Arithmetic Functions and Dirichlet Series

Adriana Cardoso
Mestrado em Matemática
Departamento de Matemática
2022

Orientador
António Machiavelo, Professor Auxiliar
Faculdade de Ciências da Universidade do Porto

Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, ______/______/_________

Agradecimentos

Primeiro, agradeço aos meus pais e ao meu irmão por sempre me terem dado a motivação e o apoio necessários para completar mais uma etapa.

De seguida, gostaria de agradecer também ao meu orientador, o Professor António Machiavelo por me ter sempre motivado e guiado e por toda a ajuda durante não só este ano, mas desde o início do meu percurso académico. Agradeço também ao meu colega Pedro Ribeiro por toda a ajuda que me prestou, desde indicar-me livros e artigos que poderiam ser úteis bem como a responder a diversas perguntas.

Por último, agradeço ainda a todos os amigos dentro e fora da faculdade, com especial atenção à Inês, à Pips, à Mariana e à Cecília, por toda a paciência e ajuda, e ainda a todos os colegas, professores e outros que me apoiaram e guiaram até aqui.

A todos vocês, o meu mais sincero obrigada.

Abstract

This dissertation aims to provide an overview of the theory of arithmetical functions and Dirichlet series, and their relation to the popular Riemann Hypothesis. The main objective of this work is to expose a direct proof of two statements equivalences to the Riemann Hypothesis, one that the growth of $M(x) := \sum_{n \leq x} \mu(n)$ is asymptotically bounded by $x^{\frac{1}{2}+\varepsilon}$, for all $\varepsilon > 0$ and the other that the growth of $L(x) := \sum_{n \leq x} \lambda(n)$ is equally asymptotically bounded by the same expression, where μ is the Möbius function and λ is the Liouville function, two very relevant arithmetical functions. Here, we also aim to provide a connection between two branches of Number theory, the Algebraic and the Analytic, as arithmetical functions are part of the former and the study of the Riemann Zeta Function often lies in the realm of the latter.

Keywords: Arithmetical Functions, Dirichlet Series, Riemann Hypothesis, Riemann's Zeta Function

Resumo

Esta dissertação pretende dar uma visão geral sobre a teoria das funções aritméticas e séries de Dirichlet e também sobre a sua relação com a famosa Hipótese de Riemann. O objetivo principal deste trabalho é expor uma prova que duas proposições equivalentes à Hipótese de Riemann: que o crescimento de $M(x) := \sum_{n \leq x} \mu(n)$ é assimptoticamente limitado por $x^{\frac{1}{2}+\varepsilon}$, para todo o $\varepsilon > 0$ e que o crescimento de $L(x) := \sum_{n \leq x} \lambda(n)$ é igualmente limitado assintoticamente pela mesma expressão, onde μ denota a função de Möbius e λ a função de Liouville duas importantes funções aritméticas. Aqui, também pretendemos mostrar uma conexão entre dois ramos da Teoria de Números, Algébrica e Analítica, dado que as funções aritméticas fazem parte da primeira enquanto o estudo de Função Zeta de Riemann usualmente está relacionado com a segunda.

Keywords: Funções Aritméticas, Séries de Dirichlet, Hipotese de Riemann, Função Zeta de Riemann

Contents

Chapter 1

Introduction

Number Theory is a branch of Mathematics devoted mainly to the study of the integers and integer-valued functions. The primary area where this dissertation is placed, Analytic Number Theory, is one of its sub-branches and can be described as the study of the integers employing tools from real and complex analysis, in particular, about estimates on size and density, as opposed to identities.

The main focus of this work lies on arithmetical functions, also known as number-theoretic functions, which are simply maps with domain in $\mathbb{N}$ and codomain $\mathbb{C}$, and their respective Dirichlet series, defined as $\sum_{n\geq 1} \frac{f(n)}{n^s}$ for an arithmetical function f and complex number s. In fact, these series are a special case of the general Dirichlet series, named after Lejeune Dirichlet (1805–1859), defined as $\sum_{n\geq 1} f(n)e^{\lambda_n s}$ where λ_n is a complex valued sequence and s a complex number. Throughout this work, we will use $s = \sigma + it$, where $\sigma, t \in \mathbb{R}$ to represent a complex number, as it is common in this area, after it being first used by the mathematician Bernhard Riemann (1826–1866).

Riemann had a great role in establishing the relation between Dirichlet series and analytic number theory when he first studied the now called Riemann Zeta function, first defined as what we now call the Dirichlet series of the arithmetical function constant equal to 1, i.e.,

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

for any s with real part greater than 1. We will see how we can then extend, analytically, this function through the whole complex plane except at $s = 1$. Riemann is also responsible for his notorious Riemann Hypothesis, a conjecture proposed in 1859 that says that the Riemann zeta function has its zeros only at the negative even integers, known as trivial zeros, and at the complex numbers with real part $\frac{1}{2}$, known as the critical line. Still unsolved at the time of writing of this dissertation, proving the conjecture validity is one of the Millennium Problems, `www.claymath.org/millennium-problems`.

Here we hope to convey the importance of arithmetical functions in this particular context, while also expound their theory. Futhermore, we will focus on two very relevant arithmetic functions, namely the Möbius function, μ, and the Liouville function, λ, that are defined as follows. Let $\nu_p(n)$ be the biggest exponent k such that $p^k \mid n$. Then, we define

$$\lambda(n) := (-1)^{\sum\limits_{p|n} \nu_p(n)}$$

and

$$\mu(n) := \begin{cases} \lambda(n), & \text{if n is square-free} \\ 0, & \text{otherwise.} \end{cases}$$

From these two functions, one defines

$$L(x) := \sum_{n \leq x} \lambda(n) \quad \text{ and } \quad M(x) := \sum_{n \leq x} \mu(n)$$

two functions that are closely related to the Riemann Hypothesis. The M function is known as the Mertens function, in honour of Franz Mertens (1840–1927).

This dissertation is organized as follows. In chapter 2, we will give some examples of well-known arithmetic functions and then explore some properties of the ring of all arithmetic functions, denoted as $\mathcal{A}$. The main point of this chapter is to expose a clearer version of the proof of [CE59] of the following theorem

Theorem. *$\mathcal{A}$ is a unique factorization domain.*

As it will be seen, this ring is isomorphic to the ring of formal Dirichlet series. We close this chapter with the study of the above mentioned λ, μ, L and Mertens functions. Our main references for this chapter were [Apo76] and [Siv89].

We move on to look into the convergence of Dirichlet series in the complex plane, in chapter 3. As it is known, for this type of series, if it converges for some complex number s, it will converge for any s_0 with real part bigger than $\mathrm{Re}(s)$. This will allow us to define some convergence abscissas, namely the absolute, uniform and conditional convergence abscissas. These numbers are closely related among themselves, as we will see. In this chapter, we follow the book [Apo76] and the articles [BH16] and [Boa97].

In the last chapter 4, our objective is to present a complete but concise proof of the following main Theorem involving the Riemann's Hypothesis.

Theorem. *The following conditions are equivalent*

1. *(**Riemann's Hypothesis**) The Zeta function has its zeros only at the negative even integers (called the trivial zeros) and complex numbers with real part 1/2.*

2. *For all* $\varepsilon > 0$, $M(x) = \sum_{n \le x} \mu(n) = \mathcal{O}(x^{\frac{1}{2}+\varepsilon})$.

3. *For all* $\varepsilon > 0$, $L(x) = \sum_{n \le x} \lambda(n) = \mathcal{O}(x^{\frac{1}{2}+\varepsilon})$.

This Theorem serves to show that the validity of the Riemann Hypothesis is closely tied to the cumulative growth of the Möbius and Liouville functions. First, we start by exposing and proving some necessary facts about the zeta function before moving to the proof of the Theorem. Here, our references were [TH87], [Edw01],[Bro17a] and [Bro17b].

At the end of this work are four appendixes with some background theory necessary to supplement the main chapters, namely in Ring Theory, Fourier Series, Summation Formulas and Complex Analysis.

Chapter 2

Arithmetic Functions

In this chapter we recall the notions of arithmetical function and Dirichlet series, giving several examples, including some of the most notorious and relevant ones. We then describe the usual ring structure on the set of all arithmetic functions and on the set of all Dirichlet series, showing that they are in fact isomorphic, and unique factorization domains. We finish this chapter by bringing attention to two important functions, the Liouville, $\lambda(n)$, and Möbius, $\mu(n)$, functions, and to their summation functions $L(x) = \sum_{n\leq x} \lambda(n)$ and $M(x) = \sum_{n\leq x} \mu(n)$. These last two functions will play an important role later in this dissertation, in chapter 4, by their relation to the Riemann's Hypothesis.

Definition 2.1. *An **arithmetical function** is a complex valued function defined on* $\mathbb{N} = \{1, 2, 3, 4, \dots\}$.

To each arithmetical function $f\colon \mathbb{N} \to \mathbb{C}$*, we associate a **formal Dirichlet series**:*

$$\mathcal{D}(f; s) := \sum_{n\geq 1} \frac{f(n)}{n^s} \quad (s \in \mathbb{C}).$$

We will denote by $\mathbf{0}$ *the identically zero arithmetical function, by* $\mathcal{A}$ *the set of all arithmetical functions, and by* $\mathcal{S}$ *the set of all Dirichlet series.*

Throughout the following section, we will use the following notation

1. $n \in \mathbb{N}$, unless stated otherwise;

2. $p \in \mathbb{N}$ is always prime, unless stated otherwise;

3. $\nu_p(n) := \max\{t \in \mathbb{N}_0 : p^t \mid n\}$ is the p-adic valuation of $n \in \mathbb{N}$;

4. s denotes a complex-valued variable of the form $s = \sigma + it$ $(\sigma, t \in \mathbb{R})$.

2.1 List of Arithmetic Functions

Here we list some of the most notorious and relevant arithmetic functions. For the ones without a universally agreed notation, we will chose one that helps to better identify the function. Also, for those with known Dirichlet series, we will display it as a formal infinite series and later in the chapter we will justify them.

- $\iota(n) := \lfloor \frac{1}{n} \rfloor = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}$

  $\mathcal{D}(\iota; s) = 1.$

  Other notations for this function are e_0 on [Siv89], e_1 on [Bor20] δ on [McC86] and i and I on [Apo76].

- $\mathbf{m}$ denotes the constant function equal to $m \in \mathbb{C}$,

  $\mathcal{D}(\mathbf{1}; s) = \sum_{n \geq 1} \frac{1}{n^s} = \zeta(s).$

  Other notations for the function $\mathbf{1}$ are U, u on [Apo76] and I on [Siv89].

- $e_\alpha(n) := n^\alpha$, $(\alpha \in \mathbb{C})$,

  $\mathcal{D}(e_0; s) = \zeta(s)$ and $\mathcal{D}(e_\alpha; s) = \zeta(s - \alpha)$.

  Other notations for this function are N^α on [Apo76], id_α on [Bor20] and ζ_α on [McC86].

- $\omega(n) := \begin{cases} \sum_{p|n} 1, & n > 1, \\ 0, & n = 1, \end{cases}$ the number of different prime factors of n.

- $\Omega(n) := \sum_{p|n} \nu_p(n)$, the number of prime factors of n with multiplicity.

- $\lambda(n) := (-1)^{\Omega(n)}$, the **Liouville function**.

  $\mathcal{D}(\lambda; s) = \frac{\zeta(2s)}{\zeta(s)}.$

- $\mu(n) := \begin{cases} \lambda(n), & \text{if n is square-free,} \\ 0, & \text{otherwise,} \end{cases}$ the **Möbius function**,

  $\mathcal{D}(\mu; s) = \frac{1}{\zeta(s)}.$

- $d^*(n) = 2^{\omega(n)}$, the number of unitary divisors of n, i.e. the numbers d such that $d \mid n$ and $(d, \frac{n}{d}) = 1$, which is the same as the number of square-free divisors.

- $q_k(n) := \begin{cases} 0, & \text{if } \exists \text{ prime } p \text{ s.t. } p^k \mid n, \\ 1, & \text{otherwise} \end{cases}$, the characteristic function of the set of k-free numbers,

  Note that $q_2 = \mu^2 = |\mu|$.

  Other notation for this function is μ_k on [Bor20].

- $\varsigma_k(n) := \begin{cases} 1, & \text{if } \exists\, a \text{ s.t. } a^k = n, \\ 0, & \text{otherwise,} \end{cases}$ the characteristic function of the set of k-full numbers,

  $\varsigma_1 = e_0$,

  $\mathcal{D}(\varsigma_k; s) = \zeta(ks)$.

  Other notations are χ_k and s_k on [Bor20].

- $\Lambda(n) = \begin{cases} \log(p), & \text{if } n = p^k \text{ for some } p \text{ prime, } k \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$ the **Van Mangoldt function**;

- $\Lambda_k(n) = \sum_{d|n} \mu(d)(\log(\frac{n}{d}))^k$, the **generalized Van Mangoldt function**,

  $\Lambda_1 = \Lambda$.

- $\tau_k(n) = \sum_{d|n} \tau_{k-1}(d)$ and $\tau_1 = 1$, the **Dirichlet-Piltz divisor function**,

  $\tau := \tau_2$.

  Other notation for this family of function is d_k on [Siv89].

- $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$, divisor functions,

  $\sigma_0(n) = \tau(n)$, number of divisors of n,

  $\sigma_1(n) =: \sigma(n)$, sum of the divisors of n.

- $\phi(n) := \sum_{\substack{m \leq n \\ (m,n)=1}} 1 = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$, the **Euler's totient function**.

- $J_k(n) := \sum_{\substack{(m_1,\ldots,m_k) \in \mathbb{N}^k \\ m_i \leq n \\ (m_1,\ldots,m_k,n)=1}} 1$, the **Jordan totient function**,

  $J_1 = \phi$.

- $\Psi(n) := n \prod_{p|n} \left(1 + \frac{1}{p}\right)$, the **Dedekind totient function**.

- $\Psi_k(n) := n^k \prod_{p|n} \left(1 + \frac{1}{p^k}\right)$, the **generalized Dedekind totient function**,

  $\Psi_1 = \Psi$.

- A **Dirichlet character modulo** q, $\chi : \mathbb{N} \to \mathbb{C}$ is a map satisfying:

$$\begin{aligned} &\chi(a) = \chi(a \mod q);\\ &\chi(ab) = \chi(a)\chi(b);\\ &\text{If } (a, q) > 1, \text{ then } \chi(a) = 0. \end{aligned}$$

  There are $\varphi(q)$ characters modulo q and the set formed by them is a group isomorphic to the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$.

  The identity element of this group is called the principal character modulo q and is usually denoted by χ_0. Thus, χ_0 is defined for all $a \in \mathbb{Z}$ by

$$\chi_0(a) = \begin{cases} 1, & \text{if } (a, q) = 1,\\ 0, & \text{otherwise.} \end{cases}$$

  Also, for any χ, the **Dirichlet L-function** associated to χ is defined by

$$L(s, \chi) := \mathcal{D}(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

2.2 The Ring of Arithmetic Functions

The set of all arithmetic functions has a natural ring structure if we consider a particular product, the Dirichlet convolution, inspired by the product of two series. Here, we will describe this structure in some detail.

Definition 2.2. *An arithmetical function f is said to be* ***multiplicative*** *if one has:*

$$(m, n) = 1 \implies f(mn) = f(m)f(n).$$

If $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$, we say that f is ***completely multiplicative****.*

If we multiply two Dirichlet series, we see that

$$\sum_{a \geq 1} \frac{f(a)}{a^s} \cdot \sum_{b \geq 1} \frac{g(b)}{b^s} = \sum_{n \geq 1} \frac{\sum_{ab=n} f(a)g(b)}{n^s}.$$

This provides a motivation for the following definition.

Definition 2.3. *Let f and g be two arbitrary arithmetical functions.*

*Their **sum** is the function $f+g$ and their **product** is the function fg, defined as, respectively:*

$$(f+g)(n) := f(n)+g(n), \quad (fg)(n) := f(n)g(n),$$

*and their **Dirichlet product** or **Dirichlet convolution** is the arithmetical function defined by:*

$$(f*g)(n) := \sum_{d|n} f(d)g\left(\frac{n}{d}\right) = \sum_{ab=n} f(a)g(b).$$

Proposition 2.4. *If f, g are multiplicative functions, then $f*g$ is also multiplicative.*

Proof. Notice first that when $(m,n)=1$,

$$\{d : d \mid mn\} = \{d_1 d_2 : (d_1 \mid m) \wedge (d_2 \mid n)\}$$

and if $d_1 \mid m$ and $d_2 \mid n$, then $(d_1, d_2) = 1$.Therefore $f(d_1 d_2) = f(d_1)f(d_2)$. Hence:

$$\begin{aligned}(f*g)(mn) = \sum_{d|mn} f(d)g\left(\frac{mn}{d}\right) &= \sum_{\substack{d_1|m \\ d_2|n}} f(d_1)f(d_2)g\left(\frac{m}{d_1}\right)g\left(\frac{n}{d_2}\right) \\ &= \sum_{d_1|m} f(d_1)g\left(\frac{m}{d_1}\right)\cdot\sum_{d_2|n} f(d_2)g\left(\frac{n}{d_2}\right) = (f*g)(m)\cdot(f*g)(n).\end{aligned}$$

□

Theorem 2.5. *$(\mathcal{A}, +, *)$ is a commutative ring, and its group of units is $\mathcal{A}^* = \{f : f(1) \neq 0\}$.*

Proof. It is trivial to verify that $(\mathcal{A}, +)$ is an abelian group, and it is also very easy to see that $*$ is associative, distributive with respect to $+$, commutative, and that ι is the identity element.

As for the existence of an inverse, f^{-1}, for every arithmetical function f that satisfies $f(1) \neq 0$, one proceeds by complete induction, observing that

$$f^{-1}(1) = \frac{1}{f(1)},$$

and that, for $n > 1$,

$$f * f^{-1}(n) = 0 \iff f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d|n \\ d<n}} f\left(\frac{n}{d}\right) f^{-1}(d),$$

which allows us to define $f^{-1}(n)$, assuming f^{-1} defined for all numbers less than n. □

2.2.1 The Ring of Arithmetic Functions is a Unique Factorization Domain

We will now show that $\mathcal{A}$ is a unique factorization domain, following the proof in [CE59].

Let us define the following function

$$\begin{aligned} N\colon \mathcal{A} &\longrightarrow \mathbb{N}_0 \\ \mathbf{0} &\longmapsto 0 \\ \mathbf{0} \neq f &\longmapsto \min\{n\colon f(n) \neq 0\}. \end{aligned}$$

We will show that this is a valuation in the sense defined as follows.

Definition 2.6. *Let R be a ring. A **valuation** on R is a real-valued function $v\colon R \to \mathbb{R}_0^+$ satisfying the following three properties:*

- *For all $x \in R$, $v(x) \geq 0$ and $v(x) = 0 \implies x = 0$;*
- *For all $x, y \in R$, $v(xy) = v(x)v(y)$;*
- *For all $x, y \in R$, $v(x+y) \geq \min\{v(x), v(y)\}$.*

Let us see now that N is indeed a valuation.

By definition, we have $N(f) = 0 \iff f = \mathbf{0}$. Let $f, g \in \mathcal{A}$. If $f = \mathbf{0}$ or $g = \mathbf{0}$, it is trivial that $f * g = 0$, and thus $N(f * g) = 0$.

Suppose that $N(f) = a > 0, N(g) = b > 0$. For $n < ab$, if d is a divisor of n, then $d \geq a$ implies $\frac{n}{d} < b$, so $g\left(\frac{n}{d}\right) = 0$ and $d < a$ implies $f(d) = 0$. So, we have

$$(f * g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right) = 0.$$

So $0 \neq N(f * g) \geq ab$. We have the equality, because, using the same argument as above for the last equality

$$(f * g)(ab) = \sum_{xy = ab} f(x)g(y) = f(a)g(b) \neq 0.$$

Thus, N has the property of $N(f * g) = N(f)N(g)$.

It only remains to be seen that $N(f+g) \geq \min\{N(f), N(g)\}$, for all $f, g \in \mathcal{A}$. Let us suppose, without loss of generality, that $N(f) \leq N(g)$. So for $n < N(f)$, we have both $f(n) = 0$ and $g(n) = 0$ and thus

$$N(f+g) = \min\{n\colon f(n) + g(n) \neq 0\} \geq N(f) = \min\{N(f), N(g)\}.$$

Lemma 2.7. *If f, g are elements of $\mathcal{A}$, if $N(f) \neq N(g)$, then $N(f+g) = \min\{N(f), N(g)\}$.*

Proof. From the definition of N, we already have that the valuation of the sum is greater than the minimum. Without loss of let us suppose $\min\{N(f), N(g)\} = N(f)$.

$$(f+g)(N(f)) = f(N(f)) + g(N(f)) \\ = f(N(f)) + 0 \qquad = f(N(f)) \neq 0.$$

Thus $N(f+g) \leq N(f)$. Therefore we have the equality. □

This valuation will help us better identify some elements of $\mathcal{A}$, as we have that if $N(f)$ is a prime on $\mathbb{N}$, then f is irreducible on $\mathcal{A}$, by the second propriety of valuations, and that if $N(f) = 1$ then $f \in \mathcal{A}^*$, as the units of $\mathcal{A}$ are exactly the functions with $f(1) \neq 0$.

Theorem 2.8. *$\mathcal{A}$ is an integral domain.*

Proof. Using the valuation N, it follows that

$$f * g = \mathbf{0} \iff N(f*g) = 0 \iff N(f)N(g) = 0 \\ \iff N(f) = 0 \text{ or } N(g) = 0 \iff f = \mathbf{0} \text{ or } g = \mathbf{0}.$$

□

Lemma 2.9. *$\mathcal{A}$ has the Ascending Chain Condition on Principal ideals (definition A.6).*

Proof. Let

$$(f_0) \subsetneq (f_1) \subsetneq \dots$$

be a infinite ascending chain of principal ideals of $\mathcal{A}$. This means there is, for all i, an element $g_i \in \mathcal{A} \setminus (\mathcal{A}^* \cup \{\mathbf{0}\})$ such that $f_i g_i = f_{i-1}$.

Let us show this chain has to end eventually, using the valuation N. From $f_i g_i = f_{i-1}$, we see that $N(f_{i-1}) = N(f_i)N(g_i)$ with $N(g_i) > 1$ (note that g_i is neither a unit nor zero). So $N(f_i) < N(f_{i-1})$. Because N takes values in $\mathbb{N}$, the chain can not continue infinitely, and therefore has to end. □

By the Lemma A.8, we have now shown that $\mathcal{A}$ is a factorization domain. We only need to show the uniqueness of factorization up to order and units.

Let us call $f \in \mathcal{A} \setminus (\mathcal{A}^* \cup \{\mathbf{0}\})$ **normal** if it has unique factorization and **abnormal** if it does not. It is clear that a irreducible element is normal.

Lemma 2.10. *Let α be an abnormal element with minimal norm $N(\alpha)$ and*

$$\alpha = \sigma_1 * \dots * \sigma_m = \tau_1 * \dots * \tau_n$$

two essentially different factorizations of α into irreducibles. Then necessarily $m = n = 2$ and $\sigma_1, \sigma_2, \tau_1, \tau_2$ all have the same norm.

Proof. Let α be an abnormal element as stated in the Lemma. First note that $m, n > 1$, since a irreducible is normal. Moreover no σ_j is the associate of any τ_i, for if so, cancellation would produce an abnormal element of norm less that $N(\alpha)$.

Without loss of generality, we may assume

$$N(\sigma_1) \le N(\sigma_2) \le \cdots \le N(\sigma_m),$$
$$N(\tau_1) \le N(\tau_2) \le \cdots \le N(\tau_n),$$
$$N(\sigma_1) \le N(\tau_1).$$

Then $N(\sigma_1 * \tau_1) = N(\sigma_1)N(\tau_1) \le N(\tau_1)N(\tau_1) \le N(\tau_1)N(\tau_2) \le N(\alpha)$. If any of these $\le$ is a $<$, we have $N(\sigma_1 * \tau_1) < N(\alpha)$ which we will see leads to a contradiction.

Let us suppose $N(\sigma_1 * \tau_1) < N(\alpha)$. Consider $\beta = \alpha - \sigma_1 * \tau_1$. If $\beta = 0$, we would have $\alpha = \sigma_1 * \tau_1$ and thus $\sigma_2 \ldots \sigma_m = \tau_1$ and, since τ_1 is irreducible, we would have $m = 2$ and τ_1 and σ_2 are associates, see definition A.2, which is a contradiction. So $\beta \neq 0$. Also β is not a unit since $\sigma_1 \mid \beta$. From the definition of N, the lemma 2.7 and the assumption $N(\sigma_1 * \tau_1) < N(\alpha)$, it follows that $N(\beta) = N(\sigma_1 * \tau_1) \le N(\alpha)$.

Hence β is normal. However the non-associates σ_1, τ_1 both divide β, so we must have $\sigma_1 * \tau_1 \mid \beta$, as β is normal. This because, as *beta* has a unique factorization, both σ_1 and τ_1 must be associate of (at least) one of the irreducibles in the factorization. It follows that $\sigma_1 * \tau_1 \mid \alpha$ and thus $\sigma_1 * \cdots * \sigma_m = \alpha = \sigma_1 * \tau_1 * \gamma$, which in turn imply that $\sigma_2 * \cdots * \sigma_m = \tau_1 * \gamma$. But $N(\sigma_2 * \cdots * \sigma_m) < N(\alpha)$, so $\sigma_2 * \cdots * \sigma_m$ has to be normal and τ_1 must be associated with some σ_i, which is a contradiction.

So we conclude that $N(\sigma_1 * \tau_1) = N(\alpha)$, i.e., $N(\sigma_1 * \tau_1) = N(\sigma_1)N(\tau_1) = N(\tau_1)N(\tau_1) = N(\tau_1)N(\tau_2) = N(\alpha)$. This implies that $N(\sigma_1) = N(\tau_1) = N(\tau_2)$ and $n = 2$. Let us define $M := N(\sigma_1)$ so we can easily see that $M^2 = N(\alpha) = N(\sigma_1) \ldots N(\sigma_m) \ge M^m$. We may now conclude that $m = 2$ and $N(\sigma_2) = M$. □

We have now seen that if the uniqueness fails in $\mathcal{A}$, we should have an element of the form $\alpha * \beta = \gamma * \delta$, where $\alpha, \beta, \gamma, \delta$ are irreducibles of norm M and α not associated with either γ or δ.

Proposition 2.11. *$\mathcal{A}$ is isomorphic to the ring of formal power series with infinite indeterminates $F_\omega := \mathbb{C}[[x_1, x_2, \ldots]]$, with the usual formal power series operations.*

Proof. First, let us order all primes of $\mathbb{N}$, $p_1, p_2, \ldots$ Any $n \in \mathbb{N}$ can be written uniquely as $n = p_1^{a_1} p_2^{a_2} \ldots$ and thus uniquely identified with the infinite vector $(a_1, a_2, \ldots)$, with entries in $\mathbb{N}_0$, only finitely many being non-zero. Conversely, each such vector is uniquely identified to a natural number.

We can now identify each $\alpha \in \mathcal{A}$ with a formal power series in a countably infinite number of indeterminates with coefficients in $\mathbb{C}$, as follows.

$$\begin{aligned} P\colon \mathcal{A} &\longrightarrow F_\omega \\ \alpha &\longmapsto P(\alpha) = \sum_{n\in\mathbb{N}} \alpha(n) x_1^{a_1} x_2^{a_2} \ldots, \end{aligned}$$

where $n = p_1^{a_1} p_2^{a_2} \ldots$. Note that any term of that sum only has a finite number of indeterminates, although infinitely many x_i may occur (in terms with non-zero coefficients) in the same series.

It is easy to see that

$$\begin{aligned} &P(\iota) = 1, \\ &P(\alpha + \beta) = P(\alpha) + P(\beta), \\ &P(\alpha * \beta) = P(\alpha)P(\beta), \end{aligned}$$

the last one following from definition of Dirichlet convolution. So P is a ring isomorphism. ☐

Remark. Note that, by this isomorphism, the units of F_ω correspond to the series with non-zero constant term (corresponding to $n = 1$ on the sum).

Set $F_\ell := \mathbb{C}[[x_1, \ldots, x_\ell]]$, for $\ell in \mathbb{N}$. It is known [Lan02, Theorem 9.3] that these domains F_l are unique factorization domains. As in the previous case, the units of F_ℓ are the series with non-zero constant term.

Let $A \in F_\omega$ be the series associated with $\alpha \in \mathcal{A}$, that is,

$$A := A(x_1, x_2, \ldots) = P(\alpha)$$

and let

$$(A)_\ell := A(x_1, \ldots, x_\ell, 0, 0, \ldots)$$

be the series obtained from A by deleting all its terms involving any x_i with $i > \ell$. It is easy to see that $(AB)_\ell = (A)_\ell (B)_\ell$ and $(A + B)_\ell = (A)_\ell + (B)_\ell$. Therefore, the map

$$\begin{aligned} \pi_\ell\colon F_\omega &\longrightarrow F_\ell \\ A &\longmapsto (A)_\ell \end{aligned}$$

is a ring homomorphism of F_ω onto F_ℓ.

In the same manner, we can also define a ring homomorphism, for $m \geq \ell$, from F_m onto F_ℓ. We will use the same notation for this homomorphism, as one can say that F_ω "contains" F_m, for all m. Also note that $\pi_\ell((A)_\ell) = (A)_\ell$, that is, $\pi_\ell : F_\ell \to F_\ell$ is just the identity map.

Lemma 2.12. *If $A \in F_\omega \setminus (F_\omega^* \cup \{0\})$, then there is some minimal $L(A)$ for which $(A)_\ell$ is neither zero nor a unit of F_ℓ, for all $\ell \geq L(A)$.*

Proof. Since A is not a unit, its constant term is zero, but as $A \neq 0$, the series must contain some product $x_1^{a_1} x_2^{a_2} \dots$ with some non-zero coefficient and $(a_1, a_2, \dots) \neq (0, 0 \dots)$. If in this term the last indeterminate with non-zero exponent is x_k, then $(A)_k \neq 0$. So there is a minimal $L := L(A)$ such that for all $\ell \geq L$, $(A)_\ell \neq 0$.

By construction, for $k \geq \ell$, $(A)_k = (A)_\ell + S$, where S is a series of F_k such that each term has at least one x_i with $i > \ell$. So, for all $\ell \geq L$, $(A)_\ell$ is neither zero nor a unit. □

Lemma 2.13. *Let $A \in F_\omega \setminus (F_\omega^* \cup \{0\})$. If there is a $\ell \geq L(A)$ such that $(A)_\ell$ is irreducible in F_ℓ, then for all $m \geq \ell$, $(A)_m$ is irreducible in F_m. Hence there is a minimal integer, $P(A) \geq L(A)$ such that, $(A)_\ell$ is irreducible in F_ℓ for all $\ell \geq P(A)$. Also, A is irreducible in F_ω.*

Proof. Fix $m \geq \ell$. Let us suppose $(A)_m$ is reducible. So there are non-zero elements $R_m, S_m \in F_m \setminus F_m^*$ such that $(A)_m = R_m S_m$. Thus, we can write $(A)_\ell = ((A)_m)_\ell = (R_m)_\ell (S_m)_\ell$. But as R_m, S_m were not units, $(R_m)_\ell, (S_m)_\ell$ cannot be units and therefore $(A)_\ell$ is reducible, a contradiction.

So there is a minimal integer $P := P(A)$ as stated. Let us now suppose A is reducible in F_ω, i.e., there are R, S non-zero and non-units of F_ω such that $A = RS$. But then $(A)_P = (R)_P (S)_P$, both $(R)_P, (S)_P$ are non-zero and non-units of F_P, and $(A)_P$ is reducible, which is a contradiction. □

Definition 2.14. *Let $A \in F_\omega$ be an irreducible. We say A is **finitely irreducible** if exists $\ell \geq L(A)$ such that $(A)_\ell$ is irreducible.*

By the previous Lemma, this is the same as saying for all $\ell \geq P(A)$, $(A)_\ell$ is irreducible.

We have shown that, for $A \setminus (F_\omega^* \cup \{0\})$, if there is any $m \geq L(A)$ such that $(A)_m$ is irreducible, then A is irreducible. The only remaining possibility is that for $A \setminus (F_\omega^* \cup \{0\})$, we have $(A)_\ell$ reducible in F_ℓ for all $\ell \geq L(A)$. We will show that, in this case, such A is reducible in F_ω.

Definition 2.15. *For all $\ell \in \mathbb{N}$ or $\ell = \omega$ (here, ω represent the first infinite ordinal number), we call **true factor** of $A_\ell \in F_\ell$ to a $R_\ell \in F_\ell \setminus F_\ell^*$ if there exists $S_\ell \in F_\ell \setminus F_\ell^*$ such that $A_\ell = R_\ell S_\ell$. We say $R_\ell S_\ell$ is a **true factorization** of A_ℓ.*

*We shall call any chain $[R_M, R_{M+1}, \dots, R_N]$ of true factors of the corresponding A_ℓ, for $\ell = M+1, \dots, N$, **telescopic** if for each ell, we have $R_{\ell-1} = \pi_{\ell-1}(R_\ell)$.*

Proposition 2.16. *All irreducibles of F_ω are finitely irreducible.*

Proof. Let $A \in F_\omega \setminus (F_\omega^* \cup \{0\})$, with $L := L(A)$ and suppose that $\forall \ell \geq L$, there are $R_\ell, S_\ell \in F_\ell \setminus F_\ell^*$ such that

$$(A)_\ell = R_\ell S_\ell, \tag{2.1}$$

or in other words, suppose A is reducible for all $\ell \geq L$.

Observe that $(A)_m = R_m S_m$, for some $m > L$ implies that $(A)_{m-1} = \pi_{m-1}((A)_m) = \pi_{m-1}(R_m)\pi_{m-1}(S_m)$ and so on down to L. The chain of true factors $[\pi_L(R_m), \ldots, \pi_{m-1}(R_m), R_m]$ is telescopic, since π_i are homomorphisms. Then, from the original assumption (2.1), we have the existence of a sequence of telescopic chains of true factors

$$\begin{aligned}
&\kappa_0 = [R_L] \\
&\kappa_1 = [\pi_L(R_{L+1}), R_{L+1}] \\
&\kappa_2 = [\pi_L(R_{L+2}), \pi_{L+1}(R_{L+2}), R_{L+2}] \\
&\quad\vdots \\
&\kappa_n = [\pi_L(R_{L+n}), \ldots, \pi_{L+n-j}(R_{L+n}), \ldots, \pi_{L+n-1}(R_{L+n}), R_{L+n}] \\
&\quad\vdots
\end{aligned}$$

We want to prove the existence of $R, S \in F_\omega$ such that $A = RS$. Suppose we have an infinite chain of true factors of $(A)_{L+j}$ for $j \in \{0, 1, \ldots\}$

$$\kappa^* = [R_L^*, R_{L+1}^*, R_{L+2}^*, \ldots]$$

which is telescopic throughout. Its existence implies the existence of one infinite chain $[S_L^*, S_{L+1}^*, \ldots]$ such that it is also telescopic, since

$$\begin{aligned}
R_{L+j-1}^* S_{L+j-1}^* &= (A)_{L+j-1} = \pi_{L+j-1}((A)_{L+j}) \\
&= \pi_{L+j-1}(R_{L+j}^* S_{L+j}^*) \\
&= \pi_{L+j-1}(R_{L+j}^*)\pi_{L+j-1}(S_{L+j}^*) \\
&= R_{L+j-1}^* \pi_{L+j-1}(S_{L+j}^*) \\
\implies S_{L+j-1}^* &= \pi_{L+j-1}(S_{L+j}^*).
\end{aligned}$$

Such telescopic infinite chain κ^* will define unambiguously a series R in F_ω by taking the coefficients of R_{l+j}^*, for an appropriate $j \in \mathbb{N}_0$ in each step. As the chain is infinite and telescopic, this is possible for any term $x_1^{a_1} x_2^{a_2} \ldots x_m^{a_m}$. Now that we have both R, S we can show that $A = RS$ as above, by comparing each coefficient using a suitable j and the equality $(A)_{L+j} = R_j^* S_j^*$. Thus, if we prove existence of such κ^*, we would prove the existence of $R, S \in F_\omega$ such that $A = RS$.

Since unique factorization holds in F_l, there are only a finite number of classes of associates into which the true factors of any $(A)_\ell$ can fall. Hence by the pigeon-hole principle, an infinite set of chains κ_i such that all have their first entry equivalent to some true factor T_0 of $(A)_L$. Let us choose such chain and call it K_0, that is

$$K_0 = [\pi_L(R_{i_0}), \dots, R_{i_0}] = \kappa_{i_0 - L}$$

where $i_0 \geq L$. Note that this is a finite chain, but we are only interested in the first entry.

Now, of this infinite set, there is an infinite subset of chains whose second entry is equivalent to some other true factor T_1 of $(A)_{L+1}$. Again, let us choose one and call it K_1, that is

$$K_1 = [\pi_L(R_{i_1}), \pi_{L+1}(R_{i_1}), \dots, R_{i_1}]$$

for some $i_1 \geq L$. Repeating this process for each $\ell \geq L$ and $(A)_\ell$ we will have a sequence of telescopic chains

$$\begin{aligned} K_0 &= [\pi_L(R_{i_0}), \dots, R_{i_0}] \\ K_1 &= [\pi_L(R_{i_1}), \pi_{L+1}(R_{i_1}), \dots, R_{i_1}] \\ K_2 &= [\pi_L(R_{i_2}), \pi_{L+1}(R_{i_2}), \pi_{L+2}(R_{i_2}), \dots, R_{i_2}] \\ &\vdots \end{aligned}$$

such that $\pi_{L+j}(R_{i_n}) \sim T_j$ for all $j \geq 0$, for all $n \geq j$.

We can now construct the telescopic infinite chain κ^* working only with the main diagonal and the diagonal below it, as follows. Define $R_0^* = \pi_L(R_{i_0})$. Since $\pi_L(R_{i_1}) \sim T_0 \sim R_0^*$ in F_L, as per choice of K_1, there is a unit U_L of F_L such that

$$\begin{aligned} R_0^* &= \pi_L(R_{i_1})U_L \\ &= \pi_L(R_{i_1})\pi_L(U_L) && \text{(as } \pi_\ell : F_\ell \to F_\ell \text{ is the identity)} \\ &= \pi_L(\pi_{L+1}(R_{i_1})U_L) && \text{(as } K_1 \text{ is telescopic)} \end{aligned}$$

Define $R_1^* := \pi_{L+1}(R_{i_1})U_L$ in F_{L+1} and note that R_1^* is a true factor of $(A)_{L+1}$, because $R_1^* \sim \pi_{L+1}(R_{i_1}) \sim T_1$ in F_{L+1}, and that $\pi_L(R_1^*) = R_0^*$.

Let $m \in \mathbb{N}$. Now let us assume we have R_m^* defined, such that $R_m^* \sim \pi_{L+m}(R_{i_m}) \sim T_m$ in F_{L+m}, and that $\pi_L(R_m^*) = R_{m-1}^*$. Since $\pi_{L+m}(R_{i_{m+1}}) \sim T_m$ in F_{L+m}, by definition of the sequence of chains K_n, there is a unit U_{L+m} of F_{L+m} such that

$$\begin{aligned} R_m^* &= \pi_{L+m}(R_{i_{m+1}})U_{L+m} \\ &= \pi_{L+m}(R_{i_{m+1}})\pi_{L+m}(U_{L+m}) && \text{(as } \pi_\ell : F_\ell \to F_\ell \text{ is the identity)} \\ &= \pi_{L+m}(\pi_{L+m+1}(R_{i_{m+1}})U_{L+m}) && \text{(as } K_{m+1} \text{ is telescopic)} \end{aligned}$$

Now, let us define

$$R^*_{m+1} := \pi_{L+m+1}(R_{i_{m+1}})U_{L+m}$$

in F_{L+m+1} and note that R^*_{m+1} is a true factor of $(A)_{L+m+1}$, because $R^*_{m+1} \sim \pi_{L+m+1}(R_{i_{m+1}}) \sim T_{m+1}$ in F_{L+m+1}, and that $\pi_{L+m}(R^*_{m+1}) = R^*_m$.

So, by induction, we have defined an infinite telescopic chain with the properties we wanted, and A is indeed reducible. Thus, all irreducibles of F_ω are finitely irreducible. □

Now that we have identified all irreducibles in F_ω, we can prove the uniqueness of factorization in this domain (and thus, by isomorphism, in $\mathcal{A}$).

Theorem 2.17. *$\mathcal{A}$ is a unique factorization domain.*

Proof. We have already seen that it is possible to factor any non-zero non-unit element of $\mathcal{A}$ into irreducibles. We will now show this factorization is unique up to order and units.

Let us suppose the uniqueness fails in F_ω. By Lemma 2.10 and Proposition 2.11, we must have a series of the form $AB = CD$, where A, B, C, D are irreducibles in F_ω and A is not associated with either C or D. Since all irreducibles are of finite type, taking $P := \max\{P(A), P(B), P(C), P(D)\}$, we have that $(A)_\ell, (B)_\ell, (C)_\ell, (D)_\ell$ are all irreducible in F_ℓ for all $\ell \geq P$ and that they satisfy the equation

$$(A)_\ell(B)_\ell = (AB)_\ell = (CD)_\ell = (C)_\ell(D)_\ell.$$

Since F_ℓ is a unique factorization domain, $(A)_\ell$ must be associated with either $(C)_\ell$ or $(D)_\ell$, for all $\ell \geq P$. Hence there must be a infinite subset $S \subset \mathbb{N}$ such that $\forall s \in S\ (A)_s \sim (C)_s$ in F_s or $\forall s \in S\ (A)_s \sim (D)_s$ in F_s.

Without loss of generality, let us assume the first possibility. Thus, for all $s \in S$, there is $U_s \in F^*_s$ such that $(A)_s = U_s(C)_s$. If $s < t$ are two integers of S, we have

$$U_s(C)_s = (A)_s = ((A)_t)_s = (U_t(C)_t)_s = (U_t)_s(C)_s$$

and U_t is an extension of U_s by terms that involve at least one indeterminate x_i with $t \geq i > s$. Thus the sequence $(U_s)_{s\in S}$ defines a unit U of F_ω and $A = UC$ by the same argument used in showing $A = RS$ in the previous Proposition. But then $A \sim C$ in F_ω, contradiction.

Thus, F_ω, and therefore $\mathcal{A}$, is a UFD. □

Proposition 2.18. *$\mathcal{A}$ is not a principal ideal domain.*

Proof. In a UFD, one can always define a greater common divisor, **gcd**, for any two elements. Let f, g be two primes in $\mathcal{A}$. Then, of course, we have $\gcd(f, g) = \iota$.

Now, suppose $\mathcal{A}$ is a PID. Then the Bézout identity holds, i.e., there are $h, k \in \mathcal{A}$ such that $\iota = h * f + k * g$. Let us evaluate this equality at $n = 1$. Note that, as f, g are primes, $f(1) = 0$ and $g(1) = 0$, and hence

$$1 = \iota(1) = h(1)f(1) + k(1)g(1) = 0,$$

a contradiction. So $\mathcal{A}$ is not a PID. □

The definition of the Dirichlet convolution immediately yields:

$$\mathcal{D}(f; s) \cdot \mathcal{D}(g; s) = \mathcal{D}(f * g; s)$$

It is then clear that the map $\mathcal{A} \to \mathcal{S}$ given by $f \mapsto \mathcal{D}(f; s)$ is a ring isomorphism. Therefore the ring $(\mathcal{S}, +, \cdot)$ is a unique factorization domain.

2.2.2 Möbius inversion

We will now describe some inversion results, the Möbius inversion, the inverse of a completely multiplicative function and one more particular one, all in which the Möbius function also plays an important role.

Lemma 2.19. *If f is a multiplicative function, then:*

$$(f * e_0)(n) = \prod_{p|n} \left(1 + f(p) + f(p^2) + \cdots + f(p^{\nu_p(n)})\right)$$

Proof. This is a direct consequence of unique factorization in $\mathbb{Z}$. □

Proposition 2.20. *The Möbius function is the inverse of e_0, i.e., $\mu * e_0 = \iota$.*

Proof. It follows from the previous Lemma that $(\mu * e_0)(n) = \prod_{p|n}(1 + \mu(p))$, which is zero unless $n = 1$. □

Note that it follows from Proposition 2.20 and definition 2.3 that

$$\sum_{n \geq 1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

Proposition 2.21 (Möbius inversion). *For all $f \in \mathcal{A}$,*

$$F(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).$$

Proof. This is an immediate consequence of the previous Proposition, as the above equivalence is the same as

$$F = f * e_0 \iff f = F * \mu. \qquad \square$$

Proposition 2.22. *If $f \neq \mathbf{0}$ is completely multiplicative, then $f^{-1} = \mu f$.*

Proof. First, as f is completely multiplicative, we have for all $n \in \mathbb{N}$,

$$f(n) = f(1 \cdot n) = f(1)f(n),$$

and therefore $f(1) = 1$, since $f(n) \neq 0$ for some n. Put $g = \mu f$. Then

$$\begin{aligned}(f * g)(n) &= \sum_{d|n} \mu(d)f(d)f\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d)f(n) \\ &= f(n)\sum_{d|n} \mu(d) = f(n)\iota(n) = \iota(n),\end{aligned}$$

because $f(n)\iota(n) = 0$ for all $n \neq 0$ and $f(1)\iota(1) = 1$. $\square$

Lemma 2.23. *For all $k \in \mathbb{N}$, we have that $\varsigma_k * q_k = e_0$.*

Proof. This can be seen as follows. For $n, k \in \mathbb{N}$, let

$$m_k(n) = \max\{t^k : t \in \mathbb{N} \ \wedge \ t^k \mid n\}.$$

Note that if n is k^{th}-power-free, then $m_k(n) = 1$, and that $n/m_k(n)$ is k^{th}-power-free. Now, for any divisor d of n, one has $\varsigma_k(d)q_k(n/d) \neq 0$ if and only if d is a k^{th}-power and n/d is k^{th}-power-free, and there is only one such divisor, namely $d = m_k(n)$. $\square$

Proposition 2.24. *For all $f \in \mathcal{A}$,*

$$f(n) = \sum_{j^2|n} g\left(\frac{n}{j^2}\right) \iff g(n) = \sum_{j^2|n} \mu(j)f\left(\frac{n}{j^2}\right).$$

Proof. We have that

$$f(n) = \sum_{j^2|n} g\left(\frac{n}{j^2}\right) = \sum_{d|n} \varsigma_2(d)g\left(\frac{n}{d}\right) = (\varsigma_2 * g)(n).$$

By Proposition 2.20 and Lemma 2.23, we have

$$\begin{aligned}f = \varsigma_2 * g &\iff f * q_2 = g * e_0 \iff \\ \iff f * q_2 * \mu = g * \iota &\iff f * q_2 * \mu = g\end{aligned}$$

Define

$$\overline{\mu}(n) = q_2 * \mu(n) = \begin{cases} \mu(k), & \text{if } n = k^2 \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $g = \overline{\mu} * f$ and therefore

$$g(n) = \sum_{d|n} \overline{\mu}(d) f\left(\frac{n}{d}\right) = \sum_{j^2|n} \mu(j) f\left(\frac{n}{j^2}\right)$$

So $f = \varsigma_2 * g \iff g = \overline{\mu} * f$, which shows the claim. □

2.3 Liouville and Möbius Functions

We shift our focus to two important arithmetic functions, the Liouville and Möbius functions. After we study some interesting results involving them, for the purpose of exploration, we will show some relation between them. To conclude this chapter, we will exhibit and prove a proposition on the relation between their summation functions

$$L(x) := \sum_{n \le x} \lambda(n) \quad M(x) := \sum_{n \le x} \mu(n),$$

which will be used in chapter 4, as their asymptotic growth is closely related the Riemann's Hypothesis.

Proposition 2.25. *We have that* $\lambda * e_0 = \varsigma_2$.

Proof. Since λ is completely multiplicative,

$$\begin{aligned} \sum_{d|n} \lambda(d) =& \prod_{p|n} \left(1 + \lambda(p) + \lambda(p)^2 + \cdots + \lambda(p)^{\nu_p(n)}\right) \\ =& \prod_{p|n} \frac{1 - \lambda(p)^{\nu_p(n)+1}}{1 - \lambda(p)} = \prod_{p|n} \frac{1 - (-1)^{\nu_p(n)+1}}{2} = \varsigma_2(n). \end{aligned}$$

□

Lemma 2.26. *For any sequence of complex numbers* $x_1, x_2, \ldots, x_n$*, we have*

$$\sum_{k \le n} \lambda(k)(x_k + x_{2k} + \cdots + x_{\lfloor \frac{n}{k} \rfloor k}) = \sum_{j \le \sqrt{n}} x_{j^2}. \tag{2.2}$$

Proof. Rearranging the terms, by factoring out the x_t, we have

$$\sum_{k \le n} \lambda(k) \left(\sum_{t=1}^{\lfloor \frac{n}{k} \rfloor} x_{kt} \right) = \sum_{t \le n} x_t \left(\sum_{t|k} \lambda(k) \right).$$

And, by Proposition 2.25

$$\sum_{t\leq n} x_t\,\varsigma_2(t) = \sum_{j\leq\sqrt{n}} x_{j^2}.$$

□

Lemma 2.27. *For any sequence of complex numbers $x_1, x_2, \ldots, x_n$, we have*

$$\sum_{k\leq n} \mu(k)\left(x_k + x_{2k} + \cdots + x_{\lfloor \frac{n}{k}\rfloor k}\right) = x_1. \tag{2.3}$$

Proof. Rearraging the sums, we get

$$\sum_{k\leq n} \mu(k)\left(\sum_{t=1}^{\lfloor \frac{n}{k}\rfloor} x_{kt}\right) = \sum_{t\leq n} x_t \left(\sum_{t\mid k} \mu(k)\right)$$
$$= \sum_{t\leq n} x_t\,\iota(t) = x_1 \qquad \text{(by Proposition 2.20).}$$

□

Proposition 2.28. *For all $n\in\mathbb{N}$ one has*

$$\sum_{k\leq n} \lambda(k)\left\lfloor\frac{n}{k}\right\rfloor = \left\lfloor\sqrt{n}\right\rfloor, \tag{2.4}$$

$$\sum_{k\leq n} \mu(k)\left\lfloor\frac{n}{k}\right\rfloor = 1. \tag{2.5}$$

Proof. Both equalities come from the last two Lemmas, applied to the sequence $x_k = 1$ for all k. □

Definition-Proposition 2.29. *For $x \geq 0$, we define*

$$\delta_x = \begin{cases} 0, & \lfloor x\rfloor \text{ even}, \\ 1, & \lfloor x\rfloor \text{ odd}. \end{cases} \tag{2.6}$$

We have that

$$\delta_x = \frac{1-(-1)^{\lfloor x\rfloor}}{2} = \lfloor x\rfloor - 2\left\lfloor\frac{x}{2}\right\rfloor, \tag{2.7}$$

and also that

$$\sum_{1\leq n\leq x} (-1)^{n+1} = \delta_{\lfloor x\rfloor}. \tag{2.8}$$

Proof. The first equality of (2.7) is easy.

$$\frac{1-(-1)^{\lfloor x\rfloor}}{2} = \begin{cases} \frac{1-1}{2}, & \lfloor x\rfloor \text{ even} \\ \frac{1-(-1)}{2}, & \lfloor x\rfloor \text{ odd} \end{cases} = \begin{cases} 0, & \lfloor x\rfloor \text{ even,} \\ 1, & \lfloor x\rfloor \text{ odd.} \end{cases}$$

For the second one, recall that $\{x\} = x - \lfloor x\rfloor$ and note that

$$\lfloor x\rfloor - 2\left\lfloor \frac{x}{2}\right\rfloor = \lfloor x\rfloor - 2\left\lfloor \frac{\lfloor x\rfloor}{2} + \frac{\{x\}}{2}\right\rfloor.$$

Now let us first consider x with $\lfloor x\rfloor$ even. Then $\frac{\lfloor x\rfloor}{2}$ is an integer and as $\{x\} < 1$ we have that $\frac{\{x\}}{2} < \frac{1}{2}$. Hence

$$\left\lfloor \frac{\lfloor x\rfloor}{2} + \frac{\{x\}}{2}\right\rfloor = \frac{\lfloor x\rfloor}{2}$$

and therefore

$$\lfloor x\rfloor - 2\left\lfloor \frac{\lfloor x\rfloor}{2} + \frac{\{x\}}{2}\right\rfloor = \lfloor x\rfloor - 2\frac{\lfloor x\rfloor}{2} = 0.$$

For x with $\lfloor x\rfloor$ odd, we have that $\frac{\lfloor x\rfloor}{2}$ is not an integer, but $\frac{\lfloor x\rfloor-1}{2}$ is and thus

$$\left\lfloor \frac{\lfloor x\rfloor}{2} + \frac{\{x\}}{2}\right\rfloor = \left\lfloor \frac{\lfloor x\rfloor - 1}{2} + \frac{1}{2} + \frac{\{x\}}{2}\right\rfloor$$

and, thus, as $\frac{\{x\}}{2} + \frac{1}{2} < 1$, we have that

$$\left\lfloor \frac{\lfloor x\rfloor}{2} + \frac{\{x\}}{2}\right\rfloor = \frac{\lfloor x\rfloor - 1}{2},$$

and therefore

$$\lfloor x\rfloor - 2\left\lfloor \frac{\lfloor x\rfloor}{2} + \frac{\{x\}}{2}\right\rfloor = \lfloor x\rfloor - 2\frac{\lfloor x\rfloor - 1}{2} = 1.$$

As for (2.8),

$$\begin{aligned} \sum_{1\le n\le x} (-1)^{n+1} &= 1 - 1 + 1 - 1 + \cdots + (-1)^{\lfloor x\rfloor+1} \\ &= (1-1) + (1-1) + \cdots + (-1)^{\lfloor x\rfloor+1} \\ &= \begin{cases} 0, & \lfloor x\rfloor \text{ even,} \\ 1, & \lfloor x\rfloor \text{ odd.} \end{cases} \end{aligned}$$

□

Proposition 2.30. *For any $n \in \mathbb{N}$, we have*

$$\sum_{k \leq n} \lambda(k) \left((-1)^k + (-1)^{2k} + \cdots + (-1)^{\lfloor \frac{n}{k} \rfloor k}\right) = -\delta_{\sqrt{n}}. \tag{2.9}$$

Proof. By Lemma 2.26 applied the sequence $x_k = (-1)^k$ for all k,

$$\begin{aligned}\sum_{k \leq n} \lambda(k) \left((-1)^k + (-1)^{2k} + \cdots + (-1)^{\lfloor \frac{n}{k} \rfloor k}\right) &= \sum_{j \leq \sqrt{n}} (-1)^{j^2} \\ &= \sum_{j \leq \sqrt{n}} (-1)^j \\ &= -\sum_{j \leq \sqrt{n}} (-1)^{j+1} = -\delta_{\sqrt{x}}.\end{aligned}$$

□

Proposition 2.31. *Recall that $L(x) = \sum_{n \leq x} \lambda(n)$. We have, for all $N \in \mathbb{N}$,*

$$\sum_{t=1}^{N} (-1)^t L\left(\frac{N}{t}\right) = 2\left\lfloor \sqrt{\frac{N}{2}} \right\rfloor - \left\lfloor \sqrt{N} \right\rfloor. \tag{2.10}$$

Proof.

$$\begin{aligned}\sum_{t=1}^{N} (-1)^t L\left(\frac{N}{t}\right) &= \sum_{t \leq N} (-1)^t \sum_{n \leq \frac{N}{t}} \lambda(n) \\ &= \sum_{t \leq N} \sum_{nt \leq N} (-1)^t \lambda(n) \\ &= \sum_{n \leq N} \lambda(n) \left(\sum_{nt \leq N} (-1)^t \right) \\ &= \sum_{n \leq N} \lambda(n) \left(-\delta_{\lfloor \frac{N}{n} \rfloor}\right) && \text{(by equation (2.8))} \\ &= \sum_{n \leq N} \lambda(n) \left(2\left\lfloor \frac{N}{2n} \right\rfloor - \left\lfloor \frac{N}{n} \right\rfloor\right) && \text{(by equation (2.7))} \\ &= 2\sum_{n \leq N} \lambda(n) \left\lfloor \frac{N}{2n} \right\rfloor - \sum_{n \leq N} \lambda(n) \left\lfloor \frac{N}{n} \right\rfloor\end{aligned}$$

If $\frac{N}{2} < n \leq N$, we have $\left\lfloor \frac{N}{2n} \right\rfloor = 0$ and thus we can simply the expression as

$$2\sum_{n \leq N} \lambda(n) \left\lfloor \frac{N}{2n} \right\rfloor - \sum_{n \leq N} \lambda(n) \left\lfloor \frac{N}{n} \right\rfloor = 2\sum_{n \leq \frac{N}{2}} \lambda(n) \left\lfloor \frac{N}{2n} \right\rfloor - \sum_{n \leq N} \lambda(n) \left\lfloor \frac{N}{n} \right\rfloor.$$

Finally, using equation (2.4)

$$\sum_{t=1}^{N}(-1)^t L\left(\frac{N}{t}\right) = 2\left\lfloor\sqrt{\frac{N}{2}}\right\rfloor - \left\lfloor\sqrt{N}\right\rfloor.$$

□

Lemma 2.32. *For any $n \geq 1$, we have*

$$\lambda(n) = \sum_{j^2|n}\mu\left(\frac{n}{j^2}\right) \qquad \text{and} \qquad \mu(n) = \sum_{j^2|n}\mu(j)\lambda\left(\frac{n}{j^2}\right). \tag{2.11}$$

Proof. Write $n = td^2$, where $t, d \in \mathbb{N}$ and t is square-free.

If $j^2 \mid n$, as t is square-free, we must have $j^2 \mid d^2$, and therefore $j \mid d$. So $j^2 \mid n \iff n = j^2k^2t$ for some $k \in \mathbb{N}$, and $\mu\left(\frac{n}{j^2}\right) \neq 0 \iff j = d$. We also have that

$$\lambda(t) = \lambda(d^2)\lambda(t) = \lambda(n) = \lambda\left(\frac{n}{j^2}\right)\lambda(j^2) = \lambda\left(\frac{n}{j^2}\right)\lambda(j)^2 = \lambda\left(\frac{n}{j^2}\right).$$

Thus $\lambda(n) = \lambda(t)$ and

$$\sum_{j^2|n}\mu\left(\frac{n}{j^2}\right) = \mu\left(\frac{n}{d^2}\right) = \mu(t) = \lambda(t) = \lambda(n),$$

which shows the first equality. Also, using the fact that $\lambda(t) = \lambda(n/j^2)$

$$\begin{aligned}\sum_{j^2|n}\mu(j)\lambda\left(\frac{n}{j^2}\right) &= \sum_{j^2|n}\mu(j)\lambda(t) \\ &= \lambda(t)\sum_{j|d}\mu(j) = \lambda(t)\iota(d) \quad \text{(by Proposition 2.20)} \\ &= \begin{cases}\lambda(t), & \text{if } d = 1 \iff n \text{ is square-free,} \\ 0, & \text{otherwise}\end{cases} = \mu(n).\end{aligned}$$

□

Proposition 2.33. *Recall that $L(x) = \sum_{n\leq x}\lambda(n)$ and $M(x) = \sum_{n\leq x}\mu(n)$. One has*

$$L(n) = \sum_{j\leq\sqrt{n}} M\left(\frac{n}{j^2}\right) \qquad \text{and} \qquad M(n) = \sum_{j\leq\sqrt{n}}\mu(j)L\left(\frac{n}{j^2}\right). \tag{2.12}$$

Proof. Using the previous Lemma, we have

$$
\begin{aligned}
L(n) &= \sum_{t\le n}\lambda(t) = \sum_{t\le n}\sum_{j^2|t}\mu\left(\frac{t}{j^2}\right)\\
&= \sum_{j^2\le n}\sum_{kj^2\le n}\mu\left(k\right) && \text{(by rearranging the sums with } t=kj^2\text{)}\\
&= \sum_{j\le\sqrt{n}} M\left(\frac{n}{j^2}\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
M(n) &= \sum_{t\le n}\mu(t) = \sum_{t\le n}\sum_{j^2|t}\mu(j)\lambda\left(\frac{t}{j^2}\right)\\
&= \sum_{j^2\le n}\sum_{kj^2\le n}\mu(j)\lambda\left(k\right) && \text{(by rearranging the sums as above)}\\
&= \sum_{j\le\sqrt{n}} \mu(j)L\left(\frac{n}{j^2}\right).
\end{aligned}
$$

□

Chapter 3

Convergence of Dirichlet Series

While power series converge in discs, the convergence of Dirichlet series happens in half-planes, as we will see in this chapter. Furthermore, the former converges both simply and absolutely inside its disc of convergence and converges uniformly in any compact inside the disc, whereas the Dirichlet series may have different half-planes of convergence for each type. Harald Bohr was the first mathematician to study, on [Boh13a] and [Boh13b], the relationship between the following three abscissas of convergence

$$\sigma_a := \inf\left\{\sigma \in \mathbb{R}\colon \sum_{n\geq 1} \frac{f(n)}{n^\sigma} \text{ converges absolutely}\right\},$$

$$\sigma_c := \inf\left\{\sigma \in \mathbb{R}\colon \sum_{n\geq 1} \frac{f(n)}{n^\sigma} \text{ converges}\right\},$$

$$\sigma_u := \inf\left\{\sigma \in \mathbb{R}\colon \sum_{n\geq 1} \frac{f(n)}{n^{\sigma+it}} \text{ converges uniformly for } t \in \mathbb{R}\right\}.$$

As absolute convergence implies uniform and simple convergence, and uniform convergence implies simple convergence, we have that $\sigma_c \leq \sigma_u \leq \sigma_a$. We will see that $\sigma_a - \sigma_c$ and $\sigma_a - \sigma_u$ are bounded, and that if we assume that the respective arithmetical function f is multiplicative, then $\sigma_u = \sigma_a$.

Let $s = \sigma + it \in \mathbb{C}$ with $\sigma, t \in \mathbb{R}$ be a complex number in the course of this chapter. It is worth noting that

$$|n^s| = \left|n^\sigma e^{it\log(n)}\right| = n^\sigma,$$

as this implies that if a Dirichlet series converges for some $s_0 = \sigma_0 + it_0$, then it converges for $s = \sigma_0 + it$ for all $t \in \mathbb{R}$.

Throughout this chapter, f is an arithmetic function and $\mathcal{D}(f;s)$ denotes its Dirichlet series, as defined above in Definition 2.1.

3.1 Absolute Convergence

Recall that a series converges absolutely if the series of the absolute values of its terms is finite. We start this section by seeing that the absolute convergence happens in a half-plane. After we will prove a uniqueness theorem, stating that if Dirichlet series are equal in a sequence tending to infinity, then their respective arithmetic functions are the same.

Lemma 3.1. *If $\mathcal{D}(f;s)$ converges absolutely for $s = a+ib$, then it converges absolutely for all $s = \sigma + it$ with $\sigma \geq a$.*

Proof. As $\sigma \geq a$, $|n^s| = n^\sigma \geq n^a = \left|n^{a+ib}\right|$, so

$$\left|\frac{f(n)}{n^s}\right| \leq \left|\frac{f(n)}{n^{a+ib}}\right|$$

and the result follows by the comparison test. □

Proposition 3.2. *Suppose that exists an s such that $\sum_{n\geq 1}\left|\frac{f(n)}{n^s}\right|$ diverges and also another s such that it converges.*

*Then there exists a real number σ_a, called the **abscissa of absolute convergence**, such that the series $\mathcal{D}(f;s)$ converges absolutely if $\sigma > \sigma_a$ but does not converge absolutely if $\sigma < \sigma_a$.*

Proof. Let $D = \{\sigma \in \mathbb{R} \colon \sum_{n\geq 1}\left|\frac{f(n)}{n^s}\right| \text{ diverges for } s = \sigma + it\}$ be the set of all real numbers such that the series does not converge absolutely.

It is not empty by hypothesis and by the previous Lemma it is bounded above (or else it would diverge for all s). So D has a supremum, which we call σ_a. If $\sigma < \sigma_a$ then $\sigma \in D$. If $\sigma > \sigma_a$, then $\sigma \notin D$, so it converges absolutely. □

Remark. By convention, if the series converges absolutely everywhere, we define $\sigma_a = -\infty$ and if never converges absolutely we define $\sigma_a = \infty$.

3.1.1 Examples

- If f is bounded, then $\mathcal{D}(f;s)$ converges absolutely for $\sigma > 1$, so $\sigma_a \leq 1$.
- The Riemann Zeta Function $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ has $\sigma_a = 1$, since we know it diverges for $s = 1$.

- For any character, the Dirichlet L-function $L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$ has $\sigma_a = 1$; because if χ is the principal character, then $s = 1$ is a pole of $L(s,\chi)$ and therefore $\sigma_a = 1$. If χ is not the principal character, given to the periodic and completely multiplicative nature of the function, we will have $|\chi(n)| \in \{0,1\}$ for all n; in fact, if $\chi(n) \neq 0$, then $(n,q) = 1$ and by Euler's Theorem we then have $n^{\phi(q)} \equiv 1 \mod q$, which implies $\chi(n)^{\phi(q)} = \chi(n^{\phi(q)}) = \chi(1) = 1$, therefore $|\chi(n)| = \chi_0(n)$.
- The series $\sum_{n=1}^{\infty} \frac{n^n}{n^s}$ diverges for all s, so it has $\sigma_a = \infty$.
- The series $\sum_{n=1}^{\infty} \frac{1}{n^n n^s}$ converges absolutely for all s so it has $\sigma_a = -\infty$.

Remark. Assuming $\mathcal{D}(f;s)$ converges absolutely for $\sigma > \sigma_a$, we can define a function $F(s) := \sum\limits_{n=1}^{\infty} \frac{f(n)}{n^s}$ for s with $\sigma > \sigma_a$.

Lemma 3.3. *If $N \geq 1$ and $c > \sigma_a$, for $s = \sigma + it$ such that $\sigma \geq c$ we have*

$$\left|\sum_{n=N}^{\infty} \frac{f(n)}{n^s}\right| \leq N^{-(\sigma-c)} \sum_{n=N}^{\infty} \frac{|f(n)|}{n^c}. \tag{3.1}$$

Proof. Since

$$\left|\sum_{n=N}^{\infty} \frac{f(n)}{n^s}\right| \leq \sum_{n=N}^{\infty} \left|\frac{f(n)}{n^s}\right| = \sum_{n=N}^{\infty} \frac{|f(n)|}{n^\sigma} \leq \sum_{n=N}^{\infty} \frac{|f(n)|}{n^c n^{\sigma-c}},$$

as $n \geq N$ and $\sigma - c \geq 0$, we have that $n^{\sigma-c} \geq N^{\sigma-c}$, and thus

$$\sum_{n=N}^{\infty} \frac{|f(n)|}{n^c n^{\sigma-c}} \leq \sum_{n=N}^{\infty} N^{-(\sigma-c)} \frac{|f(n)|}{n^c}.$$

□

Theorem 3.4. *If $F(s) = \mathcal{D}(f;s)$ is the function associated to the Dirichlet series, for $s = \sigma + it$, then*

$$\lim_{\sigma\to\infty} F(\sigma + it) = f(1)$$

uniformly for $-\infty < t < +\infty$.

Proof. As we have $F(s) = f(1) + \sum_{n\geq 2} \frac{f(n)}{n^s}$, we only need to prove that the second term tends to 0 as $\sigma \to \infty$. Choose $c > \sigma_a$. Then for $\sigma > c$ the previous Lemma gives us

$$\left|\sum_{n\geq 2} \frac{f(n)}{n^s}\right| \leq 2^{-(\sigma-c)} \sum_{n\geq 2} \frac{|f(n)|}{n^c} = M\frac{1}{2^\sigma}$$

where $M = 2^c \sum_{n\geq 2} \frac{|f(n)|}{n^c}$ is independent of σ and t. As $\frac{1}{2^\sigma} \to 0$ as $\sigma \to \infty$, the theorem follows. □

Theorem 3.5 (Uniqueness)**.** *Given two Dirichlet series, absolutely convergent for some s, consider the associated functions $F(s) = \mathcal{D}(f;s)$ and $G(s) = \mathcal{D}(g;s)$ defined for s with $\sigma > \sigma_a$. If $F(s_k) = G(s_k)$ for all $k \in \mathbb{N}$ where $(s_k)_{k\in\mathbb{N}} = (\sigma_k + it_k)_{k\in\mathbb{N}}$ is a sequence such that $\sigma_k \to \infty$ as $k \to \infty$, then $f(n) = g(n)$ for all $n \in \mathbb{N}$.*

Proof. Let $h(n) = f(n) - g(n)$ and $H(s) = F(s) - G(s)$. So we get $H(s_k) = 0$ for all k. Let us suppose that $h(n) \neq 0$ for some n and let

$$N := \min\{n \colon h(n) \neq 0\}.$$

Then

$$H(s) = \sum_{n=N}^{\infty} \frac{h(n)}{n^s} = \frac{h(N)}{N^s} + \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}.$$

Rewriting in order to $h(N)$ we have

$$h(N) = N^s H(s) - N^s \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}.$$

Now, taking $s = s_k$, we have $H(s_k) = 0$ and therefore

$$h(N) = -N^{s_k} \sum_{n=N+1}^{\infty} \frac{h(n)}{n^{s_k}}.$$

Let $c > \sigma_a$. Let us choose k such that $\sigma_k > c$ and thus, by Lemma 3.3, we have

$$|h(N)| \leq N^{\sigma_k}(N+1)^{-(\sigma_k - c)} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^c} =$$

$$= \left(\frac{N}{N+1}\right)^{\sigma_k} \left((N+1)^c \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^c}\right).$$

As $k \to \infty$, $\left(\frac{N}{N+1}\right)^{\sigma_k} \to 0$ and the second part does not depend on k. So $|h(N)| = 0$, which is a contradiction. So we must have $h(n) = 0$ for all n. ☐

Theorem 3.6. *Let $F(s) = \mathcal{D}(f;s)$ be the function associated to the Dirichlet series of f and suppose $F(s) \neq 0$ for some s such that $\sigma > \sigma_a$. Then exists a half-plane $\sigma > c$, with $c \geq \sigma_a$, where $F(s) \neq 0$ always.*

Proof. Let us suppose such half-plane does not exist. Then, for each $k \in \mathbb{N}$ we can choose s_k with $\sigma_k > k$ such that $F(s_k) = 0$. Since $\sigma_k \to \infty$ as $k \to \infty$, by the Uniqueness Theorem we have $f(n) = 0$ for all n, contradicting the hypothesis that $F(s) \neq 0$ for some s. ☐

3.2 Convergence

After the study of absolute convergence, we will now consider simple and conditional convergences. We will show that, if the Dirichlet series convergences conditionally, the series does it in an infinite vertical strip whose horizontal length is bounded.

Proposition 3.7. *If $\mathcal{D}(f;s)$ converges for $s_0 = \sigma_0 + it_0$, then it converges for all $s = \sigma + it$ such that $\sigma > \sigma_0$. Also if $\mathcal{D}(f;s)$ diverges for $s_1 = \sigma_1 + it_1$, then it diverges for all $s = \sigma + it$ such that $\sigma < \sigma_1$.*

Proof. The second statement follows from the first one.

As $\mathcal{D}(f;s_0)$ converges, the partial sums are bounded and therefore by Lemma 3.3, if $\sigma > \sigma_0$, we have

$$\left| \sum_{a<n\leq b} \frac{f(n)}{n^s} \right| \leq K a^{\sigma_0 - \sigma}$$

where K is independent of a. So, making $a \to \infty$, we can see that the series is a Cauchy series (see Definition D.3) and therefore converges by Proposition D.4. □

Theorem 3.8. *Suppose that exists an $s_0 \in \mathbb{C}$ such that $\sum_{n\geq 1} \frac{f(n)}{n^s}$ diverges and an $s_1 \in \mathbb{C}$ such that it converges. Then there exists a real number σ_c, called the* ***abscissa of convergence****, such that the series $\mathcal{D}(f;s)$ converges if $\sigma > \sigma_c$ but diverges if $\sigma < \sigma_c$.*

Remark. By the last Proposition, we must have $\mathrm{Re}(s_0) \leq \mathrm{Re}(s_1)$.

Proof. As in Proposition 3.2, we take

$$\sigma_c = \sup\{\sigma \colon \mathcal{D}(f;s) \text{ diverges for some } s = \sigma + it\}.$$

By the hypothesis made, $\sigma_c \in \mathbb{R}$. As a consequence of the previous Proposition and the fact that σ_c is a supremum, the theorem follows immediately. □

As absolute convergence implies convergence, we have that

$$\sigma_a \geq \sigma_c.$$

If $\sigma_a > \sigma_c$, there is an infinite strip $\sigma_c < \sigma < \sigma_a$ in which the series converges conditionally.

Theorem 3.9. *For any Dirichlet series with σ_c finite, we have*

$$0 \leq \sigma_a - \sigma_c \leq 1.$$

If $\sigma_c = +\infty$, then $\sigma_a = +\infty$. If $\sigma_c = -\infty$, then $\sigma_a = -\infty$.

Proof. It suffices to show that if $\mathcal{D}(f;s)$ converges for $s_0 = \sigma_0 + it_0$, then it converges absolutely for $s = \sigma + it$ with $\sigma > \sigma_0 + 1$.

Let A be an upper bound to $\left|\frac{f(n)}{n^{s_0}}\right|$, that must exist because $\frac{f(n)}{n^{s_0}}$ converges. Then we have

$$\left|\frac{f(n)}{n^s}\right| = \left|\frac{f(n)}{n^{s_0}}\right| \cdot \left|\frac{1}{n^{s-s_0}}\right| \le \frac{A}{n^{\sigma-\sigma_0}}.$$

So $\sum_{n\ge 1}\left|\frac{f(n)}{n^s}\right|$ converges for $\sigma - \sigma_0 > 1$, by comparison with $\sum_{n\ge 1}\frac{1}{n^{\sigma-\sigma_0}}$. □

Lemma 3.10. *Suppose that for some $s_0 = \sigma_0 + it_0$, the partial sums of $\mathcal{D}(f;s_0)$ are bounded, say*

$$\left|\sum_{n\le x}\frac{f(n)}{n^{s_0}}\right| \le M,$$

for all $x \ge 1$. Then for all s such that $\sigma > \sigma_0$ we have

$$\left|\sum_{a<n\le b}\frac{f(n)}{n^s}\right| \le 2Ma^{\sigma_0-\sigma}\left(1 + \frac{|s-s_0|}{\sigma-\sigma_0}\right). \tag{3.2}$$

Proof. Let $a(n) := \frac{f(n)}{n^{s_0}}$ and $A(x) := \sum_{n\le x} a(n)$. Then we have

$$\frac{f(n)}{n^s} = \frac{a(n)}{n^{s-s_0}},$$

and using the Abel's summation formula (C.1) on $a(n)$ we get

$$\begin{aligned}\sum_{a<n\le b}\frac{f(n)}{n^s} &= \sum_{a<n\le b}\frac{a(n)}{n^{s-s_0}}\\ &= A(b)b^{s_0-s} - A(a)a^{s_0-s} + (s-s_0)\int_a^b A(t)t^{s_0-s-1}dt.\end{aligned}$$

As $|A(x)| \le M$ by hypothesis, we have, for $\sigma > \sigma_0$

$$\begin{aligned}\left|\sum_{a<n\le b}\frac{f(n)}{n^s}\right| &\le Mb^{\sigma_0-\sigma} + Ma^{\sigma_0-\sigma} + |s-s_0|M\int_a^b t^{\sigma_0-\sigma-1}dt\\ &\le 2Ma^{\sigma_0-\sigma} + |s-s_0|M\left|\frac{b^{\sigma_0-\sigma} - a^{\sigma_0-\sigma}}{\sigma_0-\sigma}\right|\\ &\le 2Ma^{\sigma_0-\sigma} + |s-s_0|M\frac{2a^{\sigma_0-\sigma}}{\sigma_0-\sigma}\\ &\le 2Ma^{\sigma_0-\sigma}\left(1 + \frac{|s-s_0|}{\sigma-\sigma_0}\right).\end{aligned}$$

□

Proposition 3.11. *Let f be an arithmetical function. Suppose that the partial sums $\sum_{n\leq x} f(n)$ are bounded. Then the series converges for all $s = \sigma + it$ such that $\sigma > 0$.*

Proof. Take $s_0 = 0$ and let b tend to ∞ in the previous Lemma. Then

$$\left| \sum_{n>a} \frac{f(n)}{n^s} \right| \leq K a^{-\sigma}$$

for some constant $K > 0$. Taking $a \to \infty$, we have that teh series is a Cauchy series (see Definition D.3) and thus $\mathcal{D}(f; s)$ converges by Proposition D.4. □

3.2.1 Examples

- The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ has $\sigma_c = 1$, as $|\mathbf{1}(n)| = 1$, we have $\sigma_a = \sigma_c$.

- The series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$
has $\sigma_a = 1$ but $\sigma_c = 0$. As the partial sums $\sum_{n\leq x} (-1)^n$ are bounded, so $\sigma_c \leq 0$, but as we have that $\sigma_a = 1$, by Theorem 3.9 we must have $\sigma_c = 0$.

- The Dirichlet L-function $L(s, \chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$ for a non-principal character, has $\sigma_c = 0$, by the same argument as in the previous example, since $\sigma_a = 1$ and the partial sums $\sum_{n\leq x} \chi(n)$ are bounded. This can be seen as follows. As χ is periodic modulo q, we have
$$\sum_{n\leq x} \chi(n) = \left\lfloor \frac{x}{q} \right\rfloor \sum_{n=1}^{q} \chi(n) + R$$
where $|R| \leq |\sum_{n=1}^{q} \chi(n)|$, and since
$$\left| \sum_{n=1}^{q} \chi(n) \right| \leq \sum_{n=1}^{q} |\chi(n)| \leq \phi(q)$$
we have $|R| \leq \phi(n)$.

  Since χ is not the principal character, there exists a number n_0 with $(n_0, q) = 1$ such that $\chi(n_0) \notin \{0, 1\}$. We can consider only the non-zero terms in the sum, in other words, only the terms such that, for $n \in [1, q]$, we have $(n, q) = 1$. Also, observe that, if n runs through these values, then so does $m = nn_0$, after reducing modulo q. Thus we have

$$\chi(n_0)\sum_{n=1}^{q}\chi(n) = \sum_{\substack{1\le n\le q\\(n,q)=1}}\chi(n_0)\chi(n) = \sum_{\substack{1\le n\le q\\(n,q)=1}}\chi(n_0n) = \sum_{\substack{1\le m\le q\\(m,q)=1}}\chi(m)$$

and therefore, adding back the zero terms of the sum, we have

$$\chi(n_0)\sum_{n=1}^{q}\chi(n) = \sum_{\substack{1\le m\le q\\(m,q)=1}}\chi(m) = \sum_{m=1}^{q}\chi(m)$$

$\sum_{n=1}^{q}\chi(n) = 0$. So we have

$$\left|\sum_{n\le x}\chi(n)\right| = |R| \le \phi(q).$$

It is clear, like in the case of the Zeta function, $\sigma_c = 1$ if χ is the principal character.

Theorem 3.12. *There exists a Dirichlet series $\mathcal{D}(f;s)$, with f a completely multiplicative arithmetic function, such that $\sigma_a - \sigma_c = \alpha$ for any $\alpha \in [0,1]$.*

Proof. [BH16]. We already saw the cases $\alpha = 0$ and $\alpha = 1$.

For $0 < \alpha < 1$, consider the geometric series

$$G_\alpha(s) = \left(1 - 3^{1-\alpha-s}\right)^{-1} = \sum_{k\ge 0}\frac{3^{(1-\alpha)k}}{3^{ks}}.$$

We can rewrite $G_\alpha(s)$ as a Dirichlet series using the change of variable $n = 3^k$ and a suitable arithmetic function

$$G_\alpha(s) = \sum_{k\ge 0}\frac{3^{(1-\alpha)k}}{3^{ks}} = \sum_{n\ge 1}\frac{g_\alpha(n)}{n^s}$$

where

$$g_\alpha(n) = \begin{cases} n^{(1-\alpha)}, & \text{if } \exists k \text{ s.t. } n = 3^k,\\ 0, & \text{otherwise,}\end{cases}$$

is completely multiplicative. Since a geometric series both converges simply and absolutely inside the disk $|r| < 1$, where $r = 3^{1-\alpha-s}$ is the ratio, i.e. converges for any $s = \sigma + it$ with $\sigma > 1 - \alpha$, this Dirichlet series has $\sigma_a = \sigma_c = 1 - \alpha$.

Let χ denote a non-principal character modulus 3, as defined above in p. 8 and $L(s,\chi)$ its Dirichlet series. Recall that by definition χ is completely multiplicative. We saw that $\sigma_a(L(s,\chi)) = 1$ in the subsection 3.1.1 and that $\sigma_c(L(s,\chi)) = 0$ in the subsection 3.2.1.

Consider the Dirichlet series given by the product

$$\begin{aligned} F(s) = G_\alpha(s)L(s,\chi) &= \sum_{n\geq 1} \frac{g_\alpha(n)}{n^s} \sum_{n\geq 1} \frac{\chi(n)}{n^s} \\ &= \sum_{n\geq 1} \frac{\sum_{ab=n} g_\alpha(a)\chi(b)}{n^s}. \end{aligned}$$

Let us now study the behaviour of $g_\alpha * \chi$. We know it's multiplicative, because both g_α and $g_\alpha \chi$ are. Since we can write any number n as $n = 3^k d$, where $3 \nmid d$ and k is the biggest exponent such that $3^k \mid n$, we have that $g_\alpha * \chi(n) = (g_\alpha * \chi)(3^k)(g_\alpha * \chi)(d)$. Let us consider both factors separately. We have

$$\begin{aligned} g_\alpha * \chi(3^k) &= g_\alpha(1)\chi(3^k) + g_\alpha(3)\chi(3^{k-1}) + \cdots + g_\alpha(3^{k-1})\chi(3) + g_\alpha(3^k)\chi(1) \\ &= 0 + \cdots + 0 + g_\alpha(3^k)\chi(1) = g_\alpha(3^k) \end{aligned}$$

because $\chi(3) = 0$, and

$$\begin{aligned} g_\alpha * \chi(d) &= g_\alpha(1)\chi(d) + \sum_{\substack{j>1 \\ j|d}} g_\alpha(j)\chi\left(\frac{d}{j}\right) \\ &= 1 \cdot \chi(d) + 0 = \chi(d), \end{aligned}$$

because $g_\alpha \neq 0$ only if it's a power of 3. Let $n, m \in \mathbb{N}$, write $n = 3^k d$ and $m = 3^l j$. Then we have

$$\begin{aligned} g_\alpha * \chi(nm) &= g_\alpha * \chi\left(3^{k+l} dj\right) = (g_\alpha * \chi)\left(3^{k+l}\right)(g_\alpha * \chi)(dj) \\ &= g_\alpha\left(3^{k+l}\right)\chi(dj) = g_\alpha\left(3^k\right) g_\alpha\left(3^l\right)\chi(d)\chi(j) \\ &= g_\alpha\left(3^k\right)\chi(d) g_\alpha\left(3^l\right)\chi(j) = g_\alpha * \chi\left(3^k d\right)\, g_\alpha * \chi\left(3^l j\right) \\ &= g_\alpha * \chi(n)\, g_\alpha * \chi(m), \end{aligned}$$

that is, $g_\alpha * \chi$ is completely multiplicative.

Finally, the product of two series is absolutely convergent where both series are, so we have $\sigma_a(F) \leq 1$. But as $L(1, |\chi|)$ does not converge and $g_\alpha(1) > 0$, so F also does not and therefore $\sigma_a(F) = 1$. The product of two series converge where both series do, so we have $\sigma_c(F) \leq 1 - \alpha$, but as $G_\alpha(1-\alpha)$ diverges as an infinity sum of 1, F will also have an infinity number of coefficients with modulus 1, so we must have $\sigma_c(F) = 1 - \alpha$. □

3.3 Uniform Convergence

We say a series converges uniformly if the sequence of its partial sums converges uniformly. It is a stronger type of convergence than the one consider in the previous section, and this fact will have some consequences about the function defined by the series, namely that said function is analytic.

Let us define, as illustrated in figure 3.1,

$$\mathbf{D}(b,\delta,\varepsilon) = \{s \in \mathbb{C} : \mathrm{Re}(s) \geq b+\delta, |\arg(s-b)| \leq \frac{\pi}{2} - \varepsilon\}.$$

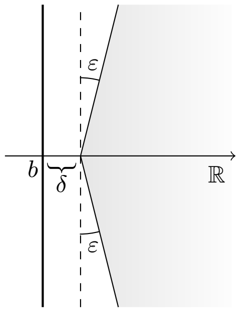

Figure 3.1: $\mathbf{D}(b,\delta,\varepsilon)$

Theorem 3.13. *Let $F(s) = \mathcal{D}(f;s)$ be the function associated to the Dirichlet series of f and $S(x) = \sum_{n\leq x} f(n)$. Suppose there exist constants $a, b > 0$ such that $|S(x)| \leq ax^b$ for any $x \geq r$, for some $r > 0$. Then the series $\mathcal{D}(f;s)$ is uniformly convergent in $\mathbf{D}(b,\delta,\varepsilon)$ for any positive δ, ε.*

In particular, $F(s)$ is analytic in the half plane $\mathrm{Re}(s) > b$.

Proof. [Jan96, Ch. IV.2] Let $s = \sigma + it$ be in $\mathbf{D}(b,\delta,\varepsilon)$. We have that $f(n) = S(n) - S(n-1)$, so we can write

$$\begin{aligned}\left|\sum_{n=u}^{v} \frac{f(n)}{n^s}\right| &= \left|\sum_{n=u}^{v} \frac{S(n)}{n^s} - \sum_{n=u-1}^{v-1} \frac{S(n)}{(n+1)^s}\right| \\ &= \left|\frac{S(v)}{v^s} - \frac{S(u-1)}{u^s} + \sum_{n=u}^{v-1} S(n)\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right)\right| \\ &\leq \left|\frac{S(v)}{v^s}\right| + \left|\frac{S(u-1)}{u^s}\right| + \sum_{n=u}^{v-1} |S(n)| \left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right|.\end{aligned}$$

As $|n^s| = n^\sigma$ and $|S(x)| \leq ax^b$, we can write, by Abel Summation Formula C.1:

$$\begin{aligned}\left|\sum_{n=u}^{v}\frac{f(n)}{n^s}\right| &\le \frac{av^b}{v^\sigma}+\frac{au^b}{u^\sigma}+\sum_{n=u}^{v-1}an^b\left|\frac{1}{n^s}-\frac{1}{(n+1)^s}\right| \\ &\le \frac{2a}{u^{\sigma-b}}+a\sum_{n=u}^{v-1}\left|\frac{n^b}{n^s}-\frac{n^b}{(n+1)^s}\right|. \qquad (u\le v).\end{aligned}$$

We have that

$$\frac{n^b}{n^s}-\frac{n^b}{(n+1)^s}\le |s|n^b\left|\int_n^{n+1}\frac{dt}{t^{s+1}}\right|\le |s|n^b\int_n^{n+1}\frac{dt}{t^{\sigma+1}}.$$

Thus, replacing it on the previous equality, we have

$$\begin{aligned}\left|\sum_{n=u}^{v}\frac{f(n)}{n^s}\right| &\le \frac{2a}{u^{\sigma-b}}+a\sum_{n=u}^{v-1}|s|n^b\int_n^{n+1}\frac{dt}{t^{\sigma+1}} \\ &\le \frac{2a}{u^{\sigma-b}}+a|s|\int_u^{\infty}\frac{dt}{t^{\sigma-b+1}} \\ &\le \frac{2a}{u^{\sigma-b}}+\frac{a|s|}{(\sigma-b)u^{\sigma-b}}.\end{aligned}$$

Defining $\theta := \arg(s-b)$, we have that $\cos\theta = \frac{\sigma-b}{|s-b|}$, and thus

$$\frac{|s|}{\sigma-b}\le\frac{|s-b|+b}{\sigma-b}\le\frac{|s-b|}{\sigma-b}+\frac{b}{\sigma-b}\le\frac{1}{\cos\theta}+\frac{b}{\sigma-b}.$$

Because $s\in\mathbf{D}(b,\delta,\varepsilon)$, we have $\sigma-b\ge\delta$ and $\theta\le\frac{\pi}{2}-\varepsilon$, which means there exists an $M>0$ such that $\frac{1}{\cos\theta}\le M$. So, for any $\varepsilon_0>0$, there exists N such that $\forall u\ge N$ and we have

$$\left|\sum_{n=u}^{v}\frac{f(n)}{n^s}\right|\le\frac{2a}{u^{\sigma-b}}+\frac{a|s|}{(\sigma-b)u^{\sigma-b}}\le\frac{2a+M+\frac{b}{\delta}}{u^{\sigma-b}}\le\varepsilon_0.$$

Hence the Cauchy criterion holds, and thus the series converges uniformly.

For any $s=\sigma+it$ such that $\sigma>b$, there exists δ and ε such that $s\in\mathcal{D}(b,\delta,\varepsilon)$, and we have $F(s)=\sum_{n\ge1}\frac{f(n)}{n^s}$, a uniformly convergent series, therefore F is analytic in $\sigma>b$. □

Definition 3.14. *If $\mathcal{D}(f;s)$ converges for some s, there exists a real number σ_u, called the* ***abscissa of uniform convergence****, such that the series $\mathcal{D}(f;s)$ uniformly converges if $\sigma>\sigma_u$ but not if $\sigma<\sigma_u$.*

It is easy to see that

$$\sigma_c\le\sigma_u\le\sigma_a.$$

Remark. The previous theorem implies that σ_u is the minimum of all possible b.

Theorem 3.15. *For any Dirichlet series, we have* $\sigma_a - \sigma_u \leq \frac{1}{2}$.

Proof. It suffices to prove that if a Dirichlet series $\mathcal{D}(f;s)$ converges uniformly for $s = \sigma_0 + it$, it converges absolutely for $s = \sigma_0 + \frac{1}{2} + \varepsilon$, for any $\varepsilon > 0$.

By the Cauchy-Schwarz inequality, we have

$$\sum_{n\geq 1} \frac{|f(n)|}{n^{\sigma_0+\frac{1}{2}+\varepsilon}} \leq \left(\sum_{n\geq 1} \frac{|f(n)|^2}{n^{2\sigma_0}}\right)^{\frac{1}{2}} \left(\sum_{n\geq 1} \frac{1}{n^{1+2\varepsilon}}\right)^{\frac{1}{2}} \tag{3.3}$$

and as we know that the second term of the product converges, we only need to show that the first also does.

Note that each partial sum $\sum_{1\leq n\leq N} \frac{f(n)}{n^s}$ is bounded at $\sigma = \sigma_0$, since uniform convergence implies convergence. More so, by hypothesis, the partial sums must be uniformly bounded on the line $\sigma = \sigma_0$, say by a constant M. Then we have, for any $t \in \mathbb{R}$

$$\begin{aligned} M^2 &\geq \left| \sum_{1\leq n\leq N} \frac{f(n)}{n^{\sigma_0+it}} \right|^2 \\ &= \sum_{1\leq n\leq N} \frac{|f(n)|^2}{n^{2\sigma_0}} + \sum_{n\neq m} \operatorname{Re}\left(\frac{f(n)}{n^{\sigma_0+it}} \overline{\left(\frac{f(m)}{m^{\sigma_0+it}}\right)} \right) \\ &= \sum_{1\leq n\leq N} \frac{|f(n)|^2}{n^{2\sigma_0}} + 2\operatorname{Re}\left(\sum_{1\leq n<m\leq N} \frac{f(n)}{n^{\sigma_0+it}} \frac{\overline{f(m)}}{m^{\sigma_0-it}} \right) \\ &= \sum_{1\leq n\leq N} \frac{|f(n)|^2}{n^{2\sigma_0}} + 2\operatorname{Re}\left(\sum_{1\leq n<m\leq N} \frac{f(n)\overline{f(m)}}{(nm)^{\sigma_0}\left(\frac{n}{m}\right)^{it}} \right). \end{aligned}$$

We can take the average value of $\frac{f(n)\overline{f(m)}}{(nm)^{\sigma_0}\left(\frac{n}{m}\right)^{it}}$ with respect to t by integrating from $-T$ to T and dividing by $2T$:

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} \frac{f(n)\overline{f(m)}}{(nm)^{\sigma_0}\left(\frac{n}{m}\right)^{it}}\,dt &= \frac{f(n)\overline{f(m)}}{(nm)^{\sigma_0}2T}\int_{-T}^{T} \left(\frac{m}{n}\right)^{it} dt \\ &= \frac{f(n)\overline{f(m)}}{(nm)^{\sigma_0}2T} \frac{2\sin\left(T\log\left(\frac{m}{n}\right)\right)}{\log\left(\frac{m}{n}\right)} \\ &= \frac{f(n)\overline{f(m)}\sin\left(T\log\left(\frac{m}{n}\right)\right)}{(nm)^{\sigma_0}T\log\left(\frac{m}{n}\right)}. \end{aligned}$$

So we have that, taking $T \to \infty$

$$\begin{aligned} M^2 &\geq \sum_{1\leq n\leq N} \frac{|f(n)|^2}{n^{2\sigma_0}} + 2\operatorname{Re}\left(\sum_{1\leq n<m\leq N} \frac{f(n)\overline{f(m)}\sin\left(T\log\left(\frac{m}{n}\right)\right)}{(nm)^{\sigma_0} T\log\left(\frac{m}{n}\right)} \right) \\ &\geq \sum_{1\leq n\leq N} \frac{|f(n)|^2}{n^{2\sigma_0}}. \end{aligned}$$

Now, as N is arbitrary, we have that $\sum_{1\leq n\leq N} \frac{|f(n)|^2}{n^{2\sigma_0}}$ converges and thus, by lemma 3.3, so does $\sum_{n\geq 1} \frac{|f(n)|}{n^{\sigma_0+\frac{1}{2}+\varepsilon}}$. □

Remark. The inequality $\sigma_a - \sigma_u \leq \frac{1}{2}$ is the best we can do, as it is possible to built an arithmetical function d such that its Dirichlet Series has $\sigma_u = \frac{1}{2}$ and $\sigma_a = 1$. The details of this construction and the proof that it has the properties desired is very technical and by that reason is omitted from this work. The full proof can be found on [Boa97].

Theorem 3.16. *There exists a Dirichlet series $\mathcal{D}(f;s)$, such that $\sigma_a - \sigma_u = \alpha$ for any $\alpha \in \left[0, \frac{1}{2}\right]$.*

Proof. Fix $\alpha \in \left[0, \frac{1}{2}\right]$. Consider the arithmetical function mention on the remark above and denote its Dirichlet series by $F(s) := \mathcal{D}(f;s)$. By construction it has $\sigma_u = \frac{1}{2}$ and $\sigma_a = 1$. Consider now the function $\zeta(s+\alpha)$, that has $\sigma_a = \sigma_u = 1-\alpha$. Then the Dirichlet series $F(s) + \zeta(s+\alpha)$ has $\sigma_a = 1$ but $\sigma_u = 1-\alpha$. So $\sigma_a - \sigma_u = \alpha$. □

Theorem 3.17. *Let $\mathcal{D}(f;s)$ be the Dirichlet series of f, where f is multiplicative. Then $\sigma_a = \sigma_u$.*

The proof can be seen in [BH16, pp. 450–452].

Chapter 4

Riemann's Hypothesis

We will first give out the necessary theory about the Riemann zeta function to then prove the main theorem that says the Riemann Hypothesis is equivalent to the growth of $M(x) := \sum_{n \leq x} \mu(n)$ and $L(x) := \sum_{n \leq x} \lambda(n)$ being asymptotically bounded by $x^{\frac{1}{2}+\varepsilon}$, for all $\varepsilon > 0$.

Recall that, we say $f(x) = \mathcal{O}(g(x))$ if there exists a positive real number M such that for any x

$$|f(x)| \leq Mg(x).$$

4.1 Riemann's Zeta Function

The following function was first introduced and studied over the real numbers by Leonhard Euler in the first half of the eighteenth century, but it gained more importance when Bernhard Riemann studied it over the complex plane in 1859.

Definition 4.1. *For any $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$ one defines*

$$\zeta(s) = \mathcal{D}(\mathbf{1}; s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

*This function can be extended to all complex values, except $s = 1$, via analytic continuation. The extended function is called the **Riemann zeta function**, while still denoted by ζ.*

Remark. It is well known, [Apo76, Theorem 11.7], that one has

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

This identity is known as Euler's product formula and it is an alternative way to express the zeta function, that shows a direct relation between the function and prime numbers.

Hypothesis (Riemann's Hypothesis)**.** *The Riemann's zeta function has its zeros only at the negative even integers (called the trivial zeros) and complex numbers with real part* $\frac{1}{2}$*.*

Definition 4.2. *For any* $s = \sigma + it \in \mathbb{C}$ *with* $\sigma > 1$ *and for* $0 < a < 1$ *a fixed real number, one defines*

$$\zeta(s,a) = \sum_{n \geq 0} \frac{1}{(n+a)^s}.$$

Once again, it is possible, via analytic continuation, to extended this function to all complex values with exception of $s = 1$*. This extended function is called* ***Hurwitz zeta function****, named after Adolf Hurwitz, who introduced it in 1882. Note that* $\zeta(s,1) = \zeta(s)$*.*

Theorem 4.3. *For any integer* $N \geq 0$ *and* $s = \sigma + it$ *such that* $\sigma > 0$*, we have*

$$\zeta(s,a) = \sum_{n=0}^{N} \frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1} - s \int_N^\infty \frac{x - \lfloor x \rfloor}{(x+a)^{s+1}}\, dx.$$

Proof. [Apo76]. Let us apply Euler's Summation formula, Proposition C.2, to $f(t) = (t+a)^{-s}$ and integers x and y.

$$\begin{aligned}\sum_{y<n\leq x} \frac{1}{(n+a)^s} &= \int_y^x \frac{1}{(t+a)^s}\, dt - s \int_y^x \frac{t - \lfloor t \rfloor}{(t+a)^{s+1}}\, dt + \frac{\lfloor x \rfloor - x}{(x+a)^s} - \frac{\lfloor y \rfloor - y}{(y+a)^s} \\ &= \int_y^x \frac{1}{(t+a)^s}\, dt - s \int_y^x \frac{t - \lfloor t \rfloor}{(t+a)^{s+1}}\, dt.\end{aligned}$$

Taking $y = N$ and letting $x \to \infty$, $\sigma > 1$, we obtain

$$\sum_{n=N+1}^{\infty} \frac{1}{(n+a)^s} = \int_N^\infty \frac{1}{(t+a)^s}\, dt - s \int_N^\infty \frac{t - \lfloor t \rfloor}{(t+a)^{s+1}}\, dt,$$

and therefore

$$\zeta(s,a) - \sum_{n=0}^{N} \frac{1}{(n+a)^s} = \frac{(N+a)^{1-s}}{s-1} - s \int_N^\infty \frac{t - \lfloor t \rfloor}{(t+a)^{s+1}}\, dt,$$

which proves the Theorem for $\sigma > 1$. If $\sigma \geq \delta > 0$, the integral $\int_N^\infty \frac{t-\lfloor t \rfloor}{(t+a)^{s+1}}\, dt$ is dominated by $\int_N^\infty \frac{1}{(t+a)^{s+1}}\, dt$, so it converges uniformly for $\sigma \geq \delta$, and hence represents an analytic function in the half-plan $\sigma > 0$. Thus, by analytic continuation, the Theorem holds for $\sigma > 0$. □

Proposition 4.4. *We have, for $\sigma > 1$,*

$$\log \zeta(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s \log n},$$

where Λ is the Van Mangoldt funtion.

Proof. Recall that

$$\Lambda(n) = \begin{cases} \log(p), & \text{if } n = p^k \text{ for some } p \text{ prime and } k \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

and that $\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$.

Then $\log \zeta(s) = -\sum_p \log(1 - p^{-s})$. Differentiating this expression, term by term, we get

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= -\sum_p \frac{\log(p) p^{-s}}{1 - p^{-s}} \\ &= \sum_p \log p \left(1 - \frac{1}{1 - p^s}\right) \\ &= \sum_p \log p \sum_{k \geq 1} p^{-ks} \\ &= \sum_p \sum_{k \geq 1} \log p (p^k)^{-s} \\ &= \sum_{n \geq 1} \Lambda(n) n^{-s}. \end{aligned}$$

Now, anti-differentiating this expression we get

$$\log \zeta(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s \log n}.$$

□

4.2 The Gamma Function

The Gamma function is an extension of the factorial function to the complex plane, except for the non-positive integers, where the function has simple poles. We will study the connection it has to the Riemann's Zeta function.

Definition 4.5. *For any $s = \sigma + it$ with $\sigma > 0$, one defines*

$$\Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}\, dx. \tag{4.1}$$

One can extend this integral function to all complex plane except zero and the negative integers by analytic continuation, to which one calls the ***gamma function****.*

Remark. The following expressions are alternative ways to write Γ (4.1), both of which extend it to $\mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$

$$\Gamma(s) = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \dots n}{s(s+1)(s+2)\dots(s+n)}(n+1)^s \tag{4.2}$$

$$\Gamma(s) = \frac{1}{s}\prod_{n\geq 1}\left(1+\frac{s}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^{s}.$$

The proof of the first one can be found in [Art64, Equation (2.7)], and the second can be obtained from the first.

Proposition 4.6. *We have the following identities:*

$$\Gamma(s+1) = s\Gamma(s), \tag{4.3}$$

$$n! = \Gamma(n+1), n \in \mathbb{N}, \tag{4.4}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, s \notin \mathbb{Z}, \tag{4.5}$$

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s). \tag{4.6}$$

Proof. The first identity comes from integration by parts of the integral expression:

$$\begin{aligned}\Gamma(s+1) &= \int_0^{+\infty} e^{-x}x^s\,dx \\ &= \left[-e^{-x}x^s\right]_0^\infty - \int_0^{+\infty} -e^{-x}sx^{s-1}\,dx = \\ &= s\int_0^{+\infty} e^{-x}x^{s-1}\,dx = s\Gamma(s).\end{aligned}$$

The second identity results from the first by induction, and the fact that $\Gamma(1) = 1$.

The third identity is known as Euler's reflection formula and it uses the product formula, for the sine function

$$\sin(\pi s) = \pi s \prod_{n\geq 1}\left(1-\frac{s^2}{n^2}\right),$$

and the limit expression of Γ (4.2) and taking $s \notin \mathbb{Z}$:

$$
\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \lim_{n\to+\infty} \frac{n!(n+1)^s}{s(s+1)(s+2)(s+3)\dots(s+n)} \\
&\qquad \frac{n!(n+1)^{1-s}}{(1-s)(-s)(1-s)(2-s)\dots(n+1-s)} \\
&= \lim_{n\to+\infty} \frac{(n!)^2(n+1)}{s(s+1)(1-s)\dots(s+n)(n-s)(n+1-s)} \\
&= \lim_{n\to+\infty} \frac{(n!)^2(n+1)}{s(n+1-s)\prod_{1\le i\le n}(i^2-s^2)} \\
&= \lim_{n\to+\infty} \frac{(n!)^2(n+1)}{s(n+1-s)\prod_{1\le i\le n} i^2(1-\frac{s^2}{i^2})} \\
&= \lim_{n\to+\infty} \frac{(n!)^2(n+1)}{s(n+1-s)(n!)^2\prod_{1\le i\le n}(1-\frac{s^2}{i^2})} \\
&= \lim_{n\to+\infty} \frac{n+1}{(n+1-s)} \frac{1}{s\prod_{1\le i\le n}(1-\frac{s^2}{i^2})} \\
&= \lim_{n\to+\infty} \frac{1}{s\prod_{1\le i\le n}(1-\frac{s^2}{i^2})} \\
&= \frac{\pi}{\sin(\pi s)}.
\end{aligned}
$$

Note that we have to assume $s \notin \mathbb{Z}$ so the left side has meaning.

The fourth identity is known as Legendre duplication formula and it comes from the beta function, which is also called the Euler integral of the first kind,

$$
B(s_1, s_2) := \int_0^1 t^{s_1-1}(1-t)^{s_2-1}\, dt = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)},
$$

the function is defined as the integral, but it is possible to show the second equality. More can be found on [Art64, Ch. 2, Equation (2.12)].

Let us evaluate the beta function at (s, s)

$$
B(s,s) = \frac{\Gamma(s)^2}{\Gamma(2s)} = \int_0^1 t^{s-1}(1-t)^{s-1}\, dt.
$$

Substituting $t = \frac{1+x}{2}$ we get

$$
\frac{\Gamma(s)^2}{\Gamma(2s)} = \frac{1}{2^{2s-1}} \int_{-1}^1 (1-x^2)^{s-1}\, dx.
$$

As the integrand is even, we get

$$
\frac{\Gamma(s)^2}{\Gamma(2s)} = \frac{2}{2^{2s-1}} \int_0^1 (1-x^2)^{s-1}\, dx.
$$

Now, let us evaluate beta at $\left(\frac{1}{2}, s\right)$

$$B\left(\frac{1}{2}, s\right) = \frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)} = \int_0^1 t^{-\frac{1}{2}}(1-t)^{s-1}\,dt.$$

Substituting $t = x^2$ we get

$$\begin{aligned}\frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)} &= \int_0^1 x^{-1}(1-x^2)^{s-1}2x\,dx \\ &= 2\int_0^1 (1-x^2)^{s-1}\,dx.\end{aligned}$$

This implies that

$$\frac{2^{2s-1}\Gamma(s)^2}{\Gamma(2s)} = \frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma\left(s+\frac{1}{2}\right)}.$$

Rearranging the terms we have

$$\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\Gamma(2s)\Gamma\left(\frac{1}{2}\right).$$

Now, evaluating equation (4.5) at $s = \frac{1}{2}$, we get $\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}}$. Therefore, we have

$$\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s).$$

□

Using the Gamma function, one can get an alternative expression for the zeta function.

Proposition 4.7. *For all $s = \sigma + it$ with $\sigma > 1$, we have*

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{x^{s-1}}{e^x-1}\,dx. \tag{4.7}$$

Proof. We have, by substituting x with nx in (4.1) and using the sum $\sum_{n\geq 1} r^{-n} = (r-1)^{-1}$:

$$\frac{\Gamma(s)}{n^s} = \int_0^{+\infty} e^{-nx}x^{s-1}\,dx.$$

Summing, we obtain

$$\begin{aligned}\Gamma(s)\sum_{n\geq 1}\frac{1}{n^s} &= \sum_{n\geq 1}\int_0^{+\infty} e^{-nx}x^{s-1}\,dx \\ &= \int_0^{+\infty}\left(\sum_{n\geq 1} e^{-nx}\right)x^{s-1}\,dx = \int_0^{+\infty}\frac{x^{s-1}}{e^x-1}\,dx.\end{aligned}$$

We can change the order of the sum and integral because, for $\sigma > 1$, the series converges absolutely. □

4.2.1 Stirling's Formula

Because of the connection between the Gamma and Zeta functions, we will need to study the former in more detail to draw some conclusions about the growth of the latter. In particular, we will need the Stirling's Formula in order to give some approximations to the Gamma function.

Proposition 4.8 (Stirling's Formula)**.** *For $-\pi+\delta \leq \arg s \leq \pi-\delta$, we have*

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + \frac{B_2}{2s} + \mathcal{O}\left(\frac{1}{s^3}\right)$$

where B_2 is the Bernoulli number (definition C.6) of order 2.

Proof. For N integer, let $S_N(s) = \sum_{n=0}^{N}\log(s+n)$. Using the Euler-Maclaurin Summation Formula C.7, with $v=0$, on the function $f(x) = \log(s+x)$, where $s>0$ is a real number, we get

$$\begin{aligned} S_N(s) &= \sum_{n=0}^{N}\log(s+n) \\ &= \int_0^N \log(s+x)\,dx + \frac{1}{2}(\log(s)+\log(s+N)) + \int_0^N \frac{\overline{B_1}(x)}{x+s}\,dx. \end{aligned}$$

Note that $\int\log(s+x)\,dx = (s+x)\log(s+x)-x$.

So we can then write

$$\begin{aligned} S_N(s) &= ((s+x)\log(s+x)-x)\Big|_0^N + \frac{1}{2}(\log(s)+\log(s+N)) \\ &\qquad + \int_0^\infty \frac{\overline{B_1}(x)}{s+x}\,dx - \int_N^\infty \frac{\overline{B_1}(x)}{s+x}\,dx \\ &= \left(N+s+\frac{1}{2}\right)\log(s+N) - N - \left(s-\frac{1}{2}\right)\log s + \int_0^\infty \frac{\overline{B_1}(x)}{s+x}\,dx - \\ &\qquad - \int_N^\infty \frac{\overline{B_1}(x)}{s+x}\,dx. \end{aligned}$$

Now, from the limit expression of the Gamma function, (4.2), we can see that

$$\begin{aligned} \log\Gamma(s) &= \lim_{N\to\infty}\left[S_{N-1}(1) - S_N(s) + s\log(N+1)\right] \\ &= \lim_{N\to\infty}\Bigg[\left(N+\frac{1}{2}\right)\log(N) + \int_0^\infty \frac{\overline{B_1}(x)}{1+x}\,dx - \int_{N-1}^\infty \frac{\overline{B_1}(x)}{1+x}\,dx \\ &\qquad - N + 1 - \left(N+s+\frac{1}{2}\right)\log(s+N) + N + \left(s-\frac{1}{2}\right)\log s \\ &\qquad - \int_0^\infty \frac{\overline{B_1}(x)}{s+x}\,dx + \int_N^\infty \frac{\overline{B_1}(x)}{s+x}\,dx + s\log(N+1)\Bigg] \end{aligned}$$

Now, we have that

$$\begin{aligned}\int_{N-1}^{\infty} \frac{\overline{B_1}(x)}{1+x}\,dx + \int_{N}^{\infty} \frac{\overline{B_1}(x)}{s+x}\,dx &= \int_{N}^{\infty} \frac{\overline{B_1}(y)}{y}\,dy + \int_{N}^{\infty} \frac{\overline{B_1}(x)}{s+x}\,dx \\ &= \int_{N}^{\infty} \frac{\overline{B_1}(x)}{x} + \frac{\overline{B_1}(x)}{s+x}\,dx \\ &= \int_{N}^{\infty} \frac{(s+x)\overline{B_1}(x)}{x} + \frac{(x)\overline{B_1}(x)}{s+x}\,dx \\ &= \int_{N}^{\infty} \frac{(s+2x)\overline{B_1}(x)}{x(s+x)}\,dx\end{aligned}$$

and as $x \to \infty$, the integrand tends to 0. Thus if we take $N \to \infty$, we have that this integral is equal to 0.

$$\begin{aligned}\log\Gamma(s) = &\left(s-\frac{1}{2}\right)\log s + 1 + \int_0^{\infty} \frac{\overline{B_1}(x)}{1+x}\,dx - \int_0^{\infty} \frac{\overline{B_1}(x)}{s+x}\,dx + \\ &+ \lim_{N\to\infty}\left[\left(N+\frac{1}{2}\right)\log(N) + s\log(N+1) - \right. \\ &\left. - \left(N+s+\frac{1}{2}\right)\log(s+N)\right] \\ = &\left(s-\frac{1}{2}\right)\log s + 1 + \int_1^{\infty} \frac{\overline{B_1}(y-1)}{y}\,dy - \int_0^{\infty} \frac{\overline{B_1}(x)}{s+x}\,dx + \\ &+ \lim_{N\to\infty}\left[\left(N+\frac{1}{2}\right)\log\left(\frac{N}{N+s}\right) + s\log\left(\frac{N+1}{N+s}\right)\right]\end{aligned}$$

Now, as $\lim_{N\to\infty}\left(N+\frac{1}{2}\right)\log\left(\frac{N}{N+s}\right) = -s$ and $\lim_{N\to\infty} s\log\left(\frac{N+1}{N+s}\right) = 0$, we have

$$\begin{aligned}\log\Gamma(s) &= \left(s-\frac{1}{2}\right)\log s + 1 + \int_1^{\infty} \frac{\overline{B_1}(y)}{y}\,dy - \int_0^{\infty} \frac{\overline{B_1}(x)}{s+x}\,dx - s \\ &= \left(s-\frac{1}{2}\right)\log s - s + A - \int_0^{\infty} \frac{\overline{B_1}(x)}{s+x}\,dx\end{aligned}$$

where $A = 1 + \int_1^{\infty} \frac{\overline{B_1}(y)}{y}\,dy$.

We can now write it as

$$\Gamma(s) = s^{s-\frac{1}{2}} e^{-s} e^{A} r(s)$$

where $r(s) = \exp\left(-\int_0^{\infty} \frac{\overline{B_1}(x)}{s+x}\,dx\right)$.

Now, substituting this expression on the identity (4.6) we have

$$s^{s-\frac{1}{2}}e^{-s}e^{A}r(s)\left(s+\frac{1}{2}\right)^{s}e^{-s-\frac{1}{2}}e^{A}r\left(s+\frac{1}{2}\right)=$$
$$=2^{1-2s}\pi^{\frac{1}{2}}(2s)^{2s-\frac{1}{2}}e^{-2s}e^{A}r(2s).$$

Cancelling equal terms on both sides, we get

$$e^{A}r(s)\left(\frac{2s+1}{2}\right)^{s}e^{-\frac{1}{2}}r\left(s+\frac{1}{2}\right)=(2\pi)^{\frac{1}{2}}s^{s}r(2s).$$

Writing in order to e^A, we have

$$e^{A}=(2\pi)^{\frac{1}{2}}\frac{s^{s}}{\frac{(2s+1)^{s}}{2^{s}}}e^{\frac{1}{2}}\frac{r(2s)}{r(s)r\left(s+\frac{1}{2}\right)}.$$

Simplifying the fraction, we have

$$e^{A}=(2\pi)^{\frac{1}{2}}\left(1+\frac{1}{2s}\right)^{-s}e^{\frac{1}{2}}\frac{r(2s)}{r(s)r\left(s+\frac{1}{2}\right)}.$$

Now, taking $s\to\infty$, we get $\left(1+\frac{1}{2s}\right)^{-s}=e^{-\frac{1}{2}}$ and

$$\lim_{s\to\infty}\frac{r(2s)}{r(s)r\left(s+\frac{1}{2}\right)}=$$
$$=\lim_{s\to\infty}\exp\left(-\int_{0}^{\infty}\frac{\overline{B_1}(x)}{2s+x}\,dx+\int_{0}^{\infty}\frac{\overline{B_1}(x)}{s+x}\,dx+\int_{0}^{\infty}\frac{\overline{B_1}(x)}{s+\frac{1}{2}+x}\,dx\right)$$
$$=\exp\left(\lim_{s\to\infty}\int_{0}^{\infty}\frac{\overline{B_1}(x)}{s+x}\,dx+\int_{0}^{\infty}\frac{\overline{B_1}(x)}{s+\frac{1}{2}+x}\,dx-\int_{0}^{\infty}\frac{\overline{B_1}(x)}{2s+x}\,dx\right)$$
$$=e^{0}.$$

Therefore,
$$e^{A}=(2\pi)^{\frac{1}{2}}e^{-\frac{1}{2}}e^{\frac{1}{2}}=(2\pi)^{\frac{1}{2}}$$

and hence
$$A=\frac{1}{2}\log(2\pi).$$

Now we have, for all $s>0$,

$$\log\Gamma(s)=\left(s-\frac{1}{2}\right)\log s-s+\frac{1}{2}\log(2\pi)-\int_{0}^{\infty}\frac{\overline{B_1}(x)}{s+x}\,dx. \tag{4.8}$$

All terms of this expression are defined throughout the complex plane except for the real negatives numbers and 0, which we are going to call the slit plane, and are analytic functions of s. This is obvious for all terms, except the integral, but we have already shown that this integral is convergent for $s > 0$ and the same argument works for all s in the slit plane. So, by analytic continuation, the above expression is valid to all complex s except the non-positive real numbers.

Now, if we integrate by parts that integral, we have

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log (2\pi) + \frac{B_2}{2s} + \frac{B_4}{4 \cdot 3 \cdot s^3} + \cdots + \\ + \cdots + \frac{B_{2v}}{2v(2v-1)s^{2v-1}} + R_{2v}$$

where $R_{2v} = - \int_0^\infty \frac{\overline{B_{2v}}(x)}{2v(s+x)^{2v}}\, dx$, or alternatively

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log (2\pi) + \frac{B_2}{2s} + \mathcal{O}\left(\frac{1}{s^3}\right).$$

□

Corollary 4.9. *For σ bounded and $|t| \to \infty$, we have*

$$|\Gamma(s)| \sim \sqrt{2\pi}|t|^{\sigma - \frac{1}{2}} e^{-\frac{|t|\pi}{2}}.$$

Proof. We only need to show this for t positive, as $\Gamma(\sigma - it) = \overline{\Gamma(\sigma + it)}$, and therefore $|\Gamma(\sigma - it)| = |\Gamma(\sigma + it)|$. Now,

$$\begin{aligned} \log |\Gamma(s)| &= \operatorname{Re} (\log \Gamma(s)) \\ &= \operatorname{Re} \left(\left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log (2\pi) + \frac{B_2}{2s} + \mathcal{O}\left(\frac{1}{s^3}\right)\right) \\ &= \left(\sigma - \frac{1}{2}\right) \log |s| + t \operatorname{Im}(\log s) - \sigma + \frac{1}{2} \log 2\pi + \mathcal{O}\left(t^{-1}\right) \\ &= \left(\sigma - \frac{1}{2}\right) \log |s| - t \arg s - \sigma + \frac{1}{2} \log 2\pi + \mathcal{O}\left(t^{-1}\right). \end{aligned}$$

As $t \to \infty$, we have $\arg s = \frac{\pi}{2} - \delta$, with $\delta \to 0$. More so, $|s| = |t| + \mathcal{O}(1)$, thus $\log |s| = \log |t|$. Also $\sigma = t \tan(\delta)$ and, as $\delta \to 0$ when $t \to \infty$, we have $\tan(\delta) = \delta + \mathcal{O}(\delta^3)$, so $\sigma = t\delta + \mathcal{O}(t\delta^3) = t\delta + \mathcal{O}(\frac{1}{t^2})$.

Adding all of these estimates together, we get

$$\begin{aligned} \log |\Gamma(s)| &= \left(\sigma - \frac{1}{2}\right) \log |s| - t \arg s - \sigma + \frac{1}{2} \log 2\pi + \mathcal{O}\left(t^{-1}\right) \\ &= \left(\sigma - \frac{1}{2}\right) \log |t| - t\left(\frac{\pi}{2} - \delta\right) - t\delta + \mathcal{O}\left(\frac{1}{t^2}\right) + \frac{1}{2} \log 2\pi + \mathcal{O}\left(t^{-1}\right) \\ &= \left(\sigma - \frac{1}{2}\right) \log |t| - t\frac{\pi}{2} + \frac{1}{2} \log 2\pi + \mathcal{O}\left(t^{-1}\right). \end{aligned}$$

So, we can deduce, for $|t| \to \infty$

$$\exp\left(\log|\Gamma(s)|\right) = \exp\left(\left(\sigma - \frac{1}{2}\right)\log|t| - t\frac{\pi}{2} + \frac{1}{2}\log 2\pi + \mathcal{O}\left(t^{-1}\right)\right),$$

and hence,

$$|\Gamma(s)| \sim \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{|t|\pi}{2}}$$

□

4.3 The Functional Equation for the Zeta Function

One of the most used ways to extend the Riemann Zeta function over all complex plane, except at its pole at $s = 1$, is through what is called its *functional equation.* In this section, we will give one of the many proofs of this functional equation.

Proposition 4.10 (Functional Equation)**.** *We have, for all $s \in \mathbb{C}$,*

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s). \tag{4.9}$$

Remark. The functional equation implies that $\zeta(s)$ has a zero at each even negative integer, collectively known as the trivial zeros of $\zeta(s)$. When s is an even positive integer, the product $\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)$ on the right is non-zero because $\Gamma(1-s)$ has a simple pole that cancels with the simple zero of the sine factor.

Proof. [Mur01]. Doing a change of variable $x = n^2x\pi$ in the identity (4.1), we have

$$\begin{aligned}\Gamma\left(\frac{1}{2}s\right) &= \int_0^{+\infty} e^{-x}x^{\frac{1}{2}s-1}\,dx\\ &= \int_0^{+\infty} e^{-n^2x\pi}(xn^2\pi)^{\frac{1}{2}s-1}n^2\pi\,dx\\ &= n^s\pi^{\frac{1}{2}s}\int_0^{+\infty} e^{-n^2x\pi}x^{\frac{1}{2}s-1}\,dx.\end{aligned}$$

Similarly to Proposition 4.7, for $\sigma > 1$, the integral and the sum converge absolutely, so we have

$$\frac{\zeta(s)\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}s}} = \int_0^{\infty} x^{\frac{1}{2}s-1}\sum_{n=1}^{\infty} e^{-n^2x\pi}\,dx. \tag{4.10}$$

Define $\psi(x) := \sum_{n=1}^{\infty} e^{-n^2x\pi}$. Note that for $x > 0$, by Corollary C.5 and by

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{-y^2\pi x}e^{-2\pi iyu}\,dy = e^{-\pi\frac{u^2}{x}}\frac{1}{\sqrt{x}},$$

we have

$$\sum_{n=-\infty}^{\infty} e^{-n^2 x\pi} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{\frac{-n^2\pi}{x}}.$$

Or, alternatively,

$$2\psi(x) + 1 = \frac{1}{\sqrt{x}} \left(2\psi\left(\frac{1}{x}\right) + 1\right).$$

Hence, we can rewrite (4.10) as

$$\begin{aligned}
\frac{\zeta(s)\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}s}} &= \int_0^\infty x^{\frac{1}{2}s-1}\psi(x)\,dx \\
&= \int_0^1 x^{\frac{1}{2}s-1}\psi(x)\,dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x)\,dx \\
&= \int_0^1 x^{\frac{1}{2}s-1}\left(\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right)dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x)\,dx \\
&= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx + \int_1^\infty x^{\frac{1}{2}s-1}\psi(x)\,dx \\
&= \frac{1}{s(s-1)} + \int_1^\infty \left(x^{-\frac{1}{2}s-\frac{1}{2}} + x^{\frac{1}{2}s-1}\right)\psi(x)\,dx.
\end{aligned}$$

As the integral converges for all s, the formula holds for all s. Replacing s by $1-s$, as the right side stays unchanged, we obtain

$$\frac{\zeta(s)\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}s}} = \frac{\zeta(1-s)\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\pi^{\frac{1}{2}-\frac{1}{2}s}}.$$

Rearranging in order to $\zeta(s)$, we get

$$\zeta(s) = \pi^s\zeta(1-s)\frac{\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}s\right)}. \tag{4.11}$$

Now using propriety (4.6) for $\frac{-s}{2}$, we have

$$\Gamma\left(\frac{-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \Gamma(-s)\,\pi^{\frac{1}{2}}2^{1+s}.$$

Multiplying both sides by $\Gamma\left(\frac{s}{2}\right)\frac{s}{2}$ we get

$$\Gamma\left(\frac{s}{2}\right)\frac{s}{2}\Gamma\left(\frac{-s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right) = \Gamma\left(\frac{s}{2}\right)\frac{s}{2}\Gamma(-s)\pi^{\frac{1}{2}}2^{1+s}.$$

Using property (4.3) on $\frac{s}{2}\Gamma\left(\frac{-s}{2}\right)$ and $s\Gamma(-s)$ we get

$$-\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right) = -2^s\Gamma\left(\frac{s}{2}\right)\pi^{\frac{1}{2}}\Gamma(1-s).$$

Rearranging the terms, we have

$$\frac{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(\frac{s}{2})} = 2^s \frac{\Gamma(1-s)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}.$$

Finally, using property (4.5), we have

$$\frac{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(\frac{s}{2})} = 2^s\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\pi^{-1}.$$

Finally, substituting the last identity on (4.11), we get the desired result. □

Corollary 4.11 (Alternative ways of expressing the functional equation)**.** *We can rewrite the functional equation as*

1. $\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$,

2. $\xi(s) = \xi(1-s)$, *where* $\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}}(s)(s-1)\Gamma\left(\frac{(s)}{2}\right)\zeta(s)$.

Proof. **(1)** Evaluate the functional equation on $s = 1-s$, we have

$$\begin{aligned}\zeta(1-s) &= 2^{1-s}\pi^{-s}\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)\zeta(s)\\ &= 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s).\end{aligned}$$

(2) As we saw on the proof of Proposition 4.10, the functional equation is equivalent to (4.11), i.e,

$$\zeta(s) = \zeta(1-s)\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

We can multiply this equation by $\frac{1}{2}\pi^{-\frac{s}{2}}s(s-1)\Gamma\left(\frac{s}{2}\right)$ to get

$$\frac{1}{2}\pi^{\frac{-s}{2}}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}\pi^{\frac{-s}{2}}\pi^{s-\frac{1}{2}}s(s-1)\Gamma\left(\frac{s}{2}\right)\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(1-s).$$

Finally, simplifying the expression we obtain, as desired,

$$\xi(s) = \xi(1-s).$$

□

4.4 Growth of the Zeta Function

As the Dirichlet series of the Möbius function and the Zeta function are inverses of one another, one can expect that the "growth" of the Zeta Function will have a role in the proof of the main theorem of this chapter, namely about the growth of $M(x)$, the summation function of μ. So, here we will study the growth of Zeta over vertical lines, first outside the critical strip and then inside the critical strip, first without any assumptions and then assuming the Riemann's Hypothesis.

Definition 4.12. *For $\sigma \in \mathbb{R}$, define*

$$\nu(\sigma) := \inf\{\alpha : \zeta(\sigma + it) = \mathcal{O}\left(t^{\alpha}\right), |t| \to \infty\}.$$

Remark. It follows from general Dirichlet Series Theory, that ν is a non-negative, non-increasing and continuous function. See [TH87, Ch. V].

Proposition 4.13. *We have that*

$$\nu(\sigma) = \begin{cases} 0, & \sigma \geq 1, \\ \frac{1}{2} - \sigma, & \sigma \leq 0. \end{cases}$$

Proof. First, for $\sigma \geq 2$, we have $|\zeta(\sigma + it)| \leq \zeta(\sigma) \leq \zeta(2)$. So, of course $\zeta(s) = \mathcal{O}(t^0)$. We choose 2 arbitrarily, it could have been any real number bigger than 1.

Now, for $1 \leq \sigma \leq 2$, we will apply Lindelöf's Bound Theorem D.9, to the function $f(s) := \frac{\zeta(s)}{\log s}$, the rectangle $[1, 2] \times [t_0, \infty]$ (where $t_0 > 0$) and affine function $\kappa(\sigma) = 0$. It's clear that $f(s)$ is holomorphic inside and on this infinite rectangle, because both $\log(s)$ and $\zeta(s)$ are holomorphic in this region (note that $s = 1$ is not in it).

Now, let us check that $|\zeta(s)| = \mathcal{O}(\log t)$ on $\sigma = 1$. Using Theorem 4.3 we can write, for $a = 1$, $N \geq 1$ (and doing a change of variable $N' = N - 1$) and $\sigma > 0$,

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^{\infty} \frac{x - \lfloor x \rfloor}{x^{s+1}}\, dx. \tag{4.12}$$

Setting $s = 1 + it$ and $N = \lfloor t \rfloor$, we have

$$\begin{aligned} |\zeta(1 + it)| &\leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\lfloor t \rfloor} + \frac{1}{t} + \sqrt{1 + t^2} \int_{\lfloor t \rfloor}^{\infty} \frac{1}{x^2}\, dx \\ &\leq \int_1^t \frac{1}{x}\, dx + \frac{\sqrt{1 + t^2}}{\lfloor t \rfloor} \\ &\leq \log(t) + \frac{t+1}{t-1} \\ &\leq 2 \log(t) = \mathcal{O}(\log t). \end{aligned}$$

Note that $|\log s| \geq \log t$, because

$$\begin{aligned} |\log s| &= \left|\log \sqrt{1+t^2} + i \arg s\right| \\ &\geq \log \sqrt{1+t^2} = \frac{1}{2} \log (1+t^2) \\ &\geq \frac{1}{2} \log t^2 \geq \log t. \end{aligned}$$

So we have, for some constants $K > 0, L > 0$

$$|f(1+it)| \leq \frac{K \log t}{|\log s|} \leq Lt^0.$$

As we have seen, $|\zeta(2+it)| \leq \zeta(2)t^0$. So, for some constant $B > 0$, we have

$$|f(2+it)| = \frac{\zeta(2)}{|\log s|} \leq Bt^0.$$

We also have, by taking $N = 1$ in identify (4.12), for $s \neq 1$ and $\sigma > 0$

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\, dx.$$

This implies that $\zeta(s) = \mathcal{O}(t^L)$ for some $L > 0$ on the infinite rectangle, because

$$|\zeta(s)| \leq \left|\frac{s}{s-1}\right| - |s| \int_1^\infty \frac{x - \lfloor x \rfloor}{x^{\sigma+1}}\, dx.$$

Thus, for $1 \leq \sigma \leq 2$ and some constants $C, D > 0$, we have

$$|f(\sigma + it)| \leq \frac{Mt^L}{|\log s|} \leq \frac{Mt^L}{\log t} \leq Ct^D.$$

Hence, we can apply Lindelöf's Theorem and we have, on the defined rectangle,

$$\frac{|\zeta(s)|}{|\log s|} \leq K,$$

and thus

$$|\zeta(s)| = \mathcal{O}(\log(t)) = \mathcal{O}(t^\varepsilon),$$

for all $\varepsilon > 0$. So the Theorem is proved for $\sigma \geq 1$.

Let us now assume $\sigma \leq 0$.First, let us consider the following estimate

$$
\begin{aligned}
\left|\sin\left(\frac{\pi(\sigma+it)}{2}\right)\right| &= \left|\sin\left(\frac{\pi\sigma}{2}\right)\cosh\left(\frac{\pi t}{2}\right) - i\cos\left(\frac{\pi\sigma}{2}\right)\sinh\left(\frac{\pi t}{2}\right)\right| \\
&\leq \left|\sin\left(\frac{\pi\sigma}{2}\right)\right|\left|\cosh\left(\frac{\pi t}{2}\right)\right| + \left|\cos\left(\frac{\pi\sigma}{2}\right)\right|\left|\sinh\left(\frac{\pi t}{2}\right)\right| \\
&\leq \left|\cosh\left(\frac{\pi t}{2}\right)\right| + \left|\sinh\left(\frac{\pi t}{2}\right)\right| \\
&\leq \frac{e^{\frac{\pi t}{2}} + e^{\frac{-\pi t}{2}}}{2} + \frac{e^{\frac{\pi t}{2}} - e^{\frac{-\pi t}{2}}}{2} = e^{\frac{\pi t}{2}}.
\end{aligned}
$$

So, we have, for fixed $\sigma \leq 0$, for some constant $K > 0$

$$
\begin{aligned}
|\zeta(\sigma+it)| &= 2^{\sigma}\pi^{\sigma-1}|\zeta(1-\sigma-it)||\Gamma(1-\sigma-it)|\left|\sin\left(\frac{\pi s}{2}\right)\right| \\
&= 2^{\sigma}\pi^{\sigma-1}|\zeta(1-\sigma+it)||\Gamma(1-\sigma+it)|\left|\sin\left(\frac{\pi s}{2}\right)\right|
\end{aligned}
$$

Using the first part of the theorem for the modulus of Zeta, the above estimate for the sine function and the estimate from Corollary 4.9, we have

$$
|\zeta(\sigma+it)| \leq K\sqrt{2\pi}t^{\sigma-\frac{1}{2}}e^{\frac{-\pi t}{2}}e^{\frac{\pi t}{2}} = \mathcal{O}\left(t^{\sigma-\frac{1}{2}}\right).
$$

So $\nu(\sigma) \leq \sigma - \frac{1}{2}$ for $\sigma \leq 0$.

Recall that $\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}}(s)(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$. Fix $\sigma \leq 0$ and let $t \to \infty$. Then, using Stirling's Formula for the Gamma function (Proposition 4.8), we have

$$
\log|\xi(\sigma+it)| = \operatorname{Re}\log\xi(\sigma+it) = \operatorname{Re}\left(\log\frac{1}{2}\pi^{-\frac{s}{2}}(s-1)s\Gamma\left(\frac{s}{2}\right)\zeta(s)\right)
$$

Using property (4.3)

$$
\begin{aligned}
\log|\xi(\sigma+it)| = \operatorname{Re}\left(\log\Gamma\left(\frac{s}{2}+1\right)\right) + \operatorname{Re}\left(\log\pi^{-\frac{s}{2}}\right) + \operatorname{Re}\log\left(\sigma-1+it\right)+ \\
+ \operatorname{Re}\left(\log(\zeta(\sigma+it))\right) \\
= \log\left|\Gamma\left(\frac{s}{2}+1\right)\right| - \frac{\sigma}{2}\log\pi + \log|s-1| + \log|\zeta(\sigma+it)|
\end{aligned}
$$

Now using the Stirling Formula and the fact that as $t \to \infty$, $\log|s| \to \log t$ and $\operatorname{Im}\left(\log\frac{s}{2}\right) = \arg\frac{s}{2} \to \frac{\pi}{2}$, we have

$$\begin{aligned}\log|\xi(\sigma+it)| &\sim \left(\frac{\sigma+1}{2}\right)\log\left|\frac{s}{2}\right| - \frac{t}{2}\operatorname{Im}\left(\log\frac{s}{2}\right) - \frac{\sigma}{2} + \frac{1}{2}\log 2\pi - \\ &\qquad - \frac{\sigma}{2}\log\pi + \log|s-1| + \log|\zeta(\sigma+it)| \\ &\sim \frac{\sigma}{2}\left(\log t - 1 - \log\pi\right) + \frac{3}{2}\log t - \frac{\pi t}{4} + \frac{1}{2}\log 2\pi + \\ &\qquad + \log|\zeta(\sigma+it)| \\ &\sim \frac{\sigma}{2}\log\frac{t}{\pi e} + \frac{3}{2}\log t - \frac{\pi t}{4} + \frac{1}{2}\log 2\pi + \log|\zeta(\sigma+it)|\end{aligned}$$

As we have $\xi(s) = \xi(1-s)$, we can write

$$0 = \log|\xi(\sigma+it)| - \log\xi(1-\sigma+it),$$

or, alternatively, using the above estimates

$$0 \sim \frac{\sigma}{2}\log\frac{t}{e\pi} - \frac{1-\sigma}{2}\log\frac{t}{e\pi} + \log|\zeta(\sigma+it)| - \log|\zeta(1-\sigma+it)|.$$

Thus, we have

$$0 \sim \left(\sigma - \frac{1}{2}\right)\log\frac{t}{e\pi} + \log\left|\frac{\zeta(\sigma+it)}{\zeta(1-\sigma+it)}\right|.$$

Therefore

$$1 \sim \left(\frac{t}{e\pi}\right)^{\sigma-\frac{1}{2}}\left|\frac{\zeta(\sigma+it)}{\zeta(1-\sigma+it)}\right|.$$

As we have already seen that $|\zeta(s)| = \mathcal{O}(\log t) = \mathcal{O}(t^{\varepsilon})$ for $\sigma \geq 1$, we have, on any vertical line $\operatorname{Re}(s) = \sigma \leq 0$, that

$$|\zeta(\sigma+it)| = \mathcal{O}\left(t^{\frac{1}{2}-\sigma}\log t\right).$$

So it follows that the exponent $\frac{1}{2} - \sigma$ is the best possible, that is, it is the infimum. □

Proposition 4.14 (Zeta critical strip bound)**.** *Let* $\kappa(\sigma) := \frac{1}{2} - \frac{1}{2}\sigma$ *and* $\varepsilon > 0$*. Then for* $t \geq 2$ *and* $0 \leq \sigma \leq 1$ *we have*

$$|\zeta(\sigma+it)| = \mathcal{O}\left(t^{\kappa(\sigma+\varepsilon)}\right)$$

Proof. We want to use once again Lindelöf's bound Theorem D.9 on the affine function $\kappa(\sigma)+\varepsilon$ and infinite rectangle $[0,1]\times[2,\infty[$ and holomorphic function $\zeta(s)$, so we need to check if the growth conditions apply.

As we have seen in Proposition 4.13, for $\sigma = 1$ we have $|\zeta(s)| = \mathcal{O}(t^{\varepsilon})$, for all $\varepsilon > 0$, i.e., for some constant B we have $|\zeta(1+it)| \leq Bt^{\varepsilon}$, and for

$\sigma = 0$, we have $|\zeta(s)| = \mathcal{O}(t^{\frac{1}{2}+\varepsilon})$, for all $\varepsilon > 0$, i.e., for some constant A, we have $|\zeta(it)| \leq At^{\frac{1}{2}+\varepsilon}$.

Now, recall that in order to use Theorem D.9 we need to check that the function admits the following growth condition, $|\zeta(s)| \leq Ct^D$ on $0 \leq \sigma \leq 1$ and $t \geq 2$, for some constants C, D. First, let us consider $\frac{1}{2} \leq \sigma \leq 1$. Once again, from Theorem 4.3 for $N = 0$ and $a = 1$ we get

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_0^\infty \frac{x - \lfloor x \rfloor}{(x+1)^{s+1}}\,dx = \frac{s}{s-1} - s\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,dx.$$

So we get

$$\begin{aligned}|\zeta(s)| &\leq \frac{|s|}{|1-s|} + |s|\int_1^\infty \frac{x - \lfloor x \rfloor}{x^{\sigma+1}}\,dx \\ &\leq \frac{\sqrt{\sigma^2+t^2}}{\sqrt{(1-\sigma)^2+t^2}} + \frac{\sqrt{\sigma^2+t^2}}{\sigma}\end{aligned}$$

As we have $\sigma^2 \leq 1$ and $0 \leq (1-\sigma)^2 \leq \frac{1}{4}$ and $t \geq 2$:

$$\begin{aligned}|\zeta(s)| &\leq \frac{\sqrt{1+t^2}}{t} + 2\sqrt{1+t^2} \\ &\leq \frac{1+t}{t} + 2(1+t) \\ &\leq \frac{3}{2} + 3t \leq 3\sqrt{2}t.\end{aligned}$$

Now, let consider σ such that $0 \leq \sigma \leq \frac{1}{2}$. From the functional equation evaluated at $s = \sigma - it$ and the Schwarz reflection principle, [Con78, Ch. IX.1], that says $\overline{\zeta(s)} = \zeta(\overline{s})$ for all $s \neq 1$, we get that

$$\begin{aligned}|\zeta(\sigma+it)| = |\zeta(\sigma-it)| &= \frac{|\zeta(1-\sigma+it)|(2\pi)^\sigma}{2|\Gamma(\sigma-it)|\left|\cos\left(\frac{\pi(\sigma-it)}{2}\right)\right|} \\ &= \frac{4(2\pi)^\sigma 3\sqrt{2}}{2\sqrt{2\pi}} \frac{t}{t^{\sigma-\frac{1}{2}}e^{-\frac{\pi t}{2}}e^{\frac{\pi t}{2}}} \\ &= \mathcal{O}\left(t^{\frac{3}{2}-\sigma}\right) = \mathcal{O}\left(t^2\right).\end{aligned}$$

So, for $t \geq 2$, $0 \leq \sigma \leq 1$ and for all $\varepsilon > 0$, we have

$$|\zeta(\sigma+it)| = \mathcal{O}\left(t^2\right).$$

So, by Theorem D.9, we have the result as desired. □

We have proved that the graph of ν between 0 and 1 lies somewhere in the grey triangle shown in the figure 4.1. We will show that, if the Riemann's Hypothesis is true, for $\sigma > \frac{1}{2}$, $\nu(\sigma) = 0$.

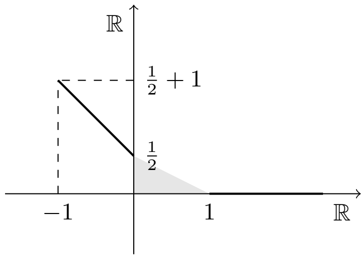

Figure 4.1: Graph of the ν function

Theorem 4.15. *Let $\varepsilon > 0$ be given and assume the Riemann's Hypothesis. Then, for all $\sigma > \frac{1}{2}$, we have, as $t \to \infty$,*

$$\zeta(s) = \mathcal{O}\left(t^{\varepsilon}\right) \quad \text{and} \quad \frac{1}{\zeta(s)} = \mathcal{O}\left(t^{\varepsilon}\right).$$

Proof. [Bro17b]. For $\sigma \geq 1 + \delta$, where $\delta > 0$, it is already proven, as $|\zeta(\sigma + it)| \leq \sum_{n \geq 1} \frac{1}{n^{\sigma}} \leq \sum_{n \geq 1} \frac{1}{n^{1+\delta}}$ which is a convergent series. Because we are assuming the Riemann's Hypothesis, $\log \zeta(s)$ is well defined on $\sigma > \frac{1}{2}$, with a pole on $s = 1$.

Now, let us prove that we have, for $0 < \delta < \frac{1}{2}$ and $\frac{1}{2} + \delta \leq \sigma \leq 1$ and

$$\log \zeta(\sigma + it) = \mathcal{O}\left((\log t)^{2-2\sigma+\lambda}\right)$$

uniformly for any $\lambda > 0$.

Fix $1 \leq \sigma_1 \leq t$. Then $\frac{1}{2} - \frac{\delta}{2} \leq \sigma_1 - \frac{1}{2} - \frac{\delta}{2}$ and $\frac{1}{2} - \delta \leq \sigma_1 - \frac{1}{2} - \delta$, so the function $\log \zeta(s)$, assuming Riemann's hypothesis, is well defined and holomorphic on the two concentric circles

$$|s - (\sigma_1 + it)| = \sigma_1 - \frac{1}{2} - \frac{\delta}{2} \text{ and } |s - (\sigma_1 + it)| = \sigma_1 - \frac{1}{2} - \delta.$$

Now, let us apply the Borel-Carathéodory Theorem D.8 to the function $\log \zeta(s)$ and the two circles.

By Lemma 4.13 and Proposition 4.14 we have, for some $A > 0$,

$$\zeta(s) = \mathcal{O}\left(t^{A}\right).$$

Thus $\operatorname{Re}(\log \zeta(s)) = \log |\zeta(s)| < A \log t$. Then, on the inner circle we have

$$\begin{aligned} |\log \zeta(s)| &\leq \frac{2(2\sigma_1 - 1 - 2\delta)}{\delta} A \log t + \frac{4\sigma_1 - 2 - 3\delta}{\delta} |\log \zeta(\sigma_1 + it)| \\ &< \frac{\tilde{A} \log t}{\delta} \quad (\text{for } \tilde{A} > 0). \end{aligned} \tag{4.13}$$

Now, let $1 < \sigma_1 \leq t$, and apply Hadamard's three circles Theorem, theorem D.7, to the circles

$$C_1 : |s - (\sigma_1 + it)| = \sigma_1 - 1 - \delta =: r_1,$$

$$C_2 : |s - (\sigma_1 + it)| = \sigma_1 - \sigma =: r_2,$$
$$C_3 : |s - (\sigma_1 + it)| = \sigma_1 - \frac{1}{2} - \delta =: r_3.$$

Recall that $\frac{1}{2} + \delta \le \sigma \le 1$. Hence we have $r_1 \le r_2 \le r_3$ and that each circle respectively passes through the points $1 + \delta + it$, $\sigma + it$ and $\frac{1}{2} + \delta + it$.

Let $M_i := \max_{s \in C_i} |\log \zeta(s)|$ for each i. By (4.13), we have that

$$M_3 \le \frac{A \log t}{\delta}.$$

By Proposition 4.4, valid for $\sigma \ge 1 + \delta$, we have on the circle C_1, that

$$M_1 \le \max_{x \ge 1+\delta} \left| \sum_{n \ge 1} \frac{\Lambda(n)}{n^x \log n} \right| \le \frac{1}{n^{1+\delta}} < \frac{A}{\delta}$$

Now Theorem D.7 gives

$$M_2^{\log \frac{r_3}{r_1}} \le M_1^{\log \frac{r_3}{r_2}} M_3^{\log \frac{r_2}{r_1}},$$

or, alternatively,

$$M_2 \le M_1^{\frac{\log \frac{r_3}{r_2}}{\log \frac{r_3}{r_1}}} M_3^{\frac{\log \frac{r_2}{r_1}}{\log \frac{r_3}{r_1}}}.$$

Let us denote

$$a := \frac{\log \frac{r_2}{r_1}}{\log \frac{r_3}{r_1}}.$$

Then $M_2 \le M_1^{1-a} M_3^a$. We can estimate a, using Lemma D.10, because $0 < 1 + \delta - \sigma < 1$, as follows

$$\begin{aligned} a =& \frac{\log\left(1 + \frac{r_2 - r_1}{r_1}\right)}{\log\left(1 + \frac{r_3 - r_1}{r_1}\right)} \\ =& \frac{\log\left(1 + \frac{1+\delta-\sigma}{\sigma_1 - 1 - \delta}\right)}{\log\left(1 + \frac{\frac{1}{2}}{\sigma_1 - 1 - \delta}\right)} \\ =& \frac{1 + \delta - \sigma}{\frac{1}{2}} + \mathcal{O}\left(\frac{1}{\sigma_1}\right) \\ =& 2 - 2\sigma + \mathcal{O}(\delta) + \mathcal{O}\left(\frac{1}{\sigma_1}\right). \end{aligned}$$

Hence

$$|\log \zeta(\sigma + it)| \le M_2 \le \left(\frac{A}{\delta}\right)^{1-a} \left(\frac{A \log t}{\delta}\right)^a = \frac{A}{\delta} (\log t)^a.$$

Now, take $\sigma_1 = \log\log t$ and $\delta = \frac{1}{\sigma_1}$. Then $\delta \to 0$ implies $a = 2 + 2\sigma + \lambda$, for some $\lambda > 0$. Thus

$$|\log \zeta(\sigma + it)| = \mathcal{O}\left(\log\log t(\log t)^{2-2\sigma+\lambda}\right) = \mathcal{O}\left((\log t)^{2-2\sigma+\lambda}\right).$$

Now, let us choose λ sufficiently small such that $2 - 2\sigma + \lambda < 1$. Thus, we have, for t sufficiently large and any given $\varepsilon > 0$,

$$\log|\zeta(s)| \leq |\log \zeta(s)| \leq \varepsilon \log t.$$

Therefore

$$-\varepsilon \log t < \log|\zeta(s)| < \varepsilon \log t.$$

Hence,

$$|\zeta(s)| < t^{\varepsilon} \text{ and } \left|\frac{1}{\zeta(s)}\right| < t^{\varepsilon}.$$

□

4.5 Equivalents to the Riemann's Hypothesis

Finally, we will prove the main theorem of this dissertation.

Theorem 4.16. *The following conditions are equivalent*

1. *(**Riemann's Hypothesis**) The Zeta function has its zeros only at the negative even integers (called the trivial zeros) and complex numbers with real part 1/2.*
2. *For all* $\varepsilon > 0$, $M(x) = \sum_{n \leq x} \mu(n) = \mathcal{O}(x^{\frac{1}{2}+\varepsilon})$.
3. *For all* $\varepsilon > 0$, $L(x) = \sum_{n \leq x} \lambda(n) = \mathcal{O}(x^{\frac{1}{2}+\varepsilon})$.

Before we get to the main theorem of this chapter, we will need the following Lemma.

Lemma 4.17. *Let* $a > 0$, $T > 0$. *Then, for* $0 < x < 1$, *we have*

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT} \frac{x^s}{s}\, ds\right| \leq \frac{x^a}{\pi T|\log(x)|}.$$

And for $x > 1$ *we have*

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT} \frac{x^s}{s}\, ds - 1\right| \leq \frac{x^a}{\pi T|\log(x)|}.$$

Proof. [Bro17a]. First assume $0 < x < 1$, let $A > a$ and apply Cauchy's Residue Theorem, Theorem D.2, to the holomorphic function $\frac{x^s}{s}$ on the rectangle $[a, A] \times [-T, T]$. As the function has no singularities inside the rectangle, we have

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{x^s}{s}\,dx = -\frac{1}{2\pi i}\left(\int_{a+iT}^{A+iT} - \int_{A-iT}^{A+iT} - \int_{a-iT}^{A-iT}\right)\frac{x^s}{s}\,ds.$$

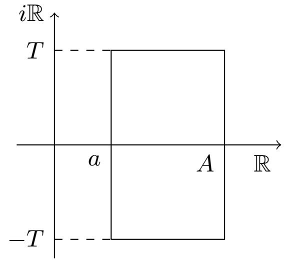

Figure 4.2: The rectangle $[a, A] \times [-T, T]$

On the right hand vertical section (the line from $A - iT$ to $A + iT$), we have $\left|\frac{x^s}{s}\right| \le \frac{x^A}{A}$ and on the horizontal sections (the lines $a - iT$ to $A - iT$ and $a + iT$ to $A + iT$), we have, for $s = \sigma \pm iT$, $\left|\frac{x^s}{s}\right| \le \frac{x^\sigma}{T}$. Thus we have, as $x < 1$,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{x^s}{s}\,dx\right| &\le \frac{1}{2\pi}\int_{-T}^{T}\frac{x^A}{A}\,dt + 2\frac{1}{2\pi}\int_a^A\frac{x^\sigma}{T}\,d\sigma\\ &\le \frac{Tx^A}{\pi A} + \frac{(x^a - x^A)}{\pi T|\log(x)|} \xrightarrow[A\to\infty]{} \frac{x^a}{\pi T\log(x)}.\end{aligned}$$

Similarly, for $x > 1$, let $A > a$ and apply Cauchy's Residue Theorem, Theorem D.2, to the function $\frac{x^s}{s}$ but this time on the rectangle $[-A, a] \times [-T, T]$, on which the function is meromorphic. Here the function has one

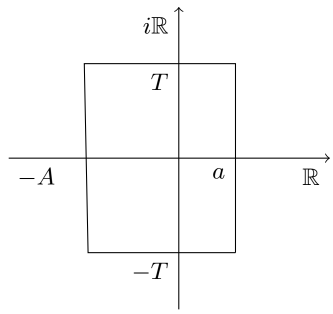

Figure 4.3: The rectangle $[-A, a] \times [-T, T]$

singularity at $s = 0$, and its residue has value 1, because it is a simple pole and $\lim_{s \to 0} x^s = 1$. So we have

$$\frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{x^s}{s}\, dx - 1 = -\frac{1}{2\pi i} \left(- \int_{-A+iT}^{a+iT} - \int_{-A+iT}^{-A-iT} + \int_{-A-iT}^{a-iT} \right) \frac{x^s}{s}\, ds.$$

And using the same calculations as in the first part of the Lemma, we get the result. □

We now have all the necessary tools for the proof of the Theorem 4.16.

Proof of Theorem 4.16. **(2)** $\iff$ **(3)**

Fix $\varepsilon > 0$. Let us suppose $|M(x)| \le Cx^{\frac{1}{2}+\varepsilon}$, for $C > 0$. Then, we have

$$\begin{aligned}
|L(x)| &\le \sum_{j^2 \le x} \left| M\left(\frac{x}{j^2}\right) \right| \quad \text{(By Proposition 2.33)} \\
&\le \sum_{j^2 \le x} C \frac{x^{\frac{1}{2}+\varepsilon}}{(j^2)^{\frac{1}{2}+\varepsilon}} \\
&\le Cx^{\frac{1}{2}+\varepsilon} \sum_{j^2 \le x} \frac{1}{j^{1+2\varepsilon}} \\
&\le Cx^{\frac{1}{2}+\varepsilon} \left(1 + \int_1^{\sqrt{x}} \frac{1}{t^{2\varepsilon+1}}\, dt \right) \\
&\le Cx^{\frac{1}{2}+\varepsilon} \left(1 + \left[\frac{t^{-2\varepsilon}}{-2\varepsilon} \right]_1^{\sqrt{x}} \right) \\
&\le Cx^{\frac{1}{2}+\varepsilon} \left(1 + \frac{1}{2\varepsilon} \left(1 - (\sqrt{x})^{-2\varepsilon} \right) \right) \\
&\le Cx^{\frac{1}{2}+\varepsilon} \left(\frac{1+2\varepsilon}{2\varepsilon} - \frac{x^{-\varepsilon}}{2\varepsilon} \right) \\
&\le \tilde{C}x^{\frac{1}{2}+\varepsilon} - \frac{C}{2\varepsilon} x^{\frac{1}{2}} \quad \left(\text{where } \tilde{C} = \frac{C(1+2\varepsilon)}{2\varepsilon} \right) \\
&\le \tilde{C}x^{\frac{1}{2}+\varepsilon}
\end{aligned}$$

Now, let us suppose $|L(x)| \le Cx^{\frac{1}{2}+\varepsilon}$, for $C > 0$. Then, we have, in the same manner by using Proposition 2.33

$$\begin{aligned}
|M(x)| &\le \sum_{j^2 \le x} |\mu(j)| \left| L\left(\frac{x}{j^2}\right) \right| \le \sum_{j^2 \le x} \left| L\left(\frac{x}{j^2}\right) \right| \\
&\le \sum_{j^2 \le x} C \frac{x^{\frac{1}{2}+\varepsilon}}{(j^2)^{\frac{1}{2}+\varepsilon}} \le \tilde{C}x^{\frac{1}{2}+\varepsilon}.
\end{aligned}$$

Therefore (2) is true if and only if (3) is true.

(2) ⇒ (1)

Let us suppose that for all $\varepsilon > 0$, we have $M(x) = \mathcal{O}\left(x^{\frac{1}{2}+\varepsilon}\right)$, that is, there exist $M > 0$ such that $|M(x)| \leq Mx^{\frac{1}{2}+\varepsilon}$. Then, by Theorem 3.13, $\sum_{n\geq 1} \frac{\mu(n)}{n^s}$ converges uniformly in $\mathcal{D}(\frac{1}{2}, \delta, \theta)$, for all $\delta > 0$ and $\theta > 0$ and $F(s) := \sum_{n\geq 1} \frac{\mu(n)}{n^s}$ is an analytic function in $\sigma > \frac{1}{2}$.

We know that $F(s)\zeta(s) = 1$, for $\sigma > 1$. Taking the analytic continuation of the zeta function, we define $f(s) := F(s)\zeta(s)$ and $g(s) := 1$ which are analytic functions in $D = \{s \in \mathbb{C} | \sigma > \frac{1}{2}\} \setminus \{1\}$. By analytic continuation, as $f = g$ in a open non-empty proper subset of D, we have that $f = g$ in all D. Therefore $\zeta(s)F(s) = 1$. As $F(s) < \infty$ in D, we have that $\zeta(s) \neq 0$ in $\frac{1}{2} < \sigma < 1$.

From the functional equation of the zeta function, (4.9), it is clear that $\zeta(s) = 0 \iff \zeta(1-s) = 0$, so there are no zeros in the strip $0 < \sigma < 1$ outside the line $\sigma = \frac{1}{2}$.

(1) ⇒ (2) ([Bro17a])

Suppose that the Riemann's Hypothesis is true. Then, for $\sigma > \frac{1}{2}$, $\frac{1}{\zeta(s)}$ is well defined and by analytic continuation for $\sigma > \frac{1}{2}$, $\sum_{n\geq 1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$.

Let $T > 0$ and $x > 0$, considered large and $x \notin \mathbb{N}$. Let us define

$$\begin{aligned}\Delta(x,T) :&= \left| M(x) - \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{x^s}{\zeta(s)}\frac{1}{s}\, ds \right| \\ &= \left| \sum_{1\leq n<x} \mu(n) - \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \sum_{n\geq 1} \mu(n)\left(\frac{x}{n}\right)^s \frac{1}{s}\, ds \right| \\ &= \left| \sum_{1\leq n<x} \mu(n) - \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \sum_{n<x} \mu(n)\left(\frac{x}{n}\right)^s \frac{1}{s}\, ds \right. \\ &\qquad \left. -\frac{1}{2\pi i}\int_{2-iT}^{2+iT} \sum_{n>x} \mu(n)\left(\frac{x}{n}\right)^s \frac{1}{s}\, ds \right|.\end{aligned}$$

Let us restrict x to of the form of an integer plus one half, so that $|\log(x/n)| \geq \frac{1}{2n}$ for $x/2 \leq n \leq 2x$ and $|\log(x/n)| \geq \log(2)$ for $x/2 > n$ or $n > 2x$. Note that $|\mu(n)| \leq 1$ for any $n \in \mathbb{N}$. Now, using Lemma 4.17 and these estimates, we have

$$\begin{aligned}\Delta(x,T) \leq &\left| \sum_{n<x} \mu(n) - \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \sum_{n<x} \mu(n)\left(\frac{x}{n}\right)^s \frac{1}{s}\, ds \right| + \\ &+ \left| \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \sum_{n>x} \mu(n)\left(\frac{x}{n}\right)^s \frac{1}{s}\, ds \right|\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n<x} \left| 1 - \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \left(\frac{x}{n}\right)^s \frac{1}{s}\, ds \right| + \sum_{n>x} \left| \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \left(\frac{x}{n}\right)^s \frac{1}{s}\, ds \right| \\
&\leq \sum_{n<x} \frac{(x/n)^2}{\pi T |\log(x/n)|} + \sum_{n>x} \frac{(x/n)^2}{\pi T |\log(x/n)|} \\
&\leq \sum_{x/2 \leq n < 2x} \frac{(x/n)^2}{\pi T |\log(x/n)|} + \sum_{x/2 < n \text{ or } n > 2x} \frac{(x/n)^2}{\pi T |\log(x/n)|} \\
&\leq \frac{x^2}{\pi T} \left(\sum_{x/2 \leq n \leq 2x} \frac{2n}{n^2} + \sum_{x/2 < n \text{ or } n > 2x} \frac{1}{n^2 \log(2)} \right) \\
&\leq \frac{x^2}{\pi T} \left(2 \sum_{x/2 \leq n \leq 2x} \frac{1}{n} + \frac{1}{\log(2)} \sum_{n \geq 1} \frac{1}{n^2} \right) \\
&\leq \frac{x^2}{\pi T} \left(2 \int_{x/2}^{2x} \frac{1}{t}\, dt + \frac{1}{\log(2)} \frac{\pi^2}{6} \right) \\
&\leq \frac{x^2}{\pi T} \left(2\log(2x) - 2\log(x/2) + \frac{\pi^2}{6\log(2)} \right) \\
&= \mathcal{O}\left(\frac{x^2}{T} \right).
\end{aligned}
$$

That is,

$$
\Delta(x, T) \leq M \frac{x^2}{T}, \tag{4.14}
$$

for some constant M. By taking T sufficiently large, we have that the integral $\frac{1}{2\pi i} \int_{2-iT}^{2+iT} \sum_{n<x} \mu(n) \left(\frac{x}{n}\right)^s \frac{1}{s}\, ds$ is a good approximation of $M(x)$, in the sense that the difference is no greater than a constant. So now we need to estimate the growth of the integral.

Now, fix $\delta > 0$ and let us integrate the function $\frac{x^s}{\zeta(s)s}$ around the rectangle $\left[\frac{1}{2} + \delta, 2\right] \times [-T, T]$.

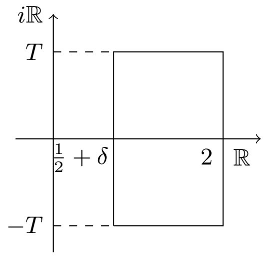

Figure 4.4: The rectangle $[\frac{1}{2} + \delta, 2] \times [-T, T]$

As $\frac{x^s}{\zeta(s)}$ does not have any singularities inside the rectangle, because we are assuming Riemann's Hypothesis, we have

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{x^s}{\zeta(s)s}\,ds=\frac{1}{2\pi i}\int_{\frac{1}{2}+\delta+iT}^{2+iT}\frac{x^s}{\zeta(s)s}\,ds$$
$$+\frac{1}{2\pi i}\int_{\frac{1}{2}-\delta+iT}^{\frac{1}{2}+\delta-iT}\frac{x^s}{\zeta(s)s}\,ds+\frac{1}{2\pi i}\int_{2-iT}^{\frac{1}{2}+\delta-iT}\frac{x^s}{\zeta(s)s}\,ds$$

On the horizontal sections, we can use the Theorem 4.15, in particular the fact that, for all $\varepsilon > 0$, $\sigma > \frac{1}{2}$ and $t \to \infty$

$$\frac{1}{|\zeta(\sigma+it)|}\leq K_\varepsilon t^\varepsilon,$$

as we are assuming the Riemann's Hypothesis is true, and also the fact that

$$\left|\frac{x^s}{s}\right|\leq\frac{x^\sigma}{|\sigma+iT|^2}\leq\frac{x^2}{T}$$

to estimate

$$\left|\int_{\frac{1}{2}+\delta+iT}^{2+iT}\frac{x^s}{\zeta(s)s}\,ds\right|\leq\tilde{K}_\varepsilon T^{\varepsilon-1}x^2.$$

For the vertical section, we start by taking $T_0 \in]0,T[$, such that Theorem 4.15 is valid for $t > T_0$. Then we have

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+\delta-iT}^{\frac{1}{2}+\delta+iT}\frac{x^s}{\zeta(s)s}\,ds\right|\leq\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+\delta-iT_0}^{\frac{1}{2}+\delta+iT_0}\frac{x^s}{\zeta(s)s}\,ds\right|+2\left|\int_{\frac{1}{2}+\delta+iT_0}^{\frac{1}{2}+\delta+iT}\frac{x^s}{\zeta(s)s}\,ds\right|,$$

Thus,

$$2\left|\int_{\frac{1}{2}+\delta+iT_0}^{\frac{1}{2}+\delta+iT}\frac{x^s}{\zeta(s)s}\,ds\right|\leq\frac{2}{2\pi}\int_{T_0}^{T}\frac{x^{\frac{1}{2}+\delta}K_\varepsilon t^\varepsilon}{t}\,dt\leq\frac{K_\varepsilon x^{\frac{1}{2}+\delta}(T^\varepsilon-T_0^\varepsilon)}{\pi\varepsilon}.$$

For the other integral, as T_0 is independent of x and by hypothesis ζ does not have zeros in the line $\sigma = \frac{1}{2}+\delta$, we have that

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+\delta-iT_0}^{\frac{1}{2}+\delta+iT_0}\frac{x^s}{\zeta(s)s}\,ds\right|\leq\frac{x^{\frac{1}{2}+\delta}}{2\pi}\int_{-T_0}^{T_0}\left|\frac{1}{\zeta\left(\frac{1}{2}+\delta+it\right)t}\right|dt\leq Kx^{\frac{1}{2}+\delta}.$$

Putting together the two estimates, we have for a constant C_ε

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+\delta-iT}^{\frac{1}{2}+\delta+iT}\frac{x^s}{\zeta(s)s}\,ds\right|\leq Kx^{\frac{1}{2}+\delta}+\frac{2K_\varepsilon}{\pi\varepsilon}x^{\frac{1}{2}+\delta}T^\varepsilon\leq Kx^{\frac{1}{2}+\delta}+C_\varepsilon x^{\frac{1}{2}+\delta}T^\varepsilon.$$

Thus, we get

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{x^s}{\zeta(s)s}\,ds\right| &\leq \frac{1}{2\pi}\left|\int_{\frac{1}{2}+\delta+iT}^{2+iT}\frac{x^s}{\zeta(s)s}\,ds\right|+\\ &\quad+\frac{1}{2\pi}\left|\int_{\frac{1}{2}-\delta+iT}^{\frac{1}{2}+\delta+iT}\frac{x^s}{\zeta(s)s}\,ds\right|+\frac{1}{2\pi}\left|\int_{2-iT}^{\frac{1}{2}+\delta-iT}\frac{x^s}{\zeta(s)s}\,ds\right|\\ &\leq 2K_\varepsilon T^{\varepsilon-1}x^2+Kx^{\frac{1}{2}+\delta}+C_\varepsilon x^{\frac{1}{2}+\delta}T^\varepsilon.\end{aligned}$$

Now, taking $T = x^2$, we have that

$$\left|\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{x^s}{\zeta(s)s}\,ds\right| \leq 2K_\varepsilon x^{2\varepsilon}+Kx^{\frac{1}{2}+\delta}+C_\varepsilon x^{\frac{1}{2}+\delta+2\varepsilon}\leq M_\varepsilon x^{\frac{1}{2}+\delta+2\varepsilon},$$

where M_ε is chosen appropriately. And, by (4.14),

$$\Delta(x,x^2)=\left|M(x)-\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{x^s}{\zeta(s)}\frac{1}{s}\,ds\right|\leq M\frac{x^2}{x^2}=M.$$

Hence, we have

$$\begin{aligned}|M(x)| &\leq \left|\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{x^s}{\zeta(s)}\frac{1}{s}\,ds\right|+\Delta(x,x^2)\\ &\leq M_\varepsilon x^{\frac{1}{2}+\delta+2\varepsilon}+M\\ &\leq \tilde{M}_\varepsilon x^{\frac{1}{2}+\delta+2\varepsilon}.\end{aligned}$$

So $M(x)=\mathcal{O}\left(x^{\frac{1}{2}+\delta+2\varepsilon}\right)$ for all large x of the form of an integer plus one half. But since $|\mu(n)|\leq 1$, $M(x)$ can change by at most 1 between these values of x, so replacing δ and ε by $\varepsilon/3$, we get for all large x

$$M(x)=\mathcal{O}(x^{\frac{1}{2}+\varepsilon}).$$

□

Appendix A

Ring Theory

The goal of this appendix is to recall some results about Ring Theory while also clarifying the notation that is used in the main body of this dissertation.

Definition A.1. *A **ring** A is a set with two binary operations addition, $+$, and multiplication, $\cdot$, satisfying the following conditions:*

- *With respect to addition, A is a commutative group;*
- *The multiplication is associative, and has a unit element;*
- *For all $x, y, z \in A$ we have*

$$(x+y)\cdot z = x\cdot z + y\cdot z \text{ and } z\cdot(x+y) = z\cdot x + z\cdot y.$$

*A **commutative ring** is a ring such that it is commutative with respect to multiplication. The set of the **units** of a ring, elements who have both a right and a left inverse, is a group under ring multiplication and it is often denoted as A^*.*

Definition A.2. *In a ring A, we say x and y in A are **associates**, and write $x \sim y$, if exists $u \in A^*$ such that $x = u \cdot y$.*

Definition A.3. *A **Integral Domain (ID)** is a nonzero commutative ring in which the product of any two nonzero elements is nonzero.*

Definition A.4. *Let A be a integral domain. We call $x \in A$ an **irreducible** element if $x \neq 0$, x is not a unit and if we write $x = y \cdot z$, with $y, z \in R$ then $y \in R^*$ or $z \in A^*$. The remaining elements that are not units, zero or irreducible are called **reducible**.*

Definition A.5. *We call an integral domain in which every ideal is principal, i.e., can be generated by a single element, a **Principal Ideal Domain**.*

Definition A.6. *A partially ordered set $(P, <)$ is said to satisfy the* ***Ascending Chain Condition (ACC)*** *if no infinite strictly ascending sequence*

$$a_1 < a_2 < a_3 < \dots$$

of elements of P exists.

Equivalently, every weakly ascending sequence

$$a_1 \leq a_2 \leq a_3 \leq \dots$$

of elements of P eventually stabilizes, meaning that there exists a positive integer n such that

$$a_n = a_{n+1} = a_{n+2} = \dots$$

We can say a ring A has the ***Ascending Chain Condition on Principal ideals (ACCP)*** *is satisfied if there is no infinite strictly ascending chain of principal ideals in the ring, or said another way, every ascending chain of principal ideals is eventually constant.*

Remark. Let x, y be two elements of the same ring. Note that if $(x) \subsetneq (y)$, then $x \in (y)$ and there is $z \in A \setminus A^* \cup \{0\}$ such that $x = zy$. Conversely, if $x = yz$, with y, z non-units, then $(x) \subsetneq (y)$.

Definition A.7. *A ring is called a* ***Unique Factorization Domain*** *if it is a integral domain and if every non-zero element has a unique factorization into irreducible elements.*

Lemma A.8. *If a integral domain satisfies the Ascending Chain Condition on Principal ideals, then every non-zero non-unit element has a factorization into irreducible elements.*

Proof. Let A be an integral domain. Suppose x is a non-zero non-unit element of A that does not have a factorization into irreducible elements.

Clearly, x is reducible and thus there are $y, z \in A$ non-units and non-zero such that $x = yz$ and such that at least one of them also cannot be factorized into irreducible elements. Let us suppose, without loss of generality, that y is that element.

We can then repeat this process infinitely and obtain an infinite chain

$$(x) \subsetneq (y) \subsetneq (y_1) \subsetneq \dots$$

But as A has the ACCP, this is a contradiction. So every element has at least one factorization into irreducible elements. □

Definition A.9. *Let A be a commutative ring, and a and b elements of A. We call $d \in A$ a* ***common divisor*** *of a and b if it divides both elements, i.e., if there are elements $x, y \in A$ such that $a = d\dot{x}$ and $b = d\dot{y}$. If d is a common divisor of a and b such that every other common divisor of the two elements divides d, we call d a* ***greatest common divisor*** *of a and b and we write $(a, b) = d$ or $\gcd(a, b) = d$.*

Proposition A.10 (Bézout's Identity)**.** *If A is a PID and a, b, d are elements of A such that d is a greatest common divisor of a and b, then there are elements x and y in A such that $ax + by = d$.*

Proof. By hypothesis the ideal $(a) + (b)$ is principal, i.e., there is $c \in A$ such that $(a) + (b) = (c)$, which means both $a \in (c)$ and $b \in (c)$, i.e., there are $x, y \in A$ such that $a = x \cdot c$ and $b = y \cdot c$.Thus c is a common divisor of a and b. But $d = \gcd(a, b)$ implies $(a) \subseteq (d)$ and $(b) \subseteq (d)$. Thus $(c) \subseteq (d)$ and we have $d \mid c$. Since d is a greatest common divisor, we must have $d\ c$ and thus $(a) + (b) = (d)$. □

Appendix B

Fourier Series

In this appendix, we will give some definitions and results on Fourier Series, following [Zyg59], necessary in appendix C.

Definition B.1. *Given a sequence $s_0, s_1, s_2, \dots$ we define, for every $k \in \mathbb{N}_0$, the sequences $S_0^k, S_1^k, \dots$ and $A_0^k, A_1^k, \dots$ by the conditions:*

$$S_n^0 = s_n, \quad S_n^1 = s_0 + s_1 + \cdots + s_n,$$
$$S_n^k = S_0^{k-1} + S_1^{k-1} + \cdots + S_n^{k-1} \, (k \in \mathbb{N}; n \in \mathbb{N}_0),$$

$$A_n^0 = 1, \quad A_n^1 = n + 1,$$
$$A_n^k = A_0^{k-1} + A_1^{k-1} \cdots + A_n^{k-1} \, (k \in \mathbb{N}; n \in \mathbb{N}_0).$$

*We say that the sequence $(s_n)_{n \in \mathbb{N}_0}$ (or the series whose partial sums are s_n) is **summable by the k-th arithmetic mean of Cesàro** or **summable** (C, k) to limit (or sum), if*

$$\lim_{n \to \infty} S_n^k / A_n^k = s.$$

Remark. Summability $(C, 0)$ is ordinary convergence. Summability (C, k) implies summability $(C, k + 1)$ to the same limit. See [Zyg59, Ch. III, Theorem 1.6]

Definition B.2. *We say a matrix*

$$M = \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,n} & \dots \\ a_{1,0} & a_{1,1} & \dots & a_{1,n} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n,0} & a_{n,1} & \dots & a_{n,n} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

*is **regular** if*

1. $\lim_{n\to\infty} a_{n,v} = 0$ *for* $v = 0, 1, \dots$;

2. *the* $N_n = \sum_{k\geq 0} |a_{n,k}|$ *are bounded;*

3. $\lim_{n\to\infty} \sum_{k\geq 0} a_{n,k} = 1$.

Definition B.3. *Let* M *be a regular matrix, as in the definition B.2. Given a sequence* $s_0, s_1, \dots$ *consider the* ***linear means***

$$\sigma_n = a_{n,0}s_0 + a_{n,1}s_1 + \dots + a_{n,k}s_k + \dots$$

generated by M*.*

Remark. The fact that M is regular implies that the convergence of the series $(\sigma_n)_n$ is bounded.

Remark. If $s_n = \sum_{k=0}^{n} u_k$, the partial sums of the series $\sum u_k$, then we can rewrite σ_n as

$$\sigma_n = \alpha_{n,0}u_0 + \alpha_{n,1}u_1 + \dots + \alpha_{n,k}u_k + \dots$$

where $\alpha_{n,k} = \sum_{j=k}^{\infty} a_{n,j}$.

Definition B.4. *Given a function* f *of period* T*, the* ***Fourier coefficients of*** f *are*

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos\left(kt\frac{2\pi}{T}\right)dt$$
$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin\left(kt\frac{2\pi}{T}\right)dt$$

Let us define the ***complex Fourier coefficients***

$$c_v = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-\frac{2\pi}{T}ivt}\,dt.$$

It is easy to see that $c_v = \frac{1}{2}(a_v - ib_v)$ *and that* $c_{-v} = \frac{1}{2}(a_v + ib_v)$ *for* $v \geq 0$.

Let us assume here, without loss of generality, that the period of f *is* $T = 2\pi$*. So*

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt,$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt.$$

Then the ***Fourier series*** *of* f *is*

$$S[f] = \sum_{-\infty}^{+\infty} c_v e^{ivx}.$$

Theorem B.5 (Fejér). *At every point x_0 at which the limits $f(x_0 \pm 0) := \lim_{h \to 0^+} f(x_0 \pm h)$ exist (and, if they are both infinite, are of the same sign), we have*

$$\frac{1}{n+1} \sum_{k=0}^{n} s_k(x_0) \xrightarrow{n \to \infty} \frac{1}{2} \left(f(x_0 + 0) + f(x_0 - 0) \right)$$

where $s_k(x_0) = \sum_{|n| \le k} c_n e^{inx_0}$.

This theorem is saying that the sequence $(s_n)_{n \in \mathbb{N}}$ of partial sums of the Fourier series of f is summable by the 1st arithmetic mean of Cesàro, that is, the linear mean of the partial sums converges to $f(x_0)$, if f is continuous at x_0. See [Zyg59, Theorem III.3.4], for the proof, as the necessary theory would be too extensive to include.

Appendix C

Summation Formulas

Here we will present and prove the Abel, Euler, Poisson and Euler-Maclaurin Summation Formulas, which are used throughout the main chapters.

Proposition C.1 (Abel Summation Formula)**.** *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of complex numbers and* $A(x) = \sum_{1\leq n\leq x} a_n$ *its partial sums.*

Fix $0 \leq y < x \in \mathbb{R}$ *and let* $f\colon [y, x] \to \mathbb{R}$ *be a continuously differentiable function. Then we have,*

$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt.$$

Proof. First, suppose that

$$\sum_{1\leq n\leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\,dt. \tag{C.1}$$

Then

$$\begin{aligned}\sum_{y<n\leq x} a_n f(n) &= \sum_{1\leq n\leq x} a_n f(n) - \sum_{1\leq n\leq y} a_n f(n)\\ &= A(x)f(x) - \int_1^x A(t)f'(t)\,dt - A(y)f(y) + \int_1^y A(t)f'(t)\,dt\\ &= A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt.\end{aligned}$$

So, it suffices to show that (C.1) is true.

Note that $A(x) = A(\lfloor x\rfloor)$, so if we fix $n \in \mathbb{N}$, we have $A(x) = A(n)$ for all $n \leq x < n+1$. Now,

$$
\begin{aligned}
\sum_{1\le n\le x} a_n f(n) &= \sum_{1\le n\le x} (A(n)-A(n-1))f(n)\\
&= \sum_{1\le n\le x} A(n)f(n) - \sum_{1\le n\le x} A(n-1)f(n)\\
&= A(\lfloor x\rfloor)f(\lfloor x\rfloor) + \sum_{1\le n\le x-1} A(n)f(n) - \sum_{1\le n\le x} A(n-1)f(n)\\
&= A(\lfloor x\rfloor)f(\lfloor x\rfloor) + \sum_{1\le n\le x-1} A(n)(f(n)-f(n+1))\\
&= A(x)f(x) + A(\lfloor x\rfloor)(f(\lfloor x\rfloor - f(x)) -\\
&\qquad - \sum_{1\le n\le x-1} A(n)\left(\int_n^{n+1} f'(t)\,dt\right)\\
&= A(x)f(x) - A(\lfloor x\rfloor)\int_{\lfloor x\rfloor}^{x} f'(t)\,dt -\\
&\qquad - \sum_{1\le n\le x-1}\left(\int_n^{n+1} A(t)f'(t)\,dt\right)\\
&= A(x)f(x) - \int_{\lfloor x\rfloor}^{x} A(t)f'(t)\,dt - \sum_{1\le n\le x-1}\int_n^{n+1} A(t)f'(t)\,dt\\
&= A(x)f(x) - \int_{\lfloor x\rfloor}^{x} A(t)f'(t)\,dt - \int_1^{\lfloor x-1\rfloor+1} A(t)f'(t)\,dt\\
&= A(x)f(x) - \int_1^{x} A(t)f'(t)\,dt.
\end{aligned}
$$

□

Proposition C.2 (Euler Summation Formula)**.** *Let f be a continuously differentiable function on the interval $[y,x]$, where $0 \le y < x$. Then we have,*

$$
\sum_{y<n\le x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t-\lfloor t\rfloor)f'(t)\,dt + f(x)(\lfloor x\rfloor - x) - f(y)(\lfloor y\rfloor - y).
$$

Proof. This is a consequence of the Abel Summation Formula, Proposition C.1. Let us use it for for $a_n = 1$ for all n. Then we have $A(x) = \lfloor x\rfloor$ and

$$
\begin{aligned}
\sum_{y\le n\le x} f(n) &= \lfloor x\rfloor f(x) - \lfloor y\rfloor f(y) - \int_y^x \lfloor t\rfloor f'(t)\,dt\\
&= \lfloor x\rfloor f(x) - \lfloor y\rfloor f(y) + \int_y^x (t-\lfloor t\rfloor)f'(t)\,dt - \int_y^x t f'(t)\,dt
\end{aligned}
$$

$$= \lfloor x \rfloor f(x) - \lfloor y \rfloor f(y) + \int_y^x (t - \lfloor t \rfloor) f'(t)\, dt$$
$$- x f(x) + y f(y) + \int_y^x f(t)\, dt.$$

Rearranging the expression by putting in evidence equal terms, the result follows.

□

Definition C.3. *For $f \in L^1(\mathbb{R})$ function, we define $\hat{f}(u)$, its **Fourier transform**, as*

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i u x}\, dx.$$

Proposition C.4 (Poisson Summation Formula)**.** *Let $f \in L^1(\mathbb{R})$. Suppose that the series $\sum_{n\in\mathbb{Z}} f(n+v)$ converges absolutely and uniformly for $v \in \mathbb{R}$ and that $\sum_{m\in\mathbb{Z}} \left|\hat{f}(m)\right| < \infty$. Then*

$$\sum_{n\in\mathbb{Z}} f(n+v) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n v}.$$

Proof. [Mur01]. Set $S_N(x_0) = \sum_{|n|\le N} c_n e^{2\pi i n x_0}$. By Fejér Theorem, Theorem B.5, we have

$$\lim_{N\to\infty} \frac{S_0(x_0) + \cdots + S_N(x_0)}{N+1} = \frac{1}{2}(f(x_0+0) + f(x_0-0)).$$

As f is continuous and S_N convergent, we get that

$$\begin{aligned}
f(x_0) &= \lim_{N\to\infty} \frac{S_0(x_0) + \cdots + S_n(x_0)}{N+1} \\
&= \lim_{N\to\infty} \frac{1}{N+1}\Big(c_0 + (c_0 + c_1 e^{2\pi i x_0} + c_{-1} e^{-2\pi i x_0}) + \cdots + \\
&\qquad + \cdots + (c_0 + \sum_{k=1}^{N} c_k e^{2\pi i k x_0} + c_k e^{-2\pi i k x_0})\Big) \\
&= \lim_{N\to\infty} \frac{(N+1)c_0}{N+1} + \frac{N(c_1 e^{2\pi i x_0} + c_{-1} e^{-2\pi i x_0})}{N+1} + \cdots + \\
&\qquad + \cdots + \frac{c_N e^{2\pi i N x_0} + c_{-N} e^{-2\pi i N x_0}}{N+1} \\
&= c_0 + \lim_{N\to\infty} \sum_{1\le n\le N} \frac{N+1-n}{N} (c_n e^{2\pi i n x_0} + c_{-n} e^{-2\pi i n x_0}) \\
&= c_0 + \sum_{n\in\mathbb{N}} (c_n e^{2\pi i n x_0} + c_{-n} e^{-2\pi i n x_0}).
\end{aligned}$$

If $\sum_{n\in\mathbb{Z}} |c_n| < \infty$ then $f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$.

Let us define $G(v) = \sum_{n\in\mathbb{Z}} f(n+v)$ is a continuous function of v of period 1. Then, we have

$$\begin{aligned} c_m &= \int_0^1 G(v)e^{2\pi imv}\,dv \\ &= \sum_{n\in\mathbb{Z}}\int_0^1 f(n+v)e^{-2\pi mv}\,dv \\ &= \sum_{n\in\mathbb{Z}}\int_n^{n+1} f(x)e^{-2\pi mx}\,dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi mx}\,dx \\ &= \hat{f}(m). \end{aligned}$$

□

Corollary C.5. *With f as above, we have*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k).$$

Proof. Take $v = 0$ on the previous proposition. □

Definition C.6. *The **Bernoulli numbers** can be defined in many different ways, here we will give a explicit formula for them:*

$$\begin{aligned} B_n^- &= \sum_{k=0}^{n}\sum_{v=0}^{k}(-1)^v\binom{k}{v}\frac{v^n}{k+1} \\ B_n^+ &= \sum_{k=0}^{m}\sum_{v=0}^{k}(-1)^v\binom{k}{v}\frac{(v+1)^n}{k+1}. \end{aligned}$$

It is possible to see that $B_n^- = B_n^+$ for all $n \neq 1$. For $n \neq 1$ odd, $B_n = 0$ and for n even, B_n is negative if n is divisible by 4 and positive otherwise.

*The **Bernoulli polynomials** can be defined by recurrence*

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k,$$

or by a generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

Note that $B_n = B_n(0) = B_n(1) = B_n$ for all n expect for $n = 1$, where $B_1^- = B_1(0) = -B_1(0) = -B_1^+$.

Proposition C.7 (Euler-Maclaurin Summation Formula)**.** *For m, n natural number and $f(x)$ a real or complex valued continuous function for real numbers x in the interval $[M, N]$, then, for $v \in \mathbb{N}_0$ such that f is $2v+1$ times continuously differentiable*

$$\sum_{i=M}^{N} f(i) = \int_M^N f(x)\,dx + \frac{f(M)+f(N)}{2} + \sum_{j=1}^{v} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(x)\Big|_M^N + R_{2v}$$

where

$$R_{2v} = \frac{1}{(2v+1)!}\int_M^N \overline{B_{2v+1}}(x) f^{(2v+1)}(x)\,dx$$

is the error term, $B_p(x)$ are the Bernoulli polynomials, $\overline{B_p}(x) = B_p(x - \lfloor x \rfloor)$ for $p > 1$ and $\overline{B_1}(x)$ agree with B_1 on the interval $[0, 1]$.

Proof. We will prove this by induction. Let us start with the case $v = 0$. Let k be an integer in $[M, N]$, and consider the integral

$$\int_k^{k+1} f(x)\,dx.$$

The integrand is of the form $u\,dv$, where $u = f(x)$ and $v = x + c$, where c is any constant. Let $c = -(k + \frac{1}{2})$, so $v = x - \lfloor x \rfloor - \frac{1}{2} = \overline{B_1}(x)$.

$$\begin{aligned}
\int_k^{k+1} f(x)\,dx &= \Big[f(x)\overline{B_1}(x)\Big]_k^{k+1} - \int_k^{k+1} f'(x)\overline{B_1}(x)\,dx \\
&= f(k+1)\overline{B_1}(k+1) - f(k)\overline{B_1}(k) - \int_k^{k+1} f'(x)P_1(x)\,dx \\
&= f(k+1)B_1(1) - f(k)B_1(0) - \int_k^{k+1} f'(x)\overline{B_1}(x)\,dx \\
&= \frac{f(k)+f(k+1)}{2} - \int_k^{k+1} f'(x)\overline{B_1}(x)\,dx.
\end{aligned}$$

We can then write

$$\begin{aligned}
\int_M^N f(x)\,dx &= \int_M^{M+1} f(x)\,dx + \cdots + \int_{N-1}^{N} f(x)\,dx \\
&= \frac{f(M)}{2} + f(M+1) + \cdots + f(N-1) + \frac{f(N)}{2} - \\
&\qquad - \int_M^N f'(x)\overline{B_1}(x)\,dx,
\end{aligned}$$

adding $\frac{f(N)+f(M)}{2}$ on both sides, we get

$$\int_M^N f(x)\,dx + \frac{f(N)+f(M)}{2} = \sum_{k=M}^{N} f(k) - \int_M^N f'(x)\overline{B_1}(x)\,dx.$$

Finally, we have

$$\sum_{k=M}^{N} f(k) = \frac{f(N)+f(M)}{2} + \int_M^N f(x)\,dx + \int_M^N f'(x)\overline{B_1}(x)\,dx.$$

Let us do the induction step next, let us assume it is true for $n-1$, that is

$$\sum_{k=M}^{N} f(k) = \int_M^N f(x)\,dx + \frac{f(M)+f(N)}{2} + \sum_{j=1}^{n-1} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(x)\Big|_M^N + \\ + \frac{1}{(2n-1)!}\int_M^N \overline{B_{2n-1}}(x) f^{(2n-1)}(x)\,dx.$$

Let us write for k integer in $[M,N]$.

$$\int_k^{k+1} \overline{B_{2n-1}}(x) f^{(2n-1)}(x)\,dx = \int_k^{k+1} u\,dv$$

then $u = f^{(2n-1)}(x)$ and $v = \frac{1}{2n}\overline{B_{2n}}(x)$. Thus, we have

$$\begin{aligned}\int_k^{k+1} \overline{B_{2n-1}}(x) f^{(2n-1)}(x)\,dx &= \left[\frac{1}{2n} f^{(2n-1)}(x)\overline{B_{2n}}(x)\right]_k^{k+1} \\ &\quad - \frac{1}{2n}\int_k^{k+1} f^{(2n)}(x)\overline{B_{2n}}(x)\,dx \\ &= \frac{f^{(2n-1)}(k+1)B_{2n}(0) - f^{(2n-1)}(k)B_{2n}(0)}{2n} - \\ &\quad - \frac{1}{2n}\int_k^{k+1} f^{(2n)}(x)\overline{B_{2n}}(x)\,dx \\ &= \frac{B_{2n}(0)(f^{(2n-1)}(k+1) - f^{(2n-1)}(k))}{2n} - \\ &\quad - \frac{1}{2n}\int_k^{k+1} f^{(2n)}(x)\overline{B_{2n}}(x)\,dx.\end{aligned}$$

Now, let us sum these integrals from M to N

$$\int_M^N \overline{B_{2n-1}}(x) f^{(2n-1)}(x)\,dx = \frac{B_{2n}(0)(f^{(2n-1)}(N) - f^{(2n-1)}(M))}{2n} - \\ - \frac{1}{2n}\int_M^N f^{(2n)}(x)\overline{B_{2n}}(x)\,dx.$$

Let us substitute this in the previous expression

$$\sum_{k=M}^{N} f(k) = \int_M^N f(x)\,dx + \frac{f(M)+f(N)}{2} + \sum_{j=1}^{n} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(x)\Big|_M^N - \\ - \frac{1}{(2n)!}\int_M^N \overline{B_{2n}}(x) f^{(2n)}(x)\,dx.$$

Now, let us repeat the same process for

$$\int_k^{k+1} \overline{B_{2n}}(x) f^{(2n)}(x)\,dx = \int_k^{k+1} u\,dv$$

where $u = f^{(2v)}(x)$, $dv = \overline{B_{2n}}(x)dx$ and $v = \frac{1}{2n+1}\overline{B_{2n+1}}(x)$. Then, we have

$$\begin{aligned}\int_k^{k+1} f^{2n}(x)\overline{B_{2n}}(x)\,dx &= \left[\frac{1}{2n+1} f^{2n}(x)\overline{B_{2n+1}}(x)\right]_k^{k+1} - \\ &\quad - \frac{1}{2n+1}\int_k^{k+1} f^{(2n+1)}(x)\overline{B_{2n+1}}(x)\,dx \\ &= \frac{f^{2n}(k+1)B_{2n+1}(0) - f^{2n}(k)B_{2n+1}(0)}{2n+1} - \\ &\quad - \frac{1}{2n+1}\int_k^{k+1} f^{(2n+1)}(x)\overline{B_{2n+1}}(x)\,dx \\ &= -\frac{1}{2n+1}\int_k^{k+1} f^{2n+1}(x)\overline{B_{2n+1}}(x)\,dx.\end{aligned}$$

Let us substitute this in the previous expression

$$\sum_{k=M}^{N} f(k) = \int_M^N f(x)\,dx + \frac{f(M)+f(N)}{2} + \sum_{j=1}^{n} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(x)\Big|_M^N + \\ + \frac{1}{(2n+1)!}\int_M^N \overline{B_{2n+1}}(x) f^{(2n+1)}(x)\,dx.$$

Hence, denoting $R_{2n} := \frac{1}{(2n+1)!}\int_M^N \overline{B_{2n+1}}(x) f^{(2n+1)}(x)\,dx$, the results follows. □

Appendix D

Complex Analysis

In this appendix, we gather some results of Complex Analysis, omitting the proof of the most well-known.

Theorem D.1 (Analytic Continuation). *Let f, g be two analytic functions in a connected open subset $\Omega \subset \mathbb{C}$. If there exists a non-empty open subset $U \subset \Omega$ such that, for all $s \in U$, $f(s) = g(s)$, then, for all $s \in \Omega$, we have $f(s) = g(s)$.*

Theorem D.2 (Cauchy's Residue Theorem). *Let Ω be an open set of $\mathbb{C}$ and S be a subset of Ω without cluster points of Ω. Let f be analytic in $\Omega \setminus S$ and K be a boundary-regular compact subset of Ω, such that ∂K does not contain any point of S. Then, S has only finitely many points in K and*

$$\frac{1}{2\pi i} \int_{\partial K} f(s)\, ds = \sum_{a \in S \cap K} \operatorname{Res}_{s=a} f(s).$$

Definition D.3. *We say a sequence $(x_n)_{n \in \mathbb{N}}$ is a **Cauchy sequence** if for all $\varepsilon > 0$, there is $N \in \mathbb{N}_0$ such that for all $m > n > N$ we have*

$$|x_m - x_n| < \varepsilon.$$

We say a series $\sum_{n=1}^{\infty} a_n$ if the sequence of partial sums $(\sum_{n=1}^{m} a_n)_{m \in \mathbb{N}}$ is a Cauchy sequence, that is, if for all $\varepsilon > 0$, there is $N \in \mathbb{N}_0$ such that for all $m > p > N$ we have

$$\left| \sum_{m < n \leq p} a_n \right| < \varepsilon.$$

Proposition D.4. *A series is convergent if and only if the partial sums $s_n := \sum_{i=1}^{n} a_i$ form a Cauchy sequence.*

Definition D.5. *A **harmonic function** is a twice continuously differentiable function $f \colon U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$, that satisfies Laplace's equation, that is,*

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

Proposition D.6 (Maximum principle for Harmonic functions)**.** *Let $f : U \to \mathbb{R}$ be an harmonic function. If K is a nonempty compact subset of U, then f restricted to K attains its maximum and minimum on the boundary of K.*

Theorem D.7 (Hadamard's Three Circles Theorem)**.** *Let $0 < r_1 \leq r_2 \leq r_3$ and $f : \mathbb{C} \to \mathbb{C}$ analytic inside and on an annulus $\Omega := \{s : r_1 \leq |s - s_0| \leq r_3\}$*

Let $M_i := \max\{|f(s)| : r_i = |s - s_0|\}$ for $i = 1, 2, 3$.

Then

$$M_2^{\log\left(\frac{r_3}{r_1}\right)} \leq M_1^{\log\left(\frac{r_3}{r_2}\right)} M_3^{\log\left(\frac{r_2}{r_1}\right)}.$$

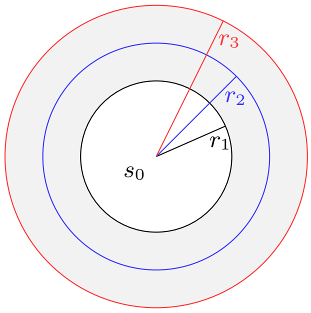

Figure D.1: The annulus Ω

Proof. [Bro17b]. We may assume $f \not\equiv 0$. So $M_i > 0$ for all i. First, let us assume that $f(s) \neq 0$ inside and on Ω. Define

$$H(s) := a \log |s - s_0| + b$$

where $a = \frac{\log M_3 - \log M_1}{\log r_3 - \log r_1}$ and $b = \frac{\log r_3 \log M_1 - \log r_1 \log M_3}{\log r_3 - \log r_1}$.

In the circle $|s - s_0| = r_3$, $H(s) = \log M_3 \geq \log |f(s)|$, while in the circle $|s - s_0| = r_1$, $H(s) = \log M_1 \geq \log |f(s)|$.

As the function $\log |f(s)|$ is harmonic inside and on Ω we can apply the Maximum Principle, Proposition D.6, and thus we have that $H(s) \geq \log |f(s)|$ throughout Ω, including the circle $r_2 = |s - s_0|$. So for s in this circle we have $H(s) = a \log r_2 + b \geq \log |f(s)|$, which means that

$$a \log r_2 + b \geq \log M_2.$$

Therefore, we have

$$\frac{(\log M_3 - \log M_1) \log r_2 + \log r_3 \log M_1 - \log r_1 \log M_3}{\log r_3 - \log r_1} \geq \log M_2.$$

Rearranging the terms, we have

$$\log M_3^{\log \frac{r_2}{r_1}} + \log M_1^{\log \frac{r_3}{r_2}} \geq \log M_2^{\log \frac{r_3}{r_1}}.$$

Thus,

$$\log\left(M_1^{\log\frac{r_3}{r_2}} M_3^{\log\frac{r_2}{r_3}}\right) \geq \log M_2^{\log\frac{r_3}{r_1}}.$$

If $f(s)$ has zeros in Ω, they must be in finite number, because f is not identically zero. Since in a neighbourhood of a zero, $\log|f(s)| \to -\infty$, we can delete an open disc of radius $\varepsilon > 0$ around each zero with ε sufficiently small so, since $H(s)$ is bounded, the inequality $H(s) \geq \log|f(s)|$ will apply on the boundary of the neighbourhoods, and thus throughout Ω minus the neighbourhoods. Letting $\varepsilon \to 0$ gives the result in this case. □

Theorem D.8 (Borel-Carathéodory)**.** *Let $f(s)$ be holomorphic on the closed disc $F := \overline{B(s_0, R)}$ and let $0 < r < R$. Then*

$$\max\{|f(s)| \colon |s - s_0| \leq r\} \leq \frac{2r}{R-r} \max\{\mathrm{Re}(f(s)) \colon |s - s_0| \leq R\} + \\ + \frac{R+r}{R-r}|f(s_0)|.$$

Proof. [Bro17b]. First, let us assume $f(s_0) = 0$. Let

$$M := \max\{\mathrm{Re}(f(s)) \colon |s - s_0| \leq R\}$$

and assume $M > 0$ (else replace f by $-f$). Let $H := \{s : \mathrm{Re}(s) \leq M\}$, so $f(F) \subset H$.

Define

$$g(s) := \frac{Rs}{s - 2M},$$

which is the composite of $s \to \frac{s}{M} - 1$ with $s \to \frac{R(s+1)}{s-1}$. So $g(0) = 0$ and g maps H into $B(0, R)$. Therefore $g(f(s_0)) = 0$ and $g(f(F)) \subset B(0, R)$. From the first statement, we get that $g \circ f(s) = (s - s_0)h(s)$, where $h(s)$ is holomorphic on Ω.

Let $0 < r < R$. By the maximum modulus principle in $B(s_0, r)$, there exist s_r with $|s_r - s_0| = r$ such that $\forall\, s$ with $|s - s_0| \leq r$, and we have

$$h(s) \leq |h(s_r)| = \frac{|(g \circ f)(s_r)|}{|s_r - s_0|} \leq \frac{R}{r}.$$

Taking $r \to R$, $|h(s)| \leq 1$. So, we have, for $|z| \leq r$

$$\frac{|Rf(s)|}{|f(s) - 2M|} = |(g \circ f)(s)| = \ abss|h(s)| \leq |s|.$$

Thus

$$\frac{|Rf(s)|}{|f(s) - 2M|} \leq |s| \leq r.$$

Therefore

$$R|f(s)| \leq r|f(s) - 2M| \leq r|f(s)| - 2Mr.$$

Finally, we get

$$(R-r)|f(s)| \leq 2Mr,$$

or, alternatively,

$$|f(s)| \leq \frac{2Mr}{R-r}.$$

Now, substituting $f(s)$ by $f(s)-f(s_0)$, we get

$$\begin{aligned}|f(s)|-|f(s_0)| &\leq |f(s)-f(s_0)| \\ &\leq \frac{2r}{R-r}\max\{\mathrm{Re}(f(s)-f(s_0))\colon |s|\leq R\} \\ &\leq \frac{2r}{R-r}(M+|f(s_0)|)\end{aligned}$$

and the result follows. □

Theorem D.9 (Lindelöf's bound)**.** *Let a, b, c be real with $a < b$ and $c > 0$. Let $\kappa(\sigma)$ be an affine function with values $\kappa(a) = \alpha, \kappa(b) = \beta$. Let $f(s)$ be holomorphic inside and on the infinite rectangle*

$$\Omega = [a, b] \times [c, \infty[$$

and such that it satisfy the bounds $|f(s)| \leq At^{\alpha}$ when $\sigma = a$ and $|f(s)| \leq Bt^{\beta}$ when $\sigma = b$. Suppose also that $f(s)$ satisfies a growth condition $|f(s)| \leq Ct^{D}$ on Ω. Then exists a constant K such that for all $s \in \Omega$

$$|f(\sigma + it)| \leq Kt^{\kappa(\sigma)}.$$

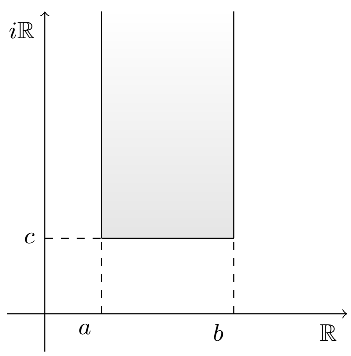

Figure D.2: The infinite rectangle $[a, b] \times [c, +\infty]$

Proof. [Bro17b]. If f is identically zero, the theorem is trivial. First let us assume $f(s) \neq 0$ for all $s \in \Omega$. From $|f(s)| \leq Ct^{d}$, we get

$$\log|f(s| \leq D\log t + \log C.$$

Let $\varepsilon > 0$ be given. Then exists T_0 such that $\forall T \geq T_0$

$$-(\kappa(\sigma) + \varepsilon)T < -D \log T - \log C$$

Let $g(s) := \log |f(s)| - (\kappa(\sigma)+\varepsilon)t$ be a harmonic function on the rectangle $[a, b] \times [c, T] \subset \Omega$. Then exists constant M such that, on the vertical line $\sigma = a$ we have

$$g(s) \leq \log A + \alpha \log t - (\alpha + \varepsilon)t \leq M,$$

on the line $\sigma = b$, we get

$$g(s) \leq \log B + \beta \log t - (\beta + \varepsilon)t \leq M,$$

on the horizontal line $t = T$ we have

$$\begin{aligned} g(s) &\leq \log |f(\sigma + iT)| - (\kappa(\sigma) + \varepsilon)T \\ &\leq D \log T + \log C - (\kappa(\sigma) + \varepsilon)T \leq M, \end{aligned}$$

where the final inequality holds for all $T \geq T_0$, and on the line $t = c$, we have

$$g(s) \leq D \log c + \log C - (\kappa(\sigma) + \varepsilon)c \leq M.$$

Thus, by the maximum modulus principle, we get $g(s) \leq M$ on Ω. Therefore, we have

$$g(s) = \log |f(s)| - (\kappa(\sigma + \varepsilon))t \leq M.$$

Hence

$$|f(s)| \leq e^M e^{\kappa(\sigma)+\varepsilon}.$$

This hold for each $\varepsilon > 0$, so setting $K = e^M$ and letting $\varepsilon \to 0$, the result follows.

Now, if $f(s)$ has zeros in Ω, they must be in finite number, because f is not identically zero. In the same manner of Theorem $D.7$, we can delete an open disc of radius $\varepsilon > 0$ around each zero with ε sufficiently small such that the inequality $g(s) \leq M$ will apply on the boundary of the neighbourhoods, and thus throughout Ω minus the neighbourhoods. Letting $\varepsilon \to 0$ gives the result in this case. □

Lemma D.10. *For a, c bounded and $b \to \infty$, we have*

$$\frac{\log\left(1 + \frac{a}{b}\right)}{\log\left(1 + \frac{c}{b}\right)} = \frac{a}{c} + \mathcal{O}\left(\frac{1}{b}\right)$$

Proof. We want to show that

$$\left| \frac{\log\left(1 + \frac{a}{b}\right)}{\log\left(1 + \frac{c}{b}\right)} - \frac{a}{c} \right| < M\frac{1}{b}$$

for all $b \geq b_0$, for some b_0 and M.

It is known that, for $|x| < 1$, we have

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots.$$

So, let us choose b_0 such that for $b \geq b_0$, we have both $\left|\frac{a}{b}\right| < 1$ and $\left|\frac{c}{b}\right| < 1$. Then we can write,

$$\frac{\log\left(1+\frac{a}{b}\right)}{\log\left(1+\frac{c}{b}\right)} = \frac{\frac{a}{b} - \frac{1}{2}\left(\frac{a}{b}\right)^2 + \frac{1}{3}\left(\frac{a}{b}\right)^3 - \dots}{\frac{c}{b} - \frac{1}{2}\left(\frac{c}{b}\right)^2 + \frac{1}{3}\left(\frac{c}{b}\right)^3 - \dots}.$$

So, we want to show that

$$\left|\frac{\frac{a}{b} - \frac{1}{2}\left(\frac{a}{b}\right)^2 + \frac{1}{3}\left(\frac{a}{b}\right)^3 - \dots}{\frac{c}{b} - \frac{1}{2}\left(\frac{c}{b}\right)^2 + \frac{1}{3}\left(\frac{c}{b}\right)^3 - \dots} - \frac{a}{c}\right| < M\frac{1}{b}.$$

First, let's simply the difference.

$$\frac{\frac{a}{b} - \frac{1}{2}\left(\frac{a}{b}\right)^2 + \frac{1}{3}\left(\frac{a}{b}\right)^3 - \dots}{\frac{c}{b} - \frac{1}{2}\left(\frac{c}{b}\right)^2 + \frac{1}{3}\left(\frac{c}{b}\right)^3 - \dots} - \frac{a}{c} =$$

$$= \frac{\frac{ac}{b} - \frac{c}{2}\left(\frac{a}{b}\right)^2 + \frac{c}{3}\left(\frac{a}{b}\right)^3 - \dots - \left(\frac{ac}{b} - \frac{a}{2}\left(\frac{c}{b}\right)^2 + \frac{a}{3}\left(\frac{c}{b}\right)^3 - \dots\right)}{\left(\frac{c}{b} - \frac{1}{2}\left(\frac{c}{b}\right)^2 + \frac{1}{3}\left(\frac{c}{b}\right)^3 - \dots\right)c} =$$

$$= \frac{\frac{1}{b^2}\left(-\frac{a^2c}{2} + \frac{1}{3}\frac{a^3c}{b} - \frac{1}{4}\frac{a^4c}{b^2} + \dots + \frac{ac^2}{2} - \frac{1}{3}\frac{ac^3}{b} + \frac{1}{4}\frac{ac^4}{b^2} - \dots\right)}{\frac{c}{b}\left(c - \frac{1}{2}\frac{c^2}{b} + \frac{1}{3}\frac{c^3}{b^2} - \dots\right)} =$$

$$= \frac{1}{b}\left(\frac{ac\left(-\frac{a}{2} + \frac{1}{3}\frac{a^2}{b} - \frac{1}{4}\frac{a^3}{b^2} + \dots + \frac{c}{2} - \frac{1}{3}\frac{c^2}{b} + \frac{1}{4}\frac{c^3}{b^2} - \dots\right)}{c\left(c - \frac{1}{2}\frac{c^2}{b} + \frac{1}{3}\frac{c^3}{b^2} - \dots\right)}\right) =$$

$$= \left(\frac{a\left[\left(-\frac{1}{2}\frac{a}{b} + \frac{1}{3}\frac{a^2}{b^b} - \frac{1}{4}\frac{a^3}{b^3} + \dots\right) + \left(\frac{1}{2}\frac{c}{b} - \frac{1}{3}\frac{c^2}{b^2} + \frac{1}{4}\frac{c^3}{b^3} - \dots\right)\right]}{c\left(1 - \frac{1}{2}\frac{c}{b} + \frac{1}{3}\frac{c^2}{b^2} - \dots\right)}\right).$$

As $b \to \infty$ we have

$$1 - \frac{1}{2}\frac{c}{b} + \frac{1}{3}\frac{c^2}{b^2} - \dots = \frac{\log\left(1+\frac{c}{b}\right)}{\frac{c}{b}} \to 1,$$

$$-\frac{1}{2}\frac{a}{b} + \frac{1}{3}\frac{a^2}{b^b} - \frac{1}{4}\frac{a^3}{b^3} + \dots = \frac{\log\left(1+\frac{a}{b}\right)}{\frac{a}{b}} - 1 \to 0,$$

$$\frac{1}{2}\frac{c}{b} - \frac{1}{3}\frac{c^2}{b^2} + \frac{1}{4}\frac{c^3}{b^3} - \ldots = -\frac{\log\left(1+\frac{c}{b}\right)}{\frac{c}{b}} + 1 \to 0.$$

Therefore, we have

$$\frac{\frac{a}{b} - \frac{1}{2}\left(\frac{a}{b}\right)^2 + \frac{1}{3}\left(\frac{a}{b}\right)^3 - \ldots}{\frac{c}{b} - \frac{1}{2}\left(\frac{c}{b}\right)^2 + \frac{1}{3}\left(\frac{c}{b}\right)^3 - \ldots} - \frac{a}{c} \to 0$$

and the desired result follows, because $\frac{1}{b}$ also converges to 0. □

Bibliography

[Apo76] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer, 1976.

[Art64] Emil Artin. *The Gamma Function.* Holt, Rinehart and Winston, 1964.

[Boa97] H. P. Boas. “The Football Player and the Infinite Series”. In: *Notices Amer. Math. Soc.* 44.11 (1997), pp. 1430–1435.

[Boh13a] H. Bohr. “Darstellung der gleichmäßigen Konvergenzabszisse einer Dirichletschen reihe $\sum_{n=0}^{\infty} \frac{a_n}{n^s}$ als Funktion der Koeffizienten der Reihe”. In: *Archiv der Mathematik und Physik* vol. 21.3 (1913), pp. 326–330.

[Boh13b] H. Bohr. “Über die gleichmäßige Konvergenz Dirichletscher Reihen”. In: *Journal fur die Reine und Angewandte Mathematik* vol. 1913.143 (1913), pp. 203–211.

[Bor20] Olivier Bordellés. *Arithmetic Tales.* Springer, 2020.

[BH16] Ole Fredrik Brevig and Winston Heap. “Convergence Abscissas for Dirichlet Series With Multiplicative Coefficients”. In: *Expositiones Mathematicae* 34.4 (2016), pp. 448–453.

[Bro17a] Kevin Broughan. *Equivalents of the Riemann Hypothesis I: Arithmetic Equivalents 1.* Cambridge University Press, 2017.

[Bro17b] Kevin Broughan. *Equivalents of the Riemann Hypothesis II: Arithmetic Equivalents 2.* Cambridge University Press, 2017.

[CE59] E. D. Cashwell and C. J. Everett. “The Ring of Number-theoretic Functions”. In: *Pacific Journal of Mathematics* 9.4 (1959), pp. 975–985.

[Con78] John B. Conway. *Functions of One Complex Variable.* 2nd ed. Springer, 1978.

[Edw01] Harold M. Edwards. *Riemann’s Zeta Function.* Dover Publications, 2001.

[Jan96] Gerald J. Janusz. *Algebraic Number Fields.* 2nd ed. American Mathematical Society, 1996.

[Lan02] Serge Lang. *Algebra*. Springer, 2002.

[McC86] P. J. McCarthy. *Introduction to Arithmetical Functions*. Springer, 1986.

[Mur01] M. Ram Murty. *Problems in Analytic Number Theory*. Springer, 2001.

[Siv89] R. Sivaramkrishnan. *Classical Theory Of Arithmetic Functions*. Marcel Dekker, 1989.

[TH87] E. C. Titchmarsh and D. R. Heath-Brown. *The Theory of the Riemann Zeta-Function*. 2nd ed. Oxford University Press, 1987.

[Zyg59] Anthony Zygmund. *Trigonometric Series*. 2nd ed. Vol. I. Cambridge University Press, 1959.