# Boundary singularities of solutions of N-harmonic equations with absorption

Rouba Borghol, Laurent Veron

# Boundary singularities of solutions of $N$-harmonic equations with absorption *

**Rouba Borghol**, **Laurent Véron**
Department of Mathematics,
University of Tours, FRANCE

*Dedicated to Jim Serrin, on his eightieth birthday*

**Abstract** We study the boundary behaviour of solutions $u$ of $-\Delta_N u + |u|^{q-1}u = 0$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ subject to the boundary condition $u = 0$ except at one point, in the range $q > N - 1$. We prove that if $q \geq 2N - 1$ such a $u$ is identically zero, while, if $N - 1 < q < 2N - 1$, $u$ inherits a boundary behaviour which either corresponds to a weak singularity, or to a strong singularity. Such singularities are effectively constructed.

## 1 Introduction

Let $\Omega$ be a domain is $\mathbb{R}^N$ $(N \geq 2)$ with a $C^2$ compact boundary $\partial\Omega$. Let $g$ be a continous real valued function and $a \in \partial\Omega$. This paper deals with the study of solutions $u \in C^1(\bar{\Omega} \setminus \{a\})$ of the problem

$$\left\{ \begin{array}{ll} -\,div\left(|Du|^{N-2}\,Du\right) + g(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}, \end{array} \right. \tag{1.1}$$

and we shall be more specifically interested in the case when $g$ has a power growth at infinity. When $N = 2$, this problem falls into the scope of the boundary singularity problem for semilinear elliptic equations. The study of the $N$-dimensional problem

$$\left\{ \begin{array}{ll} -\Delta u + g(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}, \end{array} \right. \tag{1.2}$$

has been initiated by Gmira and Véron in [7]. Among the subjects under consideration were the question of removability of isolated boundary singularities and, in the case such singularities do exist, their precise description. This seminal article was at the origin of a long series of further works by Dynkin, Kuznetsov, Le Gall, Marcus and Véron in the framework of the trace theory and, later on, the fine trace theory in the case where $g(r) = r\,|r|^{q-1}$,

*To appear in *Journal of Functional Analysis*

$q > 1$. One of the main reasons for such a large impact consists of the observation of the existence of a critical exponent $q = q^* = (N+1)/(N-1)$. If $q \geq q^*$ any solution of

$$\begin{cases} -\Delta u + |u|^{q-1} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}, \end{cases} \tag{1.3}$$

is identically zero, while if $1 < q < q^*$ it appears that there exist two possible behaviours of singular solutions near $a$, the solutions with weak singularities and the ones with the strong singular behaviour. Later on, these two types of singular solutions played a fundamental role in the description of the rough trace of positive solutions of (1.3 ).

Although the techniques needed are considerably more refined, it appeared that the description of solutions of (1.1 ) inherits the same structure as for (1.2 ). The first step is to understand the model case problem

$$\begin{cases} -\,div\left(|Du|^{N-2} Du\right) + |u|^{q-1} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}, \end{cases} \tag{1.4}$$

To this equation, we associate the homogeneous equation

$$\begin{cases} -\,div\left(|Du|^{N-2} Du\right) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}. \end{cases} \tag{1.5}$$

It is proved in [3] that for any $k > 0$ there exists a unique solution $u = u_k$ of (1.5 ) satisfying

$$u_k(x) = k\frac{\rho(x)}{|x-a|^2}(1 + \circ(1)) \quad \text{as } x \to a,\ (x-a)/|x-a| \to \sigma, \tag{1.6}$$

where $\rho(x) = \text{dist}\,(x, \partial\Omega)$. When $k = 1$, this solution plays the role of the Poisson kernel, although neither any weak formulation nor any reasonable trace theory seems to exists, and we shall denote it by $V_a^\Omega$. The behaviour (1.6 ) (up to a multiplicative constant) corresponds to *weak singularity behaviour* for (1.1 ), whenever such singularities exist. The first result we prove is the following:

**Theorem** *Let $N - 1 < q < 2N - 1 := q_c$ . Then for any $k > 0$ there exists a unique solution $u = u_{k,a}$ of problem (1.4 ) satisfying*

$$u_{k,a}(x) = k\frac{\rho(x)}{|x-a|^2}(1 + \circ(1)) \quad \textit{as } x \to a,\ (x-a)/|x-a| \to \sigma. \tag{1.7}$$

*Furthermore $u_{\infty,a} = \lim_{k\to\infty}$ exists and is a solution of (1.4 ) which satisfies*

$$\lim_{\substack{x \to a \\ \frac{x-a}{|x-a|} \to \sigma}} |x-a|^{N/(q+1-N)} u_{\infty,a}(x) = \omega(\sigma), \tag{1.8}$$

*and $\omega$ is the unique positive solution of the following quasilinear equation on the upper* *hemisphere of the unit sphere* $S^{N-1}$,

$$\begin{cases} -div_\sigma\left(\left(\beta_q^2\omega^2+|\nabla_\sigma\omega|^2\right)^{(N-2)/2}\nabla_\sigma\omega\right) \\ \qquad -\Lambda\left(\beta_q^2\omega^2+|\nabla_\sigma\omega|^2\right)^{(N-2)/2}\omega+|\omega|^{q-1}\omega=0 \quad on\ S_+^{N-1} \\ \qquad \omega=0 \quad on\ \partial S_+^{N-1}, \end{cases} \tag{1.9}$$

where $\beta_q = N/(q+1-N)$ and $\Lambda = (N-1)\beta_q^2$. The proof of the existence of $u_{k,a}$, as well as its singular behaviour, is settled upon the conformal invariance of the $N$-harmonic operator and the construction of subsolution of the same equation. Estimate (1.8 ) is proved by scaling method. The role of the critical exponent $q_c = 2N-1$ is enlighted by the following result.

**Theorem** *Let $g$ be a continuous function such that*

$$\begin{array}{ll} (i) & \liminf_{r\to\infty} g(r)/r^{q_c} > 0 \\ (ii) & \limsup_{r\to-\infty} g(r)/|r|^{q_c} < 0. \end{array} \tag{1.10}$$

*Then any function $u \in C^1(\overline{\Omega}\setminus\{a\})$ solution of (1.1 ) extends as a function $\tilde{u} \in C(\overline{\Omega})$.*

As in the semilinear case, the occurrence coincides with the case where the blow-up exponent $-\beta_q$ which is natural for equation (1.4 ) coincides with the one of the function $V_a^\Omega$ solution of (1.5 ). Finally we provide the full classification of positive solutions of problem (1.4 ).

**Theorem** *Let $N-1 < q < q_c$ and $u$ is any nonnegative solution of (1.4 ), then*
*(i) Either $u \equiv 0$,*
*(ii) Either there exists $k > 0$ such that $u = u_{k,a}$.*
*(iii) Or $u = u_{\infty,a}$.*

In the proof of (iii) the boundary Harnack inequalities that satisfies any positive solution of (1.4 ) (see [2]) play a fundamental role.

Our paper is organized as follows

1- Introduction
2- Weak and strong boundary singularities
3- The removability result
4- The classification theorem

# 2 Weak and strong boundary singularities

The construction of positive solutions of

$$-div\left(|Du|^{N-2}Du\right)+|u|^{q-1}u=0, \tag{2.1}$$

is settled upon three facts: the existence of solutions to the homogeneous equation

$$-div\left(|Du|^{N-2}Du\right)=0, \tag{2.2}$$

the conformal invariance of (2.2 ) and an a priori estimate satisfied by any solution of(2.1 ). Throughout this paper $C$ denotes a positive constant which depends only on the structural assumptions corresponding to $N$, $p$, $q$ and $\Omega$. The value of the constant may change from one occurrence to another.

**Proposition 2.1** *Let $\Omega\subset\mathbb{R}^N$ be a domain with a compact boundary and $a\in\partial\Omega$. Consider real numbers $q>p-1>0$, $A>0$ and $B\geq 0$. If $u\in C(\overline{\Omega}\setminus\{a\})\cap W^{1,p}_{loc}(\Omega)$ is a weak solution of*

$$\begin{cases} -div\left(|Du|^{p-2}Du\right)+A|u|^{q-1}u\leq B & in\ \Omega\\ \qquad\qquad\qquad\qquad\quad u\leq 0 & on\ \partial\Omega\setminus\{a\},\end{cases} \tag{2.3}$$

*it satisfies*

$$u(x)\leq\left(\frac{\lambda}{A|x-a|^p}\right)^{1/(q+1-p)}+\left(\frac{\mu B}{A}\right)^{1/q}\qquad \forall x\in\overline{\Omega}\setminus\{a\}, \tag{2.4}$$

*where $\lambda$ and $\mu$ depends on $N$, $p$ and $q$.*

*Proof.* By assumption

$$\int_\Omega\left(|Du|^{p-2}Du.D\zeta+A|u|^{q-1}u\zeta\right)dx\leq B\int_\Omega\zeta dx \tag{2.5}$$

for any $\zeta\in W^{1,p}(\Omega)$ with compact support, $\zeta\geq 0$. Let $\eta\in C^2(\mathbb{R})$ be a nonnegative function such that $0\leq\eta'\leq 1$, $\eta''\geq 0$, $\eta=\eta'=\eta''$ on $(-\infty,0]$, $0<\eta(r)\leq r$ on $(0,\infty)$. For $\epsilon>0$ we set $\eta_\epsilon(r)=\eta((r-\epsilon)_+)$. Let $\zeta\in W^{1,p}(\mathbb{R}^N\setminus\{0\})$ with compact support. Inasmuch $(\eta'_\epsilon(u))^{p-1}\zeta$ has compact support in $\Omega$ and

$$D\left((\eta'_\epsilon(u))^{p-1}\zeta\right)=(\eta'_\epsilon(u))^{p-1}D\zeta+(p-1)(\eta'_\epsilon(u))^{p-2}\eta''_\epsilon(u)\zeta Du,$$

it belongs to $W^{1,p}(\Omega)$ and is an admissible test function for (2.5 ). Thus

$$\int_\Omega\left(|Du|^{p-2}Du.D\left((\eta'_\epsilon(u))^{p-1}\zeta\right)+A|u|^{q-1}u(\eta'_\epsilon(u))^{p-1}\zeta\right)dx\leq B\int_\Omega(\eta'_\epsilon(u))^{p-1}\zeta dx,$$

and

$$|Du|^{p-2}Du.D\left((\eta'_\epsilon(u))^{p-1}\zeta\right)\geq(\eta'_\epsilon(u))^{p-1}|Du|^{p-2}Du.D\zeta=|Dv_\epsilon|^{p-2}Dv_\epsilon.D\zeta,$$

where we have set $v_\epsilon=\eta_\epsilon(u)$. Furthermore, $\eta$ can be chosen such that $r^q(\eta'_\epsilon(r))^{p-1}\geq\eta^q_\epsilon(r)$, for example if we fix $\eta(r)=r^2/2\delta$ on $(0,\delta]$ and $\eta(r)=r-\delta/2$ on $[\delta,\infty)$ for some $\delta>0$. We extend $v_\epsilon$ by 0 outside $\overline{\Omega}\setminus\{a\}$ and denote by $\tilde{v}_\epsilon$ the new function, then $\tilde{v}_\epsilon\in W^{1,p}_{loc}(\mathbb{R}^N\setminus\{a\})\cap C(\mathbb{R}^N\setminus\{a\})$ and

$$\int_\Omega\left(|D\tilde{v}_\epsilon|^{p-2}D\tilde{v}_\epsilon.D\zeta+A|\tilde{v}_\epsilon|^{q-1}\tilde{v}_\epsilon\zeta\right)dx\leq B\int_\Omega\zeta dx. \tag{2.6}$$

This means that $\tilde{v}_\epsilon$ is a weak subsolution in $\mathbb{R}^N \setminus \{a\}$. By [18, Lemma 1.3], we derive

$$\tilde{v}_\epsilon(x) \leq \left(\frac{\lambda}{A\,|x-a|^p}\right)^{1/q+1-p} + \left(\frac{\mu B}{A}\right)^{1/q} \qquad \forall x \in \mathbb{R}^N \setminus \{a\},$$

for some $\lambda > 0$ and $\mu > 0$ depending on $N$, $p$ and $q$. Letting successively $\epsilon \to 0$ and $\delta \to 0$ we obtain (2.3 ). □

When $\Omega$ is smooth we have a sharper estimate

**Proposition 2.2** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^2$ boundary and $a \in \partial\Omega$. Let $q \geq p-1 > 1$ and $a > 0$. If $u \in C(\overline{\Omega} \setminus \{a\}) \cap W^{1,p}_{loc}(\Omega)$ is a weak solution of (2.3 ) with $B = 0$, there exists $C > 0$ depending on $\Omega$, $p$ and $q$ such that*

$$u(x) \leq \frac{C\rho(x)}{\left(A\,|x-a|^{q+1}\right)^{1/(q+1-p)}} \quad \forall x \in \overline{\Omega} \setminus \{a\}, \tag{2.7}$$

*where* $\rho(x) = \operatorname{dist}(x, \partial\Omega$.

*Proof.* By translation we can assume that $a = 0$. For $\epsilon > 0$ let $v_\epsilon$ be the solution of

$$\left\{\begin{array}{l} -div\left(|Dv_\epsilon|^{p-2}\, Dv_\epsilon\right) + A\,|v|_\epsilon^{q-1}\, v_\epsilon = 0, \quad \text{in } \Omega^\epsilon = \Omega \setminus B_\epsilon \\ \qquad\qquad\qquad\qquad v_\epsilon = u_+ \quad \text{on } \partial\Omega^\epsilon. \end{array}\right. \tag{2.8}$$

By [18, Lemma 1.3] as in the proof of Proposition 2.1 and the maximum principle, there holds

$$u_+(x) \leq v_\epsilon(x) \leq \left(\frac{\lambda}{A(|x|-\epsilon)^p}\right)^{1/(q+1-p)} \qquad \forall x \in \overline{\Omega}^\epsilon.$$

Consequently $\epsilon \leq \epsilon' \Longrightarrow v_\epsilon \geq v_{\epsilon'}$. Letting $\epsilon \to 0$ and using the previous inequalities and the classical regularity results for solutions of quasilinear equations [12] we conclude that $v_\epsilon$ converges, as $\epsilon \to 0$, to some $v$ which is a nonnegative solution of

$$\left\{\begin{array}{l} -div\left(|Dv|^{p-2}\, Dv\right) + Av^q = 0, \quad \text{in } \Omega \\ \qquad\qquad\qquad v = 0 \quad \text{on } \partial\Omega \setminus \{0\}. \end{array}\right. \tag{2.9}$$

and dominate $u$. Further, if $\ell > 0$ the function $v^\ell$ defined by $v^\ell(y) = \ell^{p/(q+1-p)} v(\ell y)$ is a solution of (2.9 ) with $\Omega$ replaced by $\Omega_\ell = \ell^{-1}\Omega$. Let $x \in \overline{\Omega} \setminus \{0\}$ and $\ell = |x|$. Since

$$0 \leq v^\ell(y) \leq \left(\frac{\lambda}{A(|y|)^p}\right)^{1/(q+1-p)} \qquad \forall y \in \Omega_\ell,$$

and

$$\max\left\{\left|Dv^\ell(y)\right| : y \in \Omega_\ell \cap B_{3/2} \setminus B_{2/3}\right\} \leq M \max\left\{\left|v^\ell(z)\right| : z \in \Omega_\ell \cap B_2 \setminus B_{1/2}\right\},$$

where $M$ is uniformly bounded because the curvature of $\partial\Omega_\ell$ is bounded, we obtain that $Dv^\ell(y)$ is uniformly bounded by some constant $C$ on $\Omega_\ell \cap B_{3/2} \setminus B_{2/3}$. Because $Dv^\ell(y) = \ell^{(q+1)/(q+1-p)} Dv(\ell y)$, it follows that

$$|Dv(x)| \leq \frac{C}{A^{1/q+1-p} |x|^{(q+1)/(q+1-p)}}$$

By the mean value Theorem, and using the fact that $v$ vanishes on $\partial\Omega \setminus \{0\}$, we derive

$$v(x) \leq \frac{C\rho(x)}{A^{1/q+1-p} |x|^{(q+1)/(q+1-p)}},$$

which implies (2.7 ). $\square$.

The construction of solutions of the quasilinear equations (2.1 ) with prescribed isolated singularity on the boundary of a general $C^2$ bounded domain $\Omega$ is settled upon similar constructions when the domain is either a half space, or a ball.

**Proposition 2.3** *Assume $N-1 < q < 2N-1$ and let $H = \mathbb{R}^N_+ = \{x = (x_1, ..., x_N) : x_N > 0)\}$ and $k > 0$. Then there exists a unique positive solution $u = u^H_k \in C^1(\overline{H} \setminus \{0\})$ of (2.1 ) in $H$ which vanishes on $\partial H \setminus \{0\}$ and satisfies ,*

$$u(x) = k\frac{x_N}{|x|^2}(1 + \circ(1)) \text{ as } x \to 0. \tag{2.10}$$

*Proof.* Since the function $x \mapsto kx_N |x|^{-2}$ is $N$-harmonic in $H$ and vanishes on $\partial H \setminus \{0\}$, it is a supersolution of (2.1 ). We write spherical coordinates in $\mathbb{R}^N$ under the form

$$x = \left\{(r,\sigma) \in [0,\infty) \times S^{N-1} = (r, \sin\phi\, \sigma', \cos\phi) : \sigma' \in S^{N-2}, \phi \in [0,\pi]\right\}, \tag{2.11}$$

then

$$Du = u_r \mathbf{i} + \frac{1}{r}\nabla_\sigma u,$$

where $\mathbf{i} = x/|x|$, $\nabla_\sigma$ denotes the covariant gradient on $S^{N-1}$, and equation (2.1 ) takes the form

$$\begin{aligned} -r^{1-N}&\left(r^{N-1}\left(u_r^2 + r^{-2}|\nabla_\sigma u|^2\right)^{(N-2)/2} u_r\right)_r \\ &-r^{-2} div_\sigma . \left(\left(u_r^2 + r^{-2}|\nabla_\sigma u|^2\right)^{(N-2)/2} \nabla_\sigma u\right) + |u|^{q-1} u = 0. \end{aligned} \tag{2.12}$$

Next

$$\nabla_\sigma u = -u_\phi \mathbf{e} + \frac{1}{\sin\phi}\nabla_{\sigma'} u$$

where $\mathbf{e}$ is derived from $x/|x|$ by the rotation with angle $\pi/2$ in the plane $0, x, \mathrm{N}$ (N being the North pole), and $\nabla_{\sigma'}$ is the covariant gradient on $S^{N-2}$ and (see [3])

$$
\begin{aligned}
div_{\sigma}.&\left(\left(u_r^2+\frac{|\nabla_\sigma u|^2}{r^2}\right)^{(N-2)/2}\nabla_\sigma u\right)\\
&=\frac{1}{\sin^{N-2}\phi}\left(\sin^{N-2}\phi\left(u_r^2+\frac{u_\phi^2}{r^2}+\frac{|\nabla_{\sigma'}u|^2}{r^2\sin^2\phi}\right)^{(N-2)/2}u_\phi\right)_\phi\\
&+\frac{1}{\sin^2\phi}\,div_{\sigma'}\left(\left(u_r^2+\frac{u_\phi^2}{r^2}+\frac{|\nabla_{\sigma'}u|^2}{r^2\sin^2\phi}\right)^{(N-2)/2}\nabla_{\sigma'}u\right).
\end{aligned}
\tag{2.13}
$$

If $u$ depends only on $r$ and $\phi$, (2.1 ) takes the form

$$
\begin{aligned}
&-r^{1-N}\left(r^{N-1}\left(u_r^2+r^{-2}u_\phi^2\right)^{(N-2)/2}u_r\right)_r\\
&\quad-r^{-2}\sin^{2-N}\phi\left(\sin^{N-2}\phi\left(u_r^2+r^{-2}u_\phi^2\right)^{(N-2)/2}u_\phi\right)_\phi+|u|^{q-1}u=0.
\end{aligned}
\tag{2.14}
$$

*Step 1* We look for a local subsolution $w$ under the form

$$w(r,\sigma)=k(1-r^\alpha)r^{-1}\cos\phi \qquad r>0\,,\ \phi\in[0,\pi/2].$$

where $\alpha>0$ is to be determined. Then

$$w_r=-kr^{-2}(1+(\alpha-1)r^\alpha)\cos\phi \text{ and } w_\phi=-kr^{-1}(1-r^\alpha)\sin\phi$$

$$w_r^2+r^{-2}w_\phi^2:=P=k^2r^{-4}\left(1+2(\alpha\cos^2\phi-1)r^\alpha+r^{2\alpha}((\alpha^2-2\alpha)\cos^2\phi+1)\right)$$

$$w_{rr}=kr^{-3}(2-(\alpha-1)(\alpha-2)r^\alpha)\cos\phi \text{ and } w_{\phi\phi}=-kr^{-1}(1-r^\alpha)\cos\phi$$

$$P_r=-2k^2r^{-5}\left[2+(4-\alpha)(\alpha\cos^2\phi-1)r^\alpha+(2-\alpha)((\alpha^2-2\alpha)\cos^2\phi+1)r^{2\alpha}\right]$$

$$P_\phi=-k^2\alpha r^{\alpha-4}\left[2+(\alpha-2)r^\alpha\right]\sin 2\phi,$$

$$P_rw_r+r^{-2}P_\phi w_\phi=2k^3r^{-7}\left[2+(5\alpha-6+(2\alpha-\alpha^2)\cos^2\phi)r^\alpha+O(r^{2\alpha}))\right]\cos\phi,$$

$$
\begin{aligned}
(N-1)r^{-1}w_r&+w_{rr}+(N-2)r^{-2}\cot\phi\, w_\phi+r^{-2}w_{\phi\phi}\\
&=kr^{-3}\left[4-2N)+(2-\alpha)(N+\alpha-2)r^\alpha\right]\cos\phi.
\end{aligned}
$$

Since

$$
\begin{aligned}
-div&\left(|Dw|^{N-2}Dw\right)+w^q=Lw\\
&=-P^{(N-2)/2}\left[(N-1)r^{-1}w_r+w_{rr}+(N-2)r^{-2}\cot\phi\, w_\phi+r^{-2}w_{\phi\phi}\right]\\
&\quad-\frac{N-2}{2}P^{(N-4)/2}\left[P_rw_r+r^{-2}P_\phi w_\phi\right]+w^q,
\end{aligned}
$$

and

$$w^q=k^q(1-r^\alpha)^qr^{-q}\cos^q\phi=k^q(1-qr^\alpha+O(r^{2\alpha}))r^{-q}\cos^q\phi,$$

a straightforward computation leads to

$$
\begin{aligned}
Lw &= k^{p-1}\alpha\left[3-2N+(2+\alpha)(N-2)\cos^2\phi+O(r^\alpha)\right]P^{(N-4)/2}r^{\alpha-7}\cos\phi \\
&\qquad\qquad\qquad\qquad\qquad + k^q(1-qr^\alpha+O(r^{2\alpha}))r^{-q}\cos^q\phi \\
&= k^{p-1}\alpha\left[3-2N+(2+\alpha)(N-2)\cos^2\phi\right]r^{-(2N-1)+\alpha}\cos\phi+k^qr^{-q}\cos^q\phi \\
&\qquad - qk^qr^{-q+\alpha}\cos^q\phi+O(r^{-(2N-1)+2\alpha}\cos\phi)+O(r^{-q+2\alpha}\cos\phi).
\end{aligned}
\tag{2.15}
$$

By assumption $q<2N-1$. If we choose $\alpha<\min\{2N-1-q,1/(N-2)\}$, there exists $R\in(0,1]$ such that $Lw\leq 0$ on $H\cap B_R$.

*Step 2* Next we construct a solution $u_R$ in $B_R\cap H$ which vanishes on $\partial B_R\cap H$ and on $\partial H\setminus\{0\}$ and satisfies

$$\lim_{r\to 0}\frac{ru_R(r,\sigma)}{\cos\phi}=k. \tag{2.16}$$

Let $\ell_R=k(1-R^\alpha)R^{-1}$. Inasmuch $w-\ell_R$ is a subsolution, for any $\epsilon>0$ we can construct a nonnegative solution $u_\epsilon$ of (2.1 ) in $H\cap(B_R\setminus B_\epsilon)$ which vanishes on $H\cap\partial B_R$ and on $\partial H\cap(B_R\setminus B_\epsilon)$ and takes the value $k\epsilon^{-2}x_N$ on $H\cap\partial B_\epsilon$. By comparison

$$(w(x)-\ell_R)_+\leq u_\epsilon(x)\leq kx_N\left|x\right|^{-2}. \tag{2.17}$$

Furthermore, $\epsilon\mapsto u_\epsilon$ is increasing. Set $u=u_R=\lim_{\epsilon\to 0}u_\epsilon$, then $u$ is a solution of (2.1 ) in $H\cap B_R$ which vanishes on $\partial B_R\cap H$ and on $\partial H\setminus\{0\}$ and satisfies the same inequality (2.17 ) as $u_\epsilon$, but in whole $H\cap B_R$. This implies that (2.16 ) holds uniformly on $[0,\pi/2-\delta]$, for any $\delta>0$. In order to improve this inequality, we perform a scaling: for $r>0$, we set $u^r(x)=ru(rx)$. Then $u^r$ satisfies

$$-div\left(\left|Du^r\right|^{N-2}Du^r\right)+r^{2N-1-q}(u^r)^q=0 \tag{2.18}$$

in $H\cap B_{R/r}$ where there holds

$$k(x_N\left|x\right|^{-2}(1-r^\alpha\left|x\right|^\alpha)-\ell_R)_+\leq u^r(x)\leq kx_N\left|x\right|^{-2}. \tag{2.19}$$

Since $u^r$ is uniformly bounded for $1/2\leq|x|\leq 2$, it follows from regularity theory [12] that it is also bounded in the $C^{1,\alpha}$-topology of $2/3\leq|x|\leq 3/2$. Using Ascoli's theorem and the fact that $u^r(x)$ converges to $kx_N\left|x\right|^{-2}$ pointwise and locally uniformly, it follows that $Du^r(x)=r^2Du(rx)$ converges uniformly in $\{x\in H:2/3\leq|x|\leq 3/2\}$ to $-2kx_N\left|x\right|^{-4}x+k\left|x\right|^{-2}\mathbf{e}_N$ which is the gradient of $x\mapsto kx_n\left|x\right|^{-2}$. Using the expression of $Du$ in spherical coordinates we obtain

$$r^2u_r\mathbf{i}-ru_\phi\mathbf{e}+\frac{r}{\sin\phi}\nabla_{\sigma'}u\to -2k\sigma_N\mathbf{i}+k\mathbf{e}_N \text{ uniformly on } S_+^{N-1} \text{ as } r\to 0,$$

where $\sigma_N=\langle\sigma,\mathbf{e}_N\rangle$. Inasmuch $\mathbf{i}$, $\mathbf{e}$ and $\nabla_{\sigma'}u$ are orthogonal, the component of $\mathbf{e}_N$ is $\sin\phi$, thus

$$ru_\phi(r,\sigma',\phi)\to -k\sin\phi \text{ as } r\to 0. \tag{2.20}$$

Since

$$u(r,\sigma',\phi) = \int_{\pi/2}^{\phi} u_\phi(r,\sigma',\theta)\,d\theta, \tag{2.21}$$

the previous convergence estimate establishes (2.16 ).

*Step 3* Construction of the solution in $H$. Let $\eta$ be the truncation function introduced in the proof of Proposition 2.1, and $\eta_\epsilon(r) = \eta((r-\epsilon)_+)$. Then the function $u_{R,\epsilon}$ defined by $u_{R,\epsilon} = \eta_\epsilon \circ u_R$ in $H \cap B_R$ and zero outside, is a subsolution of (2.1 ) in $H$ which vanishes on $\partial H \setminus \{0\}$ and satisfies (2.16 ). Using the same device as in Step 2, we construct a sequence of solutions $u_\delta$ ($\delta > 0$) of (2.1 ) in $H \setminus B_\delta$ with boundary value $k\delta^{-2}x_N$ on $\partial B_\delta \cap H$, zero on $\partial H \setminus B_\delta$ and satisfies

$$u_{R,\epsilon} \le u_\delta \le kx_N\,|x|^{-2}.$$

When $\delta \to 0$, $u_\delta$ decreases and converges to some $u$ which satisfies (2.1 ) and the previous inequality. Letting successively $\epsilon \to 0$ and $\eta(r) \to r_+$ we obtain that $u$ satisfies

$$\check{u}_R(x) \le u(x) \le kx_N\,|x|^{-2} \quad \text{in } H, \tag{2.22}$$

where $\check{u}$ is the extension of $u$ by zero outside $B_R$. The proof of (2.10 ) is the same as in Step 2.

*Step 4* Uniqueness. Let $u$ and $\hat{u}$ be two solutions of (2.1 ) satisfying (2.10 ) and $\epsilon > 0$. Then $u_\epsilon = (1+\epsilon)u + \epsilon$ is a super solution which is positive of $\partial H \setminus \{0\}$. Inasmuch it dominates $\hat{u}$ both in a neighborhood of 0 and in a neighborhood of infinity, it dominates $\hat{u}$ in $H$. Letting $\epsilon \to 0$ yields to $u \ge \hat{u}$. Similarly $\hat{u} \ge u$. □

**Proposition 2.4** *Assume $N-1 < q < 2N-1$ and let $B = B_1(0)$, $a \in \partial B$ and $k > 0$. Then there exists a unique function $u = u^B_{k,a} \in C^1(\overline{B} \setminus \{a\})$ which vanishes on $\partial B \setminus \{a\}$ and satisfies (2.1 ) in $B$ and*

$$u(x) = k\frac{1-|x|}{|x-a|^2}(1+\circ(1)) \text{ as } x \to a. \tag{2.23}$$

*Proof.* With a change of coordinates, we can assume that $B$ has center $m = (0,...,0,-1/2)$ and $a$ is the origin of coordinates. We denote by $\omega$ the point $(0,...,0,-1)$ and by $\mathcal{I}_\omega$ the inversion with center $\omega$ and power 1. By this involutive transformation, the half space $H = \{x \in \mathbb{R}^N : x_N > 0\}$ is transformed into the ball $B^* = \{x \in \mathbb{R}^N : |x|^2 + x_N < 0\}$. Thus the function $x \mapsto P_k(x) = -k(|x|^2 + x_N)/2\,|x|^2$ is $N$-harmonic and positive in $B^*$, vanishes on $\partial B^* \setminus \{0\}$ and is singular at 0. Let $v_k$ be the solution of (2.1 ) in $H$ satisfying (2.10 ), and $u_k = v_k \circ \mathcal{I}_\omega$. Then $u_k \in C(\overline{B^*} \setminus \{0\})$ satisfies

$$\begin{cases} -div\left(|Du_k|^{N-2}\,Du_k\right) + |x-\omega|^{-2N}\,u_k^q = 0 & \text{in } B^* \\ u_k = 0 & \text{on } \partial B^* \setminus \{0\}. \end{cases} \tag{2.24}$$

Furthermore $u_k \le P_k$ and

$$P_k(x) = k\frac{1/4 - |x-m|^2}{2\,|x|^2} = k\frac{1/2 - |x-m|}{2\,|x|^2}(1+\circ(1)) = u_k(x)(1+\circ(1)) \tag{2.25}$$

as $x \to 0$. Inasmuch $|x-\omega| \leq 1$, $u_k$ is a subsolution of (2.1 ) in $B^*$. For $\epsilon > 0$ we construct a solution $v_\epsilon$ of (2.1 ) in $B^* \setminus B_\epsilon(0)$ with boundary value $P_k$. By the maximum principle $u_k \leq v_\epsilon \leq P_k$ in $B^* \setminus B_\epsilon(0)$. Since the sequence $\{v_\epsilon\}$ is monotone, we obtain that there exists a solution $\lim_{\epsilon\to 0} v_\epsilon := u \in C^1(\overline{B^*} \setminus \{0\})$ of (2.1 ) in $B^*$ which satisfies

$$u_k(x) \leq u(x) \leq P_k(x) \quad \text{in } B^*, \tag{2.26}$$

and

$$u(x) = k\frac{1/2 - |x-m|}{2\,|x|^2}(1 + \circ(1)). \tag{2.27}$$

We change the variables in setting $x'_N = x_N + 1/2$ and $x'_i = x_i$ $(i = 1, ..., N-1)$. We define $u'(x') = u(x)$ and denote by $a$ the point $(0, ..., 0, 1)$. Clearly $u'$ satisfies (2.1 ) in $B_{1/2}$, vanishes on $\partial B_{1/2} \setminus \{a\}$ and

$$u'(x) = k\frac{1/2 - |x|}{2\,|x - a/2|^2}(1 + \circ(1)) \text{ as } x \to a/2. \tag{2.28}$$

By the transformation $\ell \mapsto \ell^{p/(q+1-p)} u'_k(\ell x)$, where $\ell = 1/2$, we obtain a solution $u_{k,a}$ of (2.1 ) in $B$ which verifies

$$u_{k,a}(x) = 2^{N/(q+1-N)} k\frac{1 - |x|}{|x-a|^2}(1 + \circ(1)) \text{ as } x \to a. \tag{2.29}$$

Because $k$ is arbitrary, (2.23 ) follows. Uniqueness of the solution is obtained as in Proposition 2.3 with $u_\epsilon = (1+\epsilon)u$. □

**Proposition 2.5** *Assume $N - 1 < q < 2N - 1$ and let $G = \overline{B}^c$, $a \in \partial B$ and $k > 0$. Then there exists a unique function $u = u^{B^c}_{k,a} \in C^1(\overline{G} \setminus \{a\})$ which vanishes on $\partial B \setminus \{a\}$ and satisfies (2.1 ) in $G$ and*

$$u(x) = k\frac{|x| - 1}{|x-a|^2}(1 + \circ(1)) \text{ as } x \to a. \tag{2.30}$$

*Proof.* Uniqueness follows from (2.30 ) by the same method as in Proposition 2.3 and Proposition 2.4. Actually, it will be proved in Theorem 2.7. For existence we perform the inversion $\mathcal{I}^1_0$ with center 0 and power 1. It transforms the function $u^B_{k,a}$ constructed in the previous proposition into a function $v \in C^1(\overline{G} \setminus \{a\})$ which vanishes on $\partial B \setminus \{a\}$ and satisfies (2.30 ). Furthermore $v$ is solution of

$$-div\left(|Dv|^{N-2} Dv\right) + |x|^{-2N} |v|^{q-1} v = 0 \tag{2.31}$$

in $G$. Since $|x| > 1$, $v$ is a super solution for (2.1 ) in $G$. With no loss of generality we can assume that $a = (0, ...0, 1)$ and let $u^{H^1}_{k,a}$ be the solution of (2.1 ) in $H^1 = \{x = (x_1, ..., x_N : x_N > 1)\}$ satisfying (2.10 ) already constructed in Proposition 2.3. Then $v_\epsilon = \eta(u^{H^1}_{k,a})$ is a subsolution in $G$ (where $\eta_\epsilon$ has been defined in the proof of Proposition 2.1). By the same approximation as in the previous proposition, we construct an increasing sequence $\{u_\epsilon\}$ ($\epsilon > 0$) of solutions of (2.1 ) in $G \setminus B_\epsilon(a)$ which vanishes on $\partial G \setminus B_\epsilon(a)$, takes the value $v$ on $G \cap \partial B_\epsilon(a)$ and verifies $v_\epsilon \leq u_\epsilon \leq v$ in $G \setminus B_\epsilon(a)$. Letting $\epsilon \to 0$, we obtain the existence of a solution $u^*$ in $G$ which satisfies

$$\tilde{u}^{H^1}_{k,a} \leq u^* \leq v \quad \text{in } G \tag{2.32}$$

where we denote by $\tilde{u}^{H^1}_{k,a}$ the extension of $u^{H^1}_{k,a}$ by zero in $\overline{H^1}^c$. We conclude that (2.30 ) holds in $H^1$. In order to extend this convergence to whole $G$, we proceed as in the proof of Proposition 2.3, with a minor modification due to the geometry. We put the origin of coordinates at $a$, takes the same spherical coordinates and obtain again that

$$r^2 u^*_r \mathbf{i} - r u^*_\phi \mathbf{e} + \frac{r}{\sin\phi}\,\nabla_{\sigma'} u^* \to -2k\sigma_N \mathbf{i} + k\mathbf{e} \quad \text{ uniformly on } S^{N-1}_+ \text{ as } r \to 0.$$

Therefore (2.20 ) holds for any $\phi \in [0, \pi/2]$. For $r > 0$, the angle $\phi$ ranges from $\psi(r) = \cos^{-1}(-r/2)$ to 0 (here is the difference with the half-space case) and $|x|^2 \nabla u(x)$ remains bounded in this domain, by the regularity theory for quasilinear elliptic equations. Since

$$u^*(r, \sigma', \phi) = \int_{\psi(r)}^{\phi} u^*_\phi(r, \sigma', \theta)\, d\theta, \tag{2.33}$$

we derive, as in the proof of Proposition 2.3,

$$\lim_{r \to 0} u^*(r, \sigma', \phi) = k \cos\phi \quad \text{ uniformly on } [0, \pi/2]. \tag{2.34}$$

The proof that (2.30 ) holds is a particular case of Theorem 2.7. □

In a general domain we have to extend the solution through the boundary. We denote by $\dot{\rho}(x)$ the signed distance from $x \to \partial\Omega$, that is $\dot{\rho}(x) = \rho(x)$ if $x \in \Omega$ and $\dot{\rho}(x) = -\rho(x)$ if $x \in \Omega^c$. Since $\partial\Omega$ is $C^2$, there exists $\beta_0 > 0$ such that if $x \in \mathbb{R}^N$ verifies $-\beta_0 \leq \dot{\rho}(x) \leq \beta_0$, there exists a unique $\xi_x \in \partial\Omega$ such that $|x - \xi_x| = |\dot{\rho}(x)|$. Furthermore, if $\nu_{\xi_x}$ is the outward unit vector to $\partial\Omega$ at $\xi_x$, $x = \xi_x - \dot{\rho}(x)\nu_{\xi_x}$. In particular $\xi_x - \dot{\rho}(x)\nu_{\xi_x}$ and $\xi_x + \dot{\rho}(x)\nu_{\xi_x}$ have the same orthogonal projection $\xi_x$ onto $\partial\Omega$.

Let $T_{\beta_0}(\Omega) = \{x \in \mathbb{R}^N : -\beta_0 \leq \dot{\rho}(x) \leq \beta_0\}$, then the mapping $\Pi : [-\beta_0, \beta_0] \times \partial\Omega \mapsto T_{\beta_0}(\Omega)$ defined by $\Pi(\rho, \xi) = \xi - \dot{\rho}\nu(\xi)$ is a $C^2$ diffeomorphism. Moreover $D\Pi(0, \xi)(1, e) = e - \nu_\xi$ for any $e$ belonging to the tangent space $T_\xi(\partial\Omega)$ to $\partial\Omega$ at $\xi$. If $x \in T_{\beta_0}(\Omega)$, we define the reflection of $x$ through $\partial\Omega$ by $\psi(x) = \xi_x + \dot{\rho}(x)\nu_{xi_x}$. Clearly $\psi$ is an involutive diffeomorphism from $\overline{\Omega} \cap T_{\beta_0}(\Omega)$ to $\Omega^c \cap T_{\beta_0}(\Omega)$. Furthermore for any $\xi \in \partial\Omega$, $D\psi(\xi) = \mathcal{S}_{T_\xi(\partial\Omega)}$ is the symmetry with respect to the tangent space $T_\xi(\partial\Omega)$ to $\partial\Omega$ at $\xi$. If a function $v$ is defined in $\Omega \cap T_{\beta_0}(\Omega)$, we define $\tilde{v}$ in $\Omega^c \cap T_{\beta_0}(\Omega)$ by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega \cap T_{\beta_0}(\Omega) \\ -v \circ \psi(x) & \text{if } x \in \Omega^c \cap T_{\beta_0}(\Omega). \end{cases} \tag{2.35}$$

**Proposition 2.6** *Let* $v \in C^{1,\alpha}(\overline{\Omega} \cap T_{\beta_0}(\Omega) \setminus \{0\})$ *be a solution of (2.1 ) in* $\Omega \cap T_{\beta_0}(\Omega)$ *vanishing on* $\partial\Omega \setminus \{0\}$. *Then* $\tilde{v} \in C^{1,\alpha}(T_{\beta_0}(\Omega) \setminus \{0\})$ *is solution of a quasilinear equation*

$$-\sum_j \frac{\partial}{\partial x_j} \tilde{A}_j(x, D\tilde{v}) + \tilde{b}(x) |\tilde{v}|^{q-1} \tilde{v} = 0 \tag{2.36}$$

*in* $T_{\beta_0}(\Omega) \setminus \{0\}$ *where the* $\tilde{A}_j$ *and* $\tilde{b}$ *are* $C^1$ *functions defined in* $T_{\beta_0}(\Omega)$ *where they verify*

$$\begin{cases} (i) \quad \tilde{A}_j(x, 0) = 0 \\ (ii) \quad \displaystyle\sum_{i,j} \frac{\partial \tilde{A}_j}{\partial \eta_i}(x, \eta)\xi_i \xi_j \geq \Gamma |\eta|^{p-2} |\xi|^2 \\ (iii) \quad \displaystyle\sum_{i,j} \left| \frac{\partial \tilde{A}_j}{\partial \eta_i}(x, \eta) \right| \leq \Gamma |\eta|^{p-2} \\ (iv) \quad \Gamma \geq \tilde{b}(x) \geq \gamma \end{cases} \tag{2.37}$$

*for all* $x \in T_\beta(\Omega) \setminus \{0\}$ *for some* $\beta \in (0, \beta_0]$, $\eta \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$ *and some* $0 < \gamma \leq \Gamma$.

*Proof.* The assumptions (2.37 ) implies that weak solutions of (2.36 ) are $C^{1,\alpha}$, for some $\alpha > 0$ [17] and satisfy the standard a priori estimates. As it is defined the function $\tilde{v}$ is clearly $C^1$ in $T_{\beta_0}(\Omega) \setminus \{0\}$. Writing $Dv(x) = -D(\tilde{v} \circ \psi(x)) = -D\psi(x)(D\tilde{v}(\psi(x)))$ and $\tilde{x} = \psi(x) = \psi^{-1}(x)$

$$\begin{aligned} &\int_{\Omega \cap T_{\beta_0}(\Omega)} \left( |Dv|^{p-2} Dv.D\zeta + |v|^{q-1} v\zeta \right) dx \\ &\qquad = \int_{\overline{\Omega}^c \cap T_{\beta_0}(\Omega)} \left( |D\psi(D\tilde{v})|^{p-2} D\psi(D\tilde{v}).D\psi(D\zeta) + |\tilde{v}|^{q-1} \tilde{v}\zeta(\psi(\tilde{x})) \right) |D\psi| \, d\tilde{x}. \end{aligned}$$

But

$$\begin{aligned} D\psi(D\tilde{v}).D\psi(D\zeta) &= \sum_k \left( \sum_i \frac{\partial \psi_i}{\partial x_k} \frac{\partial \tilde{v}}{\partial x_i} \right) \left( \sum_j \frac{\partial \psi_j}{\partial x_k} \frac{\partial \zeta}{\partial x_j} \right) \\ &= \sum_j \left( \sum_{i,k} \frac{\partial \psi_i}{\partial x_k} \frac{\partial \psi_j}{\partial x_k} \frac{\partial \tilde{v}}{\partial x_i} \right) \frac{\partial \zeta}{\partial x_j}. \end{aligned}$$

We set $b(x) = |D\psi|$,

$$A_j(x, \eta) = |D\psi| \, |D\psi(\eta)|^{p-2} \sum_i \left( \sum_k \frac{\partial \psi_i}{\partial x_k} \frac{\partial \psi_j}{\partial x_k} \right) \eta_i, \tag{2.38}$$

and

$$A(x, \eta) = (A_1(x, \eta), ..., A_N(x, \eta)) = |D\psi| \, |D\psi(\eta)|^{p-2} \, (D\psi)^t D\psi(\eta). \tag{2.39}$$

For any $\xi \in \partial\Omega$, the mapping $D\psi_{\partial\Omega}(\xi)$ is the symmetry with respect to the hyperplane $T_\xi(\partial\Omega)$ tangent to $\partial\Omega$ at $\xi$, so $|D\psi(\xi)| = 1$. Inasmuch $D\psi$ is continuous, a lengthy but standard computation leads to the existence of some $\beta \in (0, \beta_0]$ such that (2.37 ) holds in

$T_\beta(\Omega)\cap\overline{\Omega}^c$. If we define $\tilde{A}$ (resp. $\tilde{b}$) to be $|\eta|^{p-2}\,\eta$ (resp 1) on $T_\beta(\Omega)\cap\overline{\Omega}$ and $A$ (resp. $|D\psi|$) on $T_\beta(\Omega)\cap\overline{\Omega}^c$, then inequalities (2.37 ) are satisfied in $T_\beta(\Omega)$. □

*Remark.* Notice that, similarly to the $p$-laplacian, the vector field $\tilde{A}$ is positively homogeneous with exponent $p-1$ with respect to $\eta$. Furthermore, if for $r>0$ we set $\tilde{A}^r_j(x,\eta)=\tilde{A}_j(rx,\eta)$ , then $\tilde{A}^r_j$ satisfies the same estimates (2.37 ) as $A_j$, uniformly in $T_{r^{-1}\beta}(r^{-1}\Omega)$, for $0<r\leq 1$. Furthermore

$$\lim_{r\to 0} A^r_j(x,\eta)=|\eta|^{p-2}\,\eta_j \quad \forall\eta\in\mathbb{R}^N\,,\ \forall j=1,...,N,$$

and this limit is uniform on the bounded subsets of $\mathbb{R}^N$.

**Theorem 2.7** *Let $\Omega$ be a bounded domain with a $C^2$ boundary and $a\in\partial\Omega$. Assume $N-1<q<2N-1$ and denote by $\rho(x)$ the distance from $x$ to $\partial\Omega$. Then for any $k>0$ there exists a unique function $u=u_{k,a}\in C(\overline{\Omega}\setminus\{a\})$ which vanishes on $\partial\Omega\setminus\{a\}$, is solution of (2.1 ) and satisfies*

$$u_{k,a}(x)=k\frac{\rho(x)}{|x-a|^2}(1+\circ(1)) \text{ as } x\to a. \tag{2.40}$$

*Proof.* Uniqueness follows from (2.40 ) by the same technique as in the previous propositions. For existence let $B^i_R$ be a ball of radius $R$ such that $B^i_R\subset\Omega$ and $a\in\partial B^i_R$, and let $\omega_i$ be its center. We denote by $U^i$ the solution of (2.1 ) in $B^i_R$, which vanishes on $\partial B^i_R\setminus\{a\}$ and satisfies

$$U^i(x)=k\frac{R-|x-\omega_i|}{|x-a|^2}(1+\circ(1)) \text{ as } x\to a. \tag{2.41}$$

If we set $U_\delta=\eta_\delta(U^i)$, we have already seen that $\check{U}_\delta$, the extension of $U_\delta$ by zero outside its support, is a subsolution of (2.1 ) in $\Omega$. Because $V^\Omega_a$, the $N$-harmonic function element of $C(\overline{\Omega}\setminus\{a\})$ vanishing on $\partial\Omega\setminus\{a\}$, satisfies

$$V^\Omega_a(x)=\frac{\rho(x)}{|x-a|^2}(1+\circ(1)) \text{ as } x\to a,\ x\in B^i_R, \tag{2.42}$$

there holds $kV^\Omega_a\geq\check{U}_\delta$. If $\Omega_\epsilon=\Omega\setminus\{B_\epsilon(a)\}$ ($\epsilon>0$), we construct a solution $u_\epsilon\in C(\overline{\Omega_\epsilon})$ of (2.1 ) in $\Omega_\epsilon$, which vanishes on $\partial\Omega\setminus B_\epsilon(a)$ and takes the value $kV^\Omega_a$ on $\partial B_\epsilon(a)\cap\Omega$. By the maximum principle $\epsilon\mapsto u_\epsilon$ is increasing and $\check{U}_\delta\leq u_\epsilon\leq kV^\Omega_a$ in $\Omega_\epsilon$. Letting $\epsilon\to 0$ we obtain that $u_\epsilon$ converges in the $C^1_{loc}$-topology of $\overline{\Omega}\setminus\{a\}$ to a solution $u=u_{k,a}$ of (2.1 ) in $\Omega$. It follows from the previous inequalities that

$$\check{U}_\delta(x)\leq u(x)\leq kV^\Omega_a(x) \quad \forall x\in\overline{\Omega}\setminus\{a\}. \tag{2.43}$$

In order to prove the asymptotic behaviour, we proceed as in Proposition 2.4 with the help of the reflection principle of Proposition 2.6. We fix the origin of coordinates at $a=0$ and the normal outward unit vector at $a$ to be $-\mathbf{e}_N$. If $\tilde{u}$ is the extension of $u$ by reflection through $\partial\Omega$, it satisfies

$$-\sum_j\frac{\partial}{\partial x_j}\tilde{A}_j(x,D\tilde{u})+\tilde{b}(x)\,|\tilde{u}|^{q-1}\,\tilde{u}=0 \tag{2.44}$$

in $T^\beta(\Omega)\setminus\{0\}$. For $r>0$, set $\tilde u^r(x)=r\tilde u(rx)$. Then $\tilde u^r$ is solution of

$$-\sum_j \frac{\partial}{\partial x_j}\tilde A^r_j(x,D\tilde u^r)+r^{2N-1-q}\tilde b(rx)\left|\tilde u^r\right|^{q-1}\tilde u^r=0 \tag{2.45}$$

in $T^{\beta r^{-1}}(\Omega^r)\setminus\{0\}$, where $\Omega^r:=r^{-1}\Omega$. By [3, Th 2.4] there exists $C>0$ such that

$$kV_0^\Omega(x)\le Ck\frac{\rho(x)}{|x|^2}.$$

Furthermore, for any $x\in T^\beta(\Omega)\setminus\{0\}$, $\rho(x):=\operatorname{dist}(x,\Omega)=\rho(\psi(x))$ (we recall that $\psi(x)$ is the symmetric of $x$ with respect to $\partial\Omega$ as it is defined in Proposition 2.6), and $c\,|x|\le|\psi(x)|\le c^{-1}|x|$ for some $c>0$, the same relations holds if $T^\beta(\Omega)$ is replaced by $T^{\beta r^{-1}}(\Omega^r)$ and $\rho(x)$ by $\rho_r(x):=\operatorname{dist}(x,\Omega^r)$. Since $\Omega$ is $C^2$,

$$\lim_{r\to0}\frac{\rho(rx)}{r\rho_r(x)}=1$$

uniformly on bounded subsets of $\mathbb{R}^N$. Consequently

$$\left|\tilde u^r\right|(x)\le Ckr^{-1}\frac{\rho(rx)}{|x|^2}=Ck\frac{\rho_r(x)}{|x|^2}(1+\circ(1)).$$

For $0<a<b$ fixed and $r\le r_0$ (for some $r_0\in(0,1]$) the spherical shall $\Gamma_{a,b}=\{x\in\mathbb{R}^N:a\le|x|\le b\}$ is included into $T^{\beta r^{-1}}(\Omega^r)$. By the classical regularity theory for quasilinear equations [17] and Proposition 2.6, there holds

$$\left\|D\tilde u^r\right\|_{C^\alpha(\Gamma_{2/3,3/2})}\le C_r\left\|\tilde u^r\right\|_{L^\infty(\Gamma_{1/2,2})}, \tag{2.46}$$

where $C_r$ remains bounded because $r\le1$. By Ascoli's theorem and (2.43 ) $\tilde u^r(x)$ converges to $kx_N|x|^{-2}$ in the $C^1(B_{3/2}\setminus B_{1/2})$-topology. This implies in particular

$$\lim_{r\to0}r^2D\tilde u(rx)=-2kx_Nx\,|x|^{-4}+k\,|x|^{-2}\,\mathbf{e}_N.$$

If we take in particular $|x|=1$, we derive

$$\lim_{r\to0}(r\tilde u(r,\sigma),r^2\nabla\tilde u(r,\sigma))=(k\cos\phi,-k\sin\phi\mathbf{e_N}), \tag{2.47}$$

uniformly with respect to $\sigma=(\sin\phi\,\sigma',\cos\phi)\in S^{N-2}\times[0,\pi]$. Because $\partial\Omega$ is $C^2$ there exists $\epsilon_0>0$ and a $C^2$ real valued function $h$ defined in $\Theta_{\epsilon_0}:=B_{\epsilon_0}\cap\partial H$ (we recall that $\partial H=\{x=(x',0)\}$) and an open neighborhood $\mathcal{V}_{\epsilon_0}$ of 0 such that $\partial\Omega\cap\mathcal{V}_{\epsilon_0}=\{x=(x',x_N:x_N=h(x')\}$, and $Dh(0)=0$ (this expresses the fact that $\partial H=T_0(\partial\Omega)$). If we define $\Psi$ by

$$\Psi(x)=(x',x_N-h(x'))\quad\forall x\in\mathcal{V}_{\epsilon_0}.$$

then $det(D\Psi)=1$ and $D\Psi(0)=I$. Up to replacing $\epsilon_0$ by a smaller quantity, $\Psi$ is a $C^2$ diffeomorphism from $\mathcal{V}_{\epsilon_0}$ into a neighborhood $\mathcal{V}'$ of 0 such that $\mathcal{V}_{\epsilon_0}\cap\partial\Omega)=\Theta_{\epsilon_0}$. Because dist $(\Psi(x), \partial H) = x_N - h(x')$, dist $(\Psi(x), \partial H) = \rho(x)(1 + \circ(1))$ as $x \to 0$. Thus, if we set $x = \Psi^{-1}(y)$ and $\tilde{u}(x) = u^*(y)$, (2.47 ) is equivalent to

$$\lim_{|y|\to 0} (|y|\, u^*(|y|, \sigma), |y|^2 \nabla u^*(|y|, \sigma)) = (k \cos\phi, -k \sin\phi\, \mathbf{e_N}), \tag{2.48}$$

uniformly on $S^{N-1}$, thus

$$|y|\, u^*(|y|, \sigma) = k \sin\phi\, (1 + \circ(1)) \quad \text{as} \quad |y| \to 0 \tag{2.49}$$

uniformly with respect to $\sigma \in S_+^{N-1}$, because $u^*$ vanishes on $B_{\epsilon_0} \cap \partial H \setminus \{0\}$. This implies (2.40 ). □

Clearly the mapping $k \mapsto u_{k,a}$ is increasing. As $u_k$ satisfies the estimates (2.7 ) and (2.30 ), $u_{k,a}$ converges in the $C^1_{loc}(\overline{\Omega} \setminus \{a\})$-topology, as $k \to \infty$, to some $u_{\infty,a}$, solution of (2.1 ) in $\Omega$, vanishes on $\partial\Omega \setminus \{a\}$ and satisfies

$$\lim_{x \to a} \frac{|x - a|^2\, u_{\infty,a}(x)}{\rho(x)} = \infty. \tag{2.50}$$

In order to describe the precise behaviour of $u_{\infty,a}$, we have to introduce separable solutions of (2.1 ) in $\mathbb{R}^N \setminus \{0\}$: if we look for solutions $u$ under the form $u(r,\sigma) = r^\beta \omega(\sigma)$, then $\beta = -\beta_q = -N/(q + 1 - N)$ and $\omega$ satisfies

$$-div_\sigma \left( \left( \beta_q^2 \omega^2 + |\nabla_\sigma \omega|^2 \right)^{(N-2)/2} \nabla_\sigma \omega \right) - \Lambda \left( \beta_q^2 \omega^2 + |\nabla_\sigma \omega|^2 \right)^{(N-2)/2} \omega + |\omega|^{q-1} \omega = 0 \tag{2.51}$$

on $S^{N-1}$ where $\Lambda = (N-1)\beta_q^2$. We shall denote by $\mathcal{S}_q$ the set of (always $C^{1,\alpha}$) solutions of (2.51 ). If $u$ is a separable solution of (2.1 ) in $H$ which vanishes on $\partial H \setminus \{0\}$, the function $\omega$ is a solution of (2.51 ) in $S_+^{N-1}$ which vanishes on $\partial S_+^{N-1} = S^{N-2}$. We shall denote by $\mathcal{S}_q^*$ the set of such functions and by $\mathcal{S}_{q\,+}^*$ the subset of positive solutions. We recall some simple facts

**Proposition 2.8** *(i) For any $q > N - 1$, $\mathcal{S}_q$ contains at least the three constant functions $0$ and $\pm((N-1)\beta_q^N)^{1/(q+1-N)}$.*
*(ii) For any $q \geq 2N - 1$, $\mathcal{S}_q^* = \{0\}$.*
*(iii) For any $q \in (N - 1, 2N - 1)$, $\mathcal{S}_{q\,+}^*$ contains a unique element.*

*Proof.* Assertion (i) is evident since $\Lambda > 0$. Assertion (ii), as well as the existence part of assertion (iii), can be found in [9] or [22]. Furthermore any $\omega \in \mathcal{S}_{q\,+}^*$ is positive in $S_+^{N-1}$ and verifies $\omega_\phi < 0$ by Hopf boundary lemma as the outward normal derivative on $\partial S_+^{N-1}$ is $\partial/\partial\phi$. We can construct a minimal element in $\mathcal{S}_{q\,+}^*$ in the following way: If we denote by $u_k^H$ the unique solution of (2.1 ) in $H$ which satisfies (2.10 ) and set $T_r(u_k^H)(x) = r^{\beta_q} u_k^H(rx)$ for $r > 0$, then $T_r(u_k^H)$ is a solution of (2.1 ) in $H$ which satisfies

$$T_r(u_k^H) = r^{(2N-1-q)/(q+1-N)} k \frac{x_N}{|x|^2} (1 + \circ(1)) \quad \text{as } x \to 0.$$

Thus $T_r(u_k^H) = u^H_{r^{(2N-1-q)/(q+1-N)}k}$. Furthermore, if $\omega \in \mathcal{S}^*_{q+}$, the maximum principle at 0 and at infinity (replacing $u_\omega$ by $u_\omega + \epsilon$ and letting $\epsilon \to 0$) leads to

$$u_\omega(r,\sigma) := r^{-\beta_q}\omega(\sigma) > u_k^H(r,\sigma) \quad \forall (r,\sigma) \in (0,\infty) \times S_+^{N-1}, \forall k > 0.$$

Letting $k \to \infty$ implies $u_\omega(r,\sigma) \geq u_\infty^H(r,\sigma)$ and $T_r(u_\infty^H) = u_\infty^H$ given that $2N-1-q > 0$. Then the function $u_\infty^H$ is invariant with respect to the transformation $T_r$. It is therefore self-similar, and consequently under the form $u_\infty^H(r,\sigma) = r^{-\beta_q}\underline{\omega}(\sigma)$. As a result of the previous inequality $\underline{\omega}$ is the minimal element of $\mathcal{S}^*_{q+}$. Next we denote $\delta^* = \max\{\delta \geq 0 : \delta\omega \leq \underline{\omega}\}$ and $u_{\omega,\delta^*} = \delta^* u_\omega$. Notice that $\delta^* \in (0,1]$ as $\underline{\omega} > 0$ in $S_+^{N-1}$ and satisfies Hopf boundary lemma on $\partial S_+^{N-1}$ Clearly $u_{\omega,\delta^*}$ is a subsolution for (2.1 ) and it is dominated by $u_\infty^H$ in $H$. Furthermore $\delta^*\omega \leq \underline{\omega}$ in $\overline{S_+^{N-1}}$, $\delta^*\omega_\phi \leq \underline{\omega}_\phi$ on $\partial S_+^{N-1}$, and

(i) either there exists $\sigma_0 \in S_+^{N-1}$ such that $\delta^*\omega(\sigma_0) = \underline{\omega}(\sigma_0)$,

(ii) or $\delta^*\omega < \underline{\omega}$ in $S_+^{N-1}$ and there exists $\sigma_0' \in S^{N-2}$ such that $\delta^*\omega_\phi(\sigma_0', \pi/2) = \underline{\omega}_\phi(\sigma_0', \pi/2)$.

In case (i), and as $Du_\infty^H$ never vanishes in $H$, it follows from [6, Lemma 1.3] (a variant of the strong comparison principle) that $u_{\omega,\delta^*} = \underline{u}$. This implies that $u_{\omega,\delta^*}$ is a solution, $\delta^* = 1$ and, consequently $\omega = \underline{\omega}$.

In case (ii) we follow the linearization procedure already introduced in [6]. By the mean value theorem

$$\left|Du_\infty^H\right|^{N-2} u_{\infty\, x_i} - \left|Du_{\omega,\delta^*}\right|^{N-2} u_{\omega,\delta\, x_i} = \sum_j \alpha_{ij}(u_\infty^H - u_{\omega,\delta^*})_{x_j}$$

where

$$\alpha_{ij} = \left|t_i Du_\infty^H + (1-t_i)Du_{\omega,\delta^*}\right|^{N-4} \Big(\delta_{ij}\left|t_i Du_\infty^H + (1-t_i)Du_{\omega,\delta^*}\right|^2 \\ +(N-2)\left(t_i u^H_{\infty\, x_i} + (1-t_i)u_{\omega,\delta^*\, x_i}\right)\left(t_i u^H_{\infty\, x_j} + (1-t_i)u_{\omega,\delta^*\, x_j}\right)\Big),$$

with $0 \leq t_i \leq 1$. Next $w = u_\infty^H - u_{\omega,\delta^*}$ is positive in $H$ and satisfies

$$-\sum_{ij}\left(\alpha_{ij} w_{x_j}\right)_{x_i} + cw \geq 0$$

where $c = ((u_\infty^H)^q - u^q_{\omega,\delta^*})/(u_\infty^H - u_{\omega,\delta^*}) > 0$. Notice that $(\alpha_{ij}(x))$ is the Hessian of a strictly convex function therefore it is nonnegative and that $(\alpha_{ij})(r,\sigma_0',\pi/2)$ is positive-definite. Therefore it is positive-definite in a neighborhood of $(r,\sigma_0',\pi/2)$ (independent of $r$, actually). Inasmuch $(u_\infty^H - u_{\omega,\delta^*})_{x_N} = 0$ at $(r,\sigma_0',\pi/2)$, we derive a contradiction with Hopf lemma. Therefore case (ii) cannot occur and $\omega = \underline{\omega}$. □

*Remark.* If we look for separable solutions of

$$-div\left(|Du|^{p-2}Du\right) + |u|^{q-1}u = 0, \tag{2.52}$$

in $\mathbb{R}^N$, where $q > p-1 > 0$, $p$ not necessarily equal to $N$ or to 2, under the form $u(r,\sigma) = r^\beta \omega(\sigma)$, then $\beta = \beta_{p,q} = -p/(q+1-p)$ and $\omega$ is a solution of

$$-div_\sigma \left( \left(\beta_{p,q}^2 \omega^2 + |\nabla_\sigma \omega|^2\right)^{(p-2)/2} \nabla_\sigma \omega \right) - \Lambda(p,q) \left(\beta_{p,q}^2 \omega^2 + |\nabla_\sigma \omega|^2\right)^{(p-2)/2} \omega + |w|^{q-1} \omega = 0 \tag{2.53}$$

on $S^{N-1}$ where $\Lambda(p,q) = \beta_{p,q}^{p-1}(q\beta_{p,q} - p)$. If we look for separable solutions in $H$ which vanishes on $\partial H \setminus \{0\}$ the solution $\omega$ of (2.53 ) is subject to the boundary condition $\omega = 0$ on $\partial S_+^{N-1} = S^{N-2}$. A fairly exhaustive theory of existence is developped in [22], [9]. The existence of non-trivial solution of (2.53 ) is insured as soon $\Lambda(p,q) > 0$, or equivalently $q < N(p-1)/(N-p)$ if $p < N$, and no condition if $p \geq N$. If $q \geq N(p-1)/(N-p)$ no solution exists, up to the trivial one. This is linked to the removability result proved by Vàzquez and Véron [18]. The existence of non trivial solutions of the same equation in $S_+^{N-1}$ vanishing on $\partial S_+^{N-1}$ is much more complicated. However it is proved in [22], [9] that there exists a critical exponent $q_c > p-1$ such that, if $q \geq q_c$ no non-trivial solution exists while if $p-1 < q < q_c$ there exist a unique positive solution in $S_+^{N-1}$ vanishing on $\partial S_+^{N-1}$. The uniqueness proof in the previous proposition is valid.

The next result characterizes the solution of (2.1 ) with a strong singularity on the boundary. In order to express the result, we assume that the outward normal unit vector to $\partial\Omega$ at $a$ is $-\mathbf{e}_N$.

**Theorem 2.9** *Let $\Omega$ be a bounded domain with a $C^2$ boundary and $a \in \partial\Omega$. Assume $0 < p-1 < q < 2N-1$. Then there exists a unique function $u \in C^1(\overline{\Omega} \setminus \{a\})$ which vanishes on $\partial\Omega \setminus \{a\}$, is solution of (2.1 ) in $\Omega$ and satisfies*

$$\lim_{x \to a} \frac{|x-a|^2 u(x)}{\rho(x)} = \infty. \tag{2.54}$$

*Furthermore*

$$\lim_{\substack{x \to a \\ (x-a)/|x-a| \to \sigma}} |x-a|^{\beta_q} u(x) = \omega(\sigma), \tag{2.55}$$

*locally uniformly on $S_+^{N-1}$. Finally $u = u_{\infty,a} = \lim_{k\to\infty} u_{k,a}$.*

*Proof.* We already know that $u_{\infty,a}$ satisfies (2.54 ). By translation we fix the origin 0 of coordinates at the point $a$ and we assume that $-\mathbf{e}_N$ is the outward unit vector to $\partial\Omega$ at 0. If $G$ is any $C^2$ domain in $\mathbb{R}^N$ to the boundary of which 0 belongs, we denote by $u_k^G$ the solution of (2.1 ) in $G$, which vanishes on $\partial G \setminus \{0\}$ and verifies

$$u_k^G = k \frac{\rho_G(x)}{|x|^2}(1 + \circ(1)) \quad \text{as } x \to 0, \tag{2.56}$$

where $\rho_G(x) = \text{dist}(x, G)$. When there is no ambiguity, $u_k^\Omega = u_k$. By the maximum principle $G \subset G'$ implies $u_k^G \leq u_k^{G'}$ in $G$. By dilation we can assume that there exist two balls of radius 1, $B \subset \Omega$ and $B' \subset \overline{\Omega}^c$ with respective center $b = \mathbf{e}_N$ and $b' = -b$ with the property that

$0 = \partial B \cap \partial B'$. It follows from the maximum principle, the fact that $u_k^B(x) = u_k^{B'}(\mathcal{S}(x))$ where $\mathcal{S}$ is the symmetry with respect to the hyperplane $\partial H$ and Proposition 2.4, Proposition 2.5

$$\begin{aligned}
&(i) \quad u_k^B(x) \le u_k(x) \le u_k^{B'^c}(x) \le u_k^{B'}\left(b' + \frac{x-b'}{|x-b'|^2}\right) = u_k^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \\
&\hspace{30em} \forall x \in B \\
&(ii) \quad u_k(x) \le u_k^{B'^c}(x) \le u_k^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in \Omega,
\end{aligned} \tag{2.57}$$

and similarly

$$\begin{aligned}
&(i) \quad u_k^B(x) \le u_k^H(x) \le u_k^{B'^c}(x) \le u_k^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in B \\
&(ii) \quad u_k^H(x) \le u_k^{B'^c}(x) \le u_k^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in H,
\end{aligned} \tag{2.58}$$

Letting $k \to \infty$, we obtain

$$\begin{aligned}
&(i) \quad u_\infty^B(x) \le u_\infty(x) \le u_\infty^{B'^c}(x) \le u_\infty^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in B \\
&(ii) \quad u_\infty(x) \le u_\infty^{B'^c}(x) \le u_\infty^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in \Omega,
\end{aligned} \tag{2.59}$$

as well as

$$\begin{aligned}
&(i) \quad u_\infty^B(x) \le |x|^{-\beta_q} \omega\left(\frac{x}{|x|}\right) \le u_\infty^{B'^c}(x) \le u_\infty^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in B \\
&(ii) \quad |x|^{-\beta_q} \omega\left(\frac{x}{|x|}\right) \le u_\infty^{B'^c}(x) \le u_\infty^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in H.
\end{aligned} \tag{2.60}$$

From (2.60 )-(i) and the fact that $b' = -b$, we also derive

$$|x|^{-\beta_q} \omega(x/|x|) \le u_\infty^{B'^c}(x) \le \left|\mathcal{S}\left(\frac{x+b}{|x+b|^2} - b\right)\right|^{-\beta_q} \omega\left(\frac{\mathcal{S}\left(x+b-|x+b|^2 b\right)}{\left|\mathcal{S}\left(x+b-|x+b|^2 b\right)\right|}\right). \tag{2.61}$$

But

$$\left|\mathcal{S}\left(\frac{x+b}{|x+b|^2} - b\right)\right| = \frac{|x|}{|x+b|} = |x|(1+\circ(1)) \quad \text{as } x \to 0$$

(remember that $|b| = 1$). If $x = (x_1, \dots, x_N)$, $|x+b|^2 = |x|^2 + 1 + 2x_N$ and

$$\mathcal{S}\left(x + b - |x+b|^2 b\right) = (x_1, \dots, x_N + |x|^2).$$

Thus (2.61 ) becomes

$$|x|^{-\beta_q}\,\omega(x/\,|x|) \leq u_\infty^{B'^c}(x) \leq |x|^{-\beta_q}\,|x+b|^{\beta_q}\,\omega\left(\frac{x+|x|^2\,\mathbf{e}_N}{|x|\,\sqrt{1+|x|^2+2x_N}}\right). \tag{2.62}$$

If we assume $|x|^2 = \circ(x_N)$ then $(x+|x|^2\,\mathbf{e}_N)/(|x|\,\sqrt{1+|x|^2+2x_N}) = x(1+\circ(1))/\,|x|$ as $x \to 0$, and

$$u_\infty^{B'^c}(x) = |x|^{-\beta_q}\,\omega(x/\,|x|)(1+\circ(1)). \tag{2.63}$$

If we define $\mathcal{T}$ by

$$\mathcal{T}(x) = \mathcal{S}\left(\frac{x+b}{|x+b|^2} - b\right),$$

then (2.60 )-(i) reads also as

$$\left|\mathcal{T}^{-1}(x)\right|^{-\beta_q}\omega\left(\frac{\mathcal{T}^{-1}(x)}{|\mathcal{T}^{-1}(x)|}\right) \leq u_\infty^{B}(x) \leq |x|^{-\beta_q}\,\omega\left(\frac{x}{|x|}\right). \tag{2.64}$$

Furthermore

$$\mathcal{T}^{-1}(x) = \left(\frac{x_1}{|x-b|^2},...,\frac{x_{N-1}}{|x-b|^2},\frac{1-x_N}{|x-b|^2} - 1\right) = \frac{x-|x|^2\,\mathbf{e}_N}{|x-b|^2}.$$

Then

$$\left|\mathcal{T}^{-1}(x)\right| = \left|b + \frac{x-b}{|x-b|^2}\right| = \frac{|x|}{|x-b|},$$

and

$$\left|\mathcal{T}^{-1}(x)\right|^{-\beta_q}\omega\left(\frac{\mathcal{T}^{-1}(x)}{|\mathcal{T}^{-1}(x)|}\right) = |x|^{-\beta_q}\,|x-b|^{\beta_q}\,\omega\left(\frac{x-|x|^2\,\mathbf{e}_N}{|x|\,|x-b|}\right).$$

If we assume again $|x|^2 = \circ(x_N)$ then $(x-|x|^2\,\mathbf{e}_N)/(|x|\,\sqrt{1+|x|^2-2x_N}) = x(1+\circ(1))/\,|x|$ as $x \to 0$, and

$$u_\infty^{B}(x) = |x|^{-\beta_q}\,\omega(x/\,|x|)(1+\circ(1)). \tag{2.65}$$

Combining (2.59 )-(i), (2.62 ) and (2.64 ) we obtain that

$$u_\infty(x) = |x|^{-\beta_q}\,\omega(x/\,|x|)(1+\circ(1)) \qquad \text{as } x \to 0 \tag{2.66}$$

uniformly on any subset of $\Omega$ such that $|x|^2 = \circ(x_N)$ near 0. In order to obtain the precise behaviour (2.55 ), we proceed and in the proof of Theorem 2.7. We extend $u$ by reflection through $\partial\Omega$ near 0 and denote by $\tilde{u}$ the extended function defined in $T^{\beta}(\Omega)$. For $r \in (0,1]$ we define

$$w_r := T_r(\tilde{u})(x) = r^{\beta_q}\tilde{u}(rx).$$

Then $w_r$ satisfies

$$-\sum_j \frac{\partial}{\partial x_j}\tilde{A}_j^r(x, Dw_r) + \tilde{b}(rx)\,|w_r|^{q-1}\,w_r = 0 \tag{2.67}$$

in $T^{\beta r^{-1}}(\Omega^r)$. Since $w_r$ is uniformly bounded on $\Gamma_{1/2,2}$ (by Proposition 2.1 applied to $u$ and $-u$) and the definition of the refected function), $Dw_r(u)$ is bounded in $C^\alpha(\Gamma_{2/3,3/2})$. By Ascoli's theorem $w_r$ converges in the $C^1(\Gamma_{2/3,3/2})$-topology to $x \mapsto |x|^{-\beta_q} \tilde{\omega}(x/|x|)$, where $\tilde{\omega}$ is defined from $\omega$ by reflection through the equator $\partial S_+^{N-1}$. In ordre to get rid of the boundary, we use again the $C^2$ diffeomorphism $\Psi$ which sends $B_{\epsilon_0}$ onto itself and verifies $\Psi(B_{\epsilon_0} \cap \partial\Omega) = B_{\epsilon_0} \cap \partial H$. We set $x = \Psi^{-1}(y)$ and $\tilde{u}(x) = u^*(y)$. Then

$$\lim_{|y|\to 0} (|y|^{\beta_q} u^*(|y|,\sigma), |y|^{\beta_q+1} \nabla u^*(|y|,\sigma)) = (\omega(\phi), -\omega_\phi\, \mathbf{e_N}), \tag{2.68}$$

uniformly on $S^{N-1}$, thus

$$u^*(|y|,\sigma) = |y|^{\beta_q} \omega(\phi)(1+\circ(1)) \quad \text{as} \quad |y| \to 0 \tag{2.69}$$

uniformly with respect to $\sigma \in S_+^{N-1}$, because $u^*$ vanishes on $B_{\epsilon_0} \cap \partial H \setminus \{0\}$. Actually, a stronger result than (2.55 ) follows, namely

$$u(x) = |x|^{-\beta_q} \omega(x/|x|)(1+\circ 1)) \quad \text{as } x \to 0. \tag{2.70}$$

*Mutatis mutandis*, this estimate implies uniqueness of a solution with a strong singularity as in Theorem 2.7. □

# 3 The removability result

In this section $\Omega$ is a $C^2$ domain of $\mathbb{R}^N$ and $a \in \partial\Omega$. The next result extends the removability result of Gmira-Véron [7] dealing with semilinear equations.

**Theorem 3.1** *Let $g$ be a continuous function defined on $\mathbb{R}$ which satisfies*

$$\liminf_{r\to\infty} g(r)/r^{q_c} > 0 \quad \textit{and} \quad \limsup_{r\to-\infty} g(r)/|r|^{q_c} < 0, \tag{3.1}$$

*where $q_c := 2N-1$ and let $u \in C^1(\overline{\Omega} \setminus \{a\})$ be a solution of*

$$-div\left(|Du|^{N-2} Du\right) + g(u) = 0 \quad \textit{in } \Omega \tag{3.2}$$

*which coincides with some $\phi \in C^1(\partial\Omega)$ on $\partial\Omega \setminus \{a\}$. Then $u$ extends to $\overline{\Omega}$ as a continuous function.*

*Proof.* Without any loss of generality, we can assume that $\Omega$ is bounded, $a = 0$ and $-\mathbf{e}_N$ is the outward normal vector to $\partial\Omega$ at 0. We denote by $V_0^\Omega$ the solution of (2.2 ) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies

$$V_0^\Omega(x) = \frac{\rho(x)}{|x|^2}(1+\circ(1)) \quad \text{as } x \to 0.$$

Let $M$ be the supremum of $|\phi|$ on $\partial\Omega$ and $\tilde{M} = \max\{M, (B/A)^{1/q}\}$. By assumption there exists $A > 0$ and $B \geq 0$, depending only on $g$, such that

$$-div\left(|Du|^{N-2} Du\right) + Au^{q_c} \leq B \quad \text{in } \{x \in \Omega : u(x) > 0\}. \tag{3.3}$$

If $v = u - \tilde{M}$, then $v \leq 0$ on $\partial\Omega \setminus \{0\}$ and

$$-div\left(|Dv|^{N-2}\,Dv\right) + Av^{q_c} \leq 0 \quad \text{in } \{x \in \Omega : v(x) > 0\}, \tag{3.4}$$

Using the same functions $\eta_\epsilon$ as in the proof of Proposition 2.1 we deduce that $\eta_\epsilon(v)$ satisfies the same inequality as $v$, but on whole $\Omega$. By Proposition 2.2 with $q = q_c$ and the expression of $V_0^\Omega$ it follows that

$$v(x) \leq cV^\Omega(x) \quad \forall x \in \Omega, \tag{3.5}$$

where the constant $c$ depends on $A$ and $N$. Furthermore, there exists a function $u^* \in C^1(\overline{\Omega} \setminus \{0\})$ such that $0 \leq v_+ \leq u^*(x) \leq cV_0^\Omega$ in $\Omega$, and

$$-div\left(|Du^*|^{N-2}\,Du^*\right) + A{u^*}^{q_c} = 0 \quad \text{in } \Omega. \tag{3.6}$$

As in the proof of Theorem 2.9 we extend $u^*$ through the boundary into $\tilde{u}$ and scale it by setting $T_r(\tilde{u}) := w_r(x) = r\tilde{u}(rx)$ for $r > 0$. Inasmuch all the previous a priori estimates apply (compactness), it follows that there exists a subsequence $\{r_n\}$ converging to 0 and a function $w \in C^1(\mathbb{R}^N \setminus \{0\})$ such that $w_{r_n} \to w$ in the $C^1_{loc}$-topology of $\mathbb{R}^N \setminus \{0\}$, $w$ is a solution of

$$\left\{\begin{array}{r} -div\left(|Dw|^{N-2}\,Dw\right) + Aw^{q_c} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\} \\ w \geq 0 \;\text{ in } H = \{x \in \mathbb{R}^N : x_N > 0\} \\ w = 0 \;\text{ on } \partial H \setminus \{0\}. \end{array}\right. \tag{3.7}$$

At end, (3.5 ) transforms into

$$0 \leq w(x) \leq c\frac{x_N}{|x|^2} \quad \forall x \in H. \tag{3.8}$$

For $\epsilon > 0$ we denote by $W_\epsilon$ the solution of

$$\left\{\begin{array}{r} -div\left(|DW_\epsilon|^{N-2}\,DW_\epsilon\right) + AW_\epsilon^{q_c} = 0 \quad \text{in } H \setminus B_\epsilon(0) \\ W_\epsilon = c\epsilon^{-2}x_N \;\text{ on } H \cap \partial B_\epsilon(0) \\ W_\epsilon = 0 \;\text{ on } \partial H \setminus B_\epsilon(0). \end{array}\right. \tag{3.9}$$

By the maximum principle $0 \leq w(x) \leq W_\epsilon(x) \leq cx_N\,|x|^{-2}$ for any $\epsilon > 0$, and by uniqueness, $T_r(W_\epsilon)(x) = rW_\epsilon(rx) = W_{\epsilon/r}(x)$. Furthermore $\epsilon \mapsto W_\epsilon$ is increasing. Letting $\epsilon \to 0$ we conclude that $W_\epsilon$ decreases to some $W_0$, which is a solution of

$$\left\{\begin{array}{r} -div\left(|DW_0|^{N-2}\,DW_0\right) + AW_0^{q_c} = 0 \quad \text{in } H \\ W_0 \geq 0 \;\text{ in } H \\ W_0 = 0 \;\text{ on } \partial H \setminus \{0\}, \end{array}\right. \tag{3.10}$$

by the standard regularity results, and satisfies $0 \leq w \leq W_0$. Finally, $W_0$ inherits the following scaling invariance property $T_r(W_0)(x) = W_0(x)$ for any $r > 0$. Therefore $W_0$ is a separable solution which endows the following form

$$W_0(x) = W_0(r, \sigma) = r^{-1}\omega(\sigma),$$

where $\omega$ is nonnegative on $S_+^{N-1}$ and satisfies

$$\begin{cases} -div_\sigma \left( \left(\omega^2 + |\nabla_\sigma \omega|^2\right)^{(N-2)/2} \nabla_\sigma \omega \right) - (N-1) \left(\omega^2 + |\nabla_\sigma \omega|^2\right)^{(N-2)/2} \omega + A\omega^{q_c} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } S_+^{N-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \omega = 0 \quad \text{on } \partial S_+^{N-1}. \end{cases} \tag{3.11}$$

By Proposition 2.8, $\omega = 0$. Thus $W_0 = 0 \Longrightarrow w = 0$, which implies $w_r(x) \to 0$ as $r \to 0$ and equivalently $r\tilde{u}(rx) \to 0$ in the $C^1_{loc}$-topology of $\mathbb{R}^N \setminus \{0\}$. Consequently $D\tilde{u}(x) = \circ(|x|^{-2})$ as $x \to 0$ and finally $u^*(x) = \circ(V_0^\Omega(x))$ as $x \to 0$. The maximum principle and the positivity of $u^*$ yields to $u^* \equiv 0$ and finally $u \leq \tilde{M}$ in $\Omega$. In the same way $u \geq -\tilde{M}$. Because the modulus of continuity of $u$ is uniformly bounded near 0, by the classical regularity theory of degenerate elliptic equations (see [12] for example), $u$ extends as a continuous function in whole $\overline{\Omega}$. □

# 4 The classification theorem

The next result extends some of Gmira-Véron's classification theorem [7, Sect. 4, 5 ] obtained in the study of problem (1.3 ). In the above mentioned article, the main idea was to reduce the equation to a infinite dimensional quasi-autonomous evolution system in $\mathbb{R}_+ \times S_+^{N-1}$ and to use Lyapounov-energy function. Such an approach cannot be adapted in the quasilinear case. Our method is based upon scaling and uniqueness arguments.

**Theorem 4.1** *Assume $N-1 < q < 2N-1$, $\Omega$ is a bounded domain with a $C^2$ boundary, $a \in \partial\Omega$ and $-\mathbf{e}_N$ is the outward normal unit vector to $\partial\Omega$ at $a$. Let $u \in C^1(\overline{\Omega} \setminus \{a\})$ be a positive function satisfying (2.1 ) in $\Omega$ and vanishing on $\partial\Omega \setminus \{a\}$. Then the following alternative holds.*

*(i) Either there exists $k > 0$ such that*

$$u(x) = k\frac{\rho(x)}{|x-a|^2}(1 + \circ(1)) \quad \text{as } x \to a. \tag{4.12}$$

*Furthermore $u = u_{k,a}$, the unique solution of (2.1 ) defined in Theorem 2.7.*

*(ii) Or*

$$u(x) = |x-a|^{-\beta_q}\,\omega(\sigma)(1 + \circ(1)) \quad \text{as } x \to a. \tag{4.13}$$

*where $\omega$ is the unique positive solution of (2.51 ) on $S_+^{N-1}$ which vanishes on $\partial S_+^{N-1}$, in which case $u = u_{\infty,a}$.*

*Proof.* We assume that $a = 0$ with $\nu_0 = -\mathbf{e_N}$ and define

$$k = \limsup_{x\to 0} \frac{u(x)}{V_0^\Omega(x)} = \limsup_{r\to 0} \sup_{|x|=r} \frac{u(x)}{V_0^\Omega(x)}. \tag{4.14}$$

Suppose $k = 0$. It follows from the maximum principle that for any $\epsilon > 0$ there exists a sequence $r_n \to 0$ such that $0 \le u(x) \le \epsilon V_0^\Omega(x)$ in $\Omega \setminus \{B_{r_n}(0)\}$. This fact implies the nullity of $u$. Therefore we assume that $k \ne 0$. Assume first that $k$ is finite. Then, for any $\epsilon > 0$, there exists a sequence of points $x_n$ converging to 0 such that

$$\lim_{n\to\infty} \frac{u(x_n)}{V_0^\Omega(x_n)} = k \tag{4.15}$$

and

$$\sup_{|x|\le r_n} \frac{u(x)}{V_0^\Omega(x)} \le k + \epsilon. \tag{4.16}$$

Since $u_k$ satisfies (2.40 ) with $a = 0$, the two previous relations can be replaced by

$$\begin{array}{lll} (i) & \lim_{n\to\infty} \dfrac{u(x_n)}{u_k(x_n)} = 1 \\ (ii) & \sup_{|x|\le r_n} \dfrac{u(x)}{u_k(x)} \le 1 + \epsilon. \end{array} \tag{4.17}$$

We denote $r_n = |x_n|$, $\xi_n = x_n/r_n$ and define $u_n = r_n u(r_n x)$ and $u_{k\,n} = r_n u_k(r_n x)$. By the previous arguments combining a priori estimate and regularity theory, there exist a subsequence $\{r_{n_j}\}$ and two nonnegative functions $v$ and $v'$, $N$-harmonic in $H$ and vanishing on $\partial H \setminus \{0\}$, such that $\left(u_{n_j}, u_{k\,n_j}\right)$ converges to $(v, v')$ in the $C^1_{loc}$-topology of $H = \mathbb{R}^N_+$. Clearly equality (2.40 ) implies that $r_{n_j} u_k(r_{n_j} x)$ converges to $kV_0^H(x)$ (which is defined by $kV_0^H(x) := kx_N / |x|^{-2}$) in the same topology. Since $v'$ is uniquely determined by its blow-up at 0, this implies $v' = kV_0^H$ in $H$. Furthermore there exists $\xi \in \overline{S_+^{N-1}}$ such that $\xi_{n_k} \to \xi$. If $\xi \in S_+^{N-1}$, $v(\xi) = v'(\xi)$, while, if $\xi \in \partial S_+^{N-1}$, $\partial v/\partial\nu(\xi) = \partial_{x_N} v(\xi) = \partial v'/\partial\nu(\xi)$. In both situation, the tangency conditions of the graphs of $v$ and $v'$ and the strong maximum principle implies that $v = v' = kV_0^H$. By estimate (4.17 )-(i) and the convergence properties, it follows

$$\lim_{n\to\infty} \frac{u(r_n\xi)}{u_k(r_n\xi)} = 1, \quad \text{uniformly on } |\xi| = 1.$$

Consequently, for any $\delta > 0$, there holds,

$$(1-\delta)u_k(x) \le u(x) \le (1+\delta)u_k(x) \quad \forall x \in \Omega \setminus B_{r_n},$$

for $n$ large enough, which leads to $u_k = u$. At end we consider the case $k = \infty$. Writting (2.1 ) under the form

$$-div\left(|Du|^{N-2} Du\right) + d(x)u^{N-1} = 0 \tag{4.18}$$

where $d(x) = |u|^{q+1-N}(x) \le C\,|x|^{-N}$, by (2.4 ), We use the boundary Harnack principle. By [2, Th 2.2] there exists a constant $c = c(N, q, \Omega) > 0$ such that

$$\frac{1}{c}\frac{u(y)}{\rho(y)} \le \frac{u(x)}{\rho(x)} \le c\frac{u(y)}{\rho(y)} \tag{4.19}$$

for any $x$ and $y$ in $\Omega$ such that $|x| = |y|$ be small enough. Since there exists a sequence $x_n \to 0$ such that $\lim_{n\to\infty} u(x_n)/V^\Omega(x_{n)} \to \infty$, this implies that

$$\lim_{n\to\infty} \left\{ \inf \frac{u(x)}{V^\Omega(x)} : |x| = |x_n| \right\} = \infty. \tag{4.20}$$

Thus $u$ satisfies (2.54 ); Theorem 2.9 and (2.55 ) imply that (4.13 ) holds. □

The assumption of positivity on $u$ can be weakened if a better a priori estimate is already known. The next result extends [6, Th 1.2] into the framework of boundary singularities.

**Theorem 4.2** *Assume $N - 1 < q < 2N - 1$, $\Omega$ is a bounded domain with a $C^2$ boundary, $a \in \partial\Omega$ and $-\mathbf{e}_N$ is the outward normal unit vector to $\partial\Omega$ at $a$. Let $u \in C^1(\overline{\Omega} \setminus \{a\})$ be a solution of (2.1 ) in $\Omega$ vanishing on $\partial\Omega \setminus \{a\}$ such that $u/V_a^\Omega$ is bounded in $\Omega$. Then there exists $k \in \mathbb{R}$ such that $u = u_{k,a}$.*

*Proof.* The outline of the proof are very similar to the finite case of the previous theorem. We still assume $a = 0$ and define $k$ by (4.14 ). If $k = 0$ the maximum principle implies $u \le 0$ and we return to Theorem 4.1 in the case $u \le 0$. If $k \ne 0$, $k > 0$ for example, (4.15 ) and (4.16 ) apply. By the previous scaling method we derive that $u_{n_k}$ converges to some function $v$ in the $C^1_{loc}$-topology of $H = \mathbb{R}^N_+$ which is $N$-harmonic in $H$ and vanishes on $\partial H \setminus \{0\}$. Because $r_{n_k} u_k(r_{n_k} x)$ converges to $kV_0^H$, the tangency condition of $v$ and $kV_0^H$ at some $\xi$ implies that $v = kV_0^H$. Thus $u(x) \ge 0$ for $|x| = r_{n_k}$ for $n_k$ large enough. This implies that $u \ge 0$ in $\Omega$ and we are back to Theorem 4.1. □

*Remark.* In the semilinear case of problem (1.3 ), it is proved in [7] that any signed solution $u$ which satisfies $\lim_{x\to a} |x - a|^N u(x) = 0$ has constant sign. The exponent $N$ characterize the minimal changing sign harmonic function vanishing on $\partial\Omega$, with an isolated singularity at $a$. Changing sign singular $N$-harmonic functions are constructed in [3]. In particular there exist singular $N$-harmonic functions $w$ under the form

$$w(r, \sigma) = r^{-\beta_2} \omega(\sigma)$$

where

$$\beta_2 = \frac{7N - 1 + \sqrt{N^2 + 12N + 12}}{6(N - 1)}$$

and $\omega$ is defined on $S_+^{N-1} = \{x \in S^{N-1} : x_N > 0\}$, vanishes on the equator $\partial S_+^{N-1}$, is positive on $S_+^{N-1} \cap \{x : x_{N-1} > 0\}$ and negative on $S_+^{N-1} \cap \{x : x_{N-1} < 0\}$. A natural question is therefore wether any signed solution $u$ of (2.1 ) in $\Omega$ which vanishes on $\partial\Omega \setminus \{a\}$ and satisfies $\lim_{x\to a} |x - a|^{\beta_2} u(x) = 0$ has constant sign, and can be henceforth classified through Theorem 4.1.

*Final remark.* If one replaces the $N$-harmonic operator by the $p$-harmonic operator ($p > 1$) and tries to extend the results of sections 2, 3, 4, several difficulties will appear. Even if the existence of separable singular solutions is known, the precise value of the exponent $\beta > 0$ such that $(r, \sigma) \mapsto r^{-\beta}\phi(\sigma)$ is $p$-harmonic and positive in $H$ and vanishes on $\partial H \setminus \{0\}$ is unknown but for the specific cases $N = 2$ or $p = N$ or $p = 2$. Notice that in that case the function $\phi$ satisfies the so-called *spherical p-harmonic spectral equation*

$$\begin{cases} -div_\sigma\left(\left(\beta^2\phi^2 + |\nabla_\sigma\phi|^2\right)^{(p-2)/2}\nabla_\sigma\phi\right) - \lambda\left(\beta^2\phi^2 + |\nabla_\sigma\phi|^2\right)^{(p-2)/2}\phi = 0 & \text{in } S_+^{N-1} \\ \phi = 0 & \text{on } \partial S_+^{N-1}. \end{cases} \tag{4.21}$$

where $\lambda = \beta(\beta(p-1) + p - N)$. If $p = 2$ then $\beta = N - 1$, while if $N = 2$, $\beta$ is the positive root of the equation

$$3\beta^2 + 2\frac{p-3}{p-1}\beta - 1 = 0. \tag{4.22}$$

Furthermore, up to now and due to the lack of conformal invariance, it has not been possible to construct the equivalent of the $V_a^\Omega$ in a general smooth bounded domain $\Omega$, that are positive $p$-harmonic functions in $\Omega$, vanishing on $\partial\Omega \setminus \{a\}$ and satisfying

$$\lim_{\substack{x \to a \\ (x-a)/|x-a| \to \sigma}} |x-a|^\beta u(x) = \phi(\sigma). \tag{4.23}$$

However, if $\Omega = H = \mathbb{R}_+^N$ the removability and the classification results of Sections 3 and 4 are still valid. The proofs of these theorems are developed in [1].

# References


[1] Borghol R., *Singularités au bord de solutions d'équations quasilinéaires*, Thèse de Doctorat, Univ. Tours, (2005).

[2] Bidaut-Véron M. F., Borghol R. & Véron L., *Boundary Harnack inequalities and a priori estimates of singular solutions of quasilinear equations*, Calc. Var. and P. D. E., to appear.

[3] Borghol R. & Véron L., *Boundary singularties of N-harmonic functions*, Comm. Part. Diff. Equ., to appear.

[4] Dynkin E.B. and Kuznetsov S.E., *Trace on the boundary for solutions of nonlinear differential equations*, Trans. A.M.S. **350**, 4499-4519 (1998).

[5] Dynkin E.B. and Kuznetsov S.E., *Solutions of nonlinear differential equations on a Riemannian manifold and their trace on the Martin boundary*, Trans. A.M.S. **350**, 4521-4552 (1998).

[6] Friedman A., & Véron L., *Singular solutions of some quasilinear elliptic equations*, Arch. Rat. Mech. Anal. **96**, 359-387 (1986).

[7] Gmira A.& Véron L., *Boundary singularities of solutions of some nonlinear elliptic equations*, Duke Math. J. **64**, 271-324 (1991).

[8] Kichenassamy S. & Véron L., *Singular solutions of the p-Laplace equation*, Math. Ann. **275**, 599-615 (1986).

[9] Huentutripay J., Jazar M. & Véron L., A dynamical system approach to the construction of singular solutions of some degenerate elliptic equations, J. Diff. Equ. **195**, 175-193 (2003).

[10] Krol I. N., *The behaviour of the solutions of a certain quasilinear equation near zero cusps of the boundary*, Proc. Steklov Inst. Math. **125**, 130-136 (1973).

[11] Le Gall J. F., *The brownian snake and solutions of* $\Delta u = u^2$ *in a domain*, Prob. Theory Rel. Fields **102**, 393-432 (1995).

[12] Libermann G, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12**, 1203-1219 (1988).

[13] Marcus M.& Véron L., *The boundary trace of positive solutions of semilinear elliptic equations : the subcritical case*, Arch. Rat. Mech. Anal. **144**, 201-231 (1998).

[14] Serrin J., *Local behaviour of solutions of quasilinear equations*, Acta Math. **111**, 247-302 (1964).

[15] Serrin J. & Zou H., *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189**, 79-142 (2002).

[16] Tolksdorff P., *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Part. Diff. Equ. **8**, 773-817 (1983).

[17] Tolksdorff P., *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Equ. **51**, 126-140 (1984).

[18] Vàzquez J. L. & Véron L., *Removable singularities of some strongly nonlinear elliptic equations*, Manuscripta **33**, 129-144 (1980).

[19] Véron L., *Some existence and uniqueness results for solution of some quasilinear elliptic equations on compact Riemannian manifolds*, Colloquia Mathematica Societatis János Bolyai **62**, 317-352 (1991).

[20] Véron L., *Singularities of solutions of second order quasilinear elliptic equations*, Pitman Research Notes in Math. **353**, Addison-Wesley- Longman (1996).

[21] Véron L., *Singularities of some quasilinear equations*, Nonlinear diffusion equations and their equilibrium states, II (Berkeley, CA, 1986), 333-365, Math. Sci. Res. Inst. Publ., **13**, Springer, New York (1988).

[22] Véron L., *Singular p-harmonic functions and related quasilinear equations on manifolds*, Electron. J. Differ. Equ. Conf., **8**, 133-154 (electronic) (2002).

Laboratoire de Mathématiques et Physique Théorique
CNRS UMR 6083
Faculté des Sciences
Université François-Rabelais
F37200 Tours France

borghol@univ-tours.fr
veronl@lmpt.univ-tours.fr