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Orlicz regularity of the gradient of solutions to quasilinear elliptic equations in the plane

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Abstract

Given a planar domain Ω, we study the Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x,\nabla v) = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where the higher-order term is a quasilinear elliptic operator, and f belongs to the Zygmund space $L(\log L)^{\delta}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)$ with $\beta \geq 0$ and $\delta \geq \frac{1}{2}$.

We prove that the gradient of the variational solution $v \in W_0^{1,2}(\Omega)$ belongs to the space $L^2(\log L)^{2\delta-1}(\log\log\log L)^{\beta}(\Omega)$.

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1 Introduction

In this paper we consider the following Dirichlet problem on a bounded open set $\Omega \subset \mathbb{R}^2$ with $\mathcal{C}^1$ boundary:

$$\begin{cases} -\operatorname{div} A(x,\nabla v) = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f belongs to the Zygmund space $L(\log L)^{\delta}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)$ with $\beta \geq 0$ and $\delta \geq \frac{1}{2}$. We prove that the distributional gradient of the unique solution $v \in W_0^{1,2}(\Omega)$ to (1.1) satisfies $|\nabla v| \in L^2(\log L)^{2\delta-1}(\log\log\log L)^{\beta}(\Omega)$.

Here $A : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a mapping of Leray-Lions type [1], that is,

$$\begin{aligned} &A(\cdot,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^2, \text{ and} \\ &A(x,\cdot) \text{ is continuous for almost every } x \in \Omega. \end{aligned} \tag{1.2}$$

Moreover, we assume that there exists $K \geq 1$ such that, for almost every $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^2$,

$$\begin{aligned}
&\text{(i)} \quad \bigl|A(x,\xi) - A(x,\eta)\bigr| \le K|\xi - \eta|,\\
&\text{(ii)} \quad |\xi - \eta|^2 \le K\bigl\langle A(x,\xi) - A(x,\eta), \xi - \eta\bigr\rangle, \qquad (1.3)\\
&\text{(iii)} \quad A(x,0) = 0.
\end{aligned}$$

In [2], under assumptions (1.2) and (1.3), the authors proved the existence and uniqueness of the solution to the Dirichlet problem with $f \in L^1(\Omega)$ in the grand Sobolev space $W_0^{1,2)}(\Omega)$. Precisely, $W_0^{1,2)}(\Omega)$ is the space of functions $v \in W_0^{1,1}(\Omega)$ whose gradients belong to the grand Lebesgue space $L^{2)}(\Omega)$ (see Section 2 for a definition).

Nowadays, a vast literature is available dealing with several types of a priori estimates on the gradients of solutions to equations of this kind; see, for example, [3–5].

We are interested in cases where the solution is the variational $W^{1,2}(\Omega)$ solution. The minimal assumption on f that guarantees this is $f \in L(\log L)^{\frac{1}{2}}(\Omega)$. This follows by the embedding in the plane (see [6, 7], and [8])

$$W_0^{1,2}(\Omega) \hookrightarrow \exp_2(\Omega)$$

and by the duality relation (see [9])

$$\bigl((\exp_2)(\Omega)\bigr)' = L(\log L)^{\frac{1}{2}}(\Omega).$$

In [10], the authors interpolate between the data spaces

$$L(\log L)^{\frac{1}{2}}(\Omega) \quad \text{and} \quad L(\log L)(\Omega).$$

To this aim, the following estimate was proved for $0 \le \beta \le 1$:

$$\|\nabla v\|_{L^2(\log L)^\beta(\Omega)} \le C(K,\beta)\|f\|_{L(\log L)^{\frac{(\beta+1)}{2}}(\Omega)}. \tag{1.4}$$

When f belongs to the Zygmund space $L(\log L)^{\frac{1}{2}}(\log\log L)^{\frac{\beta}{2}}(\Omega)$ for $0 \le \beta < 2$, the unique solution v to the Dirichlet problem (1.1) satisfies $|\nabla v| \in L^2(\log\log L)^\beta(\Omega)$ with the estimate

$$\|\nabla v\|_{L^2(\log\log L)^\beta(\Omega)} \le C(K,\beta)\|f\|_{L(\log L)^{\frac{1}{2}}(\log\log L)^{\frac{\beta}{2}}(\Omega)} \tag{1.5}$$

(see [11]). This generalizes a result of [12] obtained for $\beta = 1$.

Starting from the results of [11], in [13], the authors of the present paper prove an analogue of the previous result when the critical Zygmund class $L(\log L)^{\frac{1}{2}}(\Omega)$ is perturbed in a weaker way, namely with perturbations of order $\log\log\log L$. Precisely, in [13], it is proved that if $\beta \ge 0$, then

$$\|\nabla v\|_{L^2(\log\log\log L)^\beta(\Omega)} \le C(K,\beta)\|f\|_{L(\log L)^{\frac{1}{2}}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)}. \tag{1.6}$$

The aim of this paper is to extend the results of [13] to the case $f \in L(\log L)^\delta(\log\log \times \log L)^{\frac{\beta}{2}}(\Omega)$ with $\beta \ge 0$ and $\delta \ge \frac{1}{2}$, that is, to prove the following:

Theorem 1.1 *Let $A = A(x,\xi)$ satisfy* (1.2) *and* (1.3), *and let $\beta \geq 0$, $\delta \geq \frac{1}{2}$. Then, if $f \in L(\log L)^{\delta}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)$, the gradient of the unique finite energy solution $v \in W_0^{1,2}(\Omega)$ to the Dirichlet problem* (1.1) *belongs to the Orlicz space $L^2(\log L)^{2\delta-1}(\log\log\log L)^{\beta}(\Omega,\mathbb{R}^2)$, and the following estimate holds*:

$$\|\nabla v\|_{L^2(\log L)^{2\delta-1}(\log\log\log L)^{\beta}(\Omega;\mathbb{R}^2)} \leq C(K,\beta,\delta)\|f\|_{L(\log L)^{\delta}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)}.$$

In order to prove this theorem, we will find an integral expression equivalent to the Luxemburg norm in the Zygmund class (see Theorem 3.1), which is based on a method recently introduced in [14, 15].

We note that our method allows us to prove estimates (1.4) and (1.6) for any $\beta \geq 0$ (in particular, see Lemmas 2.3 and 2.4).

2 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. A function u belongs to the Lebesgue space $L^p(\Omega)$ with $1 \leq p < \infty$ if and only if

$$\|u\|_{L^p(\Omega)} = \left(\fint_{\Omega} |u|^p\,dx\right)^{\frac{1}{p}} < +\infty,$$

where $\fint_{\Omega} = \frac{1}{|\Omega|}\int_{\Omega}$.

Now we recall some useful function spaces slightly larger than the classical Lebesgue spaces.

2.1 Grand Lebesgue spaces

For $1 < p < \infty$, let us consider the class, denoted by $L^{p)}(\Omega)$, consisting of all measurable functions $u \in \bigcap_{1\leq q<p} L^q(\Omega)$ such that

$$\sup_{0<\varepsilon\leq p-1}\left\{\varepsilon\fint_{\Omega}\left|u(x)\right|^{p-\varepsilon}\right\}^{\frac{1}{p-\varepsilon}} < +\infty$$

which was introduced in [16]; $L^{p)}(\Omega)$ becomes a Banach space, the *grand Lebesgue space* $L^{p)}(\Omega)$, equipped with the norm

$$\|u\|_{L^{p)}(\Omega)} = \sup_{0<\varepsilon\leq p-1}\varepsilon^{\frac{1}{p}}\left\{\fint_{\Omega}\left|u(x)\right|^{p-\varepsilon}\right\}^{\frac{1}{p-\varepsilon}}.$$

Moreover, $\|u\|_{L^{p)}(\Omega)}$ is equivalent to

$$\sup_{0<\varepsilon\leq p-1}\left\{\varepsilon\fint_{\Omega}\left|u(x)\right|^{p-\varepsilon}\right\}^{\frac{1}{p-\varepsilon}}.$$

In general, if $0 < \alpha < \infty$, then we can define the space $L^{\alpha,p)}(\Omega)$ as the space of all measurable functions $u \in \bigcap_{1\leq q<p} L^q(\Omega)$ such that

$$\|u\|_{L^{\alpha,p)}(\Omega)} = \sup_{0<\varepsilon\leq p-1}\left\{\varepsilon^{\frac{\alpha}{p}}\|u\|_{p-\varepsilon}\right\} < +\infty.$$

2.2 Orlicz spaces

Let Ω be an open set in $\mathbb{R}^n$ with $n \geq 2$. A function $\Phi : [0, +\infty) \to [0, +\infty)$ is called a *Young function* if it is convex, left-continuous, and vanishes at 0; thus, any Young function Φ admits the representation

$$\Phi(t) = \int_0^t \phi(s)\,ds \quad \text{for } t \geq 0,$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing left-continuous function that is neither identically equal to 0 nor to ∞.

The *Orlicz space* associated to Φ, named $L^\Phi(\Omega)$, consists of all Lebesgue-measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Phi\bigl(\lambda|f|\bigr) < \infty \quad \text{for some } \lambda = \lambda(f) > 0.$$

$L^\Phi(\Omega)$ is a Banach space equipped with the Luxemburg norm

$$\|f\|_{L^\Phi(\Omega)} = \inf\left\{\frac{1}{\lambda} : \int_\Omega \Phi\bigl(\lambda|f|\bigr) \leq 1\right\}.$$

Examples of Orlicz spaces:

(1) If $\Phi(t) = t^p$ for $1 \leq p < \infty$, then $L^\Phi(\Omega)$ is the classical Lebesgue space $L^p(\Omega)$.

(2) If $\Phi(t) = t^p(\log(a+t))^q$ with either $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q \geq 0$ and where $a \geq e$, then $L^\Phi(\Omega)$ is the Zygmund space denoted by $L^p(\log L)^q(\Omega)$.

(3) If $\Phi(t) = t^p(\log(a+t))^{q_1}(\log\log\log(a+t))^{q_2}$ with either $p > 1$ and $q_1, q_2 \in \mathbb{R}$ or $p = 1$ and $q_1, q_2 \geq 0$ and where $a \geq e^{e^e}$, then $L^\Phi(\Omega)$ is the space $L^p(\log L)^{q_1}(\log\log\log L)^{q_2}(\Omega)$.

(4) If $\Phi(s) = e^{t^a} - 1$ and $a > 0$, then $L^\Phi(\Omega)$ is the space of a-exponentially integrable functions $\mathrm{EXP}_a(\Omega)$.

We denote by $\exp_a(\Omega)$ the closure of $L^\infty(\Omega)$ in $\mathrm{EXP}_a(\Omega)$.

The *Young complementary function* is given by

$$\tilde{\Phi}(t) = \int_0^t \phi^{-1}(s)\,ds,$$

where

$$\phi^{-1}(s) = \sup\bigl\{r : \phi(r) \leq s\bigr\}.$$

Moreover, the following Hölder-type inequality holds:

$$\left|\int_\Omega f(x)g(x)\,dx\right| \leq C(\Phi)\|f\|_{L^\Phi(\Omega)}\|g\|_{L^{\tilde{\Phi}}(\Omega)}$$

for $f \in L^\Phi(\Omega)$ and $g \in L^{\tilde{\Phi}}(\Omega)$.

Definition 2.1 A Young function Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$) if

$$\Phi(2s) \leq C\Phi(s)$$

for some constant $C \geq 2$ and all $s > 0$.

By the Riesz representation theorem, if Φ and $\tilde{\Phi}$ belong to the class Δ_2, then the dual space of $L^{\Phi}(\Omega)$ is $L^{\tilde{\Phi}}(\Omega)$.

Now we recall the explicit expression of the duals of some Orlicz spaces (see [17–19]).

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be an open set. If $1 < p < \infty$ and $q, q_1, q_2 \in \mathbb{R}$, then*
- $(L^p(\log L)^q(\Omega))' \cong L^{p'}(\log L)^{-\frac{q}{p-1}}(\Omega)$,
- $(L^p(\log\log\log L)^q(\Omega))' \cong L^{p'}(\log\log\log L)^{-\frac{q}{p-1}}(\Omega)$,
- $(L^p(\log L)^{q_1}(\log\log\log L)^{q_2}(\Omega))' \cong L^{p'}(\log L)^{-\frac{q_1}{p-1}}(\log\log\log L)^{-\frac{q_2}{p-1}}(\Omega)$,

where p' is the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

If $p = 1$ and $q > 0$, then
- $(L(\log L)^q(\Omega))' \cong \mathrm{EXP}_{\frac{1}{q}}(\Omega)$.

Given two Young functions Φ and Ψ, we say that Ψ *dominates* Φ *globally* (respectively *near infinity*) if there exists a constant $k > 0$ such that

$$\Phi(t) \le \Psi(kt) \quad \text{for all } t \ge 0 \text{ (respectively for all } t \ge t_0 \text{ for some } t_0 > 0);$$

moreover, Φ and Ψ are *equivalent globally* (respectively *near infinity*, $\Phi \cong \Psi$) if each dominates the other globally (respectively near infinity). If $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are the complementary Young functions of, respectively, Φ and Ψ, then Ψ dominates Φ globally (or near infinity) if and only if $\widetilde{\Phi}$ dominates $\widetilde{\Psi}$ globally (or near infinity). Similarly, Φ and Ψ are equivalent if and only if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are equivalent. We have the following result.

Theorem 2.2 *The continuous embedding $L^{\Psi}(\Omega) \hookrightarrow L^{\Phi}(\Omega)$ holds if and only if either Ψ dominates Φ globally or Ψ dominates Φ near infinity and Ω has finite measure.*

Finally, we recall the definition of the *Orlicz-Sobolev* spaces $W^{1,\Psi}(\Omega)$ and $W_0^{1,\Psi}(\Omega)$ (see [20–23]). The space $W^{1,\Psi}(\Omega)$ consists of the equivalence classes of functions u in $L^{\Psi}(\Omega)$ whose distributional gradients ∇u belong to L^{Ψ}. This is a Banach space with respect to the norm given by

$$\|u\|_{W^{1,\Psi}(\Omega)} = \|u\|_{L^{\Psi}(\Omega)} + \|\nabla u\|_{L^{\Psi}(\Omega)}.$$

As in the case of the ordinary Sobolev space, $W_0^{1,\Psi}(\Omega)$ coincides with the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\Psi}(\Omega)$.

2.3 Orlicz-Sobolev imbeddings

Lemma 2.3 *Let $\Phi(t) = \exp\{\frac{t^{\frac{1}{\delta}}}{(\log(e+\log(e+t)))^{\frac{\beta}{2\delta}}}\} - 1$ with $\beta \in \mathbb{R}$ and $\delta > 0$. Then*

$$\tilde{\Phi}(t) \cong t(\log t)^{\delta}(\log\log\log t)^{\frac{\beta}{2}}. \tag{2.1}$$

Proof Since Φ is a Young function, by definition we have

$$\Phi(t) = \int_0^t \phi(s)\,ds,$$

where ϕ is equivalent near infinity to

$$\Phi(s)\cdot\left[\frac{s^{\frac{1}{\delta}-1}}{\delta(\log\log s)^{\frac{\beta}{2\delta}}}-\frac{\beta s^{\frac{1}{\delta}-1}}{2\delta(\log s)\cdot(\log\log s)^{\frac{\beta}{2\delta}+1}}\right].$$

For large s, we have

$$\phi(s)\cong\Phi(s)\frac{s^{\frac{1}{\delta}-1}}{\delta(\log\log s)^{\frac{\beta}{2\delta}}},$$

and we will prove that, near infinity,

$$\phi(s)\cong\Phi(s). \tag{2.2}$$

We begin with the case $\delta\le 1$. Then we can state that there exists $c>1$ such that

$$\begin{aligned}\exp\left\{\frac{s^{\frac{1}{\delta}}}{(\log\log s)^{\frac{\beta}{2\delta}}}\right\}&\le\exp\left\{\frac{s^{\frac{1}{\delta}}}{(\log\log s)^{\frac{\beta}{2\delta}}}\right\}\cdot\frac{s^{\frac{1}{\delta}-1}}{\delta(\log\log s)^{\frac{\beta}{2\delta}}}\\&\le\exp\left\{\frac{(cs)^{\frac{1}{\delta}}}{(\log\log(cs))^{\frac{\beta}{2\delta}}}\right\}.\end{aligned}$$

Similarly, in the case $\delta>1$, there exists $c\in(0,1)$ such that

$$\begin{aligned}\exp\left\{\frac{(cs)^{\frac{1}{\delta}}}{(\log\log(cs))^{\frac{\beta}{2\delta}}}\right\}&\le\exp\left\{\frac{s^{\frac{1}{\delta}}}{(\log\log s)^{\frac{\beta}{2\delta}}}\right\}\cdot\frac{s^{\frac{1}{\delta}-1}}{\delta(\log\log s)^{\frac{\beta}{2\delta}}}\\&\le\exp\left\{\frac{s^{\frac{1}{\delta}}}{(\log\log s)^{\frac{\beta}{2\delta}}}\right\}.\end{aligned}$$

Hence, (2.2) is proved, and then it is not difficult to check that

$$\phi^{-1}(r)\cong(\log r)^{\delta}(\log\log\log r)^{\frac{\beta}{2}}.$$

By the definition of a complementary Young function, for large y, we obtain that

$$\tilde{\Phi}(y)=\int_0^y\phi^{-1}(r)\,dr\cong y(\log y)^{\delta}(\log\log\log y)^{\frac{\beta}{2}}.$$ □

Given a Young function Ψ such that

$$\int_0\left(\frac{r}{\Psi(r)}\right)dr<\infty,$$

we define $\Phi:[0,+\infty)\to[0,+\infty)$ as

$$\Phi(s)=\Psi\circ H_2^{-1}(s)\quad\text{for } s\ge 0, \tag{2.3}$$

where $H_2^{-1}(s)$ is the (generalized) left-continuous inverse of the function $H_2 : [0,+\infty) \to [0,+\infty)$ given by

$$H_2(r) = \left(\int_0^r \left(\frac{t}{\Psi(t)}\right) dt\right)^{\frac{1}{2}} \quad \text{for } r \geq 0. \tag{2.4}$$

In [24] and in [25], the author showed that Φ is a Young function and that the following Sobolev-Orlicz embedding theorem holds:

$$\|u\|_{L^\Phi(\Omega)} \leq C\|\nabla u\|_{L^\Psi(\Omega)}$$

for every function u in the Orlicz-Sobolev space $W^{1,\Psi}(\Omega)$. As an application, we prove an embedding theorem, which can be regarded as an extension of Lemma 2.4 in [13].

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with $\mathcal{C}^1$ boundary. Consider the Young function*

$$\Psi(t) = t^2(\log t)^{1-2\delta}(\log\log\log t)^{-\beta}$$

with $\beta \in \mathbb{R}$ and $\delta \geq \frac{1}{2}$. Then

$$W^{1,\Psi}(\Omega) \hookrightarrow L^\Phi(\Omega),$$

where

$$\Phi(s) \cong e^{s^{\frac{1}{\delta}}(\log\log s)^{-\frac{\beta}{2\delta}}}. \tag{2.5}$$

Proof By (2.4) we have that

$$H_2(r) = \left(\int_0^r \frac{(\log t)^{2\delta-1}(\log\log\log t)^\beta}{t}\, dt\right)^{\frac{1}{2}} \cong (\log r)^\delta(\log\log\log r)^{\frac{\beta}{2}}.$$

Moreover, as shown in the proof of Lemma 2.3, the inverse function $H_2^{-1}(s)$ is equivalent near infinity to

$$e^{s^{\frac{1}{\delta}}(\log\log s)^{-\frac{\beta}{2\delta}}}.$$

By (2.3) we obtain that

$$\begin{aligned}\Phi(s) &\cong e^{2s^{\frac{1}{\delta}}(\log\log s)^{-\frac{\beta}{2\delta}}}\left(s^{\frac{1}{\delta}}(\log\log s)^{-\frac{\beta}{2\delta}}\right)^{1-2\delta}\left(\log\log s^{\frac{1}{\delta}}(\log\log s)^{-\frac{\beta}{2\delta}}\right)^{-\beta}\\ &\cong e^{s^{\frac{1}{\delta}}(\log\log s)^{-\frac{\beta}{2\delta}}},\end{aligned}$$

and we conclude that

$$W^{1,\Psi}(\Omega) \hookrightarrow L^\Phi(\Omega). \qquad \square$$

Remark 2.5 The previous lemma for $\delta = \frac{1}{2}$ and $\beta = 0$ was proved in [6, 7], and [8]. The case $\beta = 0$ and $\delta > \frac{1}{2}$ is proved in [26].

3 Equivalent norm on the Zygmund spaces $L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)$

The main tool of this section is to obtain an integral expression equivalent to the Luxemburg norm in $L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)$ with $1 < q < \infty$, $\beta \geq 0$ and $\gamma > 0$.

If f is a measurable function on Ω, we set

$$|||f|||_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} = \left\{\int_0^{\varepsilon_0} \varepsilon^{\gamma-1}\|f\|^q_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}\,d\varepsilon\right\}^{\frac{1}{q}}. \tag{3.1}$$

Here $\varepsilon_0 \in]0, q-1]$ is fixed.

For $\beta = 0$, (3.1) becomes

$$|||f|||_{L^q(\log L)^{-\gamma}(\Omega)} = \left\{\int_0^{\varepsilon_0} \varepsilon^{\gamma-1}\|f\|^q_{L^{q-\varepsilon}(\Omega)}\,d\varepsilon\right\}^{\frac{1}{q}}$$

as in [15].

Theorem 3.1 *We have $f \in L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)$ if and only if*

$$|||f|||_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} < +\infty.$$

Moreover, $|||\cdot|||_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}$ is a norm equivalent to the Luxemburg one, that is, there exist constants $C_i = C_i(q,\beta,\gamma,\varepsilon_0)$, $i = 1,2$, such that, for all $f \in L^q(\log L)^{-\gamma}(\log\log \times \log L)^{-\beta}(\Omega)$,

$$\begin{aligned} C_1\|f\|_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} &\leq |||f|||_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} \\ &\leq C_2\|f\|_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}. \end{aligned}$$

Proof It is easy to check that $|||f|||_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}$, defined by (3.1), is a norm on $L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)$.

Moreover, for any measurable function f and for a.e. $x \in \Omega$, if $a \geq e^{e^e}$, then we have

$$|f|^q(a+|f|)^{-\varepsilon} \leq |f|^{q-\varepsilon} \leq 2^{q-1}\left[a^q + |f|^q(a+|f|)^{-\varepsilon}\right],$$

and so we deduce

$$\begin{aligned} |f|^q(a+|f|)^{-\varepsilon}\left(\log\log\log(a+|f|)\right)^{-\beta} &\leq |f|^{q-\varepsilon}\left(\log\log\log(a+|f|)\right)^{-\beta} \\ &\leq 2^{q-1}\left[a^q + |f|^q(a+|f|)^{-\varepsilon}\right] \\ &\quad \times \left(\log\log\log(a+|f|)\right)^{-\beta}. \end{aligned}$$

Integrating over Ω, we get

$$\begin{aligned} &\fint_\Omega |f|^q(a+|f|)^{-\varepsilon}\left(\log\log\log(a+|f|)\right)^{-\beta}\,dx \\ &\quad \leq \|f\|^{q-\varepsilon}_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)} \\ &\quad \leq 2^{q-1}a^q + 2^{q-1}\fint_\Omega |f|^q(a+|f|)^{-\varepsilon}\left(\log\log\log(a+|f|)\right)^{-\beta}\,dx. \end{aligned}$$

Then we multiply for $\varepsilon^{\gamma-1}$ and integrate between 0 and ε_0 to obtain:

$$\begin{aligned}&\int_0^{\varepsilon_0}\varepsilon^{\gamma-1}\left[\fint_\Omega |f|^q\big(a+|f|\big)^{-\varepsilon}\big(\log\log\log\big(a+|f|\big)\big)^{-\beta}\,dx\right]d\varepsilon\\&\quad\le\int_0^{\varepsilon_0}\varepsilon^{\gamma-1}\|f\|^{q-\varepsilon}_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}\,d\varepsilon\\&\quad\le 2^{q-1}a^q\frac{\varepsilon_0^\gamma}{\gamma}+2^{q-1}\int_0^{\varepsilon_0}\varepsilon^{\gamma-1}\left[\fint_\Omega |f|^q\big(a+|f|\big)^{-\varepsilon}\big(\log\log\log\big(a+|f|\big)\big)^{-\beta}\,dx\right]d\varepsilon.\end{aligned}$$

Thanks to Lemma 4.3 of [11], used with the choice $b=a+|f|$, we obtain that there exist two constant C_1, C_2, depending only on γ and ε_0, such that

$$\begin{aligned}&C_1\fint_\Omega |f|^q\big(\log\big(a+|f|\big)\big)^{-\gamma}\big(\log\log\log\big(a+|f|\big)\big)^{-\beta}\,dx\\&\quad\le\int_0^{\varepsilon_0}\varepsilon^{\gamma-1}\|f\|^{q-\varepsilon}_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}\,d\varepsilon\\&\quad\le C_2\left[1+\fint_\Omega |f|^q\big(\log\big(a+|f|\big)\big)^{-\gamma}\big(\log\log\log\big(a+|f|\big)\big)^{-\beta}\,dx\right].\end{aligned}\tag{3.2}$$

If $|||f|||_{L^q(\log\log\log L)^{-\beta}(\Omega)}$ is finite, then since

$$\|f\|^{q-\varepsilon}_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}\le\|f\|^{q}_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}+1,$$

by the first inequality in (3.2) we get that $f\in L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)$. Moreover, if $|||f|||_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}=1$, then

$$\fint_\Omega |f|^q\big(\log(a+|f|)\big)^{-\gamma}\big(\log\log\log(a+|f|)\big)^{-\beta}\,dx\le C_3,$$

where C_3 is a constant independent on f. By homogeneity, for any measurable f, we get

$$\|f\|_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}\le C_3|||f|||_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}.$$

Before proving the converse, we recall that

$$\sup_{0<\sigma\le q-1}\sigma^{\frac{\gamma}{q-\sigma}}\|f\|_{L^{q-\sigma}(\log\log\log L)^{-\beta}(\Omega)}\le C_4\|f\|_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}.\tag{3.3}$$

Indeed, if we fix $a\ge e^{e^e}$ and proceed as in Lemma 1.2 in [16], using the Hölder inequality and the inequality

$$\log^\lambda(a+t)\le\lambda^\lambda(a+t),$$

we obtain

$$\begin{aligned}&\int_\Omega |f|^{q-\sigma}\big(\log\log\log\big(a+|f|\big)\big)^{-\beta}\\&\quad=\int_\Omega\frac{|f|^{q-\sigma}(\log\log\log(a+|f|))^{-\beta+\frac{\beta(q-\sigma)}{q}-\frac{\beta(q-\sigma)}{q}}(\log(a+|f|))^{\frac{\gamma(q-\sigma)}{q}}}{(\log(a+|f|))^{\frac{\gamma(q-\sigma)}{q}}}\end{aligned}$$

$$\le \left[\int_\Omega \frac{|f|^q(\log\log\log(a+|f|))^{-\beta}}{(\log(a+|f|))^\gamma}\right]^{\frac{q-\sigma}{q}}$$
$$\times \left[\int_\Omega \big(\log\log\log\big(a+|f|\big)\big)^{(-\beta+\frac{\beta(q-\sigma)}{q})\frac{q}{\sigma}}\big(\log\big(a+|f|\big)\big)^{\frac{\gamma(q-\sigma)}{\sigma}}\right]^{\frac{\sigma}{q}}$$
$$\le \left[\int_\Omega \frac{|f|^q(\log\log\log(a+|f|))^{-\beta}}{(\log(a+|f|))^\gamma}\right]^{\frac{q-\sigma}{q}}$$
$$\times \left[\left(\frac{\gamma(q-\sigma)}{\sigma}\right)^{\frac{\gamma(q-\sigma)}{\sigma}}\int_\Omega \big(\log\log\log\big(a+|f|\big)\big)^{-\beta}\big(a+|f|\big)\right]^{\frac{\sigma}{q}}$$
$$\le \left[\int_\Omega \frac{|f|^q(\log\log\log(a+|f|))^{-\beta}}{(\log(a+|f|))^\gamma}\right]^{\frac{q-\sigma}{q}}\left[\left(\frac{\gamma(q-\sigma)}{\sigma}\right)^{\frac{\gamma(q-\sigma)}{\sigma}}\int_\Omega \big(a+|f|\big)\right]^{\frac{\sigma}{q}}.$$

Hence, elevating both sides of this inequality to the power $\frac{1}{q-\sigma}$ and then multiplying both of them by $\sigma^{\frac{\gamma}{q-\sigma}}$, we deduce

$$\left[\sigma^\gamma\int_\Omega |f|^{q-\sigma}\big(\log\log\log\big(a+|f|\big)\big)^{-\beta}\right]^{\frac{1}{q-\sigma}}$$
$$\le \left[\int_\Omega \frac{|f|^q(\log\log\log(a+|f|))^{-\beta}}{(\log(a+|f|))^\gamma}\right]^{\frac{1}{q}}\big(a|\Omega|+\|f\|_{L^1(\Omega)}\big)^{\frac{\sigma}{q(q-\sigma)}}\gamma^{\frac{\gamma}{q}}(q-\sigma)^{\frac{\gamma}{q}}\sigma^{\frac{\gamma\sigma}{q(q-\sigma)}},$$

and passing to the supremum with respect to $\sigma\in(0,q-1]$, we get formula (3.3) with

$$C_4=\gamma^{\frac{\gamma}{q}}\sup_{0<\sigma\le q-1}\left\{\big(a|\Omega|+\|f\|_{L^1(\Omega)}\big)^{\frac{\sigma}{q(q-\sigma)}}(q-\sigma)^{\frac{\gamma}{q}}\sigma^{\frac{\gamma\sigma}{q(q-\sigma)}}\right\}.$$

If $f\in L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)$, that is, if

$$\|f\|_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}<\infty \tag{3.4}$$

by (3.3), then there exists a constant C_5 independent on f such that

$$\|f\|_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}\le C_5\varepsilon^{-\frac{\gamma}{q-\varepsilon}}\|f\|_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}. \tag{3.5}$$

By (3.5) we get

$$\begin{aligned}\|f\|^q_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}&=\|f\|^{q-\varepsilon}_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}\|f\|^{\varepsilon}_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}\\&\le C_6\|f\|^{q-\varepsilon}_{L^{q-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}\|f\|^{\varepsilon}_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}.\end{aligned} \tag{3.6}$$

Hence, by (3.2) we obtain that $|||f|||_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}<+\infty$. Indeed, if

$$\|f\|_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}=1,$$

by (3.6) and (3.2) we get

$$|||f|||_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}<C_7,$$

where the constant C_7 is independent on f. By homogeneity we conclude the proof, obtaining

$$|||f|||_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} < C_7 \|f\|_{L^q(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}. \qquad \square$$

4 Proof of Theorem 1.1

In this section, before proving Theorem 1.1, we state a regularity result for elliptic equations with right-hand side in divergence form. For convenience of the reader, we recall Theorem 3.1 of [2].

Theorem 4.1 *Let $A = A(x,\xi)$ be a Leray-Lions mapping that satisfies* (1.3). *Then there exists $\sigma_0 = \sigma_0(K) > 0$ such that, for $|\sigma| \le \sigma_0$ and $\underline{\chi}_1, \underline{\chi}_2 \in L^{2-\sigma}(\Omega;\mathbb{R}^2)$, each of the two problems*

$$\begin{cases} \operatorname{div} A(x,\nabla\varphi_1) = \operatorname{div}\underline{\chi}_1 & \text{in } \Omega, \\ \varphi_1 \in W_0^{1,2-\sigma}(\Omega), \end{cases} \tag{4.1}$$

$$\begin{cases} \operatorname{div} A(x,\nabla\varphi_2) = \operatorname{div}\underline{\chi}_2 & \text{in } \Omega, \\ \varphi_2 \in W_0^{1,2-\sigma}(\Omega), \end{cases} \tag{4.2}$$

has a unique solution and

$$\|\nabla\varphi_1 - \nabla\varphi_2\|_{L^{2-\sigma}(\Omega)} \le C(K)\|\underline{\chi}_1 - \underline{\chi}_2\|_{L^{2-\sigma}(\Omega)},$$

where $C(K) > 0$ depends only on K.

Theorem 4.1 allows us to prove the following:

Theorem 4.2 *Let $A = A(x,\xi)$ be a Leray-Lions mapping that satisfies* (1.3). *Then, if $\gamma > 0$ and $\beta \ge 0$, for $i = 1,2$ and for any $\underline{\chi}_i \in L^2(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega;\mathbb{R}^2)$, there exists a unique solution φ_i to the Dirichlet problem*

$$\begin{cases} \operatorname{div} A(x,\nabla\varphi_i) = \operatorname{div}\underline{\chi}_i & \text{in } \Omega, \\ \varphi_i \in W_0^{1,1}(\Omega). \end{cases} \tag{4.3}$$

Moreover,

$$\|\nabla\varphi_1 - \nabla\varphi_2\|_{L^2(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} \le C\|\underline{\chi}_1 - \underline{\chi}_2\|_{L^2(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}, \tag{4.4}$$

where $C = C(\beta,\gamma,K) > 0$ is a positive constant that depends on the parameters K, β, and γ.

Proof By Theorem 4.1 there exists a positive constant $\sigma_0 = \sigma(K)$ such that if $|\sigma| \le \sigma_0$, then for $i = 1,2$ and for any $\underline{\chi}_i \in L^{2-\sigma}(\Omega;\mathbb{R}^2)$, problem (4.3) admits a unique solution $\varphi_i \in W_0^{1,2-\sigma}$, and

$$\|\nabla\varphi_1 - \nabla\varphi_2\|_{L^{2-\sigma}(\Omega)} \le C\|\underline{\chi}_1 - \underline{\chi}_2\|_{L^{2-\sigma}(\Omega)}, \tag{4.5}$$

where $C = C(K) > 0$ is a positive constant that depends only on the parameter K.

If $\gamma > 0$ and $\beta \geq 0$ are fixed, using Theorem 3.1, we obtain

$$\begin{aligned}
&\|\nabla\varphi_1 - \nabla\varphi_2\|^2_{L^2(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} \\
&\quad\leq C_1(\beta,\gamma)|||\nabla\varphi_1 - \nabla\varphi_2|||^2_{L^2(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} \\
&\quad= C_1(\beta,\gamma)\int_0^{\varepsilon_0}\varepsilon^{\gamma-1}\|\nabla\varphi_1 - \nabla\varphi_2\|^2_{L^{2-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}\,d\varepsilon.
\end{aligned}$$

For $\beta = 0$, by Theorem 4.1 we get

$$\|\nabla\varphi_1 - \nabla\varphi_2\|^2_{L^2(\log L)^{-\gamma}(\Omega)} \leq C_2(\gamma,K)\int_0^{\varepsilon_0}\varepsilon^{\gamma-1}\|\underline{\chi}_1 - \underline{\chi}_2\|^2_{L^{2-\varepsilon}(\Omega)}\,d\varepsilon.$$

If $\beta > 0$, then with a suitable choice of λ_0, by Theorem 3 in [13] and Theorem 4.1, we get

$$\begin{aligned}
&\|\nabla\varphi_1 - \nabla\varphi_2\|^2_{L^2(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} \\
&\quad\leq C_3(\beta,\gamma)\int_0^{\varepsilon_0}\varepsilon^{\gamma-1}\biggl[\int_0^{\lambda_0}\bigl(1+\log|\log\lambda|\bigr)^{-\beta-1}\bigl(\lambda|\log\lambda|\bigr)^{-1} \\
&\qquad\times\|\nabla\varphi_1 - \nabla\varphi_2\|^{2-\varepsilon}_{L^{2-\varepsilon-\lambda}(\Omega)}\,d\lambda\biggr]^{\frac{2}{2-\varepsilon}}\,d\varepsilon \\
&\quad\leq C_4(\beta,\gamma,K)\int_0^{\varepsilon_0}\varepsilon^{\gamma-1}\biggl[\int_0^{\lambda_0}\bigl(1+\log|\log\lambda|\bigr)^{-\beta-1}\bigl(\lambda|\log\lambda|\bigr)^{-1} \\
&\qquad\times\|\underline{\chi}_1 - \underline{\chi}_2\|^{2-\varepsilon}_{L^{2-\varepsilon-\lambda}(\Omega)}\,d\lambda\biggr]^{\frac{2}{2-\varepsilon}}\,d\varepsilon \\
&\quad\leq C_5(\beta,\gamma,K)\int_0^{\varepsilon_0}\varepsilon^{\gamma-1}\|\underline{\chi}_1 - \underline{\chi}_2\|^2_{L^{2-\varepsilon}(\log\log\log L)^{-\beta}(\Omega)}\,d\varepsilon.
\end{aligned}$$

Using again Theorem 3.1 in the last term, we have

$$\begin{aligned}
&\|\nabla\varphi_1 - \nabla\varphi_2\|^2_{L^2(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} \\
&\quad\leq C_5(\beta,\gamma,K)|||\underline{\chi}_1 - \underline{\chi}_2|||^2_{L^2(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)} \\
&\quad\leq C_6(\beta,\gamma,K)\|\underline{\chi}_1 - \underline{\chi}_2\|^2_{L^2(\log L)^{-\gamma}(\log\log\log L)^{-\beta}(\Omega)}.
\end{aligned}$$

□

Now we are in position to prove the main theorem.

Proof of Theorem 1.1 Since $L^{\widetilde{\Phi}}(\Omega) = L(\log L)^{\delta}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)$ is a subspace of $L(\log L)^{\frac{1}{2}}(\Omega)$ if $\beta \geq 0$ and $\delta \geq \frac{1}{2}$, we can ensure (as already observed) that (1.1) has a unique finite energy solution $v \in W_0^{1,2}(\Omega)$.

In order to prove Theorem 1.1, we want to apply the regularity result given by Theorem 4.2. To do this, as already showed in the papers [10, 11, 13], and [12], we need to linearize problem (1.1). We will use a linearization procedure introduced in [27] that preserves the ellipticity bounds.

For shortness, we do not give all the details of the linearization procedure, and we refer, for example, to proof of Theorem 1.1 in [11]. So we know that there exists a symmetric, definite positive, and measurable matrix-valued function $B = B(x)$ such that

$$A(x, \nabla v) = B(x)\nabla v.$$

Then, the unique finite energy solution $v \in W_0^{1,2}(\Omega)$ of (1.1) with $f \in L^{\widetilde{\Phi}}(\Omega)$ solves also the following linear problem:

$$\begin{cases} -\operatorname{div} B(x)\nabla v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

that is,

$$\int_\Omega B(x)\nabla v \nabla\varphi = \int_\Omega f\varphi, \quad \forall\varphi \in W_0^{1,2}(\Omega). \tag{4.7}$$

The case $\boldsymbol{\beta = 0}$ and $\boldsymbol{\frac{1}{2} \le \delta \le 1}$ has been proved in [10].
The case $\boldsymbol{\beta > 0}$ and $\boldsymbol{\delta = \frac{1}{2}}$ has been proved in [13].
Now, if $\boldsymbol{\beta \ge 0}$ and $\boldsymbol{\delta > \frac{1}{2}}$, then we fix $\underline{\chi} \in C^1(\overline{\Omega})$ such that

$$\|\underline{\chi}\|_{L^2(\log L)^{-(2\delta-1)}(\log\log\log L)^{-\beta}(\Omega;\mathbb{R}^2)} \le 1,$$

and we consider the unique finite energy solution φ to the linear Dirichlet problem

$$\begin{cases} -\operatorname{div} B(x)\nabla\varphi = \operatorname{div}\underline{\chi} & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $B(x)$ is the matrix given by the linearization procedure. By Theorem 4.2 we have

$$\begin{aligned} &\|\nabla\varphi\|_{L^2(\log L)^{-(2\delta-1)}(\log\log\log L)^{-\beta}(\Omega)} \\ &\quad \le C(\beta,\delta,K)\|\underline{\chi}\|_{L^2(\log L)^{-(2\delta-1)}(\log\log\log L)^{-\beta}(\Omega)} \le C(\beta,\delta,K), \end{aligned}$$

and so, using Lemma 2.4, we obtain

$$\|\varphi\|_{L^\Phi(\Omega)} \le C_1(\beta,\delta,K), \tag{4.8}$$

where $\Phi(s) \cong e^{s^{\frac{1}{\delta}}(\log\log s)^{-\frac{\beta}{2\delta}}}$, and $C_1(\beta,K)$ is another constant depending only on β, δ, and K.

Thanks to the fact that v satisfies the linear problem (4.6) and that $B(x)$ is a symmetric matrix, using Lemma 2.3 and the Hölder inequality between the complementary spaces $L^\Phi(\Omega)$ and $L^{\widetilde{\Phi}}(\Omega)$, by (4.8) we obtain that, for any $\underline{\chi} \in C^1(\overline{\Omega};\mathbb{R}^2)$ such that $\|\underline{\chi}\|_{L^2(\log L)^{-(2\delta-1)}(\log\log\log L)^{-\beta}(\Omega)} \le 1$, we have

$$\begin{aligned} \left|\int_\Omega \nabla v \cdot \underline{\chi}\right| &= \left|\int_\Omega v \operatorname{div}\underline{\chi}\right| \\ &= \left|\int_\Omega v \operatorname{div}\bigl(B(x)\nabla\varphi\bigr)\right| = \left|\int_\Omega B(x)\nabla v \cdot \nabla\varphi\right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \int_\Omega f\varphi \right| \le C_2(\beta,\delta) \|\varphi\|_{L^\Phi(\Omega)} \|f\|_{L(\log L)^\delta (\log\log\log L)^{\frac{\beta}{2}}(\Omega)} \\
&\le C_2(\beta,\delta,K) \|f\|_{L(\log L)^\delta (\log\log\log L)^{\frac{\beta}{2}}(\Omega)},
\end{aligned}
\tag{4.9}
$$

where $C_2(\beta,\delta,K)$ is a constant that depends only on β, δ, and K.

By Theorem 2.1 the dual space of $L^2(\log L)^{-(2\delta-1)}(\log\log\log L)^{-\beta}(\Omega)$ is $L^2(\log L)^{2\delta-1} \times (\log\log\log L)^\beta(\Omega)$.

Now, since $C^1(\overline{\Omega};\mathbb{R}^2)$ is dense in $L^2(\log L)^{-(2\delta-1)}(\log\log\log L)^{-\beta}(\Omega)$ (see [20], Theorem 8.20 and [23], Corollary 5), passing to the supremum in (4.9) under the conditions $\underline{\chi} \in C^1(\overline{\Omega};\mathbb{R}^2)$, $\|\underline{\chi}\|_{L^2(\log L)^{-(2\delta-1)}(\log\log\log L)^{-\beta}(\Omega;\mathbb{R}^2)} \le 1$, we obtain

$$
\|\nabla v\|_{L^2(\log L)^{2\delta-1}(\log\log\log L)^\beta(\Omega)} \le c(\beta,\delta,K)\|f\|_{L(\log L)^\delta(\log\log\log L)^{\frac{\beta}{2}}(\Omega)},
$$

as desired. □

Remark 4.3 In [27], it was proved that the linearization procedure holds in any dimension with the following ellipticity bounds:

$$
|\xi|^2 + \left|A(x,\xi)\right|^2 \le \left(K + \frac{1}{K}\right)\langle A(x,\xi),\xi\rangle, \quad \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega.
$$

We would like to point out that the linear growth of $A(x,\xi)$ with respect to ξ is absolutely essential for the previous results. The main difficulty with the n-harmonic-type equations ($n \neq 2$) is due to the lack of uniqueness for very weak solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors performed all the steps of the ideas and proofs in this research. All authors read and approved the final manuscript.

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