Correspondence Theory for Atomic Logics

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Abstract

We develop the correspondence theory for the framework of atomic and molecular logics on the basis of the work of Goranko & Vakarelov. First, we show that atomic logics and modal polyadic logics can be embedded into each other. Using this embedding, we reformulate the notion of inductive formulas introduced by Goranko & Vakarelov into our framework. This allows us to prove correspondence theorems for atomic logics by adapting their results.

1 Introduction

Atomic and molecular logics, introduced in [6], are based on Dunn's Gaggle theory [16, 17] and are a generalization of modal logics which behave as 'normal form' logics or 'paradigmatic' logics. We indeed showed in [6] that every non-classical logic such that the truth conditions of its connectives can be expressed in first-order logic is as expressive as an atomic or a molecular logic. We also proved in that article that first-order logic is as expressive as a specific atomic logic. Moreover, from a model-theoretic point of view, invariance notions for atomic and molecular logics can be defined systematically from the truth conditions of their connectives and when those are uniform we obtain automatically a van Benthem characterization theorem for the logic considered [7]. These results support formally our claim that atomic and molecular logics are somehow 'universal' [8].

The display and Hilbert calculi that we introduced in [5] are sound and complete for *basic* atomic and molecular logic. That is, we are able to axiomatize the validities of logics whose class of models or frames do not satisfy any specific conditions (such as reflexivity, transitivity, *etc.*). It constitutes an advancement because, like for their bisimulation notions, we can automatically obtain the display and Hilbert calculi of any *basic* atomic or molecular logic given by its set of connectives. However, this has limitations. Indeed, we would also like to obtain automatically a sound and complete axiomatization of any atomic or molecular logic defined on *any* class of models or frames, when it is possible. Solving this problem amounts to developing a correspondence theory for atomic and molecular logics.

Correspondence theory started to be developed with modal logic as initial object of study [40, 41, 28, 30, 23]. It investigates to what extent specific properties of the semantics can be reformulated in terms of the validity of specific formulas or inference rules. More precisely, it addresses the following kinds of questions. When does the truth of a given formula in a frame corresponds to a first-order property in that frame? When does the validity of a formula on a class of frames correspond to the fact that this class of frames satisfies a specific first-order property? Which class of frames defined by a first-order property can be defined by a formula? It has been, since its inception with modal logic, extended and applied to other non-classical logics, such as tense logic [28, 29, 30]. A general and unified theory of correspondence has also been propounded by Palmigiano & Al. [12, 26, 15]. Like Kracht, they introduced a calculus for correspondence by extending the language with nominals. This enables to compute the minimal valuation and the first-order correspondent of a formula of a given non-classical logic automatically by means of an algorithm called ALBA.

In that article, we do not intend to propose another unified correspondence theory but simply to apply well–known results for modal polyadic logics [23] to our atomic logics. Hence, our overall methodology is somehow different. In order to apply the unified correspondence techniques of Palmigiano & Al. to a given logical framework, one has to recognize the logic as a normal DLE-logic introduced in [26] or a fragment thereof (*e.g.* in [14, Example 1.2]). However, it is not always possible, for example for temporal logic.[1] Here, instead of tackling the full range of non-classical logics and developing a general correspondence theory which can be instantiated, adapted and applied to each of these non-classical logics, we have first found a way to represent uniformly, faithfully and systematically the full range of non–classical logics by means of our atomic and molecular logics and now we develop a correspondence theory for these specific 'paradigmatic' logics. This 'universal' and 'paradigmatic' aspect of our atomic and molecular logics has been shown in [6] as we said. So, indirectly, our correspondence theory for atomic logics should lead to an induced correspondence theory for any non–classical logic.

Organization of the article. We start in Section 2 by recalling some basics of group theory and in Section 3, we recall atomic logics. In Section 4, we recall the Hilbert calculus for atomic logics introduced in [5]. In Section 5, we show that modal polyadic logics and atomic logics whose connectives are all of type $(1, 1, \dots, 1)$ are equally expressive. In Section 6, based on this translation, we adapt the definitions of Goranko & Vakarelov [23] to our setting and reformulate their main correspondence results in terms of atomic logics. We end in Section 7 by discussing related works and conclude.

Note. The article is self–contained. It is the second part of a series of articles on the proof and correspondence theory of atomic and molecular logics starting with [5] and continuing with [4]. All the proofs are in the appendix.

2 Notions of group theory

We first recall some basics of group theory (see for instance [39] for more details). They will not really play a role in that article, except for the definition of $\mathcal{O}(\mathsf{C})$ by Expression (1) before Theorem 4.2, but they are recalled for the sake of completeness.

Permutations and cycles. If X is a non-empty set, a *permutation* of X is a bijection $\sigma : X \to X$. We denote the set of all permutations of X by $\mathfrak{S}_X$. In the important special case when $X = \{1, \dots, n\}$, we write $\mathfrak{S}_n$ instead of $\mathfrak{S}_X$. Note that $|\mathfrak{S}_n| = n!$, where $|Y|$ denotes the number of elements in a set Y. A permutation σ on the set $\{1, \dots, n\}$ such that $\sigma(1) = x_1, \sigma(2) =$

[1] This said, the scope of applicability of unified correspondence can be significantly widened, so that the modal connectives do not need to be normal, but can also be regular [15, 36] and monotone or without any order-theoretic properties [11].

$x_2, \ldots, \sigma(n) = x_n$ is denoted $(x_1, x_2, \ldots, x_n)$. For example, $(1, 3, 2)$ is the permutation σ such that $\sigma(1) = 1, \sigma(2) = 3$ and $\sigma(3) = 2$.

If $x \in X$ and $\sigma \in \mathfrak{S}_X$, then σ *fixes* x if $\sigma(x) = x$ and σ *moves* x if $\sigma(x) \neq x$. Let $j_1, \ldots, j_r$ be distincts integers between 1 and n. If $\sigma \in \mathfrak{S}_n$ fixes the remaining $n - r$ integers and if $\sigma(j_1) = j_2, \sigma(j_2) = j_3, \ldots, \sigma(j_{r-1}) = j_r, \sigma(j_r) = j_1$ then σ is an r*–cycle*; one also says that σ is a cycle of *length* r. Denote σ by $(j_1\ j_2\ \ldots\ j_r)$. A 2–cycle which merely interchanges a pair of elements is called a *transposition*.

Groups. A *group* $(G, \circ)$ is a non–empty set G equipped with an associative operation $\circ : G \times G \to G$ and containing an element denoted Id_G called the *neutral element* such that: $\mathrm{Id}_G \circ a = a = a \circ \mathrm{Id}_G$ for all $a \in G$; for every $a \in G$, there is an element $b \in G$ such that $a \circ b = \mathrm{Id}_G = b \circ a$. This element b is unique and called the *inverse* of a, denoted a^{-1}. The set $\mathfrak{S}_n$ with the composition operation is a group called the *symmetric group on* n *letters*.

Group actions. If X is a set and G a group, an *action of* G *on* X is a function $\alpha : G \times X \to X$ given by $(g, x) \mapsto gx$ such that: $\mathrm{Id}x = x$ for all $x \in X$; $(g_1 g_2)x = g_1(g_2 x)$ for all $x \in X$ and all $g_1, g_2 \in G$. If $x \in X$ and α an action of a group G on X, then the *orbit* of x under α is $\mathcal{O}_\alpha(x) \triangleq \{\alpha(g, x) \mid g \in G\}$. The orbits form a partition of X.

3 Atomic logics

Atomic logics are non-classical logics such that the truth conditions of their connectives are defined by first-order formulas of the form $\forall x_1 \ldots x_n(\pm_1 \mathsf{Q}_1 x_1 \vee \ldots \vee \pm_n \mathsf{Q}_n x_n \vee \pm \mathsf{R} x_1 \ldots x_n x)$ or $\exists x_1 \ldots x_n(\pm_1 \mathsf{Q}_1 x_1 \wedge \ldots \wedge \pm_n \mathsf{Q}_n x_n \wedge \pm \mathsf{R} x_1 \ldots x_n x)$ where the $\pm_i$s and $\pm$ are either empty or $\neg$. Likewise, propositional letters are defined by first-order formulas of the form $\pm \mathsf{R} x$. We will represent the structure of these formulas by means of so–called *skeletons* whose various arguments capture the different features and patterns from which they can be redefined completely. Atomic logics are also generalizations of our gaggle logics [3] with types associated to formulas.

We recall that $\mathbb{N}^*$ denotes the set of natural numbers without 0 and that for all $n \in \mathbb{N}^*$, $\mathfrak{S}_n$ denotes the group of permutations over the set $\{1, \ldots, n\}$. Permutations are generally denoted σ, τ, the identity permutation Id is sometimes denoted 1 as the neutral element of every permutation group and σ^- stands for the inverse permutation of the permutation σ. For example, the permutation $\sigma = (3, 1, 2)$ is the permutation that maps 1 to 3, 2 to 1 and 3 to 2 (see for instance [39] for more details).

Definition 3.1 (Atomic skeletons and connectives)**.** The sets of *atomic skeletons* $\mathbb{P}$ and $\mathbb{C}$ are defined as follows:

$$\begin{aligned}
\mathbb{P} &\triangleq \mathfrak{S}_1 \times \{+, -\} \times \{\forall, \exists\} \times \mathbb{N}^* \\
\mathbb{C} &\triangleq \mathbb{P} \cup \bigcup_{n \in \mathbb{N}^*} \left\{ \mathfrak{S}_{n+1} \times \{+, -\} \times \{\forall, \exists\} \times {\mathbb{N}^*}^{n+1} \times \{+, -\}^n \right\}.
\end{aligned}$$

$\mathbb{P}$ is called the set of *propositional letter skeletons* and $\mathbb{C}$ is called the set of *connective skeletons.* They can be represented by tuples $(\sigma, \pm, Æ, \overline{k}, \overline{\pm_j})$ or $(\sigma, \pm, Æ, k)$ if it is a propositional letter skeleton, where $Æ \in \{\forall, \exists\}$ is called the *quantification signature* of the skeleton, $\overline{k} = (k, k_1, \ldots, k_n) \in {\mathbb{N}^*}^{n+1}$ is called the *type signature* of the skeleton and $\overline{\pm_j} = (\pm_1, \ldots, \pm_n) \in \{+, -\}^n$ is called the *tonicity signature* of the skeleton; $(Æ, \overline{k}, \overline{\pm_j})$ is called the *signature* of the skeleton. The *arity* of a propositional letter skeleton is 0 and its *type* is k. The *arity* of a skeleton $\circledast \in \mathbb{C}$ is n, its *input types* are $k_1, \ldots, k_n$ and its *output type* is k.

A *(atomic) connective* or *propositional letter* is a symbol generally denoted $\circledast$ or p to which is associated a (atomic) skeleton. Its arity, signature, quantification signature, type signature, tonicity signature, input and output types are the same as its skeleton. By abuse, we sometimes identify a connective with its skeleton. If C is a set of atomic connectives, its set of propositional letters is denoted $\mathbb{P}(\mathsf{C})$. Propositional letters are denoted p, p_1, p_2, *etc.* and connectives $\circledast, \circledast_1, \circledast_2$, *etc.*

We need to distinguish between connectives and skeletons because in general we need a countable number of propositional letters or connectives of the same skeleton, like in some modal logics, where we need multiple modalities of the same (similarity) type/skeleton.

Definition 3.2 (Atomic language)**.** Let C be a set of atomic connectives. The *(typed) atomic language* $\mathcal{L}_\mathsf{C}$ associated to C is the smallest set that contains the propositional letters and that is closed under the atomic connectives. That is,

- $\mathbb{P}(\mathsf{C}) \subseteq \mathcal{L}_\mathsf{C}$;
- for all $\circledast \in \mathsf{C}$ of arity $n > 0$ and of type signature $(k, k_1, \ldots, k_n)$ and for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_\mathsf{C}$ of types $k_1, \ldots, k_n$ respectively, we have that $\circledast(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}_\mathsf{C}$ and $\circledast(\varphi_1, \ldots, \varphi_n)$ is of *type* k.

Elements of $\mathcal{L}_\mathsf{C}$ are called *atomic formulas* and are denoted $\varphi, \psi, \alpha, \ldots$ The *type of a formula* $\varphi \in \mathcal{L}_C$ is denoted $k(\varphi)$.

The *skeleton syntactic tree* of a formula $\varphi \in \mathcal{L}_\mathsf{C}$ is the syntactic tree of the formula φ in which the nodes labeled with subformulas of φ are replaced by the skeleton of their outermost connective.

A set of atomic connectives C is *plain* if for all $\circledast \in \mathsf{C}$ of skeleton $(\sigma, \pm, \text{Æ}, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$ there are atoms $p_1, \ldots, p_n \in \mathbb{P}$ of types $k_1, \ldots, k_n$ respectively. *In the sequel, we assume that all sets of connectives* C *are plain.*

Our assumption that all sets of connectives C considered are plain makes sense. Indeed, we want all connectives of C to appear in some formula of $\mathcal{L}_\mathsf{C}$. If C was not plain then there would be a connective of C (with input type k) which would be necessarily composed with another connective of C (of output type k), if we want such a connective to appear in a formula of $\mathcal{L}_\mathsf{C}$. Yet, in that case, we should instead view C as a set of *molecular* connectives (introduced in the next section).

Definition 3.3 (C–models)**.** Let C be a set of atomic connectives. A *C–model* is a tuple $M = (W, \mathcal{R})$ where W is a non-empty set and $\mathcal{R}$ is a set of relations over W such that each n–ary connective $\circledast \in \mathsf{C}$ of type signature $(k, k_1, \ldots, k_n)$ is associated to a $k_1 + \ldots + k_n + k$–ary relation $R_\circledast \in \mathcal{R}$.

An *assignment* is a tuple $(w_1, \ldots, w_k) \in W^k$ for some $k \in \mathbb{N}^*$, generally denoted $\overline{w}$. The set of assignments of a C–model M is denoted $\omega(M, \mathsf{C})$. A *pointed C–model* $(M, \overline{w})$ is a C–model M together with an assignment $\overline{w}$. In that case, we say that $(M, \overline{w})$ *is of type* k. The class of all pointed C–models is denoted $\mathcal{M}_\mathsf{C}$. □

Definition 3.4 (Atomic logics)**.** Let C be a set of atomic connectives and let $M = (W, \mathcal{R})$ be a C–model. We define the *interpretation function of $\mathcal{L}_C$ in M*, denoted $[\![\cdot]\!]^M : \mathcal{L}_\mathsf{C} \to \bigcup_{k \in \mathbb{N}^*} W^k$, inductively as follows: for all propositional letters $p \in \mathsf{C}$ of type k, all connectives $\circledast \in \mathsf{C}$ of skeleton $(\sigma, \pm, \text{Æ}, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$ of arity $n > 0$, for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_\mathsf{C}$,

$$\begin{aligned} [\![p]\!]^M &\triangleq \begin{cases} R_p & \text{if } \pm = + \\ W^k - R_p & \text{if } \pm = - \end{cases} \\ [\![\circledast(\varphi_1, \ldots, \varphi_n)]\!]^M &\triangleq f_\circledast([\![\varphi_1]\!]^M, \ldots, [\![\varphi_n]\!]^M) \end{aligned}$$

Permutations of $\mathfrak{S}_2$	unary signatures
$\tau_1 = (1,2)$ $\tau_2 = (2,1)$	$t_1 = (\exists, (1,1), +)$ $t_2 = (\forall, (1,1), +)$
	$t_3 = (\forall, (1,1), -)$
	$t_4 = (\exists, (1,1), -)$
Permutations of $\mathfrak{S}_3$	**binary signatures**
$\sigma_1 = (1,2,3)$ $\sigma_2 = (3,2,1)$ $\sigma_3 = (3,1,2)$ $\sigma_4 = (2,1,3)$ $\sigma_5 = (2,3,1)$ $\sigma_6 = (1,3,2)$	$s_1 = (\exists, (1,1,1), (+,+))$ $s_2 = (\forall, (1,1,1), (+,-))$ $s_3 = (\forall, (1,1,1), (-,+))$
	$s_4 = (\forall, (1,1,1), (+,+))$ $s_5 = (\exists, (1,1,1), (+,-))$ $s_6 = (\exists, (1,1,1), (-,+))$
	$s_7 = (\exists, (1,1,1), (-,-))$
	$s_8 = (\forall, (1,1,1), (-,-))$

Figure 1: Permutations of $\mathfrak{S}_2$ and $\mathfrak{S}_3$ and 'families' of unary and binary signatures

where the function $f_\circledast$ is defined as follows: for all $W_1 \in \mathcal{P}(W^{k_1}), \ldots, W_n \in \mathcal{P}(W^{k_n})$, $f_\circledast(W_1, \ldots, W_n) \triangleq \left\{\overline{w}_{n+1} \in W^k \mid \mathcal{C}^\circledast(W_1, \ldots, W_n, \overline{w}_{n+1})\right\}$ where $\mathcal{C}^\circledast(W_1, \ldots, W_n, \overline{w}_{n+1})$ is called the *truth condition* of $\circledast$ and is defined as follows:

- if Æ = ∀: "$\forall \overline{w}_1 \in W^{k_1} \ldots \overline{w}_n \in W^{k_n} \left(\overline{w}_1 \pitchfork_1 W_1 \vee \ldots \vee \overline{w}_n \pitchfork_n W_n \vee R_\circledast^{\pm\sigma} \overline{w}_1 \ldots \overline{w}_n \overline{w}_{n+1}\right)$";
- if Æ = ∃: "$\exists \overline{w}_1 \in W^{k_1} \ldots \overline{w}_n \in W^{k_n} \left(\overline{w}_1 \pitchfork_1 W_1 \wedge \ldots \wedge \overline{w}_n \pitchfork_n W_n \wedge R_\circledast^{\pm\sigma} \overline{w}_1 \ldots \overline{w}_n \overline{w}_{n+1}\right)$";

where, for all $j \in [\![1;n]\!]$, $\overline{w}_j \pitchfork_j W_j \triangleq \begin{cases} \overline{w}_j \in W_j & \text{if } \pm_j = + \\ \overline{w}_j \notin W_j & \text{if } \pm_j = - \end{cases}$ and

$R_\circledast^{\pm\sigma} \overline{w}_1 \ldots \overline{w}_{n+1}$ holds iff $\pm R_\circledast \overline{w}_{\sigma^-(1)} \ldots \overline{w}_{\sigma^-(n+1)}$ holds, with the notations $+R_\circledast \triangleq R_\circledast$ and $-R_\circledast \triangleq W^{k+k_1+\ldots+k_n} - R_\circledast$. If $\mathcal{E}_\mathsf{C}$ is a class of pointed C–models, the *satisfaction relation* $\Vdash \subseteq \mathcal{E}_\mathsf{C} \times \mathcal{L}_\mathsf{C}$ is defined as follows: for all $\varphi \in \mathcal{L}_\mathsf{C}$ and all $(M, \overline{w}) \in \mathcal{E}_\mathsf{C}$, $((M, \overline{w}), \varphi) \in \Vdash$ iff $\overline{w} \in [\![\varphi]\!]^M$. We usually write $(M, \overline{w}) \Vdash \varphi$ instead of $((M, \overline{w}), \varphi) \in \Vdash$ and we say that φ is *true* in $(M, \overline{w})$.

The logic $(\mathcal{L}_\mathsf{C}, \mathcal{E}_\mathsf{C}, \Vdash)$ is the *atomic logic associated to $\mathcal{E}_\mathsf{C}$ and C*. The logics of the form $(\mathcal{L}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$ are called *basic atomic logics*.

The $\pm$ sign in $R_\circledast^{\pm\sigma}$ is the $\pm$ sign in $(\sigma, \pm, Æ, (k, k_1, \ldots, k_n), (\pm_1, \ldots, \pm_n))$.

Example 3.5 (Modal logic)**.** An example of atomic logic is modal logic where $\mathsf{C} = \{p, \top, \bot, \wedge, \vee, \Diamond_j, \Box_j \mid j \in AGTS\}$ is such that

- $\top, \bot$ are connectives of skeletons $(\mathrm{Id}, +, \exists, 1)$ and $(\mathrm{Id}, -, \forall, 1)$ respectively;
- $\wedge, \vee, \Diamond_j, \Box_j$ are connectives of skeletons $(\sigma_1, +, s_1)$, $(\sigma_1, -, s_4)$, $(\tau_2, +, t_1)$ and $(\tau_2, -, t_2)$ respectively;
- the C-models $M = (W, \mathcal{R}) \in \mathcal{E}_\mathsf{C}$ are such that $R_\wedge = R_\vee = \{(w, w, w) \mid w \in W\}$, $R_{\Diamond_j} = R_{\Box_j}$ and $R_\top = R_\bot = W$.

With these conditions on the C–models of $\mathcal{E}_{\mathsf{C}}$, for all $(M, w) \in \mathcal{E}_{\mathsf{C}}$,

$$\begin{array}{llll} w \in & [\![\Diamond_j \varphi]\!]^M & \text{iff} & \exists v (v \in [\![\varphi]\!]^M \wedge R_{\Diamond_j} wv) \\ w \in & [\![\Box_j \varphi]\!]^M & \text{iff} & \forall v (v \in [\![\varphi]\!]^M \vee -R_{\Box_j} wv) \\ w \in & [\![\wedge(\varphi, \psi)]\!]^M & \text{iff} & \exists vu (v \in [\![\varphi]\!]^M \wedge u \in [\![\psi]\!]^M \wedge R_\wedge vuw) \\ & & \text{iff} & w \in [\![\varphi]\!]^M \wedge w \in [\![\psi]\!]^M \\ w \in & [\![\vee(\varphi, \psi)]\!]^M & \text{iff} & \forall vu (v \in [\![\varphi]\!]^M \vee u \in [\![\psi]\!]^M \vee -R_\vee vuw) \\ & & \text{iff} & w \in [\![\varphi]\!]^M \vee w \in [\![\psi]\!]^M \end{array}$$

Other examples are given in Figures 2 and 3 as well as in [3, 6].

Boolean Atomic Logics. Atomic logics do not include Boolean connectives as primitive connectives. In fact, they can be defined in terms of specific atomic connectives, as follows.

Definition 3.6 (Boolean connectives)**.** The *Boolean connectives* called *conjunctions*, *disjunctions*, *negations* and *Boolean constants* (of type k) are the atomic connectives denoted, respectively:

$$\mathbb{B} \triangleq \{\wedge_k, \vee_k, \neg_k, \top_k, \bot_k \mid k \in \mathbb{N}^*\}$$

The skeleton of $\wedge_k$ is $(1, +, \exists, (k, k, k), (+, +))$, the skeleton of $\vee_k$ is $(1, -, \forall, (k, k, k), (+, +))$, the skeleton of $\neg_k$ is $(1, +, \exists, (k, k), -)$, the skeleton of $\top_k$ is $(1, +, \exists, k)$ and the skeleton of $\bot_k$ is $(1, -, \forall, k)$.

In any C-model $M = (W, \mathcal{R})$ containing Boolean connectives, the associated relation of any $\vee_k$ or $\wedge_k$ is $R_{\wedge_k} = R_{\vee_k} \triangleq \{(\overline{w}, \overline{w}, \overline{w}) \mid \overline{w} \in W^k\}$, the associated relation of any $\neg_k$ is $R_{\neg_k} \triangleq \{(\overline{w}, \overline{w}) \mid \overline{w} \in W^k\}$ and the associated relation of any $\top_k$ or $\bot_k$ is $R_{\bot_k} = R_{\top_k} \triangleq W^k$.

We say that a set of atomic or molecular connectives C is *Boolean* when it contains all conjunctions, disjunctions, constants as well as negations $\wedge_k, \vee_k, \top_k, \bot_k, \neg_k$, for k ranging over all input types and output types of the connectives of C. The *Boolean completion* of a set of atomic or molecular connectives C is the smallest set of connectives including C which is Boolean. A *Boolean atomic or molecular logic* is an atomic or molecular logic such that its set of connectives is Boolean.

Proposition 3.7 ([5])**.** *Let C be a set of atomic connectives containing Boolean connectives. and let $M = (W, \mathcal{R})$ be a C-model. Then, for all $k \in \mathbb{N}^*$, all $\varphi, \psi \in \mathcal{L}_{\mathsf{C}}$, if $k(\varphi) = k(\psi) = k$, then*

$$\begin{array}{rcl} [\![\top_k]\!]^M & \triangleq & W^k \\ [\![\bot_k]\!]^M & \triangleq & \emptyset \\ [\![\neg_k \varphi]\!]^M & \triangleq & W^k - [\![\varphi]\!]^M \\ [\![(\varphi \wedge_k \psi)]\!]^M & \triangleq & [\![\varphi]\!]^M \cap [\![\psi]\!]^M \\ [\![(\varphi \vee_k \psi)]\!]^M & \triangleq & [\![\varphi]\!]^M \cup [\![\psi]\!]^M. \end{array}$$

It turns out that Boolean negation can also be simulated systematically at the level of atomic connectives by applying a transformation on them. The Boolean negation of a formula then boils down to taking the Boolean negation of the outermost connective of the formula. This transformation is defined as follows.

Definition 3.8 (Boolean negation)**.** Let $\circledast$ be a n–ary connective of skeleton $(\sigma, \pm, \text{Æ}, \overline{k}, \pm_1, \ldots, \pm_n)$. The *Boolean negation of* $\circledast$ is the connective $-\circledast$ of skeleton $(\sigma, -\pm, -\text{Æ}, \overline{k}, -\pm_1, \ldots, -\pm_n)$ where $-\text{Æ} \triangleq \exists$ if $\text{Æ} = \forall$ and $-\text{Æ} \triangleq \forall$ otherwise, which is associated in any C–model to the same relation as $\circledast$. If $\varphi = \circledast(\varphi_1, \ldots, \varphi_n)$ is an atomic formula, the *Boolean negation of* φ is the formula $-\varphi \triangleq - \circledast (\varphi_1, \ldots, \varphi_n)$.

Atomic Connective	Truth condition	Non–classical connective in the literature
	The existentially positive orbit	
$(\sigma_1, +, s_1)\ \varphi$	$\exists v\,(v \in \llbracket\varphi\rrbracket \wedge Rvw)$	$\Diamond^{-}\varphi$ [37] $\Diamond_{\downarrow}$ [16]
$(\sigma_2, -, s_2)\ \varphi$	$\forall v\,(v \in \llbracket\varphi\rrbracket \vee -Rwv)$	$\Box\varphi$ [31]
	The universally positive orbit	
$(\sigma_1, +, s_2)\ \varphi$	$\forall v\,(v \in \llbracket\varphi\rrbracket \vee Rvw)$	$+_{\downarrow}\varphi$ [16] [19, p. 401]
$(\sigma_2, -, s_1)\ \varphi$	$\exists v\,(v \in \llbracket\varphi\rrbracket \wedge -Rwv)$	[16]
	The existentially negative orbit	
$(\sigma_1, +, s_4)\ \varphi$	$\exists v\,(v \notin \llbracket\varphi\rrbracket \wedge Rvw)$	$?\varphi$ [16][19, p. 402] $\boxminus_1\varphi$ [16][9, Def. 10.7.7]
$(\sigma_2, +, s_4)\ \varphi$	$\exists v\,(v \notin \llbracket\varphi\rrbracket \wedge Rwv)$	$?_{\downarrow}\varphi$ [16][20] [19, p. 402] $\boxminus_2\varphi$ [9, Def. 10.7.7]
	The universally negative orbit	
$(\sigma_1, +, s_3)\ \varphi$	$\forall v\,(v \notin \llbracket\varphi\rrbracket \vee Rvw)$	$\varphi^{\perp}$ [16, 18] $\varphi^{\mathbf{o}}$ [24] $\diamond_1^{-}\varphi$ [9, Def. 10.7.2]
$(\sigma_2, +, s_3)\ \varphi$	$\forall v\,(v \notin \llbracket\varphi\rrbracket \vee Rwv)$	$\sim\varphi$ [22] ${}^{\perp}\varphi$ [16, 18] ${}^{\mathbf{o}}\varphi$ [24] $\diamond_2^{-}\varphi$ [9, Def. 10.7.2]
	The symmetrical existentially positive orbit	
$(\sigma_1, -, s_1)\ \varphi$	$\exists v\,(v \in \llbracket\varphi\rrbracket \wedge -Rvw)$	[16]
$(\sigma_2, +, s_2)\ \varphi$	$\forall v\,(v \in \llbracket\varphi\rrbracket \vee Rwv)$	$+\varphi$ [16] [19, p. 402] φ^{*} [9, Def. 7.1.19]
	The symmetrical universally positive orbit	
$(\sigma_1, -, s_2)\ \varphi$	$\forall v\,(v \in \llbracket\varphi\rrbracket \vee -Rvw)$	$\Box^{-}\varphi$ [37] $\Box_{\downarrow}$ [16]
$(\sigma_2, +, s_1)\ \varphi$	$\exists v\,(v \in \llbracket\varphi\rrbracket \wedge Rwv)$	$\Diamond\varphi$ [31]
	The symmetrical existentially negative orbit	
$(\sigma_1, -, s_4)\ \varphi$	$\exists v\,(v \notin \llbracket\varphi\rrbracket \wedge -Rvw)$	$?\varphi$ [16][9, Ex. 1.4.5] $\varphi^{\mathbf{1}}$ [24]
$(\sigma_2, -, s_4)\ \varphi$	$\exists v\,(v \notin \llbracket\varphi\rrbracket \wedge -Rwv)$	$?_{\downarrow}\varphi$ [16] [9, Ex. 1.4.5] ${}^{\mathbf{1}}\varphi$ [24]
	The symmetrical universally negative orbit	
$(\sigma_1, -, s_3)\ \varphi$	$\forall v\,(v \notin \llbracket\varphi\rrbracket \vee -Rvw)$	[16]
$(\sigma_2, -, s_3)\ \varphi$	$\forall v\,(v \notin \llbracket\varphi\rrbracket \vee -Rwv)$	$\neg_{l}\varphi$ [32, 38] $\perp\varphi$ [20]

Figure 2: The unary connectives of atomic logics of type $(1, 1)$

Atomic connective	Truth condition	Non–classical con. in the literature
The conjunction orbit		
$\varphi\ (\sigma_1, +, s_1)\ \psi$	$\exists vu\,(v \in \llbracket\varphi\rrbracket \wedge u \in \llbracket\psi\rrbracket \wedge Rvuw)$	$\varphi \circ \psi$ [33], $\varphi \otimes_3 \psi$ [2]
$\varphi\ (\sigma_2, -, s_2)\ \psi$	$\forall vu\,(v \in \llbracket\varphi\rrbracket \vee u \notin \llbracket\psi\rrbracket \vee -Rwuv)$	
$\varphi\ (\sigma_3, -, s_2)\ \psi$	$\forall vu\,(v \in \llbracket\varphi\rrbracket \vee u \notin \llbracket\psi\rrbracket \vee -Ruwv)$	$/$ [33], $\varphi \subset_2 \psi$ [2]
$\varphi\ (\sigma_4, +, s_1)\ \psi$ $= \psi\ (\sigma_1, +, s_1)\ \varphi$	$\exists vu\,(v \in \llbracket\varphi\rrbracket \wedge u \in \llbracket\psi\rrbracket \wedge Ruvw)$	
$\varphi\ (\sigma_5, -, s_3)\ \psi$ $= \psi\ (\sigma_2, -, s_2)\ \varphi$	$\forall vu\,(v \notin \llbracket\varphi\rrbracket \vee u \in \llbracket\psi\rrbracket \vee -Rwvu)$	$\backslash$ [33], $\varphi \supset_1 \psi$ [2]
$\varphi\ (\sigma_6, -, s_3)\ \psi$ $= \psi\ (\sigma_3, -, s_2)\ \varphi$	$\forall vu\,(v \notin \llbracket\varphi\rrbracket \vee u \in \llbracket\psi\rrbracket \vee -Rvwu)$	
The not–but orbit		
$\varphi\ (\sigma_1, +, s_6)\ \psi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \in \llbracket\psi\rrbracket \wedge Rvuw)$	$\varphi \succ_3 \psi$ [2]
$\varphi\ (\sigma_2, +, s_6)\ \psi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \in \llbracket\psi\rrbracket \wedge Rwuv)$	
$\varphi\ (\sigma_3, -, s_4)\ \psi$	$\forall vu\,(v \in \llbracket\varphi\rrbracket \vee u \in \llbracket\psi\rrbracket \vee -Ruwv)$	$\varphi \oplus_2 \psi$ [2]
$\varphi\ (\sigma_4, +, s_5)\ \psi$ $= \psi\ (\sigma_1, +, s_6)\ \varphi$	$\exists vu\,(v \in \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Ruvw)$	
$\varphi\ (\sigma_5, +, s_5)\ \psi$ $= \psi\ (\sigma_2, +, s_6)\ \varphi$	$\exists vu\,(v \in \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rwvu)$	$\varphi \prec_1 \psi$ [2]
$\varphi\ (\sigma_6, -, s_4)\ \psi$ $= \psi\ (\sigma_3, -, s_4)\ \varphi$	$\forall vu\,(v \in \llbracket\varphi\rrbracket \vee u \in \llbracket\psi\rrbracket \vee -Rvwu)$	
The but–not orbit		
$\varphi\ (\sigma_1, +, s_5)\ \psi$	$\exists vu\,(v \in \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rvuw)$	$\varphi \prec_3 \psi$ [2]
$\varphi\ (\sigma_2, -, s_4)\ \psi$	$\forall vu\,(v \in \llbracket\varphi\rrbracket \vee u \in \llbracket\psi\rrbracket \vee -Rwuv)$	
$\varphi\ (\sigma_3, +, s_6)\ \psi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \in \llbracket\psi\rrbracket \wedge Ruwv)$	$\varphi \succ_2 \psi$ [2]
$\varphi\ (\sigma_4, +, s_6)\ \psi$ $= \psi\ (\sigma_1, +, s_5)\ \varphi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \in \llbracket\psi\rrbracket \wedge Ruvw)$	$\varphi \obslash \psi$ [27, 35]
$\varphi\ (\sigma_5, -, s_4)\ \psi$ $= \psi\ (\sigma_2, -, s_4)\ \varphi$	$\forall vu\,(v \in \llbracket\varphi\rrbracket \vee u \in \llbracket\psi\rrbracket \vee -Rwvu)$	$\varphi \oplus \psi$ [27, 35] $\varphi \oplus_1 \psi$ [2]
$\varphi\ (\sigma_6, +, s_5)\ \psi$ $= \psi\ (\sigma_3, +, s_6)\ \varphi$	$\exists vu\,(v \in \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rvwu)$	$\varphi \oslash \psi$ [27, 35]
The stroke orbit		
$\varphi\ (\sigma_1, +, s_7)\ \psi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rvuw)$	$\varphi \mid_3 \psi$ [1, 24]
$\varphi\ (\sigma_2, +, s_7)\ \psi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rwuv)$	
$\varphi\ (\sigma_3, +, s_7)\ \psi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Ruwv)$	
$\varphi\ (\sigma_4, +, s_7)\ \psi$ $= \psi\ (\sigma_1, +, s_7)\ \varphi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Ruvw)$	
$\varphi\ (\sigma_5, +, s_7)\ \psi$ $= \psi\ (\sigma_2, +, s_7)\ \varphi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rwvu)$	$\varphi \mid_1 \psi$ [1, 24]
$\varphi\ (\sigma_6, +, s_7)\ \psi$ $= \psi\ (\sigma_3, +, s_7)\ \varphi$	$\exists vu\,(v \notin \llbracket\varphi\rrbracket \wedge u \notin \llbracket\psi\rrbracket \wedge Rvwu)$	$\varphi \mid_2 \psi$ [1, 24]

Figure 3: Some binary connectives of atomic logics of type $(1, 1, 1)$

Proposition 3.9 ([6])**.** *Let C be a set of atomic connectives such that $-\circledast \in \mathsf{C}$ for all $\circledast \in \mathsf{C}$. Let $\varphi \in \mathcal{L}_{\mathsf{C}}$ of type k and let $M = (W, \mathcal{R})$ be a C–model. Then, for all $\overline{w} \in W^k$, $\overline{w} \in [\![-\varphi]\!]^M$ iff $\overline{w} \notin [\![\varphi]\!]^M$.*

4 Hilbert calculi for atomic logics

In this section on Hilbert calculi, we define the notion of provability (deducibility) from a set of formulas, *i.e.* $\Gamma \vdash_{\mathsf{P}} \varphi$, differently, like for modal logic [10, Definition 4.4]. If $\mathsf{L} = (\mathcal{L}, E, \models)$ is a Boolean atomic logic and we have that $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$ of type k, then we say that φ is *provable from* Γ in a proof system P for $\mathcal{L}$, written $\Gamma \vdash_{\mathsf{P}} \varphi$, when $\vdash_{\mathsf{P}} \varphi$ or there are $n \in \mathbb{N}^*$ and $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_{\mathsf{P}} (\varphi_1 \wedge_k \ldots \wedge_k \varphi_n) \rightarrow_k \varphi$ (we use the abbreviation $\varphi \rightarrow_k \psi \triangleq (\neg_k \varphi \vee_k \psi)$). The notion of *strong completeness* of a Hilbert calculus is defined as usual by $\Gamma \Vdash_{\mathsf{L}} \varphi$ implies $\Gamma \vdash_{\mathsf{P}} \varphi$.

Definition 4.1. Let C be a set of atomic connectives complete for Boolean connectives. We denote by $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$ the calculus of Figure 4 restricted to the axioms and inference rules which mention the atomic connectives of C.

If C is a set of atomic connectives, we define

$$\mathcal{O}(\mathsf{C}) \triangleq (\mathsf{C} \cap \mathbb{B}) \cup \bigcup_{\circledast \in \mathsf{C} - \mathbb{B}} \{\tau_1 - \ldots - \tau_m \circledast \mid \circledast \text{ is of arity } n \text{ and } \tau_1, \ldots, \tau_m \in \mathfrak{S}_{n+1}\}. \quad (1)$$

We need to introduce these connectives because in the completeness proof, we need to apply the abstract law of residuation for any arguments j and consider the Boolean negation for each connective.

Theorem 4.2 (Soundness and strong completeness, [5])**.** *Let C be a Boolean set of atomic connectives such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. The calculus $\mathsf{GGL}^{\mathcal{H}}_{\mathsf{C}}$ is sound and strongly complete for the Boolean basic atomic logic $(\mathcal{L}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

5 Modal polyadic logic versus atomic logics

In this section, we only consider languages whose connectives are all of type $(1, 1, \ldots, 1)$ in order to ease the presentation. This said, all notions can be easily adapted to languages of arbitrary types and all the results that follow would still hold in that more general case, even if we would need to define *typed* modal polyadic formulas as we did for atomic logics from gaggle logics.

5.1 Modal polyadic logic

These definitions are taken verbatim from Goranko & Vakarelov [23], except for the last clause of Definition 5.4 (we also removed the assumption that the propositional variables are countably infinite).

Definition 5.1 (Modal similarity type)**.** A purely modal polyadic language $\mathcal{L}_\tau$ contains a set of propositional variables VAR, negation $\neg$, and a *modal similarity type* τ consisting of a set of *basic modal terms* (modalities) with pre-assigned finite arities, including a 0-ary modality ι_0, a unary one ι_1 and a binary one ι_2.

Axiom schemas:

$$\top, \neg\bot \qquad (\mathsf{A}_0)$$
$$(\varphi \to (\varphi \wedge \varphi)) \qquad (\mathsf{A}_1)$$
$$((\varphi \wedge \psi) \to \varphi) \qquad (\mathsf{A}_2)$$
$$((\varphi \to \psi) \to (\neg(\psi \wedge \chi) \to \neg(\chi \wedge \varphi))) \qquad (\mathsf{A}_3)$$

$$\neg \circledast (\varphi_1, \ldots, \varphi_n) \leftrightarrow - \circledast (\varphi_1, \ldots, \varphi_n) \qquad (\mathsf{A}_4)$$

For all $\circledast$ of skeleton $(\sigma, \pm, \overline{k}, (\exists, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$:
if $\pm_j = +$ then
$$(\circledast(\varphi_1, \ldots, \varphi_j \vee \varphi'_j, \ldots, \varphi_n) \to (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi'_j, \ldots, \varphi_n))) \qquad (\mathsf{A}_5)$$
if $\pm_j = -$ then
$$(\circledast(\varphi_1, \ldots, \varphi_j \wedge \varphi'_j, \ldots, \varphi_n) \to (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi'_j, \ldots, \varphi_n))) \qquad (\mathsf{A}_6)$$

For all $\circledast$ such that $\text{Æ}(\circledast) = \exists : \circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n) \to \varphi_j$ $\qquad (\mathsf{A}_7)$
For all $\circledast$ such that $\text{Æ}(\circledast) = \forall : \varphi_j \to \circledast(\varphi_1, \ldots, \tau_j \circledast (\varphi_1, \ldots, \varphi_n), \ldots, \varphi_n)$ $\qquad (\mathsf{A}_8)$

Inference rules:

from φ and $(\varphi \to \psi)$, infer ψ $\qquad (\mathsf{MP})$

For all $\circledast$ of skeleton $(\sigma, \pm, \overline{k}, (\forall, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$:
if $\pm_j = +$ then from φ_j, infer $\circledast (\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n)$ $\qquad (\mathsf{R}_1)$
if $\pm_j = -$ then from $\neg\varphi_j$, infer $\circledast (\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n)$ $\qquad (\mathsf{R}_2)$

For all $\circledast$ of skeleton $(\sigma, \pm, \overline{k}, (\exists, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$:
if $\pm_j = +$ then from $\varphi_j \to \psi_j$, infer $\circledast (\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \to \circledast(\varphi_1, \ldots, \psi_j, \ldots, \varphi_n)$ $\qquad (\mathsf{R}_3)$
if $\pm_j = -$ then from $\varphi_j \to \psi_j$, infer $\circledast (\varphi_1, \ldots, \psi_j, \ldots, \varphi_n) \to \circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n)$ $\qquad (\mathsf{R}_4)$

Figure 4: Calculus $\mathsf{GGL}^{\mathcal{H}}$

The intuition behind the three distinguished modalities above is simple: ι_1 will be interpreted as the constant $\top$ and its negation $\bot$; ι_1 will be interpreted as our *id* and ι_2 will be the Boolean disjunction $\vee$ and its dual the Boolean conjunction $\wedge$.

Definition 5.2 (Modal polyadic language)**.** By simultaneous mutual induction we define the set of *modal terms* $MT(\tau)$ and their *arity function* ρ, and the set of *(purely) modal formulas* $MF(\tau)$ as follows:

(MT i) Every basic modal term of $MT_0(\tau)$ is a modal term of the predefined arity;

(MT ii) Every formula containing no propositional variables (hereafter called a *constant formula*) is a 0-ary modal term;

(MT iii) If $n > 0$, $\alpha, \beta_1, \ldots, \beta_n \in MT(\tau)$ and $\rho(\alpha) = n$, then $\alpha(\beta_1, \ldots, \beta_n) \in MT(\tau)$ and $\rho(\alpha(\beta_1, \ldots, \beta_n)) = \rho(\beta_1) + \ldots + \rho(\beta_n)$.

Modal terms of arity 0 will be called *modal constants.*

(MF i) Every propositional variable is a modal formula;

(MF ii) Every modal constant is a modal formula;

(MF iii) If φ is a formula then $\neg\varphi$ is a formula;

(MF iv) If $\varphi_1, \dots, \varphi_n$ are formulas, α is a modal term and $\rho(\alpha) = n > 0$, then $[\alpha](\varphi_1, \dots, \varphi_n)$ is a modal formula.

We also use the abbreviation $\langle\alpha\rangle(\varphi_1, \dots, \varphi_n) \triangleq \neg[\alpha](\neg\varphi_1, \dots, \neg\varphi_n)$.

Definition 5.3 (Kripke τ-model)**.** A *(Kripke) τ-frame* is a structure $F = (W, \{R_\alpha \mid \alpha \in MT_0(\tau)\})$ where the relations R_α are such that:

- $R_{\iota_0} = W$, $R_{\iota_1} = \{(w, w) \mid w \in W\}$, $R_{\iota_2} = \{(w, w, w) \mid w \in W\}$;
- for every basic modal term $\alpha \in MT_0(\tau)$, $R_\alpha \subseteq W^{\rho(\alpha)+1}$.

A *(Kripke) τ-model over F* is a pair $M = (F, V)$ where $V : VAR \to \mathcal{P}(W)$ is a *valuation* of the propositional variables in F.

Note that any τ–frame is a C–frame, and vice versa.

Definition 5.4 (Modal polyadic logic)**.** The *truth condition* of a formula at a state w of a Kripke model M is defined through the following clauses: for all basic modal term α,

- $M, w \models p$ iff $w \in V(p)$;
- $M, w \models \neg\varphi$ iff not $M, w \models \varphi$;
- $M, w \models [\alpha](\varphi_1, \dots, \varphi_n)$ iff for all $u_1, \dots, u_n \in W$ such that $R_\alpha u_1 \dots u_n w$, $M, u_i \models \varphi_i$ holds for some $i \in \{1, \dots, n\}$;
- $M, w \models [\alpha(\beta_1, \dots, \beta_n)](\varphi_1^1, \dots, \varphi_1^{k_1}, \dots, \varphi_n^1, \dots, \varphi_n^{k_n})$ iff
  $M, w \models [\alpha]\Big([\beta_1](\varphi_1^1, \dots, \varphi_1^{k_1}), \dots, [\beta_n](\varphi_n^1, \dots, \varphi_n^{k_n})\Big)$.

Definition 5.5 (General τ–frame)**.** Given a τ–frame $F = (W, \{R_\alpha \mid \alpha \in MT_0(\tau)\})$, every n–ary basic modal term α defines two n–ary *operators*, $\langle\alpha\rangle$ and $[\alpha]$, on $\mathcal{P}(W)$ as follows: $[\alpha](X_1, \dots, X_n) = \{w \in W \mid R_\alpha w_1 \dots w_n w$ implies$w_1 \in X_1$ or $\dots$ or $w_n \in X_n\}$ and dually, $\langle\alpha\rangle(X_1, \dots, X_n) = -[\alpha](-X_1, \dots, -X_n) = \{w \in W \mid R_\alpha w_1 \dots w_n w$ for some $w_1 \in X_1, \dots, w_n \in X_n\}$.

If τ is a modal similarity type, a *general τ–frame* is a structure $\mathcal{F} = (W, \{R_\alpha \mid \alpha \in MT_0(\tau)\}, \mathbb{W})$ extending a τ-frame $(W, \{R_\alpha \mid \alpha \in MT_0(\tau)\})$ with a Boolean algebra $\mathbb{W}$ of subsets of W called *admissible subsets* closed under the operators $[\alpha], \langle\alpha\rangle$ for every basic modal term $\alpha \in MT_0(\tau)$. A *model over $\mathcal{F}$* is any model over the Kripke τ–frame $(W, \{R_\alpha \mid \alpha \in MT_0(\tau)\})$ with valuation of the propositional variables ranging over $\mathbb{W}$. If $\varphi \in \mathcal{L}_\tau$ and $w \in W$, we say that *φ is (locally) valid at w in $\mathcal{F}$*, denoted $\mathcal{F}, w \models \varphi$, if φ is true at w in every model over $\mathcal{F}$.

Definition 5.6 (Local d-persistence)**.** A general τ–frame $\mathcal{F} = (W, \{R_\alpha \mid \alpha \in MT_0(\tau)\}, \mathbb{W})$ is:

- *differentiated* if for every $w, v \in W$, if $w \neq v$ then there is $X \in \mathbb{W}$ such that $x \in X$ and $y \notin X$;
- *tight* if for every basic modal term $\alpha \in MT_0(\tau)$, for all $w, w_1, \dots, w_n \in W$, $R_\alpha w_1 \dots w_n w$ iff for all $X_1, \dots, X_n \in \mathbb{W}$, $w_1 \in X_1, \dots, w_n \in X_n$ imply that $w \in \langle\alpha\rangle(X_1, \dots, X_n)$;
- *compact* if every family of admissible sets in $\mathcal{F}$ with the finite intersection property has a non–empty intersection;
- *refined* if it is differentiated and tight;

- *descriptive* if it is refined and compact.

A formula $\varphi \in \mathcal{L}_\tau$ is *locally d–persistent* if for every descriptive general τ–frame $\mathcal{F} = (F, \mathbb{W})$ and all $w \in F$, $\mathcal{F}, w \models \varphi$ iff $F, w \models \varphi$.

Theorem 5.7 (Goranko & Vakarelov [23]). *Every inductive formula is locally d-persistent.*

This theorem implies that inductive formulas are *canonical*, that is, they are true on the canonical frame for a logic which includes them as theorem, and therefore their first–order frame correspondent holds in the canonical frame as well. That enables to prove the completeness of various calculi w.r.t. a given semantics.

5.2 Equi-expressivity of Boolean atomic logics and modal polyadic logics

5.2.1 Translation from modal polyadic logics to atomic logics

Definition 5.8 (Atomic connectives associated to a modal similarity type). Let τ be a modal similarity type. The set of *atomic connectives associated to* τ, denoted C_τ, is the set of basic modal terms together with the set of propositional variables VAR as well as the truth constants $\top$ and the Boolean disjunction $\vee$ and negation $\neg$. Their skeletons are defined as follows. The skeletons of the propositional variables are all $(\mathrm{Id}, +, \exists, 1)$. The skeletons of the basic modal terms α of arity n are all $(\mathrm{Id}, -, \forall, (1, 1, \ldots, 1), (+, \ldots, +))$. The skeleton of the Boolean constant $\top$ is $(\mathrm{Id}, +, \exists, 1)$, the skeleton of the Boolean connective $\vee$ is $(\mathrm{Id}, -, \forall, (1, 1, 1), (+, +))$ and the skeleton of the Boolean connective $\neg$ is $(1, +, \exists, (1, 1), -)$.

Note that $\vee = \vee_1$ and $\neg = \neg_1$ of Definition 3.6.

Definition 5.9 (C_τ–model associated to a τ–model). Let $M = (W, \{R_\alpha \mid \alpha \in MT(\tau)\}, V)$ be a τ–model. The *C_τ–model associated to M* is the C_τ–model $T(M) \triangleq (W, \{R_\alpha \mid \alpha \in MT_0(\tau)\} \cup \{R_p \mid p \in VAR\})$ where $MT_0(\tau)$ is the set of basic modal terms and for all $p \in VAR$, $R_p \triangleq V(p)$.

Definition 5.10 (Formula associated to a modal polyadic formula). Let τ be a modal similarity type and let φ be a modal formula of $MF(\tau)$. The *formula of $\mathcal{L}_{\mathsf{C}_\tau}$ associated to* φ, denoted $T(\varphi)$, is defined inductively as follows: for all basic modal term $\alpha \in \tau$ (distinct from $\iota_0, \iota_1, \iota_2$) and all $p \in VAR$,

$$
\begin{aligned}
T(p) &\triangleq p \\
T(\iota_0) &\triangleq \top \\
T([\iota_1]\varphi) &\triangleq T(\varphi) \\
T([\iota_2](\varphi, \psi)) &\triangleq \vee(T(\varphi), T(\psi)) \\
T(\neg\varphi) &\triangleq \neg T(\varphi) \\
T([\alpha](\varphi_1, \ldots, \varphi_n)) &\triangleq \alpha(T(\varphi_1), \ldots, T(\varphi_n)) \\
T([\alpha(\beta_1, \ldots, \beta_n)](\varphi_1^1, \ldots, \varphi_1^{k_1}, \ldots, \varphi_n^1, \ldots, \varphi_n^{k_n})) &\triangleq \alpha\left(T([\beta_1](\varphi_1^1, \ldots, \varphi_1^{k_1})), \ldots, T([\beta_n](\varphi_n^1, \ldots, \varphi_n^{k_n}))\right).
\end{aligned}
$$

Proposition 5.11. *Let τ be a modal similarity type, let φ be a formula of $MF(\tau)$ and let M be a τ–model. Then, for all $w \in M$, it holds that $M, w \models \varphi$ iff $T(M), w \Vdash T(\varphi)$.*

5.2.2 Translation from atomic logics to modal polyadic logics

Definition 5.12 (Modal similarity type associated to a set of atomic connectives)**.** Let C be a set of atomic connectives. The *modal similarity type associated to* C, denoted τ_{C}, consists of the orbits of C as well as ι_0, ι_1 and ι_2; the propositional variables VAR of $\mathcal{L}_{\tau_{\mathsf{C}}}$ consist of the propositional letters of C. The basic modal term associated to an orbit containing the connective $\circledast$ is denoted $\alpha_{\circledast}$.

Definition 5.13 (τ_{C}–model associated to a C–model)**.** Let C be a set of atomic connectives and let $M = (W, \mathcal{R})$ be a C–model. The *τ_{C}–model associated to M* is the τ_{C}–model $T^-(M) \triangleq (W, \{R_\alpha \mid \alpha \in MT_0(\tau)\}, V)$ such that

- for every basic modal term $\alpha \in MT_0(\tau_{\mathsf{C}})$ corresponding to the orbit of a connective $\circledast \in \mathsf{C}$ of skeleton $(\sigma, \pm, \overline{k}, (Æ, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$, we have that $R_\alpha = \begin{cases} -R_{\circledast}^{\pm\sigma} & \text{if } Æ = \forall \\ R_{\circledast}^{\pm\sigma} & \text{if } Æ = \exists \end{cases}$;
- for all $p \in VAR$, we have that $V(p) = R_p$.

Hence, every τ_{C}–frame based on $T^-(M)$ is equal to the C–frame based on M.

Definition 5.14 (Modal polyadic formula associated to an atomic formula)**.** Let C be a set of (Boolean) atomic connectives and let φ be a formula of $\mathcal{L}_{\mathsf{C}}$. The *modal polyadic formula of $MF(\tau_{\mathsf{C}})$ associated to* φ, denoted $T^-(\varphi)$, is defined inductively as follows:

$$\begin{aligned}
T^-(p) &\triangleq p \\
T^-(\top) &\triangleq \iota_0 \\
T^-(\bot) &\triangleq \neg\iota_0 \\
T^-(\neg\varphi) &\triangleq \neg T^-(\varphi) \\
T^-(\varphi \vee \psi) &\triangleq [\iota_2](T^-(\varphi), T^-(\psi)) \\
T^-(\varphi \wedge \psi) &\triangleq \neg[\iota_2](\neg T^-(\varphi), \neg T^-(\psi)) \\
T^-(\circledast(\varphi_1, \ldots, \varphi_n)) &\triangleq \begin{cases} [\alpha_{\circledast}](\pm_1 T^-(\varphi_1), \ldots, \pm_n T^-(\varphi_n)) & \text{if } Æ(\circledast) = \forall \\ \langle\alpha_{\circledast}\rangle(\pm_1 T^-(\varphi_1), \ldots, \pm_n T^-(\varphi_n)) & \text{if } Æ(\circledast) = \exists. \end{cases}
\end{aligned}$$

where for all $j \in [\![1; n]\!]$, we have that $\pm_j \triangleq \begin{cases} \neg & \text{if } \pm_j = - \\ \text{empty} & \text{if } \pm_j = + \end{cases}$.

Proposition 5.15. *Let C be a set of atomic connectives, let φ be a formula of $\mathcal{L}_{\mathsf{C}}$ and let M be a C–model. Then, for all $w \in M$, $M, w \models \varphi$ iff $T^-(M), w \Vdash T^-(\varphi)$.*

Theorem 5.16. *The class of modal polyadic logics is as expressive as the class of Boolean atomic logics whose connectives are all of type* $(1, 1, \ldots, 1)$.

6 Correspondence theory for atomic logics

Molecular logics are logics whose primitive connectives are compositions of atomic connectives. That is why we call them 'molecular', just as molecules are compositions of atoms in chemistry. We are not going to introduce them here (see the companion article [5]) but we will need to resort to the notion of molecular connective that we recall below.

Definition 6.1 (Molecular skeleton and connective)**.** The class $\mathbb{C}^*$ of *molecular skeletons* is the smallest set such that:

- $\mathbb{P} \subseteq \mathbb{C}^*$ and $\mathbb{C}^*$ contains as well, for each $k, l \in \mathbb{N}^*$, a symbol id_k^l of *type signature* (k, k), *output type* k and *arity* 1;
- for all $\circledast \in \mathbb{C}$ of type signature $(k, k_1^0, \ldots, k_n^0)$ and all $c_1, \ldots, c_n \in \mathbb{C}^*$ of output types or types (if they are propositional letters) $k_1^0, \ldots, k_n^0$ respectively, $c \triangleq \circledast(c_1, \ldots, c_n)$ is a molecular skeleton of $\mathbb{C}^*$ of *output type* k.

If $c \in \mathbb{C}^*$, we define its *decomposition tree* as follows. If $c = p \in \mathbb{P}$ or $c = id_k^l$, then its decomposition tree T_c is the tree consisting of a single node labeled with p or id_k^l respectively. If $c = \circledast(c_1, \ldots, c_n) \in \mathbb{C}^*$ then its decomposition tree T_c is the tree defined inductively as follows: the root of T_c is c and it is labeled with $\circledast$ and one sets edges between that root and the roots $c_1, \ldots, c_n$ of the decomposition trees $T_{c_1}, \ldots, T_{c_n}$ respectively.

If $c \triangleq \circledast(c_1, \ldots, c_n)$ is a molecular skeleton with output type k and $k_1, \ldots, k_m$ are the ks of the different id_k^ls which appear in $c_1, \ldots, c_n$ (in an order which follows the first appearance of the id_k^ls in the inorder traversal of the decomposition trees of $c_1, \ldots, c_n$), then the *type signature* of c is $(k, k_1, \ldots, k_m)$ and its *arity* is m. We also define the *quantification signature* $Æ(c)$ of $c = \circledast(c_1, \ldots, c_n)$ by $Æ(c) \triangleq Æ(\circledast)$.

A *molecular connective* is a symbol to which is associated a molecular skeleton. Its arity, type signature, output type, quantification signature and decomposition tree are the same as its skeleton.

The set of *atomic connectives associated to a set* C *of molecular connectives* is the set of labels different from id_k^l of the decomposition trees of the molecular connectives of C.

Universal and existential molecular connectives are essentially molecular connectives such that the quantification patterns of the quantification signatures of their successive atomic connectives are of the form $\forall \ldots \forall$ or $\exists \ldots \exists$ respectively. They essentially behave as 'macroscopic' atomic connectives of quantification signatures $\forall$ or $\exists$.

Definition 6.2 (Universal and existential molecular connective)**.** A *universal (resp. existential) molecular skeleton* is a molecular skeleton c different from any id_k^l for any $k, l \in \mathbb{N}^*$ such that $Æ(c) = \forall$ (resp. $Æ(c) = \exists$) and such that for each node of its decomposition tree labeled with $\circledast = (\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n))$ and each of its jth children labeled with some $\circledast_j \in \mathbb{C}$ such that the subtree generated by this j^{th} children contains at least one id_k^l, we have that $Æ(\circledast_j) = \pm_j Æ$. A *universal (resp. existential) molecular connective* is a molecular connective with a universal (resp. existential) skeleton.

Example 6.3. On the one hand, the molecular connective $\circledast(p, id_k^l)$ is a universal (resp. existential) molecular connective if $Æ(\circledast) = \forall$ (resp. $Æ(\circledast) = \exists$). Likewise, $\supset (id_1^1, \Box id_1^2)$ and $\otimes(\Diamond id_1^1, p)$ are universal and existential molecular connectives respectively. On the other hand, the molecular connectives $\Box\Diamond id_1^1$ and $\supset (\Box id_1^1, \Box id_1^2)$ are neither universal nor existential molecular connectives.

Just as we have tonicity signatures for atomic connectives, we can also define an adaptation of this notion for universal and existential molecular connectives, which, we repeat, are some sort of 'macroscopic' atomic connectives.

Definition 6.4 (Tonicity)**.** The *tonicity w.r.t. the* j*th argument* of a molecular connective c, denoted $tn(c, j)$, is defined inductively as follows:

- if $c = \circledast$ is an atomic connective of skeleton $\star = (\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n)) \in \mathbb{C}$, then $tn(c, j) = \pm_j$;
- if $c = \neg c'$ then $tn(c, j) = -tn(c', j)$;
- if $c = \wedge(c_1, c_2)$ or $c = \vee(c_1, c_2)$ then $tn(c, j) = tn(c_k, j_k)$, where c_k is the molecular connective c_1 or c_2 that takes the jth argument of c also as argument at place j_k;
- if $c = \circledast(c_1, \ldots, c_n)$ with $\star = (\sigma, \pm, Æ, \overline{k}, (\pm_1, \ldots, \pm_n))$ then $tn(c, j) = \pm_k tn(c_k, j_k)$ where c_k is the molecular connective of the decomposition tree of c which takes the jth argument of c as argument at place j_k.

Definition 6.5 (Positive and negative formula)**.** Let C be a set of molecular connectives and let $\varphi \in \mathcal{L}^*_{\mathsf{C}}$. The formula φ can be written as a formula of the form $c(p_1, \ldots, p_n)$ where $p_1, \ldots, p_n$ are all the propositional letters that occur in φ (possibly with repetitions) and c is a molecular connective (not necessarily belonging to C). We say that φ is *positive* (resp. *negative*) if for all $j \in [\![1; n]\!]$, $tn(c, j)(p_j) = +$ (resp. $tn(c, j)(p_j) = -$).

Definition 6.6 (Essentially universal and existential formulas)**.** An *essentially universal formula* (resp. *essentially existential formula*) is either a negative (resp. positive) formula or a formula of an atomic language of the form $c(\varphi_1, \ldots, \varphi_{i-1}, p, \varphi_{i+1}, \ldots, \varphi_n)$ where c is a universal molecular connective (resp. existential molecular connective) and for all $j \in [\![1; n]\!] \backslash \{i\}$, φ_j is either a positive formula if $tn(c, j) = -$ (resp. $tn(c, j) = +$) or a negative formula if $tn(c, j) = +$ (resp. $tn(c, j) = -$) and $p \in \mathbb{P}$ is a propositional letter in an arbitrary position $i \in [\![1; n]\!]$ but such that $tn(c, i) = +$ (resp. $tn(c, i) = -$). In that case, p is called the *head* of the essentially universal (resp. existential) formula and the formula is said to be *headed*.

Definition 6.7 (Regular formula)**.** A *regular formula* φ is a formula of an atomic language of the form $\varphi = c(\varphi_1, \ldots \varphi_n)$ where c is a universal molecular connective and such that for all $j \in [\![1; n]\!]$, φ_j is an essentially universal formula if $tn(c, j) = -$ or an essentially existential formula if $tn(c, j) = +$. The headed formulas $\varphi_1, \ldots, \varphi_n$ are called the *main components of* φ and the *heads of* φ are the heads of $\varphi_1, \ldots, \varphi_n$ (if they exist).

There might indeed be no head of φ is the φ_is are positive or negative formulas.

Definition 6.8 (Essential and inessential atom)**.** An occurrence of a propositional letter in a regular formula φ is *essential in* φ if it is the head of a main component of the formula, otherwise it is *inessential in* φ. A propositional letter in a regular formula φ is *essential in* φ if it has at least one essential occurrence in it, otherwise it is *inessential in* φ.

Definition 6.9 (Dependency digraph)**.** Given a regular formula $\varphi = c(\varphi_1, \ldots, \varphi_n)$ with main components $\{\varphi_1, \ldots, \varphi_k\}$, the *dependency digraph* of φ is a digraph $G = (V_\varphi, E_\varphi)$ where $V_\varphi = \{p_1, \ldots, p_n\}$ is the set of heads of φ and we set $p_i E_\varphi p_j$ iff p_i occurs as an inessential propositional letter in a formula from $\{\varphi_1, \ldots, \varphi_k\}$ with a head p_j. A digraph is *acyclic* if it does not contain oriented cycles.

Definition 6.10 (Inductive atomic formula)**.** An *inductive atomic formula* is a regular formula with an acyclic dependency digraph.

Our definition of inductive atomic formulas is a more or less direct reformulation of the definitions of inductive formulas by Goranko & Vakarelov [23] by means of the translations between atomic logics and their modal polyadic logics spelled out in the previous two sections.

Proposition 6.11. *For all inductive atomic formulas φ, we have that $T^{-}(\varphi)$ is equivalent to an inductive formula as defined by Goranko & Vakarelov [23]. Vice versa, for all inductive formulas ψ of a modal polyadic logic, we have that $T(\psi)$ is equivalent to an inductive atomic formula.*

Definition 6.12 (First–order language associated to a C–frame)**.** Let C be a set of atomic connectives and let $F = (W, \mathcal{R})$ be a C–frame. The associated first–order language with equality and a family of predicates $\{\mathsf{R}_{\circledast} \mid \circledast \in \mathsf{C}\}$, with arities matching those of the respective relations in C–frames, will be denoted by $\mathcal{L}_{\mathsf{C}}^{FO}$. Hereafter we will use the same symbol, $R_{\circledast}$, for the predicate $\mathsf{R}_{\circledast}$ in $\mathcal{L}_{\mathsf{C}}^{FO}$ and for the relation which interprets it in a given C–frame. A formula with a single free variable x will be denoted $\varphi(x)$.

Definition 6.13 (Local first–order correspondent)**.** Let C be a set of atomic connectives and let $\varphi \in \mathcal{L}_{\mathsf{C}}$. A first-order formula $\chi(x) \in \mathcal{L}_{\mathsf{C}}^{FO}$ is a *local first-order correspondent* of φ if for every C–frame $F = (W, \mathcal{R})$ and all $w \in W$, $F, w \Vdash \varphi$ iff $F \models \chi[w/x]$, where $F \Vdash \chi[w/x]$ denotes the first–order truth of $\chi(x)$ in F under the assignment of w to the variable x. A formula φ is *locally first–order definable* if it has a local first–order correspondent.

Theorem 6.14. *Every inductive atomic formula is locally first–order definable. Moreover, its local first–order correspondent can be computed effectively.*

We could introduce the notions of *descriptive general C–frame* and *local d–persistence* for atomic formulas which would be more or less straightforward adaptations and counterparts of definitions that can be found for example in [23] or [10]. We could then prove that every inductive atomic formula is locally d–persistent. We bypass these steps and only state the main application of these general notions, namely the Sahlqvist–like theorem originally introduced for modal logic [10, Theorem 4.42]. This theorem is now extended to atomic logics with inductive atomic formulas (instead of Sahlqvist or inductive modal formulas).

Theorem 6.15. *Let C be a Boolean set of atomic connectives such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. Let $S \subseteq \mathcal{L}_{\mathsf{C}}$ be a set of inductive atomic formulas. The calculus $\mathsf{GGL}_{\mathsf{C}}^{\mathcal{H}} + S$ is sound and strongly complete for the Boolean atomic logics $(\mathcal{L}_{\mathsf{C}}, \mathcal{E}_{\mathsf{C}}, \Vdash)$, where $\mathcal{E}_{\mathsf{C}}$ is the class of pointed C–frames defined by the first–order correspondents of the inductive atomic formulas of S.*

Example 6.16. We illustrate the translation from inductive atomic formula to first–order frame conditions with the following two examples. First, we apply the translation T^{-} then we apply the translation in the proof of Theorem 37 of [23].

We start with the formula $\varphi \triangleq ((p \otimes q) \to (q \otimes p)) = (\neg(p \otimes q) \vee (q \otimes p))$:

- $T^{-}(\varphi) = [\underbrace{i_2(\alpha_{\otimes}(i_0, i_0), i_0)}_{\alpha}](\neg p, \neg q, \neg[\alpha_{\otimes}](\neg q, \neg p))$
- $\forall PQ \forall y_1 y_2 y_3 \, [R_{\alpha} y_1 y_2 y_3 x \to \neg P(y_1) \vee \neg Q(y_2) \vee \exists y_3' y_3'' \, (R y_3' y_3'' y_3 \wedge Q(y_3') \wedge P(y_3''))]$
  As minimal valuations, we take $\sigma(P) : \lambda u.u = y_1$ and $\sigma(Q) : \lambda u.u = y_2$.
  $\forall u y_1 y_2 y_3 \, [R_{i_2} x u y_3 \wedge R y_1 y_2 u \to \exists y_3' y_3'' \, (R y_3' y_3'' y_3 \wedge y_3' = y_2 \wedge y_3'' = y_1)]$
  $\forall y_1 y_2 \, (R y_1 y_2 x \to R y_2 y_1 x)$.

Here is a second example with the formula $\varphi \triangleq (p \to (p \otimes p)) = (\neg p \vee (p \otimes p))$:

- $T^{-}(\varphi) = [i_2](\neg p, \neg[\alpha_{\otimes}](\neg p, \neg p))$

- $\forall P \forall y_1 y_2 \, [R_{i_2} x y_1 y_2 \to \neg P(y_1) \vee \exists y_2' y_2'' \, (R y_2' y_2' y_2' \wedge P(y_2') \wedge P(y_2''))]$
  As minimal valuation we take $\sigma(P) : \lambda u.u = y_1$.
  $\forall y_1 y_2 \, [x = y_1 = y_2 \to \exists y_2' y_2'' \, (R y_2' y_2'' y_2 \wedge y_2' = y_1 \wedge y_2'' = y_1)]$
  $Rxxx$.

Remark 1. Propositional letters p in a proof system can be viewed either as propositional *variables* or propositional *constants*. In the latter case, they are in fact nullary atomic connectives. As such, they should be part of any C–frame even if $p \in \mathsf{C}$. All the results that we have proved hold as well in such a case. The only difference is when we want to obtain the first-order correspondent of a formula containing a propositional constant p. In that case, the translation of the formula into second-order logic should *not* quantify over the predicates P corresponding to p. Otherwise, everything else remains the same.

7 Related work and conclusion

The DLE-logics introduced by Greco et al.. [26] are similar to our basic atomic logics. Their families $\mathcal{F}$ and $\mathcal{G}$ correspond in our framework to connectives of "quantification signatures" $\exists$ and $\forall$ respectively. Likewise, their order types correspond in our framework to "tonicity signatures". Hence, several of their notions correspond to notions introduced by Dunn's gaggle theory [16, 17].

The main difference between their and our work is that we prove the completeness of our calculi w.r.t. a Kripke-style relational semantics. We also introduce a generalized form of residuation based on the symmetric group which is novel. Unlike them, we originally introduce the Boolean negation as a primitive connective, even if one can dispose of it after proving cut elimination. An important difference between Greco & Al.'s DLE-logics and our atomic and molecular logics lies in our introduction and use of types and in the fact that we consider compositions of atomic connectives as primitive connectives. These generalizations are motivated at length in [6]. Basically, some logics/protologics cannot be represented without the use of types, such as temporal logic [6, Example 8], arrow logic, many-dimensional logics [34] and first-order logic. This use of type is crucial to represent these logics and it is also instrumental in showing that any protologic is as expressive as a molecular logic, which constitutes the main result of [6]. It complexifies the soundness and completeness proof of the present article w.r.t. the soundness and completeness proof of [3] for gaggle logics, which are actually atomic logics of type $(1, 1, \ldots, 1)$. This said, one of the main differences with the work of Palmigiano & Al. remains the fact that we are able to define automatically from the connectives of a given atomic logic (or specific molecular logics) sound and strongly complete display and Hilbert calculi in a generic fashion together with their *Kripke-style relational semantics* for which they are sound and complete. In particular, they do not provide a Kripke-style relational semantics to their DLE-logics, only an algebraic one which more or less mimics the axioms and inference rules of their DLE-logics. Our proofs of soundness and completeness w.r.t. the Kripke-style relational semantics resorts to the results of Dunn's gaggle theory and are not straightforward.

Greco et al. [26] also adapted and generalized the framework of Goranko & Vakarelov [23] dealing with the correspondence theory of polyadic modal logics to their DLE-logics, just as we did it with our atomic and molecular logics. In particular, they introduced the notion of inductive inequality which is the counterpart in their setting to the notion of Goranko & Vakarelov's inductive formula. It is therefore not surprising that we come up with notions and results which are very similar to their notions. To be more precise, the tonicity of molecular connectives corresponds to their *sign* inherited by the leaves in the *signed generation tree* [25, Definition 14];

essentially universal atomic formulas are concatenations of normal operators that behave 'like boxes' and correspond to their positive *PIA formulas* or negative *skeleton* [25, Definition 15]; essentially existential atomic formulas are compositions of normal operators that behave 'like diamonds' and correspond to their negative *PIA formulas* or positive *skeleton* [25, Definition 15]; the universal molecular connective part is the *skeleton* part of [25, Definition 15], and the several essentially existential and universal formulas attached to it are the maximal *PIA formulas* containing the *critical* occurrences (these occurrences are called essential here and in [23]). Every branch of a molecular regular formula is *good* [25, Definition 15] since the lower part of the formula is all PIA, and the upper is all skeleton.

One of the main differences is that our notions are genuine instances of the notions introduced by Goranko & Vakarelov [23] and operate and apply directly at the level of the Kripke-style relational semantics, like [23], because we set a formal connection between our atomic logics and modal polyadic logics (Theorem 5.16), unlike what has been done by Greco et al. [26] where such a formal connection is absent. It is in fact the soundness and completeness of our calculi w.r.t. a Kripke-style relational semantics which allows to import directly the results and notions of Goranko & Vakarelov [23] in our framework. Instead, only evidence is given in [13, Section 3] to the effect that, as their name suggests, inductive inequalities, which extend analytic inductive inequalities, are the distributive counterparts of and 'project over' the inductive formulas of Goranko and Vakarelov in the classical setting. Another difference with our work is that the ordering Ω on the propositional variables which is obtained in our approach constructively by a topological sort of the dependency digraph associated to a regular atomic formula is not determined in Greco et al. [25] and is only assumed to exist for inequalities to be inductive (together with an order type).

Atomic logics are logics of residuation to which types are added. Residuated logics have been extensively studied in the algebraic approach to logic [21]. However, it still remains to propose and adapt these algebraic approaches and semantics to our atomic and molecular logics and to show how a proof of completeness for atomic and molecular logics w.r.t. to our Kripke-style relational semantics can be obtained, as well as the other results in our series of articles. In that respect, the duality theory relating the algebraic and our Kripke-style relational semantics remains to be developed for atomic and molecular logics, in the spirit of the one for modal logic for example [10, Section 5] or for other non-classical logics like in Bimbo & Dunn [9].

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A Proofs of Propositions 5.11, 5.15 and Theorem 5.16

Proposition A.1. *Let τ be a modal similarity type, let φ be a formula of $MF(\tau)$ and let M be a τ–model. Then, for all $w \in M$, it holds that $M, w \models \varphi$ iff $T(M), w \Vdash T(\varphi)$.*

Proof. The proof is without specific difficulty. It suffices essentially to check the preservation of truth for every clause of Definition 5.10. □

Proposition A.2. *Let C be a set of atomic connectives, let φ be a formula of $\mathcal{L}_\mathsf{C}$ and let M be a C–model. Then, for all $w \in M$, $M, w \models \varphi$ iff $T^-(M), w \Vdash T^-(\varphi)$.*

Proof. The proof is without specific difficulty. It suffices essentially to check the preservation of truth for every clause of Definition 5.14. □

Theorem A.3. *The class of modal polyadic logics is as expressive as the class of Boolean atomic logics whose connectives are all of type $(1, 1, \ldots, 1)$.*

Proof. We use the notion of equi-expressivity introduced in [6]. Because of the two previous propositions, it suffices to prove that for all τ–models M, $T^-(T(M)) \equiv M$ and for all C–models M, $T(T^-(M)) \equiv M$ ($\equiv$ is the equivalence w.r.t. the appropriate languages). This can be easily checked. □

B Proofs of Theorems 6.14 and 6.15

Theorem B.4. *Every inductive atomic formula is locally first–order definable. Moreover, its local first–order correspondent can be computed effectively.*

Proof. If φ is an inductive atomic formula, its translation $T^-(\varphi)$ into the modal polyadic language is equivalent to an inductive formula by Proposition 6.11. Then, by [23, Theorem 37], this formula is locally first–order definable and its local first–order equivalent can be computed effectively. Since for all sets of atomic connectives C, the class of τ_C–frames is equal to the class of C–frames, the result follows by Proposition 5.15. □

Theorem B.5. *Let C be a Boolean set of atomic connectives such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. Let $S \subseteq \mathcal{L}_\mathsf{C}$ be a set of inductive atomic formulas. The calculus $\mathsf{GGL}^{\mathcal{H}}_\mathsf{C} + S$ is sound and strongly complete for the Boolean atomic logics $(\mathcal{L}_\mathsf{C}, \mathcal{E}_\mathsf{C}, \Vdash)$, where $\mathcal{E}_\mathsf{C}$ is the class of pointed C–frames defined by the first–order correspondents of the inductive atomic formulas of S.*

Proof. Soundness is proved without difficulty. As for completeness, we must prove that any $\mathsf{GGL}^{\mathcal{H}}_\mathsf{C} + S$–consistent set $\Gamma \subseteq \mathcal{L}_\mathsf{C}$ is satisfiable in a pointed C–frame of $\mathcal{E}_\mathsf{C}$. We consider the simpler case where S consists of a single formula χ in order to highlight the main ideas. The general case where S is an arbitrary set follows easily. We are going to show that the canonical frame underlying the canonical model of [5, Definition 29] (where k–$\mathsf{GGL}_{\mathsf{C}}^{\mathcal{H}}$–consistency is replaced by $\mathsf{GGL}_{\mathsf{C}}^{\mathcal{H}} + \{\chi\}$–consistency) fulfills the expected requirements. First, Γ is $\mathsf{GGL}_{\mathsf{C}}^{\mathcal{H}} + \{\chi\}$–consistent, so by [5, Lemma 3] it can be extended into a maximal $\mathsf{GGL}_{\mathsf{C}}^{\mathcal{H}} + \{\chi\}$–consistent set Γ^+. This set Γ^+ belongs to the canonical model M^c. Therefore, by the truth lemma, we have that $M^c, \Gamma^+ \Vdash \Gamma$. Now, it remains to prove that the underlying C–frame of (M^c, Γ^+) belongs to $\mathcal{E}_{\mathsf{C}}$. Let F^c be the C–frame underlying the canonical C–model M^c. Then, one can easily prove that $(F^c, \mathbb{W})$ is a descriptive general τ_{C}–frame $(F^c, \mathbb{W})$, where the carrier set of its associated algebra $\mathbb{W}$ is $\left\{ [\![\varphi]\!]^{M^c} \mid \varphi \in \mathcal{L}_{\mathsf{C}} \right\}$. The canonical model M^c is a model based on $(F^c, \mathbb{W})$. Now, χ is true at (M^c, Γ^+). Therefore, the inductive formula $T^-(\chi)$ of the modal polyadic language is true at (M^c, Γ^+) (here we view the canonical C–model as a Kripke τ_{C}–model). In fact, by closure under uniform substitution of $\mathsf{GGL}_{\mathsf{C}}^{\mathcal{H}}$, $T^-(\chi)$ is true at Γ^+ in any model based on the general τ_{C}–frame $(F^c, \mathbb{W})$. That is, $T^-(\chi)$ is true at $((F^c, \mathbb{W}), \Gamma^+)$. Now, $T^-(\chi)$ is equivalent to an inductive formula of the modal polyadic language and inductive formulas are locally d–persistent by Theorem 5.7. Therefore, $T^-(\chi)$ is true at (F^c, Γ^+) by definition of local d–persistence. Then, by [23, Theorem 37] (and our Theorem 6.14), the local first–order correspondent of χ holds at (F^c, Γ^+). That is, (F^c, Γ^+) belongs to $\mathcal{E}_{\mathsf{C}}$ and this proves the theorem since $(M^c, \Gamma^+) \Vdash \Gamma$. □