THE MATHEMATICS OF IVO ROSENBERG

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Abstract. This paper is dedicated to the memory of the distinguished scholar and friend Professor I.G .Rosenberg. We survey some of his most well known and not so known results, as well as present some new ones related to the study of maximal partial clones and their intersections.

1. Preliminaries

After a long struggle with Parkinson's disease, yet hard working and keeping a very kind spirit, Professor Ivo G. Rosenberg died on January 18, 2018. He was born in Brno (Moravia, Czech Republic), on December 13, 1934. One year before obtaining the doctoral degree, he published his Completeness Criterion [75] in 1965. In July 1966, Professor Rosenberg received his doctoral degree on his very same wedding day. Together with his wife Vlasta (known as Loty to all family friends), he moved to Saskatoon, Canada, in September 1968. After the birth of their two children, Madeleine (known as Misha) and Marc, the family moved to Montréal in September 1971. Ivo held the position of researcher with the rank of Full Professor from 1971 to 1983 at the "Centre de recherches mathématiques de l'Université de Montréal", and later he was appointed as a full Professor at the "Département de mathématiques et de statistique" from 1983 to August 2011. Ivo retired in August 2011.

Professor Ivo G. Rosenberg was an eminent scholar, brilliant mathematician, and one of the leading experts in universal algebra and discrete mathematics. His huge impact and contributions to these areas of mathematics are very hard to assess due to the great extension and ramifications of his works. According to [61], Ivo Rosenberg wrote more than 200 papers and several books, and the number of his co-authors exceeds 55, among which are the three authors of this paper.

Ivo G. Rosenberg contributed to more than 50 conference papers in the various International Symposia on Multiple-Valued Logic. Some of his contributions to these symposia are listed and discussed in [61]. He was extremely generous towards his colleagues and provided great hospitality both personally and through his position at the University of Montréal. He wrote numerous research reports and doctoral thesis reports.

For his outstanding research and contributions to mathematics, Ivo G. Rosenberg received two honorary doctorates: the first from the Technische Universit́at Wien (Vienna Polytechnic) in 2006 and the second from the Universit́at Rostock (Rostock University) in 2013.

Ivo organized several scientific meetings in Montréal and elsewhere, notably many sessions of the well known "Séminaire de Mathématiques Supérieures de l'Université de Montréal" like, for example, the "24e session: Algèbre universelle et relations" (Montréal, July 23–August 10, 1984), and the "30e session "Algèbres et ordres"

(Montréal, July 29–August 9, 1991) as well as the "premier colloque montréalais sur la combinatoire et l'informatique" (Montréal, April 27–May 2, 1987). Together with Maurice Pouzet, he organized the conference "Aspects of Discrete Mathematics and Computer Science" (Lyon, June 24–27, 1987) within the first "Entretiens du Centre Jacques-Cartier". The second "Entretiens", called "Discrete Mathematics and Computer Science" (Montréal, October 12–14, 1988), took place in Montréal the year after. It was organized by Ivo Rosenberg together with A. Achache, M. Pouzet (Lyon) and G. Hahn (Montréal). Still within his collaboration with French colleagues, Ivo organized a French-Quebec project "Ordered sets and their applications" from 1984 to 1986. Moreover, Ivo organized several colloquia such as **(1)** *Algebraic, Extremal and Metric Combinatorics*, (1986) with M. Deza and P. Frankl (Proceedings published in London Mathematical Society Lecture Notes Series, **131**), **(2)** *Algebras and Orders* (1991) with Gert Sabidussi (Proceedings published in Nato Advanced Study Institute Series, Math. and Phys. Sci., **389**), and **(3)** *Structural Theory of Automata, Semigroups and Universal Algebra*, (2003), with V. B. Kudryavtsev (Proceedings published in Nato Science Series, Math. Phys. and Chem. **207**).

Several of his students became prominent mathematicians and computer scientists, to mention a few: Anne Fearnley, Jens-Uwe Grabowski, Lucien Haddad, Simone Hazan, Sebastian Kerkhoff, Gérard Kientega, Benoît Larose, Maxime Lauzon, Florence Magnifo, Qinghe Sun, Bogdan Szczepara and Calvin Wuntcha.

In this paper we briefly survey some of the most influential work in mathematics of Professor Ivo G. Rosenberg, rather than giving an exhaustive account of his contributions. Our presentation extends that of [15]. For further information see, e.g., [34, 38, 61].

2. Clone theory

Professor Ivo G. Rosenberg has been extremely active in research and became one of the world's leading mathematicians in universal algebra and discrete mathematics. We recall some of his most well known results in this field.

2.1. **Completeness in Multiple-Valued Logic.**

Ivo became known in the 70's for his general completeness criterion in multiple-valued logic. It was first presented in 1965 [75] and the full proof was published in [76]. Rosenberg's completeness criterion is one of the most fundamental results in multiple valued logic and universal algebra, and its impact is impossible to estimate given the ever increasing number of research papers that followed it.

Let $\mathbf{k} := \{0, 1, \dots, k-1\}$, $\mathrm{Op}(\mathbf{k})$ be the set of all operations on $\mathbf{k}$ and let Π_k be the set of all projections on $\mathbf{k}$. For $X \subseteq \mathrm{Op}(\mathbf{k})$, we denote by $\mathrm{Clone}(X)$ the set of all operations on $\mathbf{k}$ that can be realized by composing functions from $X \cup \Pi_k$. A set $X \subseteq \mathrm{Op}(\mathbf{k})$ is a *clone* if $\mathrm{Clone}(X) = X$ and X is said to be *complete* if $\mathrm{Clone}(X) = \mathrm{Op}(\mathbf{k})$. Thus X is complete if and only if any function $f : \mathbf{k}^n \to \mathbf{k}$ can be realized by composing functions from $X \cup \Pi_k$. For a relation ρ on $\mathbf{k}$, let $\mathrm{Pol}\ (\rho)$ be the set of all functions $f \in \mathrm{Op}(\mathbf{k})$ preserving ρ.

As a closure system, the set of all clones constitutes a lattice under $C \wedge C' = C \cap C'$ and $C \vee C' = \mathrm{Clone}(C \cup C')$, and where $\mathrm{Op}(\mathbf{k})$ and Π_k are its greatest and least elements. It is also well known that this lattice is atomic and co-atomic, and its atoms and co-atoms are referred to as *minimal* and *maximal* clones, respectively.

For the purpose of presenting Rosenberg's classification of all maximal clones over $\mathbf{k}$, we recall some families of relations on $\mathbf{k}$. For $1 \leq h \leq k$, let

$$\iota_k^h := \{(a_1, \ldots, a_h) \in \mathbf{k} \mid a_i = a_j \text{ for some } 1 \leq i < j \leq h\}.$$

Let ρ be an h-ary relation on $\mathbf{k}$ and denote by S_h the set of all permutations on $\{0, \ldots, h-1\}$. For $\pi \in S_h$ set

$$\rho^{(\pi)} := \{(x_{\pi(1)}, \ldots, x_{\pi(h)}) \mid (x_1, \ldots, x_h) \in \rho\}.$$

The h-ary relation ρ is said to be

(1) *totally symmetric* (or simply *symmetric* in the case $h = 2$) if $\rho^{(\pi)} = \rho$ for every $\pi \in S_h$,
(2) *totally reflexive* (in case $h = 2$ *reflexive*) if $\iota_k^h \subseteq \rho$;
(3) *prime affine* if $h = 4$, $\mathbf{k} = p^m$ where p is a prime number, $m \geq 1$, $\mathbf{p} := \{0, \ldots, p-1\}$ and we can define an elementary abelian p-group $(\mathbf{k}, +)$ on $\mathbf{k}$ so that
$$\rho := \{(\vec{a}, \vec{b}, \vec{c}, \vec{d}) \in \mathbf{k}^4 \mid \vec{a} + \vec{b} = \vec{c} + \vec{d}\};$$
(4) *central*, if $\rho \neq \mathbf{k}^h$, ρ is totally symmetric, totally reflexive and $\{c\} \times \mathbf{k}^{h-1} \subseteq \rho$ for some $c \in \mathbf{k}$. Notice that, for $h = 1$, each $\emptyset \neq \rho \subset \mathbf{k}$ is central. For $h \geq 2$ such c is called a *central element* of ρ;
(5) *elementary*, if $k = h^m$, $h \geq 3$, $m \geq 1$ and
$$(a_1, a_2, \ldots, a_h) \in \rho \ \Leftrightarrow\ \forall i = 0, \ldots, m-1, (a_1^{[i]}, a_2^{[i]}, \ldots, a_h^{[i]}) \in \iota_h^h,$$
where $a^{[i]}$ ($a \in \{0, 1, \ldots, h^m - 1\}$) denotes the i-th term in the h-adic expansion;
$$a = a^{[m-1]} \cdot h^{m-1} + a^{[m-2]} \cdot h^{m-2} + \cdots + a^{[1]} \cdot h + a^{[0]};$$
(6) a *homomorphic inverse image of an h-ary relation ρ' on $\mathbf{k}'$*, if there is a surjective mapping $q : \mathbf{k} \to \mathbf{k}'$ such that
$$(a_1, \ldots, a_h) \in \rho \ \Leftrightarrow\ (q(a_1), \ldots, q(a_h)) \in \rho'$$
for all $a_1, \ldots, a_h \in \mathbf{k}$,
(7) *h-universal*, if ρ is a homomorphic inverse image of an h-ary elementary relation.

Furthermore, we denote by

- $\mathcal{C}_k$ the set of all central relations on $\mathbf{k}$;
- $\mathcal{C}_k^h$ the set of all h-ary central relations on $\mathbf{k}$;
- $\mathcal{U}_k$ the set of all non-trivial equivalence relations on $\mathbf{k}$;
- $P_{k,p}$ the set of all fixed-point-free permutations on $\mathbf{k}$ consisting of cycles of the same prime length p;
- $\mathcal{S}_{k,p}$ the set defined by
$$\mathcal{S}_{k,p} := \{s^0 \mid s \in P_{k,p}\},$$
where $s^0 := \{(x, s(x)) \mid x \in \mathbf{k}\}$ is the *graph* of s;
- $\mathcal{S}_k$ the set defined by
$$\mathcal{S}_k := \bigcup \{\mathcal{S}_{k,p} \mid p \ \text{ is a prime divisor of } \ k\};$$
- $\mathcal{M}_k$ the set of all order relations on $\mathbf{k}$ with a least and a greatest element;
- $\mathcal{M}_k^\star$ the set of all lattice orders on $\mathbf{k}$;
- $\mathcal{L}_k$ the set of all prime affine relations on $\mathbf{k}$;

- $\mathcal{B}_k$ the set of all h-universal relations, $3 \leq h \leq k-1$.

Endowed with this terminology and notation, we can now state Rosenberg's celebrated classification of maximal clones on $\mathbf{k}$.

Theorem 1. ([75, 76]) *Let $k \geq 2$. Every proper clone on $\mathbf{k}$ is contained in a maximal one. Moreover a clone M is a maximal clone over $\mathbf{k}$ if and only if $M = \mathrm{Pol}\,\rho$ for some relation $\rho \in \mathcal{C}_k \cup \mathcal{M}_k \cup \mathcal{S}_k \cup \mathcal{U}_k \cup \mathcal{L}_k \cup \mathcal{B}_k$.*

Since every clone $C \neq \mathrm{Op}(\mathbf{k})$ is contained in a maximal clone on $\mathbf{k}$, we have the following completness criterion.

Theorem 2. *A set of functions $X \subseteq \mathrm{Op}(\mathbf{k})$ is complete if and only if for every relation ρ described in Theorem 1, X contains a function f not preserving ρ.*

Several variant completeness criteria have been obtained from Theorem 1. For example, we say that $X \subseteq \mathrm{Op}(\mathbf{k})$ is *complete with constants* if the set $X \cup \{c_a \mid a \in \mathbf{k}\}$ is complete, where $c_a : \mathbf{k} \to \mathbf{k}$ is the unary constant map with value $\{a\}$. Here are two examples of sets that are complete with constants.

A *simple group* is a non-trivial group with only two normal (trivial) subgroups. A *simple ring* is a non-zero ring (i.e., there are x, y such that xy is not the zero element) with no non-trivial double-sided ideals.

Corollary 3. *(1) A finite group is complete with constants if and only if it is a simple non-abelian group.*
(2) A finite ring is complete with constants if and only if it is a simple ring.

2.2. **Minimal clones.** Recall that a clone C is said to be *minimal* if it is an atom in the lattice of all clones or, equivalently, if it is generated by any of its non-trivial functions. A non-trivial function of smallest arity in a minimal clone is called a *minimal function*. The following result is one of the deepest in the minimal clone literature. It is due to Ivo and it was first published in [79].

Theorem 4. *Let f be a minimal function of arity n over $\mathbf{k}$. Then f satisfies one of the following conditions:*
(1) $n = 1$ and f satisfies $f^2 = f$ or $f^p(x) = x$ for some prime number p,
(2) $n = 2$ and f is an idempotent function, i.e., $f(x,x) = x$,
(3) $n = 3$ and f is a majority function, i.e., $f(x,x,y) = f(x,y,x) = f(y,x,x) = x$,
(4) $n = 3$ and $f(x,y,z) = x+y+z$, where $(\mathbf{k},+)$ is an elementary 2-group, or
(5) $n > 2$ and f is an n-ary semiprojection, i.e., there ixists an $i \in \{1,\ldots,n\}$ such that

$$f(x_1,\ldots,x_n) = x_i \quad \text{whenever} \quad |\{x_1,\ldots,x_n\}| < n.$$

Many research efforts have been devoted to the study of minimal clones because, among other things, of their connection to strongly rigid relations. This is discussed in Section 3. Further fruitful applications of minimal clones and their descriptions can be found in the theory of constraint satisfaction problems (CSPs) [48] and the theory of essential arguments of functions [4, 10, 16, 81, 86].

3. Partial clones

Ivo G. Rosenberg made significant contributions to many other areas of multiple-valued logic, for example, to the theory of partial clones. We will mainly focus on one of his main contributions to this domain, namely, the completeness criterion for finite partial algebras and some related topics.

3.1. **Completeness Criterion for Finite Partial Algebras.** Denote by $\mathrm{Par}(\mathbf{k})$ the set of all partial functions on $\mathbf{k}$ and for a relation ρ over $\mathbf{k}$, we denote by $\mathrm{pPol}\,\rho$ the set of all partial functions $f \in \mathrm{Par}(\mathbf{k})$ that preserve ρ. We need to introduce some terminology to state the *Completeness Criterion for Finite Partial Algebras.*

A relation ρ on $\mathbf{k}$ *extremal* if $M := \mathrm{pPol}\,\rho$ is a maximal partial clone and if ρ is of minimal size among all relations that determine the same maximal partial clone M. Let E_h denote the set of all equivalence relations on the set $\mathbf{h}$, and let ω_h be the smallest element in E_h, i.e., $\omega_h := \{(x,x) \mid x \in \mathbf{h}\}$. For $\varepsilon \in E_h$, we write

$$\Delta_\varepsilon := \{(x_0,\dots,x_{h-1}) \in \mathbf{k}^h \mid (i,j) \in \varepsilon \Rightarrow x_i = x_j\}.$$

We often denote Δ_ε by $\Delta_{X_1,\dots,X_n}$, where $X_1,\dots,X_n$ are the nonsingleton equivalence classes of ε. Moreover, we define

$$\Gamma_k^h := \bigcup_{0 \le i < j \le h-1} \Delta_{\{i,j\}}.$$

Notice that Γ_k^h is the set of all tuples $(a_0,\dots,a_{h-1}) \in \mathbf{k}^h$ such that $a_i = a_j$ for some $0 \le i < j \le h-1$.

An h-ary relation ρ is said to be

(a) *diagonal* if there exists $\varepsilon \in E_h$ such that $\rho = \Delta_\varepsilon$,
(b) *areflexive* if $\rho \cap \Delta_\varepsilon = \emptyset$ for each $\varepsilon \in E_h$, $\varepsilon \neq \omega_h$, i.e., for all $(x_0,\dots,x_{h-1}) \in \rho$, $x_i \neq x_j$ holds for all $0 \le i < j \le h-1$,
(c) *quasi-diagonal* if $\rho = \sigma \cup \Delta_\varepsilon$ where σ is a non-empty areflexive relation, $\varepsilon \in E_h \setminus \{\omega_h\}$, and in addition, $\rho \neq \mathbf{k}^2$ if $h = 2$,
(d) *totally reflexive* if $\Gamma_k^h \subseteq \rho$.

Let σ be an h-ary relation on $\mathbf{k}$, and suppose that there is a subgroup G of S_h such that $\sigma = \sigma^{(\pi)}$ for all $\pi \in G$ and $\sigma \cap \sigma^{(\alpha)} = \emptyset$ for all $\alpha \in S_h \setminus G$. Then G is called the *group of symmetries* of the relation σ. An h-ary relation ρ is *totally symmetric* if S_h is its group of symmetries, i.e., if

$$(x_0,\dots,x_{h-1}) \in \rho \Leftrightarrow (x_{\pi(0)},\dots,x_{\pi(h-1)}) \in \rho,\ \forall \pi \in S_h.$$

The following quaternary relations on $\mathbf{k}$ play an important role in the study of maximal partial clones (see Theorem 5). Let

$$\begin{aligned}
R_1 &:= \Delta_{\{0,1\},\{2,3\}} \cup \Delta_{\{0,2\},\{1,3\}} \cup \Delta_{\{0,3\},\{1,2\}},\\
R_2 &:= \Delta_{\{0,1\},\{2,3\}} \cup \Delta_{\{0,3\},\{1,2\}},\\
R_3 &:= \Delta_{\{0,1\},\{2,3\}} \cup \Delta_{\{0,2\},\{1,3\}}, \text{ and}\\
R_4 &:= \Delta_{\{0,2\},\{1,3\}} \cup \Delta_{\{0,3\},\{1,2\}}.
\end{aligned}$$

Observe that $(x_0,x_1,x_2,x_3) \in R_2$ if and only if $[x_0 = x_1$ and $x_2 = x_3]$ or $[x_0 = x_3$ and $x_1 = x_2]$.

Now let σ be an areflexive h-ary relation and let $F \subset E_h$. Put $G_\sigma := \{\pi \in S_h \mid \sigma \cap \sigma^{(\pi)} \neq \emptyset\}$ and suppose that the h-ary relation ρ is of the form

$$\rho = \sigma \cup \Big(\bigcup_{\varepsilon \in F} (\Delta_\varepsilon)\Big).$$

Then the *model* of ρ is the h-ary relation

$$M(\rho) := \{(\pi(0),\dots,\pi(h-1)) \mid \pi \in G_\sigma\} \cup \left(\bigcup_{\varepsilon\in F}\{(x_0,\dots,x_{h-1}) \in \mathbf{h}^h \mid (i,j)\in\varepsilon \Rightarrow x_i = x_j\}\right)$$

on the set $\mathbf{h}$.

Furthermore, suppose that h, F and σ satisfy one of the following five conditions:

(1) $h \geq 2$, $F = \emptyset$ and $\sigma \neq \emptyset$, i.e., ρ is a nonempty h-ary areflexive relation;
(2) $h \geq 2$, $F = \{\varepsilon\}$ where $\varepsilon \neq \omega_h$, $\sigma \neq \emptyset$ and $\sigma \cup \Delta_\varepsilon \neq \mathbf{k}^2$, i.e., ρ is a nontrivial quasi-diagonal h-ary relation;
(3) $h = 4$ and

$$F = \{\{[0,1],[2,3]\},\{[0,3],[1,2]\},\{[0,2],[1,3]\}\},$$

i.e., $\rho = \sigma \cup R_1$, where σ is a (possibly empty) areflexive 4-ary relation;
(4) $h = 4$ and $F = \{\{[\pi(0),\pi(1)],\ [\pi(2),\pi(3)]\},\ \{[\pi(0),\pi(3)],\ [\pi(1),\ \pi(2)]\}\}$, where $\pi \in S_4$, i.e., $\rho = \sigma \cup R_i$, where $i = 2,3,4$ and σ is a (possibly empty) areflexive 4-ary relation;
(5) $h \neq 2$, $h \leq k$, $F = \bigcup_{i<j}\{i,j\}$ and $\rho \neq \mathbf{k}^h$, i.e., ρ is a totally reflexive and totally symmetric nontrivial relation.

Then the h-ary relation ρ is said to be *coherent* if

(A) the following conditions hold:
  (1) when either of conditions (1) or (2) are satisfied,

$$G_\sigma = \{\pi \in S_h \mid \sigma^{(\pi)} = \sigma\} \text{ and}$$
$$\pi(\varepsilon) := \{(\pi(x),\pi(y)) \mid (x,y)\in\varepsilon\} = \varepsilon,\ \forall\, \pi \in G_\sigma;$$

  (2) when condition (3) is satisfied,

$$G_\sigma = \{\pi \in S_4 \mid \sigma^{(\pi)} = \sigma\} = \{\pi \in S_4 \mid \pi(F) = F\} = S_4;$$

  (3) when condition (4) is satisfied,

$$G_\sigma = \{\pi \in S_4 \mid \sigma^{(\pi)} = \sigma\} = \{\pi \in S_4 \mid \pi(F) = F\};$$

  (4) when condition (5) is satisfied,

$$G_\sigma = \{\pi \in S_h \mid \sigma^{(\pi)} = \sigma\} = S_h; \text{ and}$$

(B) for every non-empty subrelation $\sigma' \subseteq \sigma$, there exists a relational homomorphism $\gamma : \mathbf{k} \to \mathbf{h}$ from σ' to $M(\rho)$ such that $(\gamma(i_0),\ \dots,\ \gamma(i_{h-1})) = (0,\ \dots,\ h-1)$ for at least one h-tuple $(i_0,\dots,i_{h-1}) \in \sigma'$.

The description of all maximal partial clones on a k-element set as given in [43] follows.

Theorem 5 ([39, 43, 36])**.** *Let $k \geq 2$. Every proper partial clone on $\mathbf{k}$ extends to a maximal one. If M is a maximal partial clone on $\mathbf{k}$, then either*

$$C = \mathcal{O}_\mathbf{k} \cup \{f \in \mathcal{P}_\mathbf{k} \mid \text{dom}\,(f) = \emptyset\}$$

or M is determined by an extremal relation on $\mathbf{k}$*. Furthermore a relation ρ is an extremal relation on* $\mathbf{k}$ *if and only if it is of one of the following types of relations:*

(1) *an h-ary areflexive or quasi-diagonal relation which is coherent and $h \geq 2$,*
(2) *an h-ary non-trivial totally reflexive and totally symmetric relation and $h \neq 2$,*
(3) *one of the quaternary relations R_1 or R_2,*
(4) *a quaternary coherent relation $\sigma \cup R_i$ where $i = 1, \ldots, 4$ and $\sigma \neq \emptyset$ is a quaternary areflexive relation.*

Remark 6. *The number of maximal partial clones on a finite set* $\mathbf{k}$ *greatly exceeds the number of maximal clones on* $\mathbf{k}$*. For example, Theorem 1 says that any order relation that is bounded defines a maximal clone, while Theorem 5 says that any non-trivial order relation defines a maximal partial clone on* $\mathbf{k}$*. Let $\mathcal{M}_k$ and $p\mathcal{M}_k$ be the families of all maximal and maximal partial, respectively, clones on* $\mathbf{k}$*. It is known (see, e.g.,* [31, 83]*) that $|\mathcal{M}_2| = 5$ and $|p\mathcal{M}_2| = 8$, $|\mathcal{M}_3| = 18$ and $|p\mathcal{M}_3| = 58$, $|\mathcal{M}_4| = 82$ and $|p\mathcal{M}_4| = 1102$, $|\mathcal{M}_5| = 634$ and $|p\mathcal{M}_5| = 325,722$, $|\mathcal{M}_6| = 15,182$ and $|p\mathcal{M}_6| = 5,242,621,816$.*

3.2. **Contributions to the understanding of the lattice of partial clones.** Besides the completeness problem for partial algebras, Ivo G. Rosenberg deeply contributed to the research on partial clones on a finite set. For space limitation, we will mention only one main contribution to the study of clones of partial Boolean functions.

Let $\mathbf{k} = \{0, 1\}$, and $\mathcal{I}(SM)$ be the interval of all partial clones containing the self-dual and monotone functions on $\{0, 1\}$. In the late 2000's it was widely believed that this interval must be finite. A breakthrough was obtained when the second author constructed an infinite set of partial clones contained in the interval $\mathcal{I}(SM)$ [35], proving that the interval is actually *infinite*.

Later, the first two authors together with Ivo G. Rosenberg proved that this interval is not just infinite, but it has the cardinality of the continuum ([13]). Ivo came with the main construction of this paper, and that construction was adapted to different situations and led to the complete solution, in the case of Boolean functions, to the following open problem in clone theory due to D. Lau:

Problem 1. *For each total clone C on* $\mathbf{k}$*, describe the interval of partial clones*

$$\mathcal{I}(C) := \{D \mid D \text{ is a partial clone on } \mathbf{k} \text{ and } D \cap O_k = C\}.$$

This problem was considered by several authors and many partial results were obtained towards its solution. It is shown in [14] that if C is a clone on $\{0, 1\}$, then the interval $\mathcal{I}(C)$ is finite for finitely many clones C, and that $\mathcal{I}(C)$ has the cardinality of the continuum, otherwise. See [14] for the complete description of clones C for which $\mathcal{I}(C)$ is finite.

4. Rigidity and projectivity

Many works of Ivo G. Rosenberg deal with several notions of rigidity and projectivity. We briefly survey a few in this section.

A relational structure $R := (A, (\rho_i)_{i \in I})$ on a set A is said to be:

- *rigid* if the identity is the only endomorphism of R;
- *strongly rigid* if the projections are the only maps of several variables which preserve R;

- *semirigid* if the only unary functions that preserve ρ are the identity and all constant maps;
- *n-projective* if the only idempotent n-ary functions that preserve R are the projections.

4.1. **Semirigidity.** Recall that R is *semirigid* if the functions that preserve R are either the projections or the constant maps (Langer and Pöschel).

Relational systems of equivalence relations are prototypes of semirigid systems, since, if a set E has at least three elements, only the constant functions and the identity preserve all equivalence relations on E. From this it follows that if a set $\{\rho_i : i \in I\}$ generates by means of joins and meets the lattice of equivalences on E then $M := (E, (\rho_i)_{i\in I})$ is semirigid. The converse does not hold. Indeed, according to Strietz [84], if E is finite with at least four elements, four equivalences are needed to generate the lattice of equivalence relations on E and Zádori [87] has described for every set E, whose size $|E|$ is finite and distinct from 2 and 4, a semirigid system made of three equivalence relations. Several constructions of semirigid systems of equivalence relations are presented in [63]. A general method of constructing semirigid systems of three equivalence relations on sets of cardinality at most the continuum and distinct from 2 and 4, is developed in [18].

4.2. **Semirigidity, orthogonality and primality.** Demetrovics, Miyakawa, Rosenberg, Simovici and Stojmenović [21] introduced the following notion: two orders ρ and τ on the same set E are *orthogonal* if (E, τ, ρ) is semirigid. Nozaki, Miyakawa, Pogosyan and Rosenberg [64] investigated the existence of a linear order orthogonal to a given finite linear order. They observed that there is always one, provided that the number of elements is not equal to 3. They proved:

Theorem 7. *The proportion $q(n)/n!$ of linear orders orthogonal to the natural order on $[n] := \{1, ..., n\}$ goes to $e^{-2} = 0.1353...$ when n goes to infinity.*

Their counting argument was based on the fact that two linear orders on the same *finite* set are orthogonal if and only if they do not have a common nontrivial interval. The notion capturing the properties of intervals of a linear order was extended a long time ago to posets, graphs and binary structures and a decomposition theory has been developed (see, e.g., [26, 27, 28, 33]). One of the terms proposed for this notion is *autonomous set*; structures with no nontrivial autonomous subsets (the building blocks in the decomposition theory) are called *prime* (or *indecomposable*). With this terminology, two linear orders ρ and τ on the same finite set V are orthogonal if and only if the binary structure $B := (V, \rho, \tau)$, called a *bichain*, is prime. This leads to results relating primality and orthogonality ([73], [88]).

The notion of primality and Theorem 7 reappeared in recent years under quite a different setting: a study of permutations motivated by the Stanley-Wilf conjecture, now settled by Marcus and Tardos [62]. This study was pursueded in many papers, and can be presented as follows. To a permutation σ on $[n]$ associate first the linear order $\leq_\sigma$ defined by $x \leq_\sigma y$ if $\sigma(x) \leq \sigma(y)$ for the natural order on $[n]$. Then, associate the bichain $B_\sigma := ([n], (\leq, \leq_\sigma))$. On the set $\mathfrak{S} := \cup_{n\in\mathbb{N}} \mathfrak{S}_n$ of all permutations, set $\sigma \leq \tau$ if B_σ is embeddable into B_τ. Say that a subset $\mathfrak{C}$ of $\mathfrak{S}$ is *hereditary* if $\sigma \leq \tau$ and $\tau \in \mathfrak{C}$ imply $\sigma \in \mathfrak{C}$. The goal is to evaluate the growth rate of the function $\varphi_{\mathfrak{C}}$ that counts, for each integer n, the number $\varphi_{\mathfrak{C}}(n)$ of permutations σ on $[n]$ which belong to $\mathfrak{C}$ (the Stanley-Wilf conjecture asserted that $\varphi_{\mathfrak{C}}$ is bounded by an exponential if $\mathfrak{C} \neq \mathfrak{S}$).

For this purpose, simple permutations were introduced. A permutation σ is *simple* if $\leq_\sigma$ and the natural order $\leq$ on $[n]$ have no nontrivial interval in common. Arbitrary permutations being obtained by means of simple permutations, the enumeration of permutations belonging to a hereditary class of permutations can be then reduced to the enumeration of simple permutations belonging to that class. This fact was illustrated in many papers: see [1, 52]; see also [5] for a survey on simple permutations, and [2], where the asymptotic result mentioned in Theorem 7 is rediscovered.

Research on classes of finite permutations led to the study of infinite bichains, in particular, of the prime ones. However, in the infinite, primality and orthogonality no longer coincide. A series of papers [82, 55, 19] that culminated in [20], described pairs of orthogonal countable chains.

4.3. **Projectivity.** Corominas [11] introduced the notion of 2-projectivity for ordered sets (posets). For posets, 2-projectivity is equivalent to n-projectivity [68]. Larose, then a student Professor Rosenberg, showed that for P with at least three elements, projectivity is equivalent to the apparently weaker notion of quasi-projectivity[1] (projections beeing replaced by quasi-projections, i.e., maps f such that $f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$) [56, 57]. With Tardif [58] he obtained the same conclusion for graph (with no loops). More results of the same nature were obtained by Hazan [47], another student of Ivo G. Rosenberg, and by Delhommé [20] for posets and reflexive graphs. In [68] it was shown that a relational structure R on a set A is 2-projective, and not n-projective for some $n > 2$, if and only if its clone is the set of functions that preserve the congruences of a 2-elementary group on a set of at least 4 elements. This leads to a proof of a variant of Arrow's Theorem [3, 66]. Later, Pouzet and Rosenberg [70], provided an alternative proof of this result on projectivity as a consequence of an extension of Rosenberg's classification of minimal clones presented in subsection 2.2.

5. Generalizations of Metric spaces

Ivo wrote several papers on generalizations of metric spaces. The motivation behind [69] was the similarity observed, notably by Quilliot (1983), between some properties of metric spaces and of ordered sets and graphs. A striking example is the similarity between the Sine-Soardi Fixed-Point Theorem for "contracting" (or non-expansive) mappings on a bounded hyperconvex metric space (1979) and the Tarski Fixed-Point Theorem for order preserving maps on a complete lattice (1955). A basic reason for a similarity between these objects is that metric spaces can be turned into binary relational structures, and that in this transformation contracting mappings become relational homomorphisms.

For a full understanding of this connexion, metric spaces with distances valued in an ordered monoid, say V, were introduced. With some mild conditions, V can be equipped with a distance d with values in V, and every metric space over V can be embedded isometrically into some power of (V, d). From this, the characterization of hyperconvex metric spaces obtained by Aronszajn and Panitchpakdi (1956) for ordinary metric spaces extends to this new type of metric spaces. From the study of the clone of contracting operations, the analogous notion of hyperconvexity is presented in terms of the extension property of contracting operations. In one of

[1]Also called *quasitriviality* or *conservativeness* in the literature.

the last papers of Ivo [50], the existence of an injective envelope of a generalized metric space is applied to show that some monoids of subsets on the free monoid are free. A generalization of metric spaces to sets equipped with a n-ary map into an ordered monoid is considered. Composition of contracting operations corresponds to the composition of Mealy automata. Examples including ordered sets, graphs and transition systems are presented. For a recent survey on these generalizations of metric spaces, see [49], and for further generalizations, see [22]. With Gérard Kientega, his student, Ivo G. Rosenberg showed [51] that generalized metric spaces can be induced by tolerance relations of a given algebra.

Ivo G. Rosenberg wrote several papers with Michel Deza, two of which on generalizations of distances to $(n+1)$-ary maps from a set into the set of reals. In [23] they introduce n-semimetrics generalizing the notion of n-metric, a fairly old notion that has received a lot of attention. A map $d : E^{n+1} \to \mathbb{R}_+$ is an *n-semimetric* on E if it is invariant under all permutations of the coordinates and satisfies the simplex inequality:

$$d(x_1, \ldots, x_{n+1}) \leq \sum_{i=1}^{n+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+2}),$$

for all $(x_1, \ldots, x_{n+2}) \in E^{n+2}$. For $n = 2$ the simplex inequality captures the fact that in $\mathbb{R}^3$, the area of a triangle face of a tetrahedron does not exceed the sum of the areas of the remaining three faces. Some examples of n-semimetrics on the Boolean m-cube $\{0,1\}^m$ are constructed and their relations to combinatorics and coding theory are mentioned.

In [24], they introduce *m-hemimetrics* (the map d above may have negative values). For small values of m and n, they considered the cones of all m-hemimetrics, of all nonnegative m-hemimetrics, and the cone generated by all partition m-hemimetrics on $\{1, \ldots, n\}$. The facets and generators for these cones are determined by computer search and, in particular, they studied the 1-skeleton and the ridge graph of these polyhedra. Note also that recent work on generalizations of distances by Marichal [53, 54] and his collaborators, the so-called n-distances, are based on the simplex inequalities.

6. Poset of orbits for a group action

Let G be a group of permutations on a set E. The set $O(G)$ of orbits $Orb(a)$ of subsets a of E can be ordered as follows: $Or(a) \leq O(b)$ if $\sigma(a) \subseteq b$ for some $\sigma \in G$. If E is finite, this poset is ranked and the function that counts the number of elements of each level is symmetric and unimodal (Livingstone and Wagner 1965). In [67] it was shown that this poset has the *strong Sperner property*, that is, for every k, the union of k of the largest level is a maximal sized k-family, a result obtained independently by Harper (1984) and Stanley (1984). In fact, Pouzet's result was slightly more general. For that, hereditary equivalence relations were introduced.

An equivalence relation $\sim$ on the set $\mathcal{P}(E)$ of subsets of a set E is called *hereditary* if $|a| = |b|$ and $|\{x \subseteq a : x \sim c\}| = |\{x \subseteq b : x \sim c\}|$, for all a, b, c whenever $a \sim b$. For instance, let R be a relational structure on a E, and set $a \sim b$ if the induced substructure $R_{\restriction a}$ and $R_{\restriction b}$ are isomorphic. Then this equivalence is hereditary. In particular, if G is a group of permutations of E, then the equivalence into orbits is hereditary. The set of equivalence classes of an hereditary equivalence has a poset structure, and if E is finite, then it decomposes into levels. It was shown that the strong Sperner property holds if this poset is symmetric. It was conjectured that in this case this poset was the poset of orbits of a group. A negative answer was given by Buchwalder [7, 8], then a student of Adrien Bondy.

7. Further research results

We finish this incomplete account of the many contributions of Ivo G. Rosenberg by briefly discussing some selected results in extremal set theory, geometry, and graph and number theory.

7.1. Extremal set theory.

Let $X_1, X_2, \dots, X_m$ be finite sets, and let $A := X_1 \cup X_2 \cup \dots \cup X_m$ be the *ground set*. Ivo G. Rosenberg and Michel Deza found several conditions that guarantee that the ground set has size at least m (the number of sets considered).

Theorem 8. [25] *Suppose that for every $i \neq j$, $|X_i \cap X_j|$ is odd $\iff$ both $|X_i|$ and $|X_j|$ are even. If $|X_1|+|X_2|+\dots+|X_m|$ has the same parity as m, then the ground set A must have at least m elements.*

Ivo G. Rosenberg collaborated with Peter Frankl on Extremal Set Theory. One of their main results is discussed in the paper "A finite set intersection theorem" (Theorem 2.6) by Peter Frankl. The paper is included in the present volume dedicated to the memory of Ivo G. Rosenberg.

7.2. Geometry and finite geometry.

Ivo also contributed to the study of (finite) geometries. To illustrate, Consider a triangle ABC and suppose that this triangle is bisected by the line joining the midpoint M of the longest side BC to the opposite vertex A. The two obtained triangles AMB and AMC are also bisected by the same procedure. The four thus obtained triangles are also bisected by the same procedure. And so forth...

Theorem 9. [80] *Let θ be an angle of any arbitrary triangle of the above sequence. Then $\theta \geq \alpha/2$.*

Together with Z. Füredi, Ivo extended the Bruijn - Erdös Theorem [6] in incidence Geometry, as follows. A *colored incidence structure* is a system $(S, \{P_1, \dots, P_t\}, \mathcal{L})$, where

(1) S is a finite set,
(2) $P_1, \dots, P_t$ are pairwise disjoint non-empty subsets of S,
(3) $\mathcal{L}$ is a collection of subsets of S, called *lines*, and

fulfilling the two following conditions:

(1) each line of $\mathcal{L}$ meets at least two of P_i and P_j,
(2) For $x \in P_i$ and $y \in P_j$, $x \neq y$, if:
    - $i \neq j$ then the pair $\{x, y\}$ belongs to exactly one line of $\mathcal{L}$, and
    - $i = j$ then $\{i, j\}$ belongs to at most one line of $\mathcal{L}$.

Notice that a projective plane is a colored incidence structure, where every set P_i is a single point. The question is then to determine how many lines are needed for a colored incidence structure to exist.

Theorem 10. [32] *Let* $(S, \{P_1, \ldots, P_t\}, \mathcal{L})$ *be a colored incidence structure such that* $|P_1| \leq |P_2| \leq \cdots \leq |P_t|$*. Then* $|\mathcal{L}| \geq 1 + |P_1| + \cdots + |P_{t-1}|$*. Furthermore, the equality* $|\mathcal{L}| = 1 + |P_1| + \cdots + |P_{t-1}|$ *holds in exactly six cases, one of which is the projective planes.*

7.3. **Graph theory.** Suppose that in a (finite) group of people, any pair of persons have exactly one common friend. Then there is always a person who is everybody's friend. A *friendship graph* is a graph where two distinct vertices have exactly one common neighbour. Finite Friendship graphs have been characterized by Erdös *et al.* in 1966.

Ivo and colleagues constructed as many non-isomorphic infinite friendship graphs as there are non-isomorphic infinite graphs with no cycle of length 4. In fact, they proved the following result.

Theorem 11. [9] *Let* G *be a graph. Then there is a graph* $Ext(G)$ *such that:*

(1) if G *is infinite then* $Ext(G)$ *is infinite with same cardinality as* G*,*
(2) if G *has chromatic number at least 3, then* $Ext(G)$ *has same chromatic number as* G*,*
(3) f G *contains no cycle of length 4, then* $Ext(G)$ *is a friendship graph.*

7.4. **A result in number theory.** In 1962, Ivo (while a graduate student) established several results in number theory, involving sums of the form

$$\sum_{m=1}^{p} \left(\frac{f(m)}{p} \right),$$

where f is a polynomial, p is an odd prime number, and $\left(\frac{a}{p}\right)$ is the *Legendre Symbol*. His results culminated in the following noteworthy theorem.

Theorem 12. [74] *Let* $p = 2(2d)t + 1$ *be a prime number. Then* p *is the sum of* $2d$ *squares, i.e., there are positive integers* $a_1, a_2, \ldots, a_{2d}$*, such that*

$$p = a_1^2 + a_2^2 + \cdots + a_{2d}^2.$$

Now if $p = 2(2d)t + 1$ and $d = 1$, then we get $p = 4t + 1$, and thus Ivo's result is a generalization of *Fermat's Christmas Theorem* that says the following: *A prime number* $p \geq 5$ *is a sum of two squares if and only if* p *has form* $4t + 1$*, for some* $t \geq 1$. For instance, 2,333,233 is a prime number and $2,333,233 = 2(6)194,436 + 1$. By Theorem 12, 2,333,233 is the sum of six squares. Indeed,

$$2,333,233 = 370^2 + 502^2 + 609^2 + 616^2 + 706^2 + 834^2.$$

8. Conclusion

In [38], several colleagues wrote heartful tributes to Ivo Rosenberg as a mathematician but especially as a wonderful human being. Ivo will always be remembered not only for his huge contributions to mathematics, but also for all his human values, for the way he treated colleagues, visitors, young researchers and graduate students. He will be remembered for his generosity, his modesty and for his extreme kindness. As our late colleague Dietlinde Lau described him, *Ivo G. Rosenberg was a real gentleman of the old school.* May his soul rest in peace.

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