geometric flow of curves, we let $f_i : [0,T] \times \bar{I} \to \mathbb{R}^n$, for some $T > 0$, represent a family of sufficiently smooth and regular curves in $\mathbb{R}^n$, i.e., $|\partial_x f_i(t,x)| \neq 0$, $\forall\,(t,x) \in [0,T]\times\bar{I}$, $\forall\, i \in \{1, ..., K\}$. Note that, at any boundary point $(t^*, x^*) \in \{0,T\}\times\bar{I}\cup [0,T]\times\partial\bar{I}$, the derivatives of f_i are defined by $\partial_t^k\partial_x^j f_i(t^*, x^*) = \lim_{(t,x)\to(t^*,x^*)} \partial_t^k\partial_x^j f_i(t,x)$, for any $k, j \in \mathbb{N}_0$. Denote by $\nabla_s\eta_i := (\partial_s\eta_i)^\perp$ the normal component of $\partial_s\eta_i$, where η_i is a vector field along f_i. By applying the first variation formula of $\mathcal{E}$ and $\mathcal{L}$ in Lemma 5.2, the gradient flow of $\mathcal{E}_\lambda$ is given by

$$(\partial_t f_i)^\perp = -\nabla_{L^2}\mathcal{E}_\lambda[f_i] = -\nabla_s^2\vec{\kappa}_i - \frac{|\vec{\kappa}_i|^2}{2}\vec{\kappa}_i + \lambda\vec{\kappa}_i, \qquad \text{in } (0,T)\times I, \tag{1.4}$$

with the initial-boundary conditions,

$$f_i(t,x^*) = p_{i-1+x^*}, \qquad (t,x^*) \in [0,T]\times\partial I, i\in\{1,...,K\}, \tag{1.5}$$

$$\tau_1(t,0) = \tau^{(0)}, \qquad \tau_K(t,1) = \tau^{(K)}, \tag{1.6}$$

$$\partial_t \tau_{i+1-x^*}(t,x^*) = [\triangle_i \vec{\kappa}]\,(t), \qquad (t,x^*) \in [0,T]\times\partial I, i\in\{1,...,K-1\}, \tag{1.7}$$

$$f_i(0,x) = f_{0,i}(x), \text{ with } \quad \Gamma_{f_0}\in C^1(\bar{I}),\ f_{0,i}(x^*) = p_{i-1+x^*}, \quad x\in\bar{I}, i\in\{1,...,K\}, \tag{1.8}$$

where $[\triangle_i \vec{\kappa}]\,(t) := \vec{\kappa}_{i+1}(t,0) - \vec{\kappa}_i(t,1)$, and $\{\tau^{(0)},\tau^{(K)}\}$ is the set of prescribed constant unit vectors. The prescribed fixed points, $p_0,...,p_K$, in the interpolation by splines are called *knot points*. At p_0 and p_K, the conditions (1.5) and (1.6), represent the case of *clamped ends*.

Below we introduce the special case in which the tangential component of the moving speed $\partial_t f_i$ vanishes, i.e.,

$$\partial_t f_i = -\nabla_s^2 \vec{\kappa}_i - \frac{|\vec{\kappa}_i|^2}{2}\vec{\kappa}_i + \lambda\vec{\kappa}_i, \qquad \text{in } (0,T)\times I,\ i\in\{1,...,K\}. \tag{1.9}$$

Definition 1.1 (GP, GS, SGP, SGS)**.** *We call the geometric problem, or GP, to mean to find a solution to (1.4)∼(1.8). Any solution, — $f = (f_1,...,f_K)$, to the geometric problem, GP, is said to be a geometric solution, GS. Similarly, to find a solution to (1.5)∼(1.9) is called the special geometric problem, SGP; while any solution $f = (f_1,...,f_K)$ to the special geometric problem, SGP, is said to be a special geometric solution, SGS.*

The special geometric solutions, SGS, play an important role in the study of long-time existence of solutions to the geometric problem, GP.

Define the boundary operators $B_{G,0}$ and $B_{G,1}$, acting on f at the boundary ∂I, by

$$B_{G,0}(f_i)_{\lfloor (t,x^*)} = f_i(t,x^*) - p_{i-1+x^*}, \qquad \forall\,(t,x^*)\in[0,T]\times\{0,1\}, i\in\{1,...,K\}, \tag{1.10}$$

and

$$\begin{cases} B_{G,1}(f_i)_{\lfloor (t,x^*)} = \tau_i(t,x^*) - \tau^{(i)}, & \forall\,(i,x^*)\in\{(1,0),(K,1)\}, \\ B_{G,1}(f_i)_{\lfloor (t,x^*)} = \tau_i(t,x^*) - \tau_{0,i-1+2x^*}(1-x^*) - \int_0^t [\Delta_{i-1+x^*}\vec{\kappa}](\tau)\,d\tau, & \\ \qquad\qquad \forall\,(i,x^*)\in\{1,...,K\}\times\{0,1\}\backslash(i,x^*)\in\{(1,0),(K,1)\}. \end{cases} \tag{1.11}$$

For any $\ell\in\mathbb{N}_0$, define the differential operators, $B^{(\ell)}_{G,0}$ and $B^{(\ell)}_{G,1}$ by

$$B^{(\ell)}_{G,0}(f_i)_{\lfloor (t,x^*)} = \partial_t^\ell \left(B_{G,0}(f_i)\right)_{\lfloor (t,x^*)}, \qquad B^{(\ell)}_{G,1}(f_i)_{\lfloor (t,x^*)} = \partial_t^\ell \left(B_{G,1}(f_i)\right)_{\lfloor (t,x^*)},$$

where $(t,x^*)\in[0,T]\times\partial I$, $i\in\{1,...,K\}$. Note that $B^{(\ell)}_{G,0}$ and $B^{(\ell)}_{G,1}$ should be understood as differential operators with respect to the space variable x by converting every ∂_t into a fourth-order differential operator in (1.9).

Definition 1.2 (The compatibility conditions to SGP (1.5)∼(1.9) at boundary)**.** *We say that the initial datum $f_0 = (f_{0,1},...,f_{0,K})$, $f_{0,i}:\bar{I}\to\mathbb{R}^n$ fulfills the compatibility conditions of order $k\in\mathbb{N}_0$ to SGP (1.5)∼(1.9) on $\partial\bar{I}$, if the following conditions are satisfied:*

- $B^{(\ell)}_{G,0}(f_{0,i})_{\lfloor (x^*)} = 0, \qquad \forall\, 4\ell-4\le k,$
- $B^{(\ell)}_{G,1}(f_{0,i})_{\lfloor (x^*)} = 0, \qquad \forall\, 4\ell-3\le k.$

where $x^* \in \partial I$, $i \in \{1, ..., K\}$.

Theorem 1 (clamped/dynamic B.C.). *Let* $\lambda \in (0, \infty)$, $\alpha \in (0, 1)$, $I = (0, 1)$, *Suppose* $f_0 = (f_{0,1}, ..., f_{0,K})$ *is an initial datum to* (1.8) *with* $f_{0,i} \in C^{5,\alpha}\left(\bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$, *and* $\Gamma_{f_0} \in C^1([0, K])$. *Assume that, for each* $i \in \{1, ..., K\}$, $f_{0,i}$, *fulfills* $0 < \mathcal{L}[f_{0,i}] < \infty$, *and the compatibility conditions of order* 1 *in Definition 1.2.*
Then, there exists a global solution to the geometric problem (1.4)∼(1.8) *with the regularity* $f_i \in C^{\frac{5+\alpha}{4}, 5+\alpha}\left([0, \infty) \times \bar{I}\right) \bigcap C^\infty\left((0, \infty) \times \bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$, *and* $\Gamma_f(t, \cdot) \in C^1([0, K])$, $\forall\, t \in [0, \infty)$. *As* $t \to \infty$, *the family of curves* $\{f(t, \cdot)\}$ *subconverges to* $f_\infty = (f_{\infty,1}, ..., f_{\infty,K})$, *which is an equilibrium configuration to the energy functional* $\mathcal{E}_\lambda$ *with clamped boundary conditions at* p_0 *and* p_K. *Moreover,* $f_{\infty,i} \in C^\infty(\bar{I})$, $\forall\, i \in \{1, ..., K\}$, *and* Γ_{f_∞} *is* C^2*-smooth, i.e.,* $\Gamma_{f_\infty} \in C^2([0, K])$.

Remark 1.1. *One reason to impose the regularity on initial datum in the Hölder space,* $C^{5,\alpha}(\bar{I})$, *is due to the boundary condition* (1.7). *Namely, if one would like continuity of* $\partial_t \tau_i$ *to hold on* $[0, T] \times \bar{I}$, *then* $f_{0,i} \in C^{5,\alpha}(\bar{I})$ *seems to be the required regularity. It also induces the required compatibility conditions of order* 1 *for* $f_{0,i}$ *at* ∂I, $\forall\, i \in \{1, ..., K\}$.

Remark 1.2 (dynamic B.C.). *As the boundary conditions at all the boundary points* 0, 1 *are also dynamic, i.e.,* (1.7), *the existence result in Theorem 1 also holds. In this case, we let* $\vec{\kappa}_1(t, 0) = 0 = \vec{\kappa}_K(t, 1)$ *in order to define the difference of curvature* $[\triangle_i \vec{\kappa}]$. *In this setting the intermediate boundary conditions at* $i \in \{1, K\}$ *are the same as* (1.7). *We skip the proof of global solutions under dynamic B.C., since the argument is along similar ideas to the ones used in the clamped case presented in this article. As there is no knot point, i.e., in the case* $\vec{\kappa}_1(t, 0) = 0 = \vec{\kappa}_K(t, 1)$ *and* $K = 1$, *the case has been treated in [4], and also in [21] but with the different setting, a second-order parabolic equation for planar curves.*

We intend to apply PDE theory to show existence of classical solutions to (1.4). Firstly, we prove short-time existence of solutions. One complication is that the parabolicity, both in (1.4) and (1.9), degenerates as one views them in the PDE setting. This problem can be addressed in mainly two approaches. Starting with the work by Hamilton [17] on Ricci flow, one can utilize a suitable integrability condition to solve degenerate parabolic equations applying the Nash-Moser implicit function theorem. Polden applied this approach to fourth-order flows like the elastica in his PhD thesis [25]. Another way is to reformulate the evolution using the action of a diffeomorphism group on the manifold making the refined problem uniformly parabolic and hence allows to apply the classical existence theorems for parabolic systems. This now well-established idea was initially applied by DeTurck [9] for the Ricci flow and is also the basis of our approach in this article to address the additional complications due to the openness of the curves in the GP and SGP and the dynamic boundary conditions at the knot points. To this aim we set up, in §2, an analytical problem (AP), whose solutions are also solutions to (1.4). The proof relies on Solonnikov's theory of parabolic systems (see [26]). The required parabolicity is obtained via the composition of the family of curves with a family of diffeomorphisms. The composition fulfills a parabolic PDE with certain boundary conditions. In other words, the family of diffeomorphisms provides a tangential reparametrization so that the PDE maintains the required full parabolicity. The short-time existence (STE) is obtained from a contraction mapping argument in suitable Hölder spaces by applying Solonnikov's theory.

To show the long-time existence (LTE) of solutions to (1.4), we show that the special geometric solutions (SGS) to (1.9) exist globally in time. The SGS is obtained from converting AS into SGS by the composition with a suitable family of diffeomorphisms, fulfilling certain first-order equations. Equally, we can convert an SGS back into AS by the composition with a family of diffeomorphisms, fulfilling a second set of first-order equations. Both compositions are discussed in §3.

To show that an SGS exists globally in time, we establish uniform bounds of the speed of parametrization of curves (see Lemma 4.5) while taking into account the compatibility conditions of any order of the initial datum. Converting an SGS into AS allows us to apply Solonnikov's theory to establish an extension of AS in (short) time.

We convert the time extended AS back into SGS to establish bounds uniform in time. The uniform bounds of the speed of parametrization of the curves can be obtained by uniform estimates for geometric terms, see §4. We follow an established approach on long-time existence of solutions to elastic flows of open curves in the literature, e.g., [5, 12, 22], to provide the long-time existence of solutions of (1.9) with dynamic boundary conditions (1.5), (1.6), (1.7). Differentiating both sides of (1.9) provides an "algebraic" structure, which offers differential inequalities for higher-order Sobolev semi-norms of the curvature. These differential inequalities are of Gronwall's type and lead to global bounds for the curvature. To derive these differential inequalities, we integrate by parts along the curve and bound lower order terms by Gagliardo-Nirenberg type interpolation inequalities. As we work on open elastic curves, the boundary terms generated from integration by parts need careful examination. In [22], we found that the difficulty in estimating the boundary terms could be avoided by working with the L^2-norm of covariant derivatives of the curvature with respective to the time variable, e.g., $\|\nabla_t^m f_i\|_{L^2}$, instead of spatial derivatives with respect to the arclength, e.g., $\|\nabla_s^m \vec{\kappa}_i\|_{L^2}$.

Compared to previous work in the literature, the dynamic boundary condition (1.7), considered in this article, generates a new difficulty as it produces terms, whose order are too high to apply the usual interpolation inequalities. To overcome this problem, we utilize the "algebraic" structure in deriving a differential equality for terms of the form

$$\mathcal{Y}_m(t) := \sum_{i=1}^{K} \int_I |\nabla_t^m f_i|^2 \, ds + \sum_{i=1}^{K-1} |\nabla_t^m \tau_i(t,1)|^2. \tag{1.12}$$

The corresponding Gronwall's differential inequality gives uniform bounds of $\mathcal{Y}_m(t)$, $\forall\, m \in \mathbb{N}$.

These uniform bounds provide the long-time existence and asymptotic behavior of the piecewise smooth solutions to the elastic flow (1.9) stated in Theorem 1. In particular, the speed of the parametrization remains uniformly bounded away from 0 and ∞ (see Lemma 4.5).

Notice that the knot points $\{p_0, ..., p_K\}$ are not necessarily distinct in Theorem 1, i.e., the condition (1.5) allows for an intersection point, $p_i = p_j$ for some $i \neq j$.

The remainder of the article is arranged as follows. In §2, we set up the analytical problem and apply Solonnikov's theory to provide classical short-time solutions. In §3, we show how to construct the family of diffeomorphisms so that one can convert either AS into SGS or SGS into AS. In §4, we provide several estimates to obtain uniform bounds on derivatives of the curvature w.r.t. arclength parameter and present the proof to extend SGS globally in time. In the Appendix §5, we collect some notation, identities, estimates, as well as previous results in literature. This is to assist the reader in keeping the article self-contained.

2 The analytical problem and the short-time existence

We already mentioned that the parabolicity of the fourth-order quasilinear PDE (1.4) degenerates. So in order to be able to apply the Solonnikov's theory of linear parabolic PDE for the short-time existence of solutions, we need to add an appropriate reparametrization to make the flow (1.4) uniformly parabolic. Namely, we need to consider the parabolicity of the evolution equation,

$$\partial_t f_i = -\nabla_s^2 \vec{\kappa}_i - \frac{|\vec{\kappa}_i|^2}{2} \vec{\kappa}_i + \lambda \vec{\kappa}_i + \varphi_i \tau_i, \tag{2.1}$$

for a suitably chosen tangential component φ_i. Note that any solution to (2.1) is also a solution to (1.4). It follows from a straightforward computation (see also [6, (A.4)]) that the normal component on the right-hand side of (2.1) fulfills

$$\vec{V}_i := -\nabla_s^2 \vec{\kappa}_i - \frac{1}{2}|\vec{\kappa}|^2 \vec{\kappa}_i + \lambda \, \vec{\kappa}_i = -\mathcal{D}(f_i) + \left\langle \mathcal{D}(f_i), \frac{\partial_x f_i}{|\partial_x f_i|} \right\rangle \frac{\partial_x f_i}{|\partial_x f_i|} = -\left(\mathcal{D}(f_i)\right)^{\perp}, \tag{2.2}$$

where $(W)^{\perp}$ denotes the normal part of a vector field W along a curve f,

$$\mathcal{D}(f_i) = \frac{\partial_x^4 f_i}{|\partial_x f_i|^4} - h(f_i), \tag{2.3}$$

$$h(f_i) = 6 \frac{\langle \partial_x f_i, \partial_x^2 f_i \rangle \partial_x^3 f_i}{|\partial_x f_i|^6} + \left[4 \frac{\langle \partial_x f_i, \partial_x^3 f_i \rangle}{|\partial_x f_i|^6} + \frac{5}{2} \frac{|\partial_x^2 f_i|^2}{|\partial_x f_i|^6} - \frac{35}{2} \frac{\langle \partial_x f_i, \partial_x^2 f_i \rangle^2}{|\partial_x f_i|^8} + \frac{\lambda}{|\partial_x f_i|^2} \right] \partial_x^2 f_i, \tag{2.4}$$

and $\lambda \in (0, \infty)$. Thus, by choosing the tangential component in (2.1) as

$$\varphi_i = -\langle \mathcal{D}(f_i), \tau_i \rangle, \tag{2.5}$$

(2.1) becomes

$$\partial_t f_i = -\mathcal{D}(f_i). \tag{2.6}$$

Notice that if $f_i : [0, T] \times [0, 1] \to \mathbb{R}^n$ fulfills

$$\partial_x f_i(t, 1) = \partial_x f_{i+1}(t, 0), \qquad \forall \, t \in [0, T], i \in \{1, ..., K-1\}, \tag{2.7}$$

then $\Gamma_f(t, \cdot) : [0, K] \to \mathbb{R}^n$ is C^1-smooth, for any fixed $t \in [0, T]$. From a direct computation, we have

$$\partial_t \tau_i = \left(\frac{\partial_t \partial_x f_i}{|\partial_x f_i|} \right)^{\perp}; \quad \vec{\kappa}_i = \left(\frac{\partial_x^2 f_i}{|\partial_x f_i|^2} \right)^{\perp}, \qquad \forall \, i \in \{1, ..., K\}. \tag{2.8}$$

Let $f = (f_1, f_2, ..., f_K)$, where $f_i : D^T \to \mathbb{R}^n$, and

$$D^T = [0, T] \times [0, 1] = [0, T] \times \bar{I}. \tag{2.9}$$

To find solutions to GP (1.4)∼(1.8), we consider the initial-boundary value problem as follows,

$$\begin{cases} \partial_t f_i = -\mathcal{D}(f_i), & \text{in } (0, T) \times (0, 1), i \in \{1, ..., K\}, \\ f_i(t, x^*) = p_{i-1+x^*}, & (t, x^*) \in [0, T] \times \{0, 1\}, i \in \{1, ..., K\}, \\ \partial_x f_1(t, 0) = \partial_x f_{0,1}(0), \quad \partial_x f_K(t, 1) = \partial_x f_{0,K}(1), & \forall \, t \in [0, T], \\ \partial_t \partial_x f_{i+1-x^*}(t, x^*) = [\Delta_i(\delta^2 f)](t), \forall \, (t, x^*) \in [0, T] \times \{0, 1\}, & i \in \{1, ..., K-1\}, \\ f_i(0, x) = f_{0,i}(x), & \forall \, x \in [0, 1], i \in \{1, ..., K\}, \end{cases} \tag{2.10}$$

where

$$[\Delta_i(\delta^2 f)](t) := \frac{\partial_x^2 f_{i+1}(t, 0)}{|\partial_x f_{i+1}(t, 0)|} - \frac{\partial_x^2 f_i(t, 1)}{|\partial_x f_i(t, 1)|}, \quad \forall \, i \in \{1, ..., K-1\}.$$

Note that the order of differentiability of the dynamic boundary conditions in (2.10) is higher than that of the parabolic equation therein. In order to apply standard theory of parabolic PDEs, we reformulate the setup in (2.10) as below. For any $i \in \{1, ..., K\}$,

$$\begin{cases} \partial_t f_i = -\mathcal{D}(f_i), & \text{in } (0,T) \times (0,1), \\ f_i(t,x^*) = p_{i-1+x^*}, & (t,x^*) \in [0,T] \times \{0,1\}, \\ \partial_x f_i(t,x^*) = b(f_i)(t,x^*), & (t,x^*) \in [0,T] \times \{0,1\}, \\ f_i(0,x) = f_{0,i}(x), & x \in [0,1], \end{cases} \tag{2.11}$$

where

$$\begin{cases} b(f_i)(t,x^*) = \partial_x f_{0,i}(x^*), & \forall\, (i,x^*) \in \{(1,0),(K,1)\}, \\ b(f_i)(t,x^*) = \partial_x f_{0,i-1+2x^*}(1-x^*) + \int_0^t [\Delta_{i-1+x^*}(\delta^2 f)](\tau) d\tau, & \\ \qquad \forall\, (i,x^*) \in \{1,...,K\} \times \{0,1\} \backslash \{(1,0),(K,1)\}. \end{cases} \tag{2.12}$$

Definition 2.1 (The analytical problems and solutions)**.** *To find a solution $f = (f_1, ..., f_K)$ fulfilling (2.11) is called the analytical problem (2.11) or AP (2.11). Any solution to (2.11) is called an analytical solution (2.11) or AS (2.11).*

Note that, in this article, solutions always mean classical solutions.

Remark 2.1 (Equivalence of solution of (2.10) and AS (2.11))**.** *Note that $f = (f_1, ..., f_K)$, $f_i : D^T \to \mathbb{R}^n$, is a solution of (2.10) fulfilling $f_i \in C^{\frac{5+\alpha}{4}, 5+\alpha}(D^T)$, $\forall\, i \in \{1, ..., K\}$, if and only if f is an AS (2.11) with the same regularity. In fact, by taking ∂_t on both sides of $\partial_x f_i(t,x^*) = b(f_i)(t,x^*)$, $\forall\, (t,x^*) \in [0,T] \times \partial I$, $i \in \{1, ..., K-1\}$, we have*

$$\partial_t \partial_x f_{i+1-x^*}(t,x^*) = [\Delta_i(\delta^2 f)](t), \qquad \forall\, (t,x^*) \in [0,T] \times \partial I, i \in \{1,...,K-1\}. \tag{2.13}$$

Conversely, by taking integration on both sides of (2.13), we have $\partial_x f_i(t,x^*) = b(f_i)(t,x^*)$, $\forall\, (t,x^*) \in [0,T] \times \partial I$, $i \in \{1, ..., K-1\}$.

Denote the boundary operators $B_{A,0}$ and $B_{A,1}$ acting on f at the boundary $[0,T] \times \partial I$ by

$$(B_{A,0}(f_i))_{\lfloor (t,x^*)} = f_i(t,x^*) - p_{i-1+x^*}, \quad (B_{A,1}(f_i))_{\lfloor (t,x^*)} = \partial_x f_i(t,x^*) - b(f_i)(t,x^*), \tag{2.14}$$

and the differential operators, $B^{(\ell)}_{A,0}$, $B^{(\ell)}_{A,1}$, $\forall\, \ell \in \mathbb{N}_0$, by

$$B^{(\ell)}_{A,0}(f_i)_{\lfloor (t,x^*)} = \left(\partial_t^\ell B_{A,0}(f_i)\right)_{\lfloor (t,x^*)}, \quad B^{(\ell)}_{A,1}(f_i)_{\lfloor (t,x^*)} = \left(\partial_t^\ell B_{A,1}(f_i)\right)_{\lfloor (t,x^*)}, \tag{2.15}$$

where $(t,x^*) \in [0,T] \times \partial I$, $i \in \{1, ..., K\}$. Note that $B^{(\ell)}_{A,0}$ and $B^{(\ell)}_{A,1}$, should be understood to be differential operators with respect to space-variable x by converting every ∂_t into a fourth-order differential operator by following the PDE in (2.11).

Definition 2.2 (The compatibility conditions to AP (2.11) at boundaries)**.** *We say that the initial datum $f_0 = (f_{0,1}, f_{0,2}, ..., f_{0,K})$, where $f_{0,i} : \bar{I} \to \mathbb{R}^n$, fulfills the compatibility conditions of order k, $k \in \mathbb{N}_0$, to AP (2.11), if the following conditions are satisfied:*

- $B^{(\ell)}_{A,0}(f_{0,i})_{\lfloor (x^*)} = 0, \qquad \forall\, 4\ell - 4 \leq k,$
- $B^{(\ell)}_{A,1}(f_{0,i})_{\lfloor (x^*)} = 0, \qquad \forall\, 4\ell - 3 \leq k,$

where $x^* \in \partial I$, $i \in \{1, ..., K\}$.

Theorem 2.2 (The short-time existence and uniqueness to AP (2.11))**.** *Let $\lambda \in (0, \infty)$, $\alpha \in (0, 1)$, $I = (0, 1)$, $K \geq 2$, and*

$$M_0 = N_0 \cdot \sum_{j=0}^{4} \left(\sum_{i=1}^{K} \|f_{0,i}\|_{C^{4,\alpha}(\bar{I})} \right)^{2j+1},$$

where $N_0 > 1$ is a sufficiently large constant. Let $f_0 = (f_{0,1}, f_{0,2}, ..., f_{0,K})$, $f_{0,i} : \bar{I} \to \mathbb{R}^n$, represent an initial datum of the AP (2.11) and fulfill the compatibility conditions of order 1 in Definition 2.2. For each $i \in \{1, ..., K\}$, assume that $f_{0,i}$ fulfills $0 < \mathcal{L}[f_{0,i}] < \infty$, and

$$\delta_0 \leq |\partial_x f_{0,i}(x)| \leq \delta_0^{-1}, \tag{2.16}$$

for some $\delta_0 \in (0, 1)$, with the regularity $f_{0,i} \in C^{5,\alpha}\left(\bar{I}\right)$. Then, there exist $t_0 = t_0(n, \delta_0, \lambda, M_0) > 0$ and $f_i \in C^{\frac{5+\alpha}{4}, 5+\alpha}(D^{t_0}) \bigcap C^\infty\left((0, t_0] \times \bar{I}\right)$, $i \in \{1, ..., K\}$, such that $f = (f_1, ..., f_K)$ is the unique solution to AP (2.11).

We leave the proof of Theorem 2.2 to §2.2.

2.1 The linear problem

We linearize AP (2.11) as follows. For any $i \in \{1, ..., K\}$, consider

$$\begin{cases} \partial_t f_i + \frac{\partial_x^4 f}{|\partial_x f_{0,i}|^4} = G(\bar{f}_i), & \text{in } (0,T) \times I, \\ f_i(t, x^*) = p_{i-1+x^*}, & \forall\, (t, x^*) \in [0,T] \times \partial I, \\ \partial_x f_i(t, x^*) = b(\bar{f}_i)(t, x^*), & \forall\, (t, x^*) \in [0,T] \times \partial I, \\ f_i(0, x) = f_{0,i}(x), & \forall\, x \in \bar{I}, \end{cases} \tag{2.17}$$

where

$$G(\bar{f}_i) := R(\bar{f}_i) + h(\bar{f}_i), \qquad \text{in } (0,T) \times I, \tag{2.18}$$

$$R(\bar{f}_i) := \left(\frac{1}{|\partial_x f_{0,i}|^4} - \frac{1}{|\partial_x \bar{f}_i|^4} \right) \partial_x^4 \bar{f}_i, \qquad \text{in } (0,T) \times I. \tag{2.19}$$

Assume that the initial datum $f_0 = (f_{0,1}, ..., f_{0,K})$ fulfills the compatibility conditions of order 0, defined in Definition 2.2, and satisfies $f_{0,i} \in C^{4,\alpha}(\bar{I})$, $\delta_0 \leq |\partial_x f_{0,i}(x)| \leq \delta_0^{-1}$, $\forall\, x \in \bar{I}, i \in \{1, ..., K\}$. Let

$$X_{f_0}^T = \left\{ f = (f_1, ..., f_K) : D^T \to \mathbb{R}^{n \times K}, f_i \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left(D^T\right), \, f_i(0, \cdot) = f_{0,i}(\cdot), \, i \in \{1, ..., K\} \right\},$$

be a subset of the Banach space associated with the norm

$$\|f\|_{X_{f_0}^T} = \sum_{i=1}^{K} \|f_i\|_{C^{\frac{4+\alpha}{4}, 4+\alpha}(D^T)}.$$

Denote by $B_M = \{f \in X_{f_0}^T : \|f\|_{X_{f_0}^T} \leq M\}$ the closed, bounded, and convex subset.

Definition 2.3 (The compatibility conditions to the linear problem, LP (2.17))**.** *We say that the initial datum $f_0 = (f_{0,1}, f_{0,2}, ..., f_{0,K})$, where $f_{0,i} : \bar{I} \to \mathbb{R}^n$, fulfills the compatibility conditions of order k, for some $k \in \mathbb{N}_0$, to LP (2.17) at the boundary ∂I, if the following conditions hold:*

- $B^{(\ell)}_{A,0}(f_{0,i})_{\lfloor(x^*)} = 0, \qquad \forall\, 4\ell-4\leq k,$
- $B^{(\ell)}_{A,1}(f_{0,i})_{\lfloor(x^*)} = 0, \qquad \forall\, 4\ell-3\leq k,$

where $x^*\in\partial I$, $i\in\{1,...,K\}$.

Lemma 2.3. *Let* $\delta_0\in(0,1)$, $M_0>0$ *be the ones given in Theorem 2.2. For any initial datum* $f_0=(f_{0,1},f_{0,2},...,f_{0,K})$ *fulfilling the compatibility conditions of order* 0 *in Definition 2.2, and* $|\partial_x f_{0,i}(x)|\geq\delta_0>0$, $\forall\, x\in\bar I$, $f_{0,i}\in C^{4,\alpha}(\bar I)$, $\forall\, i\in\{1,...,K\}$. *There exists* $T_1>0$, *such that*

$$|\partial_x \bar f_i(t,x)|\geq\frac{\delta_0}{2},\qquad \forall\,(t,x)\in D^{T_1},\ \forall\, i\in\{1,...,K\},$$

holds for any $\bar f\in X^{T_1}_{f_0}\cap B_{M_0}$.

Proof. Assume $\bar f\in X^{T_1}_{f_0}\cap B_{M_0}$ for some $T_1>0$ (to be determined later). By applying the triangle inequality, the definition of semi-norm, $[\cdot]_{\frac{\alpha}{4},t}$, the assumption $\|\bar f_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^{T_1})}\leq M_0$, $\forall\, i\in\{1,...,K\}$, and Lemma 5.9, we have

$$|\partial_x\bar f_i(t,x)|\geq|\partial_x f_{0,i}(x)|-|\partial_x\bar f_i(t,x)-\partial_x f_{0,i}(x)|\geq\delta_0-T_1^{\frac{\alpha}{4}}[\partial_x\bar f_i]_{\frac{\alpha}{4},t}\geq\delta_0-T_1^{\frac{\alpha}{4}}M_0\geq\frac{\delta_0}{2},$$

where the last inequality comes from choosing T_1 so that $T_1^{\frac{\alpha}{4}}M_0\leq\frac{\delta_0}{2}$. □

Theorem 2.4 ($C^{\frac{k+\alpha}{4},k+\alpha}$-solutions to the linear problem (2.17))**.** *Let* $\delta_0\in(0,1)$, $M_0>0$ $\lambda\in(0,\infty)$, $\alpha\in(0,1)$, *and* $K\geq 2$. *Suppose that* $f_{0,i}:\bar I\to\mathbb{R}^n$ *satisfies* (2.16), $f_{0,i}\in C^{k,\alpha}\left(\bar I\right)$, *and the compatibility condition of order* $k-4$ *to LP* (2.17), $\forall\, i\in\{1,...,K\}$, $k\in\mathbb{N}$, $k\geq 4$. *Then for any* $T\in(0,T_1]$, *where* T_1 *is given in Lemma 2.3, and any* $\bar f\in X^T_{f_0}\cap B_{M_0}$ *with the regularity* $\bar f_i\in C^{\frac{k+\alpha}{4},k+\alpha}\left(D^T\right)$, $\forall\, i\in\{1,...,K\}$, $k\in\mathbb{N}$, $k\geq 4$, *there exists a unique solution* $f\in X^T_{f_0}\cap B_{M_0}$ *to the linear problem* (2.17) *fulfilling* $f_i\in C^{\frac{k+\alpha}{4},k+\alpha}\left(D^T\right)$. *Moreover, there exists a constant* $C_0=C_0(n,\delta_0)$ *such that*

$$\begin{aligned}\|f\|_{X^T_{f_0}}\leq &C_0\left(\sum_{i=1}^K\|G(\bar f_i)\|_{C^{\frac{\alpha}{4},\alpha}(D^T)}+\sum_{i=1}^K\left\|b\left(\bar f_i\right)_{|[0,T]\times\partial I}\right\|_{C^{\frac{3+\alpha}{4}}([0,T])}\right)\\ &+C_0\left(\sum_{i=1}^K\left(|p_{i-1}|+|p_i|\right)+\sum_{i=1}^K\|f_{0,i}\|_{C^{4,\alpha}(\bar I)}\right).\end{aligned}\tag{2.20}$$

Proof. Observe that the left-hand side of the fourth-order linear PDE in (2.17) can be presented as $\mathcal{L}(x,t,\partial_x,\partial_t)f^T$,

$$\mathcal{L}(x,t,\partial_x,\partial_t)=\mathrm{diag}\ (l_{kk})^{nK}_{k=1},$$

and $l_{kk}(x,t,\partial_x,\partial_t)=\partial_t+\frac{\partial_x^4}{|\partial_x f_{0,i}(x)|^4}$ as $k=(i-1)n+j$, $j\in\{1,...,n\}$, $i\in\{1,...,K\}$. Notice that in [26, page 8] $\mathcal{L}_0$ represents the principal part of $\mathcal{L}$. Since $\mathcal{L}$ coincide with its principal part, for simplicity, we work only with $\mathcal{L}$ avoiding the usage of the notation $\mathcal{L}_0$.

We associate to these differential operators the polynomials with coefficients depending on (t,x) with the replacement of ∂_x by $\mathrm{i}\xi,\xi\in\mathbb{R}$, $\mathrm{i}=\sqrt{-1}$, and ∂_t by $p\in\mathbb{C}$. Then

$$l_{kk}(x,t,\mathrm{i}\xi,p)=p+\frac{\xi^4}{|\partial_x f_{0,i}(x)|^4},$$

as $k=(i-1)n+j$, $j\in\{1,...,n\}$, $i\in\{1,...,K\}$. In particular, for any $\lambda\in\mathbb{R}$,

$$l_{kk}(x,t,\mathrm{i}\xi\lambda,p\lambda^4)=p\lambda^4+\frac{(\mathrm{i}\xi\lambda)^4}{|\partial_x f_{0,i}(x)|^4}=\lambda^4 l_{kk}(x,t,\mathrm{i}\xi,p).$$

Define

$$L(x,t,\mathrm{i}\xi,p):=\det\mathcal{L}(x,t,\mathrm{i}\xi,p)=\prod_{m=1}^{K}\left(p+\frac{\xi^4}{|\partial_x f_{0,m}(x)|^4}\right)^n, \tag{2.21}$$

hence $L(x,t,\mathrm{i}\xi\lambda,p\lambda^4)=\lambda^{4nK}L(x,t,\mathrm{i}\xi,p)$, see [26, Eq.(1.2)]. Let

$$\hat{\mathcal{L}}(x,t,\mathrm{i}\xi,p):=L(x,t,\mathrm{i}\xi,p)\mathcal{L}^{-1}(x,t,\mathrm{i}\xi,p)=\mathrm{diag}(A_{kk})_{k=1}^{nK}, \tag{2.22}$$

with

$$A_{kk}=A_{kk}(x,t,\mathrm{i}\xi,p)=\frac{\prod\limits_{m=1}^{K}\left(p+\frac{\xi^4}{|\partial_x f_{0,m}(x)|^4}\right)^n}{p+\frac{\xi^4}{|\partial_x f_{0,i}(x)|^4}},$$

as $k=(i-1)n+j$, $j\in\{1,...,n\}$, $i\in\{1,...,K\}$. Notice that $A_{(i-1)n+1,(i-1)n+1}=A_{(i-1)n+j,(i-1)n+j}$, $j\in\{1,...,n\}$, $i\in\{1,...,K\}$. For simplicity, denote by

$$A_i:=A_{(i-1)n+1,(i-1)n+1},\qquad \forall\, i\in\{1,...,K\}. \tag{2.23}$$

• **Parabolicity condition.** For any $\xi\in\mathbb{R}$ and from (2.21), we see that the roots of the polynomial $L(x,t,\mathrm{i}\xi,p)$ with respective to the variable p are given by

$$p=-\frac{\xi^4}{|\partial_x f_{0,i}(x)|^4},\qquad \forall\, i\in\{1,...,K\},$$

with multiplicity n. From (2.16), we have $p=-\frac{\xi^4}{|\partial_x f_{0,i}(x)|^4}\le-\delta_0^4\xi^4$, $\forall\, i\in\{1,...,K\}$. So the uniform parabolicity holds (see [26, page 8]).

• **Complementary conditions on the initial datum.** Let $f_i=(f_i^1,...,f_i^n)^T$, $f_{0,i}=(f_{0,i}^1,...,f_{0,i}^n)^T$, $i\in\{1,...,K\}$. Since the initial conditions are

$$f_i^j(0,x)=f_{0,i}^j(x),\quad j\in\{1,...,n\}, i\in\{1,...,K\},$$

the associated matrix is

$$\mathcal{C}_0(x,\partial_x,\partial_t)=Id_{nK\times nK}. \tag{2.24}$$

According to [26, page 12], we need to show that the rows of the matrix $\mathcal{D}(x,p)=\mathcal{C}_0(x,0,p)\cdot\hat{\mathcal{L}}_0(x,0,0,p)$ are linearly independent modulo p^{nK}. Taking (2.22) and (2.23) together and using (2.24), we have $\mathcal{D}(x,p)=$ diag $(p^{nK-1})\in\mathbb{R}^{nK\times nK}$. Hence, the rows of $\mathcal{D}(x,p)$ are linearly independent modulo p^{nK}.

• **The polynomial M^+.** From [26, page 11] we consider the polynomial M^+ as follows. Consider the polynomial $L=L(x,t,\mathrm{i}\xi,p)$ given in (2.21). As a function of ξ the polynomial L has $2nK$ roots with positive real parts and $2nK$ roots with negative real parts, if Re $p\ge0$ and $p\ne0$ (see [26, page 11]). From the assumption on p, we may write $p=|p|e^{\mathrm{i}\theta_p}$ with $-\frac{1}{2}\pi\le\theta_p\le\frac{1}{2}\pi$, $|p|\ne0$, and let $\xi_{i,1}(x^*,p)$ and $\xi_{i,2}(x^*,p)$ be the roots of $p+\frac{\xi^4}{|\partial_x f_{0,i}(x^*)|^4}=0$ with positive imaginary parts, namely,

$$\xi_{i,1}(x^*,p)=r_ie^{\mathrm{i}\left(\frac{\theta_p}{4}+\frac{\pi}{4}\right)},\quad \xi_{i,2}(x^*,p)=r_ie^{\mathrm{i}\left(\frac{\theta_p}{4}+\frac{3\pi}{4}\right)}, \tag{2.25}$$

with $r_i(x^*,p)=\sqrt[4]{|p|}\cdot|\partial_x f_{0,i}(x^*)|$, $\mathrm{i}=\sqrt{-1}$. Now we have

$$p+\frac{\xi^4}{|\partial_x f_{0,i}(x^*)|^4}=\frac{1}{|\partial_x f_{0,i}(x^*)|^4}(\xi-\xi_{i,1}(x^*,p))(\xi-\xi_{i,2}(x^*,p))(\xi-\xi_{i,3}(x^*,p))(\xi-\xi_{i,4}(x^*,p)),$$

where $\xi_{i,3}(x^*,p)$ and $\xi_{i,4}(x^*,p)$ are the roots of $p+\frac{\xi^4}{|\partial_x f_{0,i}(x^*)|^4}=0$ with negative imaginary parts, namely,

$$\xi_{i,3}(x^*,p)=r_i e^{\mathrm{i}\left(\frac{\theta_p}{4}+\frac{5\pi}{4}\right)},\quad \xi_{i,4}(x^*,p)=r_i e^{\mathrm{i}\left(\frac{\theta_p}{4}+\frac{7\pi}{4}\right)},\quad x^*\in\partial I, i\in\{1,...,K\}.$$

Since each root has multiplicity n, we let

$$M^+(x^*,\xi,p)=\prod_{i=1}^{K}\left(\xi-\xi_{i,1}(x^*,p)\right)^n\left(\xi-\xi_{i,2}(x^*,p)\right)^n.$$

• **Complementary conditions at the boundary points** $(t,x^*)\in[0,T]\times\partial I$**.** Let $f_i=(f_i^1,...,f_i^n)^T, p_i=(p_i^1,...,p_i^n)^T$, and $b(\bar{f}_i)(t,x^*)=(b^1(\bar{f}_i)(t,x^*),...,b^n(\bar{f}_i)(t,x^*))^T$, $(t,x^*)\in[0,T]\times\partial I$, where $b(f_i)$ are defined as in (2.12). The boundary conditions (2.11) can be rewriten as

$$\begin{aligned}&f_i^j(t,x^*)=p_{i-1+x^*}^j,\\&\partial_x f_i^j(t,x^*)=b^j(\bar{f}_i)(t,x^*),\end{aligned}$$

where $(t,x^*)\in[0,T]\times\partial I$, $j\in\{1,...,n\}$, $i\in\{1,...,K\}$. Thus, from [26, page 10], we have

$$\mathcal{B}(x,t,\partial_x,\partial_t)f^T(t,x^*)=(v(\bar{f})(t,x^*))^T,\qquad(t,x^*)\in[0,T]\times\partial I,$$

where

$$\mathcal{B}(x^*,t,\partial_x,\partial_t)=\begin{pmatrix}B&0&\cdots&0\\0&B&\cdots&0\\ \cdot&\cdot&\cdots&\cdot\\0&0&\cdots&B\end{pmatrix},$$

is a $2nK\times nK$ matrix,

$$B=B(x^*,t,\partial_x,\partial_t)=\begin{pmatrix}Id_{n\times n}\\Id_{n\times n}\partial_x\end{pmatrix},$$

and

$$v(\bar{f})(t,x^*)=\left(v(\bar{f}_1)(t,x^*),...,v(\bar{f}_K)(t,x^*)\right),$$

with

$$v(\bar{f}_i)(t,x^*)=\left(p_{i-1+x^*}^1,...,p_{i-1+x^*}^n,b^1(\bar{f}_i)(t,x^*),...,b^n(\bar{f}_i)(t,x^*)\right),\quad x^*\in\partial I, i\in\{1,...,K\}.$$

Note,

$$B(x^*,t,\mathrm{i}\xi,p)=\begin{pmatrix}Id_{n\times n}\\ \mathrm{i}\xi\cdot Id_{n\times n}\end{pmatrix}.$$

According to [26, page 11], we need to show that at $(t,x^*)\in[0,T]\times\partial I$, the rows of the matrix

$$\mathcal{A}(x^*,t,\mathrm{i}\xi,p)=\mathcal{B}(x^*,t,\mathrm{i}\xi,p)\hat{\mathcal{L}}(x^*,t,\mathrm{i}\xi,p)$$

are linearly independent modulo $M^+(x^*, \xi, p)$, if $\mathrm{Re}\{p\} \geq 0$, $p \neq 0$. Notice that $\mathcal{A}$ is a diagonal block-matrix, since $\mathcal{B}$ is a diagonal block-matrix and $\hat{\mathcal{L}}$ is a diagonal matrix. Hence, to obtain the linear independency for the rows of the matrix $\mathcal{A}$, it is sufficient to consider the different blocks separately. For simplicity we consider the first $2n$ rows. We consider the rows of the $2n \times n$ matrix, since we do not need to consider the columns, which are identically zero, namely,

$$B_1 = B(x^*, t, \mathrm{i}\xi, p) \cdot A_1(x^*, t, \mathrm{i}\xi, p) = \begin{pmatrix} A_1 Id_{n\times n} \\ \mathrm{i}\xi \cdot A_1 Id_{n\times n} \end{pmatrix}, \tag{2.26}$$

where A_1 is inferred from (2.23). To check the linear independence of the rows of matrix B_1 modulo M^+, we need to show that if $\omega = (\omega^1, ..., \omega^{2n}) \in \mathbb{R}^{2n}$ fulfills

$$\omega B_1(x^*, t, \mathrm{i}\xi, p) = 0 \quad \mod \quad M^+(x^*, \xi, p), \tag{2.27}$$

then $\omega = 0$.

We rewrite (2.27) as

$$\left(\omega^j + \mathrm{i}\xi\omega^{n+j}\right)\left(p + \frac{\xi^4}{|\partial_x f_{0,1}(x^*)|^4}\right)^{n-1} \prod_{i=2}^{K} \left(p + \frac{\xi^4}{|\partial_x f_{0,i}(x^*)|^4}\right)^n = 0 \quad \mod \quad M^+(x^*, \xi, p),$$

$\forall\, j \in \{1, ..., n\}$. Divide both sides of the above equation by

$$(\xi - \xi_{1,1}(x^*, p))^{n-1} (\xi - \xi_{1,2}(x^*, p))^{n-1} \prod_{i=2}^{K} (\xi - \xi_{i,1}(x^*, p))^n (\xi - \xi_{i,2}(x^*, p))^n,$$

we obtain

$$a_1(x^*, \xi, p)\left(\omega^j + \mathrm{i}\xi\omega^{n+j}\right) = 0 \quad \mod s_1(x^*, \xi, p),$$

$\forall\, j \in \{1, ..., n\}$, where

$$a_1(x^*, \xi, p) = (\xi - \xi_{1,3}(x^*, p))^{n-1} (\xi - \xi_{1,4}(x^*, p))^{n-1} \prod_{i=2}^{K} (\xi - \xi_{i,3}(x^*, p))^n (\xi - \xi_{i,4}(x^*, p))^n,$$
$$s_1(x^*, \xi, p) = (\xi - \xi_{1,1}(x^*, p))(\xi - \xi_{1,2}(x^*, p)).$$

We see that $s_1(x^*, \xi, p)$ divides $\omega^j + \mathrm{i}\xi\omega^{n+j}$, because $s_1(x^*, \xi, p)$ can't divide $a_1(x^*, \xi, p)$. Hence we obtain

$$\omega^j + \mathrm{i}\xi\omega^{n+j} = 0 \quad \mod \ (\xi - \xi_{1,1}(x^*, p))(\xi - \xi_{1,2}(x^*, p)),$$

$\forall\, j \in \{1, ..., n\}$. From (2.25), we have $\xi_{i,2} = \mathrm{i}\xi_{i,1}$, $\forall\, i \in \{1, ..., K\}$, where $\mathrm{i} = \sqrt{-1}$. Hence,

$$\begin{cases} \omega^j + \mathrm{i}\xi_{1,1}\omega^{n+j} = 0, \\ \omega^j - \xi_{1,1}\omega^{n+j} = 0, \end{cases}$$

$\forall\, j \in \{1, ..., n\}$. Thus, we have $\omega^j = \omega^{n+j} = 0$, $\forall\, j \in \{1, ..., n\}$, which imply $\omega = 0$. The same argument can be applied to other matrices $B_i = B(x^*, t, \mathrm{i}\xi, p) \cdot A_i(x^*, t, \mathrm{i}\xi, p)$, $i \in \{2, ..., K\}$. Therefore, we have verified the complementary conditions.

To finish the proof, it remains to verify the required assumptions as one applies Solonnikov's theorem stated in Lemma 5.14. Note that from above the parabolicity condition and the complementary conditions are all verified. Moreover, since $\bar{f} \in X_{f_0}^T \cap B_{M_0}$, we have $|\partial_x \bar{f}_i(t,x)| \geq \frac{1}{2}\delta_0$, $\forall\,(t,x) \in D^{T_1}$, $\forall\, i \in \{1,...,K\}$, from using Lemma 2.3. By the assumption on the regularity of the initial data f_0, we have $\bar{f} \in X^T_{f_0} \cap B_{M_0}$. By applying Lemmas 5.9 and 5.10, we have

$$\begin{cases} 1/|\partial_x \bar{f}_i|^4 \in C^{\frac{k-1+\alpha}{4},k-1+\alpha}(D^T), \\ G(\bar{f}_i) \in C^{\frac{k-4+\alpha}{4},k-4+\alpha}(D^T), \\ b(\bar{f}_i)(.,x^*) \in C^{\frac{k+2+\alpha}{4}}([0,T]), \qquad \forall\, x^* \in \partial I, \end{cases} \tag{2.28}$$

$\forall\, T \in (0, T_1]$, where $i \in \{1,...,K\}$ and T_1 is given in Lemma 2.3. Thus, the regularity of coefficients of the linear parabolic PDE in (2.17) is assured. Note that since f_0 satisfies the compatibility conditions of order $k-4$ at boundary in Definition 2.3, by applying Lemma 5.14, there exists a unique solution f to (2.17) with the regularity $f_i \in C^{\frac{k+\alpha}{4},k+\alpha}\left(D^T\right)$, $\forall\, i \in \{1,...,K\}$, $k \in \mathbb{N}$, $k \geq 4$.

□

2.2 The proof of Theorem 2.2

The proof of Theorem 2.2 is proceeded as follows. We associate the linear equation (2.17) to AP (2.11) for each $\bar{f} \in X^T_{f_0} \cap B_M$. By applying Theorem 2.4 to (2.17), we define the operator

$$\begin{aligned} \mathcal{G} : X^T_{f_0} \cap B_M &\to X^T_{f_0} \cap B_M \\ \bar{f} &\mapsto f, \end{aligned} \tag{2.29}$$

where f is the solution to (2.17). In Step 1°, we show that $\mathcal{G} : X^{t_0}_{f_0} \cap B_{M_0} \to X^{t_0}_{f_0} \cap B_{M_0}$ is well-defined and is a strict contraction-map for some $t_0 > 0$. Thus a fixed point f of this map is an AS (2.11) with the regularity $f_i \in C^{\frac{4+\alpha}{4},4+\alpha}(D^{t_0})$. In Step 2°, we prove that $f_i \in C^{\frac{5+\alpha}{4},5+\alpha}\left(D^{t_0}\right) \bigcap C^\infty\left((0,t_0] \times \bar{I}\right)$, $\forall\, i \in \{1,...,K\}$.

In the following, we assume $T \in (0, T_1)$, where T_1 and M_0 are given in Lemma 2.3. Without loss of generality, we assume $T_1 < 1$.

Note that as f_0 satisfies the compatibility conditions of order 0 in Definition 2.2, f_0 also satisfies that of order 0 in Definition 2.3. Then, for any $\bar{f} \in X^T_{f_0} \cap B_{M_0}$ there exists a unique solution $f \in X^T_{f_0} \cap B_{M_0}$ to (2.17). Moreover, we have (2.20).

Step 1° We show that $\mathcal{G} : X^T_{f_0} \cap B_{M_0} \to X^T_{f_0} \cap B_{M_0}$ is well-defined and a strict contraction-map for some properly chosen $T > 0$.

• **Self-maps. i.e., $\exists\ T_2 \in (0,T_1)$ such that $\mathcal{G}\left(X^T_{f_0} \cap B_{M_0}\right) \subset X^T_{f_0} \cap B_{M_0}$, $\forall\, T \in (0,T_2)$.**

From (2.20), (2.18), and by the triangle inequalities in Hölder spaces, with notice that $|p_{i-1}| + |p_i| \leq 2\|f_{0,i}\|_{C^{4,\alpha}(\bar{I})}$, we have

$$\begin{aligned} \sum_{i=1}^K \|f_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)} &\leq C_0 \sum_{i=1}^K \left(\|R(\bar{f}_i)\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} + \|h(\bar{f}_i) - h(f_{0,i})\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \right) \\ &+ C_0 \sum_{i=1}^K \left\| b\left(\bar{f}_i\right)_{|[0,T]\times\partial I} \right\|_{C^{\frac{3+\alpha}{4}}([0,T])} + C_0 \sum_{i=1}^K \left(\|h(f_{0,i})\|_{C^{0,\alpha}(\bar{I})} + 3\|f_{0,i}\|_{C^{4,\alpha}(\bar{I})} \right), \end{aligned} \tag{2.30}$$

$\forall\, i \in \{1,...,K\}$. As $T \in (0,T_1)$, we apply Lemmas 5.10, 5.13, and 5.9, with the notice of $\|\bar{f}_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)} \leq M_0$, $\forall\, i \in \{1,...,K\}$, to derive

$$\|R_i\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \leq C(n) \left\| \frac{1}{|\partial_x f_{0,i}|^4} - \frac{1}{|\partial_x \bar{f}_i|^4} \right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \left\|\partial_x^4 \bar{f}_i\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \leq C T^{\frac{\alpha}{4}}, \tag{2.31}$$

$\forall\, i \in \{1, ..., K\}$, where $C = C(n, \delta_0, M_0)$. Similarly, as $T \in (0, T_1)$, we apply Lemmas 5.9 $\sim$ 5.13 to derive

$$\|h(\bar{f}_i) - h(f_{0,i})\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \leq C \sum_{k=1}^{3} \left\|\partial_x^k \bar{f}_i - \partial_x^k f_{0,i}\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)}$$

$$+ C \sum_{k=1}^{4} \left\| \frac{1}{|\partial_x f_{0,i}|^{2k}} - \frac{1}{|\partial_x \bar{f}_i|^{2k}} \right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \leq C T^{\frac{\alpha}{4}}, \quad \forall\, i \in \{1, ..., K\}, \tag{2.32}$$

where $C = C(n, \delta_0, \lambda, M_0)$.

Next, we estimate $\left\|b(\bar{f}_i)(\cdot, x^*)\right\|_{C^{\frac{3+\alpha}{4}}([0,T])}$, where $x^* \in \{0, 1\}$, $i \in \{1, ..., K\}$. Observe from (2.12) that, as $(i, x^*) \in \{(1, 0), (K, 1)\}$,

$$\left\|b(\bar{f}_i)(\cdot, x^*)\right\|_{C^{\frac{3+\alpha}{4}}([0,T])} = |\partial_x f_{0,i}(x^*)| \leq \|f_{0,i}\|_{C^{4,\alpha}(\bar{I})}, \tag{2.33}$$

while $(i, x^*) \in \{1, ..., K\} \times \{0, 1\} \backslash \{(1, 0), (K, 1)\}$,

$$\partial_t b(\bar{f}_i)(t, x^*) = [\Delta_{i-1+x^*}(\delta^2 f)](t) = \frac{\partial_x^2 \bar{f}_{i+x^*}(t, 0)}{|\partial_x \bar{f}_{i+x^*}(t, 0)|} - \frac{\partial_x^2 \bar{f}_{i-1+x^*}(t, 1)}{|\partial_x \bar{f}_{i-1+x^*}(t, 1)|}. \tag{2.34}$$

From (2.12) and by applying the triangle inequality, we obtain

$$\left\|b(\bar{f}_i)(\cdot, x^*)\right\|_{C^{\frac{3+\alpha}{4}}([0,T])} \leq \|f_{0,i-1+2x^*}\|_{C^{4,\alpha}(\bar{I})} + \left\| \int_0^t \partial_t b(\bar{f}_i)(\tau, x^*)\, d\tau \right\|_{C^{\frac{3+\alpha}{4}}([0,T])},$$

$\forall\, (i, x^*) \in \{1, ..., K\} \times \{0, 1\} \backslash \{(1, 0), (K, 1)\}$. From (2.16) and (2.34), we have

$$\sup_{t \in [0,T]} \left|\partial_t b(\bar{f}_i)(t, x^*)\right| \leq \frac{2M_0}{\delta_0}, \qquad \forall\, (i, x^*) \in \{1, ..., K\} \times \{0, 1\} \backslash \{(1, 0), (K, 1)\}.$$

Hence,

$$\left\| \int_0^t \partial_t b(\bar{f}_i)(\tau, x^*)\, d\tau \right\|_{C^{\frac{3+\alpha}{4}}([0,T])} = \sup_{t \in [0,T]} \left| \int_0^t \partial_t b(\bar{f}_i)(\tau, x^*)\, d\tau \right|$$

$$+ \sup_{t,t' \in [0,T]} \frac{\left| \int_{t'}^t \partial_t b(\bar{f}_i)(\tau, x^*)\, d\tau \right|}{|t - t'|^{\frac{3+\alpha}{4}}} \leq \frac{2M_0}{\delta_0} \cdot (T + T^{\frac{1-\alpha}{4}}) \leq \frac{4M_0}{\delta_0} \cdot T^{\frac{1-\alpha}{4}},$$

$\forall\, (i, x^*) \in \{1, ..., K\} \times \{0, 1\} \backslash \{(1, 0), (K, 1)\}$, where the last inequality comes from applying $T < T_1 < 1$. Now, we have

$$\left\|b(\bar{f}_i)(\cdot, x^*)\right\|_{C^{\frac{3+\alpha}{4}}([0,T])} \leq \frac{4M_0}{\delta_0} \cdot T^{\frac{1-\alpha}{4}} + \|f_{0,i-1+x^*}\|_{C^{4,\alpha}(\bar{I})}, \tag{2.35}$$

$\forall\, (i, x^*) \in \{1, ..., K\} \times \{0, 1\} \backslash \{(1, 0), (K, 1)\}$. Now, we conclude from (2.30) $\sim$ (2.33) and (2.35) that

$$\|f\|_{X_{f_0}^T} \leq C_0 \sum_{i=1}^{K} \left(\|h(f_{0,i})\|_{C^{0,\alpha}(\bar{I})} + 5\|f_{0,i}\|_{C^{4,\alpha}(\bar{I})}\right) + \tilde{C}_0 T^{\beta},$$

where $\beta = \min\{\frac{1-\alpha}{4}, \frac{\alpha}{4}\}$ and $\tilde{C}_0 = \tilde{C}_0(n, \delta_0, \lambda, M_0)$. Therefore,

$$\|f\|_{X^T_{f_0}} \leq C_0 \sum_{i=1}^{K} \left(\|h(f_{0,i})\|_{C^{0,\alpha}(\bar{I})} + 5\|f_{0,i}\|_{C^{4,\alpha}(\bar{I})} \right) + \tilde{C}_0 T^\beta, \tag{2.36}$$

where $C_0 = C_0(n, \delta_0)$ and $\tilde{C}_0 = \tilde{C}_0(n, \delta_0, \lambda, M_0)$ are universal constants. By applying Lemma 5.10, (2.16), and Lemma 5.9, we have

$$\begin{aligned}\sum_{i=1}^{K} \|h(f_{0,i})\|_{C^{0,\alpha}(\bar{I})} \leq & C_0(n, \delta_0) \sum_{i=1}^{K} \left(\|f_{0,i}\|^3_{C^{4,\alpha}(\bar{I})} + \|f_{0,i}\|^7_{C^{4,\alpha}(\bar{I})} + \|f_{0,i}\|^9_{C^{4,\alpha}(\bar{I})} \right) \\ \leq & C_0(n, \delta_0) \sum_{j=1}^{4} \left(\sum_{i=1}^{K} \|f_{0,i}\|_{C^{4,\alpha}(\bar{I})} \right)^{2j+1}.\end{aligned}$$

Hence

$$\sum_{i=1}^{K} \|f_i\|_{C^{\frac{4+\alpha}{4}, 4+\alpha}(D^T)} \leq C_0 \sum_{j=0}^{4} \left(\sum_{i=1}^{K} \|f_{0,i}\|_{C^{4,\alpha}(\bar{I})} \right)^{2j+1} + \tilde{C}_0 T^\beta,$$

where $C_0 = C_0(n, \delta_0) > 1$, is a sufficiently large constant and $\tilde{C}_0 = \tilde{C}_0(n, \delta_0, \lambda, M_0)$ are universal constants. Now, we conclude that $\|f\|_{X^{T_2}_{f_0}} \leq M_0$ by choosing $M_0 \in (0, \infty)$ so that

$$\frac{M_0}{2} = C_0 \sum_{j=0}^{4} \left(\sum_{i=1}^{K} \|f_{0,i}\|_{C^{4,\alpha}(\bar{I})} \right)^{2j+1} \tag{2.37}$$

and $T_2 \in (0, T_1)$ so that

$$\tilde{C}_0 T_2^\beta \leq \frac{M_0}{2}.$$

In other words, we obtain the self-map property, i.e.,

$$\mathcal{G}\left(X^T_{f_0} \cap B_{M_0} \right) \subset X^T_{f_0} \cap B_{M_0}, \qquad \forall\, T \in (0, T_2].$$

• **Contraction-maps.** We show that $\mathcal{G}$ is a contradiction-map, i.e., with $\bar{f}, \bar{g} \in X^T_{f_0} \cap B_{M_0}$ and $f = \mathcal{G}(\bar{f})$, $g = \mathcal{G}(\bar{g})$, there exists $T > 0$ such that

$$\|f - g\|_{X^T_{f_0}} \leq C\, T^\beta \left\| \bar{f} - \bar{g} \right\|_{X^T_{f_0}}, \tag{2.38}$$

where $\beta \in (0, 1)$ and $C = C(n, \delta_0, \lambda, M_0)$.

Observe that $f - g$ fulfills

$$\begin{cases} \partial_t (f_i - g_i) + \frac{\partial_x^4 (f_i - g_i)}{|\partial_x f_{0,i}|^4} = G(\bar{f}_i) - G(\bar{g}_i), & \text{in } (0,T) \times I, \\ (f_i - g_i)(t, x^*) = 0, & \forall\, (t, x^*) \in [0,T] \times \partial I, \\ \partial_x (f_i - g_i)(t, x^*) = b(\bar{f}_i)(t, x^*) - b(\bar{g}_i)(t, x^*), & \forall\, (t, x^*) \in [0,T] \times \partial I, \\ (f - g)_i(0, x) = 0, & \forall\, x \in \bar{I}, \end{cases}$$

where $i \in \{1, ..., K\}$.

By the same argument in §2.1, the linear problem is well-posed and the assumptions on the regularity of coefficients are satisfied. Since $\bar{f} = \bar{g}$ at $t = 0$ we see that the zero initial datum satisfies the compatibility condition of order zero. From applying Lemma 5.14, $f-g$ is the unique solution to the linear equation, and

$$\begin{aligned}\sum_{i=1}^{K}\|f_i-g_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)} \leq & C_0(n,\delta_0)\sum_{i=1}^{K}\left\|G(\bar f_i)-G(\bar g_i)\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \\ & + C_0(n,\delta_0)\sum_{i=1}^{K}\left\|\left(b(\bar f_i)-b(\bar g_i)\right)_{|[0,T]\times\partial I}\right\|_{C^{\frac{3+\alpha}{4}}([0,T])}.\end{aligned} \tag{2.39}$$

To obtain (2.38), we need to estimate the terms on the right-hand side of (2.39), $\forall\, i\in\{1,...,K\}$.

Note that

$$G(\bar f_i)-G(\bar g_i)=\left(R(\bar f_i)-R(\bar g_i)\right)+\left(h(\bar f_i)-h(\bar g_i)\right).$$

By applying the triangle inequality in Hölder spaces, Lemmas 5.10, 5.13, and 5.9, we find that

$$\begin{aligned}&\left\|R(\bar f_i)-R(\bar g_i)\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \leq C(n)\left\|\frac{1}{|\partial_x f_0|^4}-\frac{1}{|\partial_x \bar f_i|^4}\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)}\left\|\partial_x^4\bar f_i-\partial_x^4\bar g_i\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \\ &+C(n)\left\|\frac{1}{|\partial_x\bar g_i|^4}-\frac{1}{|\partial_x\bar f_i|^4}\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)}\left\|\partial_x^4\bar g_i\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \leq CT^{\frac{\alpha}{4}}\left\|\bar f_i-\bar g_i\right\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)},\end{aligned} \tag{2.40}$$

$\forall\, i\in\{1,...,K\}$, where $C=C(n,\delta_0,M_0)$.

By applying Lemmas 5.9 $\sim$ 5.13 and by noticing $\bar f_i=\bar g_i$ at $t=0$, we have

$$\begin{aligned}&\left\|h(\bar f_i)-h(\bar g_i)\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \leq C\sum_{k=1}^{3}\left\|\partial_x^k\bar f_i-\partial_x^k\bar g_i\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \\ &+C\sum_{k=1}^{4}\left\|\frac{1}{|\partial_x\bar g_i|^{2k}}-\frac{1}{|\partial_x\bar f_i|^{2k}}\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \leq CT^{\frac{\alpha}{4}}\left\|\bar f_i-\bar g_i\right\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)},\quad \forall\, i\in\{1,...,K\},\end{aligned} \tag{2.41}$$

where $C=C(n,\delta_0,\lambda,M_0)$.

Now, we estimate the boundary terms, $\left\|b(\bar f_i)(\cdot,x^*)-b(\bar g_i)(\cdot,x^*)\right\|_{C^{\frac{3+\alpha}{4}}([0,T])}$, where $x^*\in\{0,1\}$, $i\in\{1,...,K\}$. We observe that

$$b(\bar f_1)(\cdot,0)-b(\bar g_1)(\cdot,0)=b(\bar f_K)(\cdot,1)-b(\bar g_K)(\cdot,1)=0, \tag{2.42}$$

while $(i,x^*)\in\{1,...,K\}\times\{0,1\}\backslash\{(1,0),(K,1)\}$,

$$\begin{aligned}\partial_t b(f_i)(t,x^*)-\partial_t b(g_i)(t,x^*)=&\left[\frac{\partial_x^2\bar f_{i+x^*}(t,0)}{|\partial_x\bar f_{i+x^*}(t,0)|}-\frac{\partial_x^2\bar g_{i+x^*}(t,0)}{|\partial_x\bar g_{i+x^*}(t,0)|}\right] \\ &-\left[\frac{\partial_x^2\bar f_{i-1+x^*}(t,1)}{|\partial_x\bar f_{i-1+x^*}(t,1)|}-\frac{\partial_x^2\bar g_{i-1+x^*}(t,1)}{|\partial_x\bar g_{i-1+x^*}(t,1)|}\right].\end{aligned} \tag{2.43}$$

As $(j,x^*)\in\{1,...,K\}\times\{0,1\}\backslash\{(1,0),(K,1)\}$, we apply the triangle inequality, Lemmas 5.9, 2.3, and obtain

$$\begin{aligned}&\left\|\frac{\partial_x^2\bar f_j(t,x^*)}{|\partial_x\bar f_j(t,x^*)|}-\frac{\partial_x^2\bar g_j(t,x^*)}{|\partial_x\bar g_j(t,x^*)|}\right\|_{C^0([0,T])} \leq \left\|\frac{\partial_x^2\bar f_j(t,x^*)-\partial_x^2\bar g_j(t,x^*)}{|\partial_x\bar f_j(t,x^*)|}\right\|_{C^0([0,T])} \\ &+\left\|\partial_x^2\bar g_j(t,x^*)\left[\frac{1}{|\partial_x\bar f_j(t,x^*)|}-\frac{1}{|\partial_x\bar g_j(t,x^*)|}\right]\right\|_{C^0([0,T])} \leq C(\delta_0,M_0)\|\bar f_j-\bar g_j\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)},\end{aligned}$$

where $\bar{f}, \bar{g} \in X^T_{f_0} \cap B_{M_0}$. Thus, we have

$$\begin{aligned} &\|\partial_t b(\bar{f}_i)(t,x^*) - \partial_t b(\bar{g}_i)(t,x^*)\|_{C^0([0,T])} \\ &\leq C\left(\|\bar{f}_{i-1+x^*} - \bar{g}_{i-1+x^*}\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)} + \|\bar{f}_{i+x^*} - \bar{g}_{i+x^*}\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)}\right), \end{aligned}$$

$\forall\, (i,x^*) \in \{1,...,K\} \times \{0,1\}\backslash\{(1,0),(K,1)\}$, where $C = C(\delta_0, M_0)$. Hence,

$$\begin{aligned} &\left\|b(\bar{f}_i)(\cdot,x^*) - b(\bar{g}_i)(\cdot,x^*)\right\|_{C^{\frac{3+\alpha}{4}}([0,T])} = \sup_{t\in[0,T]}\left|\int_0^t (\partial_t b(\bar{f}_i)(\tau,x^*) - \partial_t b(\bar{g}_i)(\tau,x^*))d\tau\right| \\ &+ \sup_{t,t'\in[0,T]} \frac{\left|\int_{t'}^t (\partial_t b(\bar{f}_i)(\tau,x^*) - \partial_t b(\bar{g}_i)(\tau,x^*))d\tau\right|}{|t-t'|^{\frac{3+\alpha}{4}}} \\ &\leq (T + T^{\frac{1-\alpha}{4}}) \cdot C \cdot \left(\|\bar{f}_{i-1+x^*} - \bar{g}_{i-1+x^*}\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)} + \|\bar{f}_{i+x^*} - \bar{g}_{i+x^*}\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)}\right), \end{aligned}$$

where $C = C(\delta_0, M_0)$. Now, we derive

$$\sum_{i=1}^{K-1} \left\|b(\bar{f}_i)(\cdot,1) - b(\bar{g}_i)(\cdot,1)\right\|_{C^{\frac{3+\alpha}{4}}([0,T])} \leq (T + T^{\frac{1-\alpha}{4}}) \cdot C \sum_{i=1}^{K} \|\bar{f}_i - \bar{g}_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)}, \tag{2.44}$$

$$\sum_{i=2}^{K} \left\|b(\bar{f}_i)(\cdot,0) - b(\bar{g}_i)(\cdot,0)\right\|_{C^{\frac{3+\alpha}{4}}([0,T])} \leq (T + T^{\frac{1-\alpha}{4}}) \cdot C \cdot \sum_{i=1}^{K} \|\bar{f}_i - \bar{g}_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)}, \tag{2.45}$$

where $C = C(\delta_0, M_0)$. From (2.39)∼(2.42), (2.44), (2.45) and the choice of $T \in (0, T_2) \subset (0,1)$, we obtain

$$\sum_{i=1}^{K} \|f_i - g_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)} \leq C \cdot T^\beta \cdot \sum_{i=1}^{K} \|\bar{f}_i - \bar{g}_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)}, \tag{2.46}$$

where $\beta = \min\{\frac{1-\alpha}{4}, \frac{\alpha}{4}\}$ and $C = C(n, \delta_0, \lambda, M_0)$. Thus, (2.38) is obtained. By choosing $T_3 \in (0, T_2)$ such that $CT_3^\beta < 1$, we conclude that

$$\mathcal{G} : X^{t_0}_{f_0} \cap B_{M_0} \to X^{t_0}_{f_0} \cap B_{M_0}$$

is a self-map and also a strict contraction-map, $\forall\, t_0 \in (0, T_3]$.

We may let $t_0 = T_3(n, \delta_0, \lambda, M_0)$. By applying Banach fixed point theorem, there exists a *unique* fixed point $f \in X^{t_0}_{f_0} \cap B_{M_0}$ such that f is a solution to (2.11).

Step 2° In this step, we follow the approach in [6, Theorem 3.6 or Theorem 2.3] to show higher regularity of the analytical solutions obtained in Step 1°, i.e., $f_i \in C^{\frac{5+\alpha}{4},5+\alpha}(D^{t_0})$, and $f_i \in C^\infty\left((0,t_0] \times \bar{I}\right)$, $\forall\, i \in \{1,...,K\}$.

• $C^{\frac{5+\alpha}{4},5+\alpha}(D^{t_0})$**-smoothness.**

From the assumption, $f_0 = (f_{0,1},...,f_{0,K})$, $f_{0,i} \in C^{5,\alpha}(\bar{I})$ satisfies the compatibility conditions of order 1, defined in Definition 2.2. Note that, from Step 1°, we have $f_i \in C^{\frac{4+\alpha}{4},4+\alpha}(D^{t_0})$, $\forall\, i \in \{1,...,K\}$. Denote by $d_i = \frac{1}{|\partial_x f_i|^4} \in C^{\frac{3+\alpha}{4},3+\alpha}(D^{t_0})$ and $g_i :=$

$h(f_i) \in C^{\frac{1+\alpha}{4},1+\alpha}(D^{t_0})$, $\forall\, i \in \{1, ..., K\}$. Moreover, $b(f_i)(\cdot, x^*) \in C^{\frac{6+\alpha}{4}}([0, t_0])$, $\forall\, x^* \in \partial I$, $i \in \{1, ..., K\}$. Observe that f solves the linear parabolic PDE,

$$\begin{cases} \partial_t f_i = -d_i \cdot \partial_x^4 f_i + g_i & \text{in } (0,t_0) \times I, \\ f_i(t,x^*) = p_{i-1+x^*}, & \forall\,(t,x^*) \in [0,t_0] \times \partial I, \\ \partial_x f_i(t,x^*) = b(f_i)(t,x^*), & \forall\,(t,x^*) \in [0,t_0] \times \partial I, \\ f_i(0,x) = f_{0,i}(x), & \forall\, x \in \bar{I}, \end{cases}$$

where $i \in \{1, ..., K\}$.

Note that this is a linear parabolic PDE and the complementary conditions at the boundary are satisfied. By applying Solonnikov's theorem stated in Lemma 5.14, we conclude that $f_i \in C^{\frac{5+\alpha}{4},5+\alpha}(D^{t_0}), \forall\, i \in \{1, ..., K\}$.

• $C^\infty\left((0, t_0] \times \bar{I}\right)$**-smoothness.**

Given any $\varepsilon \in (0, t_0)$, let $\zeta = f\phi = (f_1\phi, ..., f_K\phi)$, where $\phi : [0, t_0] \to [0, 1]$, is a smooth cut-off function with $\phi(t) = 0$, as $0 \le t \le \frac{1}{4}\varepsilon$, and $\phi(t) = 1$, as $\frac{1}{2}\varepsilon \le t \le t_0$. Since $f_i \in C^{\frac{5+\alpha}{4},5+\alpha}(D^{t_0})$, we have $\zeta_i = f_i\phi \in C^{\frac{5+\alpha}{4},5+\alpha}(D^{t_0})$, $\forall\, i \in \{1, ..., K\}$. Moreover, ζ satisfies the linear parabolic equation,

$$\begin{cases} \partial_t \zeta_i = -\frac{1}{|\partial_x f_i|^4} \partial_x^4 \zeta_i + \phi \cdot h(f_i) + f_i \cdot \frac{d}{dt}\phi, & \text{in } (0,t_0) \times I, \\ \zeta_i(t,x^*) = p_{i-1+x^*}\phi(t), & \forall\,(t,x^*) \in [0,t_0] \times \partial I, \\ \partial_x \zeta_i(t,x^*) = \phi(t) b(f_i)(t,x^*), & \forall\,(t,x^*) \in [0,t_0] \times \partial I, \\ \zeta_i(0,x) = \zeta_{0,i}(x) := 0, & \forall\, x \in \bar{I}, \end{cases} \tag{2.47}$$

where $i \in \{1, ..., K\}$.

Notice that ζ_0 satisfies the compatibility conditions of any order, given in Definition 2.3. The parabolicity condition and the complementary conditions can be verified from applying the same argument in §2.1. Let $e_i := \phi \cdot h(f_i) + f_i \cdot \frac{d}{dt}\phi$, $\forall\, i \in \{1, ..., K\}$. Then, by applying Lemmas 5.9 and 5.10, we have

$$\begin{cases} -\frac{1}{|\partial_x f_i|^4} \in C^{\frac{4+\alpha}{4},4+\alpha}(D^{t_0}), \\ e_i \in C^{\frac{2+\alpha}{4},2+\alpha}(D^{t_0}), \\ p_{i-1+x^*}\phi(t) \in C^\infty\left([0,t_0]\right), & \forall\, x^* \in \partial I, \\ \phi(\cdot) b(f_i)(\cdot,x^*) \in C^{\frac{7+\alpha}{4}}\left([0,t_0]\right), & \forall\, x^* \in \partial I, \end{cases}$$

where $i \in \{1, ..., K\}$.

From applying Lemma 5.14, we have $\zeta_i \in C^{\frac{6+\alpha}{4},6+\alpha}(D^{t_0})$, $\forall\, i \in \{1, ..., K\}$, which give $f_i \in C^{\frac{6+\alpha}{4},6+\alpha}\left([\frac{\varepsilon}{2}, t_0] \times \bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$. By repeating the procedure, we obtain

$$f_i \in C^{\frac{5+\alpha}{4},5+\alpha}(D^{t_0}) \bigcap \bigcap_{k=1}^{\infty} C^{\frac{5+k+\alpha}{4},5+k+\alpha}\left(\left[\frac{2k-1}{2k}\varepsilon, t_0\right] \times \bar{I}\right), \quad \forall\, i \in \{1, ..., K\}.$$

For further details on this procedure, the reader is referred to [8, App.B.2.3]. Hence, $f_i \in C^\infty\left([\varepsilon, t_0] \times \bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$. Since $\varepsilon > 0$ can be chosen arbitrarily small, we have concluded that $f_i \in C^\infty\left((0, t_0] \times \bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$.

• **Uniqueness.** Since the (local) solutions are fixed points of contraction maps, uniqueness is the natural consequence. For the details, the reader is also referred to [6, Theorem 2.3] for the same argument.

Now we have finished the proof.

3 Converting solutions and diffeomorphisms

In this section we establish two lemmas concerning how to convert GS to SGS or AS by a family of diffeomorphisms. The first lemma provides the existence of diffeomorphisms, converting a geometric solution (GS) into a special geometric solution (SGS), while the second lemma provides the existence of diffeomorphisms, converting a SGS into an analytical solution (AS).

Lemma 3.1 (Converting GS into SGS)**.** *Let $f = (f_1, ..., f_K)$, $f_i : D^{T_0} \to \mathbb{R}^n$ be a geometric solution (GS), i.e., a solution to (1.4)∼(1.8), fulfilling $f_i \in C^{\frac{k+\alpha}{4}, k+\alpha}(D^{T_0})$, $\forall\, i \in \{1, ..., K\}$, $k \in \mathbb{N}$, $k \geq 8$. Suppose that there exist $M_0 > 0$ and $\delta_0 > 0$ such that the tangent component $\varphi_i = \langle \partial_t f_i, \partial_s f_i \rangle$, $i \in \{1, ..., K\}$, fulfills*

$$\begin{cases} |\varphi_i(t,y)| \leq M_0, & |\partial_y \varphi_i(t,y)| \leq M_0, \quad |\partial_t \varphi_i(t,y)| \leq M_0, \\ \delta_0 \leq |\partial_y f_i(t,y)| \leq M_0, & |\partial_y^2 f_i(t,y)| \leq M_0, \quad |\partial_t \partial_y f_i(t,y)| \leq M_0, \end{cases} \tag{3.1}$$

for all $(t,y) \in D^{T_0}$.

Then, there exist $\tilde{T}_0 = \tilde{T}_0(\delta_0, M_0) \in (0, T_0)$ and $\sigma_i \in C^{\frac{k-4+\alpha}{4}, k-4+\alpha}\left(D^{\tilde{T}_0}\right)$, such that $\sigma_i(t,\cdot) : \bar{I} \to \bar{I}$ is a family of diffeomorphisms, $\forall\, t \in [0, \tilde{T}_0]$, $i \in \{1, ..., K\}$, and $g = (g_1, ..., g_K)$, $g_i(t,z) = f_i(t, \sigma_i(t,z))$, consist a special geometric solution (SGS), i.e., a solution to (1.5)∼(1.9), fulfilling $g_i \in C^{\frac{k-4+\alpha}{4}, k-4+\alpha}(D^{\tilde{T}_0})$, $\forall i \in \{1, ..., K\}$.

Proof. To convert a GS to a SGS, we may apply the formula (C12) in [5, Lemma C.4] by letting the tangential components $\varphi_i^{II} = 0$ and $\varphi_i^I = \varphi_i$. The computation therein shows that we need to find a family of diffeomorphisms $\sigma_i(t,\cdot) : \bar{I} \to \bar{I}$, $\forall\, t \in [0, \tilde{T}_0]$, for some $\tilde{T}_0 > 0$, such that $g = (g_1, ..., g_K)$, with

$$g_i(t,z) = f_i(t, \sigma_i(t,z)), \tag{3.2}$$

being a SGS, where $\sigma_i(t,\cdot)$ is a family of diffeomorphisms fulfilling

$$\begin{cases} \partial_t \sigma_i(t,z) = \frac{-1}{|\partial_y f_i(t,\sigma_i(t,z))|} \varphi_i(t, \sigma_i(t,z)), \ (t,z) \in D^{\tilde{T}_0}, \\ \sigma_i(0,z) = z, \quad z \in \bar{I}, \end{cases} \tag{3.3}$$

$\forall\, i \in \{1, ..., K\}$. Note that from (3.2), (3.3) and the assumption that $f = (f_1, ..., f_K)$ is a GS, we have

$$\begin{aligned} \partial_t g_i(t,z) &= \partial_t f_i(t, \sigma_i(t,z)) + \partial_y f_i(t, \sigma_i(t,z)) \partial_t \sigma_i(t,z) \\ &= \left[-\nabla_s^2 \vec{\kappa}_i - \frac{1}{2} |\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda \vec{\kappa}_i \right] (t, \sigma_i(t,z)) + \varphi_i(t, \sigma_i(t,z)) \frac{\partial_y f_i(t, \sigma_i(t,z))}{|\partial_y f_i(t, \sigma_i(t,z))|} \\ &\quad + \frac{-\partial_y f_i(t, \sigma_i(t,z))}{|\partial_y f_i(t, \sigma_i(t,z))|} \varphi_i(t, \sigma_i(t,z)) \\ &= \left[-\nabla_s^2 \vec{\kappa}_i - \frac{1}{2} |\vec{\kappa}_i|^2 \vec{\kappa}_i + \lambda \vec{\kappa}_i \right] (t, \sigma_i(t,z)). \end{aligned}$$

Hence $g = (g_1, ..., g_K)$ is indeed a SGS, if such diffeomorphisms exist.

Below we discuss the existence and regularity of solutions to (3.3). Let

$$\theta_i : D^{T_0} \to \mathbb{R}, \quad \theta_i(t,y) = \frac{-1}{|\partial_y f_i(t,y)|} \varphi_i(t,y),$$

be the tangential component of $\partial_t f_i$, i.e., the term on the right-hand side of (3.3). Note that $f_i \in C^{\frac{k+\alpha}{4},k+\alpha}\left(D^{T_0}\right)$ implies

$$\theta_i \in C^{\frac{k-4+\alpha}{4},k-4+\alpha}\left(D^{T_0}\right), \tag{3.4}$$

and $\partial_t f(t,y^*) = 0$, $\forall\,(t,y^*) \in [0,T_0]\times\partial I$, implies

$$\theta_i(t,y^*) = 0, \qquad \forall\,(t,y^*) \in [0,T_0]\times\partial I. \tag{3.5}$$

To solve (3.3) by ODE theory on an open set, we apply Whitney's extension theorem, e.g., see [11, §3.1.3, Theorem 5] for C^1-extension. From (3.1) and Whitney's extension theorem, there exists a C^1-function $\Theta_i : \mathbb{R}^2 \to \mathbb{R}$ such that $\Theta_i = \theta_i$ on D^{T_0} with the Lipschitz constant $L_0 := L_0(\delta_0, M_0)$, and Θ_i fulfills $||\Theta_i||_{C^0(\mathbb{R}^2)} \le C||\theta_i||_{C^0(D^{T_0})} \le C_0(\delta_0, M_0)$, for some constant $C > 0$. Then, a solution to the initial value problem

$$\partial_t\sigma_i(t,z) = \Theta_i(t,\sigma_i(t,z)) \quad \sigma_i(0,z) = z,$$

restricted to $z \in \bar{I}$, is the solution to (3.3). Notice that $\Theta_i \in C^1$, for each fixed $z_0 \in \mathbb{R}$, by apply [27, Sections 1.2 and 1.6], there exist an open set $(z_0 - r_0, z_0 + r_0)$, where $r_0 = r_0(z_0) > 0$, and $T_1 := T_1(L_0, C_0, z_0) = T_1(\delta_0, M_0, z_0) \in (0, T_0)$ such that for any $z \in (z_0 - r_0, z_0 + r_0)$ there is $u_i(\cdot, z)$ solves the initial value problem,

$$\partial_t u_i(t,z) = \Theta_i(t,u_i(t,z)), \quad u_i(0,z) = z, \tag{3.6}$$

on $(-T_1, T_1)$. Moreover, the family of solutions is continuously differentiable in t and z. Let $K \supset\supset \bar{I}$ be a compact set in $\mathbb{R}$. Since K is a compact set, there exists a finite $\{z_j\}_{j\in\{1,\dots,N\}} \subset K$, and $r_j := r_j(z_j) > 0$, $\forall\, j \in \{1, ..., N\}$, such that $K \subset \bigcup\limits_{j=1}^{N} (z_j - r_j, z_j + r_j)$. Define $\tilde{T}_0 = \tilde{T}_0(\delta_0, M_0) := \frac{1}{2} \underset{j\in\{1,\dots,N\}}{inf} T_1(\delta_0, M_0, z_j) \in (0, T_0)$. Note that $u_i(\cdot, z)$ is defined on $[-\tilde{T}_0, \tilde{T}_0]$, $\forall\, z \in K$. Thus, there exists a $\tilde{T}_0 = \tilde{T}_0(\delta_0, M_0) > 0$ such that the map

$$\begin{aligned}\sigma_i : [-\tilde{T}_0, \tilde{T}_0] \times K &\to \mathbb{R}, \\ (t,z) &\mapsto u_i(t,z)\end{aligned}$$

is well-defined and C^1. From (3.5), $\sigma_i(t,x^*) = x^*$, $\forall\,(t,x^*) \in [0,\tilde{T}_0]\times\partial I$. By the uniqueness of the solution to (3.6) and by choosing $\sigma_i = u_i$, we have $\sigma_i : D^{\tilde{T}_0} \to \bar{I}$, and

$$\sigma_i(t,z) = z + \int_0^t \theta_i(\tau, \sigma_i(\tau,z))\,d\tau.$$

By differentiating σ_i w.r.t. variable z, we have

$$\partial_x\sigma_i(t,z) = 1 + \int_0^t \partial_y\theta_i(\tau,\sigma_i(\tau,z)) \cdot \partial_z\sigma_i(\tau,z)\,d\tau. \tag{3.7}$$

Since the right-hand side of (3.7) is continuously differentiable w.r.t. variable t, we obtain

$$\partial_t\partial_z\sigma_i(t,z) = \partial_y\theta_i(t,\sigma_i(t,z)) \cdot \partial_z\sigma_i(t,z). \tag{3.8}$$

Integration of (3.8) and the condition $\partial_z\sigma_i(0,z) = 1$ imply

$$\partial_z\sigma_i(t,z) = e^{\int_0^t \partial_y\theta_i(\tau,\sigma_i(\tau,z))\,d\tau}, \forall\,(t,z) \in D^{\tilde{T}_0} \tag{3.9}$$

Notice that, from (3.9), (3.1) and definition of θ_i, we have

$$\partial_z \sigma_i(t,z) \ge e^{-C(\delta_0,M_0)\cdot\tilde{T}_0} > 0, \qquad \forall\,(t,z) \in D^{\tilde{T}_0}.$$

Hence $\sigma_i(t,\cdot)$ is a family of diffeomorphisms, $\forall\, t \in [0,\tilde{T}_0]$, $i \in \{1,...,K\}$.

Observe that one could derive from (3.3), and (3.9) the following formulae,

$$\partial_z^{\nu}\sigma_i(t,z) = \sum_{m=1}^{\nu-1}\left[\int_0^t P_{\nu-m+1}(\partial_y^{\nu-m+1}\theta_i,...,\partial_y\theta_i,\partial_z^{\nu-m}\sigma_i,...,\partial_z\sigma_i)\,d\tau\right]\partial_z^m\sigma_i(t,z), \quad (3.10)$$

for all $\nu \ge 2$, and

$$\partial_t^{\mu}\partial_z^{\nu}\sigma_i(t,z) = Q_{(\mu,\nu)}(\partial_t^{\mu-1}\partial_y^{\nu}\theta_i,...,\theta_i,\partial_t^{\mu-1}\partial_z^{\nu}\sigma_i,...,\partial_z\sigma_i), \quad \forall\,\mu\ge 1,\nu\ge 0, \qquad (3.11)$$

where $P_{\nu-m+1}(z_1,z_2,\cdots)$ and $Q_{(\mu,\nu)}(z_1,z_2,\cdots)$ represent polynomials of $z_1,z_2,\cdots$. The regularity, $\sigma_i \in C^{\frac{k-4+\alpha}{4},k-4+\alpha}(D^{\tilde{T}_0})$, can be obtained from an induction argument based on (3.9), (3.10), (3.11), and (3.4).

□

Lemma 3.2 (Converting SGS into AS)**.** *Let $f_0 = (f_{0,1},...,f_{0,K})$, $f_{0,i} : \bar{I} \to \mathbb{R}^n$, fulfills the compatibility conditions of order 1 to SGP to (1.5)∼(1.9). Let $f = (f_1,...,f_K)$, $f_i : D^T \to \mathbb{R}^n$ be a SGS to (1.5)∼(1.9) fulfilling $f_i \in C^{\frac{5+\alpha}{4},5+\alpha}(D^T)$, $\forall\, i \in \{1,...,K-1\}$, and*

$$\begin{cases} \delta_0 \le |\partial_x f_i(t,x)| \le \delta_0^{-1}, & \\ |\partial_x^{\ell} f_i(t,x)| \le M_0, & \ell \in \{2,3,4,5\}, \end{cases} \qquad (3.12)$$

$\forall\,(t,x) \in D^T$, for some positive constants $0 < \delta_0 \le 1$ and M_0.

Then, there exist $t_2 = t_2(\delta_0,\lambda,M_0,\sum_{i=1}^{K}\|\eta_{0,i}\|_{C^{4,\alpha}(\bar{I})}) > 0$, and functions $\eta_i : D^{t_2} \to \mathbb{R}$, $i \in \{1,...,K\}$, such that

$$\begin{aligned} &(i) && \eta_i \in C^{\frac{5+\alpha}{4},5+\alpha}(D^{t_2}) \textstyle\bigcap C^{\infty}\left((0,t_2]\times\bar{I}\right) \\ &(ii) && \eta_i(t,\cdot) : \bar{I} \to \bar{I} \text{ is a diffeomorphism}, && \forall\, t \in [0,t_2], i \in \{1,...,K\}, \\ &(iii) && \frac{\delta_0^{2K-2}}{2} \le \partial_x\eta_i(0,x) = \partial_x\eta_{0,i}(x) \le \frac{2}{\delta_0^{2K-2}}, && \forall\, x \in \bar{I}, \end{aligned} \qquad (3.13)$$

and $\tilde{f} = (\tilde{f}_1,...,\tilde{f}_K)$ defined by $\tilde{f}_i(t,\eta_i(t,x)) = f_i(t,x)$ is an AS (2.10) with the initial datum $\tilde{f}_{0,i} = f_{0,i}\circ\eta_{0,i}^{-1}$, fulfilling the regularity $\tilde{f}_i \in C^{\frac{5+\alpha}{4},5+\alpha}(D^{t_2})\bigcap C^{\infty}\left((0,t_2]\times\bar{I}\right)$.

Proof. **Step 1° (converting SGS into AS by composition with diffeomorphisms η)** We show below that if $f = (f_1,...,f_K)$ is a SGS, then $\tilde{f} = (\tilde{f}_1,...,\tilde{f}_K)$, $\tilde{f}_i = f_i\circ\eta_i^{-1}$, is an AS, where $\eta_i^{-1}(t,\cdot)$ represents the inverse of a diffeomorphism $\eta_i(t,\cdot)$, and $\eta = (\eta_1,...,\eta_K)$, $\eta_i : D^T \to \bar{I}$, is a solution to an initial-boundary value problem of fourth-order parabolic PDE in (3.26)∼(3.29) with initial condition η_0 fulfilling the compatibility conditions of order 1 in Definition 3.1. It is shown in §5.1.1 in Appendix that as f is a SGS, then $\tilde{f}$ satisfies

$$\partial_t\tilde{f}_i + \mathcal{D}(\tilde{f}_i) = -\frac{\partial_x f_i}{\partial_x\eta_i}\left(\partial_t\eta_i + \mathcal{D}_{f_i}(\eta_i)\right), \qquad (3.14)$$

where

$$\mathcal{D}_{f_i}(\eta_i) = \frac{\partial_x^4 \eta_i}{|\partial_x f_i|^4} - H_{f_i}(\eta_i), \tag{3.15}$$

$$\begin{aligned} H_{f_i}(\eta_i) = & \frac{6\langle \partial_x^2 f_i, \partial_x f_i\rangle}{|\partial_x f_i|^6} \cdot \partial_x^3 \eta_i + \frac{\langle \mathcal{D}(f_i), \partial_x f_i\rangle}{|\partial_x f_i|^2} \cdot \partial_x \eta_i \\ & + \left[\frac{4\langle \partial_x^3 f_i, \partial_x f_i\rangle}{|\partial_x f_i|^6} + \frac{5}{2}\frac{|\partial_x^2 f_i|^2}{|\partial_x f_i|^6} - \frac{35}{2}\frac{\langle \partial_x^2 f_i, \partial_x f_i\rangle^2}{|\partial_x f_i|^8} + \frac{\lambda}{|\partial_x f_i|^2}\right] \cdot \partial_x^2 \eta_i, \end{aligned} \tag{3.16}$$

are linear differential operators.

Since we want to convert a SGS into an AS, we first look at the boundary conditions in (2.10). Namely, for the boundary conditions involving the first-order derivatives in (2.10), we apply (5.1), and obtain

$$\begin{cases} \partial_y \tilde{f}_1(t,0) = \partial_y \tilde{f}_{0,1}(0) \\ \partial_y \tilde{f}_K(t,1) = \partial_y \tilde{f}_{0,K}(1) \end{cases} \quad \text{iff} \quad \begin{cases} \partial_x \eta_1(t,0) - \frac{|\partial_x f_1(t,0)|}{|\partial_x f_{0,1}(0)|}\partial_x \eta_{0,1}(0) = 0 \\ \partial_x \eta_K(t,1) - \frac{|\partial_x f_K(t,1)|}{|\partial_x f_{0,K}(1)|}\partial_x \eta_{0,K}(1) = 0 \end{cases}, \tag{3.17}$$

$\forall\, t \in (0,T)$.

For the next boundary conditions in (2.10), we apply (5.11) and (5.12) in §5.1.1 in Appendix and obtain

$$\begin{cases} \partial_t \partial_y \tilde{f}_i(t,1) - \left[\frac{\partial_y^2 \tilde{f}_{i+1}(t,0)}{|\partial_y \tilde{f}_{i+1}(t,0)|} - \frac{\partial_y^2 \tilde{f}_i(t,1)}{|\partial_y \tilde{f}_i(t,1)|}\right] \\ = -\frac{\partial_x f_i(t,1)}{(\partial_x \eta_i(t,1))^2}\left[\partial_t \partial_x \eta_i(t,1) - L_{f_i}(\eta_i)(t,1) - \langle B(f_i)(t,1), \tau_i(t,1)\rangle \partial_x \eta_i(t,1)\right] \\ \quad - \left[\partial_x \eta_i(t,1) - \frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\partial_x \eta_{i+1}(t,0)\right] F_{i,0}(\eta_i,\eta_{i+1})(t), \qquad i \in \{1,...,K-1\}, \\ \partial_t \partial_y \tilde{f}_{i+1}(t,0) - \left[\frac{\partial_y^2 \tilde{f}_{i+1}(t,0)}{|\partial_y \tilde{f}_{i+1}(t,0)|} - \frac{\partial_y^2 \tilde{f}_i(t,1)}{|\partial_y \tilde{f}_i(t,1)|}\right] \\ = -\frac{\partial_x f_{i+1}(t,0)}{(\partial_x \eta_{i+1}(t,0))^2}\left[\partial_t \partial_x \eta_{i+1}(t,0) - L_{f_{i+1}}(\eta_{i+1})(t,0) - \langle B(f_{i+1})(t,0), \tau_{i+1}(t,0)\rangle \partial_x \eta_{i+1}(t,0)\right] \\ \quad - \left[\partial_x \eta_i(t,1) - \frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\partial_x \eta_{i+1}(t,0)\right] F_{i,1}(\eta_i,\eta_{i+1})(t), \qquad i \in \{1,...,K-1\}, \end{cases} \tag{3.18}$$

where

$$\begin{cases} L_{f_1}(\eta_1)(t,0) = L_{f_K}(\eta_K)(t,1) = 0, \\ L_{f_i}(\eta_i)(t,1) = |\partial_x f_i(t,1)| \left[\frac{\partial_x^2 \eta_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|^2} - \frac{\partial_x^2 \eta_i(t,1)}{|\partial_x f_i(t,1)|^2}\right], \qquad \forall\, i \in \{1,...,K-1\}, \\ L_{f_{i+1}}(\eta_{i+1})(t,0) = |\partial_x f_{i+1}(t,0)| \left[\frac{\partial_x^2 \eta_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|^2} - \frac{\partial_x^2 \eta_i(t,1)}{|\partial_x f_i(t,1)|^2}\right], \ \forall\, i \in \{1,...,K-1\}, \end{cases} \tag{3.19}$$

$$\begin{cases} B(f_1)(t,0) = \frac{\partial_t \partial_x f_1(t,0)}{|\partial_x f_1(t,0)|}, \qquad B(f_K)(t,1) = \frac{\partial_t \partial_x f_K(t,1)}{|\partial_x f_K(t,1)|}, \\ B(f_i)(t,1) = \frac{\partial_t \partial_x f_i(t,1)}{|\partial_x f_i(t,1)|} - \left[\frac{\partial_x^2 f_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|^2} - \frac{\partial_x^2 f_i(t,1)}{|\partial_x f_i(t,1)|^2}\right], \qquad \forall\, i \in \{1,...,K-1\}, \\ B(f_{i+1})(t,0) = \frac{\partial_t \partial_x f_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|} - \left[\frac{\partial_x^2 f_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|^2} - \frac{\partial_x^2 f_i(t,1)}{|\partial_x f_i(t,1)|^2}\right], \ \forall\, i \in \{1,...,K-1\}, \end{cases} \tag{3.20}$$

and

$$\begin{cases} F_{i,0}(\eta_i,\eta_{i+1})(t) = \frac{|\partial_x f_{i+1}(t,0)|}{\partial_x \eta_i(t,1)\partial_x \eta_{i+1}(t,0)} \cdot \\ \qquad \left[\frac{\partial_x^2 f_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|^2} - \frac{\tau_{i+1}(t,0)\partial_x^2 \eta_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|^2}\left(\frac{|\partial_x f_{i+1}(t,0)|}{\partial_x \eta_{i+1}(t,0)} + \frac{|\partial_x f_i(t,1)|}{\partial_x \eta_i(t,1)}\right)\right], \\ F_{i,1}(\eta_i,\eta_{i+1})(t) = \frac{|\partial_x f_i(t,1)|}{\partial_x \eta_i(t,1)\partial_x \eta_{i+1}(t,0)} \cdot \\ \qquad \left[\frac{\partial_x^2 f_i(t,1)}{|\partial_x f_i(t,1)|^2} - \frac{\tau_i(t,1)\partial_x^2 \eta_i(t,1)}{|\partial_x f_i(t,1)|^2}\left(\frac{|\partial_x f_{i+1}(t,0)|}{\partial_x \eta_{i+1}(t,0)} + \frac{|\partial_x f_i(t,1)|}{\partial_x \eta_i(t,1)}\right)\right]. \end{cases} \tag{3.21}$$

Observe from (3.14), (3.17), and (3.18) that in order to convert SGS into AS, we need to set up a proper IBVP for η. Note that the boundary conditions involving derivatives ∂_x are

$$\partial_x\eta_1(t,0) = \frac{|\partial_x f_1(t,0)|}{|\partial_x f_{0,1}(0)|}\partial_x\eta_{0,1}(0), \qquad \partial_x\eta_K(t,1) = \frac{|\partial_x f_K(t,1)|}{|\partial_x f_{0,K}(1)|}\partial_x\eta_{0,K}(1), \tag{3.22}$$

$$\partial_t\partial_x\eta_i(t,1) = L_{f_i}(\eta_i)(t,1) + \langle B(f_i)(t,1), \tau_i(t,1)\rangle\partial_x\eta_i(t,1), \qquad i \in \{1, ..., K-1\}, \tag{3.23}$$

$$\partial_t\partial_x\eta_i(t,0) = L_{f_i}(\eta_i)(t,0) + \langle B(f_i)(t,0), \tau_i(t,0)\rangle\partial_x\eta_i(t,0), \qquad i \in \{2, ..., K\}, \tag{3.24}$$

$$\partial_x\eta_i(t,1) = \frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\partial_x\eta_{i+1}(t,0), \qquad i \in \{1, ..., K-1\}. \tag{3.25}$$

The linear parabolic PDE for η_i, $i \in \{1, ..., K\}$, is

$$\partial_t\eta_i = -\frac{1}{|\partial_x f_i|^4}\partial_x^4\eta_i + H_{f_i}(\eta_i), \qquad \text{in } (0,T)\times(0,1). \tag{3.26}$$

We also impose the initial-boundary conditions,

$$\eta_i(0,x) = \eta_{0,i}(x), \qquad \forall\, x \in [0,1], i \in \{1, ..., K\}, \tag{3.27}$$

$$\eta_i(t,x^*) = x^*, \qquad \forall\, (t,x^*) \in [0,T]\times\{0,1\},\ i \in \{1, ..., K\}. \tag{3.28}$$

Below, we show that the boundary conditions (3.22)∼(3.25) can be replaced by

$$\partial_x\eta_i(t,x^*) = b_{f_i}(\eta_i)(t,x^*), \qquad (t,x^*) \in [0,T]\times\{0,1\},\ i \in \{1, ..., K\}, \tag{3.29}$$

where

$$\begin{cases} b_{f_i}(\eta_i)(t,x^*) = \frac{|\partial_x f_i(t,x^*)|}{|\partial_x f_{0,i}(x^*)|}\partial_x\eta_{0,i}(x^*), & \forall\,(i,x^*) \in \{(1,0),(K,1)\}, \\ b_{f_i}(\eta_i)(t,x^*) = \frac{|\partial_x f_{0,i}(x^*)|}{|\partial_x f_{0,i-1+2x^*}(1-x^*)|}\partial_x\eta_{0,i-1+2x^*}(1-x^*) \\ \qquad + \int_0^t [L_{f_i}(\eta_i)(\tau,x^*) + \langle B(f_i)(\tau,x^*), \tau_i(\tau,x^*)\rangle\partial_x\eta_i(\tau,x^*)]\, d\tau, \\ \qquad\qquad \forall\,(i,x^*) \in \{1, ..., K\}\times\{0,1\}\backslash\{(1,0),(K,1)\}. \end{cases} \tag{3.30}$$

Now we prove the replacement of (3.22)∼(3.25) by (3.29). As $(i,x^*) \in \{(1,0),(K,1)\}$, (3.29) is (3.22). As $(i,x^*) \in \{1, ..., K\}\times\{0,1\}\backslash\{(1,0),(K,1)\}$, we take ∂_t on both sides of (3.29) and then obtain (3.23) and (3.24). It remains to show that (3.29) implies (3.25).

To prove it, we first claim that,

$$\partial_t v_i(t) = \langle B(f_i)(t,1), \tau_i(t,1)\rangle v_i(t), \tag{3.31}$$

where

$$v_i(t) = \partial_x\eta_i(t,1) - \frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\partial_x\eta_{i+1}(t,0), \quad \forall\, i \in \{1, ..., K-1\}.$$

In fact, from (3.29), the definitions of $B(f_i)$ and $L_{f_i}(\eta_i)$ in (3.20) and (3.19) respectively, and $\tau_{i+1}(t,0)=\tau_i(t,1)$, $\forall\, i\in\{1,...,K-1\}$, we have

$$\begin{aligned}
&\partial_t\left(\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\partial_x\eta_{i+1}(t,0)\right)=\partial_t\left(\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\right)\partial_x\eta_{i+1}(t,0)+\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\partial_t\partial_x\eta_{i+1}(t,0)\\
=&\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\langle\frac{\partial_t\partial_x f_i(t,1)}{|\partial_x f_i(t,1)|}-\frac{\partial_t\partial_x f_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|},\tau_{i+1}(t,0)\rangle\partial_x\eta_{i+1}(t,0)\\
&+\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\left[L_{f_{i+1}}(\eta_{i+1})(t,0)+\langle B(f_{i+1})(t,0),\tau_{i+1}(t,0)\rangle\partial_x\eta_{i+1}(t,0)\right]\\
=&\langle B(f_i)(t,1),\tau_i(t,1)\rangle\cdot\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\cdot\partial_x\eta_{i+1}(t,0)+L_{f_i}(\eta_i)(t,1)\\
=&\partial_t\partial_x\eta_i(t,1)-\langle B(f_i)(t,1),\tau_i(t,1)\rangle\left[\partial_x\eta_i(t,1)-\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\partial_x\eta_{i+1}(t,0)\right],
\end{aligned}$$

$\forall\, i\in\{1,...,K-1\}$. By moving terms in the equality above, (3.31) is confirmed.

By solving the first-order ODE (3.31), we obtain the solution

$$v_i(t)=e^{\int_0^t\langle B(f_i)(\tau,1),\tau_i(\tau,1)\rangle\,d\tau}v_i(0),$$

$\forall\, i\in\{1,...,K-1\}$. Note that $v_i(0)=0$, $\forall\, i\in\{1,...,K-1\}$, come from applying (3.29), (3.30) as $t=0$ therein, and $\eta_i(0,\cdot)=\eta_{0,i}(\cdot)$. In fact, it corresponds to the proper choice of the initial conditions $\eta_{0,i}$, $\forall\, i\in\{1,...,K-1\}$ so that $v_i(0)=0$ holds. Thus, we conclude (3.25).

Step 2° (well-posedness of the linear problem of the IBVP (3.26)∼(3.29))

Similar to the IBVP for the AP (2.11), we first consider the linearized IBVP of (3.26)∼(3.29): $\forall\, i\in\{1,...,K\}$,

$$\begin{cases}\partial_t\eta_i+\frac{\partial_x^4\eta_i}{|\partial_x f_i|^4}=H_{f_i}(\tilde\eta_i), & \text{in } (0,T)\times I,\\ \eta_i(t,x^*)=x^*, & \forall\,(t,x^*)\in[0,T]\times\partial I,\\ \partial_x\eta_i(t,x^*)=b_{f_i}(\tilde\eta_i)(t,x^*), & \forall\,(t,x^*)\in[0,T]\times\partial I,\\ \eta_i(0,x)=\eta_{0,i}(x), & \forall\, x\in\bar I,\end{cases}\tag{3.32}$$

where $\tilde\eta_i$ is given.

The left-hand side of the fourth-order PDE in (3.32) can be written as $\mathcal{L}(x,t,\partial_x,\partial_t)\eta^T$ where $\eta=(\eta_1,...,\eta_K)$,

$$\mathcal{L}(x,t,\partial_x,\partial_t)=\operatorname{diag}\,(l_{ii})_{i=1}^K,$$

and $l_{ii}(x,t,\partial_x,\partial_t)=\partial_t+\frac{\partial_x^4}{|\partial_x f_i(x)|^4}$, $\forall\, i\in\{1,...,K\}$. By using the same notation in §2.1, we have

$$l_{ii}(x,t,\mathrm{i}\xi,p)=p+\frac{\xi^4}{|\partial_x f_i(x)|^4},\qquad \forall\, i\in\{1,...,K\}.$$

Define

$$L(x,t,\mathrm{i}\xi,p):=\det\mathcal{L}(x,t,\mathrm{i}\xi,p)=\prod_{i=1}^K\left(p+\frac{\xi^4}{|\partial_x f_i(x)|^4}\right).\tag{3.33}$$

Let

$$\begin{aligned}\hat{\mathcal{L}}(x,t,\mathrm{i}\xi,p):=&L(x,t,\mathrm{i}\xi,p)\mathcal{L}^{-1}(x,t,\mathrm{i}\xi,p)=\prod_{i=1}^K\left(p+\frac{\xi^4}{|\partial_x f_i(x)|^4}\right)\cdot\ \operatorname{diag}\ \left(l_{ii}^{-1}\right)_{i=1}^K\\ =&\operatorname{diag}(A_{ii})_{i=1}^K,\end{aligned}$$

where

$$A_{ii}(x,t,\mathrm{i}\xi,p)=\frac{L(x,t,\mathrm{i}\xi,p)}{p+\frac{\xi^4}{|\partial_x f_i(x)|^4}},\qquad \forall\, i\in\{1,...,K\}.$$

• **Parabolicity condition.** For any $\xi\in\mathbb{R}$ and from (3.33), we see that the roots (in the variable p) of the polynomial $L(x,t,\mathrm{i}\xi,p)$ are given by

$$p=-\frac{\xi^4}{|\partial_x f_i(x)|^4},\quad \forall\, i\in\{1,...,K\}.$$

From (3.12), $p=-\frac{\xi^4}{|\partial_x f_i(x)|^4}\le -\delta_0^4\xi^4$, $\forall\, i\in\{1,...,K\}$. Hence, the uniform parabolicity holds (See [26, page 8]).

• **Complementary conditions on the initial datum η_0.** The conditions can be obtained in the same way as in that of §2.1.

• **The polynomial M^+.** From [26, page 11], we consider the polynomial M^+ as follows. Namely, consider the polynomial $L=L(x,t,\mathrm{i}\xi,p)$ given in (3.33). Let $p=|p|e^{\mathrm{i}\theta_p}, -\frac{1}{2}\pi\le\theta_p\le\frac{1}{2}\pi$, and $\xi_{i,1}(x^*,p),\xi_{i,2}(x^*,p)$ be roots of $p+\frac{\xi^4}{|\partial_x f_i(x^*)|^4}=0$, $x^*\in\partial I$, with positive imaginary parts, i.e.,

$$\xi_{i,1}(x^*,p)=r_ie^{\mathrm{i}\left(\frac{\theta_p}{4}+\frac{\pi}{4}\right)},\quad \xi_{i,2}(x^*,p)=r_ie^{\mathrm{i}\left(\frac{\theta_p}{4}+\frac{3\pi}{4}\right)},$$

where $r_i(x^*,p)=\sqrt[4]{|p|}\cdot|\partial_x f_i(x^*)|$, $\mathrm{i}=\sqrt{-1}$.
Let

$$M^+(x^*,\xi,p):=\prod_{i=1}^{K}(\xi-\xi_{i,1}(x^*,p))(\xi-\xi_{i,2}(x^*,p)).$$

• **Complementary conditions at the boundary points $x^*\in\partial I$.** The boundary conditions of (3.32) can be presented as

$$\mathcal{B}(x,t,\partial_x,\partial_t)\eta(t,x^*)^T=(x^*,...,x^*,b(f_1)(\tilde\eta_1)(t,x^*),....,b(f_K)(\tilde\eta_K)(t,x^*))^T,\quad x^*\in\partial I,$$

where

$$\mathcal{B}(x^*,t,\partial_x,\partial_t)=\begin{pmatrix} Id_{K\times K}\\ Id_{K\times K}\cdot\partial_x\end{pmatrix},$$

as a $2K\times K$ matrix. Hence

$$\mathcal{B}(x^*,t,\mathrm{i}\xi,p)=\begin{pmatrix} Id_{K\times K}\\ \mathrm{i}\xi\cdot Id_{K\times K}\end{pmatrix}.$$

By applying the same argument as before in the proof of (2.26), it is straightforward to verify that the rows of the matrix

$$\mathcal{A}(x^*,t,\mathrm{i}\xi,p)=\mathcal{B}(x^*,t,\mathrm{i}\xi,p)\hat{\mathcal{L}}(x^*,t,\mathrm{i}\xi,p)$$

are linearly independent modulo the polynomial $M^+(x^*,\xi,p)$ for $\mathrm{Re}\{p\}\ge 0$ and $p\ne 0$. Since the procedure is similar to that of in the proof of Theorem 2.4, we leave the detail to the reader.

Step 3° (the compatibility conditions of order 1 to the IBVP (3.26)$\sim$(3.29), and construction of initial datum η_0)

Observe from the previous step that if η is a solution to the initial-boundary value problem (3.26)∼(3.29) where $\eta_i \in C^{\frac{5+\alpha}{4},5+\alpha}(D^T)$ is a diffeomorphism, for some $T>0$ and $\forall\, i \in \{1,...,K\}$, then a SGS f with $f_i \in C^{\frac{5+\alpha}{4},5+\alpha}(D^T)$ can be converted into an AS $\tilde{f}$ with $\tilde{f}_i \in C^{\frac{5+\alpha}{4},5+\alpha}(D^T)$, $\forall\, i \in \{1,...,K\}$. To achieve the goal, it is required to impose the compatibility conditions of order 1 to the IBVP (3.26)∼(3.29).

Define the boundary operators $B_{f,0}$ and $B_{f,1}$ acting on η at the boundary ∂I by

$$(B_{f_i,0}(\eta_i))_{\lfloor(t,x^*)} = \eta_i(t,x^*) - x^*, \qquad (B_{f_i,1}(\eta_i))_{\lfloor(t,x^*)} = \partial_x\eta_i(t,x^*) - b_{f_i}(\eta_i)(t,x^*),$$

and the (higher-order) boundary operators, $B^{(\ell)}_{f_i,0}$ and $B^{(\ell)}_{f_i,1}$ acting on η at the boundary ∂I by

$$B^{(\ell)}_{f_i,0}(\eta_i)_{\lfloor(t,x^*)} = \left(\partial_t^\ell B_{f_i,0}(\eta_i)\right)_{\lfloor(t,x^*)}, \qquad B^{(\ell)}_{f_i,1}(\eta_i)_{\lfloor(t,x^*)} = \left(\partial_t^\ell B_{f_i,1}(\eta_i)\right)_{\lfloor(t,x^*)},$$

where $(t,x^*) \in [0,T] \times \partial I$, and $\ell \in \mathbb{N}_0$. Note the boundary operators, $B^{(\ell)}_{f_i,0}$ and $B^{(\ell)}_{f_i,1}$, should be understood as differential operators with respect to ∂_x by following the PDE (3.26).

Definition 3.1. *We say that the initial datum $\eta_0 = (\eta_{0,1},...,\eta_{0,K})$, $\eta_{0,i} : \bar{I} \to \bar{I}$, $i \in \{1,...,K\}$, fulfills the compatibility conditions of order k, $k \in \mathbb{N}_0$, to the IBVP (3.26)∼(3.29) on $\bar{I}$, if the following conditions are satisfied:*

- $B^{(\ell)}_{f_{0,i},0}(\eta_{0,i})_{\lfloor(x^*)} = 0, \qquad \forall\, 4\ell - 4 \le k,$
- $B^{(\ell)}_{f_{0,i},1}(\eta_{0,i})_{\lfloor(x^*)} = 0, \qquad \forall\, 4\ell - 3 \le k,$

where $x^* \in \partial I$, $i \in \{1,...,K\}$.

To prove the existence of the diffeomorphisms $\eta_{0,i} : \bar{I} \to \bar{I}$, where $I = (0,1)$, so that the compatibility conditions of order 1 in Definition 3.1 are fulfilled, i.e.,

$$B^{(\ell)}_{f_{0,i},J}(\eta_{0,i})_{\lfloor(x^*)} = 0, \qquad x^* \in \{0,1\}, \ell \in \{0,1\}, J \in \{0,1\}, i \in \{1,...,K\}, \tag{3.34}$$

we need to find explicit formulas of $B^{(\ell)}_{f_{0,i},J}$. Observe that (3.28) is equivalent to

$$B^{(0)}_{f_{0,i},0}(\eta_{0,i})_{\lfloor(x^*)} = 0, \qquad \forall\, x^* \in \{0,1\}, i \in \{1,...,K\}, \tag{3.35}$$

while (3.29) is equivalent to

$$B^{(0)}_{f_{0,i},1}(\eta_{0,i})_{\lfloor(x^*)} = 0, \qquad \forall\, x^* \in \{0,1\}, i \in \{1,...,K\}. \tag{3.36}$$

From (3.26), we have

$$B^{(1)}_{f_i,0}(\eta_i)_{\lfloor(t,x^*)} = -\frac{\partial_x^4\eta_i(t,x^*)}{|\partial_x f_i(t,x^*)|^4} + H_{f_i}(\eta_i)(t,x^*), \forall\,(t,x^*) \in [0,T] \times \{0,1\}, i \in \{1,...,K\}. \tag{3.37}$$

Next, we derive the formula on $B^{(1)}_{f_i,1}(\eta_i)_{\lfloor(t,x^*)}$. Note that, from (3.26), we have

$$\partial_t\partial_x\eta_i = \partial_x\partial_t\eta_i = -\frac{1}{|\partial_x f_i|^4}\partial_x^5\eta_i + 4\frac{\langle\partial_x^2 f_i, \partial_x f_i\rangle}{|\partial_x f_i|^6}\partial_x^4\eta_i + \partial_x H_{f_i}(\eta_i). \tag{3.38}$$

By the definitions of $B^{(1)}_{f_i,1}$ and $b_{f_i}(\eta_i)$, $\forall\,(t,x^*)\in[0,T]\times\{0,1\}$, $i\in\{1,...,K\}$, we have

$$
\begin{aligned}
B^{(1)}_{f_i,1}(\eta_i)_{\lfloor(t,x^*)} =&\partial_t\partial_x\eta_i(t,x^*)-\partial_t b_{f_i}(\eta_i)(t,x^*)\\
=&\partial_t\partial_x\eta_i(t,x^*)-L_{f_i}(\eta_i)(t,x^*)-\langle B(f_i)(t,x^*),\tau_i(t,x^*)\rangle\partial_x\eta_i(t,x^*). \quad (3.39)
\end{aligned}
$$

We remark that f is a SGS and f_i satisfies $\partial_t f_i=-\nabla_s^2\vec{\kappa}_i-\frac{|\vec{\kappa}_i|^2}{2}\vec{\kappa}_i+\lambda\vec{\kappa}_i$ for each $i\in\{1,...,K\}$, so $B(f_i)$ in (3.20) can be written as

$$
B(f_i)=\frac{\partial_x\vec{V}_i}{|\partial_x f_i|}-E(f_i),\quad i\in\{1,...,K\}, \tag{3.40}
$$

where $\vec{V}_i$ is defined in (2.2), and

$$
\begin{cases}
E(f_1)(t,0)=E(f_K)(t,1)=0,\\
E(f_i)(t,1)=E(f_{i+1})(t,0)=\left[\frac{\partial_x^2 f_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|^2}-\frac{\partial_x^2 f_i(t,1)}{|\partial_x f_i(t,1)|^2}\right],\quad \forall\, i\in\{1,...,K-1\}.
\end{cases} \tag{3.41}
$$

Putting (3.38), (3.39), and (3.40) all together gives

$$
B^{(1)}_{f_i,1}(\eta_i)_{\lfloor(t,x^*)}=-\frac{\partial_x^5\eta_i(t,x^*)}{|\partial_x f_i(t,x^*)|^4}+G_{f_i}(\eta_i)(t,x^*),\quad (t,x^*)\in[0,T]\times\{0,1\}, i\in\{1,...,K\}, \tag{3.42}
$$

where

$$
G_{f_i}(\eta_i)=\frac{4\langle\partial_x^2 f_i,\partial_x f_i\rangle}{|\partial_x f_i|^6}\partial_x^4\eta_i+\partial_x H_{f_i}(\eta_i)-L_{f_i}(\eta_i)-\langle\frac{\partial_x\vec{V}_i}{|\partial_x f_i|}-E_{f_i},\frac{\partial_x f_i}{|\partial_x f_i|}\rangle\partial_x\eta_i. \tag{3.43}
$$

It remains to construct initial datum $\eta_{0,i}$, $i\in\{1,...,K\}$, so that the compatibility conditions of order 1 in (3.34) are fulfilled. Observe that (3.35) and (3.36) provide the case when $\ell=0$ and $J\in\{0,1\}$. Note that (3.35) implies

$$
\eta_{0,i}(x^*)=x^*,\qquad \forall\, x^*\in\{0,1\}, i\in\{1,...,K\}. \tag{3.44}
$$

while (3.36) implies

$$
\partial_x\eta_{0,i}(1)=\frac{|\partial_x f_{0,i}(1)|}{|\partial_x f_{0,i+1}(0)|}\partial_x\eta_{0,i+1}(0),\qquad i\in\{1,...,K-1\}. \tag{3.45}
$$

As $\ell=1$ and $J=0$, (3.37) gives

$$
\partial_x^4\eta_{0,i}(x^*)=H_{f_{0,i}}(\eta_{0,i})(x^*)\cdot|\partial_x f_{0,i}(x^*)|^4,\qquad x^*\in\{0,1\}. \tag{3.46}
$$

As $\ell=1$ and $J=1$, (3.42) gives

$$
\partial_x^5\eta_{0,i}(x^*)=G_{f_{0,i}}(\eta_{0,i})(x^*)\cdot|\partial_x f_{0,i}(x^*)|^4,\qquad x^*\in\{0,1\}. \tag{3.47}
$$

The compatibility conditions of order 1 to the IBVP (3.26)$\sim$(3.29) can be obtained from (3.44)$\sim$(3.47) and letting $\partial_x^2\eta_{0,i}(x^*)=\partial_x^3\eta_{0,i}(x^*)=0$, $\forall\, x^*\in\{0,1\}$, $\partial_x\eta_{0,1}(0)=1=\partial_x\eta_{0,1}(1)$, i.e.,

$$
\begin{cases}
\eta_{0,i}(0)=0,\quad \eta_{0,i}(1)=1,\\
\partial_x\eta_{0,i}(0)=:c_i^0=c_i^1:=\partial_x\eta_{0,i}(1),\\
\partial_x^2\eta_{0,i}(0)=\partial_x^2\eta_{0,i}(1)=0,\\
\partial_x^3\eta_{0,i}(0)=\partial_x^3\eta_{0,i}(1)=0,\\
\partial_x^4\eta_{0,i}(0)=:d_i^0,\ \partial_x^4\eta_{0,i}(1)=:d_i^1,\\
\partial_x^5\eta_{0,i}(0)=:e_i^0,\ \partial_x^5\eta_{0,i}(1)=:e_i^1,
\end{cases} \tag{3.48}
$$

where

$$c_1^0 = c_1^1 = 1, \qquad c_k^0 = c_k^1 = \prod_{j=1}^{k-1} \frac{|\partial_x f_{0,j+1}(0)|}{|\partial_x f_{0,j}(1)|} > 0, \quad k \in \{2, ..., K\}, \tag{3.49}$$

$$\begin{cases} d_i^{x^*} = \langle \mathcal{D}(f_{0,i})(x^*), \partial_x f_{0,i}(x^*)\rangle \cdot |\partial_x f_{0,i}(x^*)|^2 \cdot c_i^{x^*}, & x^* \in \{0,1\}, \\ e_i^{x^*} = R(f_{0,i})(x^*) \cdot |\partial_x f_{0,i}(x^*)|^4 \cdot c_i^{x^*}, & x^* \in \{0,1\}, \end{cases} \tag{3.50}$$

$$R(f_i) = \frac{10\langle \partial_x^2 f_i, \partial_x f_i\rangle}{|\partial_x f_i|^4} \langle \mathcal{D}(f_i), \partial_x f_i\rangle + \partial_x \left(\frac{\langle \mathcal{D}(f_i), \partial_x f_i\rangle}{|\partial_x f_i|^2} \right) - \langle \frac{\partial_x \vec{V}_i}{|\partial_x f_i|} - E(f_i), \frac{\partial_x f_i}{|\partial_x f_i|}\rangle, \tag{3.51}$$

and $i \in \{1, ..., K\}$, so that all the compatibility conditions of order 1, i.e., (3.44)∼(3.47), are fulfilled. To derive (3.48), first note that (3.49) is obtained from an induction argument based on (3.45) and from letting $c_1^0 = c_1^1 = 1$. By letting $\partial_x^2 \eta_{0,i}(x^*) = \partial_x^3 \eta_{0,i}(x^*) = 0$, $\forall\, x^* \in \{0,1\}$, we obtain from (3.46) that

$$\partial_x^4 \eta_{0,i}(x^*) = \langle \mathcal{D}(f_{0,i})(x^*), \partial_x f_{0,i}(x^*))\rangle \cdot |\partial_x f_{0,i}(x^*)|^2 \cdot \partial_x \eta_{0,i}(x^*) = d_i^{x^*}. \tag{3.52}$$

By applying $\partial_x^2 \eta_{0,i}(x^*) = \partial_x^3 \eta_{0,i}(x^*) = 0$, and (3.52), to (3.43) and (3.47), we obtain

$$\begin{aligned} G_{f_{0,i}}(\eta_{0,i})(x^*) &= \frac{4\langle \partial_x^2 f_{0,i}(x^*), \partial_x f_{0,i}(x^*)\rangle}{|\partial_x f_{0,i}(x^*)|^6} \partial_x^4 \eta_{0,i}(x^*) + \frac{6\langle \partial_x^2 f_{0,i}(x^*), \partial_x f_{0,i}(x^*)\rangle}{|\partial_x f_{0,i}(x^*)|^6} \partial_x^4 \eta_{0,i}(x^*) \\ &+ \partial_x \left(\frac{\langle \mathcal{D}(f_{0,i}), \partial_x f_{0,i}\rangle}{|\partial_x f_{0,i}|^2} \right)(x^*) \partial_x \eta_{0,i}(x^*) - \langle \frac{\partial_x \vec{V}_{0,i}(x^*)}{|\partial_x f_{0,i}(x^*)|} - E_{f_{0,i}}(x^*), \frac{\partial_x f_{0,i}(x^*)}{|\partial_x f_{0,i}(x^*)|}\rangle \partial_x \eta_{0,i}(x^*) \\ =& R(f_{0,i})(x^*) \partial_x \eta_{0,i}(x^*) = R(f_{0,i})(x^*) c_i^{x^*}, \end{aligned} \tag{3.53}$$

where $R(f_i)$ is given in (3.51), and thus conclude

$$\partial_x^5 \eta_{0,i}(x^*) = e_i^{x^*}.$$

Note that, from (3.49) and (3.12), we have that $0 < \delta_0 < 1$, and

$$\delta_0^{2K-2} \le c_i^0 = c_i^1 \le \delta_0^{-(2K-2)}, \quad \forall\, i \in \{1, ..., K\}.$$

Moreover, the differentiability of $\eta_{0,i}$ at the boundary $\partial I = \{0,1\}$ are merely prescribed values for $\partial_x^k \eta_{0,i}(x^*)$ as $x^* \in \{0,1\}$ and $k \in \{0, \cdots, 5\}$. It is not hard to verify that such diffeomorphisms on $\bar{I}$ fulfilling $\frac{1}{2}\delta_0^{2K-2} \le \partial_x \eta_{0,i}(x) \le 2\delta_0^{-2K+2}$ always exist. The reader can consult the draft in [7] for the construction of diffeomorphisms similar to the case here. Hence we skip the proof of the construction here.

Step 4° (the existence of the family of diffeomorphisms)

The proof of the short-time existence to the IBVP (3.26)∼(3.29) is similar to that of the analytical problem in Theorem 2.2, thus we leave the details to §5.1.2 in Appendix. We remark here that, from §5.1.2, there exist $T_3 = T_3\left(\delta_0, \lambda, M_0, \sum_{i=1}^{K} \|\eta_{0,i}\|_{C^{4,\alpha}(\bar{I})}\right) > 0$ and a solution η to the IBVP (3.26)∼(3.29) with the regularity

$$\eta_i \in C^{\frac{5+\alpha}{4}, 5+\alpha}(D^{T_3}) \bigcap C^\infty\left((0, T_3] \times \bar{I}\right), \quad \forall\, i \in \{1, ..., K\}.$$

Since each $\eta_i(0,\cdot)=\eta_{0,i}(\cdot)$ is a diffeomorphism fulfilling (3.13), we should ensure a constant $t_2 = t_2\left(\delta_0,\lambda,M_0,\sum_{i=1}^{K}\|\eta_{0,i}\|_{C^{4,\alpha}(\bar I)}\right)\in(0,T_3)$ such that $\eta_i(t,\cdot)$ is a diffeomorphism for any fixed $t\in[0,t_2]$. This however can be achieved by applying the triangle inequality, $\|\eta_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^{T_3})}\le \tilde M_0$, $\forall\, i\in\{1,...,K\}$, assured by (5.17), and (3.13), so that

$$
\begin{aligned}
\partial_x\eta_i(t,x) \ge& \partial_x\eta_{0,i}(x)-|\partial_x\eta_i(t,x)-\partial_x\eta_{0,i}(x)| \ge \frac{\delta_0^{2K-2}}{2} - t_2^{\frac{3+\alpha}{4}}[\partial_x\eta_i]_{\frac{\alpha}{4},t} \\
\ge& \frac{\delta_0^{2K-2}}{2} - \tilde M_0 t_2^{\frac{3+\alpha}{4}} \ge \frac{\delta_0^{2K-2}}{4},
\end{aligned}
$$

where the last inequality comes from choosing $t_2<T_3$ such that

$$
\tilde M_0 t_2^{\frac{3+\alpha}{4}} \le \frac{\delta_0^{2K-2}}{4}.
$$

It can be achieved by letting

$$
t_2 = \min\left\{\left(\frac{\delta_0^{2K-2}}{4\tilde M_0}\right)^{\frac{4}{3+\alpha}}, \frac{1}{2}T_3\right\} = t_2\left(\delta_0,\lambda,M_0,\sum_{i=1}^{K}\|\eta_{0,i}\|_{C^{4,\alpha}(\bar I)}\right).
$$

Now, we may conclude that $\tilde f$ is an AS with $\tilde f_i\in C^{\frac{5+\alpha}{4},5+\alpha}(D^{t_2})$, $\forall\, i\in\{1,...,K\}$, since η is a solution to the IBVP (3.26)∼(3.29) with $\eta_i\in C^{\frac{5+\alpha}{4},5+\alpha}(D^{t_2})$, $\forall\, i\in\{1,...,K\}$.

□

4 The long-time existence

To prove the long-time existence, we need to estimate the higher-order Sobolev seminorms of curvature. We use an argument similar to the one used in [22]. Namely, we consider the evolution equation for $\nabla_t^m f_i$ and derive the equation

$$
\nabla_t\nabla_t^m f_i = -\nabla_s^4\nabla_t^m f_i + \text{ tensors of lower-order}
$$

for all $m\in\mathbb N$. The difference here is that we need to manage a way to split the boundary terms, coming from applying integration by parts in the L^2 estimates of $\nabla_t^m f_i$ (these boundary terms vanish in the case of clamped boundary conditions), so that we derive the following differential equality,

$$
\begin{aligned}
&\tfrac{d}{dt}\left\{\sum_{i=1}^{K}\int_I |\nabla_t^m f_i|^2\, ds + \sum_{i=1}^{K-1}|\nabla_t^m \tau_i(\cdot,1)|^2\right\} + 2\cdot\sum_{i=1}^{K}\int_I |\nabla_s^{4m}\vec\kappa_i|^2\, ds \qquad (4.1)\\
&= \text{ terms of lower-order.}
\end{aligned}
$$

It is sufficient to keep track only on the scaling of the terms of lesser-order, instead of computing these terms explicitly, in (4.1). In other words, we only have to know the order of the derivatives involved such that the Gagliardo-Nirenberg type interpolation inequalities still apply to (4.1) to derive the required differential inequalities.

4.1 Uniform bounds

Lemma 4.1 (Energy Identity)**.** *Suppose $f = (f_1, ..., f_K)$, $f_i : [t_0, t_1] \times \bar{I} \to \mathbb{R}^n$ is a SGS to (GP) with the regularity $f_i \in C^\infty\left([t_0, t_1] \times \bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$. Then, for any $t \in [t_0, t_1]$, we have*

$$\tfrac{d}{dt}\mathcal{E}_\lambda\,[f] = -\sum_{i=1}^{K}\int_I |\partial_t f_i|^2\; ds - \sum_{i=1}^{K-1}|\partial_t \tau_i|^2(t,1) = -\sum_{i=1}^{K}\int_I |\nabla_t f_i|^2\; ds - \sum_{i=1}^{K-1}|\nabla_t \tau_i|^2(t,1). \tag{4.2}$$

Proof. From Lemma 5.2 and (1.9), one derives on I, and $t \in [t_0, t_1]$ the equality,

$$\begin{aligned}
&\tfrac{d}{dt}\int_I \left(\tfrac{1}{2}|\vec{\kappa}_i|^2 + \lambda\right)\; ds \\
&= \int_I \langle \nabla_s^2 \vec{\kappa}_i + \tfrac{|\vec{\kappa}_i|^2}{2}\vec{\kappa}_i - \lambda \cdot \vec{\kappa}_i, \partial_t f_i\rangle\; ds + \left[\langle \vec{\kappa}_i, \nabla_s \nabla_t f_i\rangle + \langle (\lambda + \tfrac{1}{2}|\vec{\kappa}_i|^2)\tau - \nabla_s \vec{\kappa}_i, \partial_t f_i\rangle\right]_{|\partial I} \\
&= -\int_I |\partial_t f_i|^2\; ds + \langle \vec{\kappa}_i, \nabla_t \tau_i\rangle_{|\partial I} = -\int_I |\partial_t f_i|^2\; ds + \langle \vec{\kappa}_i, \partial_t \tau_i\rangle_{|\partial I},
\end{aligned} \tag{4.3}$$

where the second equality comes from applying the boundary condition (1.5); the inferred property $\partial_t f_i = \nabla_t f_i$ from (1.9), and (5.25) ; the last equality comes from using the identity $\nabla_t \tau_i = \partial_t \tau_i$ (since $\langle \partial_t \tau_i, \tau_i\rangle = \frac{1}{2}\partial|\tau_i|^2 = 0$). Hence, from (4.3) and the boundary conditions (1.5), (1.6), (1.7), one derives the energy identity,

$$\tfrac{d}{dt}\mathcal{E}_\lambda\,[f] = \tfrac{d}{dt}\sum_{i=1}^{K}\left\{\int_I \left(\tfrac{1}{2}|\vec{\kappa}_i|^2 + \lambda\right)\; ds\right\} = -\sum_{i=1}^{K}\int_I |\partial_t f_i|^2\; ds - \sum_{i=1}^{K-1}|\partial_t \tau_i|^2(t,1). \tag{4.4}$$

□

A classical theorem by John Milnor states that the total curvature of a closed curve f in $\mathbb{R}^n$ can be approximated by the limit of the total curvatures of inscribed polygons of f. Hence, the total curvature of a smooth closed curve in $\mathbb{R}^n$ is at least 2π (cf. [23]). We adapt part of the proof of Milnor's theorem into the situation in Lemma 4.2 below.

Lemma 4.2. *Let $f : I = [a, b] \to \mathbb{R}^n$ be a regular curve fulfilling $f \in C^2([a, b], \mathbb{R}^n)$. Assume $f(a) = f(b)$, then the total curvature of f is at least π, i.e.,*

$$\int_{x=a}^{b} |\vec{\kappa}|\; ds > \pi. \tag{4.5}$$

Proof. Note that $f(a) = f(b)$ implies $\int_I \tau(s)\; ds = 0$. Thus the tangent indicatrix τ can't be contained in any hemisphere, $\mathbb{S}^{n-1}_+$. Hence, the spherical diameter of the spherical curve τ is greater than one-half of the length of a great circle on the unit sphere $\mathbb{S}^{n-1}(1)$, i.e.,

$$\mathrm{dist}_{\mathbb{S}^{n-1}(1)}\left(\tau(x_1), \tau(x_2)\right) > \pi,$$

for some $x_1, x_2 \in I$. Notice that $\int_{x=a}^{b} |\vec{\kappa}|\; ds$ is equal to the length of the spherical C^1-map $\tau : I \to \mathbb{S}^{n-1}(1)$. Thus, (4.5) is obtained.

□

The formula in the following lemma could be thought as a "higher-order energy identity".

Lemma 4.3 (Higher-order energy identity)**.** *Suppose $f = (f_1, ..., f_K)$, $f_i : [t_0, t_1] \times \bar{I} \to \mathbb{R}^n$ is a SGS to GP with the regularity $f_i \in C^\infty\left([t_0, t_1] \times \bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$. Then, for any $t \in [t_0, t_1]$, the quantity $\mathcal{Y}_m(t)$, defined in (1.12), satisfies*

$$\tfrac{d}{dt}\mathcal{Y}_m(t) + 2\sum_{i=1}^{K}\int_I |\nabla_s^{4m}\vec{\kappa}_i|^2\, ds = \sum_{i=1}^{K}\sum_{\substack{[\![a,b]\!]\le[\![8m-2,4]\!]\\ c\le 4m}}\int_I P_b^a(\vec{\kappa}_i)\, ds. \tag{4.6}$$

Proof. From (5.30), (5.24), (1.9), we have

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_I |\nabla_t^m f_i|^2\, ds = \int_I \langle \nabla_t^m f_i, \nabla_t^{m+1} f_i\rangle\, ds - \int_I \frac{1}{2}|\nabla_t^m f_i|^2 \cdot \langle \vec{\kappa}_i, \partial_t f_i\rangle\, ds\\
=&\int_I \langle \nabla_t^m f_i, \nabla_t^m(-\nabla_s^2\vec{\kappa}_i - \frac{|\vec{\kappa}_i|^2}{2}\vec{\kappa}_i + \lambda\cdot\vec{\kappa}_i)\rangle\, ds - \int_I \frac{1}{2}|\nabla_t^m f_i|^2\cdot\langle\vec{\kappa}_i, \partial_t f_i\rangle\, ds\\
=&-\int_I \langle \nabla_t^m f_i, \nabla_t^m\nabla_s^2\vec{\kappa}_i\rangle\, ds - \int_I \left(\langle\nabla_t^m f_i, \nabla_t^m(\frac{|\vec{\kappa}|^2}{2}\vec{\kappa}_i - \lambda\cdot\vec{\kappa}_i)\rangle + \frac{1}{2}|\nabla_t^m f_i|^2\cdot\langle\vec{\kappa}_i,\partial_t f_i\rangle\right)\, ds.
\end{aligned} \tag{4.7}$$

By applying (5.38) with $k = 0$ and $k = 2$ therein, we have

$$\int_I \left(\langle\nabla_t^m f_i, \nabla_t^m(\tfrac{|\vec{\kappa}_i|^2}{2}\vec{\kappa}_i - \lambda\cdot\vec{\kappa}_i)\rangle + \tfrac{1}{2}|\nabla_t^m f_i|^2\cdot\langle\vec{\kappa}_i,\partial_t f_i\rangle\right)\, ds = \sum_{\substack{[\![a,b]\!]\le[\![8m-2,4]\!]\\ c\le 4m}}\int_I P_b^{a,c}(\vec{\kappa}_i)\, ds. \tag{4.8}$$

By applying (5.41) and integration by parts, we have

$$\begin{aligned}
&\int_I \langle\nabla_t^m f_i, \nabla_t^m\nabla_s^2\vec{\kappa}_i\rangle\, ds\\
&= -\int_I \langle\nabla_s\nabla_t^m f_i, \nabla_t^m\nabla_s\vec{\kappa}_i\rangle\, ds + \langle\nabla_t^m f_i, \nabla_t^m\nabla_s\vec{\kappa}_i\rangle_{|\partial I} + \sum_{\substack{[\![a,b]\!]\le[\![8m-2,4]\!]\\ c\le 4m}}\int_I P_b^{a,c}(\vec{\kappa}_i)\, ds\\
&= -\int_I \langle\nabla_t^m\tau_i, \nabla_t^m\nabla_s\vec{\kappa}_i\rangle\, ds + \langle\nabla_t^m f_i, \nabla_t^m\nabla_s\vec{\kappa}_i\rangle_{|\partial I} + \sum_{\substack{[\![a,b]\!]\le[\![8m-2,4]\!]\\ c\le 4m}}\int_I P_b^{a,c}(\vec{\kappa}_i)\, ds,
\end{aligned} \tag{4.9}$$

where the last equality comes from applying (5.40) and (5.41). Again, by applying (5.41) and integration by parts, we have

$$\begin{aligned}
&\int_I \langle\nabla_t^m\tau_i, \nabla_t^m\nabla_s\vec{\kappa}_i\rangle\, ds\\
&= -\int_I \langle\nabla_s\nabla_t^m\tau_i, \nabla_t^m\vec{\kappa}_i\rangle\, ds + \langle\nabla_t^m\tau_i, \nabla_t^m\vec{\kappa}_i\rangle_{|\partial I} + \sum_{\substack{[\![a,b]\!]\le[\![8m-2,4]\!]\\ c\le 4m-1}}\int_I P_b^{a,c}(\vec{\kappa}_i)\, ds\\
&= -\int_I \langle\nabla_t^m\vec{\kappa}_i, \nabla_t^m\vec{\kappa}_i\rangle\, ds + \langle\nabla_t^m\tau_i, \nabla_t^m\vec{\kappa}_i\rangle_{|\partial I} + \sum_{\substack{[\![a,b]\!]\le[\![8m-2,4]\!]\\ c\le 4m-1}}\int_I P_b^{a,c}(\vec{\kappa}_i)\, ds,
\end{aligned} \tag{4.10}$$

where the last equality comes from applying (5.38) with $k = 1$ and $k = 2$ therein, i.e.,

$$\nabla_t^m\vec{\kappa}_i = \nabla_s\nabla_t^m\partial_s f_i + \sum_{[\![a,b]\!]\le[\![4m-2,3]\!]} P_b^a(\vec{\kappa}_i).$$

Thus, from (4.7), (4.8), (4.9) and (4.10), we have

$$\frac{1}{2}\frac{d}{dt}\int_I |\nabla_t^m f_i|^2\, ds + \int_I |\nabla_t^m \vec{\kappa}_i|^2\, ds \tag{4.11}$$
$$= -\langle \nabla_t^m f_i, \nabla_t^m \nabla_s \vec{\kappa}_i\rangle_{|\partial I} + \langle \nabla_t^m \tau_i, \nabla_t^m \vec{\kappa}_i\rangle_{|\partial I} + \sum_{\substack{[\![a,b]\!]\le[\![8m-2,4]\!]\\ c\le 4m}} \int_I P_b^{a,c}(\vec{\kappa}_i)\, ds.$$

Hence, from applying (5.38) with $k=2$ therein to (4.11), we have

$$\frac{d}{dt}\int_I |\nabla_t^m f_i|^2\, ds + 2\cdot \int_I |\nabla_s^{4m} \vec{\kappa}_i|^2\, ds \tag{4.12}$$
$$= -2\cdot \langle \nabla_t^m f_i, \nabla_t^m \nabla_s \vec{\kappa}_i\rangle_{|\partial I} + 2\cdot \langle \nabla_t^m \tau_i, \nabla_t^m \vec{\kappa}_i\rangle_{|\partial I} + \sum_{\substack{[\![a,b]\!]\le[\![8m-2,4]\!]\\ c\le 4m}} \int_I P_b^{a,c}(\vec{\kappa}_i)\, ds.$$

Therefore, by taking the sum $\sum_{i=1}^{K}$ in (4.12) and applying the boundary conditions (1.5) $\sim$ (1.7), we obtain

$$\frac{d}{dt}\sum_{i=1}^{K}\int_I |\nabla_t^m f_i|^2\, ds + 2\cdot \sum_{i=1}^{K}\int_I |\nabla_s^{4m} \vec{\kappa}_i|^2\, ds \tag{4.13}$$
$$= -2\cdot \sum_{i=1}^{K-1} \langle \nabla_t^m \tau_i(\cdot,1), \nabla_t^m\left[\triangle_i \vec{\kappa}\right]\rangle + \sum_{i=1}^{K}\sum_{\substack{[\![a,b]\!]\le[\![8m-2,4]\!]\\ c\le 4m}} \int_I P_b^{a,c}(\vec{\kappa}_i)\, ds.$$

Note that, from (1.7), we have

$$\nabla_t^{m+1}\tau_i(\cdot,1) = \nabla_t^m\left[\triangle_i \vec{\kappa}\right], \quad \forall\, i\in\{1,...,K-1\}.$$

Thus, from (4.13), we obtain (4.6). □

Lemma 4.4 (Uniform bounds for the derivatives of curvature of SGS)**.** *Assume $f=(f_1,...,f_K)$, $f_i:[t_0,t_1]\times\bar{I}\to\mathbb{R}^n$, is a SGS with the regularity $f\in C^\infty\left([t_0,t_1]\times\bar{I}_i\right)$, $\forall\, i\in\{1,...,K\}$. Then, $\forall\, t\in[t_0,t_1]$, $i\in\{1,...,K\}$, and $\ell\in\mathbb{N}$, we have*

$$\left\|\partial_s^\ell \vec{\kappa}_i\right\|_{L^\infty(I)} \le C\left(\mathcal{Y}_{m_\ell}(t_0), \mathcal{E}_\lambda[f(t_0,\cdot)], \lambda, p_0,...,p_K, K, m_\ell, n\right), \tag{4.14}$$

where $m_\ell := [\![\frac{\ell+2}{4}]\!]+1$.

Proof. Let $\delta\in(0,1/2)$ and rewrite (4.6) as

$$\frac{d}{dt}\mathcal{Y}_m(t) + \delta\cdot\mathcal{Y}_m(t) + 2\cdot\sum_{i=1}^{K}\int_I |\nabla_s^{4m}\vec{\kappa}_i|^2\, ds \tag{4.15}$$
$$= \delta\cdot\sum_{i=1}^{K}\int_I |\nabla_t^m f_i|^2\, ds + \delta\cdot\sum_{i=1}^{K-1}|\nabla_t^m \tau_i(\cdot,1)|^2 + \sum_{i=1}^{K}\sum_{\substack{[\![a,b]\!]\le[\![8m-2,4]\!]\\ c\le 4m}} \int_I P_b^{a,c}(\vec{\kappa}_i)\, ds.$$

From this equation, we would like to derive a differential inequality for $\mathcal{Y}_m$ by estimating the terms of lesser-order.

Step 1° From the energy identity in (4.2), $\mathcal{E}_\lambda[f]$ is non-increasing as t increases and

$$\mathcal{E}[f] + \lambda\cdot\mathcal{L}[f] =: \mathcal{E}_\lambda[f(t,\cdot)] \le \mathcal{E}_\lambda[f(t_0,\cdot)].$$

Hence, as $t \in [t_0, t_1]$,

$$\sum_{i=1}^{K} \int_I |\vec{\kappa}_i(t,\cdot)|^2 \, ds \leq 2 \cdot \mathcal{E}_\lambda \left[f(t_0,\cdot)\right] \tag{4.16}$$

and

$$\mathcal{L}\left[f(t,\cdot)\right] \leq \frac{\mathcal{E}_\lambda[f(t,\cdot)]}{\lambda} \leq \frac{\mathcal{E}_\lambda[f(t_0,\cdot)]}{\lambda} =: \mathcal{L}_+. \tag{4.17}$$

Note that, from the assumption on the regularity of f_i, the tangent indicatrix, τ_i, satisfies $\tau_i(t,\cdot) \in C^\infty(I, \mathbb{S}^{n-1})$, $\forall\, i \in \{1, ..., K\}$, and $\tau_i(t,1) = \tau_{i+1}(t,0)$, $\forall\, i \in \{1, ..., K-1\}$.

Assume $p_i \neq p_{i-1}$, $\forall\, i \in \{1, ..., K\}$. Then it is obvious that

$$\mathcal{L}[f_i](t) \geq |p_i - p_{i-1}| \gneqq 0. \tag{4.18}$$

On the other hand, if we assume $p_i = p_{i-1}$, then we apply Lemma 4.2 to obtain

$$\int_I |\vec{\kappa}_i| \, ds > \pi. \tag{4.19}$$

From (4.16), (4.19) and by applying Hölder's inequality, we obtain

$$\mathcal{L}[f_i] = \int_I ds \geq \frac{\left(\int_I |\vec{\kappa}_i| \, ds\right)^2}{\int_I |\vec{\kappa}_i|^2 \, ds} \geq \frac{\pi^2}{2 \cdot \mathcal{E}_\lambda[f(t_0,\cdot)]} \gneqq 0. \tag{4.20}$$

From (4.18) and (4.20), there is a positive constant

$$\mathcal{L}_-^{(i)} = \mathcal{L}_-^{(i)}(\mathcal{E}_\lambda[f(t_0,\cdot)], p_{i-1}, p_i) = \max\left\{\frac{\pi^2}{2 \cdot \mathcal{E}_\lambda[f(t_0,\cdot)]}, |p_i - p_{i-1}|\right\}$$

such that

$$\mathcal{L}[f_i] \geq \mathcal{L}_-^{(i)} \gneqq 0. \tag{4.21}$$

Thus, we conclude from (4.21) that

$$\mathcal{L}[f] = \sum_{i=1}^{K} \mathcal{L}[f_i] \geq \sum_{i=1}^{K} \mathcal{L}_-^{(i)} =: \mathcal{L}_- = \mathcal{L}_-(\mathcal{E}_\lambda[f(t_0,\cdot)], p_0, ..., p_K) \gneqq 0. \tag{4.22}$$

Step 2° Since the unit tangent vector field τ_i satisfy $\tau_i(t,1) = \tau_{i+1}(t,0)$ for all $i \in \{1, ..., K-1\}$. Hence, we may write

$$\tau_i(t,1) - \tau^{(0)} = \sum_{j=1}^{i} \left(\tau_j(t,1) - \tau_j(t,0)\right) = \sum_{j=1}^{i} \int_I \vec{\kappa}_j \, ds, \quad \forall\, t \in [t_0, t_1].$$

Then, by taking the differentiation ∇_t^m on both side, we have

$$\nabla_t^m \tau_i(t,1) = \sum_{j=1}^{i} \int_I \nabla_t^m (\vec{\kappa}_j \, ds) = \sum_{j=1}^{i} \int_I \sum_{m_1+m_2=m} C_{m_1}^m \cdot \nabla_t^{m_1} \vec{\kappa}_j \cdot \partial_t^{m_2}(ds),$$

where $C^m_{m_1} = \frac{m!}{m_1! \cdot m_2!}$. From applying (5.38) with $k = 2$ therein, and (5.42), we have

$$\nabla_t^{m_1} \vec{\kappa}_j \cdot \partial_t^{m-m_1}(ds) = \begin{cases} \left((-1)^m \nabla_s^{4m} \vec{\kappa}_j + \sum\limits_{[\![a,b]\!] \le [\![4m-2,3]\!]} P^a_b(\vec{\kappa}_j) \right) ds, & \text{as } m_1 = m, \\ \left(\sum\limits_{[\![a,b]\!] \le [\![4m-2,3]\!]} P^a_b(\vec{\kappa}_j) \right) ds, & \text{as } m_1 \in \{0, ..., m-1\}. \end{cases}$$

Hence,

$$\sum_{m_1+m_2=m} C^m_{m_1} \cdot \nabla_t^{m_1} \vec{\kappa}_j \cdot \partial_t^{m_2}(ds) = (-1)^m \nabla_s^{4m} \vec{\kappa}_j \ ds + \sum_{[\![a,b]\!] \le [\![4m-2,3]\!]} P^a_b(\vec{\kappa}_j) \ ds,$$

where the constant $C^m_{m_1}$ has been absorbed by the notation $P^a_b(\vec{\kappa}_j)$ as $m_1 < m$. Thus,

$$\begin{aligned} \left|\nabla_t^m \tau_i(t,1)\right|^2 &\le C \cdot \sum_{j=1}^K \left(\left(\int_I \left|\nabla_s^{4m} \vec{\kappa}_j\right| \ ds \right)^2 + \sum_{[\![a,b]\!] \le [\![4m-2,3]\!]} \left(\int_I \left|P^a_b(\vec{\kappa}_j)\right| \ ds \right)^2 \right) \\ &\le C \cdot \mathcal{L}[f] \cdot \sum_{j=1}^K \int_I \left|\nabla_s^{4m} \vec{\kappa}_j\right|^2 \ ds + C \cdot \mathcal{L}[f] \cdot \sum_{j=1}^K \sum_{\substack{[\![a,b]\!] \le [\![8m-4,6]\!] \\ c \le 4m-2}} \int_I \left|P^{a,c}_b(\vec{\kappa}_j)\right| \ ds, \end{aligned}$$

where $C = C(K, m)$. Therefore, the term $\sum\limits_{i=1}^{K-1} \left|\nabla_t^m \tau_i(t,1)\right|^2$ on the R.H.S. of (4.15) can be estimated by

$$\begin{aligned} \sum_{i=1}^{K-1} \left|\nabla_t^m \tau_i(t,1)\right|^2 \le & C_0(K,m) \cdot \mathcal{L}[f] \cdot \sum_{i=1}^K \int_I \left|\nabla_s^{4m} \vec{\kappa}_i\right|^2 \ ds \\ & + C_0(K,m) \cdot \mathcal{L}[f] \cdot \sum_{i=1}^K \sum_{\substack{[\![a,b]\!] \le [\![8m-4,6]\!] \\ c \le 4m-2}} \int_I \left|P^{a,c}_b(\vec{\kappa}_i)\right| \ ds. \end{aligned} \tag{4.23}$$

Note, from applying (5.38) with $k = 0$ therein, we have

$$\sum_{i=1}^K \int_I \left|\nabla_t^m f_i\right|^2 \ ds = \sum_{i=1}^K \sum_{\substack{[\![a,b]\!] \le [\![8m-4,2]\!] \\ c \le 4m-2}} \int_I P^{a,c}_b(\vec{\kappa}_i) \ ds. \tag{4.24}$$

Hence, from (4.15), (4.23), (4.24), we obtain

$$\begin{aligned} & \frac{d}{dt} \mathcal{Y}_m(t) + \delta \cdot \mathcal{Y}_m(t) + (2 - \delta \cdot C_0(K,m) \cdot \mathcal{L}[f]) \cdot \sum_{i=1}^K \int_I |\nabla_s^{4m} \vec{\kappa}_i|^2 \ ds \\ & \le \delta \cdot C_0(K,m) \cdot \mathcal{L}[f] \cdot \sum_{i=1}^K \sum_{\substack{[\![a,b]\!] \le [\![8m-4,6]\!] \\ c \le 4m-2}} \int_I \left|P^{a,c}_b(\vec{\kappa}_i)\right| \ ds \\ & + \delta \cdot \sum_{i=1}^K \sum_{\substack{[\![a,b]\!] \le [\![8m-4,2]\!] \\ c \le 4m-2}} \int_I P^{a,c}_b(\vec{\kappa}_i) \ ds + \sum_{i=1}^K \sum_{\substack{[\![a,b]\!] \le [\![8m-2,4]\!] \\ c \le 4m}} \int_I P^{a,c}_b(\vec{\kappa}_i) \ ds. \end{aligned} \tag{4.25}$$

Step 3° From the upper bound of the total length $\mathcal{L}_+$ defined in (4.17), we may choose a sufficiently small $\delta > 0$ so that

$$\delta \cdot C_0(K,m) \cdot \mathcal{L}_+ \leq 1$$

and then (4.25) gives

$$\frac{d}{dt}\mathcal{Y}_m(t) + \delta \cdot \mathcal{Y}_m(t) + \sum_{i=1}^{K} \int_I |\nabla_s^{4m} \vec{\kappa}_i|^2 \, ds \tag{4.26}$$
$$\leq (C_0(K,m)\cdot \mathcal{L}_+)^{-1} \sum_{i=1}^{K} \sum_{\substack{[\![a,b]\!] \leq [\![8m-4,2]\!] \\ c \leq 4m-2}} \int_I P_b^{a,c}(\vec{\kappa}_i) \, ds + \sum_{i=1}^{K} \sum_{\substack{[\![a,b]\!] \leq [\![8m-2,4]\!] \\ c \leq 4m}} \int_I P_b^{a,c}(\vec{\kappa}_i) \, ds.$$

From applying the interpolation inequality (5.37), the lower bound of total length in (4.22) and the upper bound of bending energy in (4.16), we have

$$R.H.S. \text{ of } (4.26) \leq \varepsilon \cdot \sum_{i=1}^{K} \int_I |\nabla_s^{4m} \vec{\kappa}_i|^2 \, ds + C\left(\mathcal{E}_\lambda[f(t_0,\cdot)], c_0, \lambda, p_0, ..., p_K, K, m, n, \varepsilon\right),$$

where $c_0 := \max\{1, (C_0(K,m)\cdot\mathcal{L}_+)^{-1}\}$. By choosing a sufficiently small $\varepsilon \in (0,1)$, we obtain from (4.26)

$$\frac{d}{dt}\mathcal{Y}_m(t) + \delta \cdot \mathcal{Y}_m(t) \leq C_1\left(\mathcal{E}_\lambda[f(t_0,\cdot)], \lambda, p_0, ..., p_K, K, m, n\right),$$

where $\delta = \delta(\mathcal{E}_\lambda[f(t_0,\cdot)], \lambda, p_0, ..., p_K, K, m, n) > 0$. Thus, Gronwall's differential inequality implies the uniform upper bound of $\mathcal{Y}_m(t)$, i.e.,

$$\mathcal{Y}_m(t) \leq e^{\delta \cdot t_0} \mathcal{Y}_m(t_0) + \frac{C_1}{\delta}, \quad \forall\, t \in [t_0, t_1].$$

Hence, $\forall\, t \in [t_0, t_1]$, where $t_0 > 0$ is sufficiently close to 0 such that $e^{\delta \cdot t_0} < 2$, we have

$$\sum_{i=1}^{K} \|\nabla_t^m f_i\|^2_{L^2(I)}(t) \leq 2\mathcal{Y}_m(t_0) + \frac{C_1}{\delta}, \tag{4.27}$$

where $\frac{C_1}{\delta}$ only depends on $\mathcal{E}_\lambda[f(t_0,\cdot)], \lambda, p_0, ..., p_K, K, m, n$.

Step 4° For each fixed $i \in \{1, ..., K\}$ and $t \in [t_0, t_1]$, we could estimate $\|\nabla_s^{4m-2} \vec{\kappa}_i\|^2_{L^2(I)}$ by applying (5.38) with $k = 0$ therein, the interpolation inequality (5.37), the upper bound of total bending energy $\sum_{i=1}^{K} \|\vec{\kappa}_i\|^2_{L^2(I)}$ in (4.16), and the upper bound of $\|\nabla_t^m f_i\|^2_{L^2(I)}$ in (4.27) to obtain

$$\|\nabla_s^{4m-2} \vec{\kappa}_i\|^2_{L^2(I)}(t) \leq C\left(\mathcal{Y}_m(t_0), \mathcal{E}_\lambda[f(t_0,\cdot)], \lambda, p_0, ..., p_K, K, m, n\right), \tag{4.28}$$

$\forall\, t \in [t_0, t_1]$, $\forall\, m \in \mathbb{N}$. Here, we denote by

$$m = m_\ell := [\![\frac{\ell+2}{4}]\!] + 1, \quad \forall\, \ell \in \mathbb{N},$$

where the notation $[\![A]\!]$ represents the greatest integer part of real number A. It is easy to verify that $\ell < 4m_\ell - 2$. Hence, we may apply the interpolation inequality (5.37), the upper bound of total bending energy $\sum_{i=1}^{K} \|\vec{\kappa}_i\|^2_{L^2(I)}$ in (4.16), Lemma 5.7, and (4.28) to obtain

$$\left\|\nabla_s^\ell \vec{\kappa}_i\right\|^2_{L^2(I)}(t) + \left\|\partial_s^\ell \vec{\kappa}_i\right\|^2_{L^2(I)}(t) \le C\left(\mathcal{Y}_{m_\ell}(t_0), \mathcal{E}_\lambda[f(t_0,\cdot)], \lambda, p_0, ..., p_K, K, m_\ell, n\right), \tag{4.29}$$

for any $t \in [t_0, t_1]$, $i \in \{1, ..., K\}$, and $\ell \in \mathbb{N}$.

For any differentiable function $g_i : \bar{I} \to \mathbb{R}^n$, it is easy to see that

$$\tilde{g}_i(s) := g_i(s) - \left(\int_{\sigma\in I} d\sigma\right)^{-1} \left(\int_{\sigma\in I} g_i(\sigma)\, d\sigma\right) \tag{4.30}$$

satisfies $\int_I \tilde{g}_i(s)\, ds = 0$ and hence a direct computation gives

$$\|\tilde{g}_i\|_{L^\infty(I)} \le c(n) \cdot \|\partial_s \tilde{g}_i\|_{L^1(I)}. \tag{4.31}$$

By letting $g_i = \partial_s^{\ell-1}\vec{\kappa}_i$ in (4.30) and (4.31), we derive

$$\left\|\partial_s^{\ell-1}\vec{\kappa}_i\right\|_{L^\infty(I)} \le c(n) \cdot \left\|\partial_s^{\ell}\vec{\kappa}_i\right\|_{L^1(I)} + \left(\int_I ds\right)^{-1} \cdot \left\|\partial_s^{\ell-1}\vec{\kappa}_i\right\|_{L^1(I)}. \tag{4.32}$$

By applying Hölder's inequality to the R.H.S. of (4.32), we obtain

$$\left\|\partial_s^{\ell-1}\vec{\kappa}_i\right\|_{L^\infty(I)} \le c(n) \cdot \left(\int_I ds\right)^{1/2} \left\|\partial_s^{\ell}\vec{\kappa}_i\right\|_{L^2(I)} + \left(\int_I ds\right)^{-1/2} \left\|\partial_s^{\ell-1}\vec{\kappa}_i\right\|_{L^2(I)}. \tag{4.33}$$

From applying the uniform bounds of total length in (4.17) and (4.21), and from the estimates in (4.29), we obtain from (4.33) that

$$\left\|\partial_s^{\ell-1}\vec{\kappa}_i\right\|_{L^\infty(I)} \le C\left(\mathcal{Y}_{m_\ell}(t_0), \mathcal{E}_\lambda[f(t_0,\cdot)], \lambda, p_0, ..., p_K, K, m_\ell, n\right), \tag{4.34}$$

which gives a uniform upper bound of $\left\|\partial_s^{\ell-1}\vec{\kappa}_i\right\|_{L^\infty(I)}$ for any $\ell \in \mathbb{N}$. □

To show that the regularity of SGS $f_i : [t_0, t_1) \times \bar{I} \to \mathbb{R}^n$ could be extended to $t = t_1$, i.e., $f_i \in C^\infty\left([t_0, t_1] \times \bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$. it remains to prove that, as $t \to t_1$, the parametrization speed $\gamma_i(t,x) = |\partial_x f_i(t,x)|$ stays uniformly bounded away from 0 and its derivatives stay uniformly bounded, i.e., $|\partial_x^\ell \gamma_i(t,x)| \le M_i^\ell$, $M_i^\ell \in (0,\infty)$, $\forall\, \ell \in \mathbb{N}$. The idea of the proof in the following lemma follows that in [10, Theorem 3.1].

Lemma 4.5 (Uniform bounds for the parametrization speed of SGS)**.** *Assume $f = (f_1, ..., f_K)$, $f_i : [t_0, t_1] \times \bar{I} \to \mathbb{R}^n$ is a SGS with the same regularity as those given in Lemma 4.4. Then, $\forall\, t \in [t_0, t_1]$, $\forall\, i \in \{1, ..., K\}$, and $\forall\, \ell \in \mathbb{N}$, we have*

$$\|\gamma_i(t_0,\cdot)\|_{L^\infty(\bar{I})} \cdot e^{-C_i(t_1-t_0)} \le \|\gamma_i(t,\cdot)\|_{L^\infty(\bar{I})} \le \|\gamma_i(t_0,\cdot)\|_{L^\infty(\bar{I})} \cdot e^{C_i(t_1-t_0)}, \tag{4.35}$$

$$\|\partial_x^\ell \gamma_i(t,\cdot)\|_{L^\infty(\bar{I})} \le M_i^\ell, \tag{4.36}$$

where

$$C_i = C_i\left(\mathcal{Y}_1(t_0), \mathcal{Y}_2(t_0), \mathcal{E}_\lambda[f_0], \lambda, p_0, ..., p_K, K, n\right) \in (0,\infty),$$

and $M_i^\ell \in (0,\infty)$ depends on $\|\gamma_i(t_0,\cdot)\|_{L^\infty(\bar{I})}, ..., \|\partial_x^\ell \gamma_i(t_0,\cdot)\|_{L^\infty(\bar{I})}$, $\mathcal{Y}_{m_1}(t_0), ..., \mathcal{Y}_{m_{\ell+3}}(t_0)$, $\mathcal{E}_\lambda[f(t_0,\cdot)]$, λ, $p_0, ..., p_K$, K, n, $|t_1 - t_0|$.

Proof. From (1.9) and a direct computation, we have

$$\partial_t \gamma_i = -\langle \vec{\kappa}_i, \vec{V}_i \rangle \gamma_i. \tag{4.37}$$

By integration for fixed x in (4.37), we have

$$\gamma_i(t,x) = \gamma_i(t_0,x) \cdot e^{-\int_{t_0}^{t} \langle \vec{\kappa}_i(\tau,x), \vec{V}_i(\tau,x)\rangle \, d\tau}, \qquad \forall\, x \in \bar{I}. \tag{4.38}$$

From (4.34) and Lemma 5.7, we obtain

$$\left\| \langle \vec{\kappa}_i, \vec{V}_i \rangle \right\|_{L^\infty(\bar{I})} \le C_i \left(\mathcal{Y}_1(t_0), \mathcal{Y}_2(t_0), \mathcal{E}_\lambda[f_0], \lambda, p_0, ..., p_K, K, n \right), \tag{4.39}$$

From (4.39) and (4.38), we conclude (4.35).

Note that, for any vector field $h_i : \bar{I} \to \mathbb{R}^n$ and $\ell \in \mathbb{N}$,

$$\partial_x^\ell h_i = \gamma_i^\ell \partial_s^\ell h_i + \sum_{k=1}^{\ell-1} P_{\ell-1}(\gamma_i, ..., \partial_x^{\ell-k}\gamma_i) \partial_s^k h_i \tag{4.40}$$

where $P_{\ell-1}$ a polynomial of degree at most $\ell - 1$. A bound of $\left\| \partial_x^\ell \vec{\kappa}_i \right\|_{L^\infty(\bar{I})}$ follows from taking $h_i = \vec{\kappa}_i$ in (4.40), from uniform bounds of $\left\| \partial_s^k \vec{\kappa}_i \right\|_{L^\infty(\bar{I})}$, and from uniform bounds of $\left\| \partial_x^k \gamma_i \right\|_{L^\infty(\bar{I})}$, $\forall\, k \in \{0,1,...,\ell\}$. Thus it remains to prove that $\left\| \partial_x^\ell \gamma_i \right\|_{L^\infty(\bar{I})}$ is uniformly bounded, $\forall\, \ell \in \mathbb{N}$. Assume inductively that

$$\left\| \partial_x^k \gamma_i \right\|_{L^\infty(I)} \le C_i \left(\mathcal{Y}_{m_1}(t_0), ..., \mathcal{Y}_{m_{k+3}}(t_0), \mathcal{E}_\lambda[f(t_0,\cdot)], \lambda, p_0, ..., p_K, K, m_1, ..., m_{k+3}, n, |t_0 - t_1| \right)$$

$\forall\, k \in \{0, ..., \ell - 1\}$. Then, by applying (4.40), (4.29),(4.34), and (4.35), we obtain

$$\left\| \partial_x^\ell \langle \vec{\kappa}_i, \vec{V}_i \rangle \right\|_{L^\infty(I)} \le C_i, \tag{4.41}$$

where C_i depends on $\left\| \gamma_i(t_0,\cdot) \right\|_{L^\infty(I)}, ..., \left\| \partial_x^{\ell-1}\gamma_i(t_0,\cdot) \right\|_{L^\infty(I)}$, $\mathcal{Y}_{m_1}(t_0), ..., \mathcal{Y}_{m_{\ell+3}}(t_0)$, $\mathcal{E}_\lambda[f(t_0,\cdot)]$, λ, $p_0, ..., p_K, K$, $m_1, ..., m_{\ell+3}, n, |t_0 - t_1|$. By differentiating (4.37) ℓ-times with respect to x, we obtain

$$\partial_t \partial_x^\ell \gamma_i = -\langle \vec{\kappa}_i, \vec{V}_i \rangle \partial_x^\ell \gamma_i - \sum_{0 \le k \le \ell-1} c(\ell,k) \cdot \partial_x^{\ell-k} \langle \vec{\kappa}_i, \vec{V}_i \rangle \cdot \partial_x^k \gamma_i$$

for some coeficients $c(\ell,k)$, which in turn implies (4.36), by applying (4.41) and inductive hypothesis in the linear ODE, like the type of $Y_i'(t) = m_i(t) \cdot Y_i(t) + \ell_i(t)$.

□

Lemma 4.6 (Rigidity in the parametrization of SGS). *Assume that both $f = (f_1, ..., f_K)$, $g = (g_1, ..., g_K)$, $f_i, g_i : [t_0, t_1] \times \bar{I} \to \mathbb{R}^n$ are SGS to (1.5)∼(1.9) representing the same family of curves with the regularity $f_i, g_i \in C^\infty\left([t_0,t_1] \times \bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$. Suppose that, for some $t_* \in [t_0, t_1]$, there exists diffeomorphisms $\chi_i : \bar{I} \to \bar{I}$, $\forall\, i \in \{1, ..., K\}$, such that $f_i(t_*, x) = g_i(t_*, \chi_i(x))$, $\forall\, x \in \bar{I}$. Then, $f_i(t,x) = g_i(t, \chi_i(x))$, $\forall\, (t,x) \in [t_0,t_1] \times \bar{I}$, $\forall\, i \in \{1, ..., K\}$.*

Proof. From the assumption, we may let $f_i(t,x) = g_i(t, \chi_i(t,x))$, $\forall\, (t,x) \in [t_0,t_1] \times \bar{I}$, where $\chi_i \in C^\infty\left([t_0,t_1] \times \bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$. Then,

$$\begin{aligned} \partial_t f_i(t,x) &= \partial_t g_i(t, \chi_i(t,x)) + \partial_{\chi_i} g_i(t, \chi_i(t,x)) \cdot \partial_t \chi_i(t,x) \\ &= \vec{V}_{g_i}(t, \chi_i(t,x)) + \partial_{\chi_i} g_i(t, \chi_i(t,x)) \cdot \partial_t \chi_i(t,x). \end{aligned} \tag{4.42}$$

Since the tangential component of $\partial_t f_i$ is null, it forces that $\partial_t \chi_i \equiv 0$. Thus, together with the assumption, we have $\chi_i(t,x) = \chi_i(t_*,x) =: \chi_i(x)$, $\forall\, (t,x) \in [t_0,t_1] \times \bar{I}$. □

4.2 Proof of the long-time existence

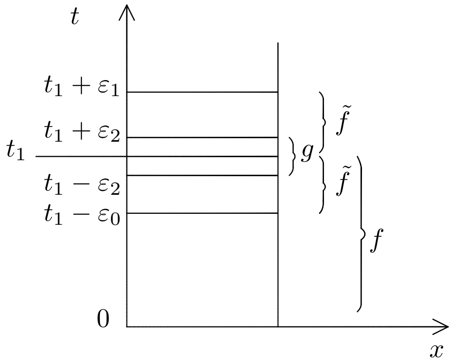

Figure 2: A sketch of the construction with AS and SGS in different time intervals: both f and g are SGS; while $\tilde{f}$ is an AS.

Below we give an argument by contradiction. Namely, we assume on the contrary that $t_1' = t_{\max} < \infty$ is the maximum time to the long-time existence of SGS. In fact, the argument below will show that the speed of parametrization of any SGS remains strictly positive for any $t \in [t_0, +\infty)$. Similar argument also appears in [7].

Step 1° (convert SGS f into AS $\tilde{f}$ on the time interval $[t_1 - \varepsilon_0, t_1]$ for some $\varepsilon_0 > 0$.)

Let $t_1 < t_1'$ be sufficiently close to t_1' and $\varepsilon_0 > 0$. To convert SGS $f = (f_1, ..., f_K)$ into AS $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_K)$ on the time interval $[t_1 - \varepsilon_0, t_1]$, we need to find a family of diffeomorphisms $\eta_i : \bar{I} \to \bar{I}$, $\forall\, i \in \{1, ..., K\}$, so that $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_K)$, $\tilde{f}_i = f_i \circ \eta_i^{-1}$, is an AS. The required argument is provided by Lemma 3.2. Note that Lemmas 4.4 and 4.5 provide the required uniform bounds on δ_0, $\tilde{\delta}_0$, and M_0, so that $\tilde{f}$ becomes AS on the closed time interval $[t_1 - \varepsilon_0, t_1]$.

Step 2° (extend the AS $\tilde{f}$ by obtaining an AS $\tilde{f} : [t_1, t_1 + \varepsilon_1]$, $\varepsilon_1 > 0$.)

Note that, from Theorem 2.2, $\tilde{f}(t_1, \cdot)$ fulfills the compatibility conditions of any order, defined in Definition 2.2. Now, we choose $\tilde{f}(t_1, \cdot)$ as the initial datum to the AP (2.11). Then, we apply Theorem 2.2 to obtain an AS over the time interval $[t_1, t_1 + \varepsilon_1]$, which is still denoted by $\tilde{f}$, such that $\tilde{f}_i \in C^{\frac{5+\alpha}{4}, 5+\alpha}([t_1, t_1 + \varepsilon_1] \times \bar{I})$, $\forall\, i \in \{1, ..., K\}$. The regularity $\tilde{f}_i \in C^\infty([t_1, t_1 + \varepsilon_1] \times \bar{I})$ is obtained from applying boot-strapping argument and the linear theory in Theorem 2.4. Notice that, from Theorem 2.4, ε_1 is uniformly bounded away from 0, i.e., independent of the choice of ε_0, $t_1 + \varepsilon_1 > t_1'$ can be achieved by choosing a sufficiently small $\varepsilon_0 > 0$ (see [6] for similar argument). Note that $\tilde{f}_i \in C^\infty([t_1, t_1 + \varepsilon_1] \times \bar{I})$ and $\tilde{f}_i \in C^\infty([t_1 - \varepsilon_0, t_1] \times \bar{I})$, $\forall\, i \in \{1, ..., K\}$, imply that the smooth solution $\tilde{f}_i$ is extended smoothly from $[t_1 - \varepsilon_0, t_1]$ to $[t_1 - \varepsilon_0, t_1 + \varepsilon_1]$, $\forall\, i \in \{1, ..., K\}$. Therefore, we obtain an AS $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_K)$, $\tilde{f}_i : [t_1 - \varepsilon_0, t_1 + \varepsilon_1] \times \bar{I} \to \mathbb{R}^n$ such that $\tilde{f}_i \in C^\infty([t_1 - \varepsilon_0, t_1 + \varepsilon_1] \times \bar{I})$, $\forall\, i \in \{1, ..., K\}$.

Step 3° (obtain a SGS g with $g_i \in C^\infty((t_1 - \varepsilon_2, t_1 + \varepsilon_2] \times \bar{I})$ from the AS $\tilde{f}$ with $\tilde{f}_i \in C^\infty((t_1 - \varepsilon_0, t_1 + \varepsilon_1] \times \bar{I})$.)

By applying Lemma 3.1, we obtain a family of diffeomorphisms $\sigma_i(t, \cdot) : \bar{I} \to \bar{I}$, $\forall\, t \in [t_1 - \varepsilon_2, t_1 + \varepsilon_2]$, for some $\varepsilon_2 \in (0, min\{\varepsilon_0, \varepsilon_1\})$, such that

$$g_i(t, z) = \tilde{f}_i(t, \sigma_i(t, z)), \qquad \forall\, i \in \{1, ..., K\},$$

consist a SGS g.

Step 4° (the existence of diffeomorphisms $\chi_i : \bar{I} \to \bar{I}$, $\forall\, i \in \{1, ..., K\}$, s.t. $f_i(t, x) = g_i(t, \chi_i(x))$ for $t \in [t_1 - \varepsilon_2, t_1]$.)

From the previous steps, we can write $f_i(t_1, x) = g_i(t_1, \chi_i(x))$ for some diffeomorphisms $\chi_i : \bar{I} \to \bar{I}$, $i \in \{1, ..., K\}$. By applying the rigidity of parametrization of SGS in Lemma 4.6, we conclude that

$$f_i(t, x) = g_i(t, \chi_i(x)), \qquad \forall\, (t, x) \in [t_1 - \varepsilon_2, t_1] \times \bar{I}. \tag{4.43}$$

Denote by

$$\tilde{g}_i(t, x) = g_i(t, \chi_i(x)), \qquad \forall\, (t, x) \in [t_1 - \varepsilon_2, t_1 + \varepsilon_2] \times \bar{I}, \tag{4.44}$$

$i \in \{1, ..., K\}$. Since $\tilde{g}$ is obtained from time-independent reparametrization of a SGS g, it is easy to verify that $\tilde{g}$ is also a SGS. Notice that from (4.43) and (4.44), we conclude

$$\tilde{g}_i(t, x) = f_i(t, x), \qquad \forall\, (t, x) \in [t_1 - \varepsilon_2, t_1] \times \bar{I},\ i \in \{1, ..., K\}.$$

Hence, the SGS f is extended beyond t_1', if t_1 was chosen sufficiently close to t_1' so that $t_1 + \varepsilon_2 > t_1'$. Now we arrive a contradiction to the assumption that $t_1' = t_{\max}$ is the maximum time to the long-time existence.

Step 5° Asymptotic behavior.

On the asymptotic behavior of the flow, we choose a subsequence of curves $f(t, \cdot) = (f_1(t, \cdot), ..., f_K(t, \cdot))$, so that each $f_i(t_j, \cdot)$ converges smoothly to $f_{\infty,i}(\cdot)$ as $t_j \to \infty$. Let

$$u\,(t) := \sum_{i=1}^{K} \int_I |\partial_t f_i|^2 \, ds.$$

By applying (4.27), we derive the inequality,

$$\left| \frac{d}{dt} u\,(t) \right| \le C\left(\mathcal{Y}_1(t_0), \mathcal{Y}_2(t_0), \mathcal{E}_\lambda[f(t_0, \cdot)], \lambda, p_0, ..., p_K, K, n\right), \forall\, t \in [t_0, \infty).$$

On the other hand, the energy identity in (4.2) implies that $u\,(t) \in L^1\left([t_0, \infty)\right)$. Therefore $u\,(t) \to 0$ as $t \to \infty$, which implies that f_∞ is independent of t. Therefore, from the equation of elastic flow (1.9), f_∞ is an equilibrium of $\mathcal{E}_\lambda$ on I with the uniform bound of any higher-order derivatives in (4.34), i.e., $f_{\infty,i} \in C^\infty(\bar{I})$, $\forall\, i \in \{1, ..., K\}$. Besides, from the boundary condition in (1.7), Γ_{f_∞} is C^2-smooth. Notice that, from Lemma 4.5, the speed of parametrization also remains uniformly bounded away from 0 and ∞, as $t \to \infty$. Thus, the smoothness of Γ_{f_∞} applies not only geometrical (differentiation w.r.t. arclength parameter) but also analytical (differentiation w.r.t. x).

5 Appendix

5.1 Supporting materials for the diffeomorphisms converting SGS to AS

5.1.1 Some formulae related to change of variables

Let $f = (f_1, ..., f_K)$ be a SGS fulfilling (1.5)~(1.9) with initial condition $f_0(0, x) = f_0(x)$, where $f_0 = (f_{0,1}, ..., f_{0,K})$. Denote by $\tilde{f}_i = \tilde{f}_i(t, y)$, $y = \eta_i(t, x)$, $f_i = f_i(t, x)$,

$\tilde{f}_i(t,\eta_i(t,x)) = f_i(t,x)$, where $\eta_i(t,\cdot)$ is a diffeomorphism, $\forall\, i$. A straightforward computation shows that

$$
\begin{aligned}
\partial_y \tilde{f}_i =& \frac{\partial_x f_i}{\partial_x \eta_i}, && (5.1)\\
\partial_y^2 \tilde{f}_i =& \frac{1}{(\partial_x\eta_i)^2}\cdot \partial_x^2 f_i - \frac{\partial_x^2\eta_i}{(\partial_x\eta_i)^3}\cdot \partial_x f_i, && (5.2)\\
\partial_y^3 \tilde{f}_i =& \frac{1}{(\partial_x\eta_i)^3} - \frac{3\partial_x^2\eta_i}{(\partial_x\eta_i)^4}\cdot \partial_x^2 f_i + \left(\frac{3(\partial_x^2\eta_i)^2}{(\partial_x\eta_i)^5} - \frac{\partial_x^3\eta_i}{(\partial_x\eta_i)^4}\right)\cdot \partial_x f_i, && (5.3)\\
\partial_y^4 \tilde{f}_i =& \frac{1}{(\partial_x\eta_i)^4}\cdot \partial_x^4 f_i - \frac{6\partial_x^2\eta_i}{(\partial_x\eta_i)^5}\cdot \partial_x^3 f_i - \left(\frac{4\partial_x^3\eta_i}{(\partial_x\eta_i)^5} - \frac{15(\partial_x^2\eta_i)^2}{(\partial_x\eta_i)^6}\right)\cdot \partial_x^2 f_i \\
&+\left(\frac{10\partial_x^3\eta_i\cdot\partial_x^2\eta_i}{(\partial_x\eta_i)^6} - \frac{15(\partial_x^2\eta_i)^3}{(\partial_x\eta_i)^7} - \frac{\partial_x^4\eta_i}{(\partial_x\eta_i)^5}\right)\cdot \partial_x f_i, && (5.4)\\
\partial_t \tilde{f}_i =& \partial_t f_i - \frac{\partial_t \eta_i}{\partial_x\eta_i}\cdot \partial_x f_i, && (5.5)\\
\partial_t\partial_y \tilde{f}_i =& \frac{\partial_t\partial_x f_i}{\partial_x\eta_i} - \frac{\partial_t\eta_i}{(\partial_x\eta_i)^2}\cdot \partial_x^2 f_i + \frac{\partial_t\eta_i\cdot\partial_x^2\eta_i}{(\partial_x\eta_i)^3}\cdot\partial_x f_i - \frac{\partial_t\partial_x\eta_i}{(\partial_x\eta_i)^2}\cdot\partial_x f_i. && (5.6)
\end{aligned}
$$

- **Deriving** (3.14). Note that from (1.9) and (2.2), a SGS f fulfills

$$\partial_t f_i + \mathcal{D}(f_i) = \langle \mathcal{D}(f_i), \tau_i\rangle \frac{\partial_x f_i}{|\partial_x f_i|}, \text{ in } (0,T)\times I, \forall\, i\in\{1,...,K\}. \tag{5.7}$$

From (5.1)$\sim$(5.4) and a complex but straightforward computation, we can verify that

$$
\begin{aligned}
\mathcal{D}(\tilde{f}_i) =& \mathcal{D}(f_i) - \frac{\partial_x f_i}{\partial_x\eta_i}\left[\frac{\partial_x^4\eta_i}{|\partial_x f_i|^4} - \frac{6\langle\partial_x^2 f_i, \partial_x f_i\rangle}{|\partial_x f_i|^6}\cdot\partial_x^3\eta_i\right]\\
&+\frac{\partial_x f_i}{\partial_x\eta_i}\cdot\left[\frac{4\langle\partial_x^3 f_i,\partial_x f_i\rangle}{|\partial_x f_i|^6} + \frac{5}{2}\frac{|\partial_x^2 f_i|^2}{|\partial_x f_i|^6} - \frac{35}{2}\frac{\langle\partial_x^2 f_i,\partial_x f_i\rangle^2}{|\partial_x f_i|^8} + \frac{\lambda}{|\partial_x f_i|^2}\right]\cdot\partial_x^2\eta_i\\
=& \mathcal{D}(f_i) - \frac{\partial_x f_i}{\partial_x\eta_i}\left[\frac{\partial_x^4\eta_i}{|\partial_x f_i|^4} - H_{f_i}(\eta_i) + \frac{\langle\mathcal{D}(f_i),\tau_i\rangle}{|\partial_x f_i|}\partial_x\eta_i\right], && (5.8)
\end{aligned}
$$

where $H_{f_i}(\eta_i)$ is defined in (3.16). From (5.5), (5.7) and (5.8), we have

$$
\begin{aligned}
\partial_t\tilde{f}_i + \mathcal{D}(\tilde{f}_i) &= \left(\partial_t f_i - \frac{\partial_t\eta_i}{\partial_x\eta_i}\cdot\partial_x f_i\right) + \left[\mathcal{D}(f_i) - \frac{\partial_x f_i}{\partial_x\eta_i}\left(\frac{\partial_x^4\eta_i}{|\partial_x f_i|^4} - H_{f_i}(\eta_i) + \frac{\langle\mathcal{D}(f_i),\tau_i\rangle}{|\partial_x f_i|}\partial_x\eta_i\right)\right]\\
&= (\partial_t f_i + \mathcal{D}(f_i)) - \frac{\partial_x f_i}{\partial_x\eta_i}\left[\partial_t\eta_i + \frac{\partial_x^4\eta_i}{|\partial_x f_i|^4} - H_{f_i}(\eta_i) + \frac{\langle\mathcal{D}(f_i),\tau_i\rangle}{|\partial_x f_i|}\partial_x\eta_i\right]\\
&= -\frac{\partial_x f_i}{\partial_x\eta_i}\left[\partial_t\eta_i + \frac{\partial_x^4\eta_i}{|\partial_x f_i|^4} - H_{f_i}(\eta_i)\right]. && (5.9)
\end{aligned}
$$

Now, (3.14) is verified.

- **Deriving** (3.18). As f is a SGS, it is easy to verify from (1.7) and (2.8) that the normal components in (3.20) vanish, i.e.,

$$B(f_i)(t,x^*) = \langle B(f_i)(t,x^*), \tau_i(t,x^*)\rangle\tau_i(t,x^*), \qquad \forall\,(t,x^*)\in[0,T]\times\{0,1\}, i\in\{1,...,K\}. \tag{5.10}$$

It follows from (5.6), (5.2), (3.28), (5.10) and the requirement $\tau_{i+1}(t,0)=\tau_i(t,1)$, $\forall\, i\in\{1,...,K-1\}$, that

$$\begin{aligned}
&\partial_t\partial_y\tilde{f}_i(t,1)-\left[\frac{\partial_y^2\tilde{f}_{i+1}(t,0)}{|\partial_y\tilde{f}_{i+1}(t,0)|}-\frac{\partial_y^2\tilde{f}_i(t,1)}{|\partial_y\tilde{f}_i(t,1)|}\right]\\
=&\left(\frac{\partial_t\partial_x f_i(t,1)}{\partial_x\eta_i(t,1)}-\frac{\partial_t\partial_x\eta_i(t,1)}{(\partial_x\eta_i(t,1))^2}\cdot\partial_x f_i(t,1)\right)-\frac{\partial_x^2 f_{i+1}(t,0)}{(\partial_x\eta_{i+1}(t,0))^2}\cdot\frac{\partial_x\eta_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|}\\
&+\frac{\partial_x f_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|}\frac{\partial_x^2\eta_{i+1}(t,0)}{(\partial_x\eta_{i+1}(t,0))^2}+\frac{\partial_x^2 f_i(t,1)}{(\partial_x\eta_i(t,1))^2}\frac{\partial_x\eta_i(t,1)}{|\partial_x f_i(t,1)|}-\frac{\partial_x f_i(t,1)}{|\partial_x f_i(t,1)|}\frac{\partial_x^2\eta_i(t,1)}{(\partial_x\eta_i(t,1))^2}\\
=&\frac{|\partial_x f_i(t,1)|}{\partial_x\eta_i(t,1)}\left[\frac{\partial_t\partial_x f_i(t,1)}{|\partial_x f_i(t,1)|}-\left(\frac{\partial_x^2 f_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|^2}-\frac{\partial_x^2 f_i(t,1)}{|\partial_x f_i(t,1)|^2}\right)\right]\\
&-\frac{\partial_x f_i(t,1)}{(\partial_x\eta_i(t,1))^2}\left[\partial_t\partial_x\eta_i(t,1)-\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|^2}\cdot\partial_x^2\eta_{i+1}(t,0)+\frac{\partial_x^2\eta_i(t,1)}{|\partial_x f_i(t,1)|}\right]\\
&+\left[\frac{|\partial_x f_i(t,1)|}{\partial_x\eta_i(t,1)}-\frac{|\partial_x f_{i+1}(t,0)|}{\partial_x\eta_{i+1}(t,0)}\right]\frac{\partial_x^2 f_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|^2}\\
&+\frac{\tau_i(t,1)\partial_x^2\eta_{i+1}(t,0)}{|\partial_x f_{i+1}(t,0)|^2}\left[\frac{|\partial_x f_{i+1}(t,0)|^2}{(\partial_x\eta_{i+1}(t,0))^2}-\frac{|\partial_x f_i(t,1)|^2}{|\partial_x\eta_i(t,1)|^2}\right]\\
=&\frac{|\partial_x f_i(t,1)|}{\partial_x\eta_i(t,1)}B(f_i)(t,1)-\frac{\partial_x f_i(t,1)}{(\partial_x\eta_i(t,1))^2}\cdot[\partial_t\partial_x\eta_i(t,1)-L_{f_i}(\eta_i)(t,1)]\\
&-\left[\partial_x\eta_i(t,1)-\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\partial_x\eta_{i+1}(t,0)\right]F_{i,0}(\eta_i,\eta_{i+1})(t),\\
=&-\frac{\partial_x f_i(t,1)}{(\partial_x\eta_i(t,1))^2}\cdot[\partial_t\partial_x\eta_i(t,1)-L_{f_i}(\eta_i)(t,1)-\langle B(f_i)(t,1),\tau_i(t,1)\rangle\partial_x\eta_i(t,1)]\\
&-\left[\partial_x\eta_i(t,1)-\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\partial_x\eta_{i+1}(t,0)\right]F_{i,0}(\eta_i,\eta_{i+1})(t),
\end{aligned}\tag{5.11}$$

where $L_{f_i}(\eta_i)$, $B(f_i)$, and $F_{i,0}(\eta_i,\eta_{i+1})$ are defined in (3.19), (3.20), and (3.21), respectively.

Similarly, for all $i\in\{1,...,K-1\}$, we have

$$\begin{aligned}
&\partial_t\partial_y\tilde{f}_{i+1}(t,0)-\left[\frac{\partial_y^2\tilde{f}_{i+1}(t,0)}{|\partial_y\tilde{f}_{i+1}(t,0)|}-\frac{\partial_y^2\tilde{f}_i(t,1)}{|\partial_y\tilde{f}_i(t,1)|}\right]\\
=&-\frac{\partial_x f_{i+1}(t,0)}{(\partial_x\eta_{i+1}(t,0))^2}\left[\partial_t\partial_x\eta_{i+1}(t,0)-L_{f_{i+1}}(\eta_i)(t,0)-\langle B(f_{i+1})(t,0),\tau_{i+1}(t,0)\rangle\partial_x\eta_{i+1}(t,0)\right]\\
&-\left[\partial_x\eta_i(t,1)-\frac{|\partial_x f_i(t,1)|}{|\partial_x f_{i+1}(t,0)|}\partial_x\eta_{i+1}(t,0)\right]F_{i,1}(\eta_i,\eta_{i+1})(t).
\end{aligned}\tag{5.12}$$

Now, (3.18) is proved.

5.1.2 The argument on the contraction-map in the proof of Lemma 3.2

The proof on the existence of the family of diffeomorphisms η in Lemma 3.2 is proceeded as follows. In Step 1° below, the linear problem (3.32) is well-posed so that we can apply Solonnikov's theory (see Theorem 5.14) to derive existence of solutions to the linear equation. Hence, we are able to define the operators

$$\begin{aligned}
\mathcal{G}: X_{\eta_0}^T\cap B_{\tilde{M}_0}&\to X_{\eta_0}^T\cap B_{\tilde{M}_0}\\
\bar{\eta}&\mapsto\eta
\end{aligned}$$

where $\tilde{M}_0$ is given by (5.17) below and η is the solution to (3.32).
In Step 2° below, we show that $\mathcal{G} : X^{T_3}_{\eta_0} \cap B_{\tilde{M}_0} \to X^{T_3}_{\eta_0} \cap B_{\tilde{M}_0}$ is well-defined and is a contraction-map for some $T_3 = T_3\left(\delta_0, \lambda, M_0, \tilde{M}_0\right) > 0$. Then a fixed point η of this map is a solution to the IBVP (3.26)∼(3.29), with the regularity $\eta_i \in C^{\frac{4+\alpha}{4}, 4+\alpha}(D^{T_3})$.
In Step 3° below, we prove $\eta_i \in C^{\frac{5+\alpha}{4}, 5+\alpha}\left(D^{T_3}\right) \bigcap C^\infty\left((0, T_3] \times \bar{I}\right)$, $\forall\, i \in \{1, ..., K\}$.

Step 1° Let $\eta_0 = (\eta_{0,1}, ..., \eta_{0,K})$, $\eta_{0,i} \in C^{4,\alpha}(\bar{I})$, and η_0 fulfill the compatibility conditions of order 1 to the IBVP (3.26)∼(3.29). For any $T > 0$, let

$$X^T_{\eta_0} := \left\{\eta = (\eta_1, ..., \eta_K), \eta_i : D^T \to \mathbb{R} : \eta_i \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left(D^T\right), \eta_i(0, \cdot) = \eta_{0,i}(\cdot), i \in \{1, ..., K\}\right\}$$

be a subset in the Banach space associated with the norm

$$\|\eta\|_{X^T_{\eta_0}} := \sum_{i=1}^K \|\eta_i\|_{C^{\frac{4+\alpha}{4}, 4+\alpha}(D^T)}.$$

Denote by $B_M = \{\eta \in X^T_{\eta_0} : \|\eta\|_{X^T_{\eta_0}} \le M\}$ the closed, bounded, and convex subset.

By applying the same argument as in Lemma 2.3, for any $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_K) \in X^T_{\eta_0} \cap B_{\tilde{M}_0}$, there exists $T_1 \in (0, 1)$ such that

$$|\partial_x \tilde{\eta}_i(t, x)| \ge \frac{\delta_0^{2K-2}}{4}, \qquad \forall\, (t, x) \in D^{T_1},\ \forall\, i \in \{1, ..., K\}.$$

Let $\tilde{M}_0$ be the one given by (5.17), and $C_0 = C_0(\delta_0, \lambda, M_0) > 1$ be a sufficiently large constant. Then, for any $\tilde{\eta} \in X^{T_1}_{\eta_0} \cap B_{\tilde{M}_0}$, there exists unique solution of the linear problem equation (3.32) $\eta \in X^{T_1}_{\eta_0} \cap B_{\tilde{M}_0}$, with the regularity $\eta_i \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left(D^{T_1}\right)$, $\forall\, i \in \{1, ..., K\}$. This can be achieved by applying the same argument in the proof of Theorem 2.4. Moreover, there is a constant $C_0 = C_0(\delta_0)$ such that, for any $T \in (0, T_1]$, we have

$$\begin{aligned}\sum_{i=1}^K \|\eta_i\|_{C^{\frac{4+\alpha}{4}, 4+\alpha}(D^T)} \le & C_0\left(\sum_{i=1}^K \|H_{f_i}(\tilde{\eta}_i)\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)} + \sum_{i=1}^K \left\|b_{f_i}\left(\tilde{\eta}_i\right)_{|[0,T]\times\partial I}\right\|_{C^{\frac{3+\alpha}{4}}([0,T])}\right), \\ & + C_0\left(K + \sum_{i=1}^K \|\eta_{0,i}\|_{C^{4,\alpha}(\bar{I})}\right),\end{aligned} \tag{5.13}$$

where $H_{f_i}(\tilde{\eta}_i)$ and $b_{f_i}(\tilde{\eta}_i)$ are given by (3.16) and (3.30), respectively. Since the argument is similar to that in the proof of Theorem 2.4, we skip the details.

Step 2° We show that $\mathcal{G} : X^T_{\eta_0} \cap B_{\tilde{M}_0} \to X^T_{\eta_0} \cap B_{\tilde{M}_0}$ is well-defined and is a contraction-map for some $T > 0$.

• **Self-maps.** We show that $\exists\ T_2 \in (0, T_1)$ with $\mathcal{G}\left(X^T_{\eta_0} \cap B_{\tilde{M}_0}\right) \subset X^T_{\eta_0} \cap B_{\tilde{M}_0}$, $\forall\, T \in (0, T_2)$.
By applying (3.12), and Lemmas 5.9, 5.10, 5.12, we have

$$\begin{aligned}\|H_{f_i}(\tilde{\eta}_i)_i - H_{f_i}(\eta_{0,i})\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)} \le & \sum_{j=1}^3 C_0(\delta_0, \lambda, M_0)\|\partial_x^j \tilde{\eta}_i - \partial_x^j \eta_{0,i}\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)} \\ \le & C_0(\delta_0, \lambda, M_0, \tilde{M}_0) T^{\frac{\alpha}{4}}, \forall\, i \in \{1, ..., K\},\end{aligned} \tag{5.14}$$

where $H_{f_i}(\eta_{0,i})$ is given by replacing η_i by $\eta_{0,i}$ in (3.16).

By applying the same argument in (2.33), and (2.35), we find

$$\sum_{i=1}^{K} \|b_{f_i}(\tilde{\eta}_i)(\cdot, x^*)\|_{C^{\frac{3+\alpha}{4}}([0,T])} \le C_0(\delta_0, M_0, \tilde{M}_0)T^{\frac{1-\alpha}{4}} + C_0(\delta_0)\sum_{i=1}^{K} \|\eta_{0,i}\|_{C^{4,\alpha}(\bar{I})}, \tag{5.15}$$

for all $x^* \in \{0, 1\}$. By using (3.12), and Lemmas 5.9, 5.10, we derive

$$\|H_{f_i}(\eta_{0,i})\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \le C_0(\delta_0, \lambda, M_0)\|\eta_{0,i}\|_{C^{4,\alpha}(\bar{I})}, \quad \forall\, i \in \{1, ..., K\}. \tag{5.16}$$

From the triangle inequality, (5.13)∼(5.16), and $1 \le \|\eta_{0,i}\|_{C^{4,\alpha}(\bar{I})}$, $\forall\, i \in \{1, .., K\}$, we have

$$\sum_{i=1}^{K} \|\eta_i\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \le C_0 \sum_{i=1}^{K} \|\eta_{0,i}\|_{C^{4,\alpha}(\bar{I})} + \tilde{C}_0\left(\delta_0, \lambda, M_0, \tilde{M}_0\right) T^{\beta},$$

where $\beta = \min\{\frac{1-\alpha}{4}, \frac{\alpha}{4}\}$, $C_0 = C_0\,(\delta_0, \lambda, M_0) > 1$ is a sufficiently large constant. By letting $\tilde{M}_0 \in (0, \infty)$ with

$$\frac{\tilde{M}_0}{2} = C_0 \sum_{i=1}^{K} \|\eta_{0,i}\|_{C^{4,\alpha}(\bar{I})}, \tag{5.17}$$

and $T_2 = T_2\left(\delta_0, \lambda, M_0, \tilde{M}_0\right) \in (0, T_1)$ with $\tilde{C}_0 T_2^{\beta} \le \frac{\tilde{M}_0}{2}$, we conclude that $\|\eta\|_{X_{\eta_0}^{T_2}} \le \tilde{M}_0$. Thus, we obtain the self-map property, i.e., $\mathcal{G}\left(X_{\eta_0}^{T_2} \cap B_{\tilde{M}_0}\right) \subset X_{\eta_0}^{T_2} \cap B_{\tilde{M}_0}$, $\forall\, T \in (0, T_2]$.

• **Contraction-maps.** We show that $\mathcal{G}$ is a contraction-map, i.e., with $\tilde{\eta}, \tilde{\zeta} \in X_{\eta_0}^{T} \cap B_{\tilde{M}_0}$ and $\eta = \mathcal{G}(\tilde{\eta})$, $\zeta = \mathcal{G}(\tilde{\zeta})$, there exists $T > 0$ such that

$$||\eta - \zeta||_{X_{\eta_0}^{T}} \le C\, T^{\beta}\, ||\tilde{\eta} - \tilde{\zeta}||_{X_{\eta_0}^{T}}, \tag{5.18}$$

where $\beta \in (0, 1)$ and $C = C(\delta_0, \lambda, M, \tilde{M}_0)$.

It is easy to verify that $\eta - \zeta$ satisfies the following. For any $i \in \{1, ..., K\}$,

$$\begin{cases} \partial_t(\eta_i - \zeta_i) + \frac{\partial_x^4(\eta_i - \zeta_i)}{|\partial_x f_i|^4} = H_{f_i}(\tilde{\eta}_i) - H_{f_i}(\tilde{\zeta}_i) & \text{in } (0, T) \times I, \\ (\eta_i - \zeta_i)(t, x^*) = 0 & \forall\,(t, x^*) \in [0, T] \times \partial I, \\ \partial_x(\eta_i - \zeta_i)(t, x^*) = b_{f_i}(\tilde{\eta}_i)(t, x^*) - b_{f_i}(\tilde{\zeta}_i)(t, x^*), & \forall\,(t, x^*) \in [0, T] \times \partial I, \\ (\eta_i - \zeta_i)(0, x) = 0, & \forall\, x \in \bar{I}. \end{cases}$$

The same as before, the linear problem is well-posed and the regularity assumptions on the coefficients are satisfied. We see that the zero initial datum for $\eta - \zeta$ satisfies the compatibility conditions of order zero. By applying Lemma 5.14, $\eta - \zeta$ is the unique solution of the linear equation, and

$$\begin{aligned} \sum_{i=1}^{K} \|\eta_i - \zeta_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)} \le & C_0 \sum_{i=1}^{K} \left\|H_{f_i}(\tilde{\eta}_i) - H_{f_i}(\tilde{\zeta}_i)\right\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \\ & + C_0 \sum_{i=1}^{K} \left\|\left(b_{f_i}(\tilde{\eta}_i) - b_{f_i}(\tilde{\zeta}_i)\right)_{|[0,T]\times\partial I}\right\|_{C^{\frac{3+\alpha}{4}}([0,T])}. \end{aligned} \tag{5.19}$$

From (3.12), and Lemmas 5.9, 5.10, 5.12, we have

$$\begin{aligned} \|H_{f_i}(\tilde{\eta}_i) - H_{f_i}(\tilde{\zeta}_i)\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \le & \sum_{j=1}^{3} C_0(\delta_0, \lambda, M_0)\|\partial_x^j \tilde{\eta}_i - \partial_x^j \tilde{\zeta}_i\|_{C^{\frac{\alpha}{4},\alpha}(D^T)} \\ \le & C_0(\delta_0, \lambda, M_0)\|\tilde{\eta}_i - \tilde{\zeta}_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)} T^{\frac{\alpha}{4}} \end{aligned} \tag{5.20}$$

$\forall\, i\in\{1,...,K\}$. By applying the same argument in (2.42) $\sim$(2.45), we have

$$\sum_{i=1}^{K}\left\|b_{f_i}(\tilde{\eta}_i)(\cdot,x^*)-b_{f_i}(\tilde{\zeta}_i)(\cdot,x^*)\right\|_{C^{\frac{3+\alpha}{4}}([0,T])}\le C_0(\delta_0,M_0,\tilde{M}_0)\sum_{i=1}^{K}\|\tilde{\eta}_i-\tilde{\zeta}_i\|_{C^{\frac{4+\alpha}{4},4+\alpha}(D^T)}T^{\frac{1-\alpha}{4}},\tag{5.21}$$

$\forall\, x^*\in\{0,1\}$. Combining (5.19)$\sim$(5.21), we obtain (5.18) by letting $\beta=\min\{\frac{1-\alpha}{4},\frac{\alpha}{4}\}$. By choosing $T_3=T_3\left(\delta_0,\lambda,M_0,\tilde{M}_0\right)\in(0,T_2)$ such that $CT_3^{\beta}<1$, we conclude that $\mathcal{G}: X_{\eta_0}^{T_3}\cap B_{\tilde{M}_0}\to X_{\eta_0}^{T_3}\cap B_{\tilde{M}_0}$ is a strict contraction-map. By applying the Banach fixed-point theorem, there exists η with the regularity $\eta_i\in C^{\frac{4+\alpha}{4},4+\alpha}(D^{T_3})$, $\forall\, i\in\{1,...,K\}$, which is also a solution to the IBVP (3.26)$\sim$(3.29).

Step 3°. In this step, we show that $\eta_i\in C^{\frac{5+\alpha}{4},5+\alpha}(D^{T_3})\bigcap C^{\infty}\left((0,T_3]\times\bar{I}\right)$, $\forall\, i\in\{1,...,K\}$.

• **$C^{\frac{5+\alpha}{4},5+\alpha}(D^{T_3})$-smoothness.** Note that, we have $\eta_i\in C^{\frac{4+\alpha}{4},4+\alpha}(D^{T_3})$, $\forall\, i\in\{1,...,K\}$. It is easy to verify the regularity: $\forall\, x^*\in\partial I$, $i\in\{1,...,K\}$, $d_i:=\frac{1}{|\partial_x f_i|^4}\in C^{\frac{3+\alpha}{4},3+\alpha}(D^{T_3})$; $e_i:=H_{f_i}(\eta_i)\in C^{\frac{1+\alpha}{4},1+\alpha}(D^{T_3})$; and $g_i(\cdot,x^*):=b_{f_i}(\eta_i)(\cdot,x^*)\in C^{\frac{6+\alpha}{4}}([0,T_3])$. Notice that η solves the linear parabolic PDE,

$$\begin{cases}\partial_t\eta_i=-d_i\cdot\partial_x^4 f_i+e_i & \text{in } (0,T_3)\times I,\\ \eta_i(t,x^*)=x^*, & \forall\,(t,x^*)\in[0,T_3]\times\partial I,\\ \partial_x\eta_i(t,x^*)=g_i(t,x^*), & \forall\,(t,x^*)\in[0,T_3]\times\partial I,\\ \eta_i(0,x)=\eta_{0,i}(x), & \forall\, x\in\bar{I},\end{cases}\tag{5.22}$$

$\forall\, i\in\{1,...,K\}$. Moreover, $\forall\, i\in\{1,...,K\}$, $\eta_{0,i}$ satisfies the compatibility conditions of order 1 to the IBVP (3.26)$\sim$(3.29), hence, $\eta_{0,i}$ also satisfies the compatibility conditions of order 1 to the linear parabolic PDE (5.22). By applying Lemma 5.14, we conclude that $\eta_i\in C^{\frac{5+\alpha}{4},5+\alpha}(D^{T_3}), \forall\, i\in\{1,...,K\}$.

• **$C^{\infty}\left((0,T_3]\times\bar{I}\right)$-smoothness.** We use the cut-off function method to prove $\eta_i\in C^{\infty}\left((0,T_3]\times\bar{I}\right)$, which is the same as the corresponding part in the proof of Theorem 2.2. Thus, we skip the details of proof.

5.2 Technical lemmas from literature

Lemma 5.1 ([5, Lemma 3.1])**.** *Suppose ϕ is any normal field along f and $f:[0,T)\times I\to\mathbb{R}^n$ is a time dependent curve satisfying $\partial_t f=V+\varphi\tau$, where V is the normal velocity and $\varphi=\langle\tau,\partial_t f\rangle$. Then the following formulae hold.*

$$\nabla_s\phi=\partial_s\phi+\langle\phi,\vec{\kappa}\rangle\,\tau,\tag{5.23}$$

$$\partial_t\,(ds)=(\partial_s\varphi-\langle\vec{\kappa},V\rangle)\,ds,\tag{5.24}$$

$$\partial_t\partial_s-\partial_s\partial_t=(\langle\vec{\kappa},V\rangle-\partial_s\varphi)\,\partial_s,\tag{5.25}$$

$$\partial_t\tau=\nabla_s V+\varphi\vec{\kappa},\tag{5.26}$$

$$\partial_t\phi=\nabla_t\phi-\langle\nabla_s V+\varphi\,\vec{\kappa},\phi\rangle\,\tau,\tag{5.27}$$

$$\nabla_t\vec{\kappa}=\nabla_s^2 V+\langle\vec{\kappa},V\rangle\,\vec{\kappa}+\varphi\nabla_s\vec{\kappa},\tag{5.28}$$

$$(\nabla_t\nabla_s-\nabla_s\nabla_t)\,\phi=(\langle\vec{\kappa},V\rangle-\partial_s\varphi)\,\nabla_s\phi+\langle\vec{\kappa},\phi\rangle\,\nabla_s V-\langle\nabla_s V,\phi\rangle\,\vec{\kappa}.\tag{5.29}$$

Notice that the formula of integration by parts for the covariant differentiation ∇_s is still applicable. This is because that, as ψ_1,ψ_2 are normal vector fields along a smooth curve, one has

$$\partial_s\left\langle\psi_1,\psi_2\right\rangle=\left\langle\nabla_s\psi_1,\psi_2\right\rangle+\left\langle\psi_1,\nabla_s\psi_2\right\rangle.\tag{5.30}$$

Lemma 5.2. *Suppose $f : I = [a,b] \to \mathbb{R}^n$ is a smooth curve in $\mathbb{R}^n$. Then for any perturbation of f, $f_\varepsilon(x) = f(x) + \varepsilon \cdot W(x)$, where $W \in C^\infty(I)$, one has the following formulae:*

$$\frac{d}{d\varepsilon}\lfloor_{\varepsilon=0}\mathcal{L}[f_\varepsilon] = -\int_I \langle \vec{\kappa}, W\rangle \, ds + [\langle \tau, W\rangle]_a^b ,$$

$$\frac{d}{d\varepsilon}\lfloor_{\varepsilon=0}\mathcal{E}[f_\varepsilon] = \int_I \langle \nabla_s^2\vec{\kappa} + \frac{|\vec{\kappa}|^2}{2}\vec{\kappa}, W\rangle \, ds$$
$$+ \left[\langle \tau, W\rangle \cdot \frac{|\vec{\kappa}|^2}{2} + \langle \vec{\kappa}, \nabla_s(W - \langle W, \tau\rangle \tau)\rangle - \langle \nabla_s\vec{\kappa}, W\rangle\right]_a^b .$$

Proof of Lemma 5.2. The proof is based on a direct computation by applying (5.24), (5.28), (5.30) and integration by parts. The reader can also find the details of this computation in the literature (e.g., [19]). □

For normal vector fields $\phi_1, ..., \phi_k$ along f, we denote by $\phi_1 * \cdots * \phi_k$ a term of the type

$$\phi_1 * \cdots * \phi_k = \begin{cases} \langle \phi_{i_1}, \phi_{i_2}\rangle \cdots \langle \phi_{i_{k-1}}, \phi_{i_k}\rangle & \text{, for } k \text{ even,} \\ \langle \phi_{i_1}, \phi_{i_2}\rangle \cdots \langle \phi_{i_{k-2}}, \phi_{i_{k-1}}\rangle \cdot \phi_{i_k}, & \text{for } k \text{ odd,} \end{cases}$$

where $i_1, ..., i_k$ is any permutation of $1, ..., k$. Slightly more generally, we allow some of the ϕ_i to be functions, in which case the $*$-product reduces to multiplication. For a normal vector field ϕ along f, we denote by $P_b^{a,c}(\phi)$ any linear combination of terms of the type $\nabla_s^{i_1}\phi * \cdots * \nabla_s^{i_b}\phi$ with coefficients bounded by a universal constant, where $a = i_1 + \cdots + i_b$ is the total number of derivatives and $max\{i_j\} \le c$. Notice that the following formulae hold:

$$\begin{cases} \nabla_s\left(P_{b_1}^{a_1,c_1}(\phi) * P_{b_2}^{a_2,c_2}(\phi)\right) = \nabla_s P_{b_1}^{a_1,c_1}(\phi) * P_{b_2}^{a_2,c_2}(\phi) + P_{b_1}^{a_1,c_1}(\phi) * \nabla_s P_{b_2}^{a_2,c_2}(\phi), \\ P_{b_1}^{a_1,c_1}(\phi) * P_{b_2}^{a_2,c_2}(\phi) = P_{b_1+b_2}^{a_1+a_2,max\{c_1,c_2\}}(\phi), \; \nabla_s P_{b_2}^{a_2,c_2}(\phi) = P_{b_2}^{a_2+1,c_2+1}(\phi). \end{cases}$$

In order to simplify the terminology of summation in the lemma below, we introduce the notation,

$$\sum_{\substack{[\![a,b]\!] \le [\![A,B]\!] \\ c \le C}} P_b^{a,c}(\vec{\kappa}) := \sum_{a=0}^{A} \sum_{b=1}^{2A+B-2a} \sum_{c=0}^{C} P_b^{a,c}(\vec{\kappa}), \tag{5.31}$$

where $[\![a,b]\!] := 2a + b$. For our convenience, let's call $[\![a,b]\!]$ the order of $P_b^{a,c}(\vec{\kappa})$ and $\sum_{\substack{[\![a,b]\!] \le [\![A,B]\!] \\ c \le A}} P_b^{a,a}(\vec{\kappa})$ is replaced by $\sum_{[\![a,b]\!] \le [\![A,B]\!]} P_b^{a}(\vec{\kappa})$. Hence, (5.31) stands for the sum of $P_b^{a,c}(\vec{\kappa})$ with order no greater than that of $P_B^{A,C}(\vec{\kappa})$.

Remark 5.3. *For simplicity, we might use the notation $P_b^a(\phi)$ instead of $P_b^{a,a}(\phi)$.*

Lemma 5.4 ([1, Theorem 5.2])**.** *Let Ω be an interval in $\mathbb{R}$ and $u \in W^{m,p}(\Omega)$ for some $p \in [1,\infty)$, $m \in \mathbb{N}$. Then for each $\epsilon_0 > 0$ there exists finite constants K and K', each depending on m, p, ϵ_0, such that*

$$\|u\|_{W^{j,p}} \le K \left(\epsilon \, \|D^m u\|_{L^p} + \epsilon^{-j/(m-j)} \, \|u\|_{L^p}\right), \tag{5.32}$$

$$\|u\|_{W^{j,p}} \le K' \left(\epsilon \, \|u\|_{W^{m,p}} + \epsilon^{-j/(m-j)} \, \|u\|_{L^p}\right), \tag{5.33}$$

$$\|u\|_{W^{j,p}} \le 2K' \, \|u\|_{W^{m,p}}^{j/m} \, \|u\|_{L^p}^{(m-j)/m}, \tag{5.34}$$

for any $j \in \{0, 1, ..., m-1\}$ and $\epsilon \in (0, \epsilon_0)$. Here, $\|u\|_{L^p} := \left(\int_\Omega |u|^p\right)^{1/p}$ is the L^p-norm, and $\|u\|_{W^{m,p}} := \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_{L^p}^p\right)^{1/p}$ is the standard Sobolev norm.

Below are interpolation inequalities for non-closed curves, which are modified from [10]. Note that in this article we still follow the notation in [10] to use the scale invariant Sobolev norms:

$$\|\vec{\kappa}\|_{k,p} := \sum_{i=0}^{k} \left\|\nabla_s^i \vec{\kappa}\right\|_p, \ \left\|\nabla_s^i \vec{\kappa}\right\|_p := \mathcal{L}\,[f]^{i+1-1/p} (\int_I \left|\nabla_s^i \vec{\kappa}\right|^p \ ds)^{1/p}.$$

Note that using scale invariant Sobolev norms is convenient as working with inequalities in geometric flows since domain of functions also depends on time.

Lemma 5.5 ([5, Lemma 3.8]). *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}^n$ be a smooth curve. Then for any $k \in \mathbb{N}_0$, $p \geq 2$ and $0 \leq i < k$, we have*

$$\left\|\nabla_s^i \vec{\kappa}\right\|_p \leq c \,\|\vec{\kappa}\|_2^{1-\alpha} \|\vec{\kappa}\|_{k,2}^{\alpha}\,, \tag{5.35}$$

where $\alpha = (i + \frac{1}{2} - \frac{1}{p})/k$ and $c = c(n, k, p)$.

Lemma 5.6 ([5, Lemma 3.10]). *Let $f : I \to \mathbb{R}^n$ be a smooth regular curve. For any $a, c, \ell \in \mathbb{N}_0$, $b \in \mathbb{N}$, $b \geq 2$, $c \leq \ell + 2$ and $a < 2(\ell + 2)$ we find*

$$\int_I |P_b^{a,c}(\vec{\kappa})| \, ds \leq C\mathcal{L}[f]^{1-a-b} \|\vec{\kappa}\|_2^{b-\gamma} \|\vec{\kappa}\|_{\ell+2,2}^{\gamma}\,, \tag{5.36}$$

with $\gamma = (a + \frac{1}{2}b - 1)/(\ell + 2)$ and $C = C(n, \ell, a, b)$. Further if $a + \frac{1}{2}b < 2\ell + 5$, then for any $\varepsilon > 0$

$$\int_I |P_b^{a,c}(\vec{\kappa})| \, ds \leq \varepsilon \int_I |\nabla_s^{\ell+2} \vec{\kappa}|^2 \, ds + C\varepsilon^{-\frac{\gamma}{2-\gamma}} (\|\vec{\kappa}\|_{L^2}^2)^{\frac{b-\gamma}{2-\gamma}} + C\mathcal{L}[f]^{1-a-\frac{b}{2}} \|\vec{\kappa}\|_{L^2}^b\,, \tag{5.37}$$

with $C = C(n, \ell, a, b)$.

Lemma 5.7 ([5, Lemma 3.4]). *We have the identities*

$$\partial_s \vec{\kappa} = \nabla_s \vec{\kappa} - |\vec{\kappa}|^2 \tau,$$
$$\partial_s^m \vec{\kappa} = \nabla_s^m \vec{\kappa} + \tau \sum_{\substack{[[a,b]] \leq [[m-1,2]] \\ c \leq m-1, \ b \ \text{even}}} P_b^{a,c}(\vec{\kappa}) + \sum_{\substack{[[a,b]] \leq [[m-2,3]] \\ c \leq m-2 \ b \ \text{odd}}} P_b^{a,c}(\vec{\kappa}), \quad \text{for } m \geq 2\,.$$

Lemma 5.8 ([22, Lemma 8]). *Suppose $f = (f_1, ..., f_K)$, $f_i : [0, T) \times I \to \mathbb{R}^n$ is a smooth solution of (1.9). Denote by $\phi_i^\ell := \nabla_s^\ell \vec{\kappa}_i$. Then, for any $\ell \in \mathbb{N}_0$, and $m \in \mathbb{N}$, we have the following formulae*

$$\nabla_t^m \partial_s^k f_i - (-1)^m \nabla_s^{4m-2+k} \vec{\kappa}_i = \sum_{[\![a,b]\!] \leq [\![4m-4+k,3]\!]} P_b^a(\vec{\kappa}_i), \qquad k \in \{0, 1, 2\}, \tag{5.38}$$

$$\nabla_t^m P_\nu^\mu(\vec{\kappa}_i) = \sum_{[\![a,b]\!] \leq [\![4m+\mu,\nu]\!]} P_b^a(\vec{\kappa}_i), \tag{5.39}$$

$$\nabla_t^m \partial_s f_i - \nabla_s \nabla_t^m f_i = \sum_{[\![a,b]\!] \leq [\![4m-3,3]\!]} P_b^a(\vec{\kappa}_i), \tag{5.40}$$

$$\nabla_t^m \nabla_s^k \phi_i^\ell - \nabla_s^k \nabla_t^m \phi_i^\ell = \sum_{[\![a,b]\!] \leq [\![4m+k+\ell-2,3]\!]} P_b^a(\vec{\kappa}_i), \qquad k \in \mathbb{N}, \tag{5.41}$$

$$\partial_t^m (ds) = \left(\sum_{[\![a,b]\!] \leq [\![4m-2,2]\!]} P_b^a(\vec{\kappa}_i) \right) ds. \tag{5.42}$$

Proof. The proof from (5.38) to (5.41) has been shown in [22]. Hence, we only prove (5.42) here. The proof is an induction argument. As $m = 1$, one proves (5.42) by applying (5.24) and (1.9). Suppose that (5.42) holds for $m = k$, where k is any positive integer. Then,

$$\begin{aligned}\partial_t^{k+1}(ds) =&\partial_t\left(\partial_t^k(ds)\right) = \partial_t\left(\left(P_2^{4k-2}(\vec{\kappa}_i) + \cdots + P_2^0(\vec{\kappa}_i)\right)ds\right)\\ =&\partial_t\left(P_2^{4k-2}(\vec{\kappa}_i) + \cdots + P_2^0(\vec{\kappa}_i)\right)ds + \left(P_2^{4k-2}(\vec{\kappa}_i) + \cdots + P_2^0(\vec{\kappa}_i)\right)\partial_t(ds)\\ =&\left(P_2^{4k+2}(\vec{\kappa}_i) + \cdots + P_2^0(\vec{\kappa}_i)\right)ds = \left(\sum_{[\![a,b]\!] \le [\![4k+2,2]\!]} P_b^a(\vec{\kappa}_i)\right)ds,\end{aligned}$$

where the last equality comes from applying (5.39) and (5.24). □

In the following lemmas, we always assume that D^T is domain given in (2.9). Suppose that $v_i, w_i : D^T \to \mathbb{R}^n$, $\forall\, i \in \{1, ..., K\}$.

Lemma 5.9. *([6, Remark B.1]) For $m \le k$, $m, k \in \mathbb{N}_0$, we have*

$$C^{\frac{k+\alpha}{4}, k+\alpha}(D^T) \subset C^{\frac{m+\alpha}{4}, m+\alpha}(D^T), \qquad \forall\, i \in \{1, ..., K\}$$

and if $v_i \in C^{\frac{k+\alpha}{4}, k+\alpha}(D^T)$, then $\partial_x^\ell v_i \in C^{\frac{k-\ell+\alpha}{4}, k-\ell+\alpha}(D^T)$ for all $0 \le \ell \le k$, so that

$$\left\|\partial_x^\ell v_i\right\|_{C^{\frac{k-\ell+\alpha}{4}, k-\ell+\alpha}(D^T)} \le \|v_i\|_{C^{\frac{k+\alpha}{4}, k+\alpha}(D^T)}, \qquad \forall\, i \in \{1, ..., K\}.$$

In particular at each fixed $x \in \bar{I}$, $\forall\, i \in \{1, ..., K\}$, we have $\partial_x^\ell v_i(\cdot, x) \in C^{s,\beta}([0, T])$ with $s = [\frac{k-\ell+\alpha}{4}]$ and $\beta = \frac{k-\ell+\alpha}{4} - s$.

Lemma 5.10. *([6, Lemma B.2]) For $k \in \mathbb{N}_0$, $\alpha, \beta \in (0, 1)$, and $T > 0$, we have*

(1) if $v_i, w_i \in C^{\frac{k+\alpha}{4}, k+\alpha}(D^T)$, $\forall\, i \in \{1, ..., K\}$, then

$$\|v_i w_i\|_{C^{\frac{k+\alpha}{4}, k+\alpha}(D^T)} \le C(n)\, \|v_i\|_{C^{\frac{k+\alpha}{4}, k+\alpha}(D^T)}\, \|w_i\|_{C^{\frac{k+\alpha}{4}, k+\alpha}(D^T)}, \quad \forall\, i \in \{1, ..., K\}.$$

(2) if $v_i \in C^{\frac{\alpha}{4}, \alpha}(D^T)$, $\forall\, i \in \{1, ..., K\}$, and $v_i(t, x) \ne 0$ for all $(t, x) \in D^T$, $\forall\, i \in \{1, ..., K\}$, then

$$\left\|\frac{1}{|v_i|}\right\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)} \le \left\|\frac{1}{|v_i|}\right\|^2_{C^0(D^T)} C\, \|v_i\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)}, \quad \forall\, i \in \{1, ..., K\}.$$

Similar statements are true for functions in $C^{k,\beta}([0, T])$ and $C^{k,\beta}(\bar{I})$, $\forall\, i \in \{1, ..., K\}$.

Lemma 5.11. *([6, Lemma B.3]) For $k \in \mathbb{N}_0$, $\alpha, \beta \in (0, 1)$, and $T > 0$, we have*

(1) if a vector-field $v_i \in C^{\frac{\alpha}{4}, \alpha}(D^T; \mathbb{R}^n)$, $\forall\, i \in \{1, ..., K\}$, then

$$\|\,|v_i|\,\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)} \le C(n)\, \|v_i\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)}, \quad \forall\, i \in \{1, ..., K\}.$$

(2) for $v_i, w_i \in C^{\frac{\alpha}{4}, \alpha}(D^T; \mathbb{R}^n)$, $\forall\, i \in \{1, ..., K\}$, we have

$$\begin{aligned}&\|\,|v_i| - |w_i|\,\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)}\\ \le& C(n)\left\|\frac{1}{|v_i| + |w_i|}^2\right\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)}\left(\|v_i\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)} + \|w_i\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)}\right)^2 \|v_i - w_i\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)},\end{aligned}$$

for all $i \in \{1, ..., K\}$. Similar statements are true for functions in $C^{k,\beta}([0, T])$ and $C^{k,\beta}(\bar{I})$, $\forall\, i \in \{1, ..., K\}$.

Lemma 5.12. *([6, Lemma B.5]) Let* $0 < T < 1$ *and* $v_i \in C^{\frac{4+\alpha}{4}, 4+\alpha}(D^T; \mathbb{R}^n)$, $\forall\, i \in \{1, ..., K\}$, *such that* $v_i(0, x) = 0$ *for any* $x \in \bar{I}$, $\forall\, i \in \{1, ..., K\}$, *then*

$$\left\| \partial_x^\ell v_i \right\|_{C^{\frac{m+\alpha}{4}, m+\alpha}(D^T)} \leq C(m) T^\beta \left\| v_i \right\|_{C^{\frac{4+\alpha}{4}, 4+\alpha}(D^T)}, \qquad \forall\, i \in \{1, ..., K\},$$

for all $\ell, m \in \mathbb{N}_0$ *such that* $\ell + m < 4$. *Here* $\beta = \min\{\frac{1-\alpha}{4}, \frac{\alpha}{4}\} \in (0, 1)$; *more precisely for* $\ell \geq 1$ *then* $\beta = \frac{\alpha}{4}$.

Lemma 5.13. *([6, Lemma 3.4]) Let* $f_{0,i} \in C^{4,\alpha}([0, 1])$, $\forall\, i \in \{1, ..., K\}$, $\bar{f}, \bar{g} \in X^T_{f_0}$, *and* δ_0 *as defined in* (2.16). *Then, for* $m \in \mathbb{N}$ *and any* $T \leq T_1$ *(with* T_1 *as defined in Lemma 2.3) we have*

$$\left\| \frac{1}{|\partial_x f_{0,i}|^m} - \frac{1}{|\partial_x \bar{f}_i|^m} \right\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)} \leq C T^{\frac{\alpha}{4}}, \forall\, i \in \{1, ..., K\},$$

and with $C = C(n, m, \delta_0, \|\bar{f}_i\|_{C^{\frac{4+\alpha}{4}, 4+\alpha}(D^T)}, \|f_{0,i}\|_{C^{4,\alpha}(\bar{I})})$ *as well as*

$$\left\| \frac{1}{|\partial_x \bar{f}_i|^m} - \frac{1}{|\partial_x \bar{g}_i|^m} \right\|_{C^{\frac{\alpha}{4}, \alpha}(D^T)} \leq C T^{\frac{\alpha}{4}} \|\bar{f}_i - \bar{g}_i\|_{C^{\frac{4+\alpha}{4}, 4+\alpha}(D^T)}, \forall\, i \in \{1, ..., K\},$$

$\forall\, i \in \{1, ..., K\}$, *and with* $C = C\left(n, m, \delta_0, \|\bar{f}_i\|_{C^{\frac{4+\alpha}{4}, 4+\alpha}(D^T)}, \|\bar{g}_i\|_{C^{\frac{4+\alpha}{4}, 4+\alpha}(D^T)}\right)$.

Let us recall the theorem on the existence of solutions of linear parabolic equations on Hölder space, namely [26, Theorem 4.9, page 121].

Let $Q = \Omega \times [0, T]$ be a cylindrical domain in the space $\mathbb{R}^{n+1}$ and let Ω be a domain with a smooth boundary S in the space $\mathbb{R}^n$. The side surface of the cylinder Q we denote by Γ; $\Gamma = S \times [0, T]$. In the cylinder Q we consider systems parabolic (boundary value problem) of m equations with constant coefficients containing m unknown functions $u_1, ..., u_m$

$$\begin{cases} \sum_{j=1}^m l_{kj}(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}) u_j(x, t) = v_k(x, t) \quad (k = 1, ..., m), \\ \sum_{j=1}^m B_{qj}(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}) u_j(x, t)|_\Gamma = \Phi_q(x, t) \quad (q = 1, ..., m), \\ \sum_{j=1}^m C_{\alpha j}(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}) u_j(x, t)|_{t=0} = \varphi_\alpha(x, t) \quad (\alpha = 1, ..., m), \end{cases} \tag{5.43}$$

where the l_{kj}, B_{qj} are linear differential operators with coefficients which depend on t and x, $C_{\alpha j}$ are linear differential operators with coefficients which depend on t, $v_k(x, t)$, Φ_q, φ_α are specified functions. The functions l_{kj}, B_{qj} are polynomials in t and x while $C_{\alpha j}$ are polynomials in x.

Let $s_k, t_j \in \mathbb{Z}$, $k, j \in \{1, ..., m\}$ such that degree of the polynomials $l_{kj}(t, x, p\lambda^{2b}, \mathrm{i}\xi\lambda)$ with respect to the variable λ at each point $(t, x) \in Q$ does not exceed $s_k + t_j$ and if $s_k + t_j < 0$ then $l_{kj} = 0$. Let $\sum_{j=1}^m (s_j + t_j) = 2br$, $r > 0$. Let β_{qj}, $\gamma_{\alpha j}$ be the degree of the polynomials $B_{qj}(x, t, \mathrm{i}\xi\lambda, p\lambda^{2b})$, $C_{\alpha j}(x, \mathrm{i}\xi\lambda, p\lambda^{2b})$ with respect to λ, respectively. If $B_{qj} = 0$, $C_{\alpha j} = 0$, take for β_{qj}, $\gamma_{\alpha j}$ any integer. Define $\sigma_q = max\{\beta_{qj} - t_j : j \in \{1, ..., m\}\}$, $\rho_\alpha = max\{\gamma_{\alpha j} - t_j : j \in \{1, ..., m\}\}$.

We write (5.43) by

$$\begin{cases} \mathcal{L}u = v, \\ \mathcal{B}u|_\Gamma = \Phi, \mathcal{C}u|_{t=0} = \phi. \end{cases} \tag{5.44}$$

Define B^0_{qj}, $C^0_{\alpha j}$ are principal part of B_{qj}, $C_{\alpha j}$. Let $\mathcal{B}^0 := (B^0_{qj})$ and $\mathcal{C}^0 := (C^0_{\alpha j})$. Assume that $\mathcal{B}^0$ and $\mathcal{C}^0$ satisfy complementary condition at $(t, x) \in \Gamma$ and $x \in \Omega$, respectively. We have the following lemma.

Lemma 5.14 ([26, Theorem 4.9, page 121])**.** *Let l be a positive, noninteger number satisfying $l > max\{0, \sigma_1, ..., \sigma_{br}\}$. Let $S \in C^{l+t_{max}}$, and let the coefficients of the operators l_{kj} belong to the class $C^{\frac{l-s_k}{2b}, l-s_k}(Q)$, those of the operator $C_{\alpha j}$ to the class $C^{l-\rho}(\Omega)$ and those the operator B_{qj} to the class $C^{\frac{l-\sigma_k}{2b}, l-\sigma_k}(\Gamma)$.*
If $v_j \in C^{\frac{l-s_k}{2b}, l-s_k}(Q)$, $\phi_\alpha \in C^{l-\rho_\alpha}(\Omega)$, $\Phi_q \in C^{\frac{l-\sigma_k}{2b}, l-\sigma_k}(\Gamma)$ and if compatibility condition of order $l' = [l]$ are fullfied, then problem (5.44) has a unique solution $u = (u_1, ..., u_m)$ with $u_j \in C^{\frac{l+t_j}{2b}, l+t_j}(Q)$ for which the inequality

$$\sum_{j=1}^{m} \|u_j\|_{C^{\frac{l+t_j}{2b}, l+t_j}(Q)} \leq C \left(\sum_{j=1}^{m} \|v_j\|_{C^{\frac{l-s_j}{2b}, l-s_j}(Q)} + \sum_{\alpha=1}^{r} \|\phi_\alpha\|_{C^{l-\rho_\alpha}(\Omega)} + \sum_{q=1}^{br} \|\Phi_q\|_{C^{\frac{l-\sigma_k}{2b}, l-\sigma_k}(\Gamma)} \right)$$

is valid.

Acknowledgement. This work has been partially supported by the research grant of the National Science Council of Taiwan (NSC-100-2115-M-003-003), the National Center for Theoretical Sciences at Taipei, and the Max-Planck-Institut für Mathematik in den Naturwissenschaften in Leipzig. C.-C. L. and H.S. would like to thank Prof. Dr. Stephan Luckhaus, Prof. Dr. Felix Otto and Prof. Dr. Angela Stevens for their support and hospitality during the preparation of this article in Leipzig. T.D.T. received financial support from Taiwan MoST 108-2115-M-003-003-MY2.

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