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Tuza's Conjecture for Threshold Graphs *†

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Tuza famously conjectured in 1981 that in a graph without $k+1$ edge-disjoint triangles, it suffices to delete at most $2k$ edges to obtain a triangle-free graph. The conjecture holds for graphs with small treewidth or small maximum average degree, including planar graphs. However, for dense graphs that are neither cliques nor 4-colourable, only asymptotic results are known. Here, we confirm the conjecture for threshold graphs, i.e. graphs that are both split graphs and cographs, and for co-chain graphs with both sides of the same size divisible by 4.

Keywords: Tuza's conjecture, packing, covering, threshold graphs, co-chain graphs

1 Introduction

If we can "pack" at most k disjoint objects of some type in a given graph, how many elements do we need to "cover" all appearances of such an object in the graph? Erdős and Pósa famously proved that if a graph contains at most k pairwise vertex-disjoint cycles, then there is a set of at most $f(k)$ vertices that intersects every cycle [8]. While the exact best value of function f is yet unknown, the asymptotic behaviour was recently determined to be $f(k) = \Theta(k \log k)$ [5].

In this paper, we focus on edge-disjoint triangles; we refer the interested reader to [16] for a dynamic survey on other objects. For a graph G, we call every family of pairwise edge-disjoint triangles a *triangle packing*, and every subset of edges intersecting all triangles in G a *triangle hitting*. We denote by $\mu(G)$ the maximum size of a triangle packing in G, and by $\tau(G)$ the minimum size of a triangle hitting in G. Trivially, there is a set of at most $3\mu(G)$ edges that intersect every triangle. We are concerned with improving that bound, following Tuza's conjecture from 1981.

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Conjecture 1 (Tuza [17])**.** *For any graph G it holds $\tau(G) \leq 2\mu(G)$.*

Conjecture 1, if true, is tight for K_4 and K_5. Gluing together copies of K_4 and K_5 along vertices, it is easy to build an infinite family of connected graphs for which Conjecture 1 is tight. However, for larger cliques, it is known that the ratio $\tau(K_p)/\mu(K_p)$ tends to $3/2$ as p increases [9]. In addition, Haxell and Rödl [11] proved that $\tau(G) \leq 2\mu(G) + o(|V(G)|^2)$ for any graph G, meaning Conjecture 1 is asymptotically true when $\tau(G)$ is quadratic with respect to $|V(G)|$. Those seem to indicate that Conjecture 1 should be easier for dense graphs than for sparse graphs. Conversely, it is asymptotically tight in some classes of dense graphs [2]. If we focus on *hereditary graph* classes (i.e. classes that contain every induced subgraph of a graph in the class), the conjecture has only been confirmed for a few graph classes. Those classes include most notably graphs of treewidth at most 6 [4], 4-colourable graphs [1], and graphs with maximum average degree less than 7 [15].

A good candidate for an interesting dense hereditary graph class is the class of *split graphs*, i.e. graphs whose vertex set can be partitioned into two sets: one that induces a clique, the other inducing an independent set. However, Conjecture 1 remains a real challenge even when restricted to split graphs. Another good candidate for an interesting dense hereditary graph class is the class of *cographs*, i.e. graphs with no induced path on four vertices. As an initial step, we focus on graphs that are both split graphs and cographs, i.e. *threshold* graphs. While this may seem like a small step, it is arguably the first dense hereditary superclass of cliques where the conjecture is confirmed.

Theorem 1. *If G is a threshold graph, then $\tau(G) \leq 2\mu(G)$.*

In the latter part of the paper, we show that similar tools with more involved analysis can be used to verify Conjecture 1 also for specific co-chain graphs. A graph G is a *co-chain graph* (or sometimes alternatively called *co-difference graph*) if its vertex set can be partitioned into two sets K_1 and K_2 such that $G[K_1]$ and $G[K_2]$ are cliques and there is an ordering $c_1, \dots, c_n$ on the vertices of K_1 and an ordering $d_1, \dots, d_m$ on the vertices of K_2 with $N[c_{i+1}] \subseteq N[c_i]$ for all $1 \leq i < n$ and $N[d_i] \subseteq N[d_{i+1}]$ for all $1 \leq i < m$. We call (K_1, K_2) a *co-chain representation* of G. We say that G is an *even balanced* co-chain graph if additionally K_1 and K_2 are of the same size that is divisible by four.

Theorem 2. *If G is an even balanced co-chain graph, then $\tau(G) \leq 2\mu(G)$.*

Theorem 2 can be seen as a very first step towards attacking Conjecture 1 on (mixed) unit interval graphs as those graphs can be modelled as a *concatenation* of co-chain graphs. That is, vertices of graph G are partitioned into r cliques $C_1, \dots, C_r$ where each (C_i, C_{i+1}) induce a co-chain graph and G contains no other edges; see [12, 13] for more details. The simplest object for further study might be a k-path, which can be viewed as a concatenation of well-structured same-sized co-chain graphs.

Finally, it is worth mentioning that Conjecture 1 is known to hold as soon as we consider *multi*-packing [6], and in particular it holds in its fractional relaxation [14]. Another angle of attack consists of lowering the bound of 3 step by step for all graphs. The best, and in fact only, such bound is slightly under 2.87 [10].

1.1 Preliminaries

All graphs in this paper are undirected and simple. Let $G = (V, E)$ be a graph. By the *size* of a graph G (alt. $|G|$), we always mean the number of its vertices. For all $v \in V$ the set $N(v) := \{u \mid \{u, v\} \in E\}$ is called the *neighbourhood* of v and $N[v] := N(v) \cup \{v\}$ is its *closed neighbourhood*. A *matching* in G is a set of edges $M \subseteq E$ such that every vertex of G is incident to at most one edge of M. A vertex $v \in V$

is *complete* to $A \subseteq V, v \notin A$ if v is adjacent to all vertices in A. Disjoint sets $A, B \subseteq V$ are *complete* to each other if E contains all edges between A and B. Any omitted definitions can be found in the book by Diestel [7].

Let us first recall the following well-known property (chromatic index of a clique).

Lemma 3. *The edge set of a clique K on k vertices can be decomposed into k edge disjoint maximal matchings for k odd and $k-1$ edge disjoint maximal matchings for k even.*

Proof: If k is even, we may identify the vertices of K with the set $\{0, 1, \ldots, k-1\}$ and consider matchings

$$M_i = \{\{0, i\}\} \cup \{\{a, b\} \mid a \neq b,\ ab \neq 0,\ a + b \equiv 2i \pmod{k-1}\}$$

for $1 \leq i \leq k-1$. These matchings are edge disjoint and cover the entire edge set of K (cf. Fig. 1). Removing any vertex (along with all incident edges) yields a desired matching decomposition into $k-1$ matchings of the edge set of the clique of $k-1$ vertices. □

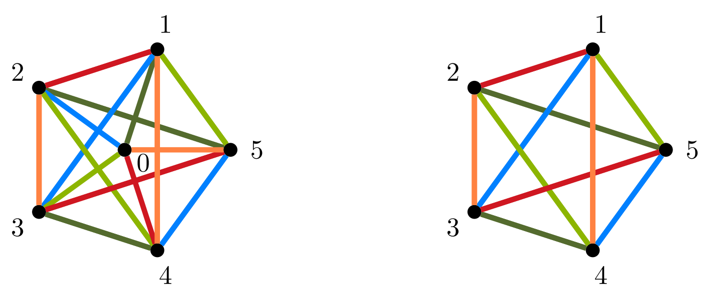


Fig. 1: The decomposition of edges of a 6-vertex clique into 5 matchings and the corresponding decomposition of a 5-vertex clique.

A graph $G = (V, E)$ is a *star* if $V = \{c, s_1, \ldots, s_k\}$ and $E = \{\{c, s_i\} | 1 \leq i \leq k\}$; the vertex c is called the *center vertex* of the star. A graph G is a *complete split graph* if its vertex set can be partitioned into sets K and S, such that S is independent, K induces a clique, and K and S are complete to each other.

The following lemma describes how to pack triangles in complete split graphs. As it is very central to our proofs later, we include a proof here.

Lemma 4 ([9])**.** *Let K be a clique, S an independent set such that they are complete to each other and $|K| = |S| = k$. Then we can find an (optimal) triangle packing TP of size $\binom{k}{2}$ such that:*

1. *It uses all edges from K and each triangle in TP contains exactly one edge from K.*
2. *If k is odd, the remaining edges (not used in TP) create a matching between K and S, otherwise they create a star with its center vertex in S. Moreover, we can choose the unused matching and the center vertex of the unused star arbitrarily.*

Proof: Consider a graph G composed of a clique K' complete to an independent set S' with $|K'| = k$ and $|S'| = k-1$, where k is even. By Lemma 3, K can be decomposed into $k-1$ edge disjoint (perfect) matchings of size $k/2$. Each such matching fully joined to a different vertex in S' yields a family of $k/2$ edge disjoint triangles (see Fig. 2). The collection of all $k-1$ such joins is a decomposition of the entire edge set of G into triangles.

Removing any vertex u from K' yields a balanced graph with both sides of odd size, in which edges not packed into triangles (participating in triangles whose vertex u got removed) create a matching between $K' - u$ and S'. On the other hand, by adding a single vertex v to S', we get a balanced graph with both sides of even size, in which unpacked edges form a star (with v being its center vertex). □

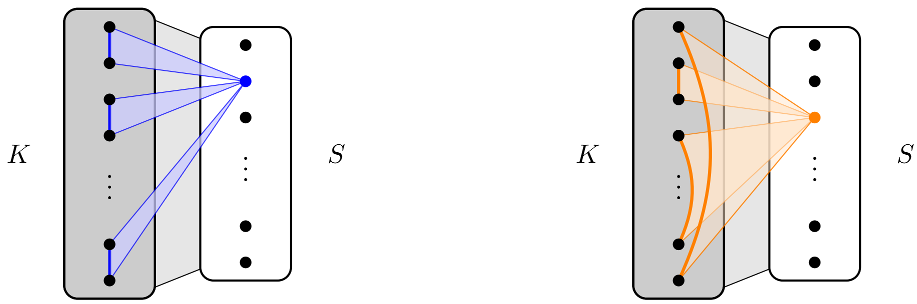


Fig. 2: Full joins of matchings in K with vertices in S as families of triangles.

Corollary 5. *Let K be a clique and S an independent set such that they are complete to each other.*

(a) If $|S| < |K|$, then we can find a triangle packing of size $|S| \cdot \lfloor |K|/2 \rfloor$.

(b) If $|S| \geq |K|$, then we can find a triangle packing of size $\binom{|K|}{2}$.

Proof: If $|S| < |K|$, we take arbitrary $|S|$ edge-disjoint maximal matchings in K whose existence follows from Lemma 3 and assign them to different vertices in S. The full join of each such pair consists of $\lfloor |K|/2 \rfloor$ edge-disjoint triangles.

If $|S| \geq |K|$, we can derive the statement from Lemma 4: it is enough to take any $|K|$-element subset S' of S. □

We say that we *pack edges of K with vertices of S* when we use triangle packings from Corollary 5. The following lemma describes tightly how many edge-disjoint triangles can be packed in a clique.

Lemma 6 ([9]). *The optimal triangle packing for K_n with $n = 6x + i, 0 \leq i \leq 5$ is $(\binom{n}{2} - k)/3$ where k is the number of not covered edges and*

- *$k = 0$ for $i = 1, 3$,*
- *$k = 4$ for $i = 5$,*
- *$k = \frac{n}{2}$ for $i = 0, 2$,*

- $k = \frac{n}{2} + 1$ *for* $i = 4$.

Observe, that we can always hit all the triangles in a clique by leaving a bipartite graph with partitions of as equal size as possible and removing the rest. Therefore, the optimal triangle hitting in a clique consists of at most half the edges.

2 Threshold graphs

A graph $G = (V, E)$ is a *threshold graph* if its vertex set can be partitioned into two sets $K = \{c_1, \ldots, c_k\}$ and $S = \{u_1, \ldots, u_s\}$ such that $G[K]$ is a clique and $G[S]$ is an independent set in G, and $N[c_{i+1}] \subseteq N[c_i]$ for all $1 \leq i < k$ and $N(u_i) \subseteq N(u_{i+1})$ for all $1 \leq i < s$. We identify K with the clique $G[K]$ and say $G = (K \cup S, E)$ is a threshold graph with given *threshold representation* (K, S).

The threshold representation of a threshold graph may not be unique. We prove that it can be chosen such that the clique contains a vertex which is not adjacent to any vertex of the independent set.

Lemma 7. *For every threshold graph* $G = (V, E)$ *there exists a threshold representation* (K, S) *such that there is a vertex* $v \in K$ *with* $N(v) \cap S = \emptyset$.

Proof: We fix a threshold representation (K, S) of G. Suppose for all $v \in K$ holds $N(v) \cap S \neq \emptyset$. Then, since G is a threshold graph, there is a vertex $w \in S$ such that $N(w) = K$. We obtain a new threshold representation (K', S') of G with $K' := K \cup \{w\}$ and $S' := S \setminus \{w\}$. Since S is an independent set, w has no neighbours in S'. □

We can now prove that Conjecture 1 holds for all threshold graphs.

Proof of Theorem 1: Let $G = (K \cup S, E)$ be a threshold graph with $K = \{c_1, \ldots, c_k\}$ and $S = \{u_1, \ldots, u_s\}$ such that $N(c_k) \cap S = \emptyset$. By Lemma 7, such a representation exists. Let $r \in \{1, \ldots, s\}$ be chosen minimal such that $\{c_1, \ldots, c_{\lceil k/2 \rceil}\} \subseteq N(u_r)$ and let X be the subset $\{u_r, \ldots, u_s\}$ of S (see Fig. 3). Note that X is complete to the set $\{c_1, \ldots, c_{\lceil k/2 \rceil}\}$. We distinguish two cases, based on the parity

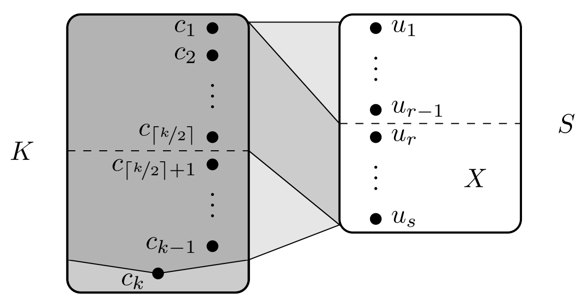
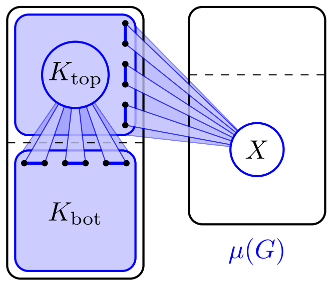


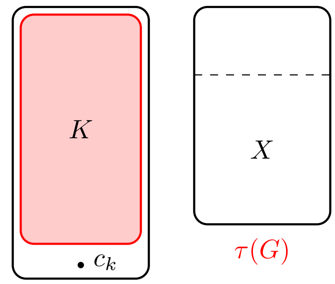
Fig. 3: The structure of threshold graph G.

of k. First, we focus on the case that k is even. In this case we consider two cliques K_{top} and K_{bot} of equal size, induced by vertices $\{c_1, \ldots, c_{k/2}\}$ and $\{c_{k/2+1}, \ldots, c_k\}$, respectively.

We construct a triangle packing TP of G using Corollary 5 as follows: we pack the edges of K_{bot} with vertices in K_{top}, and the edges of K_{top} with vertices in X (see Fig. 4(a)).



(a) triangle packing



(b) triangle hitting

Fig. 4: The (a) triangle packing and (b) triangle hitting providing the bounds for $|X| \geq k/2$.

If $|X| \geq \frac{k}{2}$, then TP is a triangle packing of size $2\binom{k/2}{2}$. On the other hand, a triangle hitting of size $\binom{k-1}{2}$ can be obtained by taking all edges from K except those incident to c_k (see Fig. 4(b)). Thus, we obtain a lower bound on the triangle packing and an upper bound on the triangle hitting yielding:

$$\tau(G) \leq \binom{k-1}{2} = \frac{k-2}{2} \cdot (k-1) \leq \frac{k-2}{2} \cdot k = 4\binom{k/2}{2} \leq 2\mu(G).$$

If $|X| < \frac{k}{2}$, then TP is of size at least

$$\binom{k/2}{2} + |X| \cdot \left\lfloor \frac{k}{4} \right\rfloor \geq \binom{k/2}{2} + |X| \left(\frac{k}{4} - \frac{1}{2}\right).$$

On the other hand, the edges inside K_{top} and inside K_{bot} together with all edges between S and K_{bot} build a triangle hitting of G (cf. Fig. 5(b)) of size at most

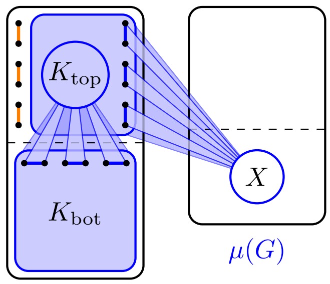
$$2\binom{k/2}{2} + |X| \left(\frac{k}{2} - 1\right).$$

Indeed, recall that c_k does not have any neighbours in S, therefore we have at most $|X| \left(\frac{k}{2} - 1\right)$ edges between X and K_{bot}, and by definition of X, there are no vertices in K_{bot} having neighbours in $S \setminus X$. Thus, we again obtain a lower bound on the triangle packing and an upper bound on the triangle hitting yielding:

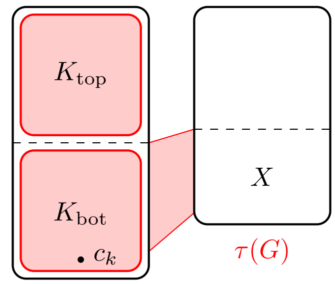
$$\tau(G) \leq 2\binom{k/2}{2} + |X| \left(\frac{k}{2} - 1\right) = 2\binom{k/2}{2} + 2|X| \left(\frac{k}{4} - \frac{1}{2}\right) \leq 2\mu(G).$$

We are left with the case that k is odd. We consider the cliques K_{top} and K_{bot} induced by sets $\{c_1, \ldots, c_{(k+1)/2}\}$ and $\{c_{(k+1)/2+1}, \ldots, c_k\}$, respectively.

Again, we look at the size of X and in case it is large, we can derive a similar argument as in the previous case, using Corollary 5. More precisely, assume that $|X| \geq \frac{k+1}{2}$. Then we pack the edges of K_{bot} into $\binom{(k-1)/2}{2}$ triangles with vertices in K_{top}, and the edges of K_{top} into $\binom{(k+1)/2}{2}$ triangles with



(a) triangle packing



(b) triangle hitting

Fig. 5: The (a) triangle packing and (b) triangle hitting providing the bounds when $|X| < {}^k/_2$.

vertices in X. Together, this gives a triangle packing of size

$$\binom{\frac{k+1}{2}}{2} + \binom{\frac{k-1}{2}}{2} = \frac{(k-1)^2}{4}.$$

The triangle hitting again consists of all edges from K except those adjacent to c_k, therefore has size $\binom{k-1}{2}$ (recall Fig. 4). These two bounds together yield:

$$\tau(G) \leq \binom{k-1}{2} = \frac{k-1}{2} \cdot (k-2) \leq \frac{(k-1)^2}{2} \leq 2\mu(G).$$

It remains to consider the case $|X| < \frac{k+1}{2}$. In order to find a triangle packing, we define K'_{top} and K'_{bot} to be induced by $\{c_1, \ldots, c_{(k-1)/2}\}$ and $\{c_{(k+1)/2}, \ldots, c_k\}$, respectively (so $K'_{\text{top}} = K_{\text{top}} \setminus \{c_{(k+1)/2}\}$ is of size $\frac{k-1}{2}$ and $K'_{\text{bot}} = K_{\text{bot}} \cup \{c_{(k+1)/2}\}$ is of size $\frac{k+1}{2}$). We build a triangle packing analogously to before, using Corollary 5. The edges of K'_{bot} can be packed into $\lfloor \frac{(k+1)/2}{2} \rfloor \cdot \frac{k-1}{2} \geq \frac{k-1}{4} \cdot \frac{k-1}{2}$ triangles with vertices in K'_{top}. Moreover, $\min\{|X| \cdot \lfloor \frac{k-1}{4} \rfloor, \binom{(k-1)/2}{2}\} \geq |X| \frac{k-3}{4}$ edges of K'_{top} can be packed into triangles with vertices in X (see Fig. 6(a)). This gives a triangle packing of size at least

$$\frac{k-1}{2} \cdot \frac{k-1}{4} + |X| \frac{k-3}{4}.$$

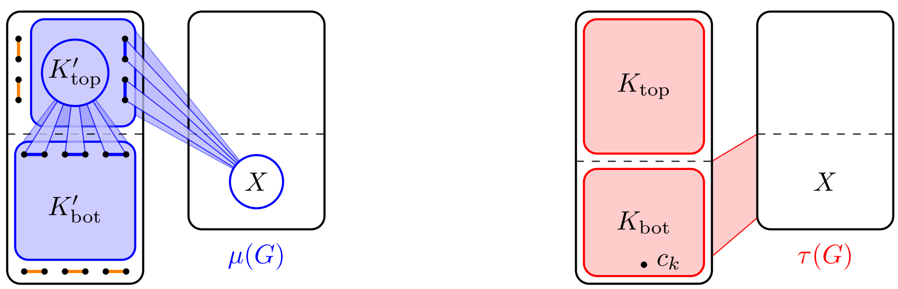
To find a triangle hitting, we again consider the partition of K into K_{top} and K_{bot}. We take all edges inside K_{top} and inside K_{bot} together with all edges between S and K_{bot} (see Fig. 6(b)). Again, recall that $c_k \in K_{\text{bot}}$ does not have any neighbours in S, and there are no vertices in K_{bot} having neighbours in $S \setminus X$. Thus, this yields a triangle hitting of size at most.

$$\binom{\frac{k+1}{2}}{2} + \binom{\frac{k-1}{2}}{2} + |X| \frac{k-3}{2}.$$

Therefore, we obtain the following which concludes the proof:

$$\begin{aligned}\tau(G) &\leq \binom{\frac{k+1}{2}}{2} + \binom{\frac{k-1}{2}}{2} + |X|\frac{k-3}{2} \\ &= \frac{(k-1)^2}{4} + |X|\frac{k-3}{2} = 2\cdot\frac{k-1}{2}\cdot\frac{k-1}{4} + 2|X|\frac{k-3}{4} \leq 2\mu(G).\end{aligned}$$

□



(a) triangle packing (b) triangle hitting

Fig. 6: In (a) the triangle packing and in (b) the triangle hitting providing the bounds for $|K|$ odd and $|X| < (k+1)/2$.

3 Even balanced co-chain graphs

In this section we prove Theorem 2. To this end let G be an even balanced co-chain graph and (K_1, K_2) its *co-chain representation*. Recall that K_1 and K_2 are of same size which is divisible by 4, for the rest of the section let $|K_1| = |K_2| = 2\ell$ for ℓ even. We identify K_1 and K_2 with the cliques $G[K_1]$ and $G[K_2]$. See Fig. 7 for an illustration.

We prove that Tuza's conjecture holds for this graph class.

Proof of Theorem 2: Note that in the case $\ell = 2$ we get an 8-vertex graph which is either a clique, or has average degree less than 7, so this case is covered by [15]. Therefore in the following we assume that $\ell \geq 4$.

Similarly to threshold graphs, we use $K_1^{\text{top}}, K_1^{\text{bot}}$ for the top and the bottom half of K_1, respectively, and similarly $K_2^{\text{top}}, K_2^{\text{bot}}$ for the top and the bottom half of K_2. Let $X_1 \subseteq K_1, X_2 \subseteq K_2$ be the sets defined as follows: $c \in X_1$ if $K_2^{\text{bot}} \subseteq N[c]$, and $d \in X_2$ if $K_1^{\text{top}} \subseteq N[d]$. See Fig. 7 for an illustration. We denote $x_1 = |X_1|$ and $x_2 = |X_2|$. By definition, $x_1 \geq \ell$ implies that the set $X_1 \supseteq K_1^{\text{top}}$ is complete to K_2^{bot}. Consequently, $x_2 \geq \ell$. Similarly, $x_2 \geq \ell$ implies $x_1 \geq \ell$. Therefore, $x_1 \geq \ell$ if and only if $x_2 \geq \ell$. We assume without loss of generality throughout the entire proof that $x_1 \geq x_2$. We split the analysis into two main cases.

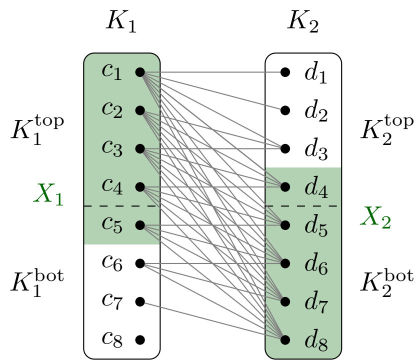


Fig. 7: An example of an even balanced co-chain graph with $\ell = 4$ (omitting the edges inside the cliques K_1 and K_2).

3.1 *The case* $x_1, x_2 \leq \ell$

In this case $X_1 \subseteq K_1^{\text{top}}$ and $X_2 \subseteq K_2^{\text{bot}}$. Suppose there is an edge cd with $c \in K_1 \setminus X_1$ and $d \in K_2^{\text{top}}$, then c is adjacent to all the vertices in K_2^{bot} and so $c \in X_1$, which yields a contradiction. Similarly, there are no edges between K_1^{bot} and $K_2 \setminus X_2$. In particular, there are no edges between K_2^{top} and K_1^{bot}.

We choose a triangle hitting TH obtained by taking all edges within K_1^{top}, K_2^{top}, K_1^{bot}, and K_2^{bot}, as well as edges between X_1 and K_2^{bot}, and between X_2 and K_1^{top} as illustrated in Fig. 8. Observe now that in the graph $G - \mathsf{TH}$ vertices in X_1 only have neighbours in the independent set $K_1^{\text{bot}} \cup K_2^{\text{top}}$, vertices in $K_1^{\text{top}} \setminus X_1$ only have neighbours in the independent set $K_1^{\text{bot}} \cup K_2^{\text{bot}} \setminus X_2$, while vertices in K_1^{bot} only have neighbours in the independent set $K_1^{\text{top}} \cup X_2$. Therefore the set TH is indeed a triangle hitting of G.

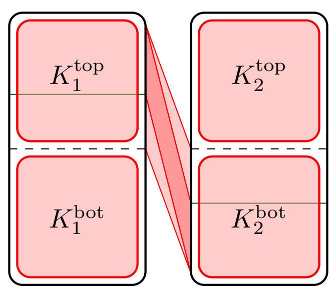


Fig. 8: The triangle hitting used in the case $x_1, x_2 \leq \ell$.

Therefore,

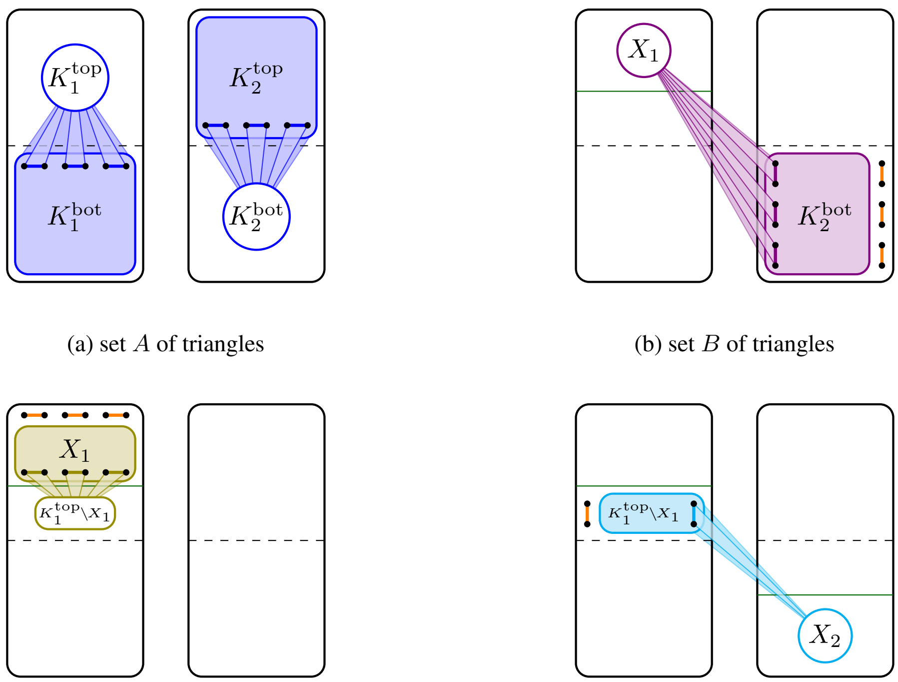
$$\tau(G) \leq |\mathsf{TH}| = 4\binom{\ell}{2} + \ell x_1 + \ell x_2 - x_1 x_2 = 4\binom{\ell}{2} + \ell x_1 + (\ell - x_1) x_2.$$

Indeed, we note that we counted edges between X_1 and X_2 once in term ℓx_1 and once in term ℓx_2 which we compensate by subtracting the last term $x_1 x_2$.

Let us now create a sufficiently large triangle packing. First, we pack all edges of K_1^{bot} with vertices in K_1^{top} and also all edges of K_2^{top} with vertices in K_2^{bot}; we denote the set of these triangles by A (see Fig. 9(a)). By Lemma 4, A contains $2\binom{\ell}{2}$ triangles. Observe that $2|A| - |\mathsf{TH}| = -\ell x_1 - (\ell - x_1)x_2$. First, we sort out the single case where $x_1 = \ell$, and, in consequence, $x_2 = \ell$ by definition of X_1 and X_2 together with the assumption that $x_2 \leq \ell$.

3.1.1 The subcase $x_1 = x_2 = \ell$

In this case, $|\mathsf{TH}| = 4\binom{\ell}{2} + \ell^2$. As $K_1^{\mathrm{top}} \cup K_2^{\mathrm{bot}}$ is a clique, by Lemma 6 we can pack at least $\frac{1}{3}\left(\binom{2\ell}{2} - \ell - 1\right)$ triangles in it. Together with A, we obtain a triangle packing TP. If $\ell \geq 5$, then $2\mathsf{TP} - \mathsf{TH} \geq \frac{2}{3}\left(\binom{2\ell}{2} - \ell - 1\right) - \ell^2 = \frac{1}{3}\left(\ell^2 - 4\ell - 2\right) \geq 0$. If $\ell = 4$, Lemma 6 gives us a stronger bound without the term -2, leading to $2\mathsf{TP} - \mathsf{TH} \geq \frac{1}{3}\left(\ell^2 - 4\ell\right) = 0$. Both cases imply $2\mu(G) \geq \tau(G)$.



(a) set A of triangles (b) set B of triangles

(c) set C of triangles (d) set D of triangles

Fig. 9: Triangles in (a) A, (b) B, (c) C, and (d) D in the case $x_1, x_2 \leq \ell$.

3.1.2 The subcase $x_1, x_2 < \ell$

Now, we consider the remaining case where $x_1 < \ell$, and, in consequence, $x_2 < \ell$.

We choose a triangle packing TP as follows (see Fig. 9). We take the set A of triangles as defined before. Recall that $2|A| - |\mathsf{TH}| = -\ell x_1 - (\ell - x_1)x_2$. We create a set B of triangles by packing edges of K_2^{bot} with vertices in X_1. By Corollary 5(a) and as $x_1 < \ell$, B is of size $\ell/2 \cdot x_1$. We create another set of triangles C by packing edges of X_1 with vertices of $K_1^{\text{top}} \setminus X_1$. Next, let D be the set of triangles created by packing edges of $K_1^{\text{top}} \setminus X_1$ with vertices in X_2. It is clear that all triangles in $\mathsf{TP} = A \cup B \cup C \cup D$ are mutually edge-disjoint, therefore TP is indeed a triangle packing.

Let us first settle the case that x_1 is even. As $2(|A| + |B|) - |\mathsf{TH}| = -(\ell - x_1)x_2$ if $x_1 < \ell$, it remains to show that $2\,|\mathsf{TP} \setminus (A \cup B)| = 2(|C| + |D|) \geq (\ell - x_1)x_2$.

If $\ell - x_1 > x_2$, then $2|D| = (\ell - x_1)x_2$ by Corollary 5(a). So, assume that $\ell - x_1 \leq x_2$. Consequently, $\ell - x_1 \leq x_1$ and thus $\ell/2 \leq x_1$. If $x_1 = \ell/2$, then, by $x_1 \geq x_2 \geq \ell/2$, we have $x_2 = \ell/2$ as well. Thus, as $\ell \geq 4$, $2(|C| + |D|) - (\ell - x_1)x_2 = 4\binom{\ell/2}{2} - \ell^2/4 = \ell(\ell - 4)/4 \geq 0$. For $\ell - x_1 < x_1$ we get $2|C| = x_1(\ell - x_1) \geq x_2(\ell - x_1)$. Therefore, we always have $2\,|C \cup D| \geq (\ell - x_1)x_2$ for even x_1, and so $2\mu(G) \geq 2\mathsf{TP} \geq \mathsf{TH} \geq \tau(G)$.

In case x_1 is odd, we add one additional triangle to our triangle packing as follows. Note that if there is no edge between K_1^{bot} and K_2^{bot}, then all edges between K_1^{top} and K_2^{top} hit all triangles between K_1 and K_2, therefore taking these edges instead of edges between K_1^{top} and K_2^{bot} creates a triangle hitting TH' of size at most $4\binom{\ell}{2} + x_1\ell$ as all the edges between K_1^{top} and K_2^{top} have one endpoint in X_1. As $x_1 < \ell$, we obtain $2\mu(G) \geq 2(|A| + |B|) \geq |\mathsf{TH}'| \geq \tau(G)$. Thus we can assume that there is at least one edge uv with $u \in K_1^{\text{bot}}$ and $v \in K_2^{\text{bot}}$.

Note in particular that $v \in X_2$ as every edge between K_1^{bot} and K_2^{bot} has one endpoint in X_2. Observe that $|K_1^{\text{top}} \setminus X_1| = \ell - x_1$ is odd, so there exists an unpacked matching between $K_1^{\text{top}} \setminus X_1$ and X_2 (not containing edges used in triangles from set D). Indeed, each maximal matching in $K_1^{\text{top}} \setminus X_1$ constructed according to Lemma 3 omits a different vertex $u_1 \in K_1^{\text{top}} \setminus X_1$, so after the matching is fully joined with a vertex $u_2 \in X_2$, as in Corollary 5, the edge u_1u_2 remains unpacked. A collection of all such edges gives the desired matching. Let $w \in K_1^{\text{top}} \setminus X_1$ be a vertex such that wv is an edge of the mentioned unpacked matching. Finally, as ℓ is even, a star with center in K_1^{top} is not used in any triangle in A, by Lemma 4. Note that the center of this star can be chosen arbitrarily among vertices of K_1^{top} by Lemma 4; let us choose w to be the center. Therefore, uvw is a triangle which is edge-disjoint with every triangle in $A \cup B \cup C \cup D$ and we may set $\mathsf{TP}^{\text{odd}} = \mathsf{TP} \cup \{uvw\}$ for odd x_1.

Recall that $2(|A| + |B|) - |\mathsf{TH}| = -(\ell - x_1)x_2$. Similarly as before, we need to prove that

$$2\left|\mathsf{TP}^{\text{odd}} \setminus (A \cup B)\right| = 2(|C| + |D| + 1) \geq (\ell - x_1)x_2.$$

If $\ell - x_1 \leq x_2$, then again $\ell - x_1 \leq x_1$ and thus $\ell/2 \leq x_1$. The case $\ell/2 = x_1$ can be handled exactly as in the even case. So assume further $\ell - x_1 < x_1$, then using Corollary 5 we obtain $2(|C|+|D|) = (x_1-1)(\ell-x_1)+2\binom{\ell-x_1}{2} = (x_1-1)(\ell-x_1)+(\ell-x_1)(\ell-x_1-1) = (\ell-x_1)(\ell-2)$. Consequently, $2(|C|+|D|+1)-(\ell-x_1)x_2 = 2+(\ell-x_1)(\ell-2-x_2)$. Observe that, for $x_2 \leq \ell-2$, we already get $(\ell - x_1)(\ell - 2 - x_2) \geq 0$. We have $x_1 = \ell - 1$ because x_1 is odd and ℓ is even. For $x_2 = \ell-1$, we have $x_1 = \ell-1$ because $x_2 \leq x_1 < \ell$. Thus $2+(\ell-x_1)(\ell-2-x_2) = 2+1\cdot(-1) \geq 0$. Therefore, we obtain $2\,(|C| + |D| + 1) \geq (\ell - x_1)x_2$.

If $\ell - x_1 > x_2$, then $2|D| = (\ell - x_1 - 1)x_2 = (\ell - x_1)x_2 - x_2$. Hence in this case, D alone does not suffice as it is missing x_2 triangles. We therefore need $2|C| + 2 \geq x_2$. We use Corollary 5 to analyse the size of C.

If $x_1 \leq \ell - x_1$, then $2|C| + 2 - x_2 \geq x_1(x_1 - 1) - x_2 + 2 \geq (x_2 - 1)^2 + 1 \geq 1$ as $x_1(x_1 - 1) \geq x_2(x_2 - 1)$. If $x_1 > \ell - x_1$, then, $2|C| + 2 - x_2 = (x_1 - 1)(\ell - x_1) - x_2 + 2 \geq x_1 - x_2 + 1 \geq 1$, as $\ell - x_1 \geq 1$ and $x_1 \geq x_2$. So in both cases we obtain $2|C| + 2 \geq x_2 + 1 \geq x_2$.

We conclude that $2\mu(G) \geq 2\mathsf{TP}^{\text{odd}} \geq \mathsf{TH} \geq \tau(G)$.

3.2 The case $x_1 > \ell$ and $x_2 \geq \ell$

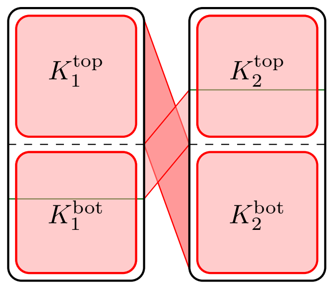


Fig. 10: The triangle hitting used in the case $x_1 > \ell$ and $x_2 \geq \ell$.

We choose a triangle hitting TH obtained by taking all edges within K_1^{top}, K_1^{bot}, K_2^{top} and K_2^{bot} as well as edges between K_1^{top} and K_2^{bot} and between K_1^{bot} and K_2^{top} (cf. Fig. 10). The graph $G - \mathsf{TH}$ is bipartite, thus TH is indeed a triangle hitting in G. We have

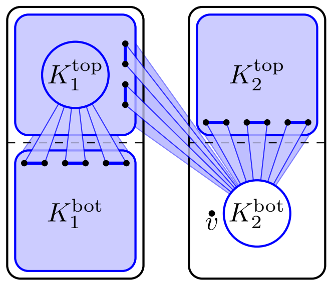
$$|\mathsf{TH}| = 4\binom{\ell}{2} + \ell^2 + \left|E\left(K_2^{\text{top}}, K_1^{\text{bot}}\right)\right| \leq 3\ell^2 - 2\ell + (x_1 - \ell)(x_2 - \ell).$$

We choose a triangle packing TP as follows. Pack all edges of K_2^{top} with vertices of K_2^{bot}, all edges of K_1^{top} with vertices in K_2^{bot} and all edges of K_1^{bot} with vertices in K_1^{top}. This gives a set A' of $3\binom{\ell}{2}$ triangles (see Fig. 11(a)). By the second part of Lemma 4 there exists $v \in K_2^{\text{bot}}$ such that edges between v and $K_2^{\text{top}} \cup K_1^{\text{top}}$ are not used in A'. Additionally, define a set B' of triangles obtained by packing edges from K_2^{bot} with vertices of $X_1 \cap K_1^{\text{bot}}$ (see Fig. 11(b)). Then $|B'| = \frac{\ell}{2}(x_1 - \ell)$ if $x_1 \neq 2\ell$ and $|B'| = \binom{\ell}{2}$ (by Corollary 5(b)) if $x_1 = 2\ell$. Finally, let C' be the set of triangles using v and any maximal matching between K_1^{top} and $X_2 \cap K_2^{\text{top}}$ (see Fig. 11(c)). Since K_1^{top} is complete to $X_2 \cap K_2^{\text{top}}$, we obtain $|C'| = x_2 - \ell$. It is clear that $\mathsf{TP} = A' \cup B' \cup C'$ is a triangle packing.

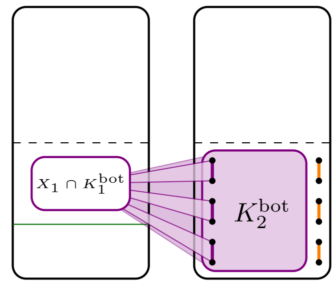
If $x_1 < 2\ell$, then

$$\begin{aligned} 2|\mathsf{TP}| - |\mathsf{TH}| &\geq 3\ell(\ell - 1) + \ell(x_1 - \ell) + 2(x_2 - \ell) - 3\ell^2 + 2\ell - (x_1 - \ell)(x_2 - \ell) \\ &= (x_1 - \ell - 1)(2\ell - x_2) + x_2 - \ell \geq 0. \end{aligned}$$

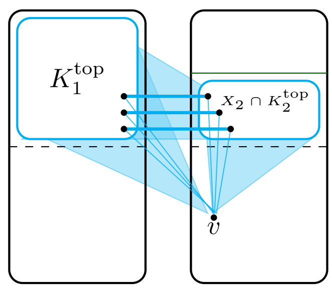
The last inequality follows as $x_1 \geq \ell + 1$.



(a) set A' of triangles



(b) set B' of triangles



(c) set C' of triangles

Fig. 11: Triangles in (a) A', (b) B', and (c) C' in the case $x_1 > \ell$ and $x_2 \geq \ell$.

If $x_1 = 2\ell$, then we similarly get

$$\begin{aligned} 2\,|\mathsf{TP}| - |\mathsf{TH}| &\geq 3\ell(\ell-1) + \ell(\ell-1) + 2(x_2-\ell) - 3\ell^2 + 2\ell - \ell\,(x_2-\ell) \\ &= (\ell-2)\,(2\ell-x_2) \geq 0. \end{aligned}$$

We conclude that indeed $2\mu(G) \geq \tau(G)$. □

References

[1] S. Aparna Lakshmanan, Cs. Bujtás, and Zs. Tuza. Small edge sets meeting all triangles of a graph. *Graphs Combin.*, 28(3):381–392, April 2011. doi:10.1007/s00373-011-1048-8.

[2] J. D. Baron and J. Kahn. Tuza's conjecture is asymptotically tight for dense graphs. *Comb. Probab. Comput.*, 25(5):645–667, 2016. doi:10.1017/S0963548316000067.

[3] M. Bonamy, Ł. Bożyk, A. Grzesik, M. Hatzel, T. Masařík, J. Novotná, and K. Okrasa. Tuza's conjecture for threshold graphs. In J. Nešetřil, G. Perarnau, J. Rué, and O. Serra, editors, *Extended Abstracts EuroComb 2021*, pages 765–771, Cham, 2021. Springer International Publishing. doi:10.1007/978-3-030-83823-2_122.

[4] F. Botler, C. G. Fernandes, and J. Gutiérrez. On Tuza's conjecture for triangulations and graphs with small treewidth. *Discrete Math.*, 344(4):112281, 2021. doi:10.1016/j.disc.2020.112281.

[5] W. Cames van Batenburg, T. Huynh, G. Joret, and J.-F. Raymond. A tight Erdős-Pósa function for planar minors. *Adv. Comb.*, 2019:33, 2019. Id/No 2. doi:10.19086/aic.10807.

[6] P. Chalermsook, S. Khuller, P. Sukprasert, and S. Uniyal. Multi-transversals for triangles and the Tuza's conjecture. In *Proceedings of SODA 2020*, pages 1955–1974. SIAM, 2020. doi:10.5555/3381089.3381210.

[7] R. Diestel. *Graph theory*, volume 173. Berlin: Springer, 5th edition, 2017.

[8] P. Erdős and L. Pósa. On independent circuits contained in a graph. *Can. J. Math.*, 17:347–352, 1965. doi:10.4153/CJM-1965-035-8.

[9] T. Feder and C. S. Subi. Packing edge-disjoint triangles in given graphs. *Electron. Colloquium Comput. Complex.*, 19:13, 2012. URL: http://eccc.hpi-web.de/report/2012/013.

[10] P. E. Haxell. Packing and covering triangles in graphs. *Discrete Math.*, 195(1–3):251–254, 1999. doi:10.1016/S0012-365X(98)00183-6.

[11] P. E. Haxell and V. Rödl. Integer and fractional packings in dense graphs. *Combinatorica*, 21(1):13–38, January 2001. doi:10.1007/s004930170003.

[12] P. Heggernes, D. Meister, and C. Papadopoulos. A new representation of proper interval graphs with an application to clique-width. *Electron. Notes Discret. Math.*, 32:27–34, 2009. doi:10.1016/j.endm.2009.02.005.

[13] J. Kratochvíl, T. Masařík, and J. Novotná. U-bubble model for mixed unit interval graphs and its applications: The MaxCut problem revisited. *Algorithmica*, 83(12):3649–3680, December 2021. doi:10.1007/s00453-021-00837-4.

[14] M. Krivelevich. On a conjecture of tuza about packing and covering of triangles. *Discrete Math.*, 142(1):281–286, 1995. doi:10.1016/0012-365X(93)00228-W.

[15] G. J. Puleo. Tuza's conjecture for graphs with maximum average degree less than 7. *Eur. J. Comb.*, 49:134–152, 2015. doi:10.1016/j.ejc.2015.03.006.

[16] J.-F. Raymond. Dynamic Erdős-Pósa listing. Available at https://perso.limos.fr/~jfraymon/Erd%C5%91s-P%C3%B3sa/.

[17] Zs. Tuza. A conjecture: Finite and infinite sets, Eger, Hungary 1981, A. Hajnal, L. Lovász, V. T. Sós. In *Proc. Colloq. Math. Soc. J. Bolyai*, volume 37, page 888, 1981.