The balance between diffusion and absorption in semilinear parabolic equations

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Abstract

Let $h:[0,\infty)\mapsto[0,\infty)$ be continuous and nondecreasing, $h(t)>0$ if $t>0$, and m,q be positive real numbers. We investigate the behavior when $k\to\infty$ of the fundamental solutions $u=u_k$ of $\partial_t u-\Delta u^m+h(t)u^q=0$ in $\Omega\times(0,T)$ satisfying $u_k(x,0)=k\delta_0$. The main question is wether the limit is still a solution of the above equation with an isolated singularity at $(0,0)$, or a solution of the associated ordinary differential equation $u'+h(t)u^q=0$ which blows-up at $t=0$.

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1 Introduction

Let m and q positive parameters and $h:[0,\infty)\mapsto[0,\infty)$ a nondecreasing continuous. If one consider a reaction-diffusion equation such as

$$\partial_t u-\Delta u^m+h(t)u^q=0 \tag{1.1}$$

($u>0$ for simplicity) in a cylindrical domain $Q^T=\mathbb{R}^N\times(0,T)$ ($N\geq 1$), the behaviour of u is subject to two competing features: the diffusion associated to the partial differential operator, here $-\Delta$, and the absorption which is represented by the term $h(t)u^q$. When $q>1$ and $h(t)>0$ for $t>0$, the absorption term is strong enough in order positive solution to satisfy an universal bound

$$0\leq u(x,t)\leq U_h(t)=\left((q-1)\int_0^t h(s)\,ds\right)^{-1/(q-1)} \tag{1.2}$$

for every $(x,t)\in Q^T$. In addition, the function U_h which appears above is a particular solution of (1.1). The associated diffusion equation

$$\partial_t v-\Delta v^m=0 \tag{1.3}$$

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admits fundamental solutions $v = v_k$ $(k > 0)$ which satisfy $v_k(x, 0) = k\delta_0$ if $m > (N-2)_+/N$. If

$$\int_0^T \int_{B_R} h(t) v_k^q \, dx \, dt < \infty, \quad B_R := \{|x| < R\}, \tag{1.4}$$

for any $R \in (0, \infty]$, it is shown that (1.1) admits fundamental solutions $u = u_k$ in Q^T which satisfy initial condition $u_k(x, 0) = k\delta_0$. The maximum principle holds and therefore the mapping $k \mapsto u_k$ is increasing. If $h > 0$ on $(0, \infty)$ then due to universal bound(1.2) there exists $u_\infty = \lim_{k\to\infty} u_k$, and u_∞ is a solution of (1.1) in Q^T. A natural question is whether u_∞ admits a singularity only at the origin $(0, 0)$ or at other points too. Actually, in the last case it will imply $u_\infty \equiv U$ since the following alternative occurs:

(i) either $u_\infty = U$. (*complete initial blow-up*);

(ii) or u_∞ is a solution singular at $(0, 0)$ and such that $\lim_{t\to 0} u(x, t) = 0$ for all $x \neq 0$. (*single-point initial blow-up*).

This phenomenon is observed for the first time by Marcus and Véron. They considered the semilinear equation

$$\partial_t u - \Delta u + h(t) u^q = 0 \tag{1.5}$$

and proved [8, Prop. 5.2]

Theorem 1.1 *If $h(t) = e^{-\kappa/t}$ $(\kappa > 0)$, then the complete initial blow-up occurs.*

However they raised the question whether this type of degeneracy of the absorption is sharp or not. The method of [8] relies on the construction of subsolutions associated to very singular solutions of equations

$$\partial_t u - \Delta u + c_\epsilon t^\alpha u^q = 0 \tag{1.6}$$

for suitable $\alpha > 0$ and $c_\epsilon > 0$, and on the study of asymptotics of these solutions. One the main result of present paper states that if the degeneracy of the absorption terms is lightly smaller respectivelly to Th. 1.1, then localization occurs.

Theorem 1.2 *If $h(t) = \exp(-\omega(t)/t)$, where ω is continuous, nondecreasing and satisfies*

$$\int_0^1 \frac{\sqrt{\omega}(s)}{s} ds < \infty, \tag{1.7}$$

then u_∞ has single-point initial blow-up at $(0, 0)$.

The method of the proof is totally different from the one of Marcus and Véron and based upon local energy estimates in the spirit of the famous Saint-Venant 's principle (see [5, 12, 13]). Using appropriate test functions we prove by induction that the energy of the fundamental solutions u_k remains uniformly locally bounded in $\overline{Q}^T \setminus \{(0, 0)\}$.

In the case of equation

$$\partial_t u - \Delta u + h(t)(e^u - 1) = 0 \tag{1.8}$$

the same type of phenomenon occurs, but at a different scale of degeneracy. We prove the following

Theorem 1.3 *1) If $h(t) = e^{-e^{\kappa/t}}$ for some $\kappa > 0$, then the complete initial blow-up occurs.*

2) If $h(t) = e^{-e^{\omega(t)/t}}$ for some $\omega \in C(0, \infty)$ positive, nondecreasing and satisfying (1.7), then u_∞ has single-point initial blow-up at $(0, 0)$.

In this paper we also extend the study of equation (1.1) to the case $m \neq 1$. The situation differs completely corresponding to $m > 1$, the porous media equation with slow diffusion, and to $(N-2)_+/N < m < 1$, the fast diffusion equation. Concerning the porous media equation, we prove

Theorem 1.4 *If $q > m > 1$ and h is nondecreasing and satisfies $h(t) = O(t^{(q-m)/(m-1)})$ as $t \to 0$, then $u_\infty \equiv U_h$.*

We give two proofs. The first one, valid only in the subscritical case $1 < m < q < m + 2/N$, is based upon the construction of suitable subsolutions, as in the semilinear case. The second one, based upon scaling transformations, is valid in all the cases $q + 1 > 2m > 2$ where the u_k exists. It reduces to proving that the equation

$$-\Delta\Psi - \Psi^{1/m} + \Psi^{q/m} = 0 \quad \text{in } \mathbb{R}^N$$

admits only one positive solution, the constant 1. The localization counter part is as follows,

Theorem 1.5 *Assume $q > m > 1$, in Equation (1.1). If $h(t) = t^{(q-m)/(m-1)}\omega^{-1}(t)$ with $\omega(t) \to 0$ as $t \to 0$, and*

$$\int_0^1 \omega^\theta(s)\frac{ds}{s} < \infty \tag{1.9}$$

where

$$\theta = \frac{m^2 - 1}{[N(m-1) + 2(m+1)](q-1)},$$

then u_∞ has single-point initial blow-up at $0, 0)$.

Actually, the method is applicable to a much more general class of equations.

In the fast diffusion case there is always localization.

Theorem 1.6 *Assume $(N-2)_+/N < m < 1$ and $q > 1$, in Equation (1.1). Then*

$$u_\infty(x,t) \leq \min\left\{U_h(t), C_*\left(\frac{t}{|x|^2}\right)^{1/1-m)}\right\} \tag{1.10}$$

where

$$C_* = \left(\frac{(1-m)^3}{2m(mN+2-N}\right)^{1/(1-m)}.$$

This type of problem has an elliptic counterpart which is initiated in [10] where the following question is considered: suppose Ω is a C^2 bounded domain in $\mathbb{R}^N$, $q > 1$ and $h \in C(0,\infty)$ is positive. What is the limit, when $k \to \infty$ of the solutions (when they exist) $u = u_k$ of the following problem

$$\begin{cases} -\Delta u + h(\rho(x))u^q = 0 & \text{in } \Omega \\ u = k\delta_0 & \text{in } \partial\Omega, \end{cases} \tag{1.11}$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. It is proved in [10] that, if $h(t) = e^{-1/t}$, then $u_\infty(:= \lim_{k\to\infty} u_k)$ is the maximal solution of the equation in Ω, that is the function which satisfies

$$\begin{cases} -\Delta u + h(\rho(x))u^q = 0 & \text{in } \Omega \\ \lim_{\rho(x)\to 0} u(x) = \infty. \end{cases} \tag{1.12}$$

On the contrary, if $h(t) = t^\alpha$, for $\alpha > 0$ and $1 < q < (N + 1 + \alpha)/(N - 1)$, it is proved in [11] that u_∞ has an isolated singularity at 0, and vanishes everywhere outside 0. In a forthcoming article we shall study this localization of singularity phenomenon for the complete nonlinear elliptic problem, replacing the powers by more general functions, and the ordinary Laplacian by the p-Laplacian operator.

Our paper is organized as follows: §1 Introduction. In §2 we study sufficient conditions of complete initial blow-up for semilinear heat equation. In §3 we prove sharp sufficient condition of existence of single point initial blow-up for heat equation with power nonlinear absorption. In §4 local energy method from §3 is adapted to the heat equation with nonpower absorption nonlinearity. §5 deals with porous media equation with power nonlinear absorption, §6 — the fast diffusion equation with nonlinear absorption.

2 Complete initial blow-up for semilinear heat equation

We recall the standard result concerning the existence of a fundamental solution $u = u_k$ $(k > 0)$ to the following problem

$$\begin{cases} \partial_t u - \Delta u + g(x, t, u) = 0 \quad \text{in } Q^T = \mathbb{R}^N \times (0, T) \\ u(x, 0) = k\delta_0. \end{cases} \tag{2.1}$$

If v is defined in Q^T, we denote by $\tilde{g}(v)$ the function $(x, t) \mapsto g(x, t, v(x, t))$. By a solution we mean a function $u \in L^1_{loc}(\overline{Q}^T)$ such that $\tilde{g}(u) \in L^1_{loc}(\overline{Q}^T)$, which verifies

$$\iint_{Q^T} (-u\partial_t\phi - u\Delta\phi + \tilde{g}(u)\phi)\, dxdt = k\phi(0, 0), \tag{2.2}$$

for any $\phi \in C_0^{2,1}(\mathbb{R}^N \times [0, T) \times \mathbb{R})$. We denote by $E(x, t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$ the fundamental solution of the heat equation in Q^∞, by $B_R(a)$ an open ball of center a and radius R, and $B_R(0) = B_R$. The following result is classical

Theorem 2.1 *Let $g \in C(\mathbb{R}^N \times [0, T] \times \mathbb{R})$ such that $g(x, t, r) \geq 0$ on $\mathbb{R}^N \times [0, T] \times \mathbb{R}_+$, and assume that $g = g_1 + g_2$ where g_1 and g_2 are respectively nondecreasing and locally Lipschitz continuous with respect to the r-variable functions. Let $k > 0$ be such that*

$$\int_0^T \int_{B_R} g(x, t, kE(x, t))dxdt < \infty. \tag{2.3}$$

for any $R > 0$. Then there exists a solution $u = u_k$ to problem (2.1). Furthermore, if $g_2 = 0$, then u_k is unique.

Function $g(x, t, r) = e^{-\kappa/t}|r|^{q-1}r$, with $\kappa > 0$ and $q > 1$, satisfies (2.3). Thus the problem

$$\begin{cases} \partial_t u - \Delta u + e^{-\kappa/t}\, |u|^{q-1}\, u = 0 \quad \text{in } Q^\infty \\ u(x, 0) = k\delta_0. \end{cases} \tag{2.4}$$

admits a unique solution. The next result is proved in [8], but we recall the proof both for the sake of completeness and to present the key-lines of the method in a simple case.

Theorem 2.2 *For $k > 0$, let u_k denote the solution of (2.4) in Q^∞. Then $u_k \uparrow U_S$ as $k \to \infty$, where*

$$U_S(t) = \left((q-1)\int_0^t e^{-\kappa/s}ds\right)^{1/(1-q)}, \quad \forall t > 0. \tag{2.5}$$

Proof. *Case 1.* $1 < q < 1 + 2/N$. For any $\epsilon > 0$, $u_k = u$ satisfies

$$\partial_t u - \Delta u + e^{-\kappa/\epsilon}u^q \geq 0 \tag{2.6}$$

on Q^ϵ. Therefore if $v = v_k$ is the solution of

$$\left\{\begin{array}{l} \partial_t v - \Delta v + e^{-\kappa/\epsilon}v^q = 0 \quad \text{in } Q^\infty \\ v(x,0) = k\delta_0, \end{array}\right. \tag{2.7}$$

there holds $u_k \geq v_k$. Passage to the limit $k \to \infty$, yields

$$\lim_{k\to\infty} u_k := u_\infty \geq v_\infty = \lim_{k\to\infty} v_k \ \text{ in } Q^\epsilon. \tag{2.8}$$

If we write $v_\infty(x,t) = e^{\kappa/\epsilon(q-1)}t^{-1/(q-1)}f(x/\sqrt{t})$, then f is radial and satisfies

$$\left\{\begin{array}{l} f'' + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right)f' + \dfrac{1}{q-1}f - f^q = 0 \quad \text{ on } (0,\infty), \\ f'(0) = 0\,,\ \lim_{r\to\infty} r^{2/q-1)}f(r) = 0. \end{array}\right.$$

Furthermore the asymptotics of f is given in [2],

$$f(r) = Cr^{2/(q-1)-N}e^{-r^2/4}(1 + \circ(1)))\,, \ \text{ as } r \to \infty,$$

for some $C = C(N,q) > 0$. Therefore

$$f(r) \geq \tilde{C}(r+1)^{2/(q-1)-N}e^{-r^2/4} \quad \forall r \geq 0, \tag{2.9}$$

for some $\tilde{C} = \tilde{C}(N,q) > 0$. If we take $t = \epsilon$, we derive from (2.8)

$$u_\infty(x,t) \geq e^{\kappa/t(q-1)}t^{-1/(q-1)}f(x/\sqrt{t}) \ \text{ in } \mathbb{R}^N. \tag{2.10}$$

Let $0 < \ell < 2\sqrt{\kappa/(q-1)}$. Inequalities (2.9) and (2.10) imply

$$u_\infty(x,t) \geq \tilde{C}t^{-1/(q-1)}e^{(\kappa/(q-1)-\ell^2/4)t^{-1}}, \quad \forall x \in \bar{B}_\ell. \tag{2.11}$$

Therefore $\lim_{t\to 0} u_\infty(x,t) = \infty\,, \quad \forall x \in \bar{B}_\ell$. We pick some point x_0 in B_ℓ. Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (2.4) with initial value $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ $(0 < \sigma < \ell - |x_0|)$, the previous inequality implies

$$u_\infty(x,t) \geq u_\infty(x - x_0, t).$$

Reversing the role of 0 and x_0 yields to

$$u_\infty(x,t) = u_\infty(x - x_0, t).$$

If we iterate this process we derive

$$u_\infty(x,t) = u_\infty(x - y, t)\,, \quad \forall y \in \mathbb{R}^N. \tag{2.12}$$

Since $u_{k\delta_y}$ is radial with respect to y, (2.12) implies that $u_\infty(x,t)$ is independent of x and therefore it is solution of

$$\left\{\begin{array}{l} z' + e^{-\kappa/t}z^q = 0 \quad \text{ on } (0,\infty) \\ \lim_{t\to 0} z(t) = \infty. \end{array}\right. \tag{2.13}$$

Thus $u_\infty = U_S$ where U_S is defined by (2.5).

Case 2. $q \geq 1 + 2/N$. Let $\alpha > 0$ such that $q < q_{c,\alpha} = 1 + 2(1+\alpha)/N$. We write $e^{-\kappa/t} = t^\alpha \tilde{h}(t)$ with $\tilde{h}(t) = t^{-\alpha}e^{-\kappa/t}$. The function $\tilde{h}$ is increasing on $(0, \kappa/\alpha]$ and we extend it by $\tilde{h}(0) = 0$. Let $0 < \epsilon \leq \kappa/\alpha$, then the solution $u = u_k$ of (2.4) verifies

$$\partial_t u - \Delta u + \tilde{h}(\epsilon)t^\alpha u^q \geq 0,$$

in $\mathbb{R}^N \times (0, \epsilon]$. As in Case 1, u is bounded from below on $\mathbb{R}^N \times (0, \epsilon]$ by $\left(\tilde{h}(\epsilon)\right)^{-1/(q-1)} v_\infty$ where $v_\infty = v$ is is the very singular solution of

$$\partial_t v - \Delta v + t^\alpha v^q = 0. \tag{2.14}$$

Then $v_\infty(x,t) = t^{-(1+\alpha)/(q-1)} f_\alpha(|x|/\sqrt{t})$, and $f_\alpha = f$ satisfies

$$\left\{\begin{array}{l} f'' + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right) f' + \dfrac{1+\alpha}{q-1} f - f^q = 0 \quad \text{ on } (0,\infty), \\ f'(0) = 0\,,\ \lim_{r\to\infty} r^{2(1+\alpha)/q-1)} f(r) = 0. \end{array}\right.$$

The asymptotics of f_α is given in [9]

$$f_\alpha(r) = Cr^{2(1+\alpha)/(q-1)-N} e^{-r^2/4}(1 + \circ(1)) \quad \text{as } r \to \infty,$$

thus

$$f_\alpha(r) \geq \tilde{C}(1+r)^{2(1+\alpha)/(q-1)-N} e^{-r^2/4} \quad \forall r \in \mathbb{R}_+.$$

Consequently

$$u(x,t) \geq \tilde{C}e^{(\kappa/(q-1)-\ell^2/4)t^{-1}}\,, \quad \forall x \in \bar{B}_\ell. \tag{2.15}$$

Taking again $0 < \ell < 2\sqrt{\kappa/(q-1)}$, we derive

$$\lim_{t\to 0} u(x,t) = \infty\,, \quad \forall x \in \bar{B}_\ell.$$

As in the Case 1, it yields to $u_\infty(x,t) = u_\infty(x-y,t)$ for any $y \in \mathbb{R}^N$, and finally $u_\infty(x,t) = U_S(t)$. □

Next we consider Cauchy problem for diffusion equation with an exponential type absorption term

$$\left\{\begin{array}{l} \partial_t u - \Delta u + h(t)e^u = 0 \quad \text{in } Q^\infty \\ u(x,0) = k\delta_0 \end{array}\right. \tag{2.16}$$

where $h \in C(\mathbb{R}_+)$ is nonnegative. Theorem 2.1 yields the following existence result:

Proposition 2.3 *Assume h satisfies*

$$\lim_{t\to 0} t^{N/2} \ln h(t) = -\infty. \tag{2.17}$$

Then for any $k > 0$ problem (2.16) admits a unique solution $u = u_k$. Furthermore

$$u_k(x,t) \leq V_S(t) := -\ln\left(\int_0^t h(s)ds\right) \quad \forall (x,t) \in Q^\infty. \tag{2.18}$$

Notice that estimate (2.18) is a consequence of the fact that V_S satisfies the associated O.D.E.

$$y' + h(t)e^y = 0 \quad \text{in } (0, \infty),$$

with infinite initial value. Our main result concerning nonexistence of localized singularities for equation (2.16) is

Theorem 2.4 *Let $h(t) = e^{-e^{\sigma/t}}$ for some $\sigma > 0$ and any $t > 0$. Then $u_k \uparrow V_S$ as $k \to \infty$.*

Proof. Step 1. Construction of an approximate very singular solution. For $n > 1$ and $c_n > 0$ to be defined later on, let $v = V_n$ be the very singular solution of

$$\partial_t v - \Delta v + c_n t^{\alpha_n} v^n = 0. \tag{2.19}$$

The necessary and sufficient condition for the existence of a V_n is

$$n < 1 + N(\alpha_n + 1)/2.$$

This function is obtained in the form

$$V_n(x,t) = t^{-(1+\alpha_n)/(n-1)} F(x/\sqrt{t}),$$

where F solves

$$\Delta F + \frac{1}{2}\xi.DF + \frac{1+\alpha_n}{n-1}F - c_n F^n = 0.$$

We fix

$$\frac{1+\alpha_n}{n-1} = 1 + \frac{N}{2} \Longleftrightarrow \alpha_n = (2+N)(n-1)/2 - 1, \tag{2.20}$$

and set

$$f_n = c_n^{1/(n-1)} F.$$

Then f_n solves

$$\Delta f_n + \frac{1}{2}\xi.Df_n + \frac{N+2}{2} f_n - f_n^n = 0.$$

We prove that f_n has an asymptotic expansion essentially independent of n, in the following form

$$f_n(\xi) \geq \delta(|\xi|^2 + 1)e^{-|\xi|^2/4} \Longrightarrow V_n(x,t) \geq \delta c_n^{-1/(n-1)} t^{-2-N/2}(|x|^2 + t)e^{-|x|^2/4t} \tag{2.21}$$

It order to see that, we put

$$\tilde{f}_n = \left(\frac{2}{N+2}\right)^{1/(n-1)} f_n$$

then

$$\Delta \tilde{f}_n + \frac{1}{2}\xi.D\tilde{f}_n + \frac{N+2}{2}\tilde{f}_n - \frac{N+2}{2}\tilde{f}_n^n = 0.$$

By the maximum principle $0 \leq \tilde{f}_n \leq 1$ so that $0 \leq \tilde{f}_n^{n'} \leq \tilde{f}_n^n$ for $n' > n$. Thus

$$\Delta \tilde{f}_n + \frac{1}{2}\xi.D\tilde{f}_n + \frac{N+2}{2}\tilde{f}_n - \frac{N+2}{2}\tilde{f}_n^{n'} \geq 0,$$

which implies that $\tilde{f}_n$ is a subsolution of the equation for $\tilde{f}_{n'}$ and therefore,

$$n' > n \Longrightarrow \tilde{f}_n \leq \tilde{f}_{n'} \Longleftrightarrow f_n \leq \left(\frac{N+2}{2}\right)^{(n'-n)/(n-1)(n'-1)} f_{n'}. \tag{2.22}$$

In the particular case $n = n^* = (N+4)/(N+2)$, the equation falls into the scoop of Brezis-Peletier-Terman study since it can also be written in the form

$$\Delta f_{n^*} + \frac{1}{2}\xi.Df_{n^*} + \frac{1}{n^*-1}f_{n^*} - f_{n^*}^{n^*} = 0.$$

and their asymptotic expansion applies (with $2/(n^*-1) - N = 2$) as $|\xi| \to \infty$:

$$f_{n^*}(\xi) = C\,|\xi|^2\, e^{-|\xi|^2/4}(1+\circ(1)) \Longrightarrow f_{n^*}(\xi) \geq \delta_*(|\xi|^2+1)e^{-|\xi|^2/4} \quad \forall \xi. \tag{2.23}$$

Combining (2.22) with $n = n^*$ and n' replaced by n, and (2.23), we get

$$f_n(\xi) \geq \delta_* \left(\frac{2}{N+2}\right)^{(n-n^*)/(n-1)(n^*-1)} (|\xi|^2+1)e^{-|\xi|^2/4} \quad \forall \xi. \tag{2.24}$$

Since $n \mapsto (2/(N+2)^{(n-n^*)/(n-1)(n^*-1)}$ is bounded from below independently of $n > n^*$, we get (2.21).

Step 2. Some estimates from below for a related problem. In order to have $v_n \leq u$ in the range of value of u, which is

$$u(t) \leq V_S(t) = -\ln\left(\int_0^t h(s)ds\right) \quad \forall t > 0, \tag{2.25}$$

we need $v = v_n$ to be a subsolution near $t = 0$ of the equation that u verifies. Furthermore this can be done up to some bounded function. It is sufficient to have

$$c_n t^{\alpha_n}(x^n+1) \geq h(t)e^x, \quad \forall t \in (0,\tau_n],\ x \in [0, V_S(t)] \tag{2.26}$$

where τ_n has to be defined. In particular, at the end points of the interval,

$$\begin{cases} (i)\ c_n t^{\alpha_k} \geq h(t) \\ (ii)\ c_n t^{\alpha_n}\left(\ln^n\left(\dfrac{1}{\int_0^t a(s)ds}\right)+1\right) \geq \dfrac{h(t)}{\int_0^t h(s)\,ds}. \end{cases} \tag{2.27}$$

We write (2.26) in the form

$$\frac{e^x}{1+x^n} \leq \frac{c_n t^{\alpha_n}}{h(t)}, \tag{2.28}$$

and set

$$\phi(x) = \frac{e^x}{1+x^n}.$$

Then

$$\phi'(x) = e^x \frac{1+x^n - nx^{n-1}}{(1+x^n)^2}.$$

The sign of ϕ' is the same as the one of $\psi(x) = 1 + x^k - nx^{n-1}$, a function which decreasing then increasing, is positive near 0, vanishes somewhere between 0 and 1 and again between $n-1$ and n. The first maximum of ϕ is less than $e/2$. This is not important in (2.28) since we can always assume that the minimum of $c_k t^{\alpha_k}/h(t)$ is larger than $e/2$. Therefore, it is sufficient to have

$$\frac{e^{V_S(t)}}{1+V_S^n(t)} \leq \frac{c_n t^{\alpha_n}}{h(t)}, \tag{2.29}$$

in order to have (2.28). This is exactly (2.27)-ii. If we express $h(t)$ in the form

$$h(t) = -\omega'(t)e^{-\omega(t)},$$

then (2.27)-ii is equivalent to

$$c_n t^{\alpha_n} \left(\omega^n(t) + 1\right) \geq -\omega'(t). \tag{2.30}$$

Since

$$\omega^n(t) + 1 \geq 2^{1-n}(\omega(t) + 1)^n,$$

we associate the following O. D. E. on $\mathbb{R}_+$

$$c_n t^{\alpha_n} = 2^{1-n} \frac{-\eta'}{(\eta + 1)^n},$$

the maximal solution of which is

$$\eta(t) = \frac{1}{2} \left(\frac{1}{c_n(n-1)}\right)^{1/(n-1)} t^{-(\alpha_n+1)/(n-1)} = \frac{1}{2} \left(\frac{1}{c_n(n-1)}\right)^{1/(n-1)} t^{-1-N/2}.$$

If we write ω in the form

$$\omega(t) = e^{\alpha(t)},$$

with $\alpha(0) = \infty$, $\alpha' < 0$, then (2.27)-ii becomes

$$c_n t^{\alpha_n} \left(e^{n\alpha(t)} + 1\right) \geq -\alpha'(t)e^{\alpha(t)},$$

and this inequality is ensured provided

$$c_n t^{\alpha_n} e^{(n-1)\alpha(t)} \geq -\alpha'(t) \Longleftrightarrow c_n \geq -\alpha'(t)e^{(1-n)\alpha(t)-\alpha_n \ln t} = -t\alpha'(t)e^{(1-n)\left(\alpha(t)+2^{-1}(N+2)\ln t\right)}, \tag{2.31}$$

by replacing α_n by its value. Next we fix

$$\alpha(t) = \alpha_\sigma(t) = \frac{\sigma}{t} \quad \forall t > 0 \tag{2.32}$$

where $\sigma > 0$ is a parameter, thus

$$-t\alpha'(t)e^{(1-n)\left(\alpha(t)+2^{-1}(N+2)\ln t\right)} = e^{(1-n)\sigma/t-\left(2^{-1}(n-1)(N+2)+1\right)\ln t} = e^{\rho(t)}.$$

In order to have (2.31) it is sufficient to have the monotonicity of the function ρ and

$$\rho'(t) = \frac{\sigma(n-1)}{t^2} - \frac{n(N+2) - N}{2t}$$

Then there exist $\gamma > 0$, independent of k and σ such that $\rho'(t) > 0$ on $(0, \sigma\gamma]$. Consequently, inequality (2.31) is ensured on $(0, \epsilon] \subset (0, \sigma\gamma]$ as soon as

$$c_n \geq e^{\rho(\epsilon)} = e^{(1-n)\sigma/\epsilon - 2^{-1}(n(N+2)-N)\ln \epsilon}. \tag{2.33}$$

Step 3. Complete initial blow-up for a related problem. Assume now

$$h(t) = \tilde{\sigma}t^{-2}e^{\tilde{\sigma}t^{-1} - e^{\tilde{\sigma}/t}} \tag{2.34}$$

for some $\tilde{\sigma} > 0$. For $n > 2$, we fix $\epsilon < \tilde{\sigma}\gamma$ and take $c_n = e^{\rho(\epsilon)}$. On $(0, \epsilon]$ we have

$$c_n t^{\alpha_n}(e^{n\alpha(t)} + 1) \geq -\alpha'(t)e^{\alpha(t)}.$$

Therefore, if $u = u_k$ is the solution of (2.16) with $h(t)$ given by (2.34), it satisfies $u(t) \leq V_S(t)$, where V_S is given by (2.25), and

$$\partial_t u - \Delta u + c_n t^{\alpha_n}(u^n + 1) \geq 0 \quad \text{in } Q^\epsilon.$$

Therefore u is larger that the solution $v = \tilde{v}_k$ of

$$\partial_t v - \Delta v + c_n t^{\alpha_n}(v^n + 1) = 0 \quad \text{in } Q^\epsilon,$$

with $\tilde{v}_k(0) = k\delta_0$. Furthermore $\tilde{v}_k \geq v_k - c_n t^{\alpha_n+1}/(\alpha_n + 1)$, where $v = v_k$ solves

$$\partial_t v - \Delta v + c_n t^{\alpha_n} v^n = 0 \quad \text{in } Q^\epsilon,$$

with $v_k(0) = k\delta_0$. If we let $k \to \infty$, we derive from (2.21) and by replacing $c_n = e^{\rho(\epsilon)}$ by its precise value $e^{(1-n)\sigma/\epsilon - 2^{-1}(n(N+2)-N)\ln\epsilon}$, that

$$u_\infty(x,t) \geq V_n(x,t) - \frac{c_n t^{\alpha_n+1}}{\alpha_n+1} \geq \delta t^{-2-N/2}(|x|^2 + t)e^{\frac{\sigma}{\epsilon} + \frac{(n(N+2)-N\ln\epsilon}{n-1} - \frac{|x|^2}{4t}}$$

on $(0, \epsilon]$. In particular

$$u_\infty(x,\epsilon) \geq \delta\epsilon^{-2-N/2}(|x|^2 + \epsilon)e^{\frac{\sigma}{\epsilon} + \frac{(n(N+2)-N\ln\epsilon}{n-1} - \frac{|x|^2}{4\epsilon}}. \tag{2.35}$$

Taking $|x|^2 < \sigma/4$ yields to

$$\lim_{\epsilon\to 0} \epsilon^{-2-N/2}(|x|^2 + \epsilon)e^{\frac{\sigma}{\epsilon} + \frac{(n(N+2)-N\ln\epsilon}{n-1} - \frac{|x|^2}{4\epsilon}} = \infty.$$

Thus

$$\lim_{\epsilon\to 0} u_\infty(x,\epsilon) = \infty, \quad \forall x \in B_{\sqrt{\sigma}/2}.$$

As in the proof of Theorem 2.2, it implies $u_\infty = V_S$.

Step 4. End of the proof. Since for any $\sigma > \tilde{\sigma} > 0$ there exists an interval $(0, \theta]$ on which

$$\tilde{\sigma} t^{-2} e^{\sigma' t^{-1} - e^{\sigma'/t}} \geq e^{-e^{\sigma/t}},$$

any solution of (2.16) with $h(t)$ given by (2.34) is a subsolution in Q^θ of the same equation with $h(t) = e^{-e^{-\sigma/t}}$. This implies the claim. □

3 Single point initial blow-up for semilinear heat equation

We consider the following Cauchy problem

$$\begin{cases} \partial_t u - \Delta u + h(t)\,|u|^{q-1}\,u = 0 & \text{in } Q^\infty \\ u(x,0) = k\delta_0. \end{cases} \tag{3.1}$$

The first result dealing with the localization of the blow-up that we prove is the following.

Theorem 3.1 *Assume $h(t) = e^{-\omega(t)/t}$ where $\omega \in C([0,\infty))$ is positive, nondecreasing function which satisfies $\omega(s) \geq s^{\alpha_0}$ for some $\alpha_0 \in [0,1)$ and any $s > 0$, and the following Dini like condition holds:*

$$\int_0^1 \frac{\sqrt{\omega(s)}}{s} ds < \infty. \tag{3.2}$$

Then u_k always exists and $u_\infty := \lim_{k\to\infty} u_k$ has a point-wise singularity at $(0,0)$.

Proof. The proof is based on the study of asymptotic properties as $k \to \infty$ of solutions $u = u_k$ of the regularized Cauchy problem

$$\begin{cases} u_t - \Delta u + h(t)|u|^{q-1}u = 0 \quad \text{in } Q^T, \\ u(x,0) = u_{0,k}(x) = M_k^{1/2} k^{-N/2} \delta_k(x) \quad \forall x \in \mathbb{R}^N, \end{cases} \tag{3.3}$$

where $\delta_k \in C(\mathbb{R}^N)$, $\operatorname{supp} \delta_k \subset \left\{|x| \le k^{-1}\right\}$, $\delta_k \rightharpoonup \delta(x)$ weakly in the sense of measures as $k \to \infty$ and $\{M_k\}$ is some sequence tending to ∞ as $k \to \infty$ fast enough so that

$$M_k^{1/2} k^{-N/2} \to \infty \text{ as } k \to \infty. \tag{3.4}$$

Without loss of generality we will suppose that

$$\|\delta_k(x)\|^2_{L_2(\mathbb{R}^N)} \le c_0 k^N \quad \forall\, k \in \mathbb{N}, \quad c_0 = \text{const}. \tag{3.5}$$

Our method of analysis is some variant of the local energy estimates method (also called Saint-Venant principle), developed, particulary, in [12, 13, 15–17] (see also review in [5]). Let introduce the families of subdomains

$$\begin{aligned} \Omega(\tau) &= \mathbb{R}^N \cap \{|x| > \tau\} \quad \forall\, \tau > 0, \\ Q^r(\tau) &= \Omega(\tau) \times (0, r) \quad \forall\, r \in (0, T), \\ Q_r(\tau) &= \Omega(\tau) \times (r, T) \quad \forall\, r \in (0, T). \end{aligned}$$

Step 1. The local energy framework. We fix arbitrary $k \in \mathbb{N}$ and consider solution $u = u_k$ of (3.3), but for convenience we will denote it by u. Firstly we deduce some integral vanishing properties of solution u in the family of subdomains $Q_r := \mathbb{R}^N \times (r, T)$. Multiplying (3.3) by $u(x,t) \exp\left(-\dfrac{t-r}{1+T-r}\right)$ and integrating in Q_r, we get

$$\begin{aligned} &\left(2 \exp\left(\frac{T-r}{1+T-r}\right)\right)^{-1} \int_{\mathbb{R}^N} |u(x,T)|^2\, dx \\ &\qquad + \int_{Q_r} \left(|D_x u|^2 + h(t)|u|^{q+1}\right) \exp\left(-\frac{t-r}{1+T-r}\right) dxdt \\ &\qquad\quad + \frac{1}{1+T-r} \int_{Q_r} |u|^2 \exp\left(-\frac{t-r}{1+T-r}\right) dxdt \\ &\qquad\qquad = 2^{-1} \int_{\Omega(\tau)} |u(x,r)|^2\, dx + 2^{-1} \int_{\mathbb{R}^N \setminus \Omega(\tau)} |u(x,r)|^2 dx, \end{aligned} \tag{3.6}$$

where $\tau > 0$ is arbitrary parameter. Using Hölder's inequality, it is easy to check that

$$\int_{\mathbb{R}^N \setminus \Omega(\tau)} |u(x,r)|^2\, dx \le c \tau^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} \left(\int_{\mathbb{R}^N \setminus \Omega(\tau)} |u(x,r)|^{q+1} h(r)\, dx \right)^{\frac{2}{q+1}}. \tag{3.7}$$

Here and further we will denote by c, c_i different positive constants which do not depend on parameters k, τ, r, but the precise value of which may change from one ocurrence to another. Let us consider now the energy functions

$$I_1(r) = \int_{Q_r} |D_x u|^2\, dx\, dt, \quad I_2(r) = \int_{Q_r} h(t)|u(x,t)|^{q+1}\, dxdt, \quad I_3(r) = \int_{Q_r} |u|^2\, dxdt. \tag{3.8}$$

It is easy to check that

$$-\frac{dI_2(r)}{dr} = \int_{\mathbb{R}^N} h(r)|u(x,r)|^{q+1}\,dx \geq \int_{\mathbb{R}^N\setminus\Omega(\tau)} h(r)|u(x,r)|^{q+1}\,dx \quad \forall\,\tau > 0.$$

Therefore it follows from (3.6) and (3.7)

$$\int_{\mathbb{R}^N} |u(x,T)|^2\,dx + I_1(r) + I_2(r) + I_3(r) \leq c\tau^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} \left(-I_2'(r)\right)^{\frac{2}{q+1}} + c\int_{\Omega(\tau)} |u(x,r)|^2\,dx$$
$$\forall\,\tau > 0,\ \forall\, r : 0 < r < T. \quad (3.9)$$

Next we introduce additional energy functions

$$f(r,\tau) = \int_{\Omega(\tau)} |u(x,r)|^2\,dx, \quad E_1(r,\tau) = \int_{Q^r(\tau)} |D_x u|^2\,dxdt, \quad E_2(r,\tau) = \int_{Q^r(\tau)} |u|^2\,dxdt. \quad (3.10)$$

Now we deduce some vanishing estimates of these energy functions. Let μ be some nondecreasing smooth function defined on $(0,\infty)$, $\mu(\tau) > 0$ for $\tau > 0$ (a more precise definition will be fixed later on). Then multiplying the equation (3.3) by $u(x,t)\exp(-\mu^2(\tau)t)$ and integrating in domain $Q^r(\tau)$ with $\tau > k^{-1}$ (remember that $\operatorname{supp} u_{0,k} \subset \left\{|x| < k^{-1}\right\}$) we deduce easily

$$\begin{aligned} 2^{-1} f_{\mu,r}(\tau) + J_{\mu,r}(\tau) := 2^{-1}&\int_{\Omega(\tau)} |u(x,r)|^2 \exp(-\mu^2(\tau)r)\,dx + \\ &\int_{Q^r(\tau)} \left(|\nabla_x u|^2 + \mu^2(\tau)|u|^2\right)\exp(-\mu^2(\tau)t)\,dxdt \\ \leq \mu(\tau)^{-1} &\int_{\partial\Omega(\tau)\times(0,r)} \left(|\nabla_x u|^2 + \mu^2(\tau)|u|^2\right)\exp(-\mu^2(\tau)t)\,dsdt \quad \forall\,\tau > k^{-1}. \end{aligned} \quad (3.11)$$

Clearly there holds

$$\begin{aligned} \frac{dJ_{\mu,r}(\tau)}{d\tau} = &-\int_{\partial\Omega(\tau)\times(0,r)} \left(|\nabla_x u|^2 + \mu^2(\tau)|u|^2\right)\exp(-\mu^2(\tau)t)\,dsdt \\ &+ \int_{Q^r(\tau)} 2\mu\mu'(\tau)|u|^2 \exp(-\mu^2(\tau)t)\,dxdt \\ &- 2\int_{Q^r(\tau)} \mu\mu'(\tau)t\left(|\nabla_x u|^2 + \mu^2(\tau)|u|^2\right)\exp(-\mu^2(\tau)t)\,dxdt. \end{aligned}$$

Since $\mu'(\tau) > 0$, it follows from (3.11),

$$2^{-1} f_{\mu,r}(\tau) + J_{\mu,r}(\tau) \leq \mu(\tau)^{-1}\left[-\frac{d}{d\tau}J_{\mu,r}(\tau) + 2\int_{Q^r(\tau)} \mu(\tau)\mu'(\tau)|u|^2\exp(-\mu^2(\tau)t)\,dxdt\right]. \quad (3.12)$$

If we suppose

$$1 - \frac{2\mu'(\tau)}{\mu^2(\tau)} \geq 2^{-1}, \quad (3.13)$$

we derive from (3.12)

$$f_{\mu,r}(\tau) + J_{\mu,r}(\tau) \leq -2\mu(\tau)^{-1}\frac{dJ_{\mu,r}(\tau)}{d\tau}.$$

It is easy to check that this last inequality is equivalent to

$$\frac{\mu(\tau)}{2}\exp\left(\int_{\tau_1}^{\tau}\frac{\mu(s)}{2}\,ds\right)f_{\mu,r}(\tau)\le -\frac{d}{d\tau}\left(J_{\mu,r}(\tau)\exp\left(\int_{\tau_1}^{\tau}\frac{\mu(s)}{2}\,ds\right)\right)\quad \forall\,\tau>\tau_1>k^{-1}.$$

By integrating this inequality and using monotonicity of the function $f_{\mu,r}(\tau)$ we get

$$f_{\mu,r}(\tau_2)\int_{\tau_1}^{\tau_2}\frac{\mu(\tau)}{2}\exp\left(\int_{\tau_1}^{\tau}\frac{\mu(s)}{2}\,ds\right)d\tau+J_{\mu,r}(\tau_2)\exp\left(\int_{\tau_1}^{\tau_2}\frac{\mu(s)}{2}\,ds\right)\le J_{\mu,r}(\tau_1)\quad \forall\,\tau_2>\tau_1>k^{-1}.$$

Since

$$\frac{\mu(\tau)}{2}\exp\left(\int_{\tau_1}^{\tau_2}\frac{\mu(s)}{2}\,ds\right)=\frac{d}{d\tau}\left(\exp\left(\int_{\tau_1}^{\tau}\frac{\mu(s)}{2}\,ds\right)\right),$$

it follows from last the relation

$$f_{\mu,r}(\tau_2)\left[\exp\left(\int_{\tau_1}^{\tau_2}\frac{\mu(s)}{2}\,ds\right)-1\right]+J_{\mu,r}(\tau_2)\exp\left(\int_{\tau_1}^{\tau_2}\frac{\mu(s)}{2}\,ds\right)\le J_{\mu,r}(\tau_1)\quad \forall\,\tau_2>\tau_1>k^{-1}. \tag{3.14}$$

Now we have to define $\mu(\tau)$. Let $\varepsilon>0$ and

$$\mu(\tau)=\varepsilon r^{-1}(\tau-k^{-1})\quad \forall\,\tau>k^{-1}. \tag{3.15}$$

One can easily verify that condition (3.13) is equivalent to

$$\tau\ge k^{-1}+2\varepsilon^{-1/2}r^{1/2}. \tag{3.16}$$

Now from (3.14) follow two inequalities

$$A(\tau_2):=\int_{Q^r(\tau_2)}\left(|\nabla_x u|^2+\frac{\varepsilon^2(\tau_2-k^{-1})^2}{r^2}|u|^2\right)dxdt\le A(\tau_1)$$
$$\times\exp\left[-\frac{\varepsilon\left((\tau_2-k^{-1})^2-(\tau_1-k^{-1})^2\right)}{4r}+\frac{\varepsilon^2(\tau_2-k^{-1})}{r}\right]$$
$$\forall\,\tau_2>\tau_1>k^{-1}+2\varepsilon^{-1/2}r^{1/2}, \tag{3.17}$$

and

$$f(r,\tau_2)\le A(\tau_1)\left[\exp\left(\frac{\varepsilon\left((\tau_2-k^{-1})^2-(\tau_1-k^{-1})^2\right)}{4r}\right)-1\right]^{-1}\exp\left(\frac{\varepsilon^2(\tau_2-k^{-1})^2}{r}\right)$$
$$\forall\,\tau_2>\tau_1>k^{-1}+2\varepsilon^{-1/2}r^{1/2}. \tag{3.18}$$

In particular, for $\varepsilon=8^{-1}$ we obtain from (3.17) and (3.18),

$$\int_{Q^r(\tau)}\left(|\nabla_x u|^2+\frac{(\tau-k^{-1})^2}{64r^2}|u|^2\right)dxdt\le e\exp\left(-\frac{(\tau-k^{-1})^2}{64r}\right)\int_{Q^r(\tau_0^{(k)})}\left(|\nabla_x u|^2+\frac{|u|^2}{2r}\right)dxdt$$
$$\forall\,\tau\ge\tau_0^{(k)}(r):=k^{-1}+4\sqrt{2}\sqrt{r}, \tag{3.19}$$

and

$$f(r,\tau)\le\frac{e^2}{e-1}\exp\left(-\frac{(\tau-k^{-1})^2}{64r}\right)\int_{Q^r(\tau_0^{(k)})}\left(|\nabla_x u|^2+\frac{u^2}{2r}\right)dxdt\quad \forall\,\tau\ge\widetilde{\tau}_0^{(k)}(r):=k^{-1}+8\sqrt{r}. \tag{3.20}$$

In order to have an estimate from above of the last factor in the right-hand side of (3.19), (3.20), we return to the equation satisfied by u, multiply it by the test function $u_k(x,t)\exp(-t)$ and integrate over the domain $Q^r = \mathbb{R}^N \times (0,r)$. As result of standard computations we obtain, using (3.5),

$$\int_{\mathbb{R}^N} |u_k(x,r)|^2\,dx + \int_{Q^r} \left(|\nabla_x u_k|^2 + |u_k|^2 + h(t)|u_k|^{q+1}\right) dxdt \\ \leq \overline{c}\,\|u_{0,k}\|^2_{L_2(\mathbb{R}^N)} \leq cM_k \to \infty \text{ as } k \to \infty,\ \forall\, r \leq T. \quad (3.21)$$

Due to (3.20), (3.21) it follows from (3.9)

$$\int_{\mathbb{R}^N} |u(x,T)|^2\,dx + I_1(r) + I_2(r) + I_3(r) \\ \leq c_1 \tau^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_2'(r))^{\frac{2}{q+1}} + c_2 M_k r^{-1} \exp\left(-\frac{(\tau - k^{-1})^2}{64r}\right) \quad \forall\, \tau \geq \widetilde{\tau}_0^{(k)}(r). \quad (3.22)$$

Relationships (3.19), (3.20) due to (3.21) yield:

$$f(r,\tau) + E_1(r,\tau) + \frac{(\tau - k^{-1})^2}{64r^2} E_2(r,\tau) \leq c_2\, M_k r^{-1} \exp\left(-\frac{(\tau - k^{-1})^2}{64r}\right) \quad \forall\, \tau > \widetilde{\tau}_0^{(k)}(r). \quad (3.23)$$

Step 2. The first round of computations. Next we construct some sequences $\{\tau_j\},\ \{r_j\},\ j = k, k-1, \ldots, 1$. First we explicit the choice of M_k from condition (3.3), let namely

$$M_k = e^{e^k}. \quad (3.24)$$

Then we choose $\tau_k,\ r_k$ such that the following relation is true,

$$c_2\, r_k^{-1} \exp\left(-\frac{\tau_k^2}{64 r_k}\right) M_k = M_k^{\varepsilon_0}, \quad 0 < \varepsilon_0 < e^{-1} \quad (3.25)$$

where c_2 is from (3.22), (3.23). As consequence of (3.25) and (3.24) we get

$$\tau_k = 8 r_k^{1/2} \left[(1-\varepsilon_0)e^k + \ln r_k^{-1} + \ln c_2\right]^{1/2}. \quad (3.26)$$

In inequality (3.22) we fix $\tau = \tau_k + k^{-1}$, then due to definition (3.25) it follows from (3.22),

$$\int_{\mathbb{R}^N} |u(x,T)|^2\,dx + I_1(r) + I_2(r) + I_3(r) \\ \leq c_1 (k^{-1} + \tau_k)^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_2'(r))^{\frac{2}{q+1}} + M_k^{\varepsilon_0} \quad \forall\, r : 0 < r \leq r_k. \quad (3.27)$$

$I_1(r),\ I_2(r),\ I_3(r)$ are nonincreasing functions which satisfy, due to global a' priori estimate (3.21),

$$I_1(0) + I_2(0) + I_3(0) \leq cM_k. \quad (3.28)$$

Let us define the number r_k by

$$r_k = \sup\left\{r : I_1(r) + I_2(r) + I_3(r) \geq 2M_k^{\varepsilon_0}\right\}. \quad (3.29)$$

Then it follows from (3.27) the following differential inequality

$$I_1(r) + I_2(r) + I_3(r) + \int_{\mathbb{R}^N} |u(x,T)|^2\,dx \leq 2c_1 (\tau_k + k^{-1})^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_2'(r))^{\frac{2}{q+1}} \quad \forall\, r \leq r_k. \quad (3.30)$$

Solving it, we get

$$I_1(r) + I_2(r) + I_3(r) \le c_3(\tau_k + k^{-1})^N H(r)^{-\frac{2}{q-1}} \quad \forall\, r \le r_k, \tag{3.31}$$

where

$$H(r) = \int_0^r h(s)\, ds \quad \text{and } c_3 = \left(\frac{2}{q-1}\right)^{2/(q-1)} (2c_1)^{(q+1)/(q-1)}$$

Next we will use more specific functions

$$h(t) = \exp\left(-\frac{\omega(t)}{t}\right),$$

where $\omega(t)$ is nondecreasing and satisfies the following technical assumption

$$t^{\alpha_0} \le \omega(t) \le \omega_0 = \text{const} \quad \forall\, t : 0 < t < t_0,\ 0 \le \alpha_0 < 1. \tag{3.32}$$

It is easy to show by integration by parts the following relation

$$\int_0^r \exp\left(-\frac{a\omega(t)}{t}\right) dt \ge \frac{1-\delta(r)}{(1-\alpha_0)a} \cdot \frac{r^2}{\omega(r)} \exp\left(-\frac{a\omega(r)}{r}\right) \quad \forall\, r > 0,$$

where $\delta(r) \to 0$ if $r \to 0$. Therefore

$$H(r) \ge \overline{c}\frac{r^2}{\omega(r)} h(r),\ \overline{c} = \text{const} > 0. \tag{3.33}$$

As a consequence we derive from (3.31), using (3.26),

$$I_1(r) + I_2(r) + I_3(r) \le c_4 \left[8r_k^{\frac{1}{2}} \left((1-\varepsilon_0)e^k + \ln r_k^{-1} + \ln c_2\right)^{\frac{1}{2}} + k^{-1}\right]^N \times \frac{\omega(r)^{\frac{2}{q-1}}}{r^{\frac{4}{q-1}}} \exp\left(\frac{2\omega(r)}{(q-1)r}\right) \quad \forall\, r \le r_k. \tag{3.34}$$

Comparing (3.29) and estimate (3.34) we deduce that r_k satisfies

$$r_k \le b_k, \tag{3.35}$$

where b_k is solution of equation

$$c_4 \left[8b_k^{\frac{1}{2}} \left((1-\varepsilon_0)e^k + \ln b_k^{-1} + \ln c_2\right)^{\frac{1}{2}} + k^{-1}\right]^N \omega(b_k)^{\frac{2}{q-1}} b_k^{-\frac{4}{q-1}} \exp\left(\frac{2\omega(b_k)}{(q-1)b_k}\right) = 2M_k^{\varepsilon_0} = 2\exp(\varepsilon_0 e^k).$$

This equation may be rewritten in the form

$$\ln c_4 + \frac{2}{q-1} \ln\left(\frac{\omega(b_k)}{b_k}\right) + \frac{2}{q-1} \cdot \frac{\omega(b_k)}{b_k} + N \ln\left[8b_k^{\frac{N(q-1)-4}{2(q-1)N}} \left((1-\varepsilon_0)\exp k + \ln b_k^{-1} + \ln c_2\right)^{\frac{1}{2}} + k^{-1} b_k^{-\frac{2}{(q-1)N}}\right] = \ln 2 + \varepsilon_0 e^k \quad \forall\, k \in \mathbb{N}. \tag{3.36}$$

Since $s^{-1}\ln s \to 0$ as $s \to \infty$, it follows from equality (3.36) that

$$(1 + c\gamma(k))\varepsilon_0 e^k \geq A_k + \frac{2}{q-1}\frac{\omega(b_k)}{b_k}$$
$$:= N \ln\left[8 b_k^{\frac{N(q-1)-4}{2(q-1)N}} \left((1-\varepsilon_0)e^k + \ln b_k^{-1} + \ln c_2\right)^{\frac{1}{2}} + k^{-1} b_k^{-\frac{2}{N(q-1)}}\right]$$
$$+ \frac{2}{q-1}\frac{\omega(b_k)}{b_k} \geq (1-\gamma(k))\varepsilon_0 e^k \quad \forall\, k \in \mathbb{N}, \quad (3.37)$$

where $0 < \gamma(k) < 1,\ \gamma(k) \to 0$ as $k \to \infty$. Keeping in mind condition (3.32), we obtain easily

$$\frac{\omega(b_k)}{b_k} \geq b_k^{-(1-\alpha_0)}, \quad |A_k| \leq c\,(|\ln b_k| + k) \quad \forall\, k \in \mathbb{N}. \quad (3.38)$$

Due to properties (3.38), it follows from (3.37)

$$c e^k > \frac{\omega(b_k)}{b_k} \geq d_1 e^k \quad \forall\, k \in \mathbb{N},\ d_1 > 0. \quad (3.39)$$

As a consequence of (3.39), (3.38) we obtain also

$$\ln b_k^{-1} \leq ck \quad \forall\, k \in \mathbb{N}. \quad (3.40)$$

Now using estimate (3.39) we are able to obtain suitable upper estimate of τ_k. Thanks to (3.35), (3.39) and (3.40) we deduce from (3.26)

$$\tau_k \leq c b_k^{1/2} \exp\left(\frac{k}{2}\right) \leq c \exp\left(\frac{k}{2}\right)\left(\frac{\omega(b_k)}{d_1 \exp k}\right)^{1/2} = \frac{c}{d_1^{1/2}}\omega(b_k)^{1/2}.$$

Using again estimate (3.39) and the monotonicity of the function $\omega(s)$, we deduce from the above relation

$$\tau_k \leq c\left[\omega\left(\frac{\omega_0}{d_1 e^k}\right)\right]^{1/2}, \quad \omega_0 \text{ is from (3.32).} \quad (3.41)$$

Therefore, from inequalities (3.23) and (3.34), definitions (3.25), (3.29) and property (3.35), we derive the following estimates

$$I_1(r_k) + I_2(r_k) + I_3(r_k) \leq 2M_k^{\varepsilon_0} \quad \text{where } r_k \text{ is from (3.35), (3.29),} \quad (3.42)$$

$$f(r_k, \tau_k + k^{-1}) + E_1(r_k, \tau_k + k^{-1}) + \frac{\tau_k^2}{64 r_k^2} E_2(r_k, \tau_k + k^{-1}) \leq M_k^{\varepsilon_0}, \quad (3.43)$$

where τ_k is from (3.26), (3.41). Because $\varepsilon_0 < e^{-1}$, it follows from definition (3.24) of sequence M_k that

$$3M_k^{\varepsilon_0} < cM_{k-1} \quad \forall\, k \geq k_0(c), \quad (3.44)$$

where $c > 0$ is arbitrary constant. Therefore, adding estimates (3.42) and (3.43), we obtain thanks to (3.44) and the fact that $\tau_k \gg r_k$ (which follows from (3.25)), the inequality

$$f(r_k, \tau_k + k^{-1}) + \sum_{i=1}^{3} I_i(r_k) + \sum_{i=1}^{2} E_i(r_k, \tau_k + k^{-1}) < cM_{k-1} \quad \forall\, k \geq k_0(c). \quad (3.45)$$

Step 3. The second round of computations. Next we introduce the terms $r_{k-1},\ \tau_{k-1}$. Firstly we come back to inequality (3.14). Fixing here the function

$$\mu(t) = \varepsilon r^{-1}(\tau - k^{-1} - \tau_k) \quad \forall\,\tau > k^{-1} + \tau_k \tag{3.46}$$

instead of (3.15) and using estimates (3.16)–(3.20), we obtain

$$\begin{gathered}\int_{Q^r(\tau)} \left(|\nabla_x u|^2 + \frac{(\tau - k^{-1} - \tau_k)^2 |u|^2}{64 r^2} \right) dxdt \\ \leq e \exp\left(-\frac{(\tau - k^{-1} - \tau_k)^2}{64 r} \right) \int_{Q^r(\tau_0^{(k-1)}(r))} \left(|\nabla_x u|^2 + \frac{|u|^2}{2r} \right) dxdt \\ \forall\,\tau > \tau_0^{(k-1)}(r) := k^{-1} + \tau_k + 4\sqrt{2}\sqrt{r},\end{gathered} \tag{3.47}$$

and

$$\begin{gathered}f(r,\tau) \leq \frac{e^2}{e-1} \exp\left(-\frac{(\tau - k^{-1} - \tau_k)^2}{64 r} \right) \int_{Q^r(\tau_0^{(k-1)}(r))} \left(|\nabla_x u|^2 + \frac{|u|^2}{2r} \right) dxdt \\ \forall\,\tau \geq \tilde{\tau}_0^{(k-1)} := k^{-1} + \tau_k + 8\sqrt{r}.\end{gathered} \tag{3.48}$$

The integral term in the right-hand side of (3.47), (3.48) is estimated now by using estimate (3.45) obtained in the first round of computation. So, we have

$$\begin{gathered}\int_{Q^r(\tau_0^{(k-1)}(r))} \left(|\nabla_x u|^2 + \frac{u^2}{2r} \right) dxdt \leq (2r)^{-1} \left[\sum_{i=1}^{3} I_i(r_k) + \sum_{i=1}^{2} E_i(r_k, \tau_k + k^{-1}) \right] \leq c(2r)^{-1} M_{k-1} \\ \forall\,k > k_0(c),\ \forall\,r \geq r_k.\end{gathered} \tag{3.49}$$

Using this estimate we deduce from (3.47) and (3.48)

$$\begin{gathered}f(r,\tau) + E_1(r,\tau) + \frac{(\tau - \tau_k - k^{-1})^2}{64 r^2} E_2(r,\tau) \leq c_2 r^{-1} M_{k-1} \exp\left(-\frac{(\tau - \tau_k - k^{-1})^2}{64 r} \right) \\ \forall\,\tau \geq \tilde{\tau}_0^{(k-1)}(r).\end{gathered} \tag{3.50}$$

This estimate is similar to estimate (3.23) from first round. Now we have to deduce the analogue of estimate (3.31). For this we return to the starting relation (3.9), where we now estimate last term in right-hand side by estimate (3.48), using additionally (3.49). As a result we have

$$\begin{gathered}\sum_{i=1}^{3} I_i(r) \leq c_1 \tau^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_2'(r))^{\frac{2}{q+1}} + c_2 M_{k-1} r^{-1} \exp\left(-\frac{(\tau - \tau_k - k^{-1})^2}{64 r} \right) \\ \forall\,r \geq r_k,\ \forall\,\tau \geq \tilde{\tau}_0^{(k-1)}(r),\end{gathered} \tag{3.51}$$

which is analogous of estimate (3.22) from first round. Next we define the numbers τ_{k-1} and r_{k-1} by inequalities analogous to (3.26) and (3.29),

$$c_2 r_{k-1}^{-1} M_{k-1} \exp\left(-\frac{\tau_{k-1}^2}{64 r_{k-1}} \right) = M_{k-1}^{\varepsilon_0}, \quad 0 < \varepsilon_0 < e^{-1} \tag{3.52}$$

$$r_{k-1} = \sup\{ r : I_1(r) + I_2(r) + I_3(r) \geq 2 M_{k-1}^{\varepsilon_0} \}. \tag{3.53}$$

Now combining inequalities (3.30) and (3.44), and using definitions (3.52), (3.53), we obtain the following differential inequality

$$\sum_{i=1}^{3} I_i(r) \leq 2c_1(\tau_{k-1} + \tau_k + k^{-1})^{\frac{N(q-1)}{q+1}} h(r)^{-\frac{2}{q+1}} (-I_2'(r))^{\frac{2}{q+1}} \quad \forall\, r \leq r_{k-1}. \tag{3.54}$$

Solving this differential inequality, we obtain an estimate similar to (3.31). Using property (3.33) we arrive to

$$\sum_{i=1}^{3} I_i(r) \leq c_4(\tau_{k-1} + \tau_k + k^{-1})^N \frac{\omega(r)^{\frac{2}{q-1}}}{r^{\frac{4}{q-1}}} \exp\left(\frac{2\omega(r)}{(q-1)r}\right) \quad \forall\, r \leq r_{k-1}. \tag{3.55}$$

As in first round we express from (3.52) τ_{k-1} as function $\tau_{k-1}(r_{k-1})$ (the analogue of (3.26))

$$\tau_{k-1} = 8r_{k-1}^{1/2}[(1-\varepsilon_0)\exp(k-1) + \ln r_{k-1}^{-1} + \ln c_2]^{1/2}. \tag{3.56}$$

Inserting this expression of τ_{k-1} into (3.55) and then comparing the obtained inequality with definition (3.53), we deduce an estimate similar to (3.35),

$$r_{k-1} \leq b_{k-1}, \tag{3.57}$$

where b_{k-1} is solution of equation

$$\begin{aligned} &c_4\left[8b_{k-1}^{1/2}\left((1-\varepsilon_0)\exp(k-1) + \ln b_k^{-1} + \ln c_2\right)^{1/2} + \tau_k + k^{-1}\right]^N \\ &\qquad \times \frac{\omega(b_{k-1})^{\frac{2}{q-1}}}{b_{k-1}^{\frac{4}{q-1}}} \exp\left(\frac{2\omega(b_{k-1})}{(q-1)b_{k-1}}\right) = 2M_{k-1}^{\varepsilon_0} = 2\exp(\varepsilon_0 \exp(k-1)). \end{aligned} \tag{3.58}$$

From (3.50), and due to definition (3.52), it follows

$$f(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) + \frac{\tau_{k-1}^2}{64 r_{k-1}} E_2(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) + E_1(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) \leq M_{k-1}^{\varepsilon_0}. \tag{3.59}$$

From (3.55), due to (3.56), (3.57), (3.58), it follows

$$I_1(r_{k-1}) + I_2(r_{k-1}) + I_3(r_{k-1}) \leq 2M_{k-1}^{\varepsilon_0}. \tag{3.60}$$

Summing (3.59), (3.60) and using property (3.44), we deduce new global *a priori* estimate (the analogous of (3.45)) which is the main starting information for the next round of computation

$$f(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) + \sum_{i=1}^{3} I_i(r_{k-1}) + \sum_{i=1}^{2} E_i(r_{k-1}, \tau_{k-1} + \tau_k + k^{-1}) \leq cM_{k-2}. \tag{3.61}$$

We are ready now for the next round of computations, introducing the function

$$\mu(t) = \varepsilon r^{-1}(\tau - k^{-1} - \tau_k - \tau_{k-1}) \quad \forall\, \tau > k^{-1} + \tau_k + \tau_{k-1}$$

instead of (3.46) and estimate (3.61) instead of (3.45). We realize j rounds of such computations. As result we obtain

$$f\left(r_{k-j}, \sum_{l=0}^{j} \tau_{k-l} + k^{-1}\right) + \sum_{i=1}^{3} I_i(r_{k-j}) + \sum_{i=1}^{2} E_i\left(r_{k-j}, \sum_{l=0}^{j} \tau_{k-l} + k^{-1}\right) \leq cM_{k-j-1}, \tag{3.62}$$

which was our main aim.

Step 4. The control of $r_{k-j}, \sum_{l=0}^{j} \tau_{k-l}$ *as* $j \to k$ *with arbitrary* $k \in \mathbb{N}$. It is clear that $r_{k-j},\ \tau_{k-j}$ are defined by the conditions (see (3.52), (3.53))

$$c_2 r_{k-j}^{-1} M_{k-j} \exp\left(-\frac{\tau_{k-j}^2}{64 r_{k-j}}\right) = M_{k-j}^{\varepsilon_0}, \quad 0 < \varepsilon_0 < e^{-1}. \tag{3.63}$$

$$r_{k-j} = \sup\{r : I_1(r) + I_2(r) + I_3(r) \geq 2M_{k-j}^{\varepsilon_0}\}. \tag{3.64}$$

Similarly to (3.56)–(3.58) we deduce that

$$\tau_{k-j} = 8 r_{k-j}^{1/2} \left[(1-\varepsilon_0)e^{k-j} + \ln r_{k-j}^{-1} + \ln c_2\right]^{1/2}, \tag{3.65}$$

$$r_{k-j} \leq b_{k-j}, \tag{3.66}$$

where b_{k-j} satisfies

$$c_4 \left[8 b_{k-j}^{1/2}\left((1-\varepsilon_0)e^{k-j} + \ln b_{k-j}^{-1} + \ln c_2\right)^{1/2} + \sum_{i=0}^{j-1} \tau_{k-i} + k^{-1}\right]^N \times \frac{\omega(b_{k-j})^{\frac{2}{q-1}}}{b_{k-j}^{\frac{4}{q-1}}} \exp\left(\frac{2\omega(b_{k-j})}{(q-1)b_{k-j}}\right) = 2M_{k-j}^{\varepsilon_0} = 2\exp(\varepsilon_0 e^{k-j}). \tag{3.67}$$

In the first round of computations we have obtained the upper estimate (3.41) for τ_k. Let us suppose by induction that the following estimate is true

$$\tau_{k-i} \leq c\left[\omega\left(\frac{\omega_0}{d_1 \exp(k-i)}\right)\right]^{1/2} \quad \forall\, i \leq j-1. \tag{3.68}$$

We have to prove that estimate (3.68) holds also for $i = j$. Obviously condition (3.67) is equivalent to (see (3.36))

$$\ln c_4 + \frac{2}{q-1}\ln\left(\frac{\omega(b_{k-j})}{b_{k-j}}\right) + \frac{2}{q-1}\cdot\frac{\omega(b_{k-j})}{b_{k-j}} + A_k^{(j)} = \ln 2 + \varepsilon_0 e^{k-j}, \tag{3.69}$$

where

$$A_k^{(j)} = N \ln\left[b_{k-j}^{\frac{N(q-1)-4}{2(q-1)N}}\left((1-\varepsilon_0)e^{k-j} + \ln(b_{k-j}^{-1}) + \ln c_2\right)^{1/2} + \frac{k^{-1} + \sum_{i=0}^{j-1}\tau_{k-i}}{b_{k-j}^{\frac{2}{(q-1)N}}}\right].$$

Because of the induction assumption (3.68)

$$\sum_{i=0}^{j-1}\tau_{k-i} \leq c\sum_{i=0}^{j-1}\left[\omega\left(\frac{\omega_0}{d_1\exp(k-i)}\right)\right]^{1/2} \leq c\int_0^1 \frac{\omega(s)^{1/2}}{s}\,ds := cL,$$

therefore

$$|A_k^{(j)}| \leq c\left(|\ln b_{k-j}| + (k-j) + \ln L\right). \tag{3.70}$$

From (3.69) due to (3.70) we derive easily

$$ce^{k-j} \geq \frac{\omega(b_{k-j})}{b_{k-j}} \geq d_1 e^{k-j} \quad \forall\, j : k-j \geq k_0 = k_0(L), \tag{3.71}$$

where $k_0 < \infty$ do not depend on k. From (3.71) it follows in particular

$$\ln b_{k-j}^{-1} \leq c(k-j) \quad \forall\, j : k-j \geq k_0. \tag{3.72}$$

Thanks to (3.66) and properties (3.71), (3.72), we derive from (3.65),

$$\begin{aligned}\tau_{k-j} &\leq 8 b_{k-j}^{1/2} \left((1-\varepsilon_0) e^{k-j} + \ln b_{k-j}^{-1} + \ln c_2 \right)^{1/2} \\ &\leq c b_{k-j}^{1/2} \exp\left(\frac{k-j}{2}\right) \leq \frac{c}{d_1^{1/2}} [\omega(b_{k-j})]^{1/2} \quad \forall\, j : k-j \geq k_0(L).\end{aligned} \tag{3.73}$$

Using again estimate (3.71) and monotonicity of $\omega(s)$ we deduce from (3.73)

$$\tau_{k-j} \leq c \left[\omega \left(\frac{\omega_0}{d_1 e^{k-j}} \right) \right]^{1/2} \qquad \forall\, j : k-j \geq k_0(L). \tag{3.74}$$

Thus, we have proved by induction estimate (3.68), for arbitrary $k-j \geq k_0(L)$ with $r_i,\ \tau_i$ satisfying (3.66), (3.67) and (3.74).

Step 5. Completion of the proof. We fix now $n > k_0(L)$ and take $j = k-n$ in (3.62). This leads to

$$f\left(r_n, \sum_{l=0}^{k-n} \tau_{k-l} + k^{-1}\right) + \sum_{i=1}^{3} I_i(r_n) + \sum_{i=1}^{2} E_i\left(r_n, \sum_{l=0}^{k-n} \tau_{k-l} + k^{-1}\right) \leq c M_{n-1} \quad \forall\, n > k_0(L). \tag{3.75}$$

Next we have

$$\sum_{l=0}^{k-n} \tau_{k-l} \leq \sum_{i=n}^{\infty} \tau_i \leq c \sum_{i=n}^{\infty} \left[\omega\left(\frac{\omega_0}{d_1 \exp i}\right)\right]^{1/2} \leq c \int_0^{\frac{\omega_0}{d_1 \exp(n-1)}} \frac{\omega(s)^{1/2}}{s}\, ds \to 0 \text{ as } n \to \infty. \tag{3.76}$$

Therefore, for arbitrary small $\delta > 0$, we can find and fix $n = n(\delta) < \infty$ such that from (3.75) follows uniform with respect to $k \in \mathbb{N}$ *a priori* estimate,

$$\sup_{t>0} \int_{|x|>\delta} |u_k(x,t)|^2\, dx + \int_0^T \int_{|x|>\delta} \left(|\nabla_x u_k|^2 + |u_k|^2\right) dxdt \leq C = C(\delta) < \infty \quad \forall\, k \in \mathbb{N}. \tag{3.77}$$

Since $u_k(x,0) = 0 \ \ \forall\, |x| > k^{-1} \ \ \forall\, k \in \mathbb{N}$, it follows from (3.77) that $u_\infty(x,0) = 0\ \forall\, x \neq 0$, which ends the proof. □

4 Regional initial blow-up for equation with exponential absorption.

The local energy method we have used in the proof of Theorem 3.1 is based on the sharp interpolation theorems for functional Sobolev spaces, which are natural tool for the study of solutions of equations with power nonlinearities. Here we propose the adaptation of mentroned method to the equations with nonpower nonlinearities.

Thus, we consider the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u + h(t)(e^u - 1) = 0 \quad \text{in } Q^\infty \\ u(x,0) = k\delta_0, \end{cases} \tag{4.1}$$

Theorem 4.1 *Assume $h(t) = e^{-e^{\omega(t)/t}}$ where $\omega \in C([0,\infty))$ satisfies the same asumptions as in Theorem 3.1. Then solution u_k always exists and $u_\infty := \lim_{k\to\infty} u_k$ has a point-wise singularity at $(0,0)$.*

Proof. We will consider the family $u_k(x,t)$ of solutions of regularized problems:

$$\begin{cases} u_t - \Delta u + h(t)(e^u - 1) = 0 \quad \text{in } Q^T, \\ u(x,0) = u_{0,k}(x) = M_k^{1/2} k^{-N/2} \delta_k(x) \quad \forall x \in \mathbb{R}^N, \end{cases} \tag{4.2}$$

where δ_k is nonnegative, continuous with compact support in $B_{k^{-1}}$, satisfies estimate (3.5) and converges weakly to δ_0 as $k \to \infty$, $\{M_k\}$ satisfies condition (3.2). Let us introduce the energy functions (we omit index k in u_k):

$$I_{1,0}(r) = \int_{Q_r} |\nabla_x u|^2 \, dxdt, \quad I_q(r) = (q!)^{-1} \int_{Q_r} h(t)|u|^{q+1} \, dxdt, \quad I_{3,0}(r) = \int_{Q_r} |u|^2 \, dxdt. \tag{4.3}$$

Multiplying (4.2) by $u(x,t)\exp\left(-\dfrac{t-r}{1+T-r}\right)$, integrating in Q_r and using equality

$$s(e^s - 1) = \sum_{q=1}^{\infty} \frac{s^{q+1}}{q!},$$

we obtain easily

$$I_{1,0}(r) + \sum_{l=1}^{\infty} I_l(r) + I_{3,0}(r) \le c(q!)^{2/(q+1}\tau^{N(q-1)/(q+1)} h(r)^{-2/(q+1)} (-I_q'(r))^{2/(q+1)}$$
$$+ c\int_{\Omega(\tau)} |u(x,r)|^2 \, dx \quad \forall\, \tau > 0, \ \forall\, r : 0 < r < T, \ \forall\, q \in \mathbb{N}. \tag{4.4}$$

We introduce the additional energy functions

$$f(r,\tau) \text{ from (3.10)}, \quad E_{1,0}(r,\tau) = \int_{Q^r(\tau)} |D_x u|^2 \, dxdt, \quad E_{2,0}(r,\tau) = \int_{Q^r(\tau)} |u|^2 \, dxdt. \tag{4.5}$$

Instead of (3.21) we derive the following global *a priori* estimate:

$$\int_{\mathbb{R}^N} |u_k(x,r)|^2 \, dx + \int_{Q^r} \left(|\nabla_x u|^2 + |u_k|^2 + h(t) \sum_{l=1}^{\infty} \frac{|u_k|^{l+1}}{l!} \right) dxdt$$
$$\le \overline{c}\, \|u_{0,k}\|^2_{L_2(\mathbb{R}^N)} \le cM_k \quad \forall\, r < T. \tag{4.6}$$

Using estimate (4.6) instead of (3.21) in a similar way as in the proof of Theorem 3.1, we obtain the following inequality, analogous to (3.23),

$$f(r,\tau)+E_{1,0}(r,\tau)+\frac{(\tau-k^{-1})^2}{64r^2}E_{2,0}(r,\tau)+$$
$$\leq c_2M_kr^{-1}\exp\left(-\frac{(\tau-k^{-1})^2}{64r}\right)\quad\forall\,\tau\geq\widetilde{\tau}_0^{(k)}(r)=k^{-1}+8\sqrt{r}. \quad (4.7)$$

Using this estimate we deduce from (4.4)

$$I_{1,0}(r)+\sum_{l=1}^{\infty}I_l(r)+I_{3,0}(r)\leq c(q!)^{\frac{2}{q+1}}\tau^{\frac{N(q-1)}{q+1}}h(r)^{-\frac{2}{q+1}}(-I_q'(r))^{\frac{2}{q+1}}$$
$$+c_2M_kr^{-1}\exp\left(-\frac{(\tau-k^{-1})^2}{64r}\right)\quad\forall\,\tau\geq\widetilde{\tau}_0^{(k)}(r),\ \forall\,q\in\mathbb{N}. \quad (4.8)$$

Next, we define the numbers $\tau_k,\ r_k$. Firstly, set

$$r_k:=\sup\left\{r:I_{1,0}(r)+\sum_{l=1}^{\infty}I_l(r)+I_{3,0}\geq 2M_k^{\varepsilon_0}\right\},\quad 0<\varepsilon_0<e^{-1}. \quad (4.9)$$

Then we fix the sequence $\{M_k\}$ by (3.24) again and τ_k by inequalities (3.25), (3.26). Thanks to these definitions we derive the following series of inequalities from relations (4.8)

$$I_{1,0}(r)+\sum_{l=1}^{\infty}I_l(r)+I_{3,0}(r)\leq 2c_1(q!)^{\frac{2}{q+1}}(\tau_k+k^{-1})^{\frac{N(q-1)}{q+1}}h(r)^{-\frac{2}{q+1}}(-I_q'(r))^{\frac{2}{q+1}}\quad\forall\,q\in\mathbb{N},\ \forall\,r\leq r_k. \quad (4.10)$$

Solving these differential inequalities we obtain the estimates

$$I_{1,0}(r)+\sum_{l=1}^{\infty}I_l(r)+I_{3,0}(r)\leq c_3(\tau_k+k^{-1})^N(q!)^{\frac{2}{q-1}}H(r)^{-\frac{2}{q-1}}\quad\forall\,r\leq r_k,\ \forall\,q\in\mathbb{N}, \quad (4.11)$$

where $H(r)$ is from (3.31). We have now to optimize estimate (4.11) with respect to parameter q. By integration by parts, it is easy to check the following inequality

$$H(r)\geq\overline{c}\frac{r^2}{\omega(r)}\exp\left(-\frac{\omega(r)}{r}\right)h(r)\quad\forall\,r>0,\ \overline{c}>0. \quad (4.12)$$

Using Stirling formula $q!\sim\left(\frac{q}{e}\right)^q$ and estimate (4.12), we deduce from (4.11)

$$I_{1,0}(r)+\sum_{l=1}^{\infty}I_l(r)+I_{3,0}(r)\leq c_4(\tau+k^{-1})^NF_q(r)\quad\forall\,r\leq r_k, \quad (4.13)$$

where

$$F_q(r)=q^2\omega(r)^{\frac{2}{q-1}}r^{-\frac{4}{q-1}}\exp\left(\frac{2}{q-1}\cdot\frac{\omega(r)}{r}\right)\exp\left[\frac{2}{q-1}\exp\left(\frac{\omega(r)}{r}\right)\right].$$

Fixing here the optimal value of the parameter q:

$$q=\widetilde{q}:=\left[2\exp\left(\frac{\omega(r)}{r}\right)\right],$$

where $[a]$ denotes the enteger part of a, we obtain easily

$$F_{\widetilde{q}}\leq c\exp\left(\frac{2\omega(r)}{r}\right).$$

Therefore it follows from (4.13),

$$I_{1,0}(r)+\sum_{l=1}^{\infty}I_l(r)+I_{3,0}(r)\le c_5(\tau_k+k^{-1})^N\exp\left(\frac{2\omega(r)}{r}\right)\quad\forall\, r\le r_k. \tag{4.14}$$

Comparing now definition (4.9) of r_k and estimate (4.14), and using additionally the expression (3.26) of τ_k, we obtain

$$r_k\le b_k, \tag{4.15}$$

where b_k is defined by the equation

$$c_5\left[8b_k^{1/2}((1-\varepsilon_0)e^k+\ln b_k^{-1}+\ln c_2)^{1/2}+k^{-1}\right]^N\exp\left(\frac{2\omega(b_k)}{b_k}\right)$$
$$=2M_k^{\varepsilon_0}=2\exp(\varepsilon_0\exp k),\quad 0<\varepsilon_0<e^{-1}. \tag{4.16}$$

By an analysis similar to Step 2 in the proof of Theorem 3.1, we obtain estimates (3.37)–(3.40) for b_k. Then we prove the validity of estimate (3.41) for τ_k. As a consequence of estimates (4.7), (4.14), thanks to to definitions (3.26), (4.9) of $\tau_k,\ r_k$ and the previous estimates of $\tau_k,\ r_k$, we get

$$I_{1,0}(r)+\sum_{l=1}^{\infty}I_l(r)+I_{3,0}(r)\le 2M_k^{\varepsilon_0},$$
$$f(r_k,\tau_k+k^{-1})+E_{1,0}(r_k,\tau_k+k^{-1})+\frac{\tau_k^2}{64r_k^2}E_{2,0}(r_k,\tau_k+k^{-1})\le M_k^{\varepsilon_0}.$$

Summing these inequalities, and using definition of $\{M_k\}$ and property $\tau_k\gg r_k$, we obtain an analogue of estimate (3.45), namely,

$$f(r_k,\tau_k+k^{-1})+I_{1,0}(r_k)+\sum_{l=1}^{\infty}I_l(r_k)+I_{3,0}(r_k)+E_{1,0}(r_k,\tau_k+k^{-1})+E_{2,0}(r_k,\tau_k+k^{-1})\le cM_{k-1}. \tag{4.17}$$

Using (4.17) as global *a priori* estimate instead of (4.6) and providing a second round of computations similar to (3.46)–(3.57) we derive a second global *a priori* estimate analogous to (3.61),

$$f(r_{k-1},\tau_{k-1}+\tau_k+k^{-1})+I_{1,0}(r_{k-1})+\sum_{l=1}^{\infty}I_l(r_{k-1})+I_{3,0}(r_{k-1})$$
$$+E_{1,0}(r_{k-1},\tau_{k-1}+\tau_k+k^{-1})+E_{2,0}(r_{k-1},\tau_{k-1}+\tau_k+k^{-1})\le cM_{k-2}.$$

Repeating such rounds j-times we derive a corresponding analogue of relation (3.62). It is easy to see that estimate (3.76) for constructed shifts τ_{k-i} remains valid. This fact, similar to what was used in the proof of Theorem 3.1, yields to the conclusion. □

5 The porous media equation with absorption

In this section we consider the following problem dealing with fundamental solutions of the porous media equation with time dependent absorption,

$$\begin{cases}\partial_t u-\Delta(|u|^{m-1}u)+h(t)|u|^{q-1}u=0 & \text{in } Q^T\\ u(x,0)=k\delta_0.\end{cases} \tag{5.1}$$

It is standard to assume that $h \geq 0$ is a continuous function and m, q are positive real numbers. By a solution we mean a function $u \in L^1_{loc}(Q^T)$ such that $u^m \in L^1_{loc}(Q^T)$, $hu^q \in L^1_{loc}(Q^T)$ and

$$\iint_{Q^T} \left(-u\partial_t\phi - |u|^{m-1}u\Delta\phi + h(t)|u|^{q-1}u\phi\right) dxdt = k\phi(0,0) \tag{5.2}$$

for any $\phi \in C_0^{2,1}(\mathbb{R}^N \times [0,T))$. If $h \equiv 0$ and $m > (N-2)_+/N$ this problem admits a solution for any $k > 0$. When $m > 1$ this solution has the following form

$$B_k(x,t) = t^{-\ell}\left(C_k - \frac{(m-1)\ell}{2mN}\frac{|x|^2}{t^{2\ell/N}}\right)_+^{1/(m-1)}, \tag{5.3}$$

where

$$\ell = \frac{N}{N(m-1)+2} \quad \text{and } C_k = a(m,N)k^{2(m-1)\ell/N}. \tag{5.4}$$

Since B_k is a supersolution for problem (5.1), a sufficient condition for existence (and uniqueness) of u_k is

$$\iint_{Q^T} B_k^q(x,t)h(t)dxdt < \infty. \tag{5.5}$$

By the change of variable $y = t^{\ell/N}x$ this condition is independent of $k > 0$ and we have

Proposition 5.1 *Assume $m > 1$, $q > 0$. If*

$$\int_0^1 h(t)t^{\ell-\ell q}dt < \infty, \tag{5.6}$$

then problem (5.1) admits a unique positive solution $u = u_k$. In the particular case where $h(t) = O(t^\alpha)$ ($\alpha \geq 0$), the condition is

$$\alpha > \frac{N(q-m)-2}{N(m-1)+2}. \tag{5.7}$$

We recall that if $q > 1$ and $m > (N-2)_+/N$, any solution of the porous media equation with absorption is bounded from above by the maximal solution U_h expressed by

$$U_h(t) = \left((q-1)\int_0^t h(s)\,ds\right)^{-1/(q-1)}. \tag{5.8}$$

Theorem 5.2 *Assume $q+1 > 2m > 2$ and $h \in C((0,\infty))$ is nondecreasing, positive and satisfies $h(t) = O(t^{(q-m)/(m-1)})$ as $t \to 0$. Then for any $k > 0$ u_k exists and $\lim_{k\to\infty} u_k := u_\infty = U_h$.*

Proof. We first notice that

$$q+1 > 2m > 2 \Longrightarrow q > m > 1 \quad \text{and} \quad \frac{q-m}{m-1} > \frac{N(q-m)-2}{N(m-1)+2}.$$

Step 1. Case $q < m + 2/N$. In this range of value we know [14] that there exists a nonnegative very singular solution $v = v_\infty$ to

$$\partial_t v - \Delta v^m + v^q = 0 \quad \text{in } Q^T, \tag{5.9}$$

and $v_\infty = \lim_{k\to} v_k$, where the v_k are solutions of the same equation with initial data $k\delta_0$. Furthermore, v_∞ is unique [6], radial with respect to x and has the following form

$$v_\infty(x,t) = t^{-1/(q-1)}F(|x|/t^{(q-m)/2(q-1)}),$$

where F solves

$$\begin{cases} (F^m)'' + \dfrac{N-1}{\eta}(F^m)' + \dfrac{q-m}{2(q-1)}\eta F' + \dfrac{1}{q-1}F - F^q = 0 \quad \text{in } (0,) \\ F'(0) = 0 \text{ and } \lim_{\eta\to\infty}\eta^{2/(q-m)}F(\eta) = 0. \end{cases} \tag{5.10}$$

Actually F has compact support in $[0,\xi_0]$ for some $\xi_0 > 0$. Let $\gamma = (q-m)/(m-1)$, then for any $\epsilon > 0$, $u = u_\infty$ satisfies, for some $c > 0$,

$$\partial_t u - \Delta u^m + c\epsilon^\gamma u^q \geq 0 \quad \text{in } Q^\epsilon.$$

If we set $w_\epsilon(x,t) = a^\theta v_\infty(x,at)$ with $\theta = 1/(m-1)-$ and $a = \epsilon^{-1}c^{-(q-1)/(q-m)}$, then

$$\partial_t w_\epsilon - \Delta w_\epsilon^m + c\epsilon^\gamma w_\epsilon^q = 0 \quad \text{in } Q^T.$$

By comparison $u_\infty \geq w_\epsilon$ in Q^ϵ. If we take in particular $t = \epsilon$, it implies

$$u_\infty(x,t) \geq c^{-1/(q-m)}t^{-1/(m-1)}v_\infty(x, c^{-(m-1)/(q-m)}) = c^{-1}t^{-1/(m-1)}F(c^{(m-1)/2(q-1)}|x|) \tag{5.11}$$

If $|x| < \xi_c = c^{-(m-1)/2(q-1)}\xi_0$, we derive that $\lim_{t\to 0} u_\infty(x,t) = \infty$, locally uniformly in B_{ξ_c}. This implies $u_\infty = U_h$.

Step 2. Case $q \geq m + 2/N$. We give an alternative proof valid for all q. We first observe that it is sufficient to prove the result when $h(t)$ is replaced by t^γ. If we look for a family of transformations $u \mapsto T_\ell(u)$ under the form

$$T_\ell(u)(x,t) = \ell^\alpha u(\ell^\beta x, \ell t) \quad \forall (x,t) \in Q^\infty,\ \forall \ell > 0$$

which leaves the equation

$$\partial_t u - \Delta|u|^{m-1}u + t^\gamma|u|^{q-1}u = 0 \tag{5.12}$$

invariant, we find $\alpha = (1+\gamma)/(q-1)$ and $\beta = (q-m-\gamma(m-1))/2(q-1)$. Due to the value of γ, we have $\beta = 0$. Because of uniqueness and the value of the initial mass

$$T_\ell(u_k) = u_{\ell^\alpha k} \quad \forall \ell > 0,\ \forall k > 0 \Longrightarrow T_\ell(u_\infty) = u_\infty \quad \forall \ell > 0. \tag{5.13}$$

Therefore

$$\ell^\alpha u_\infty(x,\ell t) = u_\infty(x,t) \quad \forall (x,t) \in Q^\infty,\ \forall \ell > 0.$$

In particular, if we take $\ell = t^{-1}$,

$$u_\infty(x,t) = t^{-\alpha}u_\infty(x,1) = t^{-\alpha}\phi(x).$$

Plugging this decomposition into (5.12) yields to

$$-\alpha t^{-\alpha-1}\phi - t^{-\alpha m}\Delta\phi^m + t^{\gamma-\alpha q}\phi^q = 0,$$

where all the exponents of t coincide since

$$\alpha m = \frac{m}{m-1}\ ,\ \alpha q - \gamma = \frac{m}{m-1} \ \text{ and } \ \alpha + 1 = \frac{m}{m-1}.$$

Therefore ϕ is a positive and radial (as the u_k are) solution of

$$-\alpha\phi - \Delta\phi^m + \phi^q = 0 \quad \text{in } \mathbb{R}^N.$$

Setting $\psi = \phi^m$ yields to

$$-\Delta\psi - \frac{1}{m-1}\psi^{1/m} + \psi^{q/m} = 0 \quad \text{in } \mathbb{R}^N. \tag{5.14}$$

Clearly $\psi = \psi_0 = (m-1)^{-m/(q-1)}$ is a solution. By a standard variation of the Keller-Osserman estimate, any solution is bounded from above by ψ_0. Putting $\tilde{\psi}(x) = A\psi(a)$, it is easy to find $A > 0$ and $a > 0$ such that

$$-\Delta\tilde{\psi} - \tilde{\psi}^{1/m} + \tilde{\psi}^{q/m} = 0 \quad \text{in } \mathbb{R}^N, \tag{5.15}$$

with $0 \leq \tilde{\psi} \leq 1$. Writting $\tilde{\psi}$ as a solution of an ODE, we derive

$$\tilde{\psi}(r) = \tilde{\psi}(0) + \int_0^r s^{1-n} \int_0^s (\tilde{\psi}^{q/m} - \tilde{\psi}^{1/m})\sigma^{n-1} ds \quad \forall r > 0.$$

If $\tilde{\psi}^{q/m}$ is not constant with value 1, the right-hand side of the above inequality is decreasing with respect to r, and the only possible nonnegative limit is 0, by La Salle principle. Thus

$$\tilde{\psi}'' + \frac{N-1}{r}\tilde{\psi}' + \frac{1}{2}\tilde{\psi}^{1/m} \leq 0$$

for $r \geq r_0$, large enough. If $N = 2$, we set $\tau = \ln r$, $\Psi(\tau) = \tilde{\psi}(r)$ and get

$$\Psi'' + \frac{1}{2}e^{2\tau}\Psi^{1/m} \leq 0$$

for $\tau \geq \ln r_0$. The concavity of Ψ yields a contradiction. If $N \geq 3$, we set $\tau = r^{N-2}/(N-2)$ and $\Psi(\tau) = r^{N-2}\tilde{\psi}(r)$. Then Ψ satisfies

$$\Psi'' + c_N\tau^{(4-N)/(N-2)-1/m}\Psi^{1/m} \leq 0.$$

Again the concavity yields a contradiction. In any case we obtain that $\Psi = 1$, or, equivalently $\psi = \psi_0$ and finally, $u_\infty = t^{-1/(m-1)}\psi_0^{1/m}$. □

Theorem 5.3 *Assume $q > m > 1$ and $h \in C((0,\infty))$ is nondecreasing, positive. If $h(t) = t^{(q-m)/(m-1)}\omega^{-1}(t)$ with $\omega(t) \to 0$ as $t \to 0$, and*

$$\int_0^1 \omega^\theta(s)\frac{ds}{s} < \infty, \tag{5.16}$$

where

$$\theta = \frac{m^2-1}{[N(m-1)+2(m+1)](q-1)},$$

then $u_\infty := \lim_{k\to\infty} u_k$ has a point-wise singularity at $(0,0)$

Proof. The structure of the proof is similar to the one of Theorem 3.1. We study the asymptotic behaviour as $k \to \infty$ of solutions $u = u_k(x,t)$ of the regularized Cauchy problem

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + h(t)|u|^{q-1}u = 0 \quad \text{in } Q^T \\ u(x,0) = u_{0,k}(x) = M_k^{\frac{1}{m+1}} k^{-\frac{mN}{m+1}} \delta_k(x) \qquad x \in \mathbb{R}^N, \end{cases} \tag{5.17}$$

where δ_k is as in Theorem 3.1. Let us rewrite problem (5.17) in the form

$$\begin{cases} (|v|^{p-1}v)_t - \Delta v + h(t)|v|^{g-1}v = 0, \quad \text{in } Q^T \\ v = v_k = |u|^{m-1}u, \ p = 1/m, \ g = q/m \\ |v(x,0)|^{p-1}v(x,0) = |v_{0,k}|^{p-1}v_{0,k} := u_{0,k}(x) = M_k^{\frac{p}{p+1}} k^{-\frac{N}{p+1}} \delta_k(x). \end{cases} \tag{5.18}$$

Without loss of generality we may suppose

$$\|\delta_k(x)\|^{\frac{p+1}{p}}_{L_{\frac{p+1}{p}}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\delta_k(x)|^{\frac{p+1}{p}} \, dx \le c_0 k^{\frac{N}{p}} \qquad \forall\, k \in \mathbb{N}. \tag{5.19}$$

Now sequence $\{M_k\}$ is such that

$$M_k^{\frac{p}{p+1}} k^{-\frac{N}{p+1}} \to \infty \quad \text{as} \quad k \to \infty. \tag{5.20}$$

Step 1. The local energy framework. Consider the following energy functions

$$I_1(\tau) = \int_{Q_r} |\nabla_x v|^2 \, dxdt, \qquad I_2(\tau) = \int_{Q_r} h(t)|v|^{g+1} \, dxdt, \qquad I_3(\tau) = \int_{Q_r} |v|^{p+1} \, dxdt. \tag{5.21}$$

Analogously to (3.9) we deduce the inequality

$$\int_{\mathbb{R}^N} |v(x,T)|^{p+1} \, dx + I_1(r) + I_2(r) + I_3(r) \le c\tau^{\frac{N(g-p)}{g+1}} h(r)^{-\frac{p+1}{g+1}} (-I_2'(r))^{\frac{p+1}{g+1}} + c\int_{\Omega(\tau)} |v(x,r)|^{p+1} \, dx$$
$$\forall\, \tau > 0, \ \forall\, r : 0 < r < T. \tag{5.22}$$

This inequality will control the spreading of energy with respect to the r-variable (the time direction). As to vanishing property of energy in variable τ, we will use the finite speed propagation of support property for porous media equation with slow diffusion. In the domain $Q^{(r)}(\tau)$ we will use the energy function $E_1(r,\tau) = \int_{Q^{(r)}(\tau)} |\nabla_x v|^2 \, dxdt$ from (3.12). Since $\operatorname{supp} v(\cdot,0) = \operatorname{supp} v_k(\cdot,0) = \operatorname{supp} v_{0,k} = \{x : |x| < k^{-1}\}$, multiplying equation (5.18) on $v(x,t)$ and integrating in the domain $Q^{(r)}(\tau)$, $\tau \ge k^{-1}$, we obtain after simple computations (see, for example [1, 4]) the following differential inequality

$$\int_{\Omega(\tau)} |v(x,r)|^{p+1} \, dx + E_1(r,\tau) \le c r^{\frac{(p+1)(1-\theta_1)}{p+1-(1-\theta_1)(1-p)}} \Big(-\frac{d}{d\tau} E_1(r,\tau) \Big)^{\frac{p+1}{p+1-(1-\theta_1)(1-p)}}, \tag{5.23}$$

$$\forall\, \tau \ge k^{-1}, \ \forall\, r > 0 \quad \text{where} \ \ \theta_1 = \frac{N(1-p)+(p+1)}{N(1-p)+2(p+1)}, \ 1-\theta_1 = \frac{p+1}{N(1-p)+2(p+1)}.$$

Solving this inequality and keeping in mind that $E_1(r,\tau) \ge 0 \ \forall\, r > 0, \ \forall\, \tau > 0$, we deduce easily

$$v(x,r) \equiv 0 \quad \forall\, x : |x| > k^{-1} + c_0 r^{1-\theta_1} E_1(r,k^{-1})^{\frac{(1-\theta_1)(1-p)}{1+p}} := k^{-1} + c_0\chi(r), \quad \forall\, r > 0. \tag{5.24}$$

Here the constant $c_0 > 0$ depends on the parameters of the problem under consideration, but do not on r and k. Analogously to (3.25) we deduce the following global *a priori* estimate

$$\int_{Q^{(r)}} (|\nabla_x v|^2 + r^{-1}|v|^{p+1} + h(t)|v|^{g+1})\, dxdt \leq c\, \|v_{0,k}\|^{p+1}_{L_{p+1}(\mathbb{R}^N)}. \tag{5.25}$$

Thus, due to (5.18)–(5.20), it follows from (5.25)

$$E_1(r, 0) \leq cM_k \qquad \forall\, r > 0. \tag{5.26}$$

Next we come back to the inequality (5.22). Due to (5.24) it ensues from (5.22) the inequality

$$I_1(r) + I_2(r) + I_3(r) \leq c(k^{-1} + \chi(r))^{\frac{N(g-p)}{g+1}} h(r)^{-\frac{p+1}{g+1}} (-I_2'(r))^{\frac{p+1}{g+1}} \qquad \forall\, r > 0. \tag{5.27}$$

Remark that due to (5.26) we have

$$\chi(r) \leq c_1 r^{1-\theta_1}\, M_k^{\frac{(1-\theta_1)(1-p)}{1+p}}. \tag{5.28}$$

Step 2. The first round of computations. Now we have to define $\tau_k,\ r_k$. First we impose the relation

$$\tau_k \geq c_1 r_k^{1-\theta_1}\, M_k^{\frac{(1-\theta_1)(1-p)}{1+p}}, \qquad c_1 \text{ is from (5.28)}. \tag{5.29}$$

Then (5.27) yields to

$$I(r) := I_1(r) + I_2(r) + I_3(r) \leq c(k^{-1} + \tau_k)^{\frac{N(g-p)}{g+1}} h(r)^{-\frac{p+1}{g+1}} (-I'(r))^{\frac{p+1}{g+1}} \qquad \forall\, r : 0 < r < r_k. \tag{5.30}$$

Solving this differential inequality we get the estimate

$$I(r) \leq \frac{c(k^{-1} + \tau_k)^N}{\left(\int_0^r h(s)\, ds\right)^{\frac{p+1}{g-p}}} \qquad \forall\, r : 0 < r < r_k. \tag{5.31}$$

Remember that the function $h(s)$ has the form $h(s) = s^{(g-1)/(1-p)}\omega(s)^{-1}$, therefore estimate (5.31) yields to

$$I(r) \leq \frac{c_2 \omega(r)^{\frac{p+1}{g-p}} (k^{-1} + \tau_k)^N}{r^{\frac{p+1}{1-p}}} \qquad \forall\, r : 0 < r \leq r_k. \tag{5.32}$$

Thus, as second relation, which defines our pair $\tau_k,\ r_k$, we suppose the condition

$$\frac{c_2 \omega(r_k)^{\frac{p+1}{g-p}} (k^{-1} + \tau_k)^N}{r_k^{\frac{p+1}{1-p}}} \leq cM_{k-1}, \quad c \text{ is from (5.26)}. \tag{5.33}$$

Moreover, we will find the pair $\tau_k,\ r_k$ such that the following property holds

$$k^{-1} + \tau_k \leq 1. \tag{5.34}$$

Then the next inequality is a sufficient condition for validity of (5.33):

$$c_2 \omega(r_k)^{\frac{p+1}{g-p}} r_k^{-\frac{p+1}{1-p}} \leq cM_{k-1}, \quad c \text{ is from (5.26)}, \tag{5.35}$$

and we can define r_k by equality

$$r_k := \left(\frac{c_2}{c}\right)^{\frac{1-p}{p+1}} \omega(r_k)^{\frac{1-p}{g-p}} M_{k-1}^{-\frac{1-p}{p+1}}. \tag{5.36}$$

Now we have to choose the sequence $\{M_k\}$. Namely, we set

$$M_k := e^k \qquad \forall\, k \in \mathbb{N}, \tag{5.37}$$

and we define τ_k, in accordance with assumption (5.29), by

$$\tau_k = c_1 r_k^{1-\theta_1} M_k^{\frac{(1-\theta_1)(1-p)}{1+p}}, \quad c_1 \text{ is from (5.28)}. \tag{5.38}$$

Further, due to (5.36) an (5.37), it follows from (5.38),

$$\tau_k = c_1 (r_k^{p+1} M_k^{1-p})^{\frac{1}{N(1-p)+2(p+1)}} = c_1 \Big[\Big(\frac{c_2}{c}\Big)^{1-p} \omega(r_k)^{\frac{(1-p)(p+1)}{g-p}} M_{k-1}^{-(1-p)} M_k^{1-p} \Big]^{\frac{1}{N(1-p)+2(p+1)}}$$
$$= c_1 \Big(\frac{ec_2}{c}\Big)^{\frac{(1-\theta_1)(1-p)}{1+p}} \omega(r_k)^S, \tag{5.39}$$

where $S = \frac{(1-\theta_1)(1-p)}{g-p} = \frac{(1-p)(p+1)}{(g-p)[N(1-p)+2(p+1)]}$. From definition (5.36) and because of (5.37) and (3.43), there holds

$$r_k \le \Big(\frac{c_2}{c}\Big)^{\frac{1-p}{p+1}} \omega_0^{\frac{1-p}{g-p}} \exp\Big(-\frac{1-p}{p+1}(k-1)\Big) := c_3 \exp\Big(-\frac{1-p}{p+1}\, k\Big), \tag{5.40}$$

and $r_k \to 0$ as $k \to \infty$. Therefore, since $\omega(s) \to 0$ as $s \to 0$, it follows from (5.39) that $\tau_k \to 0$ as $k \to \infty$. Consequently we can suppose k so large that condition (5.34) is satisfied. Thus, we have pair $(\tau_k,\ r_k)$ for large $k \in \mathbb{N}$.

Step 3. The second round of computations. As a starting global *a priori* estimate of solution we will use now, instead of (5.25), (5.26), the following estimate

$$I_1(r_k) = \int_{\substack{\{t \ge r_k,\\ x \in \mathbb{R}^N\}}} |\nabla_x v|^2 \, dxdt \le I(r_k) \le c M_{k-1}, \tag{5.41}$$

which follows from (5.32), due to definition (5.33), (5.36) of r_k. Using property (5.24), estimate (5.28) and property (5.29), it ensues from (5.41)

$$E_1(r, k^{-1} + \tau_k) \le I_1(r) \le I_1(r_k) < c M_{k-1} \qquad \forall\, r \ge r_k. \tag{5.42}$$

Since $v(x, r_k) = 0 \ \forall\, x : |x| \ge k^{-1} + \tau_k$ we deduce similarly to (5.23)

$$\int_{\Omega(\tau)} |v(x, r_k + r)|^{p+1} \, dx + E_1(r_k + r, k^{-1} + \tau_k + \tau) \le c r^{\frac{(p+1)(1-\theta_1)}{(p+1)-(1-\theta_1)(1-p)}}$$
$$\times \Big(-\frac{d}{d\tau} E_1(r_k + r, k^{-1} + \tau_k + \tau) \Big)^{\frac{p+1}{p+1-(1-\theta_1)(1-p)}} \qquad \forall\, r > 0,\ \forall\, \tau > 0. \tag{5.43}$$

Solving this differential inequality, we obtain

$$v(x, r_k + r) \equiv 0 \quad \forall\, x : |x| \ge k^{-1} + \tau_k + c_0 \chi_1(r), \tag{5.44}$$

where $\chi_1(r) := r^{1-\theta_1} E_1(r_k + r, k^{-1} + \tau_k)^{\frac{(1-\theta_1)(1-p)}{1+p}}$ $\forall\, r \ge 0$. But (5.42) implies

$$\chi_1(r) \le c_1 r^{1-\theta_1} M_{k-1}^{\frac{(1-\theta_1)(1-p)}{1+p}}. \tag{5.45}$$

Now we define $\tau_{k-1},\ r_{k-1}$. In the same way as (5.29) we impose

$$\tau_{k-1} \geq c_1 r_{k-1}^{1-\theta_1} M_{k-1}^{\frac{(1-\theta_1)(1-p)}{1+p}}. \tag{5.46}$$

Similarly to (5.30)–(5.32) we deduce

$$I(r) \leq \frac{c_2 \omega(r)^{\frac{p+1}{g-p}} (k^{-1} + \tau_k + \tau_{k-1})^N}{r^{\frac{p+1}{1-p}}} \qquad \forall\, r : 0 < r \leq r_k + r_{k-1}. \tag{5.47}$$

The second relation for defining the pair $\tau_{k-1},\ r_{k-1}$ is analogous to(5.33)

$$\frac{c_2 \omega(r_k + r_{k-1})^{\frac{p+1}{g-p}} (k^{-1} + \tau_k + \tau_{k-1})^N}{(r_k + r_{k-1})^{\frac{p+1}{1-p}}} \leq c M_{k-2}, \quad c \text{ is from (5.26)}. \tag{5.48}$$

Supposing that

$$k^{-1} + \tau_k + \tau_{k-1} \leq 1, \tag{5.49}$$

we can define r_{k-1} by the following analogue of (5.36)

$$r_k + r_{k-1} := \left(\frac{c_2}{c}\right)^{\frac{1-p}{p+1}} \omega(r_k + r_{k-1})^{\frac{1-p}{g-p}} M_{k-2}^{-\frac{1-p}{p+1}}. \tag{5.50}$$

And in accordance with (5.46) let us define τ_{k-1} by

$$\tau_{k-1} = c_1 r_{k-1}^{1-\theta_1} M_{k-1}^{\frac{(1-\theta_1)(1-p)}{1+p}}. \tag{5.51}$$

Due to (5.50) we have

$$\begin{aligned} \tau_{k-1} &\leq c_1 \left[(r_k + r_{k-1})^{p+1} M_{k-1}^{1-p} \right]^{\frac{1}{N(1-p)+2(p+1)}} \\ &\leq c_1 \left[\left(\frac{c_2}{c}\right)^{1-p} \omega(r_k + r_{k-1})^{\frac{(1-p)(p+1)}{g-p}} M_{k-2}^{-(1-p)} M_{k-1}^{1-p} \right]^{\frac{1}{N(1-p)+2(p+1)}} \\ &= c_1 \left(\frac{e c_2}{c}\right)^{\frac{(1-\theta_1)(1-p)}{1+p}} \omega(r_k + r_{k-1})^S, \end{aligned}$$

where S is from (5.39). Notice that, due to (5.47), (5.48), we have also

$$I_1(r_k + r_{k-1}) \leq I(r_k + r_{k-1}) \leq c M_{k-2}, \tag{5.52}$$

and, analogously to (5.42),

$$E_1(r, k^{-1} + \tau_k + \tau_{k-1}) \leq I_1(r) \leq I_1(r_k + r_{k-1}) \leq c M_{k-2} \qquad \forall\, r \geq r_k + r_{k-1}. \tag{5.53}$$

Step 4. Completion of the proof. Estimates (5.52), (5.53) we can use instead of (5.41), (5.42) for third round of computations. After j such rounds we deduce that

$$I_1\left(\sum_{i=0}^{j} r_{k-i}\right) \leq I\left(\sum_{i=0}^{j} r_{k-i}\right) \leq c M_{k-j}, \tag{5.54}$$

$$E_1\left(r, k^{-1} + \sum_{i=0}^{j} \tau_{k-i}\right) \leq I_1(r) \leq I_1\left(\sum_{i=0}^{j} r_{k-i}\right) \leq c M_{k-j} \qquad \forall\, r \geq \sum_{i=0}^{j} r_{k-i}, \tag{5.55}$$

where

$$\tau_{k-i} \le c_1\Big(\frac{ec_2}{c}\Big)^{\frac{(1-\theta_1)(1-p)}{1+p}}\omega\Big(\sum_{l=0}^{i} r_{k-l}\Big)^S, \tag{5.56}$$

with the same S as in (5.39), and

$$\sum_{l=0}^{i} r_{k-l} = \Big(\frac{c_2}{c}\Big)^{\frac{1-p}{p+1}}\omega\Big(\sum_{l=0}^{i} r_{k-l}\Big)^{\frac{1-p}{g-p}} M_{k-i-1}^{-\frac{1-p}{p+1}}. \tag{5.57}$$

Estimates (5.54) will remain true as long as the following analogue of relation (5.49) is valid

$$k^{-1} + \sum_{i=0}^{j} \tau_{k-i} \le 1.$$

Now we will check this condition. Due to (3.32), it follows from (5.57)

$$\sum_{l=0}^{i} r_{k-l} \le \Big(\frac{c_2}{c}\Big)^{\frac{1-p}{p+1}}\omega_0^{\frac{1-p}{g-p}} M_{k-i-1}^{-\frac{1-p}{p+1}} := CM_{k-i-1}^{-\frac{1-p}{p+1}} = C\exp\Big(-\frac{1-p}{p+1}(k-i-1)\Big).$$

Therefore, from (5.56), it follows

$$\tau_{k-i} \le c_1\Big(\frac{ec_2}{c}\Big)^{\frac{(1-\theta_1)(1-p)}{1+p}}\omega\Big(C\exp\Big(-\tfrac{(1-p)(k-i-1)}{p+1}\Big)\Big)^S := C_1\Big[\omega\Big(C\exp\Big(-\tfrac{(1-p)(k-i-1)}{p+1}\Big)\Big)\Big]^S.$$

Thus we have, using in particular the monotonicity of function $\omega(s)$,

$$\begin{aligned}\sum_{i=0}^{j}\tau_{k-i} &\le C_1\sum_{i=0}^{j}\Big[\omega\Big(C\exp\Big(-\frac{(1-p)(k-i-1)}{p+1}\Big)\Big)\Big]^S\\ &\le C_1\int_{k-j-1}^{k}\Big[\omega\Big(C\exp\Big(-\frac{(1-p)s}{p+1}\Big)\Big)\Big]^S ds = \frac{C_1(p+1)}{1-p}\int_{A_1}^{A_2}\frac{\omega(s)^S}{s}\,ds,\\ &A_1 = C\exp\big(-\frac{1-p}{p+1}k\big),\quad A_2 = C\exp\big[-\frac{1-p}{p+1}(k-j-1)\big].\end{aligned} \tag{5.58}$$

Due to condition (5.16) and estimate (5.58) we can find $k_0 \in \mathbb{N}$, which depends on parameters of problem under consideration, but does not depend on $k \in \mathbb{N}$, such that

$$\sum_{i=0}^{k-k_0}\tau_{k-i} + k^{-1} \le 1 \qquad \forall\, k \in \mathbb{N}.$$

At end, our estimates (5.54)–(5.57) are true for all $j \le k-k_0$. Therefore the proof of Theorem 5.3 follows from estimates (5.54)–(5.57), in the same way as Theorem 3.1 from estimates (3.75)–(3.77). □

6 The fast diffusion equation with absorption

When $(1-2/N)_+ < m < 1$, it is known that the mere fast diffusion equation

$$\partial_t v - \Delta v^m = 0 \quad \text{in } Q^\infty \tag{6.1}$$

admits a particular fundamental positive solution with initial data $k\delta_0$ $(k > 0)$ called the Barenblatt -Zeld'dovich-Kompaneets solution, expressed by

$$B_k(x,t) = t^{-\ell}\left(C_k + \frac{(1-m)\ell}{2mN}\frac{|x|^2}{t^{2\ell/N}}\right)^{-1/(1-m)}, \tag{6.2}$$

where ℓ and C_k are given in (5.4). The main feature of this expression is that $\lim_{k\to\infty} C_k = 0$, therefore

$$\lim_{k\to\infty} B_k(x,t) = W(x,t) := C_*\left(\frac{t}{|x|^2}\right)^{1/1-m)}, \tag{6.3}$$

where

$$C_* = \left(\frac{(1-m)^3}{2m(mN+2-N)}\right)^{1/(1-m)}.$$

This solution has a persisting singularity and is called a razor blade [18]. It has also the property that

$$\lim_{t\to 0} W(x,t) = 0 \quad \forall x \neq 0.$$

This phenomenon is at the origin of the work of Chasseigne and Vàzquez on extended solutions of the fast diffusion equation [3]. Concerning problem (5.1), Proposition 5.1 is still valid provided $m > (1+2/N)_+$. We shall denote by $u = u_k$ the solutions of (5.1). Furthermore estimate (5.8) holds. Combining this with the fact that the B_k are super solutions for the u_k, we derive the following

Theorem 6.1 *Assume* $(1-2/N)_+ < m < 1$ *and* $h \in C(0,\infty)$ *is positive. Assume also that* (5.6) *holds. Then* $u_\infty := \lim_{k\to\infty} u_k$ *has a point-wise singularity at* $(0,0)$ *and the following estimate is verified*

$$u_\infty(x,t) \leq \min\left\{C_* t^{-\ell}\left(\frac{|x|^2}{t^{2\ell/N}}\right)^{-1/(1-m)}, \left((q-1)\int_0^t h(s)\,ds\right)^{-1/(q-1)}\right\} \tag{6.4}$$

Remark. The profile of u_∞ near $(x,t) = (0,0)$ is completely unknown. In particular a very chaleng-ing question could be to give precise estimates on the quantity $\min\{W(x,t), U_h(t)\} - u_\infty(x,t)$.

References

[1] Antontsev S. N., *On the localization of solutions of nonlinear degenerate elliptic and parabolic equations*, Dokl. Akad. Nauk. SSSR, **260** (1981), 1289–1293 (translin. Soviet. Math. Dokl. **24** (1981)).

[2] Brezis H., Peletier L. A. and Terman D., *A very singular solution of the heat equation with absorption*, Arch. Rat. Mech. Anal **95**, 185-209 (1986). (1985).

[3] Chasseigne E. and Vàzquez J. L. *Theory of extended solutions for fast-diffusion equations in optimal classes of data. Radiation from singularities*, Arch. Rat. Mech. Anal **164**, 133-187 (2002).

[4] Diaz J. I., Veron L., *Local vanishing properties of elliptic and parabolic quasilinear equations*, Trans. Amer. Math. Soc. **290** (1985), 787–814.

[5] Galaktionov V. A. and Shishkov A.E., *Saint-Venant's principle in blow-up for higher-order quasilinear parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A **133** (2003), 1075–1119.

[6] Kamin S. and Véron L., *Existence and uniqueness of the very singular solution of the porous media equation with absorption*, J. Anal. Mathématque**51** (1988), 245–258.

[7] Marcus M. and Véron L., *Initial trace of positive solutions of some nonlinear parabolic equations*, Comm. Part. Diff. Equ. **24** (1999), 1445–1499.

[8] Marcus M. and Véron L., *Initial trace of positve solutions to semilinear parabolic inequalities*, Adv. Nonlinear Studies **2** (2002), 395–436.

[9] Marcus M. and Véron L., *Semilinear parabolic equations with measure boundary data and isolated singularities*, J. Anal. Math. **85** (2001), 245–290.

[10] Marcus M. and Véron L.,*Boundary trace of positive solutions of nonlinear elliptic inequalities* Ann. Scu. Norm. Sup. Pisa **5**, 481-533 (2004)

[11] Marcus M. and Véron L.,*The boundary trace and generalized boundary value problem for semilinear elliptic equations with a strong absorption*, Comm. Pure Appl. Math. **56** (6), 689-731 (2003).

[12] Oleinik O. A. and Iosif'yan G. A., *An analogue of Saint-Venant's principle and the uniqueness of solutions of boundary-value problem for parabolic equations in unbounded domains*, Russ. Math. Surv. **31** (1976), 153–178.

[13] Oleinik O. A. and Radkevich E.V., *Method of introducing of a parameter in evolution equation*, Russian Math. Survey **33** (1978), 7–74.

[14] Peletier L. A. and Terman D., *A very singular solution of the porous media equation with absorption*, J. Diff. Equ. **65** (1986), 396–410.

[15] Shishkov A.E. and Shchelkov A. G.,*Dynamics of the support of energy solutions of mixed problems for quasi-linear parabolic equations of arbitrary type*, Izvestiya: Mathematics **62**:3 (1998), 601–626.

[16] Shishkov A. E., *Propagation of perturbation in a singular Cauchy problem for degenerate quasilinear patabolic equations*, Sbornik: Mathematics **187**:9 (1996), 1391–1440.

[17] Shishkov A. E., *Dead cores and instanteous compactification of the support of energy solutions of quasilinear parabolic equations of arbitrary order*, Sbornik: Mathematics **190**:12 (1999), 1843–1869.

[18] Vàzquez J. L. and Véron L., *Different kinds of singular solutions of nonlinear parabolic equations*, in: Nonlinear Problems in Applied Mathematics, SIAM, Philadelphia, PA, 1996, pp. 240–249