Boundary singularities of N -harmonic functions

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1 Introduction

Let Ω be a domain is $\mathbb{R}^N$ $(N \geq 2)$ with a C^2 compact boundary $\partial\Omega$. A function $u \in W^{1,p}_{loc}(\Omega)$ is p-harmonic if

$$\int_\Omega |Du|^{p-2} \langle Du, D\phi\rangle \, dx = 0 \tag{1.1}$$

for any $\phi \in C^1_0(\Omega)$. Such functions are locally $C^{1,\alpha}$ for some $\alpha \in (0,1)$. In the case $p = N$, the function u is called N-harmonic. The N-harmonic functions play an important role as a natural extension of classical harmonic functions. They also appear in the theory of bounded distortion mappings [8]. One of the main properties of the class of N-harmonic functions is its invariance by conformal transformations of the space $\mathbb{R}^N$. This article is devoted to the study of N-harmonic functions which admit an isolated boundary singularity. More precisely, let $a \in \partial\Omega$ and $u \in W^{1,N}_{loc}(\Omega) \cap C(\overline{\Omega} \setminus \{a\})$ be a N-harmonic function vanishing on $\partial\Omega \setminus \{a\}$, then u may develop a singularity at the point a. Our goal is to show the existence of such singular solutions, and then to classify all the positive N-harmonic functions with a boundary isolated singularity. We denote by $\mathbf{n_a}$ the outward normal unit vector to Ω at a The main result we prove are presented below:

There exists a unique positive N-harmonic function $u = u_{1,a}$ in Ω, vanishing on $\partial\Omega \setminus \{a\}$ such that

$$\lim_{\substack{x \to a \\ \frac{x-a}{|x-a|} \to \sigma}} |x-a|\, u(x) = -\langle \sigma, \mathbf{n_a}\rangle \tag{1.2}$$

uniformly on $S^{N-1} \cap \overline{\Omega} = \{\sigma \in S^{N-1} : \langle \sigma, \mathbf{n_a}\rangle < 0\}$.

The functions $u_{1,a}$ plays a fundamental role in the description of all the positive singular N-harmonic functions since we the next result holds

Let u be a positive N-harmonic function in Ω, vanishing on $\partial\Omega \setminus \{a\}$. Then there exists $k \geq 0$ such that

$$u = k u_{1,a}. \tag{1.3}$$

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When u is no longer assumed to be positive we obtain some classification results provided its growth is limited as shows the following

Let u be a N-harmonic function in Ω, vanishing on $\partial\Omega\setminus\{a\}$ and verifying

$$|u| \leq M u_{1,a},$$

for some $M \geq 0$. Then there exists $k \in \mathbb{R}$ such that

$$u = k u_{1,a}. \tag{1.4}$$

In the last section we give a process to construct p-harmonic regular functions ($p > 1$) or N-harmonic singular functions as product of one variable functions. Starting from the existence of p-harmonic functions in the plane under the form $u(x) = u(r, \sigma) = r^\beta \omega(\theta)$ (see [5]), our method, by induction on N, allows us to produce separable solutions of the *spherical p-harmonic spectral equation*

$$-div_\sigma \left(\left(\beta^2 v^2 + |\nabla_\sigma v|^2 \right)^{(p-2)/2} \nabla_\sigma v \right) = \lambda_{N,\beta} \left(\beta^2 v^2 + |\nabla_\sigma v|^2 \right)^{(p-2)/2} v. \tag{1.5}$$

on S^{N-1}, where $\lambda_{N,\beta} = \beta\left(N - 1 + (\beta-1)(p-1)\right)$. This equation equation is naturally associated to the existence of p-harmonic functions under the form $u(x) = |x|^\beta\, v(x/|x|)$. As a consequence, we express p-harmonic functions under the form of a product of N-explicit functions of one real variable. *If we represent $\mathbb{R}^N$ as the set of $\{x = (x_1, ..., x_N)\}$ where $x_1 = r \sin\theta_{N-1} \sin\theta_{N-2} ... \sin\theta_2 \sin\theta_1$, $x_2 = r \sin\theta_{N-1} \sin\theta_{N-2} ... \sin\theta_2 \cos\theta_1$, ..., $x_{N-1} = r \sin\theta_{N-1} \cos\theta_{N-2}$ and $x_N = r\cos\theta_{N-1}$ with $\theta_1 \in [0, 2\pi]$ and $\theta_k \in [0, \pi]$, for $k = 2, ..., N-1$, then, for any integer k the function*

$$u(x) = (r \sin\theta_{N-1} \sin\theta_{N-2} ... \sin\theta_2)^{\beta_k} \omega_k(\theta_1) \tag{1.6}$$

is p-harmonic in $\mathbb{R}^N$, in which expression $\beta_k > 1$ is an algebraic number depending on k and ω_k is a π/k-antiperiodic solutions of a completely integrable homogeneous differential equation. Moreover N-harmonic singular functions are also obtained under the form

$$u(x) = r^{-\beta_k} (\sin\theta_{N-1} \sin\theta_{N-2} ... \sin\theta_2)^{\beta_k} \omega_k(\theta_1). \tag{1.7}$$

Our paper is organized as follows: 1- Introduction. 2- Construction of fundamental singular N-harmonic functions. 3- The classification theorem. 4- Separable solutions of the p-harmonic spectral problem.

2 Construction of fundamental singular N-harmonic functions

We denote by $\mathcal{H}_N$ the group of conformal transformations in $\mathbb{R}^N$. This group is generated by homothethies, inversion and isometries. Our first result is classical, but we repeat the proof for the sake on completeness.

Proposition 2.1 *Let u be a N-harmonic function in a domain $G \subset \mathbb{R}^N$ and $h \in \mathcal{H}_N$. Then $u_h = u \circ h$ is N-harmonic in $h^{-1}(G)$.*

Proof. Because for any $p > 1$ the class of p-harmonic functions is invariant by homothethies and isometries, it is sufficient to prove the result if h is the inversion $\mathcal{I}_0^1$ with center the origin in $\mathbb{R}^N$ and power 1. We set $y = \mathcal{I}_0^1(x)$ and $v(y) = u(x)$. For any $i = 1, ..., N$

$$u_{x_i}(x) = \sum_j \left(\delta_{ij} |x|^{-2} - 2|x|^{-4} x_i x_j\right) v_{y_j}(y).$$

Then

$$|Du|^2(x) = |x|^{-4} |Dv|^2(y) = |y|^4 |Dv|^2(y).$$

If ϕ is a test function, we denote similarly $\psi(y) = \phi(x)$, thus

$$\langle Du, D\phi \rangle = |x|^{-4} \langle Dv, D\psi \rangle = |y|^4 \langle Dv, D\psi \rangle,$$

and

$$\int_G |Du|^{N-2} \langle Du, D\phi \rangle \, dx = \int_{\mathcal{I}_0^1(G)} |y|^{2N} |Dv|^{N-2} \langle Dv, D\psi \rangle \left|D\mathcal{I}_0^1\right| dy$$

Because $\left|D\mathcal{I}_0^1\right| = |\det(\partial x_i/\partial y_j)| = |y|^{-2N}$, the result follows. □

Proposition 2.2 *Let $N \geq 2$, $B = B_1(0)$ and $a \in \partial B$. Then there exists a unique positive N-harmonic function U^i in B which vanishes on $\partial B \setminus \{a\}$ and satisfies*

$$U^i(x) = \frac{1 - |x|}{|x - a|^2}(1 + \circ(1)) \quad \text{as } x \to a. \tag{2.1}$$

Proof. We first observe that the coordinates functions x_i are N-harmonic and positive in the half-space $H_i = \{x \in \mathbb{R}^N : x_i > 0\}$ and vanishes on ∂H_i. Therefore, the functions $\chi_i(x) = x_i/|x|^2$ are also N-harmonic and singular at 0. Without loss of generality we can assume that a is the origin of coordinates, and that B is the ball with radius 1 and center $(-1, 0, ..., 0)$. Let ω be the point with coordinates $(-2, 0, ..., 0)$. By the inversion $\mathcal{I}_\omega^4$, a is invariant and B is transformed into the half space H_1. Since χ_1 is N-harmonic in H_1, the function

$$x \mapsto \chi_1 \circ \mathcal{I}_\omega^4(x) = -\frac{|x|^2 + 2x_1}{2|x|^2}$$

is N-harmonic and positive in $B = \{x : |x|^2 + 2x_1 < 0\}$, vanishes on ∂B and is singular at $x = 0$. If we set $x_1' = x_1 + 1$, $x_i' = x_i$ for $i = 2, ..., N$ and $U^i(x') = \chi_1 \circ \mathcal{I}_\omega^4(x)$, then the x' coordinates of a are $(1, 0, ..., 0)$ and

$$U^i(x') = \frac{1 - |x'|^2}{2|x' - a|^2} = \frac{1 - |x'|}{|x' - a|^2}(1 + \circ(1)) \quad \text{as } x' \to a.$$

Let $\tilde{U}^i$ be another positive N-harmonic function in B which verifies (2.1) and vanishes on $\partial B \setminus \{a\}$. Thus, for any $\delta > 0$, $(1 + \delta)\tilde{U}^i$, is positive, N-harmonic, and $U^i - (1 + \delta)\tilde{U}^i$ is negative near a. By the maximum principle, $U^i \leq (1 + \delta)\tilde{U}^i$. Letting $\delta \to 0$, and permuting U^i and $\tilde{U}^i$ yields $\tilde{U}^i = U^i$. □

By performing the inversion $\mathcal{I}_0^1$, we derive the dual result

Proposition 2.3 *Let $N \geq 2$, $G = B_1^c(0)$ and $a \in \partial B$. Then there exists a unique positive N-harmonic function U^e in G which vanishes on $\partial B \setminus \{a\}$ and satisfies*

$$U^e(x) = \circ(\ln|x|) \quad as \ |x| \to \infty, \tag{2.2}$$

and

$$U^e(x) = \frac{|x|-1}{|x-a|^2}(1+\circ(1)) \quad as \ x \to a. \tag{2.3}$$

Proof. The assumption (2.2) implies that the function $U = U^e \circ \mathcal{I}_0^1$, which is N-harmonic in $B \setminus \{0\}$ verifies

$$U(x) = \circ(\ln(1/|x|)) \quad \text{near } 0.$$

By [9], 0 is a removable singularity and thus U can be extended as a positive N-harmonic function in B which satisfies (2.1). This implies the claim. □

We denote by $\dot\rho(x)$ the signed distance from x to $\partial\Omega$. Since $\partial\Omega$ is C^2, there exists $\beta_0 > 0$ such that if $x \in \mathbb{R}^N$ verifies $-\beta_0 \leq \dot\rho(x) \leq \beta_0$, there exists a unique $\xi_x \in \partial\Omega$ such that $|x - \xi_x| = |\dot\rho(x)|$. Furthermore, if ν_{ξ_x} is the outward unit vector to $\partial\Omega$ at ξ_x, $x = \xi_x - \dot\rho(x)\nu_{\xi_x}$. In particular $\xi_x - \dot\rho(x)\nu_{\xi_x}$ and $\xi_x + \dot\rho(x)\nu_{\xi_x}$ have the same orthogonal projection ξ_x onto $\partial\Omega$.

Let $T_{\beta_0}(\Omega) = \{x \in \mathbb{R}^N : -\beta_0 \leq \dot\rho(x) \leq \beta_0\}$, then the mapping $\Pi : [-\beta_0, \beta_0] \times \partial\Omega \mapsto T_{\beta_0}(\Omega)$ defined by $\Pi(\rho, \xi) = \xi - \rho\nu(\xi)$ is a C^2 diffeomorphism. Moreover $D\Pi(0,\xi)(1,e) = e - \nu_\xi$ for any e belonging to the tangent space $T_\xi(\partial\Omega)$ to $\partial\Omega$ at ξ. If $x \in T_{\beta_0}(\Omega)$, we define the reflection of x through $\partial\Omega$ by $\psi(x) = \xi_x + \dot\rho(x)\nu_{\xi_x}$. Clearly ψ is an involutive diffeomorphism from $\overline{\Omega} \cap T_{\beta_0}(\Omega)$ to $\Omega^c \cap T_{\beta_0}(\Omega)$, and $D\psi(x) = I$ for any $x \in \partial\Omega$. If a function v is defined in $\Omega \cap T_{\beta_0}(\Omega)$, we define $\tilde v$ in $T_{\beta_0}(\Omega)$ by

$$\tilde v(x) = \left\{\begin{array}{ll} v(x) & \text{if } x \in \Omega \cap T_{\beta_0}(\Omega) \\ -v \circ \psi(x) & \text{if } x \in \Omega^c \cap T_{\beta_0}(\Omega). \end{array}\right. \tag{2.4}$$

Lemma 2.4 *Assume that $0 \in \partial\Omega$. Let $v \in C^{1,\alpha}(\overline{\Omega} \cap T_{\beta_0}(\Omega) \setminus \{0\})$ be a solution of (1.1) in $\Omega \cap T_{\beta_0}(\Omega)$ vanishing on $\partial\Omega \setminus \{0\}$. Then $\tilde v \in C^{1,\alpha}(T_\beta(\Omega) \setminus \{0\})$ is solution of a quasilinear equation*

$$\sum_j \frac{\partial}{\partial x_j} \tilde A_j(x, D\tilde v) = 0 \tag{2.5}$$

in $T_\beta(\Omega) \setminus \{0\}$ where the $\tilde A_j$ are C^1 functions defined in $T_\beta(\Omega)$ where they verify

$$\left\{\begin{array}{ll} (i) & \tilde A_j(x,0) = 0 \\ (ii) & \displaystyle\sum_{i,j} \frac{\partial \tilde A_j}{\partial \eta_i}(x,\eta)\xi_i\xi_j \geq \Gamma |\eta|^{p-2} |\xi|^2 \\ (iii) & \displaystyle\sum_{i,j} \left|\frac{\partial \tilde A_j}{\partial \eta_i}(x,\eta)\right| \leq \Gamma |\eta|^{p-2} \end{array}\right. \tag{2.6}$$

for all $x \in T_\beta(\Omega) \setminus \{0\}$ for some $\beta \in (0, \beta_0]$, $\eta \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$ and some $\Gamma > 0$.

Proof. The assumptions (2.6) implies that weak solutions of (2.5) are $C^{1,\alpha}$, for some $\alpha > 0$ [12] and satisfy the standard a priori estimates. As it is defined the function $\tilde{v}$ is clearly C^1 in $T_{\beta_0}(\Omega) \setminus \{0\}$. Writing $Dv(x) = -D(\tilde{v} \circ \psi(x)) = -D\psi(x)(D\tilde{v}(\psi(x)))$ and $\tilde{x} = \psi(x) = \psi^{-1}(x)$

$$\int_{\Omega \cap T_\beta(\Omega)} |Dv|^{p-2} Dv.D\zeta dx$$
$$= \int_{\overline{\Omega}^c \cap T_\beta(\Omega)} |D\psi(D\tilde{v})|^{p-2} D\psi(D\tilde{v}).D\psi(D\zeta) \, |D\psi| \, d\tilde{x}.$$

But

$$D\psi(D\tilde{v}).D\psi(D\zeta) = \sum_k \left(\sum_i \frac{\partial \psi_i}{\partial x_k} \frac{\partial \tilde{v}}{\partial x_i} \right) \left(\sum_j \frac{\partial \psi_j}{\partial x_k} \frac{\partial \zeta}{\partial x_j} \right)$$
$$= \sum_j \left(\sum_{i,k} \frac{\partial \psi_i}{\partial x_k} \frac{\partial \psi_j}{\partial x_k} \frac{\partial \tilde{v}}{\partial x_i} \right) \frac{\partial \zeta}{\partial x_j}.$$

We set $b(x) = |D\psi|$,

$$A_j(x,\eta) = |D\psi| \, |D\psi(\eta)|^{p-2} \sum_i \left(\sum_k \frac{\partial \psi_i}{\partial x_k} \frac{\partial \psi_j}{\partial x_k} \right) \eta_i, \tag{2.7}$$

and

$$A(x,\eta) = (A_1(x,\eta), ..., A_N(x,\eta)) = |D\psi| \, |D\psi(\eta)|^{p-2} \, (D\psi)^t D\psi(\eta). \tag{2.8}$$

For any $\xi \in \partial\Omega$, the mapping $D\psi_{\partial\Omega}(\xi)$ is the symmetry with respect to the hyperplane $T_\xi(\partial\Omega)$ tangent to $\partial\Omega$ at ξ, so $|D\psi(\xi)| = 1$. Inasmuch $D\psi$ is continuous, a lengthy but standard computation leads to the existence of some $\beta \in (0, \beta_0]$ such that (2.6) holds in $T_\beta(\Omega) \cap \overline{\Omega}^c$. If we define $\tilde{A}$ to be $|\eta|^{p-2} \eta$ on $T_\beta(\Omega) \cap \overline{\Omega}$ and A on $T_\beta(\Omega) \cap \overline{\Omega}^c$, then inequalities (2.6) are satisfied in $T_\beta(\Omega)$. □

These three results allows us to prove our main result

Theorem 2.5 *Let Ω be an open subset of $\mathbb{R}^N$ with a compact C^2 boundary, $\rho(x) = \text{dist}\,(x, \partial\Omega)$ and $a \in \partial\Omega$. Then there exists one and only one positive N-harmonic function u in Ω, vanishing on $\partial\Omega \setminus \{a\}$ verifying*

$$\lim_{\substack{x \to a \\ \frac{x-a}{|x-a|} \to \sigma}} |x-a| \, u(x) = -\langle \sigma, \mathbf{n_a} \rangle \tag{2.9}$$

uniformly on $S^{N-1} \cap \overline{\Omega}$, and

$$u(x) = \circ(\ln|x|)) \quad as \ |x| \to \infty, \tag{2.10}$$

if Ω is not bounded.

Proof. Uniqueness follows from (2.9) by the same technique as in the previous propositions. *Step 1 (Existence).* If Ω is not bounded, we perform an inversion $\mathcal{I}_m^{|m-a|^2}$ with center some $m \in \Omega$. Because of (2.10), the new function $u \circ \mathcal{I}_m^{|m-a|^2}$ is N-harmonic in $\Omega' = \mathcal{I}_m^{|m-a|^2}(\Omega)$ and satisfies (2.9). Thus we are reduced to the case were Ω is bounded. Since Ω is C^2, it satisfies the interior and exterior sphere condition at a. By dilating Ω, we can assume that the exterior and interior tangent spheres at a have radius 1. We denote them by

$B_1(\omega^e)$ and $B_1(\omega^i)$, their respective centers being $\omega^i = a - \mathbf{n_a}$ and $\omega^e = a + \mathbf{n_a}$. We set $V^i(x) = U^i(x - \omega^i)$ and $V^e(x) = U^e(x - \omega^e)$ where U^i and U^e are the two singular N-harmonic functions described in Proposition 2.2 and Proposition 2.3, respectively in $B_1(\omega^i)$ and $B_1^c(\omega^e)$, with singularity at point a. For $\epsilon > 0$, we put $\Omega_\epsilon = \Omega \setminus B_\epsilon(a)$, $\Sigma_\epsilon = \Omega \cap \partial B_\epsilon(a)$ and $\partial^*\Omega_\epsilon = \partial\Omega \cap B_\epsilon^c(a)$. Let u_ϵ be the solution of

$$\begin{cases} div(|Du_\epsilon|^{N-2} Du_\epsilon) = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } \partial^*\Omega_\epsilon \\ u_\epsilon = V^e & \text{on } \Sigma_\epsilon. \end{cases} \tag{2.11}$$

This solution is obtained classicaly by minimisation of a convex functional over a class of functions with prescribed boudary value on $\partial\Omega_\epsilon$. For any $x \in B_1(\omega^i)$, there holds

$$\text{dist}\,(x, \partial B_1(\omega^e)) = |x - \omega^e| - 1 \geq \text{dist}\,(x, \partial\Omega) \geq \text{dist}\,(x, \partial B_1(\omega^i)) = 1 - \left|x - \omega^i\right|.$$

thus

$$V^i(x) \leq V^e(x) \quad \forall x \in B_1(\omega^i),$$

by using (2.1), (2.3) and the maximum principle. Therefore

$$V^i(x) \leq u_\epsilon(x) \leq V^e(x) \quad \forall x \in B_1(\omega^i) \cap \Omega_\epsilon$$

and

$$u_\epsilon(x) \leq V^e(x) \quad \forall x \in \Omega_\epsilon.$$

Finally, for $0 < \epsilon' < \epsilon$, $u_{\epsilon'}\big|_{\Sigma_\epsilon} \leq V^e\big|_{\Sigma_\epsilon} = u_\epsilon\big|_{\Sigma_\epsilon}$. Thus

$$u_{\epsilon'}(x) \leq u_\epsilon(x) \quad \forall x \in \Omega_\epsilon.$$

The sequence $\{u_\epsilon\}$ is increasing with ϵ. By classical a priori estimates concerning quasilinear equations, it converges to some positive N-harmonic function u in Ω which vanishes on $\partial\Omega \setminus \{a\}$ and verifies

$$V^i(x) \leq u(x) \quad \forall x \in B_1(\omega^i),$$

and

$$u(x) \leq U^e(x) \quad \forall x \in \Omega.$$

This implies

$$\frac{1 - |x - \omega_i|^2}{2\,|x - a|^2} \leq u(x) \quad \forall x \in B_1(\omega^i), \tag{2.12}$$

$$u(x) \leq \frac{|x - \omega_e|^2 - 1}{2\,|x - a|^2} \quad \forall x \in \Omega, \tag{2.13}$$

By scaling we can prove the following estimate

$$u(x) \leq C\frac{\rho(x)}{|x - a|^2} \quad \forall x \in \Omega. \tag{2.14}$$

for some $C > 0$: for simplicity we can assume that a is the origin of coordinates and, for $r > 0$ set $u_r(y) = u(ry)$. Clearly u_r is N-harmonic in Ω/r and

$$\max\{|Du_r(y)| : y \in \Omega/r \cap (B_{3/2} \setminus B_{2/3})\} \leq C \max\{|u_r(z)| : z \in \Omega/r \cap (B_2 \setminus B_{1/2})\},$$

where C, which depends on the curvature of $\partial\Omega/r$, remains bounded as long as $r \leq 1$. Since $Du_r(y) = rDu(ry)$, we obtain by taking $ry = x$, $|y| = 1$ and using (2.13) with general a,

$|Du(x)| \leq C\,|x-a|^{-2}$. By the mean value theorem, since u vanishes on $\partial\Omega \setminus \{a\}$, (2.14) holds.

Step 2. In order to give a simple proof of the estimate (2.9), we fix the origin of coordinates at $a = 0$ and the normal outward unit vector at a to be $-\mathbf{e}_N$. If $\tilde{u}$ is the extension of u by reflection through $\partial\Omega$, it statisfies (2.5) in $T_\beta(\Omega)\backslash\{0\}$ (see lemma 2.4). For $r > 0$, set $\tilde{u}^r(x) = r\tilde{u}(rx)$. Then $\tilde{u}^r$ is solution of

$$\sum_j \frac{\partial}{\partial x_j}\tilde{A}_j(rx, D\tilde{u}^r) = 0 \tag{2.15}$$

in $T_{\beta/r}(\Omega/r)\backslash\{0\}$. By the construction of $\tilde{A}_j(x,\eta)$, we can note that

$$\lim_{r\to 0}\tilde{A}^j(rx,\eta) = |\eta|^{p-2}\eta_j, \qquad \forall \eta \in \mathbb{R}^N.$$

Furthermore, for any $x \in T_\beta(\Omega)\backslash\{0\}$, $\rho(x) = \rho(\psi(x))$ and $c\,|x| \leq |\psi(x)| \leq c^{-1}\,|x|$ for some $c > 0$, the estimate (2.14) holds if u is replaced by $\tilde{u}^r$, Ω by $T_{\beta/r}(\Omega/r)$ and $\rho(x)$ by $\rho_r(x) := \mathrm{dist}(x, \Omega/r)$ i.e.

$$|\tilde{u}^r(x)| \leq C|x|^{-2}\rho_r(x)\ \forall x \in T_{\beta/r}(\Omega/r).$$

For $0 < a < b$ fixed and for some $0 < r_0 \leq 1$ the spherical shall $\Gamma_{a,b} = \{x \in \mathbb{R}^N : a \leq |x| \leq b\}$ is included into $T_{\beta/r}(\Omega/r)$ for all $0 < r \leq r_0$. By the classical regularity theory for quasilinear equations [12] and lemma 2.4 , there holds

$$\|D\tilde{u}^r\|_{C^\alpha(\Gamma_{2/3,3/2})} \leq C_r\,\|\tilde{u}^r\|_{L^\infty(\Gamma_{1/2,2})},$$

where C_r remains bounded because $r \leq 1$. By Ascoli's theorem, (2.12) and (2.14), $\tilde{u}^r(x)$ converges to $x_N|x|^{-2}$ in the $C^1(\Gamma_{2/3,3/2})$-topology. This implies in particular that $r^2D\tilde{u}(rx)$ converges uniformly in $\Gamma_{2/3,3/2}$ to $-2x_N|x|^{-4}x + |x|^{-2}\mathbf{e}_N$. Using the expression of $D\tilde{u}$ in spherical coordinates we obtain

$$r^2\tilde{u}_r\mathbf{i} - r\tilde{u}_\phi\mathbf{e} + \frac{r}{\sin\phi}\nabla_{\sigma'}\tilde{u} \to\ -2\sigma_N\mathbf{i} + \mathbf{e}_N \text{ uniformly on } S^{N-1} \text{ as } r \to\ 0,$$

where $\cos\phi = x_N|x|^{-1}$, $\mathbf{i} = x/|x|$, $\mathbf{e}$ is derived from $x/|x|$ by a rotation with angle $\pi/2$ in the plane $0, x, N$ (N being the North pole), and $\nabla_{\sigma'}$ is the covariant gradient on S^{N-2}. Inasmuch $\mathbf{i}$, $\mathbf{e}$ and $\nabla_{\sigma'}$ are orthogonal, the components of $\mathbf{e}_N$ are $\cos\phi, \sin\phi$ and 0, thus

$$r\tilde{u}_\phi(r,\sigma',\phi) \to\ -\sin\phi \text{ as } r \to\ 0.$$

Since

$$\tilde{u}(r,\sigma',\phi) = \int_{\pi/2}^{\phi}\tilde{u}_\phi(r,\sigma',\theta)d\theta,$$

the previous convergence estimate establishes (2.9). □

Definition 2.6 *We shall denote by $u_{1,a}$ the unique positive N-harmonic function satisfying (2.9), and call it the fundamental solution with a point singularity at a.*

3 The classification theorem

In this section we characterize all the positive N-harmonic functions vanishing on the boundary of a domain except one point. The next statement is an immediate consequence of Theorem 2.5 and [2, Th. 2.11].

Theorem 3.1 . *Let Ω be a bounded domain with a C^2 boundary and $a \in \partial\Omega$. If u is a positive N-harmonic function in Ω vanishing on $\partial\Omega \setminus \{a\}$, there exists $M \geq 0$ such that*

$$u(x) \leq M u_{1,a}(x) \quad \forall x \in \Omega \tag{3.1}$$

In the next theorem, which extends [2, Th. 2.13], we characterize all the signed N-harmonic functions with a moderate growth near the singular point.

Theorem 3.2 . *Let Ω be a bounded domain with a C^2 boundary and $a \in \partial\Omega$. Assume that $u_{1,a}$ has only a finite number of critical points in Ω. If u is a N-harmonic function in Ω vanishing on $\partial\Omega \setminus \{a\}$ verifying $|u(x)| \leq M u_{1,a}(x)$ for some $M > 0$ and any $x \in \Omega$, there exists $k \in [-M, M]$ such that $u = k u_{1,a}$.*

Proof. We define k as the minimum of the ℓ such that $u \leq \ell u_{1,a}$ in Ω. Without any loss of generality we can assume $k > 0$. Then either the tangency of the graphs of the functions u and $k u_{1,a}$ is achieved in $\overline{\Omega} \setminus \{a\}$, or it is achieved asymptotically at the singular point a. In the first case we considered two sub-cases:

(i) The coincidence set G of u and $k u_{1,a}$ has a connected component ω isolated in Ω. In this case there exists a smooth domain $\mathcal{U}$ such that $\overline{\omega} \subset \mathcal{U}$ and $\delta > 0$ such that $k u_{1,a} - u \geq \delta$ on $\partial\mathcal{U}$. The maximum principle implies that $k u_{1,a} - u \geq \delta$ in $\mathcal{U}$, a contradiction.

(ii) In the second sub-case any connected component ω of the coincidence set touches $\partial\Omega \setminus \{a\}$, or the two graphs admits a tangency point on $\partial\Omega \setminus \{a\}$. If $m \in \omega \cap \partial\Omega \setminus \{a\}$ or is such a tangency point, the regularity theory implies $\partial u(m)/\partial \mathbf{n}_m = k u_{1,a}(m)/\partial \mathbf{n}_m$. By Hopf boundary lemma, $u_{1,a}(m)/\partial \mathbf{n}_m < 0$. By the mean value theorem, the function $w = k u_{1,a} - u$ satisfies an equation

$$Lw = 0 \tag{3.2}$$

which is elliptic and non degenerate near m (see [3], [4]), it follows that w vanishes in a neighborhood of m and the two graphs cannot be tangent only on $\partial\Omega \setminus \{a\}$. Assuming that $\omega \neq \Omega$, let $x_0 \in \Omega \setminus \omega$ such that $\operatorname{dist}(x_0, \omega) = r_0 < \rho(x_0) = \operatorname{dist}(x_0, \partial\Omega$, and let $y_0 \in \omega$ be such that $|x_0 - y_0| = r_0$. Since $u_{1,a}$ has at most a finite number of critical points, we can choose x_0 such that y_0 is not one of these critical points. By assumption $w = k u_{1,a} - u$ is positive in $B_{r_0}(x_0)$ and vanishes at a boundary point y_0. Since the equations are not degenerate at y_0 there holds

$$k \partial u_{1,a}(y_0)/\partial\nu - \partial u(y_0)/\partial\nu < 0$$

where $\nu = (y_0 - x_0)/r_0$, which contradicts the fact that the two graphs are tangent at y_0.

Next we are reduced to the case where the graphs of u and $k u_{1,a}$ are separated in Ω and asymptotically tangent at the singular point a. There exists a sequence $\{\xi_n\} \subset \Omega$ such that $\lim_{n\to\infty} u(\xi_n)/u_{1,a}(\xi_n) = k$. We set $|x_n - a| = r_n$, $u_n(y) = r_n u(a + r_n y)$ and $v_n(y) = r_n u_{1,a}(a + r_n y)$. Both u_n and v_n are N-harmonic in $\Omega_n = (\Omega - a)/r_n$. The functions u_n and v_n are locally uniformly bounded in $\overline{\Omega}_n \setminus \{0\}$. It follows, by using classical regularity results, that, there exists sub-sequences, such that $\{u_{n_k}\}$ and $\{v_{n_k}\}$ converge respectively to U and V in the C^1_{loc}-topology of $\overline{\Omega}_{n_k} \setminus \{0\}$. The functions U and V are N-harmonic in $H \approx \mathbb{R}^N_+ = \{x = (x_1, x_2, ..., x_N) : x_N > 0\}$ and vanish on $\partial H \setminus \{0\}$. Since it can be assumed that $(\xi_{n_k} - a)/r_{n_k} \to \xi$, there holds $U \leq kV$ in H, $U(\xi) = kV(\xi)$, if $\xi \in H$, and $\partial U(\xi)/\partial x_N = k\partial V(\xi)/\partial x_N > 0$, if $\xi \in \partial H$ (notice that $|\xi| = 1$). If $\xi \in \partial H$, Hopf lemma applies to V at ξ and, using the same linearization with the linear operator L as in the previous proof, it yields to $U = kV$. If $\xi \in H$, we use the fact that $|Du_{1,a}(x)| \geq \beta > 0$ for $|x - a| \leq \alpha$ for some $\beta, \alpha > 0$. Thus $|Dv_n(\xi)| \geq \beta$. The non-degeneracy of V and the strong maximum principle lead again to $U = kV$. Whatever is the position of ξ, the equality between U and kV and the convergence in C^1_{loc} leads to the fact that for any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $n \geq n_\epsilon$ implies

$$(k - \epsilon)u_{1,a}(x) \leq u(x) \leq (k + \epsilon)u_{1,a}(x) \quad \forall x \in \Omega \cap \partial B_{r_n}(a).$$

By the comparison principle between N-harmonic functions this inequality holds true in $\Omega \setminus \partial B_{r_n}(a)$. Since $r_n \to 0$ and ϵ is arbitrary, this ends the proof. □

Remark. The assumption that $u_{1,a}$ has only isolated critical points in Ω is clearly satisfied in the case of a ball, a half-space or the complementary of a ball where no critical point exists. It is likely that this assumption always holds but we cannot prove it. However the Hopf maximum principle for p-harmonic functions (see [11]) implies that $u_{1,a}$ cannot have local extremum in Ω.

4 Separable solutions of the p-harmonic spectral problem

In this section we present a technique for constructing signed N-harmonic functions, regular or singular, as a product of functions depending only on one real variable. Some of the results were sketched in [16]. The starting point is the result of Krol [5] dealing with the existence of 2-dimensional separable p-harmonic functions (the construction of singular separable p-harmonic functions was performed in [4]).

Theorem 4.1 (Krol) *Let $p > 1$. For any positive integer k there exists a unique $\beta_k > 0$ and $\omega_k : \mathbb{R} \mapsto \mathbb{R}$, with least antiperiod π/k, of class C^∞ such that*

$$u_k(x) = |x|^{\beta_k} \omega_k(x/|x|) \tag{4.1}$$

is p-harmonic in $\mathbb{R}^2$; β_k is the unique root ≥ 1 of

$$(2k - 1)X^2 - \frac{pk^2 + (p - 2)(2k - 1)}{p - 1}X + k^2 = 0. \tag{4.2}$$

(β_k, ω_k) is unique up to translation and homothety over ω_k.

This result is obtained by solving the homogeneous differential equation satisfied by $\omega_k = \omega$:

$$-\left(\left(\beta^2\omega^2 + \omega_\theta^2\right)^{(p-2)/2} \omega_\theta\right)_\theta = \beta\left(1 + (\beta - 1)(p - 1)\right)\left(\beta^2\omega^2 + \omega_\theta^2\right)^{(p-2)/2} \omega. \tag{4.3}$$

In the particular case $k = 1$, then $\beta_1 = 1$ and $\omega_1(\theta) = \sin\theta$. For the other values of k the β_k are algebraic numbers and the ω_k are not trigonometric functions, except if $p = 2$. More generally, if one looks for p-harmonic functions in $\mathbb{R}^N \setminus \{0\}$ under the form $u(x) = u(r, \sigma) = r^\beta v(\sigma)$, $r = |x| > 0$, $\sigma = x/|x| \in S^{N-1}$, one obtains that v verifies

$$-div_\sigma\left(\left(\beta^2 v^2 + |\nabla_\sigma v|^2\right)^{(p-2)/2} \nabla_\sigma v\right) = \lambda_{N,\beta}\left(\beta^2 v^2 + |\nabla_\sigma v|^2\right)^{(p-2)/2} v \tag{4.4}$$

on S^{N-1}, where $\lambda_{N,\beta} = \beta\,(N-1+(\beta-1)(p-1))$ and div_σ and ∇_σ are respectively the divergence and the gradient operators on S^{N-1} (endowed with the Riemaniann structure induced by the imbedding of the sphere into $\mathbb{R}^N$). This equation, called the *spherical p-harmonic spectral problem*, is the natural generalization of the spectral problem of the Laplace-Beltrami operator on S^{N-1}. Since it does not correspond to a variational form (except if $p=2$), it is difficult to obtain solutions. In the range of $1<p\leq N-1$, Krol proved in [5] the existence of solutions of (4.4), not on the whole sphere, but on a spherical cap (which reduced (4.4) to an non-autonomous nonlinear second order differential equation). His methods combined ODE estimates and shooting arguments. Later on, Tolksdorf [11] introduced an entirely new method for proving the existence of solutions on any C^2 spherical domain S, with Dirichlet boundary conditions. Only the case $\beta>0$ was treated in [11], and, by a small adaptation of Tolksdorf approach, the case $\beta>0$ was considered in [16]. We develop below a method which allows to express solutions as product of explicit one variable functions.

4.1 The 3-D case

Let $(r,\theta,\phi)\in(0,\infty)\times[0,2\pi]\times[0,\pi]$ be the spherical coordinates in $\mathbb{R}^3$

$$\left\{\begin{array}{l} x_1 = r\cos\theta\sin\phi \\ x_2 = r\sin\theta\sin\phi \\ x_3 = r\cos\phi \end{array}\right.$$

Then (4.4) turns into

$$\begin{aligned}-\frac{1}{\sin\phi}\left[\sin\phi\left(\beta^2v^2+v_\phi^2+\frac{v_\theta^2}{\sin^2\phi}\right)^{(p-2)/2}v_\phi\right]_\phi-\frac{1}{\sin^2\phi}\left[\left(\beta^2v^2+v_\phi^2+\frac{v_\theta^2}{\sin^2\phi}\right)^{(p-2)/2}v_\theta\right]_\theta\\ =\beta\,(2+(\beta-1)(p-1))\left(\beta^2v^2+v_\phi^2+\frac{v_\theta^2}{\sin^2\phi}\right)^{(p-2)/2}v\end{aligned} \tag{4.5}$$

We look for a function v under the form

$$v(\theta,\phi)=(\sin\phi)^\beta\omega(\theta) \tag{4.6}$$

then

$$\beta^2v^2+v_\phi^2+\frac{v_\theta^2}{\sin^2\phi}=(\sin\phi)^{2\beta-2}(\beta^2\omega^2+\omega_\theta^2),$$

$$\frac{1}{\sin^2\phi}\left[\left(\beta^2v^2+v_\phi^2+\frac{v_\theta^2}{\sin^2\phi}\right)^{(p-2)/2}v_\theta\right]_\theta=(\sin\phi)^{(\beta-1)(p-1)-1}\left((\beta^2\omega^2+\omega_\theta^2)^{(p-2)/2}\omega_\theta\right)_\theta,$$

$$\left(\beta^2v^2+v_\phi^2+\frac{v_\theta^2}{\sin^2\phi}\right)^{(p-2)/2}v=(\sin\phi)^{(\beta-1)(p-1)+1}(\beta^2\omega^2+\omega_\theta^2)^{(p-2)/2}\omega,$$

and

$$\begin{aligned}&\frac{1}{\sin\phi}\left[\sin\phi\left(\beta^2v^2+v_\phi^2+\frac{v_\theta^2}{\sin^2\phi}\right)^{(p-2)/2}v_\phi\right]_\phi\\ &=\beta(\sin\phi)^{(\beta-1)(p-1)-1}\left[((\beta-1)(p-1)+1)-\sin^2\phi\,((\beta-1)(p-1)+2)\right](\beta^2\omega^2+\omega_\theta^2)^{(p-2)/2}\omega.\end{aligned}$$

It follows that ω satisfies the same equation (4.3). The next result follows immediately from Theorem 4.1

Theorem 4.2 *Assume $N = 3$ and $p > 1$. Then for any positive integer k there exists a p-harmonic function u in $\mathbb{R}^3$ under the form*

$$u(x) = u(r, \theta, \phi) = r^{\beta_k}(\sin\phi)^{\beta_k}\omega_k(\theta) \tag{4.7}$$

where β_k and ω_k are as in Theorem 4.1.

In the case $p = 3$ we can use the conformal invariance of the 3-harmonic equation in $\mathbb{R}^3$ to derive

Theorem 4.3 *Assume $p = N = 3$. Then for any positive integer k there exists a p-harmonic function u in $\mathbb{R}^3 \setminus \{0\}$ under the form*

$$u(x) = u(r, \theta, \phi) = r^{-\beta_k}(\sin\phi)^{\beta_k}\omega_k(\theta) \tag{4.8}$$

where β_k and ω_k are as in Theorem 4.1 with $p = 3$.

As a consequence of Theorem 4.3 we obtain signed 3-harmonic functions under the form (4.7) in the half space $\mathbb{R}^3_+ = \{x : x_2 > 0\}$, vanishing on $\partial\mathbb{R}^3_+ \setminus \{0\}$, with a singularity at $x = 0$. They correspond to even integers k. The extension to general smooth domains Ω is a deep chalenge. In the particular case $k = 1$, we have seen that $\beta_1 = 1$ and $\omega_1(\theta) = \sin\theta = x_2$, that we already know.

4.2 The general case

We assume that $N > 3$ and write the spherical coordinates in $\mathbb{R}^N$ under the form

$$x = \left\{(r, \sigma) \in (0, \infty) \times S^{N-1} = (r, \sin\phi\,\sigma', \cos\phi) : \sigma' \in S^{N-2}, \phi \in [0, \pi]\right\}. \tag{4.9}$$

The main result concerning separable p-harmonic functions is the following.

Theorem 4.4 *Let $N > 3$ and $p > 1$. For any positive integer k there exists p-harmonic functions in $\mathbb{R}^N$ under the form*

$$u(x) = u(r, \sigma', \phi) = (r\sin\phi)^{\beta_k}\, w(\sigma'). \tag{4.10}$$

where β_k is the unique root ≥ 1 of (4.2) and w is solution of (4.15) with $\beta = \beta_k$. Furthermore, if $p = N$ there exists a singular N-harmonic function under the form

$$u(x) = u(r, \sigma', \phi) = r^{-\beta_k}(\sin\phi)^{\beta_k}\, w(\sigma'). \tag{4.11}$$

Proof. We first recall (see [17] for details) that the $SO(N)$ invariant unit measure on S^{N-1} is $d\sigma = a_N \sin^{N-2}\phi\, d\sigma'$ for some $a_N > 0$, and

$$\nabla_\sigma v = -v_\phi \mathbf{e} + \frac{1}{\sin\phi}\nabla_{\sigma'} v.$$

where $\mathbf{e}$ is derived from $x/|x|$ by the rotation of center 0 angle $\pi/2$ in the plane going thru 0, $x/|x|$ and the north pole. The weak formulation of (4.4) expresses as

$$\int_0^\pi \int_{S^{N-2}} \left(\beta^2 v^2 + v_\phi^2 + \frac{1}{\sin^2\phi}|\nabla_{\sigma'} v|^2\right)^{(p-2)/2} \left(v_\phi\zeta_\phi + \frac{1}{\sin^2\phi}\nabla_{\sigma'} v.\nabla_{\sigma'}\zeta\right)\sin^{N-2}\phi\, d\sigma'\, d\phi$$
$$= \lambda_{N,\beta}\int_0^\pi \int_{S^{N-2}} \left(\beta^2 v^2 + v_\phi^2 + \frac{1}{\sin^2\phi}|\nabla_{\sigma'} v|^2\right)^{(p-2)/2} v\,\zeta \sin^{N-2}\phi\, d\sigma'\, d\phi \tag{4.12}$$

or, equivalently

$$
\begin{aligned}
-\frac{1}{\sin^{N-2}\phi}&\left[\sin^{N-2}\phi\left(\beta^2v^2+v_\phi^2+\frac{1}{\sin^2\phi}\left|\nabla_{\sigma'}v\right|^2\right)^{(p-2)/2}v_\phi\right]_\phi\\
&-\frac{1}{\sin^2\phi}div_{\sigma'}\left[\left(\beta^2v^2+v_\phi^2+\frac{1}{\sin^2\phi}\left|\nabla_{\sigma'}v\right|^2\right)^{(p-2)/2}\nabla_{\sigma'}v\right]\\
&\qquad=\lambda_{N,\beta}\left(\beta^2v^2+v_\phi^2+\frac{1}{\sin^2\phi}\left|\nabla_{\sigma'}v\right|^2\right)^{(p-2)/2}v
\end{aligned}
\tag{4.13}
$$

where $div_{\sigma'}$ is the divergence operator acting on vector fields on S^{N-2}. We look again for p-harmonic functions under the form

$$
u(r,\sigma)=u(r,\sigma',\phi)=r^\beta v(\sigma',\phi)=r^\beta\sin^\beta\phi\,w(\sigma'). \tag{4.14}
$$

Then

$$
\left(\beta^2v^2+v_\phi^2+\frac{1}{\sin^2\phi}\left|\nabla_{\sigma'}v\right|^2\right)^{(p-2)/2}=(\sin\phi)^{(\beta-1)(p-2)}\left(\beta^2w^2+\left|\nabla_{\sigma'}w\right|^2\right)^{(p-2)/2},
$$

thus

$$
\begin{aligned}
&\frac{1}{\sin^{N-2}\phi}\left[\sin^{N-2}\phi\left(\beta^2v^2+v_\phi^2+\frac{1}{\sin^2\phi}\left|\nabla_{\sigma'}v\right|^2\right)^{(p-2)/2}v_\phi\right]_\phi\\
&=\beta(\sin\phi)^{(\beta-1)(p-1)-1}\left((N-2+(\beta-1)(p-1))-(N-1+(\beta-1)(p-1))\sin^2\phi\right)\\
&\times\left(\beta^2w^2+\left|\nabla_{\sigma'}w\right|^2\right)^{(p-2)/2}w,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\sin^2\phi}div_{\sigma'}&\left[\left(\beta^2v^2+v_\phi^2+\frac{1}{\sin^2\phi}\left|\nabla_{\sigma'}v\right|^2\right)^{(p-2)/2}\nabla_{\sigma'}v\right]\\
&=(\sin\phi)^{(\beta-1)(p-1)-1}div_{\sigma'}\left[\left(\beta^2w^2+\left|\nabla_{\sigma'}w\right|^2\right)^{(p-2)/2}\nabla_{\sigma'}w\right]
\end{aligned}
$$

Finally w satisfies

$$
-div_{\sigma'}\left[\left(\beta^2w^2+\left|\nabla_{\sigma'}w\right|^2\right)^{(p-2)/2}\nabla_{\sigma'}w\right]=\lambda_{N-1,\beta}\left(\beta^2w^2+\left|\nabla_{\sigma'}w\right|^2\right)^{(p-2)/2}w \tag{4.15}
$$

on S^{N-2}, which is the desired induction. $\square$

In order to be more precise, we can completely represent the preceding solutions by introducing the generalized Euler angles in $\mathbb{R}^N=\{x=(x_1,...,x_N)\}$

$$
\left\{
\begin{array}{l}
x_1=r\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_2\sin\theta_1\\
x_2=r\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_2\cos\theta_1\\
.\\
.\\
.\\
x_{N-1}=r\sin\theta_{N-1}\cos\theta_{N-2}\\
x_N=r\cos\theta_{N-1}
\end{array}
\right.
\tag{4.16}
$$

where $\theta_1 \in [0, 2\pi]$ and $\theta_k \in [0, \pi]$, for $k = 2, ..., N-1$. Notice that θ_{N-1} is the variable ϕ in the representation (4.9). The above theorem combined with the induction process yields to the following.

Theorem 4.5 *Let $N > 3$ and $p > 1$. For any positive integer k there exists p-harmonic functions in $\mathbb{R}^N$ under the form*

$$u(x) = (r \sin\theta_{N-1} \sin\theta_{N-2} ... \sin\theta_2)^{\beta_k} \omega_k(\theta_1) \tag{4.17}$$

where (β_k, ω_k) are obtained in Theorem 4.1. Furthermore, if $p = N$ there exists a singular N-harmonic function under the form

$$u(x) = r^{-\beta_k} (\sin\theta_{N-1} \sin\theta_{N-2} ... \sin\theta_2)^{\beta_k} \omega_k(\theta_1). \tag{4.18}$$

References

[1] Borghol R., *Singularités au bord de solutions d'équations quasilinéaires*, Thèse de Doctorat, Univ. Tours (in preparation).

[2] Bidaut-Véron M. F., Borghol R. & Véron L., *Boundary Harnack inequalities and a priori estimates of singular solutions of quasilinear equations*, Calc. Var. and P. D. E., to appear.

[3] Friedman A., & Véron L., *Singular solutions of some quasilinear elliptic equations*, Arch. Rat. Mech. Anal. **96**, 359-387 (1986).

[4] Kichenassamy S. & Véron L., *Singular solutions of the p-Laplace equation*, Math. Ann. **275**, 599-615 (1986).

[5] Krol I. N., *The behavior of the solutions of a certain quasilinear equation near zero cusps of the boundary*, Proc. Steklov Inst. Math. **125**, 130-136 (1973).

[6] Libermann G, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12**, 1203-1219 (1988).

[7] Manfredi J. & Weitsman A., *On the Fatou Theorem for p-Harmonic Functions*, Comm. P. D. E. **13**, 651-668 (1988).

[8] Rešetnjak, Ju. *Spatial mappings with bounded distortion* (Russian), Sibirsk. Mat. Ž. **8**, 629-658 (1967) .

[9] Serrin J., *Local behaviour of solutions of quasilinear equations*, Acta Math. **111**, 247-302 (1964).

[10] Serrin J. & Zou H., *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189**, 79-142 (2002).

[11] Tolksdorff P., *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Part. Diff. Equ. **8**, 773-817 (1983).

[12] Tolksdorff P., *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Equ. **51**, 126-140 (1984).

[13] Trudinger N., *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20**, 721-747 (1967).

[14] Véron L., *Some existence and uniqueness results for solution of some quasilinear elliptic equations on compact Riemannian manifolds*, Colloquia Mathematica Societatis János Bolyai **62**, 317-352 (1991).

[15] Véron L., *Singularities of solutions of second order quasilinear elliptic equations*, Pitman Research Notes in Math. **353**, Addison-Wesley- Longman (1996).

[16] Véron L., *Singularities of some quasilinear equations*, Nonlinear diffusion equations and their equilibrium states, II (Berkeley, CA, 1986), 333-365, Math. Sci. Res. Inst. Publ., **13**, Springer, New York (1988).

[17] Vilenkin N. *Fonctions spéciales et théorie de la représentation des groupes*, Dunod, Paris (1969).

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