Diffusion versus absorption in semilinear parabolic equations

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Partial Differential Equations/*Équations aux dérivées partielles*

Diffusion versus absorption in semilinear parabolic problems[1]

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Abstract. We study the limit, when $k \to \infty$, of the solutions $u = u_k$ of (E) $\partial_t u - \Delta u + h(t)u^q = 0$ in $\mathbb{R}^N \times (0,\infty)$, $u_k(.,0) = k\delta_0$, with $q > 1$, $h(t) > 0$. If $h(t) = e^{-\omega(t)/t}$ where $\omega > 0$ satisfies to $\int_0^1 \sqrt{\omega(t)}t^{-1}dt < \infty$, the limit function u_∞ is a solution of (E) with a single singularity at $(0,0)$, while if $\omega(t) \equiv 1$, u_∞ is the maximal solution of (E). We examine similar questions for equations such as $\partial_t u - \Delta u^m + h(t)u^q = 0$ with $m > 1$ and $\partial_t u - \Delta u + h(t)e^u = 0$.

Diffusion versus absorption dans des problèmes paraboliques semi-linéaires

Résumé. Nous étudions la limite, quand $k \to \infty$, des solutions $u = u_k$ de (E) $\partial_t u - \Delta u + h(t)u^q = 0$ dans $\mathbb{R}^N \times (0,\infty)$, $u_k(.,0) = k\delta_0$ avec $q > 1$, $h(t) > 0$. Nous montrons que si $h(t) = e^{-\omega(t)/t}$ où $\omega > 0$ vérifie $\int_0^1 \sqrt{\omega(t)}t^{-1}dt < \infty$, la fonction limite u_∞ est une solution of (E) avec une singularité isolée en $(0,0)$, alors que si $\omega(t) \equiv 1$, u_∞ est la solution maximale de (E). Nous examinons des questions semblables pour des équations des type suivants $\partial_t u - \Delta u^m + h(t)u^q = 0$ avec $m > 1$ et $\partial_t u - \Delta u + h(t)e^u = 0$.

Version française abrégée

Soit $q > 1$ et $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ une fonction continue, croissante telle que $h(t) > 0$ pour $t > 0$. Il est facile de vérifier que toute solution positive u de

$$(1) \qquad \partial_t u - \Delta u + h(t)u^q = 0 \quad \text{dans } \mathbb{R}^N \times]0,+\infty[$$

satisfait à

$$(2) \qquad u(x,t) \leq U(t) := \left((q-1)\int_0^t h(s)\,ds\right)^{-1/(q-1)} \qquad \forall (x,t) \in \mathbb{R}^N \times]0,+\infty[.$$

Si $h \in L^1(0,1,t^{Nq/2}dt)$, il est classique que pour tout $k > 0$ il existe une unique solution (dite fondamentale) $u = u_k$ de (1) sur $\mathbb{R}^N \times]0,+\infty[$ vérifiant $u_k(.,0) = k\delta_0$. Par le principe du maximum $k \mapsto u_k$ est croissant et deux cas peuvent se produire:
(i) ou bien $u_\infty = \lim_{k\to\infty} u_k = U$. *Explosion initiale complète.*
(ii) ou bien u_∞ est une solution de (1) singulière en $(0,0)$ vérifiant $\lim_{t\to 0} u_\infty(x,t)$

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$= 0$ pour tout $x \neq 0$. *Explosion initiale ponctuelle.*

Theorem 1. *(I) Si $h(t) = e^{-\sigma/t}$ pour un $\sigma > 0$, alors $u_\infty = U$.*
(II) Si $h(t) = e^{-\omega(t)/t}$ où ω est monotone croissante sur $]0, +\infty[$ et vérifie, pour un $\alpha \in [0, 1[$, $\inf\{\omega(t)/t^\alpha : 0 < t \leq 1\} > 0$ et $\int_0^1 \sqrt{\omega(t)}\, t^{-1}\, dt < \infty$, alors u_∞ a une explosion initiale ponctuelle.

Dans le cas de l'équation

$$\partial_t u - \Delta u + h(t)e^u = 0 \quad \text{dans } \mathbb{R}^N \times]0, +\infty[, \tag{3}$$

toute solution u satisfait à

$$u(x,t) \leq \tilde{U}(t) := -\ln\left(\int_0^t h(s)\, ds\right) \quad \forall (x,t) \in \mathbb{R}^N \times]0, +\infty[, \tag{4}$$

et l'existence d'une solution fondamentale $u = u_k$ est assurée si $h(t) = e^{-b(t)}$ avec $\lim_{t\to+\infty} t^{N/2} b(t) = +\infty$.

Theorem 2.*(I) Si $h(t) = e^{-e^{\sigma/t}}$ pour un $\sigma > 0$, alors $u_\infty = \tilde{U}$.*
(II) Si $h(t) = e^{-e^{\omega(t)/t}}$ où ω vérifie les conditions du Théorème 1, alors u_∞ a une explosion initiale ponctuelle.

Nos méthodes nous permettent ausi de traiter l'équation des milieux poreux avec absorption.

Main results

Let $q > 1$ and $h : (0, \infty) \mapsto (0, \infty)$ be a continuous nondecreasing function. It is easy to prove that any positive solution u of

$$\partial_t u - \Delta u + h(t)u^q = 0 \quad \text{dans } \mathbb{R}^N \times (0, +\infty) \tag{1}$$

verifies

$$u(x,t) \leq U(t) := \left((q-1)\int_0^t h(s)\, ds\right)^{-1/(q-1)} \quad \forall (x,t) \in \mathbb{R}^N \times (0, \infty). \tag{2}$$

If $h \in L^1(0, 1, t^{Nq/2}dt)$, it is classical that, for any $k > 0$, there exists a unique solution (called fundamental) $u = u_k$ of (1) sur $\mathbb{R}^N \times (0, \infty)$ such that $u_k(., 0) = k\delta_0$. By the maximum principle $k \mapsto u_k$ is increasing and the following alternative occurs:
(i) either $u_\infty = \lim_{k\to\infty} u_k = U$. *Complete initial blow-up.*
(ii) or u_∞ is a solution of (1) singular at $(0, 0)$ such that
$\lim_{t\to 0} u_\infty(x,t) = 0$ for all $x \neq 0$. *Single-point initial blow-up.*

Theorem 1. *(I) If $h(t) = e^{-\sigma/t}$ for some $\sigma > 0$, then $u_\infty = U$.*
(II) If $h(t) = e^{-\omega(t)/t}$ where ω is nondecreasing on $(0, +\infty)$ and verifies, for some $\alpha \in [0, 1)$, $\inf\{\omega(t)/t^\alpha : 0 < t \leq 1\} > 0$ and

$$\int_0^1 \frac{\sqrt{\omega(t)}\, dt}{t} < +\infty, \tag{3}$$

then u_∞ has single-point initial blow-up.

Concerning equation

$$\partial_t u - \Delta u + h(t)e^u = 0 \quad \text{dans } \mathbb{R}^N \times (0, +\infty), \tag{4}$$

any solution u verifies

$$u(x,t) \le \tilde{U}(t) := -\ln\left(\int_0^t h(s)\,ds\right) \quad \forall (x,t) \in \mathbb{R}^N \times (0,+\infty). \tag{5}$$

and the existence of a fundamental solution $u = u_k$ is ensured if $h(t) = e^{-b(t)}$ where $\lim_{t\to+\infty} t^{N/2} b(t) = +\infty$.

Theorem 2. *(I) If $h(t) = O(e^{-e^{\sigma/t}})$ for some $\sigma > 0$, then $u_\infty = \tilde{U}$.*
(II) If $h(t) = e^{-e^{\omega(t)/t}}$ where ω satisfies the conditions of Theorem 1, then u_∞ has single-point initial blow-up.

Our methods apply to equations of porous media type

$$\partial_t u - \Delta u^m + h(t)u^q = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{6}$$

with $m > 1$, $q > 1$ and $h : (0,\infty) \mapsto (0,\infty)$ is nondecreasing. As above, any positive solution satisfies (2). If $h \in L^1((0,1; t^{-(q-1)/(m-1+2N^{-1})}dt)$, for any $k > 0$ there exists a solution $u = u_k$ of (6) such that $u_k(.,0) = k\delta_0$. Since $k \mapsto u_k$ is increasing, the same alternative as in case of (1) occurs concerning u_∞.

Theorem 3. *Assume $q > m > 1$. (I) If $h(t) = O(t^{(q-m)/(m-1)})$, then $u_\infty = U$.*
(II) If $h(t) = t^{(q-m)/(m-1)}\omega^{-1}(t)$ where ω is nondecreasing and positive on $(0,+\infty)$ and verifies

$$\int_0^1 \frac{\omega^\theta(t)\,dt}{t} < +\infty, \tag{7}$$

where

$$\theta = \frac{m^2-1}{(N(m-1)+2(m+1))\,(q-1)},$$

then u_∞ has single-point initial blow-up.

Sketch of the proofs. The complete initial blow-up results are proved by constructing local subsolutions by modifying the very singular solutions of some related equations. Since for equation (1), the proof is already given in [3] we shall outline the (more complicated) construction for equation (4).

Lemma 4. *If $h(t) = \sigma t^{-2} e^{\sigma t^{-1} - e^{-\sigma/t}}$ for some $\sigma > 0$, complete initial blow-up occurs for equation (4).*

Proof. Writing $h(t) = e^{-a(t)}$ is is first observed that fundamental solutions u_k of (4) exist for all $k > 0$ if $\lim_{t\to 0} t^{N/2}a(t) = \infty$. For $\ell > 1$, let $v = v_{\infty,\ell}$ be the very singular solution of

$$\partial_t v - \Delta v + ct^{\alpha_\ell} v^\ell = 0 \tag{8}$$

in $\mathbb{R}^N \times (0,\infty)$, where α_ℓ and c are positive constants. The choice of $\alpha_\ell = (N+2)/(\ell-1)/2 - 1$ ensures the existence of $v_{\infty,\ell}$. Furthermore, if we write

$$v_{\infty,\ell}(x,t) = \left(\frac{2c}{N+2}\right)^{1/(\ell-1)} t^{-(1+N/2)} f_\ell(x/\sqrt{t}),$$

then $f_\ell(\eta) \le 1$ for $\eta \in \mathbb{R}^N$ and

$$\Delta f_\ell + \frac{1}{2} Df_\ell.\eta + \frac{N+2}{2} f_\ell - f_\ell^\ell = 0. \tag{9}$$

By the maximum principle $0 < f_\ell < f_{\ell'} \leq 1$ for $\ell' > \ell > 1$. For the particular choice $\ell^* = (N+4)/(N2)$, we can use the expression of the asymptotic expansion of the very singular solution given in [1],

$$f_{\ell^*}(\eta) = C|\eta|^2 e^{-|\eta|^2/4}(1 + \circ(1)) \text{ as } |\eta| \to \infty,$$

from which follows $f_\ell(\eta) \geq f_{\ell^*}(\eta) \geq \delta^*(|\eta|^2 + 1)e^{-|\eta|^2/4}$ for some $\delta^* > 0$, any $\eta \in \mathbb{R}^N$ and $\ell \geq \ell^*$. Thus there exists $\delta > 0$ depending only on N such that

$$v_{\infty,\ell}(x,t) \geq \delta c^{1/(\ell-1)} t^{-1-N/2}(|x|^2 + t)e^{-|x|^2/4t} \quad \forall (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{10}$$

Because any positive solution u of (4) satisfies (5), we have to prove that we can fix c and $\tau > 0$ such that

$$ct^{\alpha_\ell}(\rho^\ell + 1) \geq h(t)e^\rho \quad \forall (t,\rho) \in (0,\tau] \times [0, \tilde{V}(t)]. \tag{11}$$

Writing h under the form $h(t) = -\omega'(t)e^{\omega(t)}$ where $\omega(t) = e^{\gamma(t)}$ and γ is a positive decreasing C^1 function, infinite at $t = 0$, we first notice that it is sufficient to prove this inequality for $\rho = \tilde{U}(t)$, and in that case

$$ct^{\alpha_\ell}(e^{\ell\gamma(t)} + 1) \geq -\gamma'(t)e^{\gamma(t)} \quad \forall t \in (0,\tau]. \tag{12}$$

We take now $\gamma(t) = \sigma/t$, and prove that there exists $\beta > 0$, depending only on N such that, for any $0 < \tau \leq \beta\sigma$, estimate (11) holds with

$$c = e^{(1-\ell)\sigma/\tau - 2^{-1}(\ell(N+2)-N)\ln\tau}.$$

The maximum principle and (11) imply that for any $\ell > 1$ and $k > 0$ the solutions $u = u_k$ of (4) and $v = \tilde{v}_k$ of

$$\partial_t v - \Delta v + ct^{\alpha_\ell}(v^\ell + 1) = 0$$

with initial data $k\delta_0$ verifies $0 \leq \tilde{v}_{k,\ell} \leq u_k$, on $(0,\tau]$. Therefore $v_{\infty,\ell} \leq u_\infty + ct^{\alpha_\ell+1}/(\alpha_\ell + 1)$ on $(0,\tau]$ leads to

$$u_\infty(x,\tau) \geq \delta\tau^{-1-N/2}(|x|^2 + \tau)e^{\frac{4\sigma-|x|^2}{4\tau} - \frac{\ell(N+2)-N}{2(1-\ell)}\ln\tau}$$

Thus $\lim_{\tau\to 0} u_\infty(x,\tau) = \infty$, locally uniformly in $B_{2\sqrt{\sigma}}$, which implies the result.

The proof of Theorem 2 follows from the fact that for any $\sigma > \sigma' > 0$ there exists an interval $(0,\theta]$ where $\sigma' t^{-2} e^{\sigma' t^{-1} - e^{-\sigma'/t}} \geq e^{-e^{\sigma/t}}$.

The single-point initial blow-up is proved by local energy methods. Because of their high degree of technicality we shall just give a short sketch of them in the simplest case of Theorem 1. For $k > 0$, let $u_k = u$ be the solution of the next result.

$$\left\{\begin{array}{l} \partial_t u - \Delta u + h(t)\,|u|^{q-1}\,u = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_{0,k}(x) = M_k^{1/2} k^{-N/2} \eta_k(x) \quad \forall x \in \mathbb{R}^N, \end{array}\right. \tag{13}$$

where $\eta_k \in C(\mathbb{R}^N)$ is nonnegative, has compact support in $B_{k^{-1}}$, converges weakly to δ_0 as $k \to \infty$, and $\{M_k\}$ satisfies $\lim_{k\to\infty} k^{-N/2} M_k = \infty$. Furthermore it can be assumed that $\|\eta_k\|_{L^2} \leq c_0 k^{N/2}$. The single-point initial blow-up will be a consequence of

Lemma 5. *For any $\delta > 0$ there exists $C = C(\delta)$ such that:*

$$\sup_{t\in[0,1]} \int_{|x|\geq\delta} u_k^2(x,t)dx + \int_0^1 \int_{|x|\geq\delta} (|\nabla u_k|^2 + u_k^2)dx\,dt \leq C \quad \forall k \in \mathbb{N}. \tag{14}$$

Proof. For $r \in (0,1)$, $\tau \geq 0$ we set $\Omega(\tau) = \{x \in \mathbb{R}^N : |x| > \tau\}$, $Q^r(\tau) = \Omega(\tau)\times(0,r]$, $Q_r(\tau) = \Omega(\tau) \times (r,1)$ and $Q_r = \mathbb{R}^N \times (r,1)$, and denote

$$I_1(r) = \iint_{Q_r} |\nabla u|^2\,dx\,dt,\ I_2(r) = \iint_{Q_r} u^2 dx\,dt, I_3(r) = \iint_{Q_r} h(t)\,|u|^{q+1}\,dx\,dt.$$

If we multiply the equation by $u(x,t)e^{(r-t)/(2-r)}$, integrate on Q_r and use Hölder's inequality, we get, since h is nondecreasing,

$$\begin{aligned}\int_{\mathbb{R}^N} u^2(x,1)dx + I_1(r) + I_2(r) + I_3(r) &\leq c \int_{\mathbb{R}^N} u^2(x,r)\,dx \\ &\leq c\tau^{\frac{N(q-1)}{q+1}} h(r)^{\frac{-2}{q+1}}(-I_3'(r))^{\frac{2}{q+1}} + c\int_{\Omega(\tau)} u^2(x,r)dx.\end{aligned} \tag{15}$$

Let $\tau \mapsto \mu(\tau)$ be a smooth decreasing function, we define

$$E_1^\mu(r,\tau) = \iint_{Q^r(\tau)} \Big(|\nabla u|^2 + \mu^2 u^2(x,t)\Big)\,e^{-\mu^2 t}dx\,dt,$$

$$E_2(r,\tau) = \iint_{Q^r(\tau)} u^2 dx\,dt \quad \text{and} \quad f_\mu(r,\tau) = \sup\{e^{-\mu^2 t}\int_{\Omega(\tau)} u^2(x,t)dx : 0 \leq t \leq r\}$$

and $f(r) = f_0(r,0)$. Then we introduce a parameter in the equation as in [4] by multiplying it by $u(x,t)\exp(-\mu^2(\tau)t)$ and integrating in the domain $Q^r(\tau)$ with $\tau > k^{-1}$ $Q^r(\tau)$ and $\tau > k^{-1}$. After some simple computations we deduce:

$$f_\mu(r,\tau) + 2E_1^\mu(r,\tau) \leq \frac{2}{\mu}\int_0^r \int_{\partial\Omega(\tau)} \Big(|\nabla u|^2 + \mu^2 u^2(x,t)\Big)\,e^{-\mu^2 t}dS\,dt \qquad \forall\,\tau > k^{-1}.$$

Assuming $1 - 2\mu'/\mu^2 > 1/2$, we deduce from last inequality:

$$f_\mu(r,\tau) + E_1^\mu(r,\tau) \leq -\frac{2}{\mu(\tau)}\frac{dE_1^\mu(r,\tau)}{d\tau} \qquad \forall\,\tau > k^{-1},$$

and by integration

$$f_\mu(r,\tau_2)\left(e^{\int_{\tau_1}^{\tau_2}\frac{\mu(\tau)\,d\tau}{2}} - 1\right) + E_1^\mu(r,\tau_2)e^{\int_{\tau_1}^{\tau_2}\frac{\mu(\tau)\,d\tau}{2}} \leq E_1^\mu(r,\tau_1) \quad \forall \tau_2 > \tau_1 > k^{-1}.$$

The choice $\mu(\tau) = r^{-1}(\tau - k^{-1})/8$ $(\tau > k^{-1})$ yields to

$$\begin{aligned}&\int_{\Omega(\tau)} u^2(x,r)dx + \iint_{Q^r(\tau)}\left(|\nabla_x u|^2 + \frac{(\tau-k^{-1})^2}{64r^2}u^2\right)dx\,dt \leq c_1 e^{-\frac{(\tau-k^{-1})^2}{64r}} \\ &\times \iint_{Q^r(\tau_0^k)}\left(|\nabla_x u|^2 + \frac{u^2}{2r}\right)dx\,dt \quad \forall\tau \geq \tilde{\tau}_0^k := k^{-1} + 8\sqrt{r} > \tau_0^k := k^{-1} + 4\sqrt{2r}.\end{aligned} \tag{16}$$

We will need standard global energy estimate of solution of problem (13) too:

$$\begin{aligned}&\int_{\mathbb{R}^N} |u(x,r)|^2\,dx + \int_{Q^r} (|\nabla_x u|^2 + |u|^2 + h(t)|u|^{q+1})\,dx\,dt \\ &\qquad\leq c\|u_{0,k}\|^2_{L_2(\mathbb{R}^N)} \leq \bar{c}M_k \qquad \forall\, r > 0.\end{aligned} \tag{17}$$

Estimating the right-hand side terms in (15) and (16) by (17), we derive:
(18)

$$(i)\ \sum_{i=1}^{3} I_i(r) \le c_1 \tau^{\frac{N(q-1)}{q+1}} h(r)^{\frac{-2}{q+1}} (-I_3'(r))^{\frac{2}{q+1}} + \frac{c_2 M_k}{r} e^{-(\tau - k^{-1})^2/64r} \quad \forall \tau \ge \tilde{\tau}_0^k(r)$$

$$(ii)\ f_0(r,\tau) + E_1^0(r,\tau) + \frac{\tau - k^{-1}}{64r^2} E_2(r,\tau) \le \frac{c_2 M_k}{r} e^{-(\tau-k^{-1})^2/64r} \qquad \forall \tau \ge \tilde{\tau}_0^k(r).$$

Next we choose $M_k = e^{e^k}$, fix $\epsilon_0 \in (0, e^{-1})$ and define a pair $(r_k,\ \tau_k)$ by the following relations: $r_k = \sup\{r : I_1(r) + I_2(r) + I_3(r) > 2M_k^{\epsilon_0}\}$; $c_2 r_k^{-1} \exp(-\frac{\tau_k^2}{64 r_k}) M_k = M_k^{\epsilon_0}$ $\Leftrightarrow \tau_k = 8\sqrt{r_k(1-\epsilon_0)e^k + \ln(c_2/r_k)}$. Taking $\tau = \tau_k + k^{-1}$ in (18)-i and solving the corresponding O.D.E. yields the estimate:

$$(19) \qquad \sum_{i=1}^{3} I_i(r) \le c_3(\tau_k + k^{-1})(H(r))^{-2/(q-1)} \quad \forall r \le r_k\ ,\ H(r) = \int_0^r h(s)ds.$$

If we write $h(t) = e^{-\omega(t)/t}$, the assumption $\inf\{\omega(t)/t^\alpha : 0 < t \le 1\} > 0$ implies that $H(r) \ge c_0 e^{-\omega(r)/r} r^2/\omega(r)$ and, replacing τ_k by its expression, (19) turns into

$$\sum_{i=1}^{3} I_i(r) \le c_4(\sqrt{r_k(1-\epsilon_0)e^k + \ln(c_2/r_k)} + k^{-1})^N \left(\frac{\omega(r)e^{-\omega(r)/r}}{r^2}\right)^{2/(q-1)} \qquad \forall r \le r_k.$$

Thus $r_k \le b_k$, where b_k is solution of equation:

$$c_4\left(\sqrt{r_k(1-\epsilon_0)e^k + \ln(c_2/b_k)}\ + k^{-1}\right)^N \left(\frac{\omega(b_k)e^{-\omega(b_k)/b_k}}{b_k^2}\right)^{2/(q-1)} = 2M_k^{\epsilon_0} = 2e^{\epsilon_0 e^k}.$$

From this inequality using additionally assumption on $\omega(t)$, we obtain inequalities: $c_5 e^k \ge \omega(b_k)/b_k \ge c_6 e^k,\ c_6 > 0$; $b_k \ge e^{-c_7 k},\ c_7 > 0$. These inequalities yield:

$$(20) \qquad \tau_k \le c_8 \sqrt{\omega(c_9 e^{-k})}\ .$$

Using the definition of r_k, inequality (18)-ii, the fact that $3M_k^{\epsilon_0} \le \bar{c} M_{k-1}\ \forall k \ge k_0(\bar{c})$ ($\bar{c}$ is from (17), $0 < \epsilon_0 < e^{-1}$), we deduce the main result of first round of computations:

$$(21) \qquad \sum_{i=1}^{3} I_i(r_k) + f_0(r_k, \tau_k + k^{-1}) + \sum_{i=1}^{2} E_i(r_k, \tau_k + k^{-1}) \le 3M_k^{\epsilon_0} \le \bar{c} M_{k-1}.$$

Next we organize the second round of estimates with $\mu(\tau) = (\tau - \tau_k - k^{-1})/8$, r_{k-1} and τ_{k-1} be defined similarly as r_k and τ_k, up to the change of indices, using obtained estimate (21) instead of (17). As result we derive:

$$(22)\ \sum_{i=1}^{3} I_i(r_{k-1}) + f_0(r_{k-1}, \tau_k + \tau_{k-1} + k^{-1}) + \sum_{i=1}^{2} E_i(r_{k-1}, \tau_k + \tau_{k-1} + k^{-1}) \le \bar{c} M_{k-2}.$$

Fixing arbitrary $n > k_0(\bar{c})$ and repeating the above described round of computations $k - n$ times, we obtain:
(23)

$$\sum_{i=1}^{3} I_i(r_n) + f_0\left(r_n, \sum_{j=0}^{k-n} \tau_{k-j} + k^{-1}\right) + \sum_{i=1}^{2} E_i\left(r_n, \sum_{j=0}^{k-n} \tau_{k-j} + k^{-1}\right) \le \bar{c} M_{n-1},$$

and, since by induction τ_{k-j} satisfies (20) with k replaced by $k-j$, we obtain

$$\sum_{j=0}^{k-n} \tau_{k-j} \leq c_8 \sum_{j=0}^{k-n} \sqrt{\omega(c_9 e^{-(k-j)})} \leq c_{10} \int_{c_9 e^{-k}}^{c_9 e^{-n}} \frac{\sqrt{\omega(s)} ds}{s}. \tag{24}$$

We denote $\tau^*(n) = \lim_{k\to\infty} c_8 \sum_{j=0}^{k-n} \sqrt{\omega(c_9 e^{-(k-j)})}$. We derive from (23) by letting $k \to \infty$,

$$\sup_{0<t\leq r_n} \int_{|x|\geq\tau^*(n)} u^2(x,t)dx + \int_0^{r_n} \int_{|x|\geq\tau^*(n)} (|Du|^2 + u^2)dx\, dt \leq \bar{c} M_{n-1}. \tag{25}$$

Due to assumption (4) $\tau^*(n) \to 0$ as $n \to \infty$, therefore inequality (25) implies the result.

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