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Boundary singularities of positive solutions of some nonlinear elliptic equations

Singularités au bord de solutions positives d'équations elliptiques non-linéaires

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Abstract

We study the behavior near x_0 of any positive solution of (E) $-\Delta u = u^q$ in Ω which vanishes on $\partial\Omega \setminus \{x_0\}$, where $\Omega \subset \mathbb{R}^N$ is a smooth domain, $q \geq (N+1)/(N-1)$ and $x_0 \in \partial\Omega$. Our results are based upon *a priori* estimates of solutions of (E) and existence, non-existence and uniqueness results for solutions of some nonlinear elliptic equations on the upper-half unit sphere. *To cite this article: M.-F. Bidaut-Véron, A.C. Ponce, L. Véron, C. R. Acad. Sci. Paris, Ser. I XXX (2006).*

Résumé

Nous étudions le comportement quand x tend vers x_0 de toute solution positive de (E) $-\Delta u = u^q$ dans Ω qui s'annule sur $\partial\Omega \setminus \{x_0\}$, où $\Omega \subset \mathbb{R}^N$ est un domaine régulier, $q \geq (N+1)/(N-1)$ et $x_0 \in \partial\Omega$. Nos résultats sont fondés sur des estimations a priori des solutions de (E), et des résultats d'existence, de non existence et d'unicité de solutions de certaines équations elliptiques non linéaires sur la demi-sphère unité. *Pour citer cet article : M.-F. Bidaut-Véron, A.C. Ponce, L. Véron, C. R. Acad. Sci. Paris, Ser. I XXX (2006).*

Version française abrégée Soit Ω un ouvert régulier de \mathbb{R}^N , $N \geq 2$, tel que $0 \in \partial\Omega$. Étant donné $q > 1$, nous considérons une fonction $u \in C^2(\Omega) \cap C(\bar{\Omega} \setminus \{0\})$ qui vérifie (3). Nous nous intéressons à la description du comportement de u au voisinage de 0.

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Nous distinguerons les trois valeurs critiques de q données par (4). Si $1 < q < q_1$, le comportement en 0 des solutions est décrit dans [4] ; aussi supposons-nous le plus souvent $q \geq q_1$. Si u est une solution de (3) dans \mathbb{R}_+^N de la forme $u(x) = u(r, \sigma) = r^{-2/(q-1)}\omega(\sigma)$, alors ω vérifie l'équation (6). Dans ce cas, nous avons le résultat suivant :

Théorème 0.1 (i) Si $1 < q \leq q_1$, le problème (3) n'admet aucune solution.

(ii) Si $q_1 < q < q_3$, (3) admet une unique solution, notée ω_0 .

(iii) Si $q \geq q_3$, (3) n'admet aucune solution.

Le résultat d'unicité décrit en (ii) est en fait un cas particulier d'un résultat plus général :

Théorème 0.2 Pour tous $q > 1$ et $\lambda \in \mathbb{R}$, il existe au plus une solution positive de (7).

Ce résultat demeure si, dans (7), S_+^{N-1} est remplacé par une boule dans \mathbb{R}^N , et Δ' par le laplacien ordinaire.

Par simplicité, nous pouvons supposer que $\partial\mathbb{R}_+^N$ est l'hyperplan tangent à Ω en 0. Le théorème ci-dessous donne une classification des singularités isolées du problème (3) :

Théorème 0.3 Soit $q \geq q_1$, avec $q \neq q_2$. Supposons que la solution u du problème (3) vérifie

$$0 \leq u(x) \leq C|x|^{-2/(q-1)} \quad \forall x \in \Omega \cap B_a(0), \quad (1)$$

pour $C, a > 0$. Si $q_1 \leq q < q_3$, ou bien u est continue en 0, ou bien

$$u(r, \sigma) = \begin{cases} r^{-(N-1)} (\log(1/r))^{\frac{1-N}{2}} (k_N \sigma_1 + o(1)) & \text{si } q = q_1, \\ r^{-2/(q-1)} (\omega_0(\sigma) + o(1)) & \text{si } q_1 < q < q_3, \end{cases} \quad (2)$$

lorsque $r \rightarrow 0$, uniformément par rapport à $\sigma \in S_+^{N-1}$; k_N est une constante qui dépend seulement de N . Si $q \geq q_3$, u est continue en 0.

L'estimation a priori (1) est obtenue pour $q_1 \leq q < q_2$:

Théorème 0.4 Si $q_1 \leq q < q_2$, toute solution u de (3) vérifie (1) pour $C = C(N, q, \Omega) > 0$.

Les démonstrations détaillées sont présentées dans [2].

1. Introduction and main result

Let Ω be a smooth open subset of \mathbb{R}^N , $N \geq 2$, such that $0 \in \partial\Omega$ and let $q > 1$. Assume that $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a solution of

$$\begin{cases} -\Delta u = u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \quad (3)$$

Our goal in this paper is to describe the behavior of u in a neighborhood of 0.

This problem has similar features with the case where $x_0 \in \Omega$, which has been studied by Gidas-Spruck [7]. In our case, we encounter three critical values of q in describing the local behavior of u :

$$q_1 := \frac{N+1}{N-1}, \quad q_2 := \frac{N+2}{N-2} \quad \text{if } N \geq 3 \quad \text{and} \quad q_3 := \frac{N+1}{N-3} \quad \text{if } N \geq 4. \quad (4)$$

When $1 < q < q_1$, it is proved in [4] that for every solution u of (3) there exists $\alpha \geq 0$ (depending on N and u) such that

$$u(x) = \alpha |x|^{-N} \rho(x) (1 + o(1)) \quad \text{as } x \rightarrow 0, \quad (5)$$

where $\rho(x) = \text{dist}(x, \partial\Omega)$, $\forall x \in \Omega$. For this reason, we shall mainly restrict ourselves to $q \geq q_1$.

Let us first consider the case where $\Omega = \mathbb{R}_+^N$ and we look for solutions of (3) of the form $u(x) = u(r, \sigma) = r^{-2/(q-1)} \omega(\sigma)$, where $r = |x|$ and $\sigma \in S_+^{N-1}$. An easy computation shows that ω must satisfy

$$\begin{cases} -\Delta' \omega = \ell_{N,q} \omega + \omega^q & \text{in } S_+^{N-1}, \\ \omega \geq 0 & \text{in } S_+^{N-1}, \\ \omega = 0 & \text{on } \partial S_+^{N-1}, \end{cases} \quad (6)$$

where Δ' denotes the Laplacian in S^{N-1} and $\ell_{N,q} = \frac{2(N-q(N-2))}{(q-1)^2}$. Concerning equation (6), we prove

Theorem 1.1 (i) *If $1 < q \leq q_1$, then (6) admits no positive solution.*

(ii) *If $q_1 < q < q_3$, then (6) admits a unique positive solution.*

(iii) *If $q \geq q_3$, then (6) admits no positive solution.*

One of the main ingredients in the proof of Theorem 1.1 (ii) is the following

Theorem 1.2 *If $q > 1$ and $\lambda \in \mathbb{R}$, then there exists at most one positive solution of*

$$\begin{cases} -\Delta' v = \lambda v + v^q & \text{in } S_+^{N-1}, \\ v = 0 & \text{on } \partial S_+^{N-1}. \end{cases} \quad (7)$$

Remark 1 We emphasize that in Theorem 1.2 we *do not* assume that q is subcritical. The conclusion above remains valid if, in (7), S_+^{N-1} is replaced by $B_1 \subset \mathbb{R}^N$ and Δ' by the usual Laplacian in \mathbb{R}^N . Theorem 1.2 extends a previous result of Kwong-Li [8].

We now return to the case where $\Omega \subset \mathbb{R}^N$ is an arbitrary smooth set such that $0 \in \partial\Omega$. For simplicity, we may assume that $\partial\mathbb{R}_+^N$ is the tangent hyperplane of Ω at 0. Using Theorem 1.2, we provide a classification of isolated singularities of solutions of (3):

Theorem 1.3 *Let $q \geq q_1$, $q \neq q_2$, and let u be a solution of (3). Assume that u satisfies*

$$0 \leq u(x) \leq C |x|^{-2/(q-1)} \quad \forall x \in \Omega \cap B_a(0), \quad (8)$$

for some $C, a > 0$. If $q_1 \leq q < q_3$, then either u is continuous at 0 or

$$u(r, \sigma) = \begin{cases} r^{-(N-1)} (\log(1/r))^{-\frac{1-N}{2}} (k_N \sigma_1 + o(1)) & \text{if } q = q_1, \\ r^{-2/(q-1)} (\omega_0(\sigma) + o(1)) & \text{if } q_1 < q < q_3, \end{cases} \quad (9)$$

as $r \rightarrow 0$, uniformly with respect to $\sigma \in S_+^{N-1}$; k_N denotes a constant depending only on N and ω_0 is the unique positive solution of (6).

If $q \geq q_3$, then u is continuous at 0.

Remark 2 We do not know whether Theorem 1.3 is true when $q = q_2$. In this case, the equation is conformally invariant and thus other techniques are required. If $\Omega = \mathbb{R}_+^N$, then it can be proved that any solution of (3) depends only on the variables $r = |x|$ and $\theta = \cos^{-1}(x_1/|x|)$.

The next result establishes the existence of an *a priori* estimate for the solutions of (3). According to Theorem 1.4 below, assumption (8) is always fulfilled when $q_1 \leq q < q_2$:

Theorem 1.4 *Let $q_1 \leq q < q_2$ and let u be a solution of (3). Then,*

$$0 \leq u(x) \leq C \rho(x) |x|^{-2/(q-1)-1} \quad \forall x \in \Omega \cap B_1(0), \quad (10)$$

where C depends on N , q and Ω .

Remark 3 According to the Doob Theorem [6], any positive superharmonic function v in Ω satisfies $\int_{\Omega} |\Delta v| \rho < \infty$ and admits a boundary trace, which is a Radon measure on $\partial\Omega$. If u is a solution of (3), then its trace must be of the form $k\delta_{x_0}$, for some $k \geq 0$. We may have $k > 0$ if $1 < q < q_1$ (see [1]), but k is necessarily equal to 0 if $q \geq q_1$. Indeed, by the maximum principle, u satisfies $u \geq kP_{\Omega}(x, 0)$, where P_{Ω} denotes the Poisson potential of Ω . Since $u^q \in L^1_{\rho}(\Omega)$ (by the Doob Theorem), we must have $k = 0$ if $q \geq q_1$.

Detailed proofs will appear in [2].

2. Sketch of the proofs

Proof of Theorem 1.1. Assertion (i) is proved by multiplying (6) by $\phi(\sigma) = \sigma_1$. Note that ϕ is the first eigenfunction of $-\Delta'$ on S_+^{N-1} , with eigenvalue $\lambda_1 = N - 1$. Integrating the resulting expression over S_+^{N-1} , and using the fact that $1 < q \leq q_1 \implies \ell_{N,q} \geq \lambda_1$, we obtain (i).

The existence part in (ii) is obtained by using the Mountain Pass Theorem; the uniqueness is a consequence of Theorem 1.2.

Assertion (iii) can be deduced from the following Pohožaev-type identity:

Proposition 2.1 *Assume $N \geq 4$ and $q > 1$. Then, any solution of (7) satisfies*

$$\begin{aligned} \frac{N-3}{q+1} (q-q_3) \int_{S_+^{N-1}} |\nabla' v|^2 \phi \, d\sigma - \frac{(N-1)(q-1)}{q+1} \left(\lambda + \frac{N-1}{q-1} \right) \int_{S_+^{N-1}} v^2 \phi \, d\sigma = \\ = - \int_{\partial S_+^{N-1}} |\nabla' v|^2 \, d\tau. \end{aligned}$$

This identity is obtained by computing the divergence of the vector field $P = \langle \nabla' \phi, \nabla' v \rangle \nabla' v$, where ∇' is the gradient on S^{N-1} , and then using the fact that the first eigenfunction satisfies $D^2 \phi + \phi g_0 = 0$, where g_0 is the tensor of the standard metric on S^{N-1} . In order to establish (iii), it suffices to observe that $\ell_{N,q} \leq -\frac{N-1}{q-1} \iff q \geq q_3$.

Proof of Theorem 1.2. We first notice that any positive solution of (7) depends only on the variable $\theta = \cos^{-1}(x_1/|x|) \in [0, \pi/2]$; this follows from a straightforward adaptation of the Gidas-Ni-Nirenberg moving plane method to S_+^{N-1} (see [9]). Thus, v satisfies

$$\begin{cases} v'' + (N-2) \cot \theta v' + \lambda v + v^q = 0 & \text{in } (0, \pi/2), \\ v'(0) = 0, \quad v(\pi/2) = 0. \end{cases} \quad (11)$$

Let $w(\theta) := \sin^{\alpha} \theta v(\theta)$, where $\alpha > 0$. By choosing $\alpha = 2(N-2)/(q+3)$, then w satisfies

$$(w'(\pi/2))^2 = \int_0^{\pi/2} G'(\theta) w^2(\theta) \, d\theta, \quad (12)$$

where G is a function of the form $G(\theta) = \sin^{\beta'} \theta (\alpha_1 \sin^2 \theta + \alpha_2)$; the parameters $\alpha_1, \alpha_2, \beta' \in \mathbb{R}$ can be explicitly computed in terms of λ, N and q .

Assume, by contradiction, that v_1 and v_2 are two distinct solutions of (11). Then,

$$\int_0^{\pi/2} v_1 v_2 (v_2^{q-1} - v_1^{q-1}) \, d\theta = 0. \quad (13)$$

Therefore, their graphs must intersect at some $\theta_0 \in (0, \pi/2)$. We claim that v_1 and v_2 intersect at least twice in $(0, \pi/2)$. If there is only one intersection point, then it can be shown that there exists $\gamma \geq 0$ such

that the function $\theta \mapsto G'(\theta)(w_2^2(\theta) - \gamma w_1^2(\theta))$ never vanishes in $(0, \pi/2)$. We then let $L(t) := (t^2 - \gamma)^{-1}$, $\forall t \in \mathbb{R} \setminus \{\gamma\}$. By (12) and the Mean Value Theorem, there exists $\theta_1 \in (0, \pi/2)$ such that

$$L\left(\frac{w_2'(\pi/2)}{w_1'(\pi/2)}\right) = \frac{\int_0^{\pi/2} G'(\theta) w_1^2(\theta) d\theta}{\int_0^{\pi/2} G'(\theta) [w_2^2(\theta) - \gamma w_1^2(\theta)] d\theta} = L\left(\frac{w_2(\theta_1)}{w_1(\theta_1)}\right).$$

Since L is injective in \mathbb{R}_+ , this implies

$$\frac{w_2'(\pi/2)}{w_1'(\pi/2)} = \frac{w_2(\theta_1)}{w_1(\theta_1)}. \quad (14)$$

On the other hand, by the Sturm-Liouville Theory, the function $\theta \mapsto w_2(\theta)/w_1(\theta)$ is (strictly) monotone. L'Hôpital's Rule yields a contradiction as we let $\theta \rightarrow \pi/2$. Therefore, v_1 and v_2 must intersect at least twice. This fact leads to another contradiction by using the Shooting Method (see [8]). Thus, $v_1 = v_2$ in $(0, \pi/2)$.

Remark 4 The method above follows the lines of the proof of Kwong-Li [8]. The main difference is that we use an alternative argument based on the Mean Value Theorem in order to deduce (14). In [8], they have to assume that the exponent q is subcritical.

Proof of Theorem 1.3. It follows from methods developed in [7] and [3]. For simplicity, we shall assume that $a = 1$ and $\partial\Omega \cap B_1 = \partial\mathbb{R}_+^N \cap B_1$. We set

$$w(t, \sigma) = r^{2/(q-1)} u(r, \sigma), \quad t = \log(1/r) \in (0, \infty) \times S_+^{N-1} := Q.$$

Then, w satisfies

$$w_{tt} - \left(N - 2\frac{q+1}{q-1}\right) w_t + \Delta' w + \ell_{N,q} w + w^q = 0 \quad \text{in } Q \quad (15)$$

and w vanishes on $(0, \infty) \times \partial S_+^{N-1}$. Since w is uniformly bounded on Q , standard *a priori* estimates for elliptic problems yield

$$|\partial_t^k \nabla'^j w| \leq M_{k,j} \quad \text{in } (1, \infty) \times S_+^{N-1}$$

for any integers $k, j \geq 0$, where ∇'^j stands for the covariant derivative on S^{N-1} . Thus, the trajectory $\mathcal{T}_w = \{w(t, \cdot) : t \geq 1\}$ is relatively compact in $C^2(S_+^{N-1})$. Multiplying (15) by w_t and integrating over S_+^{N-1} , we obtain

$$\frac{d}{dt} H(t) = \left(N - 2\frac{q+1}{q-1}\right) \int_{S_+^{N-1}} w_t^2 d\sigma, \quad (16)$$

where

$$H(t) := \frac{1}{2} \int_{S_+^{N-1}} \left(w_t^2 - |\nabla' w|^2 - \ell_{N,q} w^2 + \frac{2}{q+1} w^{q+1} \right) d\sigma.$$

Since $q \neq q_2$, we know that $N - 2(q+1)/(q-1) \neq 0$. Thus, iterated energy estimates imply that $w_t(t, \cdot), w_{tt}(t, \cdot) \rightarrow 0$ in $L^2(S_+^{N-1})$ as $t \rightarrow \infty$. Therefore, the limit set Γ_w of \mathcal{T} is a connected subset of the set of solutions of (6). By Theorem 1.1, we deduce that

$$\Gamma_w = \begin{cases} \{0\} & \text{if } q = q_1 \text{ or } q \geq q_3, \\ \{0\} \text{ or } \{\omega_0\} & \text{if } q_1 < q < q_3. \end{cases}$$

Then, a linearization argument as in [3] leads to the conclusion if $q > q_1$.

We now consider the case $q = q_1$; we borrow some ideas from [1] and [11]. We first prove, by ODE techniques, that

$$X(t) := \int_{S_+^{N-1}} w(t, \cdot) \phi d\sigma \leq Ct^{-(N-1)/2}. \quad (17)$$

Using (8) and the boundary Harnack inequality (see [5]), we derive

$$0 \leq w(t, \sigma) \leq Ct^{-(N-1)/2} \quad \text{in } (1, \infty) \times S_+^{N-1}. \quad (18)$$

Set $\eta(t, \sigma) := t^{(N-1)/2}w(t, \sigma)$. We verify as above that the limit set Γ_η in $C^2(\overline{S_+^{N-1}})$ of the trajectory \mathcal{T}_η of η is an interval of the form $\{\kappa\phi : 0 \leq \kappa_0 \leq \kappa \leq \kappa_1\}$. In order to show that \mathcal{T}_η is reduced to a single point, we prove that $\|r(t, \cdot)\|_{L^2} \leq Ct^{-1}$, where

$$r(t, \cdot) := \eta(t, \cdot) - z(t)\phi \quad \text{and} \quad z(t) = \int_{S_+^{N-1}} \eta(t, \cdot)\phi \, d\sigma.$$

Writing the equation satisfied by z as a non-homogeneous second order linear ODE, we prove that either $z(t) \rightarrow 0$, which implies that u is continuous at 0, or $z(t) \rightarrow \tilde{k}_N$ as $t \rightarrow \infty$, for some constant depending only on N .

Proof of Theorem 1.4. It is an application of the Doubling Lemma Method introduced in [10], from which we derive the following local estimate:

Lemma 2.1 *Let $1 < q < q_2$ and let u be a solution of (3). Then, for every $x_0 \in \partial\Omega \setminus \{0\}$ and $0 < R < |x_0|$, we have*

$$0 \leq u(x) \leq C(R - |x - x_0|)^{-2/(q-1)} \quad \forall x \in B_R(x_0) \cap \Omega, \quad (19)$$

for some constant $C > 0$ depending only on Ω .

Apply this lemma with $x_0 \in \partial\Omega \setminus \{0\}$ and $R = |x_0|/2$. Using elliptic regularity theory, we obtain

$$0 \leq u(x) \leq C\rho(x)|x|^{-2/(q-1)-1} \quad \forall x \in \Omega \text{ such that } 0 < \rho(x) < |x|/2.$$

If $\rho(x) \geq |x|/2$, then we use Gidas-Spruck's internal estimates (see [7]). We thus obtain (10).

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