Display and Hilbert Calculi for Atomic and Molecular Logics

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Abstract

Sound and strongly complete display calculi for basic atomic and molecular logics are introduced with a Kripke-style relational semantics. These logics are based on Dunn's gaggle theory and generalize modal logics. We also provide sound and strongly complete Hilbert calculi for basic atomic logics with a Kripke-style relational semantics. All these calculi can be automatically computed from the definition of the connectives constituting a basic atomic or molecular logic, yet with some restrictions on the class of molecular logics.

1 Introduction

Gaggle logics were introduced in [2]. Their definition was directly based on and inspired by Dunn's gaggle theory [13, 14]. The main motivation for introducing them was to provide a basic logical framework that would be expressive and accurate enough so as to deal with and represent as faithfully and adequately as possible any non-classical logic, in the spirit of the "universal logic" paradigm [9, 10]. However, since then, we realized that in order to achieve this objective, we need to generalize them. More precisely, one needs to be able to take as primitive, connectives which are compositions of basic gaggle connectives and one needs to add types to the formulas of our gaggle logics. These types correspond semantically to sizes of tuples of states in a model.

This led us to introduce in [5] atomic and molecular logics. They behave as 'normal forms' for logics. We indeed showed in [5] that every non-classical logic such that the truth conditions of its connectives are expressible in first-order logic is as expressive as an atomic or a molecular logic. We also proved that first-order logic is as expressive as a specific atomic logic. Moreover, from a model-theoretic point of view, invariance notions for atomic and molecular logics can be defined systematically from the truth conditions of their connectives and when those are uniform we obtain automatically a van Benthem characterization theorem for the logic considered [6]. These results support formally our claim that atomic and molecular logics are somehow 'universal'.

In a sense, atomic (and molecular) logics are a generalization and a 'realization' into a logical framework of Dunn's Gaggle theory. Given the close connections between gaggle theory and display calculi [32], if one wants to develop the proof theory of these logics, it is natural to start with the investigation of display calculi, using the results of Dunn's gaggle theory to obtain the display rules. Our main contribution in this article is to introduce display and Hilbert calculi for basic atomic and molecular logics which are sound and complete *w.r.t. a Kripke-style relational semantics*. An important feature of our approach is that, like for their bisimulation notions, all our calculi can be automatically computed from the definition of the connectives of the atomic or molecular logics. As for display calculi of molecular logics, they are restricted to molecular connectives which are universal or existential. As for Hilbert calculi of (Boolean) atomic logics, the languages considered should include the closure of the connectives under an (appropriate) action group, that is their orbit under this action.

Organization of the article. We start in Section 2 by recalling some basics of group theory. These prerequisites will play a role in the definition of our display calculi. In Sections 3, 4 and 5 we introduce our atomic and molecular logics. In Section 6, we recall common logical notions and terminology. In Sections 7 and 8 we introduce our display and Hilbert calculi respectively. We end in Section 9 by discussing related work and conclude.

Note. The article is self–contained. It is the first part of a series of articles on the proof and correspondence theory of atomic and molecular logics. This series continues with [4, 3]. All the proofs are in the appendix.

2 Notions of group theory

We first recall some basics of group theory (see for instance [34] for more details).

Permutations and cycles. If X is a non-empty set, a *permutation* of X is a bijection $\sigma : X \to X$. We denote the set of all permutations of X by $\mathfrak{S}_X$. In the important special case when $X = \{1, \dots, n\}$, we write $\mathfrak{S}_n$ instead of $\mathfrak{S}_X$. Note that $|\mathfrak{S}_n| = n!$, where $|Y|$ denotes the number of elements in a set Y. A permutation σ on the set $\{1, \dots, n\}$ such that $\sigma(1) = x_1, \sigma(2) = x_2, \dots, \sigma(n) = x_n$ is denoted $(x_1, x_2, \dots, x_n)$. For example, $(1, 3, 2)$ is the permutation σ such that $\sigma(1) = 1, \sigma(2) = 3$ and $\sigma(3) = 2$.

If $x \in X$ and $\sigma \in \mathfrak{S}_X$, then σ *fixes* x if $\sigma(x) = x$ and σ *moves* x if $\sigma(x) \neq x$. Let $j_1, \dots, j_r$ be distincts integers between 1 and n. If $\sigma \in \mathfrak{S}_n$ fixes the remaining $n - r$ integers and if $\sigma(j_1) = j_2, \sigma(j_2) = j_3, \dots, \sigma(j_{r-1}) = j_r, \sigma(j_r) = j_1$ then σ is an *r–cycle*; one also says that σ is a cycle of *length* r. Denote σ by $(j_1\ j_2\ \dots\ j_r)$. A 2–cycle which merely interchanges a pair of elements is called a *transposition*.

Two permutations $\sigma, \tau \in \mathfrak{S}_X$ are *disjoint* if every x moved by one is fixed by the other. A family of permutations $\sigma_1, \sigma_2, \dots, \sigma_n$ is *disjoint* if each pair of them is disjoint. Every permutation $\sigma \in \mathfrak{S}_n$ is either a cycle or a product of disjoint cycles. Moreover, this factorization is unique except for the order in which the factors occur.

Groups. A *group* $(G, \circ)$ is a non–empty set G equipped with an associative operation $\circ : G \times G \to G$ and containing an element denoted Id_G called the *neutral element* such that: $\mathrm{Id}_G \circ a = a = a \circ \mathrm{Id}_G$ for all $a \in G$; for every $a \in G$, there is an element $b \in G$ such that $a \circ b = \mathrm{Id}_G = b \circ a$. This element b is unique and called the *inverse* of a, denoted a^{-1}. The set $\mathfrak{S}_n$ with the composition operation is a group called the *symmetric group on n letters*.

A non–empty subset S of a group G is a *subgroup* of G if $s \in S$ implies $s^{-1} \in S$ and $s, t \in S$ imply $s \circ t \in S$. In that case, S is also a group in its own right.

If X is a subset of a group G, then the smallest subgroup of G containing X, denoted by $\langle X \rangle$, is called the *subgroup generated by X*. For example, $\mathfrak{S}_n = \langle (1\ 2), (2\ 3), \dots, (i\ i+1), \dots, (n-1\ n) \rangle = \langle (n\ 1), (n\ 2), \dots, (n\ n-1) \rangle = \langle (n-1\ n), (1\ 2\ \dots\ n) \rangle$. $\mathfrak{S}_n$ is also generated by $(1\ 2)$ and 3–cycles. For $n \geq 3$, the *alternating group* $\mathfrak{A}_n$ is the subgroup of $\mathfrak{S}_n$ generated by the n–cycles of $\mathfrak{S}_n$.

In fact, if X is non–empty, then $\langle X\rangle$ is the set of all the words on X, that is, elements of G of the form $x_1^{\pm_1}x_2^{\pm_2}\dots x_n^{\pm_n}$ where $x_1,\dots,x_n \in X$ and $\pm_1,\dots,\pm_n$ are either -1 or empty.

Free groups and free products. If X is a subset of a group F, then F is a *free group* with *basis* X if, for every group G and every function $f: X \to G$, there exists a unique homomorphism $\varphi: F \to G$ extending f. One can prove that a free group with basis X always exists and that X generates F. We therefore use the notation $F = \langle X\rangle$ also for free groups.

If G and H are groups, the *free product* of G and H is a group P and homomorphisms j_G and j_H such that, for every group Q and all homomorphisms $f_G: G \to Q$ and $f_H: H \to Q$, there exists a unique homomorphism $\varphi: P \to Q$ with $\varphi j_G = f_G$ and $\varphi j_H = f_H$. Such a group always exists and it is unique modulo isomorphism, we denote it $G * H$. This definition can be generalized canonically to the case of a finite number of groups $G_1,\dots,G_n$, yielding the free product $G_1 * \dots * G_n$.

Group actions. If X is a set and G a group, an *action of* G *on* X is a function $\alpha: G\times X \to X$ given by $(g,x) \mapsto gx$ such that: $\mathrm{Id}x = x$ for all $x \in X$; $(g_1g_2)x = g_1(g_2x)$ for all $x \in X$ and all $g_1, g_2 \in G$. If $x \in X$ and α an action of a group G on X, then the *orbit* of x under α is $\mathcal{O}_\alpha(x) \triangleq \{\alpha(g,x) \mid g \in G\}$. The orbits form a partition of X.

Fact 2.1. *If α is an action of G on a set X and H is a subgroup of G, then the restriction of α to H, denoted α_H, is also an action of H on the set X.*

Definition 2.2. Let G and H be two groups. If α and β are actions of G and H on a set X, then the *free action* $\alpha * \beta$ is the mapping $\alpha * \beta : G * H \times X \to X$ given by $\alpha * \beta(g,x) \triangleq \alpha(g_1, \beta(h_1,\dots,\alpha(g_n,\beta(h_n,x))))$, where $g = g_1h_1\dots g_nh_n$ is the factorization of g in the free group $G * H$.

This definition can be generalized canonically to the case of a finite number of actions $\alpha_1,\dots,\alpha_n$, yielding the mapping $\alpha_1 * \dots * \alpha_n$.

Proposition 2.3. *If $\alpha_1,\dots,\alpha_n$ are actions of $G_1,\dots,G_n$ on a set X respectively, then the mapping $\alpha_1 * \dots * \alpha_n$ is an action of the (free) group $G_1 * \dots * G_n$ on X.*

3 Atomic logics

The truth conditions of the connectives of atomic logics are defined by first-order formulas of the form $\forall x_1 \dots x_n(\pm_1\mathsf{P}_1x_1 \vee \dots \vee \pm_n\mathsf{P}_nx_n \vee \pm\mathsf{R}x_1\dots x_nx)$ or $\exists x_1\dots x_n(\pm_1\mathsf{P}_1x_1 \wedge \dots \wedge \pm_n\mathsf{P}_nx_n \wedge \pm\mathsf{R}x_1\dots x_nx)$ where $\pm_i$ is either empty or $\neg$. We will represent the structure of these formulas by means of so–called *skeletons* whose various arguments capture the different features that allow us to redefine them completely.

Definition 3.1 (Atomic skeletons)**.** The sets of *atomic skeletons* $\mathbb{P}$ and $\mathbb{C}$ are defined as follows:

$$\mathbb{P} \triangleq \mathfrak{S}_1 \times \{+,-\} \times \{\forall,\exists\} \times \mathbb{N}^*$$
$$\mathbb{C} \triangleq \mathbb{P} \cup \bigcup_{n\in\mathbb{N}^*} \{\mathfrak{S}_{n+1} \times \{+,-\} \times \{\forall,\exists\} \times \mathbb{N}^* \times {\mathbb{N}^*}^n \times \{+,-\}^n\}\,.$$

$\mathbb{P}$ is called the set of *atom skeletons* and $\mathbb{C}$ is called the set of *connective skeletons*. They can be represented by tuples $(\sigma, \pm, Æ, \overline{k}, \mp_j)$ or $(\sigma,\pm,Æ,k)$ if it is a propositional letter skeleton, where $Æ \in \{\forall,\exists\}$ is called the *quantification signature* of the skeleton, $\overline{k} = (k, k_1,\dots,k_n) \in {\mathbb{N}^*}^{n+1}$ is called the *type signature* of the skeleton and $\overline{\mp_j} = (\pm_1, \ldots, \pm_n) \in \{+,-\}^n$ is called the *tonicity signature* of the skeleton; $(Æ, \overline{k}, \overline{\mp_j})$ is called the *signature* of the skeleton. The *arity* of a skeleton $\circledast \in \mathbb{C}$ is n, its *input types* are $k_1, \ldots, k_n$ and its *output type* is k. The set of n–ary connective skeletons, for $n > 0$, is denoted $\mathbb{C}_n$.

We define the mapping $\cdot : \{+,-\} \times \{+,-\} \to \{+,-\}$ by $\cdot(+,-) = \cdot(-,+) = -$ and $\cdot(-,-) = \cdot(+,+) = +$ ($(\{+,-\},\cdot)$ is therefore isomorphic to the group $\mathbb{Z}/2\mathbb{Z}$). For better readability, for all $\pm, \pm' \in \{+,-\}$, we write $\pm\pm'$ for $\cdot(\pm,\pm')$. Informally, $\forall$ is associated with $+$ and $\exists$ is associated with $-$. We formalize this association with the function $\pm : \{\forall,\exists\} \to \{+,-\}$ defined by $\pm(\forall) \triangleq +, \pm(\exists) \triangleq -$ and the inverse function $Æ : \{+,-\} \to \{\forall,\exists\}$ defined by $Æ(+) \triangleq \forall, Æ(-) \triangleq \exists$. Also, we define the function $+ : \{\forall,\exists\} \to \{\forall,\exists\}$ by $+(\forall) \triangleq \forall$ and $+(\exists) \triangleq \exists$ and the function $- : \{\forall,\exists\} \to \{\forall,\exists\}$ by $-(\forall) \triangleq \exists$ and $-(\exists) \triangleq \forall$. For better readability, we write $+\forall, +\exists, -\forall, -\exists$ instead of $+(\forall), +(\exists), -(\forall), -(\exists)$.

Definition 3.2 (Action of the symmetric group)**.** Let $n \in \mathbb{N}^*$. We define the function $\alpha_n : \mathfrak{S}_{n+1} \times \mathbb{C}_n \to \mathbb{C}_n, (\tau, \star) \mapsto \tau\star$ inductively as follows. Let $\star = \big(\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n)\big) \in \mathbb{C}_n$ and let $c \in \mathfrak{S}_{n+1}$.

- If c is the transposition $r_j = (j\ n+1)$, then

$$r_j\star \triangleq \big((j\ n+1) \circ \sigma, -\pm_j \pm, -\pm_j Æ, \overline{k}, (-\pm_j \pm_1, \ldots, \pm_j, \ldots, -\pm_j \pm_n)\big)\,. \tag{1}$$

  The connective r_j is called the *residual of* $\star$ *w.r.t. its* j^{th} *argument*.

- If c is the cycle $(j_1\ j_2\ \ldots\ j_k\ n+1)$, then $c\star \triangleq r_{j_1}(r_{j_2} \ldots (r_{j_k}\star))$, where $r_j \triangleq (j\ n+1)$ for all j.

- If c is a cycle fixing $n+1$, then

$$c\star \triangleq \big(c \circ \sigma, \pm, Æ, \overline{k}, \big(\pm_{c(1)}, \pm_{c(2)}, \ldots, \pm_{c(n)}\big)\big)\,. \tag{2}$$

Finally, if τ is an arbitrary permutation of $\mathfrak{S}_{n+1}$, it can be factorized into a product of disjoint cycles $\tau = c_1c_2\ldots c_k$ and this factorization is unique (modulo its order) [34]. So, we define $\tau\star \triangleq c_1(c_2 \ldots (c_k\star))$.

Our definition is based on cycles and not on transpositions because the decomposition of any permutation into disjoint cycles is unique modulo its order, unlike its decomposition into transpositions. The mapping α_n is well-defined [2].

Proposition 3.3. *For all $n \in \mathbb{N}^*$, the mapping $\alpha_n : \mathfrak{S}_{n+1} \times \mathbb{C}_n \to \mathbb{C}_n$ is a group action of $\mathfrak{S}_{n+1}$ on $\mathbb{C}_n$.*

Definition 3.4 (Atomic connectives)**.** An *(atomic) connective* or *propositional letter* is a symbol to which is associated a connective skeleton or a propositional letter skeleton respectively. Its arity, signature, quantification signature, type signature, tonicity signature, input and output types are the same as its skeleton. By abuse, we sometimes identify a connective with its skeleton. If C is a set of atomic connectives, its set of atoms is denoted $\mathbb{P}(\mathsf{C})$. If $\circledast$ is a connective, its skeleton is denoted $\star$.

Every set of connectives C is partitioned into a set of *orbits* $\mathcal{O}$ such that for all connectives $\circledast, \circledast'$ belonging to the same orbit $\mathcal{O}$ of skeletons $\star = (\sigma, \pm, \overline{k}, \overline{\mp_j})$ and $\star' = (\sigma', \pm', \overline{k}', \overline{\mp_j}')$, there are $\tau_1, \ldots, \tau_n \in \mathfrak{S}_{n+1}$ such that $\tau_1 - \ldots \tau_{n-1} - \tau_n\star = \star'$. In that case, we also write that $\tau_1 - \ldots \tau_{n-1} - \tau_n\circledast = \circledast'$ and $\circledast'$ is sometimes defined by this relation.

Propositional letters are denoted p, p_1, p_2, *etc.*, connectives are denoted $\circledast, \circledast_1, \circledast_2$, *etc.* and skeletons are denoted $\star, \star_1, \star_2$, *etc.*

Definition 3.5 (Atomic language)**.** Let C be a set of atomic connectives. The *(typed) atomic language* $\mathcal{L}_{\mathsf{C}}$ associated to C is the smallest set that contains the propositional letters of C and that is closed under the atomic connectives of C while respecting the type constraints. That is,

- $\mathbb{P}(\mathsf{C}) \subseteq \mathcal{L}_{\mathsf{C}}$ and the *type* of an element of $\mathbb{P}$ is its output type k;
- for all $\circledast \in \mathsf{C}$ of arity $n > 0$ and of type signature $(k, k_1, \ldots, k_n)$ and for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$ of types $k_1, \ldots, k_n$ respectively, we have that $\circledast(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}_{\mathsf{C}}$ and $\circledast(\varphi_1, \ldots, \varphi_n)$ is of *type* k.

Elements of $\mathcal{L}_{\mathsf{C}}$ are called *formulas* and are denoted $\varphi, \psi, \alpha, \ldots$ The *type of a formula* $\varphi \in \mathcal{L}_C$ is denoted $k(\varphi)$. The set of all formulas of type k of $\mathcal{L}_{\mathsf{C}}$ is denoted $\mathcal{L}^k_{\mathsf{C}}$.

A set of atomic connectives C is *plain* if for all $\circledast \in \mathsf{C}$ of skeleton $\star = (\sigma, \pm, \text{Æ}, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$ there are atoms $p_1, \ldots, p_n \in \mathbb{P}$ of types $k_1, \ldots, k_n$ respectively. *In the sequel, we assume that all sets of connectives* C *are plain.*

Our assumption that all sets of connectives C considered are plain makes sense. Indeed, we want all connectives of C to appear in some formula of $\mathcal{L}_{\mathsf{C}}$. If C was not plain then there would be a connective of C (with input type k) which would be necessarily composed with another connective of C (of output type k), if we want such a connective to appear in a formula of $\mathcal{L}_{\mathsf{C}}$. Yet, in that case, we should instead view C as a set of *molecular* connectives (introduced in the next section).

Definition 3.6 (C–model)**.** Let C be a set of atomic connectives. A *C–model* is a tuple $M = (W, \mathcal{R})$ where W is a non-empty set and $\mathcal{R}$ is a set of relations over W such that each n–ary connective $\circledast \in \mathsf{C}$ which is not a Boolean connective of type signature $(k, k_1, \ldots, k_n)$ is associated to a $k_1 + \ldots + k_n + k$–ary relation $R_\circledast \in \mathcal{R}$ such that for all connectives $\circledast_1, \circledast_2 \in \mathsf{C}$ which belong to the same orbit $\mathcal{O}$, we have that $R_{\circledast_1} = R_{\circledast_2}$.

An *assignment* is a tuple $(w_1, \ldots, w_k) \in W^k$ for some $k \in \mathbb{N}^*$, generally denoted $\overline{w}$. A *pointed C–model* $(M, \overline{w})$ is a C–model M together with an assignment $\overline{w}$. In that case, we say that $(M, \overline{w})$ *is of type* k. The class of all pointed C–models is denoted $\mathcal{M}_{\mathsf{C}}$.

Definition 3.7 (Atomic logics)**.** Let C be a set of atomic connectives and let $M = (W, \mathcal{R})$ be a C–model. We define the *interpretation function of* $\mathcal{L}_C$ *in* M, denoted $[\![\cdot]\!]^M : \mathcal{L}_{\mathsf{C}} \to \bigcup\limits_{k \in \mathbb{N}^*} W^k$, inductively as follows: for all propositional letters $p \in \mathsf{C}$ of type k, all connectives $\circledast \in \mathsf{C}$ of skeleton $(\sigma, \pm, \text{Æ}, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$ of arity $n > 0$ and for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$,

$$
\begin{aligned}
[\![p]\!]^M &\triangleq \begin{cases} R_p & \text{if } \pm = + \\ W^k - R_p & \text{if } \pm = - \end{cases} \\
[\![\circledast(\varphi_1, \ldots, \varphi_n)]\!]^M &\triangleq f_\circledast([\![\varphi_1]\!]^M, \ldots, [\![\varphi_n]\!]^M)
\end{aligned}
$$

where the function $f_\circledast$ is defined as follows: for all $W_1 \in \mathcal{P}(W^{k_1}), \ldots, W_n \in \mathcal{P}(W^{k_n})$, $f_\circledast(W_1, \ldots, W_n) \triangleq \left\{\overline{w} \in W^k \mid \mathcal{C}^\circledast\left(W_1, \ldots, W_n, \overline{w}\right)\right\}$ where $\mathcal{C}^\circledast\left(W_1, \ldots, W_n, \overline{w}\right)$ is called the *truth condition* of $\circledast$ and is defined as follows:

- if Æ $= \forall$: "$\forall \overline{w}_1 \in W^{k_1} \ldots \overline{w}_n \in W^{k_n} \left(\overline{w}_1 \pitchfork_1 W_1 \vee \ldots \vee \overline{w}_n \pitchfork_n W_n \vee R^{\pm\sigma}_\circledast \overline{w}_1 \ldots \overline{w}_n \overline{w}\right)$";
- if Æ $= \exists$: "$\exists \overline{w}_1 \in W^{k_1} \ldots \overline{w}_n \in W^{k_n} \left(\overline{w}_1 \pitchfork_1 W_1 \wedge \ldots \wedge \overline{w}_n \pitchfork_n W_n \wedge R^{\pm\sigma}_\circledast \overline{w}_1 \ldots \overline{w}_n \overline{w}\right)$";

where, for all $j \in [\![1; n]\!]$, $\overline{w}_j \pitchfork_j W_j \triangleq \begin{cases} \overline{w}_j \in W_j & \text{if } \pm_j = + \\ \overline{w}_j \notin W_j & \text{if } \pm_j = - \end{cases}$ and $R^{\pm\sigma}_\circledast \overline{w}_1 \ldots \overline{w}_{n+1}$ holds iff $\pm R_\circledast \overline{w}_{\sigma^-(1)} \ldots \overline{w}_{\sigma^-(n+1)}$ with the notations $+R_\circledast \triangleq R_\circledast$ and $-R_\circledast \triangleq W^{k+k_1+\ldots+k_n} - R_\circledast$.

Permutations of $\mathfrak{S}_2$	unary signatures
$\tau_1 = (1, 2)$	$t_1 = (\exists, (1, 1), +)$
$\tau_2 = (2, 1)$	$t_2 = (\forall, (1, 1), +)$
	$t_3 = (\forall, (1, 1), -)$
	$t_4 = (\exists, (1, 1), -)$
Permutations of $\mathfrak{S}_3$	**binary signatures**
$\sigma_1 = (1, 2, 3)$	$s_1 = (\exists, (1, 1, 1), (+, +))$
$\sigma_2 = (3, 2, 1)$	$s_2 = (\forall, (1, 1, 1), (+, -))$
$\sigma_3 = (3, 1, 2)$	$s_3 = (\forall, (1, 1, 1), (-, +))$
$\sigma_4 = (2, 1, 3)$	$s_4 = (\forall, (1, 1, 1), (+, +))$
$\sigma_5 = (2, 3, 1)$	$s_5 = (\exists, (1, 1, 1), (+, -))$
$\sigma_6 = (1, 3, 2)$	$s_6 = (\exists, (1, 1, 1), (-, +))$
	$s_7 = (\exists, (1, 1, 1), (-, -))$
	$s_8 = (\forall, (1, 1, 1), (-, -))$

Figure 1: Permutations of $\mathfrak{S}_2$ and $\mathfrak{S}_3$ and 'families' of unary and binary signatures

If $\mathcal{E}_{\mathsf{C}}$ is a class of pointed C–models, the *satisfaction relation* $\Vdash \subseteq \mathcal{E}_{\mathsf{C}} \times \mathcal{L}_{\mathsf{C}}$ is defined as follows: for all $\varphi \in \mathcal{L}_{\mathsf{C}}$ and all $(M, \overline{w}) \in \mathcal{E}_{\mathsf{C}}$, $((M, \overline{w}), \varphi) \in \Vdash$ iff $\overline{w} \in [\![\varphi]\!]^M$. We usually write $(M, \overline{w}) \Vdash \varphi$ instead of $((M, \overline{w}), \varphi) \in \Vdash$ and we say that φ is *true* in $(M, \overline{w})$. The triple $(\mathcal{L}_{\mathsf{C}}, \mathcal{E}_{\mathsf{C}}, \Vdash)$ is a logic called the *atomic logic associated to $\mathcal{E}_{\mathsf{C}}$ and C*. The logics of the form $(\mathcal{L}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ are called *basic atomic logics*.

Example 3.8. An example of atomic logic is modal logic where $\mathsf{C} = \{p, \top, \bot, \wedge, \vee, \Diamond_j, \Box_j \mid j \in AGTS\}$ is such that

- $\top, \bot$ are connectives of skeletons $(\text{Id}, +, \exists, 1)$ and $(\text{Id}, -, \forall, 1)$ respectively;
- $\wedge, \vee, \Diamond_j, \Box_j$ are connectives of skeletons $(\sigma_1, +, s_1)$, $(\sigma_1, -, s_4)$, $(\tau_2, +, t_1)$ and $(\tau_2, -, t_2)$ respectively;
- the C-models $M = (W, \mathcal{R}) \in \mathcal{E}_{\mathsf{C}}$ are such that $R_\wedge = R_\vee = \{(w, w, w) \mid w \in W\}$, $R_{\Diamond_j} = R_{\Box_j}$ and $R_\top = R_\bot = W$.

With these conditions on the C–models of $\mathcal{E}_{\mathsf{C}}$, for all $(M, w) \in \mathcal{E}_{\mathsf{C}}$,

$$
\begin{array}{lll}
w \in [\![\Diamond_j \varphi]\!]^M & \text{iff} & \exists v (v \in [\![\varphi]\!]^M \wedge R_{\Diamond_j} wv) \\
w \in [\![\Box_j \varphi]\!]^M & \text{iff} & \forall v (v \in [\![\varphi]\!]^M \vee -R_{\Box_j} wv) \\
w \in [\![\wedge(\varphi, \psi)]\!]^M & \text{iff} & \exists vu (v \in [\![\varphi]\!]^M \wedge u \in [\![\psi]\!]^M \wedge R_\wedge vuw) \\
 & \text{iff} & w \in [\![\varphi]\!]^M \wedge w \in [\![\psi]\!]^M \\
w \in [\![\vee(\varphi, \psi)]\!]^M & \text{iff} & \forall vu (v \in [\![\varphi]\!]^M \vee u \in [\![\psi]\!]^M \vee -R_\vee vuw) \\
 & \text{iff} & w \in [\![\varphi]\!]^M \vee w \in [\![\psi]\!]^M
\end{array}
$$

Other examples are given in Figure 2 as well as in [2, 5].

Atomic Connective	Truth condition	Non–classical connective in the literature
The existentially positive orbit		
$(\sigma_1, +, s_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge Rvw)$	$\Diamond^{-}\varphi$ [31] $\Diamond_{\downarrow}$ [13]
$(\sigma_2, -, s_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee -Rwv)$	$\Box\varphi$ [25]
The universally positive orbit		
$(\sigma_1, +, s_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee Rvw)$	$+_{\downarrow}\varphi$ [13] [16, p. 401]
$(\sigma_2, -, s_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge -Rwv)$	[13]
The existentially negative orbit		
$(\sigma_1, +, s_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge Rvw)$	$?\varphi$ [13][16, p. 402] $\boxminus_1\varphi$ [13][11, Def. 10.7.7]
$(\sigma_2, +, s_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge Rwv)$	$?_{\downarrow}\varphi$ [13][17] [16, p. 402] $\boxminus_2\varphi$ [11, Def. 10.7.7]
The universally negative orbit		
$(\sigma_1, +, s_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee Rvw)$	$\varphi^{\perp}$ [13, 15] $\varphi^{\mathbf{o}}$ [22] $\diamond_1^{-}\varphi$ [11, Def. 10.7.2]
$(\sigma_2, +, s_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee Rwv)$	$\sim\varphi$ [21] ${}^{\perp}\varphi$ [13, 15] ${}^{\mathbf{o}}\varphi$ [22] $\diamond_2^{-}\varphi$ [11, Def. 10.7.2]
The symmetrical existentially positive orbit		
$(\sigma_1, -, s_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge -Rvw)$	[13]
$(\sigma_2, +, s_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee Rwv)$	$+\varphi$ [13] [16, p. 402] φ^{*} [11, Def. 7.1.19]
The symmetrical universally positive orbit		
$(\sigma_1, -, s_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee -Rvw)$	$\Box^{-}\varphi$ [31] $\Box_{\downarrow}$ [13]
$(\sigma_2, +, s_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge Rwv)$	$\Diamond\varphi$ [25]
The symmetrical existentially negative orbit		
$(\sigma_1, -, s_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge -Rvw)$	$?\varphi$ [13][11, Ex. 1.4.5] $\varphi^{\mathbf{1}}$ [22]
$(\sigma_2, -, s_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge -Rwv)$	$?_{\downarrow}\varphi$ [13] [11, Ex. 1.4.5] ${}^{\mathbf{1}}\varphi$ [22]
The symmetrical universally negative orbit		
$(\sigma_1, -, s_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee -Rvw)$	[13]
$(\sigma_2, -, s_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee -Rwv)$	$\neg_l\varphi$ [26, 33] $\perp\varphi$ [17]

Figure 2: The unary connectives of atomic logics of type $(1, 1)$

4 Molecular Logics

Molecular logics are basically logics whose primitive connectives are compositions of atomic connectives in which it is possible to repeat the same argument at different places in the connective. That is why we call them 'molecular', just as molecules are compositions of atoms in chemistry.

Definition 4.1 (Molecular skeleton and connective)**.** The class $\mathbb{C}^*$ of *molecular skeletons* is the smallest set such that:

- $\mathbb{P} \subseteq \mathbb{C}^*$ and $\mathbb{C}^*$ contains as well, for each $k, l \in \mathbb{N}^*$, a symbol id_k^l of *type signature* (k, k), *output type* k and *arity* 1;
- for all $\circledast \in \mathbb{C}$ of type signature $(k, k_1^0, \ldots, k_n^0)$ and all $c_1, \ldots, c_n \in \mathbb{C}^*$ of output types or types (if they are propositional letters) $k_1^0, \ldots, k_n^0$ respectively, $c \triangleq \circledast(c_1, \ldots, c_n)$ is a molecular skeleton of $\mathbb{C}^*$ of *output type* k.

If $c \in \mathbb{C}^*$, we define its *decomposition tree* as follows. If $c = p \in \mathbb{P}$ or $c = id_k^l$, then its decomposition tree T_c is the tree consisting of a single node labeled with p or id_k^l respectively. If $c = \circledast(c_1, \ldots, c_n) \in \mathbb{C}^*$ then its decomposition tree T_c is the tree defined inductively as follows: the root of T_c is c and it is labeled with $\circledast$ and one sets edges between that root and the roots $c_1, \ldots, c_n$ of the decomposition trees $T_{c_1}, \ldots, T_{c_n}$ respectively.

If $c \triangleq \circledast(c_1, \ldots, c_n)$ is a molecular skeleton with output type k and $k_1, \ldots, k_m$ are the ks of the different id_k^ls which appear in $c_1, \ldots, c_n$ (in an order which follows the first appearance of the id_k^ls in the inorder traversal of the decomposition trees of $c_1, \ldots, c_n$), then the *type signature* of c is $(k, k_1, \ldots, k_m)$ and its *arity* is m. We also define the *quantification signature* $Æ(c)$ of $c = \circledast(c_1, \ldots, c_n)$ by $Æ(c) \triangleq Æ(\circledast)$.

A *molecular connective* is a symbol to which is associated a molecular skeleton. Its arity, type signature, output type, quantification signature and decomposition tree are the same as its skeleton.

The set of *atomic connectives associated to a set* C *of molecular connectives* is the set of labels different from id_k^l of the decomposition trees of the molecular connectives of C.

One needs to introduce the connective id_k^l to deal with molecular connectives whose skeletons are for example of the form $\circledast(p, id_k^l)$ where $p \in \mathbb{P}$ or molecular connectives in which the same argument(s) appear at different places, like for example in $\circledast(id_k^1, \ldots, id_k^1)$ which is of arity 1.

Definition 4.2 (Molecular language)**.** Let C be a set of molecular connectives. The *(typed) molecular language* $\mathcal{L}_\mathsf{C}$ associated to C is the smallest set that contains the propositional letters and that is closed under the molecular connectives while respecting the type constraints. That is,

- the propositional letters of C belong to $\mathcal{L}_\mathsf{C}$;
- for all $\circledast \in \mathsf{C}$ of type signature $(k, k_1, \ldots, k_m)$ and for all $\varphi_1, \ldots, \varphi_m \in \mathcal{L}_\mathsf{C}$ of types $k_1, \ldots, k_m$ respectively, we have that $\circledast(\varphi_1, \ldots, \varphi_m) \in \mathcal{L}_\mathsf{C}$ and $\circledast(\varphi_1, \ldots, \varphi_m)$ is of *type* k.

Elements of $\mathcal{L}_\mathsf{C}$ are called *molecular formulas* and are denoted $\varphi, \psi, \alpha, \ldots$ The *type of a formula* $\varphi \in \mathcal{L}_C$ is denoted $k(\varphi)$. We use the same abbreviations as for the atomic language.

Definition 4.3 (Molecular logic)**.** If C is a set of molecular connectives, then a *C–model M* is a C'–model M where C' is the set of atomic connectives associated to C. The truth conditions for molecular connectives are defined naturally from the truth conditions of atomic connectives.

We define the *interpretation function of $\mathcal{L}_\mathsf{C}$ in M*, denoted $[\![\cdot]\!]^M : \mathcal{L}_\mathsf{C} \to \bigcup_{k\in\mathbb{N}^*} W^k$, inductively as follows: for all propositional letters $p \in \mathsf{C}$ of skeleton $(\sigma, \pm, Æ, k)$, all molecular connectives $\circledast(c_1, \dots, c_n) \in \mathsf{C}^*$ of arity $m > 0$ and all $k, l \in \mathbb{N}^*$, for all $\varphi, \varphi_1, \dots, \varphi_m \in \mathcal{L}_\mathsf{C}$,

$$\begin{aligned} [\![p]\!]^M &\triangleq \pm R_p \\ [\![id^l_k(\varphi)]\!]^M &\triangleq [\![\varphi]\!]^M \\ [\![\circledast(c_1, \dots, c_n)\,(\varphi_1, \dots, \varphi_m)]\!]^M &\triangleq f_\circledast\left([\![c_1(\varphi^1_1, \dots, \varphi^1_{i_1})]\!]^M, \dots, [\![c_n(\varphi^n_1, \dots, \varphi^n_{i_n})]\!]^M\right) \end{aligned}$$

where for all $j \in \{1, \dots, n\}$, the formulas $\varphi^j_1, \dots, \varphi^j_{i_j}$ are those $\varphi_1, \dots, \varphi_m$ for which there is a corresponding id^l_k in c_j (the φ^j_is appear in the same order as their corresponding id^l_ks in c_j).

If $\mathcal{E}_\mathsf{C}$ is a class of pointed C–models, the triple $(\mathcal{L}_\mathsf{C}, \mathcal{E}_\mathsf{C}, \Vdash)$ is a logic called the *molecular logic associated to $\mathcal{E}_\mathsf{C}$ and C*.

As one can easily notice, every atomic logic can be canonically mapped to an equi-expressive molecular logic: each atomic connective $\circledast$ of type signature $(k, k_1, \dots, k_n)$ of the given atomic logic has to be transformed into the molecular connective of skeleton $\circledast(id^1_{k_1}, \dots, id^n_{k_n})$. Note that the id^l_k are in fact specific atomic connectives whose associated relations are the identity relations.

Example 4.4. Modal logic with a neighborhood semantics, temporal logic and modal intuitionistic logic are examples of molecular logics, but in fact all logics such that the truth conditions of their connectives are expressible in terms of first-order formulas are molecular logics. See [5] for more details.

5 Boolean Atomic and Molecular Logics

Atomic and molecular logics do not include Boolean connectives as primitive connectives. In fact, they can be defined in terms of specific atomic connectives, as follows.

Definition 5.1 (Boolean connectives)**.** The *Boolean connectives* called *conjunctions*, *disjunctions*, *negations* and *Boolean constants* (of type k) are the atomic connectives denoted, respectively:

$$\mathbb{B} \triangleq \{\wedge_k, \vee_k, \neg_k, \top_k, \bot_k \mid k \in \mathbb{N}^*\}$$

The skeleton of $\wedge_k$ is $(1, +, \exists, (k, k, k), (+, +))$, the skeleton of $\vee_k$ is $(1, -, \forall, (k, k, k), (+, +))$, the skeleton of $\neg_k$ is $(1, +, \exists, (k, k), -)$, the skeleton of $\top_k$ is $(1, +, \exists, k)$ and the skeleton of $\bot_k$ is $(1, -, \forall, k)$.

In any C-model $M = (W, \mathcal{R})$ containing Boolean connectives, the associated relation of any $\vee_k$ or $\wedge_k$ is $R_{\wedge_k} = R_{\vee_k} \triangleq \{(\overline{w}, \overline{w}, \overline{w}) \mid \overline{w} \in W^k\}$, the associated relation of any $\neg_k$ is $R_{\neg_k} \triangleq \{(\overline{w}, \overline{w}) \mid \overline{w} \in W^k\}$ and the associated relation of any $\top_k$ or $\bot_k$ is $R_{\bot_k} = R_{\top_k} \triangleq W^k$.

We say that a set of atomic or molecular connectives C is *Boolean* when it contains all conjunctions, disjunctions, constants as well as negations $\wedge_k, \vee_k, \top_k, \bot_k, \neg_k$, for k ranging over all input types and output types of the connectives of C. The *Boolean completion* of a set of atomic or molecular connectives C is the smallest set of connectives including C which is Boolean. A *Boolean atomic or molecular logic* is an atomic or molecular logic such that its set of connectives is Boolean.

Proposition 5.2. *Let C be a Boolean set of atomic connectives and let $M = (W, \mathcal{R})$ be a C-model. Then, for all $k \in \mathbb{N}^*$, all $\varphi, \psi \in \mathcal{L}_{\mathsf{C}}$, if $k(\varphi) = k(\psi) = k$, then*

$$\begin{array}{rcl} [\![\top_k]\!]^M & \triangleq & W^k \\ [\![\bot_k]\!]^M & \triangleq & \emptyset \\ [\![\neg_k \varphi]\!]^M & \triangleq & W^k - [\![\varphi]\!]^M \\ [\![(\varphi \wedge_k \psi)]\!]^M & \triangleq & [\![\varphi]\!]^M \cap [\![\psi]\!]^M \\ [\![(\varphi \vee_k \psi)]\!]^M & \triangleq & [\![\varphi]\!]^M \cup [\![\psi]\!]^M . \end{array}$$

It turns out that Boolean negation can also be simulated systematically at the level of atomic connectives by applying a transformation on them. The Boolean negation of a formula then boils down to taking the Boolean negation of the outermost connective of the formula. This transformation is defined as follows.

Definition 5.3 (Boolean negation)**.** Let $\circledast$ be a n–ary connective of skeleton $(\sigma, \pm, Æ, \overline{k}, \pm_1, \ldots, \pm_n)$. The *Boolean negation of* $\circledast$ is the connective $-\circledast$ of skeleton $(\sigma, -\pm, -Æ, \overline{k}, -\pm_1, \ldots, -\pm_n)$ where $-Æ \triangleq \exists$ if $Æ = \forall$ and $-Æ \triangleq \forall$ otherwise, which is associated in any C–model to the same relation as $\circledast$. If $\varphi = \circledast(\varphi_1, \ldots, \varphi_n)$ is an atomic formula, the *Boolean negation of* φ is the formula $-\varphi \triangleq - \circledast (\varphi_1, \ldots, \varphi_n)$.

Proposition 5.4. *Let C be a set of atomic connectives such that $-\circledast \in \mathsf{C}$ for all $\circledast \in \mathsf{C}$. Let $\varphi \in \mathcal{L}_{\mathsf{C}}$ of type k and let $M = (W, \mathcal{R})$ be a C–model. Then, for all $\overline{w} \in W^k$, $\overline{w} \in [\![-\varphi]\!]^M$ iff $\overline{w} \notin [\![\varphi]\!]^M$.*

6 Logical generalities

These definitions are very general and apply to any kind of logical formalism. Our approach to defining logics is somehow more 'semantic' in that respect than the usual proposals [9]. It corresponds in fact to the "abstract logics" of García-Matos & Väänänen [19] and to the "rooms" of Mossakowski et al. [29].

Definition 6.1 (Logic)**.** A *logic* is a triple $\mathsf{L} = (\mathcal{L}, E, \models)$ where

- $\mathcal{L}$ is a *language* defined as a set of well-formed expressions built from a set of *connectives* C and a set of *atoms* $\mathbb{P}$;
- E is a *class of pointed models or frames*;
- $\models$ is a *satisfaction relation* which relates in a compositional manner elements of $\mathcal{L}$ to models of E by means of so-called *truth conditions.*

A $\mathcal{L}$*–consecution* is an expression of the form $\varphi \vdash \psi$, $\vdash \psi$ or $\varphi \vdash$, where $\varphi, \psi \in \mathcal{L}$. □

Our definition of a calculus and of an inference rule is taken from [28].

Definition 6.2 (Conservativity)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ and $\mathsf{L}' = (\mathcal{L}', E', \models')$ be two logics such that $\mathcal{L} \subseteq \mathcal{L}'$. We say that L' is a *conservative extension* of L when $\{\varphi \in \mathcal{L} \mid \models_{\mathsf{L}} \varphi\} = \mathcal{L} \cap \{\varphi' \in \mathcal{L}' \mid \models'_{\mathsf{L}'} \varphi'\}$. □

Definition 6.3 (Calculus and sequent calculus)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. A *calculus* P for $\mathcal{L}$ is a set of elements of $\mathcal{L}$ called *axioms* and a set of *inference rules.* Most often, one can effectively decide whether a given element of $\mathcal{L}$ is an axiom. To be more precise, an *inference* *rule* R for $\mathcal{L}$ is a relation among elements of $\mathcal{L}$ such that there is a unique $l \in \mathbb{N}^*$ such that, for all $\varphi, \varphi_1, \ldots, \varphi_l \in \mathcal{L}$, one can effectively decide whether $(\varphi_1, \ldots, \varphi_l, \varphi) \in R$. The elements $\varphi_1, \ldots, \varphi_l$ are called the *premises* and φ is called the *conclusion* and we say that φ is a *direct consequence of* $\varphi_1, \ldots, \varphi_l$ by virtue of R. Let $\Gamma \subseteq \mathcal{L}$ and let $\varphi \in \mathcal{L}$. We say that φ is *provable* (from Γ) in P or a *theorem* of P, denoted $\vdash_{\mathsf{P}} \varphi$ (resp. $\Gamma \vdash_{\mathsf{P}} \varphi$), when there is a *proof* of φ (from Γ) in P, that is, a finite sequence of formulas ending in φ such that each of these formulas is:

1. either an instance of an axiom of P (or a formula of Γ);
2. or the direct consequence of preceding formulas by virtue of an inference rule R.

If $\mathcal{S}$ is a set of $\mathcal{L}$–consecutions, this set $\mathcal{S}$ can be viewed as a language. In that case, we call *sequent calculus for* $\mathcal{L}$ a calculus for $\mathcal{S}$.

Axioms and inference rules are often represented by means of *axiom schemas* and *inference rule schemas*, that is, expressions of the following form, depending on whether we deal with formulas of $\mathcal{L}$ or $\mathcal{L}$–consecutions:

Axiom schemas:

α $\qquad\qquad$ $A \models B$

Inference rule schemas:

$$\frac{\alpha_1 \quad \ldots \quad \alpha_n}{\alpha} \qquad\qquad \frac{A_1 \models B_1 \quad \ldots \quad A_n \models B_n}{A \models B}$$

where $\alpha_1, \ldots, \alpha_n, \alpha$ are built up from *variables* often denoted $\varphi, \psi, \ldots$ and the connectives of C and, likewise, $A_1, \ldots, A_n, B_1, \ldots, B_n, A, B$ are built up from *variables* often denoted $X, Y, \ldots$ and the connectives of C. In this representation, inference rules and axioms schemas are closed by *uniform substitution*: each variable can be replaced uniformly by *any* well-formed expression of $\mathcal{L}$ and this yields an *instance* of the inference rule.

An inference rule R' is *derivable from an inference rule R in* P when there is a finite sequence of rules $R_1, \ldots, R_n$ of P all different from R' and with at least one of them equal to R, such that $R' = R_1 \circ \ldots \circ R_n$. An inference rule R is *equivalent* to another inference rule R' in P when R' is derivable from R in P and vice versa. A calculus is *equivalent* to another calculus (for the same language) when every axiom and inference rule of the first calculus is provable or derivable in the second, and vice versa.

Definition 6.4 (Truth, validity, logical consequence)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. Let $M \in E$, $\varphi \in \mathcal{L}$. If Γ is a set of formulas or inference rules, we write $M \models \Gamma$ when for all $\varphi \in \Gamma$, we have $M \models \varphi$. Then, we say that

- φ is *true* (*satisfied*) at M or M is a *model* of φ when $M \models \varphi$;
- φ is a *logical consequence* of Γ, denoted $\Gamma \models_{\mathsf{L}} \varphi$, when for all $M \in E$, if $M \models \Gamma$ then $M \models \varphi$;
- φ is *valid*, denoted $\models_{\mathsf{L}} \varphi$, when for all models $M \in E$, we have $M \models \varphi$.

Definition 6.5 (Soundness and completeness)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. Let P be a calculus for $\mathcal{L}$. Then,

- P is *sound* for the logic L when for all $\varphi \in \mathcal{L}$, if $\vdash_{\mathsf{P}} \varphi$, then $\models_{\mathsf{L}} \varphi$.
- P is *(strongly) complete* for the logic L when for all $\varphi \in \mathcal{L}$ (and all $\Gamma \subseteq \mathcal{L}$), if $\models_{\mathsf{L}} \varphi$, then $\vdash_{\mathsf{P}} \varphi$ (resp. if $\Gamma \models_{\mathsf{L}} \varphi$, then $\Gamma \vdash_{\mathsf{P}} \varphi$).

When P is a sequent calculus for a set of $\mathcal{L}$–consecutions $\mathcal{S}$, we sometimes say by abuse that P is sound and complete for $(\mathcal{L}, E, \models)$ instead of $(\mathcal{S}, E, \models)$.

7 Display calculi for atomic and molecular logics

7.1 Structures and consecutions for atomic logics

In order to provide a sound and complete calculus for an atomic logic based on a set of connectives $\mathsf{C} \subseteq \mathbb{C}$, we will need to resort to the connectives of C which are in the orbits of the free action $\alpha_n * \beta_n$ (for appropriate ns). We introduce these extra connectives in the language as *structural* connectives.

Definition 7.1 (Structural connectives)**.** *(Atomic) structural connectives* are copies of the connectives: for all sets of atomic connectives C, its associated set of structural connectives is denoted $[\mathsf{C}] \triangleq \{[\circledast] \mid \circledast \in \mathsf{C}\}$. For all atomic connectives $\circledast$, the *arity*, *signature*, *type signature*, *tonicity signature*, *quantification signature* of $[\circledast]$ are the same as $\circledast$. Structural connectives are denoted $[p], [p_1], [p_2], \ldots$ and $[\circledast], [\circledast_1], [\circledast_2], \ldots$ For each $k \in \mathbb{N}^*$, we also introduce the *(Boolean) structural connective* associated to the Boolean connectives $\wedge_k, \vee_k$, denoted $\ ,_k\ $ and often simply $\ ,\ $ by abuse and the *structural constant* associated to the constants $\top_k$ and $\bot_k$, denoted $\mathbf{I}_k$ and often simply $\mathbf{I}$ by abuse.

Definition 7.2 (Structural atomic language and consecutions)**.** Let C be a set of atomic connectives. The *structural atomic language* $[\mathcal{L}_\mathsf{C}]$ is the smallest set that contains the atomic language $\mathcal{L}_\mathsf{C}$, the structures $*\varphi$ for all $\varphi \in \mathcal{L}_\mathsf{C}$ as well as $[\mathbb{P}(\mathsf{C})]$ and that is closed under the structural connectives of $[\mathsf{C}] \cup \{\ ,_k \mid k \in \mathbb{N}^*\}$ while respecting the type constraints. Its elements are called *structures* and their *types* are defined like for formulas of $\mathcal{L}_\mathsf{C}$.

A $\mathcal{L}_C$*–consecution* (resp. $[\mathcal{L}_C]$*–consecution*) is an expression of the form $\varphi \vdash \psi$ (resp. $X \vdash Y$), where $\varphi, \psi \in \mathcal{L}^k_\mathsf{C}$ are of the same type, for some $k \in \mathbb{N}^*$ (resp. $X, Y \in [\mathcal{L}_\mathsf{C}]$ are of the same type, for some $k \in \mathbb{N}^*$). The set of all $\mathcal{L}_\mathsf{C}$–consecutions (resp. $[\mathcal{L}_\mathsf{C}]$–consecutions) is denoted $\mathcal{S}_\mathsf{C}$ (resp. $[\mathcal{S}_\mathsf{C}]$).

Elements of $\mathcal{L}_\mathsf{C}$ (resp. $[\mathcal{L}_\mathsf{C}]$ and $[\mathcal{S}_\mathsf{C}]$) are called *formulas* (resp. *structures* and *consecutions*); they are denoted $\varphi, \psi, \alpha, \ldots$ (resp. $X, Y, A, B, \ldots$ and $X \vdash Y, A \vdash B, \ldots$). □

Definition 7.3 (Boolean negation)**.** Let $X \in [\mathcal{L}]$ be a structure. The *Boolean negation of* X, denoted $*X$, is defined inductively as follows ($-\circledast$ was defined in Definition 5.3):

$$*X \triangleq \begin{cases} [-\circledast]\,(X_1, \ldots, X_n) & \text{if } X = [\circledast]\,(X_1, \ldots, X_n) \\ (*X_1\ ,\ *X_2) & \text{if } X = (X_1\ ,\ X_2) \\ \mathbf{I} & \text{if } X = \mathbf{I} \\ \varphi & \text{if } X = *\varphi \\ *\varphi & \text{if } X = \varphi \in \mathcal{L} \end{cases}$$

Definition 7.4 (Formula associated to a structure)**.** Let C be a set of atomic connectives. We define inductively the function τ_0 and τ_1 from structures of $[\mathcal{L}_\mathsf{C}]$ to formulas of $\mathcal{L}_\mathsf{C}$ as follows: for all $i \in \{0, 1\}$, all $\circledast \in \mathsf{C}$ of skeleton $(\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n))$,

$$\begin{aligned} \tau_0(\mathbf{I}) &\triangleq \top \\ \tau_1(\mathbf{I}) &\triangleq \bot \\ \tau_i(\varphi) &\triangleq \varphi \\ \tau_i(*\varphi) &\triangleq \neg\varphi \\ \tau_0\,(X\ ,_k\ Y) &\triangleq (\tau_0(X) \wedge_k \tau_0(Y)) \\ \tau_1\,(X\ ,_k\ Y) &\triangleq (\tau_1(X) \vee_k \tau_1(Y)) \\ \tau_i\,([\circledast]\,(X_1, \ldots, X_n)) &\triangleq \circledast(\tau_{i_1}(X_1), \ldots, \tau_{i_n}(X_n)) \end{aligned}$$

where for all $j \in [\![1;n]\!]$, $\tau_{i_j}(X_j) \triangleq \begin{cases} \tau_i(X_j) & \text{if } \pm_j = + \\ \tau_{1-i}(X_j) & \text{if } \pm_j = - \end{cases}$.

Then, we define the function τ from $[\mathcal{L}_{\mathsf{C}}]$–consecutions of $[\mathcal{S}_{\mathsf{C}}]$ to $\mathcal{L}_{\mathsf{C}}$–consecutions of $\mathcal{S}_{\mathsf{C}}$ as follows:

$$\tau(X \vdash Y) \quad \triangleq \quad \tau_0(X) \vdash \tau_1(Y)$$

Instead of a single structural connective $\ ,\ $, we could introduce two Boolean structural connectives $[\wedge]$, $[\vee]$ as a copy of the Boolean connectives $\wedge, \vee$, like for the other atomic connectives $\circledast$. This would not be usual but in line with our approach. This would greatly simplify the definition of the function τ since the interpretation of the structural connectives would then not be context-dependent as here. In particular one would not need two functions τ_0 and τ_1 but only one. We proceed as follows on the one hand in order to stay in line with current practice and on the other hand because it simplifies the subsequent calculus $\mathsf{GGL}_{\mathsf{C}}$ of Figure 3: we use one structural connective (,) instead of two ($[\wedge]$ and $[\vee]$). This said, it would be easily possible to adapt and rewrite the calculus $\mathsf{GGL}_{\mathsf{C}}$ with these two structural connectives $[\wedge]$ and $[\vee]$: the structural connective $\ ,\ $ would need to be replaced by $[\wedge]$ in the premise of (dr_2) and in $(\mathsf{B}\vdash), (\mathsf{Cl}\vdash), (\mathsf{K}\vdash), (\wedge\vdash), (\mathbf{I}\vdash)$ and by $[\vee]$ in the conclusion of (dr_2) and in $(\vdash\mathsf{B}), (\vdash\mathsf{Cl}), (\vdash\mathsf{K}), (\vdash\vee)$ (see below). The same comment applies to **I**: we could introduce two structural connectives $[\top]$ and $[\bot]$ instead of a single one and obtain a more uniform calculus.

Definition 7.5 (Interpretation of atomic structures and consecutions)**.** Let C be a set of atomic connectives and let $M = (W, \mathcal{R})$ be a C–model. We extend the interpretation function $[\![\cdot]\!]^M$ of $\mathcal{L}_{\mathsf{C}}$ in M to $\mathcal{L}_{\mathsf{C}}$–consecutions of $\mathcal{S}_{\mathsf{C}}$ as follows: for all $\varphi, \psi \in \mathcal{L}_{\mathsf{C}}$ and all $w \in W$, we have that $w \in [\![\varphi \vdash \psi]\!]^M$ iff if $w \in [\![\varphi]\!]^M$ then $w \in [\![\psi]\!]^M$, we have that $w \in [\![\ \vdash \psi]\!]^M$ iff $w \in [\![\psi]\!]^M$ and we have that $w \in [\![\varphi \vdash\ ]\!]^M$ iff $w \notin [\![\varphi]\!]^M$. We then extend in a natural way the interpretation function $[\![\cdot]\!]^M$ of $\mathcal{L}_{\mathsf{C}}$ in M to $[\mathcal{L}_{\mathsf{C}}]$–consecutions of $[\mathcal{S}_{\mathsf{C}}]$ as follows: for all $X \in \mathcal{L}_{\mathsf{C}}$, all $X \vdash Y \in [\mathcal{S}_{\mathsf{C}}]$ and all $w \in W$, we have that $w \in [\![X \vdash Y]\!]^M$ if, and only if, $w \in [\![\tau(X \vdash Y)]\!]^M$. If $\mathcal{E}_{\mathsf{C}}$ is a class of C–models, then the satisfaction relation $\Vdash\ \subseteq \mathcal{E}_{\mathsf{C}} \times [\mathcal{S}_{\mathsf{C}}]$ is defined like for formulas of $\mathcal{L}$.

7.2 Display calculi for Boolean atomic logics

Our calculus is defined relatively to an orbit of connectives by the free action $\alpha_n * \beta_n$. This means that if we have a basic atomic logic defined on the basis of some connectives C and if we want to obtain a sound and complete calculus for that logic, we need to consider in the proof system the following associated set of connectives. This is because in the completeness proof, we need to apply the abstract law of residuation (of Definition 3.2) for any arguments j and consider the Boolean negation for each connective.

$$\begin{aligned} \mathcal{O}(\mathsf{C}) &\triangleq (\mathsf{C} \cap \mathbb{B}) \cup \bigcup_{\circledast \in \mathsf{C} - \mathbb{B}} \mathcal{O}_{\alpha_n * \beta_n}(\circledast) \\ &= (\mathsf{C} \cap \mathbb{B}) \cup \bigcup_{\circledast \in \mathsf{C} - \mathbb{B}} \{\tau_1 - \ldots - \tau_m \circledast \mid \circledast \text{ is of arity } n \text{ and } \tau_1, \ldots, \tau_m \in \mathfrak{S}_{n+1}\} \end{aligned} \tag{3}$$

Definition 7.6. Let C be a Boolean set of atomic connectives. We denote by $\mathsf{GGL}_{\mathsf{C}}$ the calculus of Figure 3 where the introduction rules $(\vdash \circledast)$ and $(\circledast \vdash)$ are defined for the connectives $\circledast$ of C, where the rule (dr_1) is defined for the elements τ of an arbitrary set of generators of $\mathfrak{S}_{n+1}$ (for each n ranging over the arities of the connectives of C) and where the structural connectives $[\circledast]$ range over $[\mathcal{O}(\mathsf{C})]$.

Structural rules:

$$\frac{(X\ ,\ Y) \vdash U}{(Y\ ,\ X) \vdash U}\ (\mathsf{CI}\vdash) \qquad \frac{X \vdash U}{(X\ ,\ Y) \vdash U}\ (\mathsf{K}\vdash)$$

$$\frac{(X\ ,\ X) \vdash U}{X \vdash U}\ (\mathsf{WI}\vdash) \qquad \frac{\mathbf{I} \vdash U}{\vdash U}\ (\mathbf{I}\vdash)$$

$$\frac{U \vdash \varphi \quad \varphi \vdash V}{U \vdash V}\ \mathit{cut}$$

Display rules:

$$\frac{S\left([\circledast], X_1, \ldots, X_n, X_{n+1}\right)}{S\left([\tau\circledast], X_{\tau(1)}, \ldots, X_{\tau(n)}, X_{\tau(n+1)}\right)}\ (\mathsf{dr}_1) \qquad \frac{(X\ ,\ Y) \vdash Z}{X \vdash (Z\ ,\ *Y)}\ (\mathsf{dr}_2)$$

(both rules drawn with a double line)

Introduction rules:

$$\frac{}{\bot \vdash \mathbf{I}}\ (\vdash \bot) \qquad \frac{U \vdash \mathbf{I}}{U \vdash \bot}\ (\bot \vdash)$$

$$\frac{}{\mathbf{I} \vdash \top}\ (\vdash \top) \qquad \frac{\mathbf{I} \vdash U}{\top \vdash U}\ (\top \vdash)$$

$$\frac{U \vdash\ *\varphi}{U \vdash \neg\varphi}\ (\vdash \neg) \qquad \frac{*\varphi \vdash U}{\neg\varphi \vdash U}\ (\neg \vdash)$$

$$\frac{U \vdash \varphi \quad U \vdash \psi}{U \vdash (\varphi \wedge \psi)}\ (\vdash \wedge) \qquad \frac{\varphi \vdash U}{(\varphi \wedge \psi) \vdash U}\ (\wedge \vdash)$$

$$\frac{U \vdash \varphi}{U \vdash (\varphi \vee \psi)}\ (\vdash \vee) \qquad \frac{\varphi \vdash U \quad \psi \vdash U}{(\varphi \vee \psi) \vdash U}\ (\vee \vdash)$$

$$\frac{U_1 \vdash V_1 \quad \ldots \quad U_n \vdash V_n}{S\left([\circledast], X_1, \ldots, X_n, \circledast\left(\varphi_1, \ldots, \varphi_n\right)\right)}\ (\vdash \circledast) \qquad \frac{S\left([\circledast], \varphi_1, \ldots, \varphi_n, U\right)}{S\left(\circledast, \varphi_1, \ldots, \varphi_n, U\right)}\ (\circledast \vdash)$$

In rules $(\vdash \circledast)$ and $(\circledast \vdash)$, for all $\circledast \in \mathsf{C}$ of skeleton $\star = (\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n))$:

- for all $j \in [\![1; n]\!]$, we set $U_j \vdash V_j \triangleq \begin{cases} X_j \vdash \varphi_j & \text{if } \pm_j \pm (Æ) = - \\ \varphi_j \vdash X_j & \text{if } \pm_j \pm (Æ) = + \end{cases}$
  such that, in rule $(\vdash \circledast)$, for all j X_j is not empty and if φ_j is empty for some j then $\circledast(\varphi_1, \ldots, \varphi_n)$ is also empty.
- for all $c \in \{\circledast, [\circledast]\}$, $S(c, X_1, \ldots, X_n, X) \triangleq \begin{cases} c(X_1, \ldots, X_n) \vdash X & \text{if } Æ = \exists \\ X \vdash c(X_1, \ldots, X_n) & \text{if } Æ = \forall \end{cases}$.

If X is empty then $*X$ is empty and $(X\ ,\ Y)$ and $(Y\ ,\ X)$ are equal to Y.

Figure 3: Calculus $\mathsf{GGL_C}$

Theorem 7.7 (Soundness and strong completeness)**.** *Let C be a Boolean set of atomic connectives such that $\mathcal{O}(C) = C$. The calculus GGL_C is sound and strongly complete for the Boolean basic atomic logic $(\mathcal{S}_C, \mathcal{M}_C, \Vdash)$.*

• The axioms and inference rules for atoms p are special instances of the rules $(\vdash \circledast)$ and $(\circledast \vdash)$ of Figure 3. With $\circledast = p$, we have that $n = 0$ and, replacing $\circledast$ with p in $(\vdash \circledast)$, we obtain the inference rules below. Note that $(\vdash p)$ is in fact an axiom.

$$\frac{}{S([p]\,, p)}\ (\vdash p) \qquad\qquad \frac{S\left([p]\,, X\right)}{S\left(p, X\right)}\ (p \vdash)$$

where, if $\circledast$ is p or $[p]$, then $S(\circledast, X) \triangleq \begin{cases} \circledast \vdash X & \text{if } Æ = \exists \\ X \vdash \circledast & \text{if } Æ = \forall \end{cases}$.

Hence, for all $p = (\mathrm{Id}, \pm, Æ, k)$, if $Æ = \exists$ then $(\vdash p)$ and $(p \vdash)$ rewrite as follows:

$$\frac{}{[p] \vdash p}\ (\vdash p) \qquad\qquad \frac{[p] \vdash X}{p \vdash X}\ (p \vdash) \tag{4}$$

and if $Æ = \forall$ then $(\vdash p)$ and $(p \vdash)$ rewrite as follows:

$$\frac{}{p \vdash [p]}\ (\vdash p) \qquad\qquad \frac{X \vdash [p]}{X \vdash p}\ (p \vdash) \tag{5}$$

Note that in both cases, the standard axiom $p \vdash p$ is derivable by applying $(p \vdash)$ once again to $[p] \vdash p$ or $p \vdash [p]$. If $[p]$ is replaced by $\mathbf{I}$ and p by $\top$ in the first pair and if $[p]$ is replaced by $\mathbf{I}$ and p by $\bot$ in the second pair then we obtain respectively the operational rules $(\vdash \top)$, $(\top \vdash)$, $(\bot \vdash)$ and $(\vdash \bot)$ of Kracht [24] and Belnap [7]. This is meaningful since truth constants can be seen as special propositional letters, those that are always true or always false. Then, one needs, like in the calculus **DLM** of Kracht [24], to impose some conditions on these atoms by means of the structural inference rules $(\mathbf{I} \vdash)$ and $(Q \vdash)$ so that these special atoms $\top$ and $\bot$ do behave as truth constants, as intended. The reading of $\mathbf{I}$, either as $\top$ or as $\bot$, could be separated by means of two structural constants, like for the structural connective , . Alternatively, one can easily prove that adding the following axioms to our calculus GGL_C is enough to capture the standard truth constants $\top$ and $\bot$:

$$\frac{}{\bot \vdash}\ (\bot \vdash) \qquad\qquad \frac{}{\vdash \top}\ (\vdash \top)$$

Atomic logics have four different propositional letter skeletons of type 1: $(\mathrm{Id}, +, \forall, 1)$, $(\mathrm{Id}, +, \exists, 1)$, $(\mathrm{Id}, -, \forall, 1)$, $(\mathrm{Id}, -, \exists, 1)$. Their eight introduction rules are the same as the eight rules $(\mathbf{1}), (\bot), (\top), (\mathbf{0})$ of Belnap's display calculus for linear logic [8, p. 19]. Hence, with appropriate structural rules, our four propositional letter skeletons of type 1 can stand for the four propositional constants of linear logic.

• The Boolean operator $*$ transforms the structures on which it is applied. It does not function as an operator applied externally on structures, it modifies them internally. Hence, for example, for any structure $[\circledast]\,(X_1, \ldots, X_n)$, $*\,[\circledast]\,(X_1, \ldots, X_n)$ is equal to $[-\circledast]\,(X_1, \ldots, X_n)$. In that sense, it is formally different from the usual structural connective $*$ used in display logics, even if its semantic meaning is the same (it behaves as a Boolean negation). Moreover, because by Definition 7.3 $**X = X$, the following rule is a reformulation of the display rule (dr_2) (premise and conclusion are turned upside down):

$$\frac{X \vdash (Y\ ,\ Z)}{\overline{(X\ ,\ *Z) \vdash Y}}$$

• In the calculus $\mathsf{GGL}_{\mathsf{C}}$, we do not need to consider *all* permutations τ of the symmetric group $\mathfrak{S}_{n+1}$. In fact, it suffices to consider only a set of generators of $\mathfrak{S}_{n+1}$ because rules for any permutations are derivable from these rules for generators as the following proposition shows. One could naturally consider transpositions because they generate the symmetric group and correspond to residuation operations. One could consider as well other generators of the symmetric group $\mathfrak{S}_{n+1}$, such as the pair $\{(n\ n+1), (1\ 2\ \ldots\ n+1)\}$ or the set of generators $\{(1\ 2), (2\ 3), \ldots, (i\ i+1), \ldots, (n\ n+1)\}$ or $(1\ 2)$ together with the 3–cycles (see Section 2).

7.3 Display calculi for atomic logics

Until now, our display calculi are sound and complete for logics including the Boolean connectives. However, we would like to obtain calculi for plain atomic logics, without Boolean connectives. Indeed, we consider the latter to be more primitive than Boolean atomic logics because even the Boolean connectives can be seen as particular atomic connectives, interpreted over special relations (identity relations, see Example 3.8). These special relations are obtained at the proof-theroretical level by imposing the validity of Gentzen's structural rules. So, in this section, we are going to define sound and complete calculi for (plain) atomic logics, without Boolean connectives.

Before dealing with this issue, we prove that the cut rule can be eliminated from any proof of $\mathsf{GGL}_{\mathsf{C}}$. This result relies on the fact that our atomic calculi are in fact display calculi and enjoy the display property: every substructure of a consecution provable in $\mathsf{GGL}_{\mathsf{C}}$ can be displayed as the sole antecedent or consequent of a provably equivalent consecution.

Theorem 7.8 (Cut–elimination)**.** *Let C be a set of atomic connectives. The calculus $\mathsf{GGL}_{\mathsf{C}}$ is cut–eliminable: it is possible to eliminate all occurrences of the cut rule from a given proof in order to obtain a cut-free proof of the same consecution.*

As usual in proof theory and ever since Gentzen [20], the fact that the cut rule can be eliminated from any proof is of practical and theoretical importance and we easily obtain a number of significant results about our logics. This also holds in our setting.

Theorem 7.9 (Conservativity)**.** *Let C, C' be sets of atomic connectives. If $\mathsf{C} \subseteq \mathsf{C}'$ then the logic $(\mathcal{S}_{\mathsf{C}'}, \mathcal{M}_{\mathsf{C}'}, \Vdash)$ is a conservative extension of the logic $(\mathcal{S}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

Theorem 7.10 (Soundness and strong completeness)**.** *Let C be a set of atomic connectives. The calculus $\mathsf{GGL}_{\mathsf{C}}$ is sound and strongly complete for the basic atomic logic $(\mathcal{S}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

The difference between the above theorem and Theorem 7.7 is that the set of connectives C considered is not assumed to be such that $\mathsf{C} = \mathcal{O}(\mathsf{C})$ (we recall that $\mathcal{O}(\mathsf{C})$ is defined by Expression (3)). This said, all connectives of $\mathcal{O}(\mathsf{C})$ do appear in the calculus, but only as structural connectives.

Definition 7.11. Let C be a set of atomic connectives without Boolean connectives. We denote by $\mathsf{GGL}^0_{\mathsf{C}}$ the calculus of Figure 4 where the introduction rules $(\vdash \circledast)$ and $(\circledast \vdash)$ are defined for the connectives $\circledast$ of C, where the rule (dr_1) is defined for the elements τ of an arbitrary set of generators of $\mathfrak{S}_{n+1}$ (for each n ranging over the arities of the connectives of C) and where the structural connectives $[\circledast]$ range over $[\mathcal{O}(\mathsf{C})]$.

The calculus $\mathsf{GGL}^0_{\mathsf{C}}$ is in fact the calculus of Figure 3 where the structural rules and the introduction rules for the Boolean connectives have been removed and where the display rule dr_2 has been replaced by the display rule (dr'_2).

Theorem 7.12 (Soundness and strong completeness)**.** *Let C be a set of atomic connectives without Boolean connectives. The calculus $\mathsf{GGL}^0_{\mathsf{C}}$ is sound and strongly complete for the basic atomic logic $(\mathcal{S}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

Display rules:

$$\frac{S\left(\left[\circledast\right], X_1, \ldots, X_n, X_{n+1}\right)}{S\left(\left[\tau\circledast\right], X_{\tau(1)}, \ldots, X_{\tau(n)}, X_{\tau(n+1)}\right)}\ (\mathsf{dr}_1) \qquad \frac{X \vdash Y}{*Y \vdash *X}\ (\mathsf{dr}_2')$$

Introduction rules:

$$\frac{U_1 \vdash V_1 \quad \ldots \quad U_n \vdash V_n}{S\left(\left[\circledast\right], X_1, \ldots, X_n, \circledast\left(\varphi_1, \ldots, \varphi_n\right)\right)}\ (\vdash \circledast) \qquad \frac{S\left(\left[\circledast\right], \varphi_1, \ldots, \varphi_n, U\right)}{S\left(\circledast, \varphi_1, \ldots, \varphi_n, U\right)}\ (\circledast \vdash)$$

In rules $(\vdash \circledast)$ and $(\circledast \vdash)$, for all $\circledast = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n))) \in \mathsf{C}$:

- for all $j \in [\![1;n]\!]$, we set $U_j \vdash V_j \triangleq \begin{cases} X_j \vdash \varphi_j & \text{if } \pm_j \pm (Æ) = - \\ \varphi_j \vdash X_j & \text{if } \pm_j \pm (Æ) = + \end{cases}$

  such that, in rule $(\vdash \circledast)$, for all j X_j is not empty and with the convention that if φ_j is empty for some j then $\circledast(\varphi_1, \ldots, \varphi_n)$ is also empty.
- for all $\star \in \{\circledast, [\circledast]\}$, $S(\star, X_1, \ldots, X_n, X) \triangleq \begin{cases} \star(X_1, \ldots, X_n) \vdash X & \text{if } Æ = \exists \\ X \vdash \star\left(X_1, \ldots, X_n\right) & \text{if } Æ = \forall. \end{cases}$

Figure 4: Calculus $\mathsf{GGL}^0_{\mathsf{C}}$

7.4 Display calculi for molecular logics

In this section, we provide display calculi for molecular logics whose connectives are so-called 'universal' or 'existential'. Universal and existential molecular connectives are essentially molecular connectives such that the quantification patterns of the quantification signatures of their successive atomic connectives are of the form $\forall \ldots \forall$ or $\exists \ldots \exists$ respectively. They essentially behave as 'macroscopic' atomic connectives of quantification signatures $\forall$ or $\exists$.

Definition 7.13 (Universal and existential molecular connective)**.** A *universal (resp. existential) molecular skeleton* is a molecular skeleton c different from any id^l_k for any $k, l \in \mathbb{N}^*$ such that $Æ(c) = \forall$ (resp. $Æ(c) = \exists$) and such that for each node of its decomposition tree labeled with $\circledast = (\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n))$ and each of its jth children labeled with some $\circledast_j \in \mathbb{C}$ such that the subtree generated by this j^{th} children contains at least one id^l_k, we have that $Æ(\circledast_j) = \pm_j Æ$. A *universal (resp. existential) molecular connective* is a molecular connective with a universal (resp. existential) skeleton.

Example 7.14. On the one hand, the molecular connective $\circledast(p, id^l_k)$ is a universal (resp. existential) molecular connective if $Æ(\circledast) = \forall$ (resp. $Æ(\circledast) = \exists$). Likewise, $\supset (id^1_1, \Box id^2_1)$ and $\otimes(\Diamond id^1_1, p)$ are universal and existential molecular connectives respectively. On the other hand, the molecular connectives $\Box\Diamond^- id^1_1$ and $\supset (\Box id^1_1, \Box id^2_1)$ are neither universal nor existential molecular connectives.

Definition 7.15. Let C be a set of molecular connectives which are either universal or existential. The calculi $\mathsf{GGL}^{0,*}_{\mathsf{C}}$ and $\mathsf{GGL}^*_{\mathsf{C}}$ are the display calculi such that:

- the introduction rule for each molecular connective of C is the composition of the introduction rules of the atomic connectives which compose it;

- the display rules are those which are associated to each atomic connective that appears in a molecular connective of C;
- the structural rules of $\mathsf{GGL}^*_\mathsf{C}$ are the same as those of GGL_C ($\mathsf{GGL}^{0,*}_\mathsf{C}$ does not contain any).

Note that if C are atomic connectives then $\mathsf{GGL}^{0,*}_\mathsf{C}$ and $\mathsf{GGL}^*_\mathsf{C}$ are $\mathsf{GGL}^0_\mathsf{C}$ and GGL_C respectively.

Theorem 7.16 (Soundness and strong completeness)**.** *Let C be a set of molecular connectives which are all either universal or existential. If C is without Boolean connectives then the calculus $\mathsf{GGL}^{0,*}_\mathsf{C}$ is sound and strongly complete for the molecular logic $(\mathcal{S}^0_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$. If C is Boolean then the calculus $\mathsf{GGL}^*_\mathsf{C}$ is sound and strongly complete for the Boolean molecular logic $(\mathcal{S}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$.*

8 Hilbert calculi for atomic logics

In this section on Hilbert systems, we define the notion of provability (deducibility) from a set of formulas, *i.e.* $\Gamma \vdash_\mathsf{P} \varphi$, differently, like for modal logic [12, Definition 4.4]. If $\mathsf{L} = (\mathcal{L}, E, \models)$ is a Boolean atomic logic and we have that $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$ of type k, then we say that φ is *provable from* Γ in a proof system P for $\mathcal{L}$, written $\Gamma \vdash_\mathsf{P} \varphi$, when $\vdash_\mathsf{P} \varphi$ or there are $n \in \mathbb{N}^*$ and $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $\vdash_\mathsf{P} (\varphi_1 \wedge_k \dots \wedge_k \varphi_n) \to_k \varphi$ (we use the abbreviation $\varphi \to_k \psi \triangleq (\neg_k \varphi \vee_k \psi)$). The notion of *strong completeness* of a Hilbert calculus is defined like in Definition 6.5 (that is, $\Gamma \Vdash_\mathsf{L} \varphi$ implies $\Gamma \vdash_\mathsf{P} \varphi$).

Definition 8.1. Let C be a Boolean set of atomic connectives. We denote by $\mathsf{GGL}^\mathcal{H}_\mathsf{C}$ the calculus of Figure 5 restricted to the axioms and inference rules which mention the Boolean and atomic connectives of C.

Theorem 8.2 (Soundness and strong completeness)**.** *Let C be a Boolean set of atomic connectives such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. The calculus $\mathsf{GGL}^\mathcal{H}_\mathsf{C}$ is sound and strongly complete for the Boolean basic atomic logic $(\mathcal{L}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$.*

9 Related work and conclusion

The DLE-logics introduced by Greco et al.. [23] are similar to our basic atomic logics. Their families $\mathcal{F}$ and $\mathcal{G}$ correspond in our framework to connectives of “quantification signatures” $\exists$ and $\forall$ respectively. Likewise, their order types correspond in our framework to “tonicity signatures”. Hence, several of their notions correspond to notions introduced by Dunn’s gaggle theory [13, 14].

The main difference between their and our work is that we prove the completeness of our calculi w.r.t. a Kripke-style relational semantics. We also introduce a generalized form of residuation based on the symmetric group which is novel. Unlike them, we originally introduce the Boolean negation as a primitive connective, even if one can dispose of it after proving cut elimination. An important difference between Greco & Al.’s DLE-logics and our atomic and molecular logics lies in our introduction and use of types and in the fact that we consider compositions of atomic connectives as primitive connectives. These generalizations are motivated at length in [5]. Basically, some logics/protologics cannot be represented without the use of types, such as temporal logic [5, Example 8], arrow logic, many-dimensional logics [27] and first-order logic. This use of type is crucial to represent these logics and it is also instrumental in showing that any protologic is as expressive as a molecular logic, which constitutes the main result of [5]. It complexifies the soundness and completeness proof of the present article w.r.t. the soundness and completeness proof of [2] for gaggle logics, which are actually atomic logics of type $(1, 1, \dots, 1)$. This said, one

Axiom schemas:

$\top, \neg\bot$ (A$_0$)

$(\varphi \to (\varphi \wedge \varphi))$ (A$_1$)

$((\varphi \wedge \psi) \to \varphi)$ (A$_2$)

$((\varphi \to \psi) \to (\neg(\psi \wedge \chi) \to \neg(\chi \wedge \varphi)))$ (A$_3$)

$\neg \circledast (\varphi_1, \ldots, \varphi_n) \leftrightarrow - \circledast (\varphi_1, \ldots, \varphi_n)$ (A$_4$)

For all $\circledast$ of skeleton $(\sigma, \pm, \overline{k}, (\exists, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$:
if $\pm_j = +$ then
$(\circledast(\varphi_1, \ldots, \varphi_j \vee \varphi'_j, \ldots, \varphi_n) \to (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi'_j, \ldots, \varphi_n)))$ (A$_5$)
if $\pm_j = -$ then
$(\circledast(\varphi_1, \ldots, \varphi_j \wedge \varphi'_j, \ldots, \varphi_n) \to (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi'_j, \ldots, \varphi_n)))$ (A$_6$)

For all $\circledast$ such that Æ$(\circledast) = \exists : \circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n) \to \varphi_j$ (A$_7$)
For all $\circledast$ such that Æ$(\circledast) = \forall : \varphi_j \to \circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n)$ (A$_8$)

Inference rules:

from φ and $(\varphi \to \psi)$, infer ψ (MP)

For all $\circledast$ of skeleton $(\sigma, \pm, \overline{k}, (\forall, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$:
if $\pm_j = +$ then from φ_j, infer $\circledast (\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n)$ (R$_1$)
if $\pm_j = -$ then from $\neg\varphi_j$, infer $\circledast (\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n)$ (R$_2$)

For all $\circledast$ of skeleton $(\sigma, \pm, \overline{k}, (\exists, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$:
if $\pm_j = +$ then from $\varphi_j \to \psi_j$, infer $\circledast (\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \to \circledast(\varphi_1, \ldots, \psi_j, \ldots, \varphi_n)$ (R$_3$)
if $\pm_j = -$ then from $\varphi_j \to \psi_j$, infer $\circledast (\varphi_1, \ldots, \psi_j, \ldots, \varphi_n) \to \circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n)$ (R$_4$)

Figure 5: Calculus $\mathsf{GGL}^{\mathcal{H}}$

of the main differences with the work of Palmigiano & Al. remains the fact that we are able to define *automatically* from the connectives of a given atomic logic (or specific molecular logics) sound and strongly complete display and Hilbert calculi in a generic fashion together with their *Kripke-style relational semantics* for which they are sound and complete. In particular, they do not provide a Kripke-style relational semantics to their DLE-logics, only an algebraic one which more or less mimics the axioms and inference rules of their DLE-logics. Our proofs of soundness and completeness w.r.t. the Kripke-style relational semantics resorts to the results of Dunn's gaggle theory and are not straightforward.

Atomic logics are logics of residuation to which types are added. Residuated logics have been extensively studied in the algebraic approach to logic [18]. However, it still remains to propose and adapt these algebraic approaches and semantics to our atomic and molecular logics and to show how a proof of completeness for atomic and molecular logics w.r.t. to our Kripke-style relational semantics can be obtained, as well as the other results in our series of articles. In that respect, the duality theory relating the algebraic and our Kripke-style relational semantics remains to be developed for atomic and molecular logics, in the spirit of the one for modal logic for example [12, Section 5] or for other non-classical logics like in Bimbo & Dunn [11].

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A Proofs of Propositions 2.3, 3.3, 5.2 and 5.4

They are without particular difficulty, it suffices to check the definitions.

B Proofs of Theorems 7.7 and 8.2

The proof of Theorem 7.7 is similar to the proof of Theorem 8.2 and follows the same steps as in [2]. So, we only prove Theorem 8.2. The proof for that theorem needs to be changed and is different from the proof in [2] because we need to take the types into account as well as a different notion of provability/deducibility.

Theorem B.7 (Soundness and strong completeness)**.** *Let C be a set of atomic connectives such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. The calculus $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$ is sound and strongly complete for the Boolean basic atomic logic $(\mathcal{L}^{\mathbb{B}}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

In this section, the set of atomic connectives C considered is such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$ and contains a connective of input or output type 1 and a propositional letter of type 1.

We provide the soundness and completeness proofs of Theorem 8.2. We adapt the proof methods introduced in [1], based on a Henkin construction, to our more abstract and general setting. We start by the soundness proof.

Lemma B.8. *The calculus $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$ is sound for the Boolean basic atomic logic $(\mathcal{L}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

Proof. Without particular difficulty. It follows the same line as in [2] and relies on the results of Dunn's gaggle theory. □

The completeness proof uses a canonical model built up from maximal $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent sets. First, we define the notions of $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set and maximal $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set.

Definition B.9 ((Maximal) $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set)**.** Let $k \in \mathbb{N}^*$.

- A *k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set* is a subset Γ of $\mathcal{L}^k_{\mathsf{C}}$ such that there are no $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_{\mathcal{H}} \neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$. If $\varphi \in \mathcal{L}^k_{\mathsf{C}}$, we also say that φ is *k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent* when the set $\{\varphi\}$ is k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent.
- A *maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set* is a k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set Γ of $\mathcal{L}^k_{\mathsf{C}}$ such that there is no $\varphi \in \mathcal{L}^k_{\mathsf{C}}$ satisfying both $\varphi \notin \Gamma$ and $\Gamma \cup \{\varphi\}$ is k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent. □

Lemma B.10 (Cut lemma)**.** *Let Γ be a maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set. For all $\varphi \in \Gamma$ and all $\psi \in \mathcal{L}^k_{\mathsf{C}}$, if $\vdash_{\mathcal{H}} (\varphi \to \psi)$ then $\psi \in \Gamma$.*

Proof. First, we show that $\Gamma \cup \{\psi\}$ is k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent. Assume towards a contradiction that it is not the case. Then, there are $\psi_1, \ldots, \psi_m \in \Gamma$ such that $\vdash_{\mathcal{H}} \neg(\psi \wedge \psi_1 \wedge \ldots \wedge \psi_m)$ $(*)$. Then, by the axiom A_3, we have that $\vdash_{\mathcal{H}} ((\varphi \to \psi) \to (\neg(\psi \wedge \psi_1 \wedge \ldots \wedge \psi_m) \to \neg(\psi_1 \wedge \ldots \wedge \psi_m \wedge \varphi)))$. By assumption, $\vdash_{\mathcal{H}} (\varphi \to \psi)$. Therefore, by Modus Ponens, $\vdash_{\mathcal{H}} (\neg(\psi \wedge \psi_1 \wedge \ldots \wedge \psi_m) \to \neg(\psi_1 \wedge \ldots \wedge \psi_m \wedge \varphi))$. Now, applying again Modus Ponens with $(*)$, we have that $\vdash_{\mathcal{H}} \neg(\psi_1 \wedge \ldots \wedge \psi_m \wedge \varphi)$. However, since $\varphi, \psi_1, \ldots, \psi_n \in \Gamma$, we have that Γ is not k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent. This is impossible. Thus, $\Gamma \cup \{\varphi\}$ is k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent. Now, since Γ is a *maximal* k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set, this implies that $\varphi \in \Gamma$. □

Lemma B.11 (Lindenbaum lemma)**.** *Any k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set can be extended into a maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set.*

Proof. Let $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ be an enumeration of $\mathcal{L}^k_{\mathsf{C}}$ (it exists because C is countable). We define the sets Γ_n inductively as follows:

$$\Gamma_0 \triangleq \Gamma$$
$$\Gamma_{n+1} \triangleq \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is } k\text{–}\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}\text{–consistent} \\ \Gamma_n & \text{otherwise.} \end{cases}$$

Then, we define the subset Γ^+ of $\mathcal{L}$ as follows: $\Gamma^+ = \bigcup_{n \in \mathbb{N}} \Gamma_n$.

We show that Γ^+ is a maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set. Clearly, for all $n \in \mathbb{N}$, Γ_n is k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent by definition of Γ_n. So, if Γ^+ was not k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent, there would be a $n_0 \in \mathbb{N}$ such that Γ_{n_0} is not k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent, which is impossible. Now, assume towards a contradiction that Γ^+ is not a *maximal* k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set. Then, there is $\varphi \in \mathcal{L}^k_{\mathsf{C}}$ such that $\varphi \notin \Gamma^+$ and $\Gamma \cup \{\varphi\}$ is k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent. But there is $n_0 \in \mathbb{N}$ such that $\varphi = \varphi_{n_0}$. Because $\varphi \notin \Gamma^+$, we also have that $\varphi_{n_0} \notin \Gamma_{n_0+1}$. So, $\Gamma_{n_0} \cup \{\varphi_{n_0}\}$ is not k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent by definition of Γ^+. Therefore, $\Gamma^+ \cup \{\varphi\}$ is not k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent either, which is impossible. ☐

Lemma B.12. *The following formulas are provable in $\mathsf{GGL}^{\mathcal{H}}$: for all $\varphi, \varphi' \in \mathcal{L}_{\mathsf{C}}$,*

$$(\varphi \to \varphi) \tag{6}$$
$$(\neg\neg\varphi \to \varphi) \tag{7}$$
$$(\varphi \to (\varphi' \to (\varphi \wedge \varphi'))) \tag{8}$$
$$((\varphi \wedge \varphi') \to \varphi') \tag{9}$$
$$(((\varphi \vee \varphi') \wedge (\varphi \vee \neg\varphi')) \to \varphi) \tag{10}$$
$$(\varphi \to ((\varphi \wedge \neg\varphi') \vee (\varphi \wedge \varphi'))) \tag{11}$$

Proof. The set of axioms and inference rule $\mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3$ and MP is known to be a complete axiomatization of propositional logic [28]. Since Expressions (6)–(11) are all validities of propositional logic, they are also provable in $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$ (more precisely with $\mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3$ and MP). ☐

Lemma B.13. *Let $\circledast(\varphi_1, \dots, \varphi_n) \in \mathcal{L}^k_{\mathsf{C}}$ with $\star = (\sigma, \pm, \text{Æ}, (k, k_1, \dots, k_n), (\pm_1, \dots, \pm_n))$. If $\circledast(\varphi_1, \dots, \varphi_n)$ is k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent then for all $j \in [\![1; n]\!]$, $\pm_j\varphi_j$ is k_j–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent, where $\pm_j\varphi_j \triangleq \begin{cases} \varphi_j & \text{if } \pm_j = + \\ \neg\varphi_j & \text{if } \pm_j = - \end{cases}$.*

Proof. We prove it by contraposition. Assume that $\pm_j\varphi_j$ is k_j–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–inconsistent. If $\pm_j = +$ then $\vdash_{\mathcal{H}} \neg\varphi_j$. If $\pm_j = -$ then $\vdash_{\mathcal{H}} \neg\neg\varphi_j$ and therefore $\vdash_{\mathcal{H}} \varphi_j$ because $\vdash_{\mathcal{H}} \neg\neg\varphi_j \to \varphi_j$ is provable. So, in both cases, applying Rules R_5 or R_4, we obtain that $\vdash_{\mathcal{H}} -\circledast(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$. Now, by A_4, we have that $\vdash_{\mathcal{H}} -\circledast(\varphi_1, \dots, \varphi_j, \dots \varphi_n) \to \neg\circledast(\varphi_1, \dots, \varphi_j, \dots \varphi_n)$. Therefore, by MP, we have that $\vdash_{\mathcal{H}} \neg\circledast(\varphi_1, \dots, \varphi_j, \dots \varphi_n)$ and thus $\circledast(\varphi_1, \dots, \varphi_n)$ is k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–inconsistent. ☐

Definition B.14 (Canonical model)**.** Let $\mathsf{C} \subseteq \mathbb{C}$. The *canonical model associated to* C is the tuple $M^c \triangleq (W^c, \mathcal{R}^c)$ where W^c is the set of all maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent sets of $\mathcal{L}^k_{\mathsf{C}}$, for k ranging over the output types of the connectives of C, and $\mathcal{R}^c$ is a set of relations $R_{\circledast}$ over W^c, associated to the connectives $\circledast \in \mathsf{C}$ (of skeleton $\star$) and defined by:

- if $\star = p = (\text{Id}, \pm, \text{Æ}, k)$ then for all maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set Γ, $\Gamma \in R^{\pm}_p$ iff $p \in \Gamma$;

- if $\star = (\sigma, \pm, \exists, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$ then for all $j \in [\![1; n]\!]$ and all maximal k_j–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent sets Γ_j,

  $(\Gamma_1, \ldots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$ iff for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $\varphi_1 \pitchfork_1 \Gamma_1$ and … and $\varphi_n \pitchfork_n \Gamma_n$ then $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma_{n+1}$;

- if $\star = (\sigma, \pm, \forall, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$ then for all $j \in [\![1; n]\!]$ and all maximal k_j–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent sets Γ_j,

  $(\Gamma_1, \ldots, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$ iff for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma_{n+1}$ then $\varphi_1 \pitchfork_1 \Gamma_1$ or … or $\varphi_n \pitchfork_n \Gamma_n$;

where for all $j \in [\![1; n]\!]$, $\varphi_j \pitchfork_j \Gamma_j \triangleq \begin{cases} \varphi_j \in \Gamma_j & \text{if } \pm_j = + \\ \varphi_j \notin \Gamma_j & \text{if } \pm_j = - \end{cases}$. □

Lemma B.15 (Truth lemma)**.** *For all $\varphi \in \mathcal{L}^k_{\mathsf{C}}$, for all maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent sets Γ, we have that $M^c, \Gamma \Vdash \varphi$ iff $\varphi \in \Gamma$.*

Proof. By induction on φ. The base case $\varphi = p \in \mathbb{P}$ holds trivially by definition of M^c.

- Case $\top$ and $\bot$.

They hold trivially by axiom A_0.

- Case $\neg\varphi$.

Assume that $\neg\varphi \in \Gamma$ and assume towards a contradiction that it is not the case that $M^c, \Gamma \Vdash \neg\varphi$. Then, $M^c, \Gamma \Vdash \varphi$. So, by Induction Hypothesis, $\varphi \in \Gamma$. Now, $\vdash_{\mathcal{H}} \neg(\varphi \wedge \neg\varphi)$ (that is Expression (6), $\vdash_{\mathcal{H}} (\varphi \to \varphi)$) and $\neg\varphi \in \Gamma$ by assumption. Thus, Γ is not k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent, which is impossible. Therefore, $M^c, \Gamma \Vdash \neg\varphi$.

Conversely, assume that $M^c, \Gamma \Vdash \neg\varphi$. Then, it is not the case that $M^c, \Gamma \Vdash \varphi$, so, by Induction Hypothesis, $\varphi \notin \Gamma$. Since Γ is a maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent set, this implies that $\Gamma \cup \{\varphi\}$ is not k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent. So, there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_{\mathcal{H}} \neg(\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \varphi)$. Now, because of Expression (7), we have that $\vdash_{\mathcal{H}} (\neg\neg\varphi \to \varphi)$. So, by MP and axiom A_3, we have that $\vdash_{\mathcal{H}} \neg(\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \neg\neg\varphi)$. That is, $\vdash_{\mathcal{H}} ((\varphi_1 \wedge \ldots \wedge \varphi_n) \to \neg\varphi)$ $(*)$. Then, by Expression (8) and an iterative application of the cut lemma, we have that $\neg\varphi \in \Gamma$.

- Case $(\varphi \wedge \psi)$.

We prove the following fact. This will prove this induction step because $M^c, \Gamma \Vdash \varphi \wedge \psi$ iff $M^c, \Gamma \Vdash \varphi$ and $M^c, \Gamma \Vdash \psi$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$ by induction hypothesis.

Fact B.16. *For all maximal k–$\mathsf{GGL}_{\mathsf{C}}$–consistent sets Γ, $(\varphi \wedge \psi) \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$, and $(\varphi \vee \psi) \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

Proof. Assume that $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then, since $\vdash_{\mathcal{H}} (\varphi \to (\psi \to (\varphi \wedge \psi)))$, we have by a double application of the cut lemma that $(\varphi \wedge \psi) \in \Gamma$. Conversely, assume that $(\varphi \wedge \psi) \in \Gamma$. Then, since $\vdash_{\mathcal{H}} ((\varphi \wedge \psi) \to \varphi)$, we have that $\varphi \in \Gamma$ by the cut lemma. Likewise, since $\vdash_{\mathcal{H}} ((\varphi \wedge \psi) \to \psi)$ by Expression (9), we have that $\psi \in \Gamma$. The second part of the proof is proved dually using the fact proved in the previous induction step for $\neg$ that for all maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent sets Γ, it holds that $\varphi \notin \Gamma$ iff $\neg\varphi \in \Gamma$. □

Algorithm 1

Require: $(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}_{\mathsf{C}}^{k_1} \times \ldots \times \mathcal{L}_{\mathsf{C}}^{k_n}$ and a maximal k–$\mathsf{GGL}_{\mathsf{C}}$–consistent set Γ such that $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$ with $\star = (\sigma, \pm, \exists, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$.

Ensure: A n–tuple of maximal k_j–$\mathsf{GGL}_{\mathsf{C}}$–consistent sets $(\Gamma_1, \ldots, \Gamma_n)$ such that $R_{\circledast}^{\pm\sigma}\Gamma_1 \ldots \Gamma_n\Gamma$ and $\pm_1\varphi_1 \in \Gamma_1, \ldots, \pm_n\varphi_n \in \Gamma_n$.

Let $(\varphi_1^0, \ldots, \varphi_n^0), \ldots, (\varphi_1^m, \ldots, \varphi_n^m), \ldots$ be an enumeration of $\mathcal{L}_{\mathsf{C}}^{k_1} \times \ldots \times \mathcal{L}_{\mathsf{C}}^{k_n}$;

$\Gamma_1^0 := \{\pm_1\varphi_1\}; \ldots; \Gamma_n^0 := \{\pm_n\varphi_n\}$;

for all $m \geq 0$ **do**
  for all $(\pm'_1, \ldots, \pm'_n) \in \{+, -\}^n$ **do**
    if $\circledast\left((\bowtie_1 \pm_1 \Gamma_1^m) \times_1 (\pm'_1\varphi_1^m), \ldots, (\bowtie_n \pm_n \Gamma_n^m) \times_n (\pm'_n\varphi_n^m)\right) \in \Gamma$ **then**
      $\Gamma_1^{m+1} := \Gamma_1^m \cup \{(\pm_1\pm'_1)\varphi_1^m\}$;
      $\vdots$
      $\Gamma_n^{m+1} := \Gamma_n^m \cup \{(\pm_n\pm'_n)\varphi_n^m\}$;
    end if
  end for
end for

$\Gamma_1 := \bigcup_{m \geq 0} \Gamma_1^m; \ldots; \Gamma_n := \bigcup_{m \geq 0} \Gamma_n^m$;

where for all $\varphi \in \mathcal{L}_{\mathsf{C}}^k$, $\pm\varphi \triangleq \begin{cases} \varphi & \text{if } \pm = + \\ \neg\varphi & \text{if } \pm = - \end{cases}$; for all $j \in [\![1; n]\!]$, $\times_j \triangleq \begin{cases} \wedge & \text{if } \pm_j = + \\ \vee & \text{if } \pm_j = - \end{cases}$ and $\bowtie_j \pm_j \Gamma_j^m \triangleq \begin{cases} \bigwedge \left\{\varphi \mid \varphi \in \Gamma_j^m\right\} & \text{if } \pm_j = + \\ \bigvee \left\{\neg\varphi \mid \varphi \in \Gamma_j^m\right\} & \text{if } \pm_j = - \end{cases}$.

- Case $\circledast(\varphi_1, \ldots, \varphi_n)$ with $\star = (\sigma, \pm, \exists, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$.

First, we deal with the subcase Æ $= \exists$.

Assume that $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$. We have to show that $M^c, \Gamma \Vdash \circledast(\varphi_1, \ldots, \varphi_n)$, *i.e.*, there are $\Gamma_1, \ldots, \Gamma_n \in M^c$ such that $R_{\circledast}^{\pm\sigma}\Gamma_1 \ldots \Gamma_n\Gamma$ and $\Gamma_1 \pitchfork [\![\varphi_1]\!]$ and $\ldots$ and $\Gamma_n \pitchfork [\![\varphi_n]\!]$. We build these maximal k–$\mathsf{GGL}_{\mathsf{C}}^{\mathcal{H}}$–consistent sets $\Gamma_1, \ldots, \Gamma_n$ thanks to (pseudo) Algorithm 1 (because it does not terminate). This algorithm is such that if $\circledast(\bowtie_1 \pm_1 \Gamma_1, \ldots, \bowtie_n \pm_n \Gamma_n) \in \Gamma$ then for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}^k$, there are $(\pm'_1, \ldots, \pm'_n) \in \{+, -\}^n$ such that $\circledast((\bowtie_1 \pm_1 \Gamma_1^m) \times_1 (\pm'_1\varphi_1^m), \ldots, (\bowtie_n \pm_n \Gamma_n^m) \times_n (\pm'_n\varphi_n^m)) \in \Gamma$. This is due to Expressions (10), (11) of Lemma B.12 and Axioms A_5 and A_6. What happens is that each $\bowtie_j \pm_j \Gamma_j$ is decomposed into disjunctions $((\bowtie_j \pm_j \Gamma_j) \wedge \varphi_n) \vee ((\bowtie_j \pm_j \Gamma_n) \wedge \neg\varphi_n)$ and conjunctions $((\bowtie_j \pm_j \Gamma_j) \vee \varphi_n) \wedge ((\bowtie_j \pm_j \Gamma_j) \vee \neg\varphi_n)$ depending on whether $\pm_j = +$ or $\pm_j = -$. Then, each decomposition of $\bowtie_j \pm_j \Gamma_n$ is replaced in Expression $\circledast(\bowtie_1 \pm_1 \Gamma_1, \ldots, \bowtie_n \pm_n \Gamma_n)$. This is possible thanks to rules R_3 and R_4 and this yields a new expression $(*)$. This new expression $(*)$ belongs to Γ because Γ is a maximal k–$\mathsf{GGL}_{\mathsf{C}}^{\mathcal{H}}$–consistent set, by the cut lemma. Then, we decompose again $(*)$ iteratively by applying Axioms A_5 and A_6. For each decomposition, at least one disjunct belongs to Γ because $(\varphi \vee \psi) \in \Gamma$ implies that either $\varphi \in \Gamma$ or $\psi \in \Gamma$ by Fact B.16. Finally, after having decomposed each argument of $\circledast$, we obtain that there is $(\pm'_1, \ldots, \pm'_n) \in \{+,-\}^n$ such that $\circledast\left((\ltimes_1 \pm_1 \Gamma_1^m) \times_1 (\pm'_1\varphi_1^m), \ldots, (\ltimes_n \pm_n \Gamma_n^m) \times_n (\pm'_n\varphi_n^m)\right) \in \Gamma$.

Now, let $m \geq 0$ be fixed and assume that Γ_j^m is k_j–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent. Then, $\circledast((\ltimes_1 \pm_1 \Gamma_1^m) \times_1 (\pm'_1\varphi_1^m), \ldots, (\ltimes_n \pm_n \Gamma_n^m) \times_n (\pm'_n\varphi_n^m))$ is k_j–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent because it belongs to the k_j–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent set Γ_j^m. Thus, by Lemma B.13, for all $j \in [\![1;n]\!]$, if $\pm_j = +$ then $\bigwedge \Gamma_j^m \wedge \pm'_j\varphi_j^m$ is k_j–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent and if $\pm_j = -$ then $\bigwedge \Gamma_j^m \wedge (-\pm'_j)\varphi_j^m$ is k_j–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent. That is, in both cases, Γ_j^{m+1} is k_j–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent. We have proved by induction that for all $m \geq 0$, Γ_j^m is k_j–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent. Thus, $\Gamma_1, \ldots, \Gamma_n$ are k_j–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent. Moreover, for all $j \in [\![1;n]\!]$, Γ_j are *maximally* k_j–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent because by construction for all $\varphi \in \mathcal{L}_\mathsf{C}^k$ either $\varphi \in \Gamma_j$ or $\neg\varphi \in \Gamma_j$.

Finally, we prove that $R_\circledast^{\pm\sigma}\Gamma_1 \ldots \Gamma_n\Gamma$, that is, we prove that for all $\psi_1, \ldots, \psi_n \in \mathcal{L}_\mathsf{C}^k$ if $\psi_1 \pitchfork_1 \Gamma_1$ and $\ldots$ and $\psi_n \pitchfork_n \Gamma_n$ then $\circledast(\psi_1, \ldots, \psi_n) \in \Gamma$, that is, since $\Gamma_1, \ldots, \Gamma_n$ are maximally k_j–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent sets, if $\pm_1\psi_1 \in \Gamma_1$ and $\ldots$ and $\pm_n\psi_n \in \Gamma_n$ then $\circledast(\psi_1, \ldots, \psi_n) \in \Gamma$. Assume that $\pm_1\psi_1 \in \Gamma_1$ and $\ldots$ and $\pm_n\psi_n \in \Gamma_n$, we are going to prove that $\circledast(\psi_1, \ldots, \psi_n) \in \Gamma$. Now $(\psi_1, \ldots, \psi_n) \in \mathcal{L}_\mathsf{C}^{k_1} \times \ldots \times \mathcal{L}_\mathsf{C}^{k_n}$, so there is $m_0 \geq 0$ such that $(\varphi_1^{m_0}, \ldots, \varphi_n^{m_0}) = (\psi_1, \ldots, \psi_n)$. Since $\Gamma_1^{m_0+1} \subseteq \Gamma_1$ and $\ldots$ and $\Gamma_n^{m_0+1} \subseteq \Gamma_n$, we have that the tuple $(\pm'_1, \ldots, \pm'_n)$ satisfying the condition of line 8 of Algorithm 1 is $(+, \ldots, +)$, because of the way $\Gamma_1^{m_0+1}, \ldots, \Gamma_n^{m_0+1}$ are defined. So, the condition of line 8, which is fulfilled, is $\circledast\left((\ltimes_1 \pm_1 \Gamma_1^{m_0}) \times_1 \varphi_1^{m_0}, \ldots, (\ltimes_n \pm_n \Gamma_n^{m_0}) \times_n \varphi_n^{m_0}\right) \in \Gamma$. Then, for all $j \in [\![1;n]\!]$, if $\pm_j = +$ then $\vdash_\mathcal{H} \left(\left((\ltimes_j \pm_j \Gamma_j^{m_0}) \times_j \varphi_j^{m_0}\right) \to \varphi_j^{m_0}\right)$ and if $\pm_j = -$ then $\vdash_\mathcal{H} \left(\varphi_j^{m_0} \to \left((\ltimes_j \pm_j \Gamma_j^{m_0}) \times_j \varphi_j^{m_0}\right)\right)$. Therefore, applying rules R_3 and R_4, we obtain that $\vdash_\mathcal{H} \circledast\left((\ltimes_1 \pm_1 \Gamma_1^{m_0}) \times_1 \varphi_1^{m_0}, \ldots, (\ltimes_n \pm_n \Gamma_n^{m_0}) \times_n \varphi_n^{m_0}\right) \to \circledast(\varphi_1^{m_0}, \ldots, \varphi_n^{m_0})$. Since we have proved that $\circledast\left((\ltimes_1 \pm_1 \Gamma_1^{m_0}) \times_1 \varphi_1^{m}, \ldots, (\ltimes_n \pm_n \Gamma_n^{m_0}) \times_n \varphi_n^{m_0}\right) \in \Gamma$, we obtain by the cut lemma that $\circledast(\varphi_1^{m_0}, \ldots, \varphi_n^{m_0}) \in \Gamma$ as well, that is $\circledast(\psi_1, \ldots, \psi_n) \in \Gamma$.

Conversely, assume that $M^c, \Gamma \Vdash \circledast(\varphi_1, \ldots, \varphi_n)$, we are going to show that $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$. By definition, we have that there are $\Gamma_1, \ldots, \Gamma_n \in M^c$ such that $R_\circledast^{\pm\sigma}\Gamma_1 \ldots \Gamma_n\Gamma$ and $\Gamma_1 \pitchfork [\![\varphi_1]\!]$ and $\ldots$ and $\Gamma_n \pitchfork [\![\varphi_n]\!]$. By Induction Hypothesis, we have that $\varphi_1 \pitchfork_1 \Gamma_1$ and $\ldots$ and $\varphi_n \pitchfork_n \Gamma_n$. Then, by definition of $R_\circledast^{\pm\sigma}$ in Definition B.14, we have that $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$.

Second, we deal with the subcase Æ $= \forall$.

Assume that $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$. We have to show that $M^c, \Gamma \Vdash \circledast(\varphi_1, \ldots, \varphi_n)$, *i.e.* for all $\Gamma_1, \ldots, \Gamma_n \in M^c$, $(\Gamma_1, \ldots, \Gamma_n, \Gamma) \in R_\circledast^{\pm\sigma}$ or $\Gamma_1 \pitchfork_1 [\![\varphi_1]\!]$ or $\ldots$ or $\Gamma_n \pitchfork_n [\![\varphi_n]\!]$. Assume that $(\Gamma_1, \ldots, \Gamma_n, \Gamma) \notin R_\circledast^{\pm\sigma}$. Then, since $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$, we have by Definition B.14 that $\varphi_1 \pitchfork_1 \Gamma_1$ or $\ldots$ or $\varphi_n \pitchfork_n \Gamma_n$. So, by Induction Hypothesis, we have that $\Gamma_1 \pitchfork_1 [\![\varphi_1]\!]$ or $\ldots$ or $\Gamma_n \pitchfork_n [\![\varphi_n]\!]$.

Conversely, we reason by contraposition and we assume that $\circledast(\varphi_1, \ldots, \varphi_n) \notin \Gamma$. We are going to show that $M^c, \Gamma \Vdash -\circledast(\varphi_1, \ldots, \varphi_n)$ (we recall that $-\circledast$ is a connective of C), which will prove that it is not the case that $M^c, \Gamma \Vdash \circledast(\varphi_1, \ldots, \varphi_n)$ by Proposition 5.4.

Then, by Fact B.16 and because $\vdash_\mathcal{H} (\varphi \vee \neg\varphi)$, we have that $\neg\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$ or $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$. So, by assumption, $\neg\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$. Therefore, by the cut lemma, since by Axiom A_4 $\vdash_\mathcal{H} \neg\circledast(\varphi_1, \ldots, \varphi_n) \to -\circledast(\varphi_1, \ldots, \varphi_n)$ we have that $-\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$. Hence, this case boils down to the case Æ $= \exists$ because $-\circledast = (\sigma, -\pm, (\exists, (-\pm_1, \ldots, -\pm_n)))$. This case has been proved in the previous item and we thus have that $M^c, \Gamma \Vdash -\circledast(\varphi_1, \ldots, \varphi_n)$. ☐

Lemma B.17. *For all $\circledast, \circledast'$ belonging to the same orbit of C, we have that $R_\circledast = R_{\circledast'}$.*

Proof. We prove this lemma using Axioms A_7 and A_8. First, we prove that for all $\circledast, \circledast' \in \mathsf{C}$ such that there is $\tau \in \mathfrak{S}_{n+1}$ such that $\circledast = \tau\circledast'$, we have that $R_\circledast = R_{\circledast'}$. For that, it suffices to prove that for all transpositions $\tau_j = (j\ n+1)$, we have that $R_{\tau_j\circledast} = R_\circledast$ because the transpositions generate the symmetric group. Proving $R_\circledast \subseteq R_{\tau_j\circledast}$ or $R_{\tau_j\circledast} \subseteq R_\circledast$ for all $\tau_j = (j\ n+1)$ is enough, because by double inclusion we then have that $R_\circledast \subseteq R_{\tau_j\circledast} \subseteq R_{\tau_j\tau_j\circledast} = R_\circledast$ and thus $R_\circledast = R_{\tau_j\circledast}$.

• Case $\star = (\sigma, \pm, \exists, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_{j-1}, +, \pm_{j+1}, \ldots, \pm_n))$. Then, $\tau_j\star = (\tau_j\sigma, -\pm, \forall, (k, k_1, \ldots, k_n), (-\pm_1, \ldots, -\pm_{j-1}, +, -\pm_{j+1}, \ldots, -\pm_n))$.

Assume that $(\Gamma_1, \ldots, \Gamma_{n+1}) \in R^{\pm\sigma}_\circledast$. We are going to show that $(\Gamma_1, \ldots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1, \ldots, \Gamma_{n+1}) \notin R^{-\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1, \ldots, \Gamma_{j-1}, \Gamma_{n+1}, \Gamma_{j+1}, \ldots, \Gamma_n, \Gamma_j) \notin R^{-\pm\tau_j\sigma}_{\tau_j\circledast}$. Let $\varphi_1, \ldots, \varphi_n \in \mathcal{L}^k_\mathsf{C}$ and assume that $\tau_j \circledast (\varphi_1, \ldots, \varphi_n) \in \Gamma_j$ and $\varphi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\varphi_n \pitchfork \Gamma_n$ where $\varphi_i \pitchfork \Gamma_i \triangleq \begin{cases} \varphi_i \in \Gamma_i & \text{if } \pm_i = + \\ \varphi_i \notin \Gamma_i & \text{if } \pm_i = - \end{cases}$. We want to prove that $\varphi_j \in \Gamma_{n+1}$.

Since $(\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}) \in R^{\pm\sigma}_\circledast$ and $\varphi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\tau_j \circledast (\varphi_1, \ldots \varphi_n) \in \Gamma_j$ and $\ldots$ and $\varphi_n \pitchfork \Gamma_n$, we have that $M^c, \Gamma_{n+1} \Vdash \circledast (\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n)$. So, by the truth lemma, $\circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n) \in \Gamma_{n+1}$. Now, by Axiom A_7, $\circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n) \to \varphi_j$. Therefore, $\varphi_j \in \Gamma_{n+1}$ by the cut lemma.

• Case $\star = (\sigma, \pm, \exists, (k, k_1, \ldots, k_n), (-, \ldots, -))$. Then, $\tau_j\star = (\tau_j\sigma, \pm, \exists, (k, k_1, \ldots, k_n), (-, \ldots, -))$.

Assume that $(\Gamma_1, \ldots, \Gamma_{n+1}) \in R^{\pm\sigma}_\circledast$, *i.e.* for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}^k_\mathsf{C}$, if $\varphi_1 \notin \Gamma_1$ and $\ldots$ and $\varphi_n \notin \Gamma_n$ then $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma_{n+1}$ (1). We are going to show that $(\Gamma_1, \ldots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1, \ldots, \Gamma_{n+1}, \ldots \Gamma_n, \Gamma_j) \in R^{\pm\tau_j\sigma}_{\tau_j\circledast}$, *i.e.* for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}^k_\mathsf{C}$, if $\varphi_1 \notin \Gamma_1$ and $\ldots$ and $\varphi_j \notin \Gamma_{n+1}$ and $\ldots$ and $\varphi_n \notin \Gamma$, then $\tau_j \circledast (\varphi_1, \ldots, \varphi_n) \in \Gamma_j$. Assume that $\varphi_1 \notin \Gamma_1$ and $\ldots$ and $\varphi_j \notin \Gamma_{n+1}$ and $\ldots$ and $\varphi_n \notin \Gamma_n$. We want to prove that $\tau_j \circledast (\varphi_1, \ldots, \varphi_n) \in \Gamma_j$.

Since $\varphi_j \notin \Gamma_{n+1}$, we have that $\circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n) \notin \Gamma_{n+1}$ because of the cut lemma since $\circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n) \to \varphi_j$ by Axiom A_7. Then, either $\varphi_1 \in \Gamma_1$ or $\varphi_2 \in \Gamma_2$ or $\ldots$ or $\tau_j \circledast (\varphi_1, \ldots, \varphi_n) \in \Gamma_j$ or $\varphi_{j+1} \in \Gamma_{j+1}$ or $\ldots$ or $\varphi_n \in \Gamma_n$, because of (1). However, $\varphi_1 \notin \Gamma_1, \ldots, \varphi_{j-1} \notin \Gamma_{j-1}, \varphi_{j+1} \notin \Gamma_{j+1}, \ldots, \varphi_n \notin \Gamma_n$. Therefore, $\tau_j \circledast (\varphi_1, \ldots, \varphi_n) \in \Gamma_j$.

• Case $\star = (\sigma, \pm, \forall, (k, k_1, \ldots, k_n), (\pm_1, \ldots \pm_{j-1}, +, \pm_{j+1}, \ldots, \pm_n))$. Then, $\tau_j\star = (\tau_j\sigma, -\pm, \exists, (k, k_1, \ldots, k_n), (-\pm_1, \ldots, -\pm_{j-1}, +, -\pm_{j+1}, \ldots, -\pm_n))$.

Assume that $(\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}) \notin R^{\pm\sigma}_\circledast$. We are going to show that $(\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}) \in R^{-\pm\sigma}_{\tau_j\circledast}$ *i.e.* $(\Gamma_1, \ldots, \Gamma_{n+1}, \Gamma_{j+1}, \ldots, \Gamma_n, \Gamma_j) \in R^{-\pm\tau_j\sigma}_{\tau_j\circledast}$ *i.e.* for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}^k_\mathsf{C}$, if $\varphi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\varphi_j \in \Gamma_{n+1}$ and $\varphi_{j+1} \pitchfork \Gamma_{j+1}$ and $\ldots$ and $\varphi_n \pitchfork \Gamma_n$ then $\tau_j\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma_j$ where $\varphi_i \pitchfork \Gamma_i \triangleq \begin{cases} \varphi_i \in \Gamma_i & \text{if } -\pm_i = + \\ \varphi_i \notin \Gamma_i & \text{if } -\pm_i = - \end{cases}$. Assume that $\varphi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\varphi_j \in \Gamma_{n+1}$ and $\varphi_{j+1} \pitchfork \Gamma_{j+1}$ and $\ldots$ and $\varphi_n \pitchfork \Gamma_n$. We want to show that $\tau_j \circledast (\varphi_1, \ldots, \varphi_n) \in \Gamma_j$.

Since $\varphi_j \in \Gamma_{n+1}$ and $\varphi_j \to \circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n)$ by Axiom A_8, we have by the cut lemma that $\circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots \varphi_n) \in \Gamma_{n+1}$. So, $M^c, \Gamma_{n+1} \Vdash \circledast (\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n)$ by the truth lemma. That is, for all $\Gamma'_1, \ldots, \Gamma'_n \in M^c$, either $(\Gamma'_1, \ldots, \Gamma'_n, \Gamma_{n+1}) \in R^{\pm\sigma}_\circledast$ or not $\varphi_1 \pitchfork \Gamma_1$ or $\ldots$ or $\tau_j \circledast (\varphi_1, \ldots, \varphi_n) \in \Gamma_j$ or $\ldots$ or not $\varphi_n \pitchfork \Gamma_n$ ($\varphi_i \pitchfork \Gamma_i$ is defined above). Take $(\Gamma'_1, \ldots, \Gamma'_n) = (\Gamma_1, \ldots, \Gamma_n)$. Then, by assumption, $(\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}) \notin R^{\pm\sigma}_\circledast$ and $\varphi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\varphi_{j-1} \pitchfork \Gamma_{j-1}$ and $\varphi_{j+1} \pitchfork \Gamma_{j+1}$ and $\ldots$ and $\varphi_n \pitchfork \Gamma_n$. Therefore, $\tau_j \circledast (\varphi_1, \ldots, \varphi_n) \in \Gamma_j$.

• Case $\star = (\sigma, \pm, \forall, (k, k_1, \ldots, k_n), (-, \ldots, -))$. Then, $\tau_j\star = (\tau_j\sigma, \pm, \forall, (k, k_1, \ldots, k_n), (-, \ldots, -))$.

Assume that $(\Gamma_1,\ldots,\Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$, *i.e.* for all $\varphi_1,\ldots,\varphi_n \in \mathcal{L}^k_{\mathsf{C}}$, if $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma_{n+1}$ and $\varphi_1 \in \Gamma_1$ and ... and $\varphi_n \in \Gamma_n$ then $\varphi_j \notin \Gamma_j$ (2). We are going to show that $(\Gamma_1,\ldots,\Gamma_{n+1}) \notin R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1,\ldots,\Gamma_{n+1},\ldots,\Gamma_n,\Gamma_j) \notin R^{\pm\tau_j\sigma}_{\tau_j\circledast}$ *i.e.* for all $\varphi_1,\ldots,\varphi_n \in \mathcal{L}^k_{\mathsf{C}}$ if $\tau_j \circledast (\varphi_1,\ldots,\varphi_n) \in \Gamma_j$ and $\varphi_1 \in \Gamma_1$ and ... and $\varphi_n \in \Gamma_n$ then $\varphi_j \notin \Gamma_{n+1}$. Assume that $\tau_j \circledast (\varphi_1,\ldots,\varphi_n) \in \Gamma_j$ (3) and $\varphi_1 \in \Gamma_1$ and ... and $\varphi_n \in \Gamma_n$. We want to prove that $\varphi_j \notin \Gamma_{n+1}$.

Assume towards a contradiction that $\varphi_j \in \Gamma_{n+1}$. Then, by Axiom A_8 and the cut lemma, $\circledast(\varphi_1,\ldots,\tau_j \circledast (\varphi_1,\ldots,\varphi_n),\ldots,\varphi_n) \in \Gamma_{n+1}$. Now, $\varphi_1 \in \Gamma_1$ and ... and $\varphi_{j-1} \in \Gamma_{j-1}$ and $\varphi_{j+1} \in \Gamma_{j+1}$ and ... and $\varphi_n \in \Gamma_n$. So, by (2), because $(\Gamma_1,\ldots,\Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$, we have that $\tau_j \circledast (\varphi_1,\ldots,\varphi_n) \notin \Gamma_j$. This contradicts (3).

Second, we prove that $R_{\circledast} = R_{-\circledast}$. Again, it suffices to prove that $R_{\circledast} \subseteq R_{-\circledast}$.

- Case $\star = (\sigma,\pm,\exists,(k,k_1,\ldots,k_n),(\pm_1,\ldots,\pm_n))$. Then, $-\star = (\sigma,-\pm,\forall,(k,k_1,\ldots,k_n),(-\pm_1,\ldots,-\pm_n))$. $(\Gamma_1,\ldots,\Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$ iff for all $\varphi_1,\ldots,\varphi_n \in \mathcal{L}^k_{\mathsf{C}}$, if $\varphi_1 \pitchfork \Gamma_1$ and ... and $\varphi_n \pitchfork \Gamma_n$ then $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma_{n+1}$ where $\varphi_j \pitchfork \Gamma_j = \begin{cases} \varphi_j \in \Gamma_j & \text{if } \pm_j = + \\ \varphi_j \notin \Gamma_j & \text{if } \pm_j = - \end{cases}$. We are going to show that $(\Gamma_1,\ldots,\Gamma_{n+1}) \in R^{\pm\sigma}_{-\circledast}$, *i.e.* $(\Gamma_1,\ldots,\Gamma_{n+1}) \notin R^{-\pm\sigma}_{-\circledast}$ *i.e.* for all $\varphi_1,\ldots,\varphi_n \in \mathcal{L}^k_{\mathsf{C}}$, if $-\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma_{n+1}$ then $\varphi_1 \pitchfork' \Gamma_1$ or ... or $\varphi_n \pitchfork' \Gamma_n$ (1) where $\varphi_j \pitchfork' \Gamma_j = \begin{cases} \varphi_j \in \Gamma_j & \text{if } -\pm_j = + \\ \varphi_j \notin \Gamma_j & \text{if } -\pm_j = - \end{cases}$.
So, for all j, $\varphi_j \pitchfork' \Gamma_j$ is (not $\varphi_j \pitchfork \Gamma_j$). Therefore, (1) holds iff if $\circledast(\varphi_1,\ldots,\varphi_n) \notin \Gamma_{n+1}$ and $\varphi_1 \pitchfork \Gamma_1$ and ... and $\varphi_n \pitchfork \Gamma_n$ then not $\varphi_j \pitchfork \Gamma_j$ by Axiom A_4
iff if $\varphi_1 \pitchfork \Gamma_1$ and ... and $\varphi_n \pitchfork \Gamma_n$ then $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma_{n+1}$
iff $(\Gamma_1,\ldots,\Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$ which holds by assumption.

- Case $\star = (\sigma,\pm,\forall,(k,k_1,\ldots,k_n),(\pm_1,\ldots,\pm_n))$. It is proved like the previous case. □

Lemma B.18. *We recall that $M^c = (W^c,\mathcal{R}^c)$ is the canonical model. There are a C–model $M = (W,\mathcal{R})$ and bijection functions $f_k : W^k \to (W^c)^k$ for each type k of C such that for all $\varphi \in \mathcal{L}^k_{\mathsf{C}}$ and all $(w_1,\ldots,w_k) \in W^k$, we have that $M,(w_1,\ldots,w_k) \Vdash \varphi$ iff $M^c, f_k(w_1,\ldots,w_k) \Vdash \varphi$.*

Proof. Because C is plain by assumption, the languages $\mathcal{L}_{\mathsf{C}}$ and $\mathcal{L}^k_{\mathsf{C}}$ are countable for all $k \in \mathbb{N}^*$. Therefore, there are at most $2^{\aleph_0}$ maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent sets, for each $k \in \mathbb{N}^*$. We are going to show that there are in fact $2^{\aleph_0}$ maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent sets, for each $k \in \mathbb{N}^*$. To prove it, we are going to define $2^{\aleph_0}$ maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent sets.

C contains at least a connective $\circledast_0$ of skeleton $\star_0 = (\sigma,\pm,\exists,(k,k_1,\ldots,k_n),(\pm_1,\ldots,\pm_n))$ and therefore also a connective $\circledast'_0$ of skeleton $\star'_0 = (\sigma',\pm',\exists,(k_1,k,k_2,\ldots,k_n),(\pm'_1,\ldots,\pm'_n))$ because $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. Because C is sane, there are atoms $p_2,\ldots,p_n \in \mathcal{L}_{\mathsf{C}}$ of types $k_2,\ldots,k_n$ respectively. Then, let us consider the molecular connective $\circledast = \circledast_0(\circledast'_0,p_2,\ldots,p_n)$. The molecular connective $\circledast$ is of type signature (k,k) and its quantification signature is of the form $\exists\exists$. By assumption, there is a propositional letter $p_k \in \mathsf{C}$ of type k. We consider the countable language $\mathcal{L}_{\mathsf{C}_0}$, where $\mathsf{C}_0 \triangleq \{p_k,\circledast\}$:

$$\mathcal{L}_{\mathsf{C}_0} \;:\; \varphi \;::=\; p_k \;\mid\; \circledast\varphi$$

For all $S \subseteq \mathcal{L}_{\mathsf{C}_0}$, one can show that $\Gamma_S \triangleq S \cup \{-\varphi \mid \varphi \in \mathcal{L}_{\mathsf{C}_0} - S\}$ is k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent by building a C_0–model satisfying Γ_S. The connective $\circledast$ behaves as an existential modality $\Diamond$ and can be treated as such to construct the C_0–model. Moreover, each Γ_S is distinct by definition. Therefore, there are at least $2^{\aleph_0}$ distinct maximal k–$\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$–consistent sets. And thus, since it is also an upper bound, by the Schröder–Bernstein theorem, there are in fact $2^{\aleph_0}$ maximal k–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent sets.

The cardinality of the real numbers $\mathbb{R}$ is also $2^{\aleph_0}$. Using a standard argument of cardinal arithmetic, for all $k \in \mathbb{N}^*$, there are also $2^{\aleph_0}$ tuples $(w_1, \ldots, w_k)$ of real numbers of size k. Therefore, for each $k \in \mathbb{N}^*$, we can define a bijection f_k between the tuples of real numbers $(w_1, \ldots, w_k)$ of size k, that is $\mathbb{R}^k$, and the k–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent sets of the canonical model.

We define the C–model $M = (W, \mathcal{R})$ as follows. The real numbers constitute the states of the C–model M: we define $W \triangleq \mathbb{R}$. For all types k of C, a propositional letter $p \in \mathsf{C}$ of type k holds in a tuple $(w_1, \ldots, w_k)$ of k states of W iff p belongs to $f_k(w_1, \ldots, w_k)$ in W^c. For every $\circledast \in \mathsf{C}$ of skeleton $\star = (\sigma, \pm, \text{Æ}, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$, if $\mathsf{R}_\circledast$ is the $k + k_1 + \ldots + k_n$–ary relation associated to $\circledast$ in M, then for all $w_1, \ldots, w_{k+k_1+\ldots+k_n} \in W$, we set $\mathsf{R}_\circledast^{\pm\sigma} w_1 \ldots w_{k+k_1+\ldots+k_n}$ iff $R_\circledast^{\pm\sigma} f_{k_1}(w_1, \ldots, w_{k_1}) \ldots f_{k_n}(w_{k_1+\ldots+k_{n-1}+1}, \ldots, w_{k_1+\ldots+k_{n-1}+k_n}) f_k(w_{k_1+\ldots+k_n+1}, \ldots, w_{k_1+\ldots+k_n+k})$ ($R_\circledast$ is the relation of the canonical model associated to $\circledast$). Then, M is clearly a C–model and one can show by an easy induction on φ that for all $\varphi \in \mathcal{L}_\mathsf{C}^k$ and all $(w_1, \ldots, w_k) \in W^k$, we have that $M, (w_1, \ldots, w_k) \Vdash \varphi$ iff $M^c, f_k(w_1, \ldots, w_k) \Vdash \varphi$. $\square$

Completeness proof. We prove that for all sets $\Gamma \subseteq \mathcal{L}_\mathsf{C}^k$ and all $\varphi \in \mathcal{L}_\mathsf{C}^k$, if $\Gamma \Vdash \varphi$ holds then φ is provable from Γ in $\mathsf{GGL}_\mathsf{C}^\mathcal{H}$. We reason by contraposition. Assume that φ is not provable from Γ in $\mathsf{GGL}_\mathsf{C}^\mathcal{H}$. That is, there is no proof of φ in $\mathsf{GGL}_\mathsf{C}^\mathcal{H}$ from Γ. Hence, $\Gamma \cup \{\neg\varphi\}$ is k–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent. So, by Lemma B.11, it can be extended into a maximal k–$\mathsf{GGL}_\mathsf{C}^\mathcal{H}$–consistent set Γ' such that $\{\neg\varphi\} \cup \Gamma \subseteq \Gamma'$. Now, Γ' is a state of the canonical model M^c. Then, by the truth Lemma B.15, we have that $(M^c, \Gamma') \Vdash \Gamma \cup \{\neg\varphi\}$. Finally, by Lemma B.18, we have that $(M, f_k^{-1}(\Gamma')) \Vdash \Gamma \cup \{\neg\varphi\}$, with $(M, f_k^{-1}(\Gamma'))$ a pointed C–model. Therefore, it is not the case that $\Gamma \Vdash \varphi$. $\square$

C Proofs of Theorems 7.8, 7.9, 7.10 and 7.12

The proofs are the same as in [2].

D Proof of Theorem 7.16

Theorem D.6 (Soundness and strong completeness)**.** *Let C be a set of molecular connectives which are all either universal or existential. If C is without Boolean connectives then the calculus $\mathsf{GGL}_\mathsf{C}^{0,*}$ is sound and strongly complete for the (molecular) logic $(\mathcal{S}_\mathsf{C}^0, \mathcal{M}_\mathsf{C}, \Vdash)$. If C contains Boolean connectives then the calculus $\mathsf{GGL}_\mathsf{C}^*$ is sound and strongly complete for the (molecular) logic $(\mathcal{S}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$.*

Proof sketch. The proofs with and without Boolean connectives are the same. We first consider the set C' of atomic connectives associated to C and the display calculus $\mathsf{GGL}_{\mathsf{C}'}$. Every consecution $X \vdash Y$ of $\mathcal{S}_\mathsf{C}$ can be canonically translated into a consecution $t(X) \vdash t(Y)$ of $\mathcal{S}_{\mathsf{C}'}$. Since by Theorem 7.10 $\mathsf{GGL}_{\mathsf{C}'}$ is sound and strongly complete for the basic atomic logic $(\mathcal{S}_{\mathsf{C}'}, \mathcal{M}_\mathsf{C}, \Vdash)$, if $t(X) \vdash t(Y)$ is valid then it is provable in $\mathsf{GGL}_{\mathsf{C}'}$. Now, we have to show that the proof of $t(X) \vdash t(Y)$ in $\mathsf{GGL}_{\mathsf{C}'}$ can be translated into a proof of $X \vdash Y$ in $\mathsf{GGL}_\mathsf{C}^*$. Given a proof of $t(X) \vdash t(Y)$ in $\mathsf{GGL}_{\mathsf{C}'}$, the translation basically boils down to replace any sequence of inference rules of the form $R \circ S_1 \circ \ldots \circ S_n \circ R'$ where R, R' are introduction rules and $S_1, \ldots, S_n$ are structural or display rules into a sequence of inference rules of the form $R \circ R' \circ S_1' \circ \ldots \circ S_{n'}'$ proving the same consecution. This follows from the fact that $\mathsf{GGL}_{\mathsf{C}'}$ is a display calculus and enjoys the display property. We provide below an example of such a translation for the case of the strict implication of conditional logic (the strict implication $p \Rightarrow q$ is a molecular connective of the form $\Box(p \to q)$ where $\to$ is the material implication [30]). The rules R_0, R'_0 and R_2, R'_2 should follow each other.

$$
\begin{prooftree}
\AxiomC{$p \vdash p$}
\AxiomC{$q \vdash q$}
\RightLabel{R_0}
\BinaryInfC{$(p \to q) \vdash (p\,[\to]\,q)$}
\RightLabel{S_1}
\UnaryInfC{$((p \to q)\ ,\ p) \vdash q$}
\RightLabel{S_2}
\UnaryInfC{$p \vdash (*(p \to q)\ ,\ q)$}
\RightLabel{R_1}
\UnaryInfC{$(p \wedge r) \vdash (*(p \to q)\ ,\ q)$}
\RightLabel{S_4}
\UnaryInfC{$(p \to q)\ ,\ (p \wedge r)) \vdash q$}
\RightLabel{S_5}
\UnaryInfC{$(p \to q) \vdash ((p \wedge r)\,[\to]\,q)$}
\RightLabel{R_2}
\UnaryInfC{$(p \to q) \vdash ((p \wedge r) \to q)$}
\RightLabel{R'_0}
\UnaryInfC{$\Box(p \to q) \vdash\ [\Box]\,((p \wedge r) \to q)$}
\RightLabel{R'_2}
\UnaryInfC{$\Box(p \to q) \vdash \Box\,((p \wedge r) \to q)$}
\end{prooftree}
\quad \Rightarrow \quad
\begin{prooftree}
\AxiomC{$p \vdash p$}
\AxiomC{$q \vdash q$}
\RightLabel{R_0}
\BinaryInfC{$(p \to q) \vdash (p\,[\to]\,q)$}
\RightLabel{R'_0}
\UnaryInfC{$\Box(p \to q) \vdash\ [\Box]\,(p\,[\to]\,q)$}
\RightLabel{(dr_1)}
\UnaryInfC{$[\Diamond^-]\,\Box(p \to q) \vdash (p\,[\to]\,q)$}
\RightLabel{S_1}
\UnaryInfC{$([\Diamond^-]\,\Box(p \to q)\ ,\ p) \vdash q$}
\RightLabel{S_2}
\UnaryInfC{$p \vdash (*\,[\Diamond^-]\,\Box(p \to q)\ ,\ q)$}
\RightLabel{R_1}
\UnaryInfC{$(p \wedge r) \vdash (*\,[\Diamond^-]\,\Box(p \to q)\ ,\ q)$}
\RightLabel{S_4}
\UnaryInfC{$([\Diamond^-]\,\Box(p \to q)\ ,\ p \wedge r) \vdash q$}
\RightLabel{S_5}
\UnaryInfC{$[\Diamond^-]\,\Box(p \to q) \vdash ((p \wedge r)\,[\to]\,q)$}
\RightLabel{(dr_1)}
\UnaryInfC{$\Box(p \to q) \vdash\ [\Box]\,((p \wedge r)\,[\to]\,q)$}
\RightLabel{R_2}
\UnaryInfC{$\Box(p \to q) \vdash\ [\Box]\,((p \wedge r) \to q)$}
\RightLabel{R'_2}
\UnaryInfC{$\Box(p \to q) \vdash \Box((p \wedge r) \to q)$}
\end{prooftree}
$$

$\Box$