CRITICAL METRICS OF THE TRACE OF THE HEAT KERNEL ON A COMPACT MANIFOLD

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Abstract

This paper is devoted to the study of critical metrics of the trace of the heat kernel on a compact manifold. We obtain various characterizations of such metrics and investigate their geometric properties. We also give a complete classification of critical metrics on surfaces of genus zero and one.

Introduction

The kernel of the heat operator of a compact Riemannian manifold (M, g) contains a lot of geometric informations. In particular, its trace, usually denoted by $Z(t)$, determines the spectrum of the manifold (M, g). The study of this function represents an important topic in the current Riemannian geometry. Comparison results and estimates were particularly investigated. For instance, Brard and Gallot [3] showed that for any $t > 0$, the standard sphere $\mathbb{S}^n$ maximizes $Z(t)$ among all the Riemannian n-manifolds having a greater Ricci curvature. This result can be seen as an extension of the well-known Lichnerowicz-Obata comparison theorem concerning the first positive eigenvalue λ_1 of the Laplacian [6].

On the other hand, Morpurgo [15] has recently obtained a partial (and "local") result in the direction of the following conjecture : *for any $t > 0$, the standard metric of $\mathbb{S}^2$ minimizes $Z(t)$ among all the metrics of the same volume*. Evidence for this conjecture is given by the fact that this metric is a global maximizer for both the first eigenvalue [12] and the determinant of the Laplacian [17].

Concerning the torus $\mathbf{T}^2$, the same conjecture can be formulated about the flat metric g_{eq} corresponding to the equilateral (or hexagonal) lattice. Indeed, this metric is already known to be the maximizer of both the first eigenvalue [16] and the determinant of the Laplacian [17]. Moreover, Montgomery [14] showed that the metric g_{eq} minimizes $Z(t)$ among all the flat metrics of the same volume.

In this paper we investigate the critical metrics of $Z(t)$ considered as a functional on the space of metrics, or a conformal class of metrics, of fixed volume. Thus, we start in section 1 by giving the first variation formula for $Z(t)$. Metrics which are critical for $Z(t)$ at any time $t > 0$ are characterized by the equation (Theorem 2.2) :

$$d_S K(t, \cdot, \cdot) = -\frac{Z'(t)}{nV} g, \tag{1}$$

where $d_S K$ is the "mixed second derivative" of the heat kernel K and V is the Riemannian volume of (M, g). This last condition is actually equivalent to the fact that each eigenspace of the Laplacian of g admits an $L_2(g)$-orthonormal basis (f_i) such that $\sum_i df_i \otimes df_i$ is proportional to g.

An alternative formulation of the criticality condition is the following : in [4], Brard, Besson and Gallot introduced for any $t > 0$ a natural embedding Φ_t of (M, g) into the Hilbert space ℓ^2. It turns out that these embeddings are homothetic if and only if the metric g is critical for $Z(t)$ at any time t (Corollary 2.1).

The metrics of strongly harmonic manifolds and of homogeneous Riemannian spaces with irreducible isotropy representation are immediate examples of critical metrics for $Z(t)$ at any time $t > 0$. For instance, standard metrics of the sphere $\mathbb{S}^n$ and the projective spaces $\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n$ and CaP^2 are critical. Moreover, if g is critical on M, then the product metric $g \times g \times \cdots \times g$ is critical on $M \times M \times \cdots \times M$. For general products $(M_1 \times M_2, g_1 \times g_2)$ of two critical metrics, Proposition 2.1 and Corollary 2.3 give necessary and sufficient conditions for the criticality of $g_1 \times g_2$.

Replacing K by its Minakshisundaram-Pleijel asymptotic expansion in the equation (1), it turns out that each coefficient of this expansion satisfies a similar equation (Theorem 3.1). From the equations satisfied by the first two coefficients we deduce that (Corollary 3.1), if g is a critical metric, then (M, g) is Einstein, the length $|R|$ of its curvature tensor is constant and $\overline{R} = \frac{|R|^2}{n} g$ with $\overline{R}_{ab} = R_{ajkl}\ R_b^{jkl}$ (note that for an Einstein metric g, the condition $\overline{R} = \frac{|R|^2}{n} g$ is equivalent to the criticality of g for the functional $\int_M |R|^2 dv_g$).

As Einstein metrics are unique, up to dilatations, in their conformal classes (Obata's theorem, see for instance [1]), it follows that a conformal class of metrics contains, up to dilatations, at most one critical metric.

Section 4 deals with metrics which are critical for $Z(t)$ at any $t > 0$ only under conformal deformations. These metrics are characterized by the fact that their heat kernel is constant on the diagonal (Theorem 4.1):

$$K(t, x, x) = \frac{Z(t)}{V}. \tag{2}$$

This condition is equivalent to the fact that each eigenspace of the Laplacian of g is spanned by an $L_2(g)$-orthonormal basis $(f_i)_i$ such that $\sum_i f_i^2$ is constant. In particular, the metric of any Riemannian homogeneous space is critical for $Z(t)$ at any $t > 0$ under conformal deformations. Moreover, the product of two arbitrary critical metrics under conformal deformations is itself critical (Proposition 4.1).

Equation (2) implies that the coefficients of the Minakshisundaram-Pleijel expansion are constant on the diagonal of $M \times M$. In particular, if g is critical under conformal deformations, then its scalar curvature as well as the length $|W|$ of its Weyl tensor are constant (Theorem 4.2).

In the last section, we classify critical metrics on surfaces of genus 0 and 1. Indeed, on the sphere $\mathbb{S}^2$ and the real projective plane $\mathbb{R}P^2$, the standard metrics are the only critical metrics (for general deformations as well as for conformal ones because of the uniqueness of the conformal class). On the torus $\mathbf{T}^2$, the critical metrics under conformal deformations are exactly the flat ones. Among them, those corresponding to square and equilateral lattices are the only critical ones under general deformations (Theorem 5.1). For the Klein bottle we show that (Theorem 5.2) it admits no critical metric under conformal deformations.

As the first variation of the determinant of the Laplacian can be expressed in terms of the first variation of $Z(t)$, it is easy to see that (Proposition 2.2) any critical metric for $Z(t)$ at any $t > 0$ is also a critical metric for the determinant.

1 First variation formula.

Let M be a compact connected manifold without boundary. For any metric g on M, the spectrum $Sp(g)$ of the Laplacian Δ_g consists of an unbounded sequence of eigenvalues $0 = \mu_0(g) < \mu_1(g) < \mu_2(g) < \ldots < \mu_k(g) < \ldots$. The multiplicity of $\mu_i(g)$ will be denoted m_i and, for any integer $k > 0$, we set : $\nu_k = \sum_{i=1}^{k} m_i$ and $\lambda_i(g) = \mu_k(g)$ if $i \in]\nu_k, \nu_{k+1}]$. If $\{f_i\}_{i\in\mathbf{N}}$ is an orthonormal basis of $L_2(M, g)$ such that for any $i, \Delta_g f_i = \lambda_i f_i$, then the heat Kernel K_g of (M, g) is given on $(0, +\infty) \times M \times M$ by :

$$K_g(t, x, y) = \sum_{i\in\mathbf{N}} e^{-\lambda_i t} f_i(x)\ f_i(y).$$

Its trace Z_g is defined for any $t > 0$ as :

$$Z_g(t) := \int_M K_g(t, x, x)\nu_g = \sum_{i\in\mathbf{N}} e^{-\lambda_i t}.$$

Before stating the first variation formula for Z, let us recall that, to each smooth function f defined on $M \times M$ we associate its "mixed second derivative" $d_S f$ which is a symmetric tensor defined on M by :

$$(d_S f)_x(X,X) = \frac{\partial^2}{\partial\alpha\partial\beta} f(c(\alpha), c(\beta))_{|\alpha=\beta=0}$$

where c is a curve on M tangent to X at x.

Let us fix a Riemannian metric g on M. The induced Riemannian metric on $\bigotimes^k T^*M$ will be denoted by $(\,,\,)$. The corresponding L_2-scalar product on the space of covariant k-tensors will be denoted by $\langle\,,\,\rangle$. Let (g_ε) be a analytic family of Riemannian metrics on M such that $g = g_0$. Set $h = \frac{d}{d\varepsilon} g_\varepsilon|_{\varepsilon=0}$ and $Z_\varepsilon = Z_{g_\varepsilon}$.

Theorem 1.1 *For any positive t, we have*

$$\frac{d}{d\varepsilon} Z_\varepsilon(t)_{|\varepsilon=0} = t\langle d_S K_g(t) + \frac{1}{4}\ \Delta_g \overline{K}_g(t) g\ ,\ h\rangle\ .$$

with the trivial notations: $\overline{K}_g(t)(x) = K_g(t,x,x)$ and $K_g(t)(x,y) = K_g(t,x,y)$.

Proof. According to Berger [5] and Bando-Urakawa [2] there exist for each integer i an analytic family $\Lambda_i(\varepsilon)$ of real numbers, and an analytic family of smooth functions $f_{i,\varepsilon}$ such that $\Lambda_i(0) = \lambda_i(g)$ and, for any ε,

$$\Delta_{g_\varepsilon} f_{i,\varepsilon} = \Lambda_i(\varepsilon) f_{i,\varepsilon}.$$

Moreover, for any ε, the familly $\{f_{i,\varepsilon}\}_{i\in\mathbf{N}}$ can be chosen to be $L_2(g_\varepsilon)$-orthonormal. From the continuity of the functions : $\varepsilon \longmapsto \lambda_i(g_\varepsilon)$ and $\varepsilon \longmapsto \Lambda_i(\varepsilon)$, one can easily deduce that for sufficiently small ε, $Sp(g_\varepsilon) = \{\Lambda_i(\varepsilon)\}_{i\in\mathbf{N}}$, and then $Z_\varepsilon(t) = \sum_{i\in\mathbf{N}} e^{-\Lambda_i(\varepsilon)t}$. Therefore,

$$\frac{d}{d\varepsilon}\ Z_\varepsilon(t)_{|\varepsilon=0} = -t\sum_{i\geq 0} \Lambda_i'\ (0)\ e^{-\lambda_i(g)t}, \tag{3}$$

with (see [5])

$$\Lambda_i'\ (0) = -\langle df_i\ \otimes\ df_i + \frac{1}{4}\big(\Delta_g\ f_i^2\big)\ g, h\rangle,$$

where $f_i = f_{i,0}$.

On the other hand, by a straightforward calculation, we get

$$d_S K_g(t) = \sum_{i\geq 0} e^{-\lambda_i(g)t}\ df_i\ \otimes\ df_i$$

and

$$\Delta_g \overline{K}_g(t) = \sum_{i\geq 0} e^{-\lambda_i(g)t}\ \Delta(f_i^2).$$

Replacing in (3) we obtain the desired formula. □

Remark *Note that the result of Theorem 1.1 can also be deduced from the first variation of the heat Kernel obtained by Ray and Singer [18].*

2 Critical metrics.

<u>*Characterizations of criticality*</u>:

A metric g on M will be said critical for the trace of the heat Kernel at the time t if, for any volume preserving deformation (g_ε) of g, we have

$$\frac{d}{d\varepsilon}\ Z_\varepsilon(t)_{|\varepsilon=0} = 0.$$

For simplicity we will write in all the sequel "THK" for "Trace of the Heat Kernel".

Theorem 2.1 *The following conditions are equivalent:*

i) g *is critical for the THK at the time* t.

ii) $d_S K(t) + \frac{1}{4}\Delta\overline{K}(t)\cdot g = -\frac{1}{nV}\ Z'(t) g.$

iii) There exist a function φ_t *on* M *and a constant* c_t *such that* $d_S\ K(t) = \varphi_t.g$ *and* $(\frac{n-2}{4})\ \Delta\overline{K}(t) - \frac{\partial}{\partial t}\overline{K}(t) = c_t.$

For the proof of this Theorem we need the following elementary property:

Lemma 2.1 $(d_S K(t), g) = -\frac{1}{2}\Delta\overline{K}(t) - \frac{\partial}{\partial t}\overline{K}(t).$

Proof. It suffices to check that

$$(d_S K(t), g) = \sum_i e^{-\lambda_i t}\ |df_i|^2,$$

and, $\frac{\partial}{\partial t}\big(\overline{K}(t)\big)(x) = -\Delta_x\big(K(t,x,y)\big)_{|x=y} = -\sum_i e^{-\lambda_i t}\ f_i\ \Delta f_i$ □

Proof of Theorem 2.1. For a volume preserving deformation (g_ε) of g, we have $\int_M < g, h > \nu_g = 0$, where $h = \frac{d}{d\varepsilon}\ g_{\varepsilon}{}_{|\varepsilon=0}$. From the first variation formula (Theorem 1.1), the criticality of g is equivalent to the fact that there exists a constant $c(t)$ such that :

$$d_S K(t) + \frac{1}{4}\Delta\overline{K}(t)\ g = c(t) g.$$

The previous lemma tells us, after integration, that $c(t)$ must be equal to $-\frac{1}{nV}Z'(t)$. □

Remark. *An immediate consequence of Theorem 2.1 is the following sufficient condition for the criticality of* g *for the THK at time* t *: The function* $x \to K(t,x,x)$ *is constant and the tensor* $d_S K(t)$ *is proportional to* g.

Metrics which are critical at any time are characterized as follows :

Theorem 2.2 *The following conditions are equivalent:*

i) The metric g is critical for the THK at any time $t > 0$.

ii) $\forall t > 0$, $d_S K(t)$ is proportional to g and the function $x \to K(t, x, x)$ is constant on M.

iii) $\forall t > 0$, $d_S K(t)$ is proportional to g, that is $d_S K(t) = -\frac{Z'(t)}{nV} g$.

iv) For any integer $k > 0$, the eigenspace E_{μ_k} is spanned by and $L_2(g)$-orthonormal basis $\{\varphi_1, \cdots, \varphi_{m_k}\}$ satisfying:

$$\sum_{i=1}^{m_k} df_i \otimes df_i = \frac{m_k \mu_k}{nV} g.$$

Proof. (i) implies (ii) : Applying Theorem 2.1, we obtain that the function $\frac{n-2}{4} \Delta \overline{K}(t) - \frac{\partial}{\partial t} \overline{K}(t)$ depends only on t, that is

$$\frac{n-2}{4} \Delta \overline{K}(t) - \frac{\partial}{\partial t} \overline{K}(t) = -\frac{Z'(t)}{V}. \tag{4}$$

Setting $\widetilde{K}(t, x) = K(t, x, x) - \frac{1}{V} Z(t)$, this equation becomes

$$\frac{n-2}{4} \Delta_x \widetilde{K} - \frac{\partial}{\partial t} \widetilde{K} = 0.$$

After multiplication by $\widetilde{K}$ and integration over M we get

$$\left(\frac{n-2}{2}\right) \int_M |\nabla_x \widetilde{K}|^2 \nu_g = \frac{\partial}{\partial t} \int_M |\widetilde{K}|^2 \nu_g \ .$$

As the integral of $\widetilde{K}$ on M vanishes, we obtain

$$\frac{\partial}{\partial t} \int_M |\widetilde{K}|^2 \nu_g \ \geq \left(\frac{n-2}{2}\right) \lambda_1 \int_M |\widetilde{K}|^2 \nu_g,$$

which implies that $\widetilde{K}$ is identically zero (indeed, $\widetilde{K}(t, x)$ goes to zero as $t \to +\infty$). Thus the function $x \to K(t, x, x)$ is constant on M and then $\Delta \overline{K}(t) = 0$. The fact that $d_S K(t)$ is proportional to g follows immediately from this last equation and Theorem 2.1.

(ii) implies (iii) : is clear (note that from Lemma 2.1, we have $\langle d_S K(t),\ g \rangle = -Z'(t)$).

(iii) implies (iv) : Let, for any integer $k > 0, \{\varphi_{i,k}\}_{1 \leq\ i \leq\ m_k}$ be an $L_2(g)$-orthonormal basis of E_{μ_k}. It suffices to check that

$$d_S K(t) = \sum_k e^{-\mu_k t} \sum_{i=1}^{m_k} d\varphi_{i,k} \otimes d\varphi_{i,k},$$

and $Z'(t) = -\sum_k m_k\mu_k \, e^{-\mu_k t}$.

(iv) implies (i) : First, condition (iv) is as we just have seen, equivalent to the fact that, $\forall t > 0, d_S K(t) = -\frac{Z'(t)}{nV} g$. On the other hand, the identity $\sum_{i=1}^{m_k} d\varphi_i \otimes d\varphi_i = \frac{m_k\mu_k}{nV} g$ means that the map $x \to \varphi(x) = (\varphi_1(x), \cdots, \varphi_{m_k}(x))$ is a homothetic immersion from (M, g) to $\mathbb{R}^{m_k}$. By a classical argument due to Takahaski [20], we deduce that for any $k > 0$, $|\varphi|^2 = \sum_{i=1}^{m_k} \varphi_i^2$ is constant (indeed, the mean curvature vector field of the submanifold $\varphi(M)$ is proportional to $\Delta\varphi = \mu_k\varphi$. Therefore, for any $X \in T_x M, X \cdot |\varphi|^2 = 2\langle\varphi(x), d\varphi_x(X)\rangle = 0$). Hence, $x \to K(t, x, x)$ is constant and $\Delta\overline{K}(t) = 0$. The criticality of g follows from (ii) of Theorem 2.1. □

Remarks. *1) The analyticity of the Heat Kernel $K(t, x, y)$ in t shows that the criticality of a metric g for the THK on a non trivial interval of time is equivalent to its criticality at any time $t > 0$.*

2) Condition (iv) of Theorem 2.2 implies that the metric g is a critical metric for all the eigenvalues of the Laplacien (cf [10]). It also implies that the multiplicity of any of these eigenvalues is at least $n + 1$.

In [4], Brard, Besson and Gallot introduced the following embeddings of (M, g) into the Hilbert space ℓ^2 :

$$\Phi_t(x) = \left(e^{-\frac{\lambda_i}{2}t} f_i(x)\right)_{i\geq 1},$$

where $(f_i)_{i\in\mathbf{N}}$ is an $L_2(g)$-orthonormal eigenbasis and t is a positive real number. One can easily see that ([4]) :

$$(\Phi_t)^*(\mathrm{can}) = d_S K(t).$$

An immediate consequence of our Theorem 2.2 is the :

Corollary 2.1 *The metric g is critical for the THK at any time $t > 0$ if and only if for any $t > 0, \Phi_t$ is a homothetic embedding of (M, g) into a sphere of ℓ^2.*

In [9], we showed that if the first eigenspace of g contains a family of functions $\{\varphi_1, \cdots, \varphi_k\}$ such that $\sum_{i=1}^k d\varphi_i \otimes d\varphi_i$ is proportional to g, then g maximizes the first eigenvalue of the Laplacien under volume preserving conformal deformations. An immediate consequence of this fact and Theorem 2.2 is the following :

Corollary 2.2 *If g is a critical metric for THK at any time $t > 0$, then for any metric g_1 conformal to g and having the same volume, there exists a constant t_0 such that, $\forall t > t_0$:*

$$Z_1(t) \geq Z(t)$$

Consequently, the functional THK admits no local maximizer at any time $t > 0$.

Standard examples.

• Recall that a compact Riemannian manifold (M, g) is called strongly harmonic (cf [8]) if there exists a map $F : \mathbb{R}_+^* \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $\forall x, y \in M$ and $\forall t \in \mathbb{R}_+^* : K(t, x, y) = F\big(t, r(x, y)\big)$, where r is the geodesic distance w.r.t. g. Known examples of such manifolds are the rank one compact symmetric spaces $(\mathbb{S}^n, \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n, CaP^2)$, and it was conjectured that there is no other strongly harmonic manifolds beside them. It is easy to see that for such manifolds, we have :

$$d_S K(t) = \frac{\partial^2}{\partial r^2} F(t, 0) g \text{ and } K(t, x, x) = F(t, 0)$$

and thus, they are critical for the THK at any time $t > 0$.

• If (M, g) is a homogeneous compact Riemannian manifold with irreducible isotropy representation, then g is critical for the THK at any time $t > 0$. Indeed, for any $t, d_S K(t)$ is invariant under isometries and then from the irreducibility condition, it is proportional to g.

Critical product metrics

Other examples are provided products of critical metrics :

Proposition 2.1 *Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds. The product metric $g_1 \times g_2$ is critical for the THK at any time $t > 0$ if and only if g_1 and g_2 are both critical for the THK at any time $t > 0$ and $Z_1^{n_2} = Z_2^{n_1}$, where n_i is the dimension of M_i.*

Proof. For any $X = (x_1, x_2)$ and $Y = (y_1, y_2) \in M_1 \times M_2$, we have

$$K(t, X, Y) = K_1(t, x_1, y_1) K_2(t, x_2, y_2), \tag{5}$$

where K_i is the heat Kernel of (M_i, g_i) and K is that of the product manifold $(M_1 \times M_2, g_1 \times g_2)$. A direct calculation gives for any $W = (W_1, W_2) \in T_X(M_1 \times M_2)$

$$\begin{aligned} d_S K(X)(W, W) = \quad & K_1(t, x_1, x_1)\ d_S K_2(x_2)(W_2, W_2) \\ & + K_2(t, x_2, x_2)\ d_S K_1(x_1)(W_1, W_1) \\ & + \tfrac{1}{2}\big(d\overline{K}_1(t)\big)(W_1)\ \big(d\overline{K}_2(t)\big)(W_2). \end{aligned} \tag{6}$$

Suppose that g_1 and g_2 are critical and $Z_1^{n_2} = Z_2^{n_1}$. Then (Theorem 2.2), for any $i, K_i(t, x_i, x_i)$ is constant on M_i, that is $K_i(t, x_i, x_i) = \frac{Z_i(t)}{V_i}$, and $d_S K_i = -\frac{Z_i'(t)}{n_i V_i} g_i$. Replacing in (6), we obtain :

$$d_S K = -\frac{1}{V_1 V_2}\Big[\frac{Z_1 Z_2'}{n_2} g_2 + \frac{Z_2 Z_1'}{n_1} g_1\Big].$$

The logarithmic derivative of the equality $Z_1^{n_2} = Z_2^{n_1}$ yields

$$\frac{Z_1 Z_2'}{n_2} = \frac{Z_2 Z_1'}{n_1} = \frac{(Z_1 Z_2)'}{n_1 + n_2} = \frac{Z'}{n_1 + n_2}.$$

Thus $d_S K = -\frac{Z'}{(n_1+n_2)V}(g_1 + g_2)$ and the product metric is critical.

Reciprocally, suppose that the metric $g_1 \times g_2$ is critical. Then it follows from the identities (5) and (6) that $K_i(t, x_i, x_i)$ is constant on M_i and $d_S K_i$ is proportional to g_i. Therefore g_1 and g_2 are critical and for any $i, d_S K_i = -\frac{Z_i'}{n_i V_i} g_i$. Using identity (6) again, we obtain

$$-\frac{Z_1 Z_2'}{n_2 V_1 V_2} = -\frac{Z_2 Z_1'}{n_1 V_1 V_2},$$

and hence : $n_2 \frac{Z_1'}{Z_1} = n_2 \frac{Z_2'}{Z_2}$, which gives after integration (as $Z_i(t) \xrightarrow[t \to +\infty]{} 1$) $Z_1^{n_2} = Z_2^{n_1}$. □

Corollary 2.3 *With the same notations as Proposition 2.1, if $n_1 = n_2$, then the metric $g_1 \times g_2$ is critical for the THK at any time t if and only if g_1 and g_2 are both critical for the THK at any time t and $Sp(g_1) = Sp(g_2)$.*

In particular, if g is a critical metric for the THK at any time t on M, then any product $g \times g \times \cdots \times g$ is critical for the THK at any time t on $M \times M \times \cdots \times M$. This is for instance the case of the standard flat metric of the n-torus $\mathbf{T}^n = S^1 \times \cdots \times S^1$.

A remark on critical metrics of the determinant of the Laplacian:

For any complex number s such that $Re\ s > \frac{n}{2}$, the zeta function of (M, g) is given by :

$$\xi(s) = \sum_{k \geq 1} \frac{1}{\lambda_k^s} = \frac{1}{\Gamma(s)} \int_0^\infty \Big(Z(t) - \frac{1}{V}\Big) t^{s-1} dt.$$

It is well known that this function extends to a meromorphic function regular at $s = 0$ (see for instance [17]). The determinant of the Laplacian of (M, g) is then defined by the formula :

$$det(\Delta) = e^{-\xi'(o)}.$$

Now, for any fixed volume deformation g_ε og g, we have:

$$\frac{d}{d\varepsilon}\ det(\Delta_\varepsilon)_{|\varepsilon=0} = -\frac{d}{d\varepsilon}\ \xi'_\varepsilon(o)_{|\varepsilon=0}\det(\Delta).$$

For $Re\ s$ large, we can write :

$$\frac{d}{d\varepsilon}\ \xi_\varepsilon\ (s)_{|\varepsilon=0} = \frac{1}{\Gamma(s)}\ \int_0^\infty (\frac{d}{d\varepsilon}Z_\varepsilon(t)_{|\varepsilon=0})t^{s-1}dt.$$

If g is a critical metric for THK at any time $t > 0$, then for $Re\ s$ large :

$$\frac{d}{d\varepsilon}\ \xi_\varepsilon(s)_{|\varepsilon=0} = 0.$$

This equality extends meromorphically to a neighborhood of $s = 0$ to give

$$\frac{d}{d\varepsilon}\ \xi'_\varepsilon(0)_{|\varepsilon=0} = 0.$$

Therefore, we have the following

Proposition 2.2 *If g is a critical metric for THK at any time $t > 0$, then it is also critical for the functional det Δ.*

3 Geometric restrictions to criticality.

The Minakshisundaram - Pleijel asymptotic expansion of the heat Kernel [13] gives rise to a family of spectral invariants $u_p \in \mathcal{C}^\infty(M \times M)$ such that for any $(x, y) \in M \times M$ close to the diagonal and sufficiently small t :

$$K(t, x, y) = \frac{e^{-r^2(x,y)/4t}}{(4\pi t)^{\frac{n}{2}}}\ (u_0(x, y) + tu_1(x, y) + \cdots + t^p u_p(x, y) + \cdots), \quad (7)$$

where $r(x, y)$ is the geodesic distance between x and y. Thus,

$$Z(t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}(a_0 + ta_1 + \cdots + t^p a_p + \cdots)$$

where $a_i = \int_M u_i(x, x)\nu_g$.

Theorem 3.1 *If g is a critical metric for the THK at any time $t > 0$, then for any integer p:*

i) The function u_p is constant on the diagonal of $M \times M$: $u_p(x, x) = \frac{a_p}{V}$, and *ii) The tensor $d_S u_p$ is proportional to g, that is*

$$d_S u_p = -\frac{(p+1)}{nV} a_{p+1} g.$$

Proof. Property (i) follows directly from (ii) of Theorem 2.2. Now, it is known that the expansion (7) can be differentiated term by term, which leads to

$$d_S K = \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{p \geq 0} d_S(e^{-r^2/4t} u_p) t^p.$$

A straightforward calculation gives at any point $x \in M$ (see [4]):

$$\begin{aligned} d_S(e^{-r^2/4t} u_p) &= \left(-\frac{1}{4t}(d_S r^2) u_p(x,x) + d_S u_p\right) \\ &= \left(\frac{1}{2t}\,\frac{a_p}{V}\; g + d_S\; u_p\right). \end{aligned}$$

On the other hand,

$$Z'(t) = \frac{1}{(4\pi)^{\frac{n}{2}}}\; t^{-\frac{n}{2}-1} \sum_{p \geq 0} (p - \frac{n}{2}) t^p a_p.$$

As the criticality of g implies the identity $d_S K = -\frac{Z'(t)}{nV} g$ (Theorem 2.2), property (ii) follows immediately. □

The Riemann curvature tensor of (M, g) will be denoted by R and we will denote by $\overline{R}$ the symmetric 2-tensor given by :

$$\overline{R}(X, Y) = (i_X R, i_Y R),$$

where $i_X R$ is the interior product of R by X.

As a consequence of Theorem 3.1 we have the:

Corollary 3.1 *If g is a critical metric for the THK at any time $t > 0$, then:*

i) (M, g) is an Einstein manifold.

ii) $|R|^2$ is constant on M and $\overline{R} = \frac{|R|^2}{n} g$.

In particular, in dimension 2 and 3, critical metrics are of constant curvature. Note that on the torus $\mathbf{T}^4$, Einstein metrics are the flat ones (see for instance [7]). Note also that a consequence of this corollary is that a critical metric for the THK at any time $t > 0$ is also critical for the functional $\int_M |R|^2 \nu_g$ (cf [7], Corollary 4.72)

Proof of Corollary 3.1. It is known that ([6]) :

$$u_0(x,y) = \theta(x,y)^{-1/2},$$

where $r^{n-1}\theta(x,y)$ is the volume density of $\exp_x^* g$ at $\exp_x^{-1}(y)$. Following Brard, Besson and Gallot [4], we first write down the Taylor expansion of θ :

$$\theta\big(c(\alpha), c(\beta)\big) = 1 - Ric\ \big(\dot{c}\ (\alpha), \dot{c}\ (\beta)\big)\big(\frac{\alpha-\beta}{6}\big)^2 + O(|\alpha-\beta|^3),$$

where c is a unit speed geodesic starting from x, then we deduce that :

$$d_S u_0 = -\ \frac{1}{6}\ Ric.$$

The first assertion then follows from Theorem 3.1.

Assertion (ii) follows from the calculation of $d_S u_1$. Indeed, we have the following Taylor expansion for u_1 (see [19] p.594):

$$u_1\big(c(\alpha), c(\beta)\big) = \frac{scal}{6} + A\ \big(\dot{c}\ (\alpha), \dot{c}\ (\alpha)\big)(\alpha-\beta)^2 + O(|\alpha-\beta|^3),$$

where *scal* is the scalar curvature of g and A is the 2-symmetric tensor :

$$A = \frac{1}{3}\bigg(\frac{2}{5!}\ \overline{R} + \frac{(scal)^2}{12n}\big(\frac{1}{2} - \frac{1}{5n}\big)g\bigg).$$

It follows that : $d_S u_1 = -2A$. Therefore, applying Theorem 3.1, A is proportional to g and then so is $\overline{R}$. □

4 Critical metrics in a conformal class.

In this section, we consider the case of metrics which are critical only for conformal deformations. Let g be a Riemannian metric on M and $g_\varepsilon = \varphi_\varepsilon g$ a conformal deformation of g. The first variation formula (Theorem 1.1) for a such deformation reads

Corollary 4.1 *For any positive t, we have*

$$\frac{d}{d\varepsilon} Z_\varepsilon(t)_{|\varepsilon=0} = t\int_M \bigg(\frac{n-2}{4}\ \Delta\ \overline{K}(t) - \frac{\partial}{\partial t}\overline{K}(t)\bigg) u\ \nu_g,$$

where $u = \frac{d}{d\varepsilon}\varphi_{\varepsilon\,|\varepsilon=0}$

Indeed, it suffices to replace h by ug in Theorem 1.1 and use Lemma 2.1. Consequently, critical metrics for the THK at time t under conformal deformations fixing the volume are characterized by the constancy on M of the expression $\frac{n-2}{4}\ \Delta\ \overline{K}(t) - \frac{\partial}{\partial t}\ \overline{K}(t)$ (recall that the volume preservation condition implies that $\int_M u\ \nu_g = 0$).

Theorem 4.1 *The following conditions are equivalent:*

i) The metric g is critical for the THK at any time $t > 0$ under volume preserving conformal deformations.

ii) $\forall t > 0, K(t, x, x)$ is constant on M.

iii) For any integer $k > 0$, the eigenspace E_{μ_k} is spanned by an $L_2(g)$-orthonormal basis $\{\varphi_1, \dots, \varphi_{m_k}\}$ such that $\sum_{i \le m_k} \varphi_i^2$ is constant on M.

Proof. (i) $\Rightarrow$ (ii) : See the proof of (i) $\Rightarrow$ (ii) of Theorem 2.2.

(ii) $\Rightarrow$ (i) is immediate.

(iii) $\Rightarrow$ (ii) : It suffices to see that

$$K(t, x, x) = \sum_k e^{-\mu_k t} \Big(\sum_{1 \le i \le m_k} \varphi_{i,k}^2(x) \Big),$$

where $\{\varphi_{i,k}\}_{1 \le i \le m_k}$ is an $L_2(g)$-orthonormal basis of the eigenspace E_{μ_k}. $\square$

Immediate consequences of Theorem 4.1 are

Corollary 4.2 *If (M, g) is a Riemannian homogeneous space, then g is a critical metric of THK at any time $t > 0$ under volume preserving conformal deformations.*

Corollary 4.3 *Let Φ_t be the embeddings of Corollary 2.1. The metric g is critical for the THK at any $t > 0$ under volume preserving conformal deformations if and only if, for any $t > 0, \Phi_t(M, g)$ is contained in a sphere of ℓ^2.*

For the product metrics, we have the following :

Proposition 4.1 *Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds. The criticality of $g_1 \times g_2$ for the THK at any time t under volume preserving conformal deformations is equivalent to that of g_1 and g_2.*

Proof. It suffices to use equation (6) and assertion (ii) of Theorem 4.1. $\square$

As in section 3, we obtain the following curvature restrictions :

Theorem 4.2 *If g is a critical metric of THK at any time $t > 0$ under volume preserving conformal deformations, then all the functions $u_p(x, x)$ given by the Minakshisundaram-Pleijel expansion of $K(t, x, x)$, are constant on M. In particular, the scalar curvature scal and the length $|W|$ of the Weyl tensor of g are constant.*

Proof. The first assertion is an immediate consequence of Theorem 4.1. For the last assertion it suffices to recall that (see [19]) : $u_1(x,x) = \frac{scal(x)}{6}$ and $u_2(x,x) = \frac{1}{180}|W|^2 + \alpha(n)(scal)^2 + \beta(n)\Delta scal$, where $\alpha(n)$ and $\beta(n)$ are constants. □

Recall that in dimension 2, a conformal class of metrics admits, up to dilatations, only one constant curvature representative. Consequently, critical metrics on surfaces under conformal deformations are unique in their respective conformal classes. In higher dimensions, we have the following uniqueness result :

Corollary 4.4 *Let g be a critical metric of THK at any time $t > 0$ under volume preserving conformal deformations. If g satisfy at least one of the following conditions:*

i) The scalar curvature of g is non positive,

ii) (M, g) is Einstein,

iii) (M, g) is non locally conformally flat,

then g is, up to dilatations, the unique critical metric in its conformal class.

Proof. Recall that if g satisfy (i) or (ii) then the conformal class of g contains (up to dilatations) at most one metric of constant scalar curvature (see for instance [1]). On the other hand, if $\overline{g} = e^f g$, then $e^f|\overline{W}| = |W|$. Now the uniqueness of the critical metric g under assumptions (i), (ii) or (iii) follows from Theorem 4.2. □

5 Classification of critical metrics on surfaces of genus 0 and 1

According to Theorem 4.2, we only have to consider constant curvature metrics. We have already pointed out the criticality of the standard metrics of the sphere $\mathbb{S}^2$ and the real projective plane $\mathbb{R}P^2$. Thus we have the:

Corollary 5.1 *The standard metric on $\mathbb{S}^2$ (resp. $\mathbb{R}P^2$) is up to dilatations the only critical metric for the THK at any time t.*

Note that in genus 0, the uniqueness of the conformal class implies that there is no difference between the global criticality and the criticality under conformal deformations.

Let us consider now the 2-dimensional torus $\mathbf{T}^2$. Recall that any flat torus $(\mathbf{T}^2, g)$ is isometric to $(\mathbb{R}^2/\Gamma, g_\Gamma)$, where Γ is a lattice of $\mathbb{R}^2$ and g_Γ is the flat metric induced by the standard one of $\mathbb{R}^2$.

Theorem 5.1 *i) The critical metrics on* $\mathbf{T}^2$ *for the THK at any time* t *under volume preserving conformal deformations are exactly the flat ones.*

ii) There are, up to dilatations, exactly two critical metrics for the THK at any time t *: The Clifford flat metric* g_{cl} *corresponding to the square lattice* $\Gamma = \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)$*, and the equilateral flat metric* g_{eq} *corresponding to the lattice* $\Gamma = \mathbb{Z}(1,0) \oplus \mathbb{Z}(1/2, \sqrt{3/2})$*.*

Proof. Let Γ be a lattice of $\mathbb{R}^2$ and Γ^* be its dual lattice. It is known [6] that $Sp(g_\Gamma) = \{4\pi^2|\tau|^2, \tau \in \Gamma^*\}$ and the family $\{f_\tau(x) = \sqrt{\frac{2}{V}}\ cos\ 2\pi < \tau, x >$, $g_\tau(x) = \sqrt{\frac{2}{V}}\ sin\ 2\pi < \tau, x >; \tau \in \Gamma^*\}$ is a $L_2(g_\Gamma)$ orthonormal eigenbasis. Therefore ,

$$K(t,x,y) = \sum_{\tau\in\Gamma^*} e^{-4\pi^2|\tau|^2 t}(f_\tau(x)f_\tau(y) + g_\tau(x)g_\tau(y)).$$

In particular, $K(t,x,x) = \frac{2}{V}\sum_{\tau\in\Gamma^*} e^{-4\pi^2|\tau|^2 t}$, which proves the criticality of g_Γ under conformal deformations according to Theorem 4.1.

To study the global criticality of g_Γ, we need to compute $d_S K$:

$$d_S K(t) = \sum_{\tau\in\Gamma^*} e^{-4\pi^2|\tau|^2 t}(df_\tau \otimes df_\tau + dg_\tau \otimes dg_\tau).$$

But,

$$(df_\tau \otimes df_\tau + dg_\tau \otimes dg_\tau) = 4\pi^2 \tau^b \otimes \tau^b,$$

where $\tau^b(X) =< \tau, X >$. From condition (iii) of Theorem 2.2, the criticality of g_Γ is then equivalent to the fact that, for any $\tau \in \Gamma^*$, $\sum_{\sigma,|\sigma|^2=|\tau|^2} \sigma^b \otimes \sigma^b$ is proportional to g_Γ. Now, up to a dilatation, we can assume that Γ has the form : $\Gamma = \mathbb{Z}(1,0) \oplus \mathbb{Z}(a,b)$. Therefore, $\Gamma^* = \mathbb{Z}(1, -\frac{a}{b}) \oplus \mathbb{Z}(0, \frac{1}{b})$ and, for any $\tau = (p, \frac{q-pa}{b}) \in \Gamma^*$, we have

$$\tau^b \otimes \tau^b = p^2 dx^2 + \left(\frac{q-pa}{b}\right)^2 dy^2 + p\left(\frac{q-pa}{b}\right)(dy \otimes dx + dx \otimes dy).$$

Finally, g_Γ is critical if and only if, for any $\tau \in \Gamma^*$, we have

$$\sum_{(p,q)\in A_\tau} p^2 = \sum_{(p,q)\in A_\tau} \left(\frac{q-pa}{b}\right)^2$$

and

$$\sum_{(p,q)\in A_\tau} p\left(\frac{q-pa}{b}\right) = 0$$

where $A_\tau = \{(p,q) \in \mathbb{Z}^2 \ ; \ p^2 + (\frac{q-pa}{b})^2 = |\tau|^2\}$. It is easy to check that these identities are satisfied only when $(a,b) = (0,1)$ or $(a,b) = (1/2, \sqrt{3/2})$. □

Concerning the Klein bottle **K**, we have the following :

Theorem 5.2 *The Klein bottle* **K** *admits no critical metric for the THK at any time t under volume preserving conformal deformations.*

In particular, **K** admits no global critical metrics.

Proof. For any $b > 0$, let Γ_b be the lattice $\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,b)$. It is known that any flat Klein bottle $(\mathbf{K}, g)$ is homothetic to the quotient $\mathbf{K}_b$ of a flat torus $\left(\mathbb{R}^2/\Gamma_{2b}, g_{\Gamma_{2b}}\right)$ by the involution $\gamma : (x,y) \longrightarrow (-x, y+b)$. The eigenfunctions of $\mathbf{K}_b$ correspond then to those of this flat torus which are invariant under γ. After an elementary calculation, we obtain (see for instance [6]) : If $\lambda \in Sp(\mathbf{K}_b)$, then the corresponding eigenspace is spanned by the families $\{cos\ 2\pi px\ cos\ \frac{\pi qy}{b},\ cos\ 2\pi px\ sin\ \frac{\pi qy}{b}; (p,q) \in A_\lambda\}$ and $\{sin\ 2\pi px\ cos \frac{\pi qy}{b}, sin\ 2\pi px\ sin\ \frac{\pi qy}{b}; (p,q) \in\ B_\lambda\}$ where $A_\lambda = \{(p,q) \in \mathbb{Z}^2;\ 4\pi^2(p^2 + \frac{q^2}{4b^2}) = \lambda\ ;\ p \geq 0$ and q even$\}$ and $B_\lambda = \{(p,q) \in \mathbb{Z}^2;\ 4\pi^2(p^2 + \frac{q^2}{4b^2}) = \lambda;\ p > 0$ and q odd$\}$. It follows that :

$$K\big(t,(x,y),(x,y)\big) = \frac{4}{b} \sum_{\lambda \in Sp(K_b)} e^{-\lambda t} \Big(\sum_{A_\lambda} cos^2\ 2\pi px + \sum_{B_\lambda} sin^2\ 2\pi px\ \Big)$$

which is clearly not constant on the diagonal of $\mathbf{K}_b \times \mathbf{K}_b$. □

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