Flowers and Satellites

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In this paper some interesting properties of circular orbiting satellites' ground traces are pointed out. It will be shown how such properties are typical of some spherical and plane curves that look like flowers both in their name and shape. In addition, the choice of the best satellite constellations satisfying specific requirements is strongly facilitated by exploiting these properties.

1. INTRODUCTION. Since the dawning of mathematical sciences, many scientists have attempted to emulate mathematically the beauty of nature, with particular care to floral forms. The mathematician Guido Grandi gave in the Age of Enlightenment a definitive contribution; some spherical and plane curves reproducing flowers have his name: *Guido Grandi's Clelie and Rodonee*. It will be deduced that the ground traces of circular orbiting satellites are described by the same equations, simple and therefore easily reproducible, and hence employable in satellite constellation designing and in other similar problems.

2. GROUND TRACE GEOMETRY. The parametric equations describing the ground trace of any satellite in circular orbit are:

$$\left.\begin{aligned} \sin\phi &= \sin i \sin \frac{24(t-\tau_0)}{T} \\ \tan(\lambda-\lambda_0+t-\tau_0) &= \cos i \tan \frac{24(t-\tau_0)}{T} \end{aligned}\right\} \tag{1}$$

where $i \equiv$ inclination of the orbital plane; $\phi, \lambda \equiv$ geographic coordinates of the subsatellite point; $t \equiv$ time parameter, defined as angular measure of Earth rotation starting from τ_0; $\tau_0, \lambda_0 \equiv$ time and geographic longitude relative to the crossing of the satellite at the ascending node; $T \equiv$ orbital satellite period as expressed in hours.

In the specific case of a polar-orbiting satellite, equations (1) become

$$\left.\begin{aligned} \sin\phi &= \sin\mu t \\ \tan(\Delta\lambda+t) &= 0 \end{aligned}\right\} \tag{2}$$

where $\tau_0 = 0$; $\Delta\lambda = \lambda-\lambda_0$; $\mu = 24/T$.

From (2)

$$\phi = -\mu\Delta\lambda \tag{3}$$

is obtained, which represents, as μ takes different values, the family of *Guido Grandi's Clelie*.

Some famous curves, well-known because they were studied for a long time

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in past centuries, also belong to this family, namely, *Pappo's spherical spiral* and, more generally, *Archimedes' spherical spirals*.

By introducing a geocentric right-hand Cartesian axes system, with the z-axis coinciding with the Earth's polar axis and the x-axis coinciding with the ascending node of the ground trace, equations (1) can be written (see Appendix)

$$\left.\begin{aligned} x &= \cos i \sin \mu t \sin t + \cos \mu t \cos t \\ y &= \cos i \sin \mu t \cos t - \cos \mu t \sin t \\ z &= \sin i \sin \mu t \end{aligned}\right\}$$

These, in the polar orbit, become

$$\left.\begin{aligned} x &= \cos \mu t \cos t \\ y &= \cos \mu t \sin t \\ z &= \sin \mu t \end{aligned}\right\} \tag{4}$$

and this is the most convenient form to describe the *Clelia* curve.

On the assumption that μ is a rational number, by denoting with a and b two whole numbers with no common divisor and satisfying the condition $\mu = a/b$, then (4) represents a closed curve entirely described by varying t from τ_0 to $\tau_0 + 2b\pi$. The assumption that μ is a rational number is necessary because both satellite orbital and Earth rotation periods can be expressed as rational numbers,[1] so the ratio between them becomes, by multiplying both numerator and denominator by the same power of 10, a ratio between the two whole numbers a and b already defined.

The ground trace will be repeated every b Earth rotations and a satellite orbital periods, neglecting the orbital node precession.[2] In the case of polar-orbiting satellites, such a precession can be indeed considered as zero and therefore the aforesaid conclusion can be taken as true.

3. MAPPING. It is now desirable to find a simple way to map the curves in a representation where beauty and ease of reproduction are preserved. It is well known that a family of plane curves similar to the *Clelia* family is that of *Guido Grandi's Rodonee*, the second being the orthogonal projection of the first on a plane perpendicular to the z-axis. Thus, the *polar orthographic projection* will be used because of its conformity with the aforesaid requirements. In fact to plot the ground trace on such a map it is sufficient to eliminate the third equation from (4) obtaining

$$\left.\begin{aligned} x &= \cos t \cos \mu t \\ y &= \sin t \cos \mu t \end{aligned}\right\}$$

or better, by introducing a polar coordinate system (ρ, ω) with the polar axis coinciding with the x-axis

$$\left.\begin{aligned} \rho &= \cos \mu t \\ \omega &= -t \end{aligned}\right\}$$

or again

$$\rho = \cos \mu\omega \tag{5}$$

that, as already said, represents the family of the *Guido Grandi's Rodonee*, namely special epicycloids, also called *Foliate Curves*.

In such a form, the ground trace projection has the property that it will be mapped in a very simple way. Moreover, the sub-satellite point is easily located at any moment with only uncertainty as to hemisphere. This uncertainty is removed, however, by taking into account the fact that the sub-satellite point (describing the ground trace) changes hemisphere every time it touches the equatorial circumference.

Let us recall two interesting properties of the *Rodonee*.

(i) In the purely theoretical case that μ is an irrational number, the *Rodonea* passes an infinite number of times through the map centre (projection of the poles) so an infinite number of leaves are produced. On the other hand, if μ is a rational number the curve is formed by a leaves in the case that a and b are both odd, or by $2a$ leaves otherwise.

(ii) The Cartesian equation of the Rodonea is

$$\sum_{i=0}^{a-2i\geqslant 0} -1^i \binom{a}{2i} x^{a-2i} y^{2i} = \sum_{i=0}^{b-2i\geqslant 0} -1^i \binom{b}{2i} (x^2+y^2)^{(a+2b-2i)/2} [1-(x^2+y^2)]^i \quad (6)$$

It is evident that if a and b are both odd the equation degree is $a+b$ becoming $2(a+b)$ otherwise because of the necessity of squaring (6) in order to rationalize it. As a matter of fact, the results achieved are very significant: the images of the ground trace of artificial satellites in circular polar orbit on the polar orthographic map are represented by algebraic curves with rank depending only by the orbital period.

Let us consider some particular and interesting cases:

(i) *Synchronous polar orbit*. With $\mu = 1$, $a = 1$, $b = 1$ equation (6) becomes

$$x^2+y^2 = x$$

that is, the equation of a circle centred on $(\frac{1}{2}, 0)$ and having $\frac{1}{2}$ as radius (Fig. 1). Such a circumference is the projection on the equatorial plane of the *Clelia*

$$\phi = -\Delta\lambda$$

better known as '*Finestra di Viviani*'; it can be seen as the intersection of the sphere with a cylinder perpendicular to the equatorial plane, internally tangent to the sphere and having the polar axis as generatrix axis (Fig. 2).

(ii) *TRANSIT Satellite*. This is the well-known navigation satellite having $\sim$ 107 m as orbital period; that is to say that $\mu = 13{\cdot}5$, $a = 27$ and $b = 2$ when an orbital period of 106 m 40 s is chosen (Fig. 3). Figures 3 to 9 show that the images of the ground traces on the map are quite similar to flowers.

4. CONCLUSIONS. Having simple formulae for satellite ground traces is very useful for their quick visualization on a screen and to rapidly determine the orbital period that changes a given configuration to a better one. In other words these simple formulae are of interest for the choice of the best satellite constellations answering defined requirements by CAD techniques. Furthermore, such formulae allow the simple and quick location, instant by instant, of all the satellites belonging to a constellation. These formulae have been developed for a particular mapping technique and for an orbit class with very interesting applications: the circular polar one.

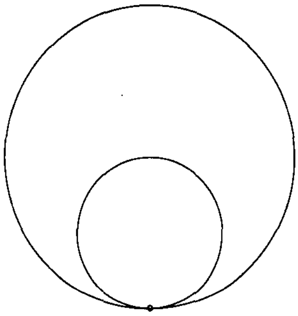


Fig. 1

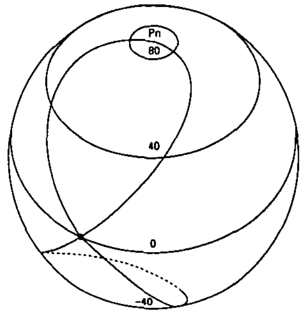
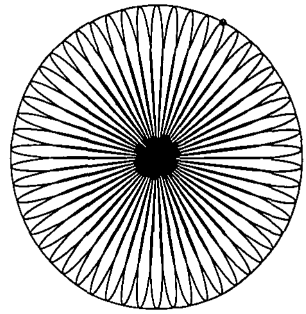


Fig. 2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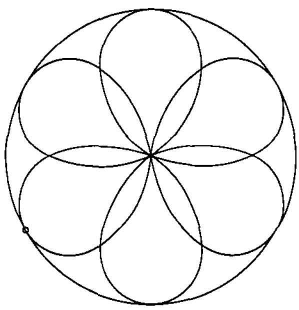
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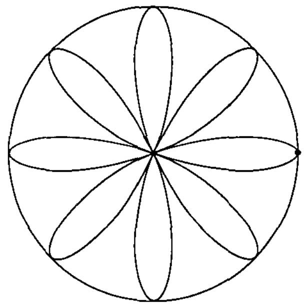
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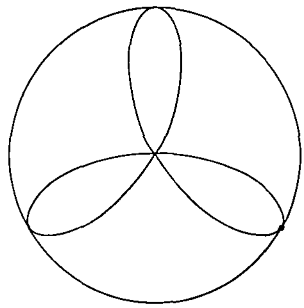
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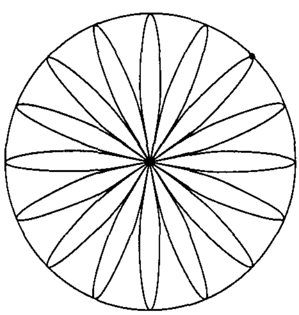
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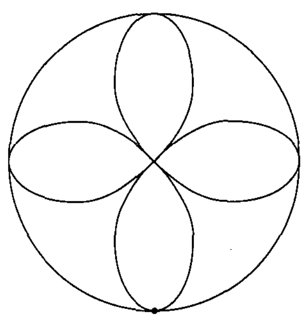
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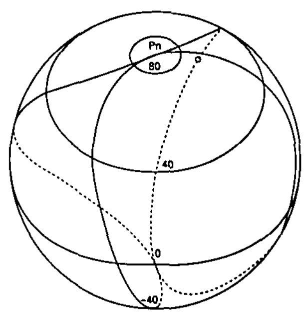

Fig. 9

NOTES

[1] The orbital period can be expressed by an irrational number but practically, because of the limit in the time measurement precision, it will be rounded off.

[2] The daily precession is given by

$$-1{\cdot}638303 * 10^{-3}\left(\frac{R}{a}\right)^2 \frac{\cos i}{(1+e^2)^2}\, n$$

where $a \equiv$ semi major orbital axis; $e \equiv$ orbital eccentricity; $n \equiv$ satellite mean motion; $R \equiv$ Earth's equatorial radius.

APPENDIX 1

Equations (1), with the conditions already specified: $\tau_0 = 0$; $\Delta\lambda = \lambda - \lambda_0$; $\mu = 24/T$ become

$$\sin\phi = \sin i \sin \mu t$$
$$\tan(\Delta\lambda + t) = \cos i \tan \mu t$$

from which, by developing the second one

$$\frac{\sin\Delta\lambda\cos t + \cos\Delta\lambda\sin t}{\cos\Delta\lambda\cos t - \sin\Delta\lambda\sin t} = \frac{\cos i\sin\mu t}{\cos\mu t};$$

$$\tan\Delta\lambda = \frac{\cos i\sin\mu t\cos t - \sin t\cos\mu t}{\cos i\sin t\sin\mu t + \cos t\cos\mu t};$$

$$\cos\Delta\lambda = \frac{\cos i\sin t\sin\mu t + \cos t\cos\mu t}{\sqrt{(\cos^2 i\sin^2\mu t + \cos^2\mu t)}}; \quad \sin\Delta\lambda = \frac{\cos i\sin\mu t\cos t - \sin t\cos\mu t}{\sqrt{(\cos^2 i\sin^2\mu t + \cos^2\mu t)}}.$$

From the first one, on the other hand

$$\cos\phi = \sqrt{(1 - \sin^2\phi)} = \sqrt{(\cos^2 i\sin^2\mu t + \cos^2\mu t)}$$

and finally, being in the right-hand Cartesian system axes

$$x = \cos\phi\cos\Delta\lambda$$
$$y = \cos\phi\sin\Delta\lambda$$
$$z = \sin\phi$$

equations (1) are obtained.

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KEY WORDS

1. Satellites 2. Projections.