Uniqueness results for convex Hamilton-Jacobi equations under $p>1$ growth conditions on data

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Abstract

Unbounded stochastic control problems may lead to Hamilton-Jacobi-Bellman equations whose Hamiltonians are not always defined, especially when the diffusion term is unbounded with respect to the control. We obtain existence and uniqueness of viscosity solutions growing at most like $o(1+|x|^p)$ at infinity for such HJB equations and more generally for degenerate parabolic equations with a superlinear convex gradient nonlinearity. If the corresponding control problem has a bounded diffusion with respect to the control, then our results apply to a larger class of solutions, namely those growing like $O(1+|x|^p)$ at infinity. This latter case encompasses some equations related to backward stochastic differential equations.

Keywords. degenerate parabolic equations, Hamilton-Jacobi-Bellman equations, viscosity solutions, unbounded solutions, maximum principle, backward stochastic differential equations, unbounded stochastic control problems.

AMS subject classifications. 35K65, 49L25, 35B50, 35B37, 49N10, 60H35.

1 Introduction

In the joint paper [13] the authors obtain a comparison result between semicontinuous viscosity solutions, neither bounded from below nor from above, growing at most quadratically in the state variable, of second order degenerate parabolic equations of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(x,t,Du,D^2u) = 0 & \text{in } I\!\!R^N \times (0,T), \\ u(x,0) = \psi(x) & \text{in } I\!\!R^N, \end{cases} \tag{1}$$

where $N \geq 1$, $T > 0$. The unknown u is a real-valued function defined in $I\!\!R^N \times [0,T]$, Du and D^2u denote respectively its gradient and Hessian matrix and ψ is a given initial condition. The Hamiltonian $H : I\!\!R^N \times [0,T] \times I\!\!R^N \times \mathcal{S}_N(I\!\!R) \to I\!\!R$ has the form

$$H(x,t,q,X) = \sup_{\alpha \in A} \left\{ -\langle b(x,t,\alpha), q\rangle - \ell(x,t,\alpha) - \text{Trace}\left[\sigma(x,t,\alpha)\sigma^T(x,t,\alpha)X\right] \right\}. \tag{2}$$

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Note that H is convex with respect to q. The key assumptions in the paper [13] are that A is an unbounded control set, the functions b and ℓ grow respectively at most linearly and quadratically with respect to both the control and the state. Instead the functions σ is assumed to grow at most linearly with respect to the state and is bounded with respect to the control. (In fact, in [13], we consider more general equations of Isaacs type by adding a concave Hamiltonian G with bounded control, see Remark 2.1. To simplify the exposition we take $G \equiv 0$ here.)

In the present work, we extend the results of [13] in two directions.

The first issue is to obtain a comparison result for unbounded solutions under the weaker assumption that the diffusion matrix σ is unbounded also with respect to the control. The main difficulty is that the Hamiltonian H may not be continuous. To illustrate this fact, consider for instance the case where $A = I\!\!R^N$, $b = \alpha$, $\sigma = |\alpha| I$ and $\ell = |\alpha|^2$. The Hamiltonian H becomes

$$\sup_{\alpha \in I\!\!R^N} \{-\langle \alpha, Du \rangle - |\alpha|^2 - \frac{|\alpha|^2}{2} \Delta u\}, \tag{3}$$

which is $+\infty$ as soon as $\Delta u < -2$. This example is motivated by the well-known Stochastic Linear Quadratic problem, see for instance Bensoussan [8], Fleming and Rishel [15], Fleming and Soner [16], Øksendal [22], Yong and Zhou [25] and the references therein for an overview of this problem. The usual way to deal with such a problem is to plug into the equation value functions V of particular form (for instance quadratic in space) for which one knows that $H(x, t, V, DV)$ is defined. It leads to some ordinary differential equations of Ricatti type which allow to identify precisely the value function (see [25]). Another way is to replace the Hamilton-Jacobi-Bellman equation by a variational inequality, see Barles [5] for instance. Our aim is to study directly the PDE (1) without any a priori knowledge on the value function. Indeed, for general datas, one does not expect explicit formula for the value function.

We overcome the above difficulty in noticing that it is possible to formulate the definition of viscosity solutions for HJB in a new way without writing the "sup" in (2), see Definition 2.1. It provides a precise definition of solutions for (1) even in cases like (3). Let us stress that it is not a new definition of viscosity solutions but only a new formulation. Using this formulation, we prove a comparison result for solutions in the class of functions growing at most like $o(1 + |x|^p)$ at infinity. It provides new results for Stochastic Linear Quadratic type problems (in this case, $p = 2$) but, unfortunately, we are not able to treat the classical Stochastic Linear Quadratic type problem with terminal cost $\psi(x) = |x|^2$ since it requires a comparison in the class $O(1 + |x|^2)$. Nevertheless, our results apply to very general datas (not only polynomials of degree 1 or 2 in (x, α)), see Example 2.1.

The second issue of our work is to extend the results of [13] for p-growth type conditions on the datas and the solutions and for more general equations with an additional nonlinearity f which is also convex with respect to the gradient and depends on u. The motivation comes from PDEs arising in the context of backward stochastic differential equations (BSDEs in short).

In the framework of BSDEs, one generally considers forward-backward systems of the form

$$\begin{cases} dX_s^{x,t} = b(X_s^{x,t}, s)ds + \sigma(X_s^{t,x}, s)dW_s, \quad t \le s \le T, \\ X_t^{t,x} = x, \end{cases} \tag{4}$$

$$\begin{cases} -dY_s^{x,t} = f(X_s^{x,t}, s, Y_s^{x,t}, Z_s^{x,t})ds - Z_s^{x,t}dW_s, \quad t \le s \le T, \\ Y_T^{x,t} = \psi(x), \end{cases} \tag{5}$$

where $(W_s)_{s\in[0,T]}$ is standard Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, with $(\mathcal{F}_t)_{t\in[0,T]}$ the standard Brownian filtration. (Note that b and σ do not depend on the control). The diffusion (4) is associated with the second-order elliptic operator L defined by

$$Lu = -\frac{1}{2}\text{Trace}(\sigma\sigma^T D^2u) - \langle b(x,t), Du\rangle.$$

The forward-backward system (4)-(5) is formally connected to the PDE

$$\begin{cases} -\dfrac{\partial u}{\partial t} + Lu - f(x,t,u,s(x,t)Du) = 0 & \text{in } I\!R^N \times (0,T) \\ u(x,T) = \psi(x) & \text{in } I\!R^N. \end{cases} \tag{6}$$

by the nonlinear Feynman-Kac formula

$$u(x,t) = Y_t^{x,t} \quad \text{for all } (x,t) \in I\!R^N \times [0,T]. \tag{7}$$

We recall that nonlinear BSDEs with Lipschitz continuous coefficients were first introduced by Pardoux and Peng [23], who proved existence and uniqueness. Their results were extended by Kobylanski [20] for bounded solutions in the case of coefficients f having a quadratic growth in the gradient. Briand and Hu [9] generalized this latter result to the case of solutions which are $O(1+|x|^p)$, as $|x| \to \infty$, with $1 \le p < 2$. In all these works, the connection with viscosity solutions to the related PDE (6) is established: u defined by (7) is a viscosity solution of (6).

Our aim is to prove the analytical counterpart of their results. More precisely, we want to prove the existence and uniqueness of the solution of (6) under the assumptions of [9].

Let us turn to a more precise description of our results. We consider equations of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(x,t,Du,D^2u) + f(x,t,u,s(x,t)Du) = 0 & \text{in } I\!R^N \times (0,T), \\ u(x,0) = \psi(x) & \text{in } I\!R^N, \end{cases} \tag{8}$$

where H is given by (2) with A unbounded, $f\colon I\!R^N \times [0,T] \times I\!R^N \to I\!R$ is continuous and convex in the gradient and s is bounded. We look for solutions with $p > 1$ growth assumptions (see (12) and (39)) and both H and f satisfies some p' growth assumptions, where $p' = p/(p-1)$ is the conjugate of p. See **(A)**, **(B)**, **(C)** for the precise assumptions. Let us mention that the typical case we want to deal with is

$$f(x,t,u,s(x,t)Du) = |s(x,t)Du|^{p'}, \quad p' > 1,$$

and the presence of x in the power-p' term is delicate to treat (especially when doubling the variables in viscosity type's proofs, see the proof of Lemma 3.2). The u-dependence in f means that $f(x,t,u,s(x,t)Du)$ may not be on the form of H and induces some technical difficulties.

Section 2 is devoted to the case with diffusion matrices σ which depend on the control in an unbounded way, see condition (9). The compensation to this condition with respect to [13] (where σ was assumed to be bounded with respect to the control and $p = 2$) is that we prove the comparison result Theorem 2.1 for semicontinuous sub- and supersolutions of (8) growing at most like $o(1+|x|^p)$ as $|x| \to \infty$ (instead of $O(1+|x|^2)$ in [13]). So far it remains an open question to know if there is uniqueness in the larger class $O(1+|x|^p)$. The proof of the comparison result relies on classical techniques of viscosity solutions. We build a suitable test-function and prove some fine estimates on the various terms which appear, the main difficulty consists in dealing with the unbounded control terms.

In Section 3, we extend the comparison result in [13] for equations with $p > 1$ growth conditions on the datas (instead of quadratic growth) with the additional nonlinearity f. One motivation to add the nonlinearity f comes from the BSDEs (where $s = \sigma$) since the main application of Theorem 3.1 is the uniqueness for the equation stated in [9] (see Example 3.2). In this case, we consider Hamiltonians H with α-bounded diffusion matrices σ, so we choose to replace (8) by the control independent PDE (38) to simplify the exposition. The control case does not present additional difficulties with respect to [13, Theorem 2.1]. The main difficulty in the proof of Theorem 3.1 is to be able to deal with solutions growing like $O(1+|x|^p)$ (which are not bounded neither from above nor from below). The strategy of proof is similar to the one used in [13] which consists essentially in the following three steps. First one computes the equation satisfied by $w_\mu = \mu u - v$, being u, v respectively the subsolution and the supersolution of the original PDE and $0 < \mu < 1$ a parameter. Then for all $R > 0$ one constructs a strict supersolution Φ_μ^R of the "linearized equation" such that $\Phi_\mu^R(x,t) \to 0$ as $R \to +\infty$. Finally one shows that $w_\mu \le \Phi_\mu^R$ and one concludes by letting first $R \to +\infty$ and then $\mu \to 1$.

A by-product of the comparison results obtained in Sections 2 and 3 and Perron's Method of Ishii [17] is the existence and uniqueness of a continuous solution to (8) which is respectively $o(1+|x|^p)$ and $O(1+|x|^p)$ as $|x| \to \infty$. However, under our general assumptions one cannot expect the existence of a solution for all times as Example 3.4 shows.

Let us compare our results with related ones in the literature for such kind of Hamilton-Jacobi equations. Uniqueness and existence problems for a class of first-order Hamiltonians corresponding to unbounded control sets and under assumptions including deterministic linear quadratic problems have been addressed by several authors, see, e.g. the book of Bensoussan [8], the papers of Alvarez [2], Bardi and Da Lio [4], Cannarsa and Da Prato [10], Rampazzo and Sartori [24] in the case of convex operators, and the papers of Da Lio and McEneaney [14] and Ishii [18] for more general operators. As for second-order Hamiltonians under quadratic growth assumptions, Ito [19] obtained the existence of locally Lipschitz solutions to particular equations of the form (1) under more regularity conditions on the data, by establishing a priori estimates on the solutions. Whereas Crandall and Lions in [12] proved a uniqueness result for very particular operators depending only on the Hessian matrix of the solution. In the case of quasilinear degenerate parabolic equations, existence and uniqueness results for viscosity solutions which may have a quadratic growth are proved in [7]. The results which are the closest to ours were obtained in the following works. Alvarez [1] addressed the case of stationary less general equations (see Example 3.1). Krylov [21] succeeded in dealing with equations encompassing the classical Stochastic Linear Quadratic problem but his assumptions are designed to handle exactly this case (cf. Example 2.1 and the discussion therein). Finally Kobylanski [20] studied also (8) under quite general assumptions on the datas but for bounded solutions. It seems to be difficult to obtain such a generality in the case of unbounded solutions since her proof is based on changes of functions of the form $u \to -e^{-u}$ which do not work for solutions which are neither bounded from below nor from above.

The rigorous connection between control problems and Hamilton-Jacobi-Bellman equations is not addressed in this paper. In the framework of unbounded controls it may be rather delicate. Some results in this direction were obtained for infinite horizon in the deterministic case by Barles [5] and in the stochastic case by Alvarez [1, 2], Krylov [21] and by the authors [13].

Finally, let us mention that the convexity of the operator with respect to the gradient is crucial in our proofs. The case of Hamiltonians which are neither convex nor concave (which, in the case of Equations (16), amounts to take both the control sets A and B unbounded) is also of interest and it is a widely open subject. Some results in this direction were obtained in [13, Section 4], for instance in the case of first order equations of the form

$$\frac{\partial u}{\partial t} + h(x,t)|Du|^2 = 0 \quad \text{in } I\!R^N \times [0,T],$$

where $h(x,t)$ may change sign and u has a quadratic growth. In a forthcoming paper we are going to investigate this issue for more general quadratic non convex-non concave equations.

Throughout the paper we will use the following notations. For all integer $N, M \geq 1$ we denote by $\mathcal{M}_{N,M}(I\!R)$ (respectively $\mathcal{S}_N(I\!R)$, $\mathcal{S}_N^+(I\!R)$) the set of real $N \times M$ matrices (respectively real symmetric matrices, real symmetric nonnegative $N \times N$ matrices). For the sake of notations, all the norms which appear in the sequel are denoting by $|\cdot|$. The standard Euclidean inner product in $I\!R^N$ is written $\langle \cdot, \cdot \rangle$. We recall that a modulus of continuity $m : I\!R \to I\!R^+$ is a nondecreasing continuous function such that $m(0) = 0$. We set $B(0,R) = \{x \in I\!R^N : |x| < R\}$. Finally for any $O \subseteq I\!R^K$, we denote by $USC(O)$ the set of upper semicontinuous functions in O and by $LSC(O)$ the set of lower semicontinuous functions in O. Given $p > 1$ we will denote by p' its coniugate, namely

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

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2 Hamilton-Jacobi-Bellman equations with unbounded diffusion in the control

In this Section we prove a comparison result for second-order fully nonlinear partial differential equations of the form (8). The main difference with respect to the result in [13] is that here we suppose that the diffusion matrix σ depends in a unbounded way in the control (see condition (9)). The compensation to the condition (9) is that we are able to get the uniqueness result in the smaller class of functions which are $o(1 + |x|^p)$ as $|x| \to \infty$ (see (12)).

We list below the main assumptions on H and f.

(A) (Assumption on H) :

(i) A is a subset of a separable complete normed space. The main point here is the possible unboundedness of A.

(ii) $b \in C(I\!\!R^N \times [0,T] \times A; I\!\!R^N)$ and there exists $C_b > 0$ such that, for all $x, y \in I\!\!R^N$, $t \in [0,T]$, $\alpha \in A$,

$$\begin{aligned} |b(x,t,\alpha) - b(y,t,\alpha)| &\leq C_b(1+|\alpha|)|x-y|, \\ |b(x,t,\alpha)| &\leq C_b(1+|x|+|\alpha|) \ ; \end{aligned}$$

(iii) $\ell \in C(I\!\!R^N \times [0,T] \times A; I\!\!R)$ and, there exist $p > 1$ and $C_\ell, \nu > 0$ such that, for all $x \in I\!\!R^N$, $t \in [0,T]$, $\alpha \in A$,

$$C_\ell(1+|x|^p+|\alpha|^p) \geq \ell(x,t,\alpha) \geq \nu|\alpha|^p - C_\ell(1+|x|^p)$$

and for every $R > 0$, there exists a modulus of continuity m_R such that for all $x, y \in B(0,R)$, $t \in [0,T]$, $\alpha \in A$,

$$|\ell(x,t,\alpha) - \ell(y,t,\alpha)| \leq (1+|\alpha|^p)\, m_R(|x-y|) \ ;$$

(iv) $\sigma \in C(I\!\!R^N \times [0,T] \times A; \mathcal{M}_{N,M}(I\!\!R))$ is Lipschitz continuous with respect to x with a constant independent of (t,α): namely, there exists $C_\sigma > 0$ such that, for all $x, y \in I\!\!R^N$ and $(t,\alpha) \in [0,T] \times A$,

$$|\sigma(x,t,\alpha) - \sigma(y,t,\alpha)| \leq C_\sigma |x-y|,$$

and satisfies for every $x \in I\!\!R^N$, $t \in [0,T]$, $\alpha \in A$,

$$|\sigma(x,t,\alpha)| \leq C_\sigma(1+|x|+|\alpha|). \tag{9}$$

(B) (Assumption on f)
$f \in C([0,T] \times I\!\!R^N \times I\!\!R \times I\!\!R^N; I\!\!R)$ and, for all $R > 0$, there exist a modulus of continuity m_R and $C_s, \hat{C} > 0$ such that, for all $t \in [0,T]$, $x, y \in I\!\!R^N$, $u, v \in I\!\!R$, $z \in I\!\!R^N$,

(i) $|f(x,t,u,z)| \leq C_f(1+|x|^p+|u|+|z|^{p'})$,
(ii) $|f(x,t,u,z) - f(y,t,u,z)| \leq m_R((1+|u|+|z|)|x-y|)$ if $|x|+|y| \leq R$,
(iii) $z \mapsto f(x,t,u,z)$ is convex,
(iv) $s \in C(I\!\!R^N \times [0,T]; \mathcal{M}_N)$, $|s(x,t) - s(y,t)| \leq C_s|x-y|$, $|s(x,t)| \leq C_s$,
(v) $|f(x,t,u,z) - f(x,t,v,z)| \leq \hat{C}|u-v|$.

The typical case we have in mind in the context of **(A)**(iv) (σ not bounded with respect to the control) is

$$\sigma(x,t,\alpha) = Q(t)x + R(t)\alpha,$$

where $Q(t)$ and $R(t)$ are matrices of suitable sizes. This case includes Linear Quadratic control problems, see Example 2.1.

Under the current hypotheses, the Hamiltonian H may be infinite (see Example 2.1) and for this reason we re-formulate the definition of viscosity solution in the following way.

Definition 2.1
(i) A function $u \in USC(I\!R^N \times [0,T])$ *is a viscosity subsolution of* (8) *if for all* $(x,t) \in I\!R^N \times [0,T]$ *and* $\varphi \in C^2(I\!R^N \times [0,T])$ *such that* $u - \varphi$ *has a maximum at* (x,t)*, we have* $u(x,t) \le \psi(x)$ *if* $t = 0$ *and, if* $t > 0$*, then*

$$\frac{\partial \varphi}{\partial t}(x,t) + H(x,t,D\varphi(x,t),D^2\varphi(x,t)) + f(x,t,u(x,t),s(x,t)D\varphi(x,t)) \le 0,$$

which is equivalent to: for all $\alpha \in A$*,*

$$\frac{\partial \varphi}{\partial t}(x,t) - \langle b(x,t,\alpha), D\varphi(x,t)\rangle - \ell(x,t,\alpha) - \mathrm{Trace}\left[\sigma(x,t,\alpha)\sigma^T(x,t,\alpha)D^2\varphi(x,t)\right] \\ + f(x,t,u(x,t),s(x,t)D\varphi(x,t)) \le 0. \qquad (10)$$

(ii) A function $u \in USC(I\!R^N \times [0,T])$ *is a viscosity supersolution of* (8) *if for all* $(x,t) \in I\!R^N \times [0,T]$ *and* $\varphi \in C^2(I\!R^N \times [0,T])$ *such that* $u - \varphi$ *has a minimum at* (x,t)*, we have* $u(x,t) \ge \psi(x)$ *if* $t = 0$ *and, if* $t > 0$*, then for all* $\eta > 0$*, there exists* $\alpha_\eta = \alpha(\eta,x,t) \in A$*, such that*

$$\frac{\partial \varphi}{\partial t}(x,t) - \langle b(x,t,\alpha_\eta), D\varphi(x,t)\rangle - \ell(x,t,\alpha_\eta) - \mathrm{Trace}\left[\sigma(x,t,\alpha_\eta)\sigma^T(x,t,\alpha_\eta)D^2\varphi(x,t)\right] \\ + f(x,t,u(x,t),s(x,t)D\varphi(x,t)) \ge -\eta. \qquad (11)$$

(iii) A locally bounded function $u : I\!R^N \times [0,T] \to I\!R$ *is a viscosity solution of* (8) *if its USC envelope* u^* *is a subsolution and its LSC envelope* u_* *is a supersolution.*

Note that (10) and (11) is only a way to write the definition of sub- and supersolutions without writing a supremum which could not exist because of assumption (9).

We say that a function $u : I\!R^N \times [0,T] \to I\!R$ is in the class $\mathcal{C}_p$ if

$$\frac{u(x,t)}{1+|x|^p} \underset{|x|\to+\infty}{\longrightarrow} 0, \quad \text{uniformly with respect to } t \in [0,T]. \qquad (12)$$

Note that $u \in \mathcal{C}_p$ if and only if, for all $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$|u(x,t)| \le M_\varepsilon + \varepsilon(1+|x|^p) \quad \text{for all } (x,t) \in I\!R^N \times [0,T].$$

In particular, for all $\lambda > 0$,

$$\sup_{x\in I\!R^N} \{u(x,t) - \lambda(1+|x|^p)\} = M_\lambda < +\infty. \qquad (13)$$

The main result of this Section is the

Theorem 2.1 *Assume* ***(A)-(B)*** *and suppose that* ψ *is a continuous function which belongs to* $\mathcal{C}_p$*. Let* $u \in USC(I\!R^N \times [0,T])$ *be a viscosity subsolution of* (8) *and* $v \in LSC(I\!R^N \times [0,T])$ *be a viscosity supersolution of* (8)*. Suppose that* U *and* V *are in the class* $\mathcal{C}_p$ *defined by* (12) *and satisfy* $u(x,0) \le \psi(x) \le v(x,0)$*. Then* $u \le v$ *in* $I\!R^N \times [0,T]$*.*

Before giving the proof of the theorem, let us state an existence result and some examples of applications. As it was already observed in [13], the question of the existence of a continuous solution to (1) is not completely obvious and in general the solutions may exist only for short time (see Example 3.4). One way to obtain the existence is to establish a link between the solution of the PDE and related control problems or BDSE systems which have a solution. By using PDE methods, in the framework of viscosity solutions, the existence is usually a consequence of the comparison principle by means of Perron's method, as soon as we can build a sub- and a super-solution to the problem. Here, the comparison principle is proved in the class of functions belonging to $\mathcal{C}_p$. Therefore, to prove the existence, it siffices to build sub- and super-solutions to (38) in $\mathcal{C}_p$. We need to strengthen **(A)**(iii) and **(B)**(i) by assuming that $\ell(\cdot,t,\alpha), f(\cdot,t,u,z) \in \mathcal{C}_p$ uniformly with respect to α, t, u, z, i.e., for all $(x,t,\alpha,u,z) \in I\!R^N \times [0,T] \times A \times I\!R \times I\!R^N$,

$$\begin{aligned} &\chi(x) \geq \ell(x,t,\alpha) \geq \nu|\alpha|^p - \chi(x), \quad |f(x,t,u,z)| \leq C_f(1+\gamma(x)+|u|+|z|^{p'}), \\ &\qquad\qquad \text{and} \quad \lim_{|x|\to+\infty} \frac{\chi(x)}{1+|x|^p}, \frac{\gamma(x)}{1+|x|^p} = 0. \end{aligned} \tag{14}$$

We have

Theorem 2.2 *Assume **(A)**–**(B)** and* (14). *For all $\psi \in \mathcal{C}_p$, there is $\tau > 0$ such that there exist a subsolution $\underline{u} \in \mathcal{C}_p$ and a supersolution $\overline{u} \in \mathcal{C}_p$ of* (8) *in $I\!R^N \times [0,T]$. In consequence, Equation* (8) *has a unique continuous viscosity solution in $I\!R^N \times [0,\tau]$ in the class $\mathcal{C}_p$.*

The proof of this theorem is postponed at the end of the section.

Example 2.1 (A Stochastic Linear Quadratic Control Problem) Consider the stochastic differential equation (in dimension 1 for sake of simplicity)

$$\begin{cases} dX_s = X_s ds + \sqrt{2}\alpha_s dW_s, \quad t \leq s \leq T, \ t \in (0,T], \\ X_0 = x \in I\!R, \end{cases}$$

where W_s is a standard Brownian motion, $(\alpha_s)_s$ is a real valued progressively measurable process and the value function is given by

$$V(x,t) = \inf_{(\alpha_s)_s} E_{tx} \left\{ \int_t^T |\alpha_s|^2 \, ds + \psi(X_T) \right\}.$$

(Note that in this case, $p = p' = 2$.) The Hamilton-Jacobi equation formally associated to this problem is

$$\begin{cases} -u_t + \sup_{\alpha \in I\!R} \{-\alpha^2(u''+1)\} - xu' = 0 & \text{in } I\!R \times (0,T], \\ u(x,T) = \psi(x) & \text{in } I\!R. \end{cases} \tag{15}$$

We observe that in this case if $u'' + 1 < 0$ then the Hamiltonian becomes $+\infty$. Nevertheless, we are able to prove comparison (15) as soon as the terminal cost $\psi \in \mathcal{C}_p$ (i.e., has a strictly sub-p growth). This is not completely satisfactory since, in the classical Linear Quadratic Control Problem, one expects to have quadratic terminal costs like $\psi(x) = |x|^2$. Let us mention that

Krylov [21] succeeded in treating this latter case. But his proof consists on some algebraic computations which rely heavily on the particular form of the datas (the datas are supposed to be polynomials of degree 1 or degree 2 in (x, α)). In our case, up to restrict slightly the growth, we are able to deal with general datas.

Remark 2.1 Theorems 2.1 and 3.1 still hold for the Isaacs equation of [13],

$$\frac{\partial u}{\partial t} + H(x,t,Du,D^2u) + G(x,t,Du,D^2u) + f(x,t,u,s(x,t)Du) = 0 \tag{16}$$

where

$$G(x,t,q,X) = \inf_{\beta \in B} \left\{ -\langle g(x,t,\beta), q\rangle - l(x,t,\beta) - \text{Trace}\left[c(x,t,\beta)c^T(x,t,\beta)X\right] \right\},$$

is a concave Hamiltonian, B is bounded, g, l, c satisfy respectively **(A)**(ii),(iii),(iv) (with bounded controls β). The case where both the control sets A and B are unbounded is rather delicate. It is the aim of a future work.

Let us turn to the proof of the comparison theorem.

Proof of Theorem 2.1. We are going to show that for every $\mu \in (0,1)$, $\mu u - v \leq 0$, in $I\!R^N \times [0,T]$. To this end we argue by contradiction assuming that there exists $(\hat{x}, \hat{t}) \in I\!R^N \times [0,T]$ such that

$$u(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) > \delta > 0. \tag{17}$$

We divide the proof in several steps.

1. The μ-equation for the subsolution. If u is a subsolution of (8), then $\tilde{u} = \mu u$ is a subsolution of

$$\begin{aligned}
&\tilde{u}_t + \sup_{\alpha \in A}\{-\text{Trace}\big(\sigma(x,t,\alpha)\sigma(x,t,\alpha)^T D^2\tilde{u}\big) + \langle b(x,t,\alpha), D\tilde{u}\rangle - \mu\ell(x,t,\alpha)\} \\
&+\mu f\big(x,t,\frac{1}{\mu}\tilde{u}(x,t),\frac{1}{\mu}s(x,t)D\tilde{u}\big) \leq 0,
\end{aligned}$$

with the initial condition $\mu u(x,0) \leq \mu\psi(x)$.

2. Test-function and estimates on the penalization terms. For all $\varepsilon > 0$, $\eta > 0$ and $\theta, L > 0$ (to be chosen later) we consider the auxiliary function

$$\Phi(x,y,t) = \mu u(x,t) - v(y,t) - e^{Lt}\big(\frac{|x-y|^2}{\varepsilon^2} + \theta(1-\mu)(1+|x|^2+|y|^2)^{p/2}\big) - \rho t.$$

Since $u, v \in \mathcal{C}_p$, the supremum of Φ in $I\!R^N \times I\!R^N \times [0,T]$ is achieved at a point $(\bar{x},\bar{y},\bar{t})$. We will drop for simplicity of notation the dependence on the various parameters. If θ and ρ are small enough we have

$$\Phi(\bar{x},\bar{y},\bar{t}) \geq \mu u(\hat{x},\hat{y}) - v(\hat{x},\hat{y}) - \theta(1-\mu)(1+2|\hat{x}|^p) - \rho\hat{t} > \frac{\delta}{2},$$

which implies

$$\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}+\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}\leq \mu u(\bar{x},\bar{t})-v(\bar{y},\bar{t}).$$

Therefore, by (13), we get

$$\begin{aligned}
& \frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}+\theta\frac{1-\mu}{2}(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2} \\
\leq\ & \sup_{(x,t)\in I\!R^N\times[0,T]}\{\mu u(x,t)-\theta\frac{1-\mu}{2}(1+|x|^p)\} \\
& +\sup_{(x,t)\in I\!R^N\times[0,T]}\{-v(x,t)-\theta\frac{1-\mu}{2}(1+|x|^p)\} \\
\leq\ & M
\end{aligned}$$

for some $0<M=M(\mu,\theta,u,v)$. Thus

$$|\bar{x}|,|\bar{y}|\leq R_{\mu,\theta} \tag{18}$$

with $R_{\mu,\theta}$ independent of ε and $|\bar{x}-\bar{y}|\to 0$ as $\varepsilon\to 0$. Up to extract a subsequence, we can assume that

$$\bar{x},\bar{y}\to x_0\in \overline{B}(0,R_{\mu,\theta}),\quad \bar{t}\to t_0\ \text{ as } \varepsilon\to 0 \tag{19}$$

Actually we can obtain a more precise estimate: we have

$$\Phi(\bar{x},\bar{y},\bar{t})\geq \max_{I\!R^N\times[0,T]}\{\mu u(x,t)-v(x,t)-e^{Lt}\theta(1-\mu)(1+2|x|^2)^{p/2}-\rho t\}:=M_{\mu,\theta}.$$

Thus

$$\liminf_{\varepsilon\to 0}\Phi(\bar{x},\bar{y},\bar{t})\geq M_{\mu,\theta}.$$

On the other hand

$$\begin{aligned}
& \limsup_{\varepsilon\to 0}\Phi(\bar{x},\bar{y},\bar{t}) \\
& \leq \limsup_{\varepsilon\to 0}\,[\mu u(\bar{x},\bar{t})-v(\bar{y},\bar{t})-e^{L\bar{t}}\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}-\rho\bar{t}]-\liminf_{\varepsilon\to 0}e^{L\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} \\
& \leq M_{\mu,\theta}-\liminf_{\varepsilon\to 0}e^{L\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}.
\end{aligned}$$

By combining the above inequalities we get, up to subsequences, that

$$\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}\to 0\ \text{ as } \varepsilon\to 0. \tag{20}$$

Note that we have

$$|\bar{x}-\bar{y}|,\ \frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}=m(\varepsilon), \tag{21}$$

where m denotes a modulus of continuity independent of ε (but which depends on θ, μ).

3. Ishii matricial theorem and viscosity inequalities. We set

$$\Theta(x,y,t) = e^{Lt}\big(\frac{|x-y|^2}{\varepsilon^2} + \theta(1-\mu)(1+|x|^2+|y|^2)^{p/2}\big) + \rho t.$$

We claim that there is a subsequence ε_n such that $\bar{t} = 0$. Suppose by contradiction that for all $\varepsilon > 0$ we have $\bar{t} > 0$. Next Steps are devoted to prove some estimates in order to obtain the desired contradiction at the end of Step 8.

By Theorem 8.3 in the User's guide [11], for every $\varrho > 0$, there exist $a_1, a_2 \in I\!R$ and $X, Y \in \mathcal{S}_N$ such that

$$\begin{aligned} (a_1, D_x\Theta(\bar{x},\bar{y},\bar{t}), X) &\in \bar{\mathcal{P}}^{2,+}(\mu u)(\bar{x},\bar{t}),\\ (a_2, -D_y\Theta(\bar{x},\bar{y},\bar{t}), Y) &\in \bar{\mathcal{P}}^{2,-}(v)(\bar{y},\bar{t}),\\ a_1 - a_2 &= \Theta_t(\bar{x},\bar{y},\bar{t}), \end{aligned}$$

and

$$-(\frac{1}{\varrho} + |M|)I \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le M + \varrho M^2$$

where $M = D^2\Theta(\bar{x},\bar{y},\bar{t})$. Note that

$$a_1 - a_2 = Le^{L\bar{t}}\big(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + \theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}\big) + \rho,$$

and, setting $p_\varepsilon = 2e^{L\bar{t}}\frac{\bar{x}-\bar{y}}{\varepsilon^2}$, $q_x = e^{L\bar{t}}p\theta(1-\mu)\bar{x}(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1}$, $q_y = -e^{L\bar{t}}p\theta(1-\mu)\bar{y}(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1}$ we have

$$D_x\Theta(\bar{x},\bar{y},\bar{t}) = p_\varepsilon + q_x \quad \text{and} \quad D_y\Theta(\bar{x},\bar{y},\bar{t}) = -p_\varepsilon - q_y,$$

and

$$M = A_1 + A_2 + A_3$$

where

$$\begin{aligned} A_1 &= \frac{2e^{L\bar{t}}}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},\\ A_2 &= e^{L\bar{t}}p\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},\\ A_3 &= e^{L\bar{t}}p(p-2)\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-2}\begin{pmatrix} x\otimes x & x\otimes y \\ x\otimes y & y\otimes y \end{pmatrix}. \end{aligned}$$

It follows

$$\begin{aligned}\langle X\xi,\xi\rangle-\langle Y\zeta,\zeta\rangle &\le \frac{2e^{L\bar t}}{\varepsilon^2}|\xi-\zeta|^2\\ &+e^{L\bar t}p\theta(1-\mu)(1+|\bar x|^2+|\bar y|^2)^{p/2-1}(|\xi|^2+|\zeta|^2)\\ &+2e^{L\bar t}p(p-2)\theta(1-\mu)(1+|\bar x|^2+|\bar y|^2)^{p/2-2}\left(\langle\xi,x\rangle^2+\langle\zeta,y\rangle^2\right)\\ &+m\left(\frac{\varrho}{\varepsilon^4}\right),\end{aligned}\tag{22}$$

where m is a modulus of continuity which is independent of ρ and ε.

We now write the viscosity inequalities satisfied by the subsolution μu and the supersolution v (recall that we assume $\bar t>0$).

For all $\alpha\in A$ we have

$$\begin{aligned}&a_1-\mathrm{Trace}(\sigma(\bar x,\bar t,\alpha)\sigma(\bar x,\bar t,\alpha)^TX)+\langle b(\bar x,\bar t,\alpha),p_\varepsilon+q_x\rangle-\mu\ell(\bar x,\bar t,\alpha)\\ &\quad+\mu f(\bar x,\bar t,u(\bar x,\bar t),\frac{1}{\mu}s(\bar x,\bar t)(p_\varepsilon+q_x))\le 0.\end{aligned}\tag{23}$$

On the other hand, for all $\eta>0$, there exists $\alpha_\eta\in A$ such that

$$\begin{aligned}&a_2-\mathrm{Trace}(\sigma(\bar y,\bar t,\alpha_\eta)\sigma^T(\bar y,\bar t,\alpha_\eta)Y)+\langle b(\bar y,\bar t,\alpha_\eta),p_\varepsilon+q_y\rangle-\ell(\bar y,\bar t,\alpha_\eta)\\ &\quad+f(\bar y,\bar t,v(\bar y,\bar t),\frac{1}{\mu}s(\bar y,\bar t)(p_\varepsilon+q_y))\ge-\eta.\end{aligned}\tag{24}$$

We set for simplicity

$$\begin{aligned}&\sigma_x:=\sigma(\bar x,\bar t,\alpha_\eta),\ \ \sigma_y=\sigma(\bar y,\bar t,\alpha_\eta)\\ &b_x=b(\bar x,\bar t,\alpha_\eta),\ \ b_y=b(\bar y,\bar t,\alpha_\eta),\ \ s_x=s(\bar x,\bar t),\ \ s_y=s(\bar y,\bar t).\end{aligned}$$

By subtracting (23) and (24) we get

$$\begin{aligned}&Le^{L\bar t}\big(\frac{|\bar x-\bar y|^2}{\varepsilon^2}+\theta(1-\mu)(1+|\bar x|^2+|\bar y|^2)^{p/2}\big)+\rho\\ \le\ &\mathrm{Trace}(\sigma_x\sigma_x^TX-\sigma_y\sigma_y^TY)+\langle b_y,p_\varepsilon+q_y\rangle-\langle b_x,p_\varepsilon+q_x\rangle\\ &-\ell(\bar y,\bar t,\alpha_\eta)+\mu\ell(\bar x,\bar t,\alpha_\eta)\\ &+f(\bar y,\bar t,v(\bar y,\bar t),s_y(p_\varepsilon+q_y))-\mu f(\bar x,\bar t,u(\bar x,\bar t),\frac{1}{\mu}s_x(p_\varepsilon+q_x))+\eta.\end{aligned}\tag{25}$$

4. Estimates of the second-order terms. From (22) and **(A)**(iv), it follows

$$\begin{aligned}
& \text{Trace}\left[\sigma_x {\sigma_x}^T X - \sigma_y {\sigma_y}^T Y\right] - m\left(\frac{\varrho}{\varepsilon^4}\right) \\
\leq\ & e^{L\bar{t}}\left(\frac{2}{\varepsilon^2}|\sigma_x - \sigma_y|^2 + p\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1}(|\sigma_x|^2+|\sigma_y|^2)\right. \\
& \qquad \left. +2p(p-2)\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-2}(|\sigma_x|^2|x|^2+|\sigma_y|^2|y|^2)\right) \\
\leq\ & 2C_\sigma^2 e^{L\bar{t}}\left(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + p(p-1)\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1}(1+|\bar{x}|^2+|\bar{y}|^2+|\alpha_\eta|^2)\right) \\
\leq\ & 2C_\sigma^2 e^{L\bar{t}}\left(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + p(p-1)\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}\right. \\
& \qquad \left. +p(p-1)\theta(1-\mu)|\alpha_\eta|^2(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1}\right).
\end{aligned}$$

By Young's inequality,

$$|\alpha_\eta|^2(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1} \leq \frac{2}{p}|\alpha_\eta|^p + \frac{p-2}{p}(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}.$$

It follows, using (21),

$$\begin{aligned}
\text{Trace}\left[\sigma_x {\sigma_x}^T X - \sigma_y {\sigma_y}^T Y\right] \ \leq\ & 4(p-1)^2 C_\sigma^2 e^{L\bar{t}}\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2} \\
& +4(p-1)C_\sigma^2 e^{L\bar{t}}\theta(1-\mu)|\alpha_\eta|^p + m(\varepsilon) + m\left(\frac{\varrho}{\varepsilon^4}\right). \quad (26)
\end{aligned}$$

5. Estimates of the drift terms. By using **(A)**(ii) and, from (21), by taking ε is small enough in order that $|\bar{x}-\bar{y}| \leq 1$, we get

$$\begin{aligned}
& \langle b_y, p_\varepsilon + q_y\rangle - \langle b_x, p_\varepsilon + q_x\rangle \\
\leq\ & \langle b_y - b_x, p_\varepsilon + q_y\rangle + \langle b_x, q_y - q_x\rangle \\
\leq\ & |b_y - b_x||p_\varepsilon| + |b_y - b_x||q_y| + |b_x||q_x - q_y| \\
\leq\ & C_b e^{L\bar{t}}\left(2(1+|\alpha_\eta|)\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + 2p\theta(1-\mu)(1+|\alpha_\eta|)|\bar{x}-\bar{y}|(1+|\bar{x}|^2+|\bar{y}|^2)^{(p-1)/2}\right. \\
& \qquad \left. +2p\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}\right) \\
\leq\ & C_b e^{L\bar{t}}\left(2\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + 4p\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}\right. \\
& \qquad \left. +m(\varepsilon)|\alpha_\eta| + \theta(1-\mu)m(\varepsilon)|\alpha_\eta|(1+|\bar{x}|^2+|\bar{y}|^2)^{(p-1)/2}\right).
\end{aligned}$$

By Young's inequality, we get

$$\begin{aligned}
& m(\varepsilon)|\alpha_\eta| + \theta(1-\mu)m(\varepsilon)|\alpha_\eta|(1+|\bar{x}|^2+|\bar{y}|^2)^{(p-1)/2} \\
\leq\ & \frac{m(\varepsilon)}{(\theta(1-\mu))^{1/(p-1)}} + \theta(1-\mu)|\alpha_\eta|^p + \theta(1-\mu)m(\varepsilon) + \theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}.
\end{aligned}$$

It follows

$$
\begin{aligned}
& \langle b_y, p_\varepsilon + q_y\rangle - \langle b_x, p_\varepsilon + q_x\rangle \\
\leq\ & (4p+1)C_b e^{L\bar t}\theta(1-\mu)(1+|\bar x|^2+|\bar y|^2)^{p/2} + C_b e^{L\bar t}\theta(1-\mu)|\alpha_\eta|^p + m(\varepsilon). \qquad (27)
\end{aligned}
$$

6. Estimates of running cost terms. Recall that we chose ε small enough in order that $|\bar x - \bar y| \leq 1$. Setting $R=1$, from **(A)**(iii), we get

$$
\begin{aligned}
\mu\ell(\bar x, \bar t, \alpha_\eta) - \ell(\bar y, \bar t, \alpha_\eta) &= (\mu-1)\ell(\bar x, \bar t, \alpha_\eta) + \ell(\bar x, \bar t, \alpha_\eta) - \ell(\bar y, \bar t, \alpha_\eta) \\
&\leq (1-\mu)|\alpha_\eta|^p \left(-\nu + \frac{m_1(|\bar x - \bar y|)}{1-\mu}\right) \\
&\quad + C_\ell(1-\mu)(1+|\bar x|^p) + m_1(|\bar x - \bar y|).
\end{aligned}
$$

Since $m_1(|\bar x - \bar y|) = m(\varepsilon)$ by (21), we obtain

$$
\mu\ell(\bar x, \bar t, \alpha_\eta) - \ell(\bar y, \bar t, \alpha_\eta) \leq (1-\mu)|\alpha_\eta|^p\,(-\nu + m(\varepsilon)) + C_\ell(1-\mu)(1+|\bar x|^p) + m(\varepsilon). \qquad (28)
$$

Note that it is the term "$-(1-\mu)\nu|\alpha_\eta|^p$" which will allow to control all the unbounded control terms in the sequel.

7. Estimates of f-terms. To simplify, we replace **(B)**(v) by the assumption that f is nondecreasing with respect to the u variable. By some changes of functions as in Lemma 3.1, we can reduce to this case without loss of generality.

We write

$$
f\big(\bar y, \bar t, v(\bar y, \bar t), s_y(p_\varepsilon + q_y)\big) - \mu\, f\big(\bar x, \bar t, u(\bar x, \bar t), \frac{1}{\mu} s_x(p_\varepsilon + q_x)\big) = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3
$$

with

$$
\begin{aligned}
\mathcal{T}_1 &= f\big(\bar y, \bar t, v(\bar y, \bar t), s_y(p_\varepsilon + q_y)\big) - f\big(\bar x, \bar t, v(\bar y, \bar t), s_y(p_\varepsilon + q_y)\big), \\
\mathcal{T}_2 &= f\big(\bar x, \bar t, v(\bar y, \bar t), s_y(p_\varepsilon + q_y)\big) - f\big(\bar x, \bar t, u(\bar x, \bar t), s_y(p_\varepsilon + q_y)\big), \\
\mathcal{T}_3 &= f\big(\bar x, \bar t, u(\bar x, \bar t), s_y(p_\varepsilon + q_y)\big) - \mu\, f\big(\bar x, \bar t, u(\bar x, \bar t), \frac{1}{\mu} s_x(p_\varepsilon + q_x)\big),
\end{aligned}
$$

and we estimate the three terms separately.

From **(B)**(ii), we have

$$
\mathcal{T}_1 \leq m_{R_{\mu,\theta}}\left((1 + |v(\bar y, \bar t)| + |s_y(p_\varepsilon + q_y)||\bar x - \bar y|\right),
$$

where $R_{\mu,\theta}$ is given by (18). Using **(B)**(iv) and the fact that $v \in \mathcal{C}_p$, we get

$$
|v(\bar y, \bar t)|, |s_y q_y| = O(R_{\mu,\theta}) \quad \text{and} \quad |s_y p_\varepsilon||\bar x - \bar y| \leq m(\varepsilon), \qquad (29)
$$

and therefore

$$
\mathcal{T}_1 \leq m(\varepsilon). \qquad (30)
$$

To deal with $\mathcal{T}_2$, we first note that

$$\begin{aligned}\mu u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) &\geq \Phi(\bar{x},\bar{y},\bar{t})\\ &\geq \Phi(\hat{x},\hat{x},\hat{t})\\ &\geq \mu u(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) - e^{L\hat{t}}\theta(1-\mu)(1+2|\hat{x}|^2)^{p/2} - \rho\hat{t}.\end{aligned}$$

Since $u(\hat{x},\hat{t}) > v(\hat{y},\hat{t})$ by (17), if we take μ close enough to 1 and ρ,θ close enough to 0, we obtain that

$$\mu u(\bar{x},\bar{t}) \geq v(\bar{x},\bar{t}).$$

From **(B)**(v) (monotonicity of f in u), it follows that

$$\begin{aligned}\mathcal{T}_2 &\leq f\big(\bar{x},\bar{t},v(\bar{y},\bar{t}),s_y(p_\varepsilon+q_y)\big) - f\big(\bar{x},\bar{t},\mu u(\bar{y},\bar{t}),s_y(p_\varepsilon+q_y)\big)\\ &\quad + f\big(\bar{x},\bar{t},\mu u(\bar{y},\bar{t}),s_y(p_\varepsilon+q_y)\big) - f\big(\bar{x},\bar{t},u(\bar{y},\bar{t}),s_y(p_\varepsilon+q_y)\big)\\ &\leq (1-\mu)|u(\bar{x},\bar{t})|\\ &\leq C_u(1-\mu)(1+|\bar{x}|^2)^{p/2},\end{aligned} \tag{31}$$

since $u \in \mathcal{C}_p$.

To estimate $\mathcal{T}_3$, we first recall the following convex inequality. If $\Psi : \mathbb{R}^N \to \mathbb{R}$ is convex and $0 < \mu < 1$, then, for all $\xi,\zeta \in \mathbb{R}^N$, we have

$$-\mu\,\Psi(\xi) + \Psi(\zeta) \leq (1-\mu)\Psi\left(\frac{\mu\xi-\zeta}{\mu-1}\right). \tag{32}$$

By **(B)**(iii) (convexity of f with respect to the gradient variable), for all $z_1,z_2 \in \mathbb{R}^N$, we obtain

$$f(x,t,u,z_1) - \mu\, f\left(x,t,u,\frac{z_2}{\mu}\right) \leq (1-\mu)\, f\left(x,t,u,\frac{z_1-z_2}{1-\mu}\right).$$

Therefore

$$\begin{aligned}\mathcal{T}_3 &\leq (1-\mu)\, f\left(\bar{x},\bar{t},u(\bar{x},\bar{t}),\frac{1}{1-\mu}(s_y(p_\varepsilon+q_y)-s_x(p_\varepsilon+q_x))\right)\\ &\leq C_f(1-\mu)\big(1+|\bar{x}|^p+|u(\bar{x},\bar{t})| + \left|\frac{s_y(p_\varepsilon+q_y)-s_x(p_\varepsilon+q_x)}{1-\mu}\right|^{p'}\big)\end{aligned} \tag{33}$$

by **(B)**(i). But

$$s_y(p_\varepsilon+q_y)-s_x(p_\varepsilon+q_x) = (s_y-s_x)p_\varepsilon + (s_y-s_x)q_y + s_x(q_y-q_x).$$

Hence for some $C > 0$ depending only on p (which may change during the computation), we have

$$\begin{aligned}\left|\frac{s_y(p_\varepsilon+q_y)-s_x(p_\varepsilon+q_x)}{1-\mu}\right|^{p'} &\leq \frac{CC_s^{p'}}{(1-\mu)^{p'}}\Big((|\bar{x}-\bar{y}||p_\varepsilon|)^{p'} + (|\bar{x}-\bar{y}||q_y|)^{p'} + |q_x-q_y|^{p'}\Big)\\ &\leq e^{p'L\bar{t}}m(\varepsilon) + CC_s^{p'}e^{p'L\bar{t}}\theta^{p'}(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2},\end{aligned}$$

by using (29). Finally, since $u \in \mathcal{C}_p$, we get from (33)

$$\mathcal{T}_3 \;\leq\; (1-\mu)C_f(1+C_u+CC_s^{p'}e^{p'L\bar{t}}\theta^{p'})(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}+e^{p'L\bar{t}}m(\varepsilon). \tag{34}$$

8. End of the case $\bar{t}>0$, choice of the various parameters. By plugging estimates (26), (27), (28), (30), (31) and (34) in (25), we get

$$\begin{aligned} Le^{L\bar{t}}\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}+\rho \;\leq\;& \Big(C_1e^{L\bar{t}}\theta+C_2+C_3e^{p'L\bar{t}}\theta^{p'}\Big)(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}\\ &+\Big(-\nu+\theta e^{L\bar{t}}(C_4+m(\varepsilon))\Big)(1-\mu)|\alpha_\eta|^p\\ &+(1+e^{p'L\bar{t}})m(\varepsilon)+m(\varrho/\varepsilon^4)+\eta, \end{aligned} \tag{35}$$

where

$$C_1=4(p-1)^2C_\sigma^2+4(p+1)C_b, \qquad C_2=C_\ell+C_u+C_f(1+C_u),$$
$$C_3=C_fCC_s^{p'}, \qquad C_4=4(p-1)C_\sigma^2+C_b,$$

are positive constants which depend only on the given datas of the problem.

Now we choose the different parameters in order to have a contradiction in the above inequality. We first assume that the final time T such that

$$T=1/L>0$$

(we will recover the result on any interval $[0,T]$ by a step-by-step argument). The main difficulty in the above estimate is to deal with the term in $|\alpha_\eta|^p$ since the control α_η is unbounded. Taking $\theta>0$ such that

$$\theta e^1(C_4+1)\leq\frac{\nu}{2},$$

we obtain that the coefficient in front of $|\alpha_\eta|^p$ is negative (we can assume that ε is small enough in order to have $m(\varepsilon)\leq 1$). Then we fix

$$L>C_1+\frac{C_2}{\theta}+C_3e^{p'-1}\theta^{p'-1} \quad\text{and}\quad \eta<\frac{\rho}{2}. \tag{36}$$

Therefore (35) implies

$$\frac{\rho}{2}\leq(1+e^{p'})m(\varepsilon)+m(\varrho/\varepsilon^4).$$

Sending first $\varrho\to 0$, we obtain a contradiction for small ε. In conclusion, up to a suitable choice of the parameters θ, L, η, the claim of the Step 3 is proved if $T\leq 1/L$.

9. Case when $\bar{t}=0$. We have just proved that there is a subsequence ε_n such that $\bar{t}=0$. Therefore for n large enough, for all $(x,t)\in I\!R^N\times[0,T]$, $T\leq 1/L$, we have

$$\begin{aligned} &\mu u(x,t)-v(x,t)-\theta(1-\mu)e^{Lt}(1+2|x|^2)^{p/2}-\rho t\\ \leq\;& \mu u(\bar{x},0)-v(\bar{y},0)-\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}-\frac{|\bar{x}-\bar{y}|^2}{\varepsilon_n^2}\\ \leq\;& (1-\mu)(|u(\bar{x},0)|-\theta(1+|\bar{x}|^2)^{p/2})+u(\bar{x},0)-v(\bar{y},0)\\ \leq\;& (1-\mu)M_\theta+u(\bar{x},0)-v(\bar{y},0) \end{aligned}$$

where M_θ is given by (13) since $u \in \mathcal{C}_p$. Since $u - v$ is upper-semicontinuous, from (19), we get

$$\limsup_{\varepsilon_n \to 0} u(\bar{x}, 0) - v(\bar{y}, 0) \leq u(x_0, 0) - v(x_0, 0) \leq 0,$$

using that $u(x_0, 0) \leq \psi(x_0) \leq v(x_0, 0)$. It follows

$$\mu u(x,t) - v(x,t) - \theta(1-\mu)e^{Lt}(1 + 2|x|^2)^{p/2} - \rho t \leq (1-\mu)M_\theta.$$

Sending $\mu \to 1$ and $\rho \to 0$, we get $u \leq v$ in $I\!R^N \times [0,T]$, $T \leq 1/L$. Noticing that L given by (36) depends only on the given constants of the problem, we recover the comparison on $[0,T]$ for any $T > 0$ by a classical step-by-step argument. It completes the proof of the theorem. □

We end with the proof of the existence result.

Proof of Theorem 2.2. The point is to build a sub- and a supersolution. We treat the case of the subsolution (the case of the supersolution being simpler). It suffices to prove that, there exists $\tau > 0$ such that, for all $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$u_\varepsilon(x,t) = -e^{\rho t}(M_\varepsilon + \varepsilon(1 + |x|^p)) \tag{37}$$

is a subsolution of (8) in $I\!R^N \times [0,\tau]$ with initial data ψ. Indeed, u_ε does not belong to $\mathcal{C}_p$ but $\underline{u} := \sup_{\varepsilon > 0} u_\varepsilon \in \mathcal{C}_p$ and $\underline{u}$ is still a subsolution.

Let $\varepsilon > 0$. Since $\psi, \ell(\cdot, t, \alpha), f(\cdot, t, u, z) \in \mathcal{C}_p$, there exists $M_\varepsilon = M_\varepsilon(\psi, \ell, f)$ such that $|\psi|, |\chi|, |\gamma| \leq M_\varepsilon + \varepsilon(1 + |x|^p)$. Let u_ε defined by (37) with this choice of M_ε. Let $\alpha \in A$. In the following computation, $C > 0$ is a constant which depends only on the given datas of the problem and may change line to line. We have, for all $(x,t) \in I\!R^N \times [0,T]$,

$$\begin{aligned}
\mathcal{L}(u_\varepsilon) \; &:= \; \frac{\partial u_\varepsilon}{\partial t} - \langle b, Du_\varepsilon \rangle - \ell - \mathrm{Trace}\left[\sigma\sigma^T D^2 u_\varepsilon\right] + f(x,t,u_\varepsilon, sDu_\varepsilon) \\
&\leq \; -\rho|u_\varepsilon| + C\varepsilon e^{\rho t}(1 + |x| + |\alpha|)|x|^{p-1} - \nu|\alpha|^p + |\chi| + C\varepsilon e^{\rho t}(1 + |x|^2 + |\alpha|^2)|x|^{p-2} \\
&\quad\; + |\gamma| + C|u_\varepsilon| + C\varepsilon^{p'} e^{p'\rho t}|x|^{p'(p-1)} \\
&\leq \; -\rho|u_\varepsilon| + C|u_\varepsilon| + C\varepsilon^{p'-1}e^{(p'-1)\rho t}|u_\varepsilon| - \frac{\nu}{2}|\alpha|^p,
\end{aligned}$$

since $p'(p-1) = p$,

$$|\alpha||x|^{p-1} + |\alpha|^2|x|^{p-2} \leq \frac{\nu}{2}|\alpha|^p + C|x|^p \quad \text{and} \quad |\chi| + |\gamma| \leq 2(M_\varepsilon + \varepsilon(1 + |x|^p)) = 2|u_\varepsilon|.$$

By choosing ρ large enough such that $\rho = C + Ce^1$ and $\tau > 0$ such that $(p'-1)\rho\tau \leq 1$, we obtain $\mathcal{L}(u_\varepsilon) \leq 0$ in $I\!R^N \times [0,\tau]$. Since $u_\varepsilon(\cdot, 0) \leq \psi$ by the choice of M_ε, we obtain that u_ε is a subsolution, which ends the proof. □

3 Equations with superlinear growth on the datas and the solutions

In this Section we extend the comparison result of [13] for equations with $p > 1$ growth conditions on the datas and on the solutions. For simplicity, we choose to consider here the following model equation where the diffusion and the drift do not depend on the control.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{Trace}(\sigma\sigma^T D^2 u) + \langle b, Du\rangle + f(x,t,u,sDu) = 0 & \text{in } I\!R^N \times [0,T], \\ u(x,0) = \psi(x) & \text{in } I\!R^N. \end{cases} \quad (38)$$

The hypothesis on the data are the following:

(C) (Asssumptions on the diffusion and the drift)

(i) $b \in C(I\!R^N \times [0,T]; I\!R^N)$ and there exists $C_b > 0$ such that, for all $x, y \in I\!R^N$, $t \in [0,T]$,

$$\begin{aligned} |b(x,t) - b(y,t)| &\leq C_b|x-y|, \\ |b(x,t)| &\leq C_b(1+|x|)\ ; \end{aligned}$$

(ii) $\sigma \in C(I\!R^N \times [0,T]; \mathcal{M}_{N,M}(I\!R))$ is Lipschitz continuous with respect to x (uniformly in t), namely, there exists $C_\sigma > 0$ such that, for all $x, y \in I\!R^N$ and $t \in [0,T]$,

$$|\sigma(x,t) - \sigma(y,t)| \leq C_\sigma |x-y|.$$

Note that σ satisfies, for every $x \in I\!R^N$, $t \in [0,T]$,

$$|\sigma(x,t)| \leq C_\sigma(1+|x|).$$

We are able to consider functions which are in a larger class than in Section 2. We say that a function $u\colon I\!R^N \times [0,T] \to I\!R$ is in the class $\tilde{\mathcal{C}}_p$ if for some $C > 0$ we have

$$|u(x,t)| \leq C(1+|x|^p), \quad \text{for all } (x,t) \in I\!R^N \times [0,T].$$

The main result of this Section if the following

Theorem 3.1 *Assume that σ and b satisfy* **(C)**, *that f satisfies* **(B)** *and that $\psi \in \tilde{\mathcal{C}}_p$. Let $u \in USC(I\!R^N \times [0,T])$ be a viscosity subsolution of* (38) *and $v \in LSC(I\!R^N \times [0,T])$ be a viscosity supersolution of* (38). *Suppose that U and V are in the class $\tilde{\mathcal{C}}_p$ and satisfy $u(x,0) \leq \psi(x) \leq v(x,0)$. Then $u \leq v$ in $I\!R^N \times [0,T]$.*

Before giving the proof of the theorem, we state an existence result and provide some examples. As observed in Section 2, we can prove the existence of solutions of (8) (at least for small time) as soon as we are able to build sub- and supersolutions in the class $\tilde{\mathcal{C}}_p$. In Example 3.4, we see that solutions may not exist for all time.

Theorem 3.2 *Assume **(B)**–**(C)**. If $K, \rho > 0$ are large enough, then $\overline{u}(x,t) = K\mathrm{e}^{\rho t}(1+|x|^2)^{p/2}$ is a viscosity supersolution of* (38) *in $I\!R^N \times [0,T]$ and there exists $0 < \tau \leq T$ such that $\underline{u}(x,t) = -K\mathrm{e}^{\rho t}(1+|x|^2)^{p/2}$ is a viscosity subsolution of* (38) *in $I\!R^N \times [0,\tau]$. In consequence, for all $\psi \in \tilde{\mathcal{C}}_p$, there exists a unique continuous viscosity solution of* (38) *in $I\!R^N \times [0,\tau]$ in $\tilde{\mathcal{C}}_p$.*

The proof is very close to the one of [13, Lemma 2.1], thus we omit it. Let us give some examples of Equations for which Theorem 3.1 applies and some examples.

Example 3.1 The typical (simple) case we have in mind is

$$u_t - \Delta u + |Du|^{p'} = -f(x,t) \quad \text{in } I\!R^N \times [0,T], \tag{39}$$

where f satisfies **(B)**. Note that (39) can be written

$$u_t - \Delta u + p \sup_{\alpha \in I\!R^N} \{\langle \alpha, Du\rangle - \frac{|\alpha|^{p'}}{p'}\} + f(x,t) = 0$$

and therefore is on the form (8). The stationary version of this equation was studied in Alvarez [1] under more restrictive assumptions on the datas and the growth of the solution. More precisely, he assumed conditions like (12) and (14).

Example 3.2 Equation (8) typically appears in the study of BSDEs where $s(x,t) = \sigma(x,t)$. In [9], Briand and Hu proved that u given by (7) is a viscosity solution of (8) for $1 \le p < 2$. Theorem 3.1 proves this solution is unique. We are able to deal with any $p > 1$ but we had to impose the regularity condition **(B)**(ii) on x for f, which is not needed for the BDSEs.

Example 3.3 As far as the coefficient f is concerned, a typical case we have in mind is

$$f(x,t,u,z) = g(x,t,u) + |z|^{p'},$$

with continuous g satisfying **(B)**(i),(ii) and (v). It leads to nonlinearities like "$g(x,t,u) + |s(x,t)Du|^{p'}$" in the equation. Note that the power-p' term depends on x via $s(x,t)$. This dependence brings an additional difficulty, see Lemma 3.2.

Example 3.4 (Deterministic Control Problem) Consider the control problem (in dimension 1 for sake of simplicity)

$$\begin{cases} dX_s = \alpha_s \, ds, \quad s \in [t,T], \ 0 \le t \le T, \\ X_t = x \in I\!R, \end{cases}$$

where the control $\alpha \in \mathcal{A}_t := L^p([t,T]; I\!R)$ and the value function is given by

$$V(x,t) = \inf_{\alpha \in \mathcal{A}_t} \{\int_t^T (\frac{|\alpha_s|^p}{p} + \rho |X_s|^p) \, ds - |X_T|^p\} \quad \text{for some } \rho > 0.$$

The Hamilton-Jacobi equation formally associated to this problem is

$$\begin{cases} -w_t + \frac{1}{p'}|w_x|^{p'} = \rho |x|^p & \text{in } I\!R \times (0,T), \\ w(x,T) = -|x|^p & \text{in } I\!R. \end{cases}$$

Looking for a solution w under the form $w(x,t) = \varphi(t)|x|^p$, we obtain that φ is a solution of the differential equation

$$-\varphi' + \frac{|\varphi|^{p'}}{p'} = \rho \ \text{ in } (0,T), \qquad \varphi(T) = -1.$$

We get

$$\int_{-1}^{\varphi(t)} \frac{p'}{|y|^{p'} - \rho p'} dy = t - T.$$

One can check that if $0 < \rho p' < 1$ and $T > \int_{-\infty}^{-1} \frac{p'}{|y|^{p'}-\rho p'} dy$, then there is $\tau \in (0,T)$ such that the solution blows up at $t = \tau$.

Let us turn to the proof of the comparison theorem.

Proof of Theorem 3.1. To avoid a lot of technicality, we start the proof with several lemmas collecting the main intermediate results. The proofs of the lemmas are postponed at the end of the section and can be skipped at first reading.

Lemma 3.1 *(Change of functions)*
Let $\tilde{u} = \mathrm{e}^{-Lt}u + h(x)$ where $h(x) = \overline{C}(1+|x|^p)$ for some constants $\overline{C}, L > 0$. Then $\tilde{u}$ is a viscosity solution of

$$\begin{cases} \tilde{u}_t - \mathrm{Trace}(\sigma(x,t)\sigma(x,t)^T D^2\tilde{u}) + \langle b(x,t), D\tilde{u}\rangle \\ \qquad\qquad + \tilde{f}(x,t,\tilde{u}-h, s(x,t)(D\tilde{u}-Dh)) = 0 & \text{in } I\!\!R^N \times (0,T], \\ \tilde{u}(x,0) = \psi(x) + h(x) & \text{for all } x \in I\!\!R^N, \end{cases} \tag{40}$$

with, for all $(x,t,v,z) \in I\!\!R^N \times [0,T] \times I\!\!R^N$,

$$\tilde{f}(x,t,v,z) = Lv + \tilde{g}(x,t) + \mathrm{e}^{-Lt} f\left(x,t,\mathrm{e}^{Lt}v, \mathrm{e}^{Lt}z\right), \tag{41}$$

where

$$\tilde{g}(x,t) = \mathrm{Trace}(\sigma(x,t)\sigma(x,t)^T D^2 h(x)) - \langle b(x,t), Dh(x)\rangle. \tag{42}$$

Moreover,

$$\tilde{f}(x,t,v,z) - \tilde{f}(x,t,v',z) \le (\hat{C} - L)(v' - v) \quad \text{if } v \le v'. \tag{43}$$

In the sequel, since $u, v, \psi \in \tilde{\mathcal{C}}_p$, we can choose $\overline{C} > 0$ such that

$$|u|, |v|, |\psi| \le \frac{\overline{C}}{2}(1 + |x|^p). \tag{44}$$

In this case, note that

$$\psi(x) + h(x) = \psi(x) + \overline{C}(1 + |x|^p) \ge 0$$

and the initial data is nonnegative in (40).

Moreover, we take

$$L > \hat{C} \quad \text{and} \quad L > 4p(p-1)NC_\sigma^2 + 4pC_b + 10\hat{C} \tag{45}$$

(the constants C_σ, C_b and $\hat{C}$ appear in **(B)**). The first condition ensures that the right-hand side of (43) is nonpositive (i.e. $v \mapsto \tilde{f}(x,t,v,z)$ is nondecreasing). The second condition appears naturally in the proof of the following lemma.

Lemma 3.2 *(A kind of linearization procedure)*
Let $\overline{C}, L > 0$ *be such that* (44) *and* (45) *hold. Let* $0 < \mu < 1$ *and set* $\tilde{w} = \mu\tilde{u} - \tilde{v}$. *Then* $\tilde{w}$ *is a viscosity subsolution of the variational inequality*

$$\begin{cases} \min\{w, \mathcal{L}[w]\} \le 0 & \text{in } I\!R^n \times (0,T), \\ w(\cdot,0) \le 0 & \text{in } I\!R^n, \end{cases} \tag{46}$$

where

$$\begin{aligned} \mathcal{L}[w] \;:=\; & \frac{\partial w}{\partial t} - \mathrm{Trace}[\sigma(x,t)\sigma^T(x,t)D^2 w] - C_b(1+|x|)|Dw| + \frac{L}{4}(1-\mu)h(x,t) \\ & -(1-\mu)\mathrm{e}^{-Lt} f\left(x,t,0,\mathrm{e}^{Lt}s(x,t)(\frac{Dw}{\mu-1} - Dh(x))\right) \end{aligned} \tag{47}$$

and h *is defined in Lemma 3.1.*

Lemma 3.3 *(An auxiliary parabolic problem)*
Consider, for any $R > 0$, *the parabolic problem*

$$\begin{cases} \varphi_t - r^2\varphi_{rr} - r\varphi_r = 0 & \textit{in } [0,+\infty) \times (0,T], \\ \varphi(r,0) = \max\{0, r-R\} & \textit{in } [0,+\infty). \end{cases} \tag{48}$$

Then (48) *has a unique solution* $\varphi_R \in C([0,+\infty) \times [0,T]) \cap C^\infty([0,+\infty) \times (0,T])$ *such that, for all* $t \in (0,T]$, $\varphi_R(\cdot,t)$ *is positive, nondecreasing and convex in* $[0,+\infty)$. *Moreover, for every* $(r,t) \in [0,+\infty) \times (0,T]$,

$$\varphi_R(r,t) \ge \max\{0, r-R\}, \quad 0 \le \frac{\partial \varphi_R}{\partial r}(r,t) \le \mathrm{e}^T \quad \textit{and} \quad \varphi_R(r,t) \underset{R\to+\infty}{\longrightarrow} 0. \tag{49}$$

For the proof of Lemma 3.3 we refer the reader to [13].

Lemma 3.4 *(Construction of a smooth strict supersolution)*
Let $\Phi(x,t) = \varphi_R(h(x), Ct)$ *where* φ_R *is given by Lemma 3.3,* $h(x) = \overline{C}(1+|x|^p)$, $\overline{C}$ *satisfies* (44) *and* $C > 0$. *Then, for* C *and* $L = L(\mu)$ *large enough, we have*

$$\mathcal{L}[\Phi(x,t)] > 0 \quad \textit{for all } (x,t) \in I\!R^N \times (0, 1/L], \tag{50}$$

where $\mathcal{L}$ *is defined by* (47).

Now, we continue the **proof of Theorem 3.1**. Consider

$$\max_{I\!R^N \times [0,1/L]} \{\tilde{w} - \Phi\}, \tag{51}$$

where $\tilde{w}$ is given by Lemma 3.2 and Φ is the function built in Lemma 3.4. From (49), for $|x|$ large enough, we have $\Phi(x,t) \ge \overline{C}(1+|x|^p) - R$. Since $\tilde{w} \le (\mu+1)\overline{C}(1+|x|^p)/2$, it follows that the maximum (51) is achieved at a point $(\bar{x},\bar{t}) \in I\!R^N \times [0,1/L]$. We can assume that $\tilde{w}(\bar{x},\bar{t}) > 0$ otherwise, arguing as in (52)-(53), we prove $\tilde{w} \le 0$ in $I\!R^N \times [0,1/L]$ and the conclusion follows.

We claim that $\bar{t}=0$. Indeed suppose by contradiction that $\bar{t}>0$. Then since $\tilde{w}$ is a viscosity subsolution of (46) with $\tilde{w}(\bar{x},\bar{t})>0$, by taking Φ as a test-function, we would have $\mathcal{L}[\Phi](\bar{x},\bar{t})\leq 0$ which contradicts the fact that Φ satisfies (50) . Thus, for all $(x,t)\in I\!\!R^N\times[0,1/L]$,

$$\tilde{w}(x,t)-\Phi(x,t)\leq \tilde{w}(\bar{x},0)-\Phi(\bar{x},0)\leq 0, \tag{52}$$

where the last inequality follows from (46) and the fact that $\Phi\geq 0$. Therefore, for every $(x,t)\in I\!\!R^N\times[0,1/L]$, we have

$$(\mu\tilde{u}-\tilde{v})(x,t)=\tilde{w}(x,t)\leq\Phi(x,t)=\varphi_R(h(x),Ct). \tag{53}$$

Letting R to $+\infty$, we get by (49), $\mu\tilde{u}-\tilde{v}\leq 0$ in $I\!\!R^N\times[0,1/L]$.

We can repeat the above arguments on $I\!\!R^N\times[1/L,2/L]$ with the same constants. By a step-by-step argument, we then prove that $\mu\tilde{u}-\tilde{v}\leq 0$ in $I\!\!R^N\times[0,T]$. Letting μ go to 1, we obtain $u\leq v$ as well which completes the proof of the theorem. □

We turn to the proof of the Lemmas 3.1, 3.2 and 3.4.

Proof of Lemma 3.1. Since

$$\begin{aligned} u&=\mathrm{e}^{Lt}(\tilde{u}-h), & u_t&=\mathrm{e}^{Lt}(\tilde{u}_t+L(\tilde{u}-h)),\\ Du&=\mathrm{e}^{Lt}(D\tilde{u}-Dh), & D^2u&=\mathrm{e}^{Lt}(D^2\tilde{u}-D^2h),\end{aligned}$$

we obtain easily that $\tilde{u}$ is a viscosity solution of (40) with $\tilde{f}$ and $\tilde{g}$ given by (41) and (42). It remains to check (43)). Take $v,v'\in I\!\!R$ such that $v\leq v'$. From **(B)**(v), we obtain

$$\begin{aligned} &\tilde{f}(x,t,v,z)-\tilde{f}(x,t,v',z)\\ \leq\ & L(v-v')+\mathrm{e}^{-Lt}\left(f(x,t,\mathrm{e}^{Lt}v,\mathrm{e}^{Lt}z)-f(x,t,\mathrm{e}^{Lt}v',\mathrm{e}^{Lt}z)\right)\\ \leq\ & -L(v'-v)+\mathrm{e}^{-Lt}\hat{C}|\mathrm{e}^{Lt}(v-v')|\leq(\hat{C}-L)(v'-v).\end{aligned}$$

It ends the proof of the lemma. □

Proof of Lemma 3.2. For $0<\mu<1$, let $\tilde{u}^\mu=\mu\tilde{u}$ and $\tilde{w}=\tilde{u}^\mu-\tilde{v}$. We divide the proof in different steps.

Step 1. A new equation for $\tilde{u}^\mu$. It is not difficult to see that, if $\tilde{u}$ is a subsolution of (40), then $\tilde{u}^\mu$ is a subsolution of

$$\left\{\begin{array}{ll} \tilde{u}^\mu_t-\mathrm{Trace}(\sigma(x,t)\sigma(x,t)^TD^2\tilde{u}^\mu)+\langle b(x,t),D\tilde{u}^\mu\rangle & \\ \qquad +\mu\,\tilde{f}(x,t,\dfrac{\tilde{u}^\mu}{\mu}-h,s(x,t)(\dfrac{D\tilde{u}^\mu}{\mu}-Dh))=0 & \text{in } I\!\!R^N\times(0,T],\\ \tilde{u}^\mu(x,0)=\mu\psi(x)+\mu h(x) & \text{for all } x\in I\!\!R^N.\end{array}\right. \tag{54}$$

Step 2. The equation for $\tilde{w}$. Let $\varphi\in C^2(I\!\!R^N\times[0,T])$ and suppose that we have

$$\max_{I\!\!R^N\times[0,T]}\tilde{w}-\varphi=(\tilde{w}-\varphi)(\bar{x},\bar{t}). \tag{55}$$

We distinguish 3 cases.

At first, if the maximum is achieved for $\bar{t} = 0$, then, writing that $\tilde{u}^\mu$ is a subsolution of (54) and $\tilde{v}$ a supersolution of (40) at $t = 0$ we obtain $\tilde{u}^\mu(\bar{x}, 0) \le \mu\psi(\bar{x}) + \mu h(\bar{x})$ and $\tilde{v}(\bar{x}, 0) \ge \psi(\bar{x}) + h(\bar{x})$. It follows that

$$\tilde{w}(\bar{x}, 0) \le (\mu - 1)(\psi(\bar{x}) + h(\bar{x})) = (\mu - 1)(\psi(\bar{x}) + \overline{C}(1 + |\bar{x}|^p)) \le 0$$

by (44). Therefore $\tilde{w}$ satisfies (46) at $(\bar{x}, 0)$.

Secondly, we suppose that $\bar{t} > 0$ and $\tilde{w}(\bar{x}, \bar{t}) \le 0$. Again, $\tilde{w}$ satisfies (46) at $(\bar{x}, \bar{t})$.

From now on, we consider the last and most difficult case when

$$\bar{t} > 0 \quad \text{and} \quad \tilde{w}(\bar{x}, \bar{t}) > 0. \tag{56}$$

Step 3. Viscosity inequalities for $\tilde{u}^\mu$ and $\tilde{v}$. This step is classical in viscosity theory. We can assume that the maximum in (55) at $(\bar{x}, \bar{t})$ is strict in the some ball $\overline{B}(\bar{x}, r) \times [\bar{t} - r, \bar{t} + r]$ (see [6] or [3]). Let

$$\Theta(x, y, t) = \varphi(x, t) + \frac{|x - y|^2}{\varepsilon^2}$$

and consider

$$M_\varepsilon := \max_{x,y \in \overline{B}(\bar{x}, r),\, t \in [\bar{t} - r, \bar{t} + r]} \{\tilde{u}^\mu(x, t) - \tilde{v}(y, t) - \Theta(x, y, t)\}.$$

This maximum is achieved at a point $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ and, since the maximum is strict, we know that

$$x_\varepsilon, y_\varepsilon \to \bar{x}, \qquad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0, \tag{57}$$

and

$$M_\varepsilon = \tilde{u}^\mu(x_\varepsilon, t_\varepsilon) - \tilde{v}(y_\varepsilon, t_\varepsilon) - \Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon) \longrightarrow (\tilde{w} - \varphi)(\bar{x}, \bar{t}) \quad \text{as } \varepsilon \to 0.$$

It means that, at the limit $\varepsilon \to 0$, we obtain some information on $\tilde{w} - \varphi$ at $(\bar{x}, \bar{t})$ which will provide the new equation for $\tilde{w}$. From (56), for ε small enough, we have

$$\tilde{u}^\mu(x_\varepsilon, t_\varepsilon) - \tilde{v}(y_\varepsilon, t_\varepsilon) > 0. \tag{58}$$

We can take Θ as a test-function to use the fact that $\tilde{u}^\mu$ is a subsolution of (54) and $\tilde{v}$ a supersolution of (40). Indeed $(x, t) \in \overline{B}(\bar{x}, r) \times [\bar{t} - r, \bar{t} + r] \mapsto \tilde{u}^\mu(x, t) - \tilde{v}(y_\varepsilon, t) - \Theta(x, y_\varepsilon, t)$ achieves its maximum at $(x_\varepsilon, t_\varepsilon)$ and $(y, t) \in \overline{B}(\bar{x}, r) \times [\bar{t} - r, \bar{t} + r] \mapsto -\tilde{u}^\mu(x_\varepsilon, t) + \tilde{v}(y, t) + \Theta(x_\varepsilon, y, t)$ achieves its minimum at $(y_\varepsilon, t_\varepsilon)$. Thus, by Theorem 8.3 in the User's guide [11], for every $\rho > 0$, there exist $a_1, a_2 \in I\!R$ and $X, Y \in \mathcal{S}_N$ such that

$$(a_1, D_x\Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon), X) \in \bar{\mathcal{P}}^{2,+}(\tilde{u}^\mu)(x_\varepsilon, t_\varepsilon), \quad (a_2, -D_y\Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon), Y) \in \bar{\mathcal{P}}^{2,-}(\tilde{v})(y_\varepsilon, t_\varepsilon),$$

$a_1 - a_2 = \Theta_t(x_\varepsilon, y_\varepsilon, t_\varepsilon) = \varphi_t(x_\varepsilon, t_\varepsilon)$ and

$$-(\frac{1}{\rho} + |M|)I \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le M + \rho M^2 \quad \text{where } M = D^2\Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon). \tag{59}$$

Setting $p_\varepsilon = 2\dfrac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}$, we have

$$D_x\Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon) = p_\varepsilon + D\varphi(x_\varepsilon, t_\varepsilon) \quad \text{and} \quad D_y\Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon) = -p_\varepsilon,$$

and

$$M = \begin{pmatrix} D^2\varphi(x_\varepsilon, t_\varepsilon) + 2I/\varepsilon^2 & -2I/\varepsilon^2 \\ -2I/\varepsilon^2 & 2I/\varepsilon^2 \end{pmatrix}.$$

Thus, from (59), it follows

$$\langle Xp, p\rangle - \langle Yq, q\rangle \le \langle D^2\varphi(x_\varepsilon, t_\varepsilon)p, p\rangle + \frac{2}{\varepsilon^2}|p-q|^2 + m\left(\frac{\rho}{\varepsilon^4}\right), \tag{60}$$

where m is a modulus of continuity which is independent of ρ and ε. In the sequel, m will always denote a generic modulus of continuity independent of ρ and ε.

Writing the subsolution viscosity inequality for $\tilde{u}^\mu$ and the supersolution inequality for $\tilde{v}$ by means of the semi-jets and subtracting the inequalities, we obtain

$$\begin{aligned}
&\varphi_t(x_\varepsilon, t_\varepsilon) \\
&-\text{Trace}\left[\sigma(x_\varepsilon, t_\varepsilon)\sigma^T(x_\varepsilon, t_\varepsilon)X\right] + \text{Trace}\left[\sigma(y_\varepsilon, t_\varepsilon)\sigma^T(y_\varepsilon, t_\varepsilon)Y\right] \\
&-\langle b(x_\varepsilon, t_\varepsilon), p_\varepsilon + D\varphi(x_\varepsilon, t_\varepsilon)\rangle + \langle b(y_\varepsilon, t_\varepsilon), p_\varepsilon\rangle \\
&+\mu\tilde{f}\left(x_\varepsilon, t_\varepsilon, \frac{\tilde{u}^\mu(x_\varepsilon, t_\varepsilon)}{\mu} - h(x_\varepsilon), s(x_\varepsilon, t_\varepsilon)(\frac{p_\varepsilon + D\varphi(x_\varepsilon, t_\varepsilon)}{\mu} - Dh(x_\varepsilon))\right) \\
&-\tilde{f}\left(y_\varepsilon, t_\varepsilon, \tilde{v}(y_\varepsilon, t_\varepsilon) - h(y_\varepsilon), s(y_\varepsilon, t_\varepsilon)(p_\varepsilon - Dh(y_\varepsilon))\right) \\
\le\ & 0
\end{aligned} \tag{61}$$

Now, we derive some estimates for the various terms appearing in (61) in order to be able ton send $\varepsilon \to 0$. The estimates for the σ and b terms are classical wheras those for the f terms are more involved.

For the sake of simplicity, for any function $g : I\!R^N \times [0,T] \to I\!R$, we set

$$g(x_\varepsilon, t_\varepsilon) = g_x \quad \text{and} \quad g(y_\varepsilon, t_\varepsilon) = g_y.$$

Step 4. Estimate of σ-terms. Let us denote by $(e_i)_{1\le i\le N}$ the canonical basis of $I\!R^N$. By using (60), we obtain

$$\begin{aligned}
\text{Trace}\left[c_x\sigma_x^T X - \sigma_y\sigma_y^T Y\right] &= \sum_{i=1}^N \langle X\sigma_x e_i, \sigma_x e_i\rangle - \langle Y\sigma_y e_i, \sigma_y e_i\rangle \\
&\le \text{Trace}\left[\sigma_x\sigma_x^T D^2\varphi(x_\varepsilon, t_\varepsilon)\right] + \frac{2}{\varepsilon^2}|\sigma_x - \sigma_y|^2 + m\left(\frac{\rho}{\varepsilon^4}\right) \\
&\le \text{Trace}\left[\sigma_x\sigma_x^T D^2\varphi(x_\varepsilon, t_\varepsilon)\right] + 2C_{\sigma,r}^2\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + m\left(\frac{\rho}{\varepsilon^4}\right) \\
&\le \text{Trace}\left[\sigma\sigma^T(\bar{x}, \bar{t})D^2\varphi(\bar{x}, \bar{t})\right] + m(\varepsilon) + m\left(\frac{\rho}{\varepsilon^4}\right),
\end{aligned} \tag{62}$$

where $C_{\sigma,r}$ is a Lipschitz constant for σ in $\bar{B}(x,r)$ and we used that σ is continuous, φ is C^2 and (57).
Step 5. Estimate of b-terms. From **(C)**, if $C_{b,r}$ is the Lipschitz constant of b in $\overline{B}(\bar{x},r)\times[\bar{t}-r,\bar{t}+r]$, then we have

$$\langle b(x_\varepsilon,t_\varepsilon)-b(y_\varepsilon,t_\varepsilon),p_\varepsilon\rangle \le C_{b,r}|x_\varepsilon-y_\varepsilon||p_\varepsilon| \le 2C_{b,r}\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2}=m(\varepsilon)$$

and

$$\langle b(x_\varepsilon,t_\varepsilon),D\varphi(x_\varepsilon,t_\varepsilon)\rangle \le C_b(1+|x_\varepsilon|)|D\varphi(x_\varepsilon,t_\varepsilon)|.$$

It follows

$$\langle b(x_\varepsilon,t_\varepsilon),p_\varepsilon+D\varphi(x_\varepsilon,t_\varepsilon)\rangle-\langle b(y_\varepsilon,t_\varepsilon),p_\varepsilon\rangle \le C_b(1+|\bar{x}|)|D\varphi(\bar{x},\bar{t})|+m(\varepsilon) \tag{63}$$

Step 6. Estimate of $\tilde{f}$-terms. We write

$$\begin{aligned}
&-\mu\tilde{f}\left(x_\varepsilon,t_\varepsilon,\frac{\tilde{u}^\mu(x_\varepsilon,t_\varepsilon)}{\mu}-h_x,s_x(\frac{p_\varepsilon+D\varphi(x_\varepsilon,t_\varepsilon)}{\mu}-Dh_x)\right)\\
&+\tilde{f}\left(y_\varepsilon,t_\varepsilon,\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y,s_y(p_\varepsilon-Dh_y)\right)\\
=\;&\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}_1 &= -\mu\tilde{f}\left(x_\varepsilon,t_\varepsilon,\frac{\tilde{u}^\mu(x_\varepsilon,t_\varepsilon)}{\mu}-h_x,s_x(\frac{p_\varepsilon+D\varphi(x_\varepsilon,t_\varepsilon)}{\mu}-Dh_x)\right)\\
&\quad+\mu\tilde{f}\left(x_\varepsilon,t_\varepsilon,\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y,s_x(\frac{p_\varepsilon+D\varphi(x_\varepsilon,t_\varepsilon)}{\mu}-Dh_x)\right),\\
\mathcal{T}_2 &= -\mu\tilde{f}\left(x_\varepsilon,t_\varepsilon,\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y,s_x(\frac{p_\varepsilon+D\varphi(x_\varepsilon,t_\varepsilon)}{\mu}-Dh_x)\right)\\
&\quad+\mu\tilde{f}\left(y_\varepsilon,t_\varepsilon,\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y,s_x(\frac{p_\varepsilon+D\varphi(x_\varepsilon,t_\varepsilon)}{\mu}-Dh_x)\right),\\
\mathcal{T}_3 &= -\mu\tilde{f}\left(y_\varepsilon,t_\varepsilon,\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y,s_x(\frac{p_\varepsilon+D\varphi(x_\varepsilon,t_\varepsilon)}{\mu}-Dh_x)\right)\\
&\quad+\tilde{f}\left(y_\varepsilon,t_\varepsilon,\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y,s_y(p_\varepsilon-Dh_y)\right).
\end{aligned}$$

We estimate $\mathcal{T}_1$. From (58), we have

$$\tilde{u}(x_\varepsilon,t_\varepsilon)=\frac{\tilde{u}^\mu(x_\varepsilon,t_\varepsilon)}{\mu}>\tilde{v}(y_\varepsilon,t_\varepsilon)+(1-\mu)\tilde{u}(x_\varepsilon,t_\varepsilon).$$

Using (43) (the monotonicity in u of $\tilde f$) and then **(B)**(v) (Lipschitz continuity in u of f), we get

$$\begin{aligned}
& \tilde f\left(x_\varepsilon, t_\varepsilon, \frac{\tilde u^\mu(x_\varepsilon,t_\varepsilon)}{\mu} - h_x, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
\geq\ & \tilde f\left(x_\varepsilon, t_\varepsilon, \tilde v(y_\varepsilon,t_\varepsilon) + (1-\mu)\tilde u(x_\varepsilon,t_\varepsilon) - h_x, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
\geq\ & \tilde f\left(x_\varepsilon, t_\varepsilon, \tilde v(y_\varepsilon,t_\varepsilon) + h_y, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
& -\hat C(1-\mu)|\tilde u(x_\varepsilon,t_\varepsilon)| - \hat C|h_x - h_y|.
\end{aligned}$$

Since h is continuous, we have $\hat C|h_x - h_y| = m(\varepsilon)$. By (44) and since $x_\varepsilon \to \bar x$, we obtain

$$\hat C(1-\mu)|\tilde u(x_\varepsilon,t_\varepsilon)| \leq \hat C\overline{C}(1-\mu)(1+|x_\varepsilon|^p) = \hat C(1-\mu)h(\bar x) + m(\varepsilon).$$

Therefore

$$\mathcal{T}_1 \leq \hat C(1-\mu)h(\bar x) + m(\varepsilon). \tag{64}$$

The estimate of $\mathcal{T}_2$ relies on **(B)**(ii). Setting $Q_\varepsilon = \mathrm{e}^{Lt_\varepsilon} s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)$ and recalling that r is defined at the beginning of Step 3, we have

$$\begin{aligned}
|\mathcal{T}_2| \ \leq\ & \mu|g(x_\varepsilon,t_\varepsilon) - g(y_\varepsilon,t_\varepsilon)| \\
& +\mu\mathrm{e}^{-Lt_\varepsilon}|f(x_\varepsilon,t_\varepsilon,\mathrm{e}^{Lt_\varepsilon}(\tilde v(y_\varepsilon,t_\varepsilon) - h_y), Q_\varepsilon) - f(y_\varepsilon,t_\varepsilon,\mathrm{e}^{-Lt_\varepsilon}(\tilde v(y_\varepsilon,t_\varepsilon) - h_y), Q_\varepsilon)| \\
\leq\ & \mu|g(x_\varepsilon,t_\varepsilon) - g(y_\varepsilon,t_\varepsilon)| \\
& +\mu\mathrm{e}^{-Lt_\varepsilon}\, m_{2r}\left((1 + |\mathrm{e}^{Lt_\varepsilon}(\tilde v(y_\varepsilon,t_\varepsilon) - h_y)| + |Q_\varepsilon|)|x_\varepsilon - y_\varepsilon|\right) \\
\leq\ & m(\varepsilon),
\end{aligned} \tag{65}$$

since g is continuous, $|x_\varepsilon - y_\varepsilon| = m(\varepsilon)$ and $p_\varepsilon|x_\varepsilon - y_\varepsilon| = |x_\varepsilon - y_\varepsilon|^2/\varepsilon^2 = m(\varepsilon)$ by (57).

Let us turn to the estimate of $\mathcal{T}_3$. We have

$$\begin{aligned}
\mathcal{T}_3 \ =\ & L(1-\mu)(\tilde v(y_\varepsilon,t_\varepsilon) - h_y) + (1-\mu)\tilde g(y_\varepsilon,t_\varepsilon) \\
& -\mu\mathrm{e}^{-Lt_\varepsilon} f\left(y_\varepsilon,t_\varepsilon,\mathrm{e}^{Lt_\varepsilon}(\tilde v(y_\varepsilon,t_\varepsilon) - h_y), \mathrm{e}^{Lt_\varepsilon}s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
& +\mathrm{e}^{-Lt_\varepsilon} f\left(y_\varepsilon,t_\varepsilon,\mathrm{e}^{Lt_\varepsilon}(\tilde v(y_\varepsilon,t_\varepsilon) - h_y), \mathrm{e}^{Lt_\varepsilon}s_y(p_\varepsilon - Dh_y)\right).
\end{aligned}$$

At first, from (44), we have

$$L(1-\mu)(\tilde v(y_\varepsilon,t_\varepsilon) - h_y) \leq -\frac{L(1-\mu)}{2}h(\bar x) + m(\varepsilon).$$

Using **(C)** (see (68) for the details), a straightforward computation gives an estimate for the continuous function $\tilde g$:

$$|\tilde g(y_\varepsilon,t_\varepsilon)| \leq (p(p-1)NC_\sigma^2 + pC_b)h(\bar x) + m(\varepsilon).$$

Now we estimate the f-terms. From **(B)**(iii) (convexity of f with respect to the gradient variable), we can apply (32) to obtain

$$\begin{aligned}
& -\mu e^{-Lt_\varepsilon} f\left(y_\varepsilon, t_\varepsilon, e^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y), e^{Lt_\varepsilon} s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
& + e^{-Lt_\varepsilon} f\left(y_\varepsilon, t_\varepsilon, e^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y), e^{Lt_\varepsilon} s_y(p_\varepsilon - Dh_y)\right) \\
\leq\ & (1-\mu) e^{-Lt_\varepsilon} f\left(y_\varepsilon, t_\varepsilon, e^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y), Q_\varepsilon\right),
\end{aligned}$$

where

$$\begin{aligned}
Q_\varepsilon &= \frac{e^{Lt_\varepsilon}}{\mu-1}\left((s_x - s_y)p_\varepsilon + s_x D\varphi(x_\varepsilon,t_\varepsilon) + s_y Dh_y - \mu s_x Dh_x\right) \\
&= e^{L\bar{t}} s(\bar{x},\bar{t})\left(\frac{D\varphi(\bar{x},\bar{t})}{\mu-1} - Dh(\bar{x})\right) + m(\varepsilon).
\end{aligned}$$

from **(B)**(iv) and (57). From **(B)**(v), (44) and the continuity of f, it follows

$$\begin{aligned}
& e^{-Lt_\varepsilon} f\left(y_\varepsilon, t_\varepsilon, e^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y), Q_\varepsilon\right) \\
\leq\ & e^{-L\bar{t}} f\left(\bar{x}, \bar{t}, 0, e^{L\bar{t}} s(\bar{x},\bar{t})\left(\frac{D\varphi(\bar{x},\bar{t})}{\mu-1} - Dh(\bar{x})\right)\right) + \frac{3\hat{C}}{2} h(\bar{x}) + m(\varepsilon).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\mathcal{T}_3 \leq\ & (1-\mu)\left(-\frac{L}{2} + p(p-1)NC_\sigma^2 + pC_b + \frac{3\hat{C}}{2}\right) h(\bar{x}) \\
& + e^{-L\bar{t}} f\left(\bar{x}, \bar{t}, 0, e^{L\bar{t}} s(\bar{x},\bar{t})\left(\frac{D\varphi(\bar{x},\bar{t})}{\mu-1} - Dh(\bar{x})\right)\right) + m(\varepsilon). \qquad (66)
\end{aligned}$$

Step 7. End of the proof. Combining (61), (62), (63), (64), (65) and (66), setting $L > 4p(p-1)NC_\sigma^2 + 4pC_b + 10\hat{C}$ and sending $\rho \to 0$ and then $\varepsilon \to 0$, we get

$$\begin{aligned}
& \varphi_t(\bar{x},\bar{t}) - \text{Trace}\left[\sigma\sigma^T(\bar{x},\bar{t}) D^2\varphi(\bar{x},\bar{t})\right] - C_b(1+|\bar{x}|)|D\varphi(\bar{x},\bar{t})| + \frac{L}{4}(1-\mu)h(\bar{x}) \\
& -(1-\mu)e^{-L\bar{t}} f\left(\bar{x}, \bar{t}, 0, e^{L\bar{t}} s(\bar{x},\bar{t})\left(\frac{D\varphi(\bar{x},\bar{t})}{\mu-1} - Dh(\bar{x})\right)\right) \\
\leq\ & 0,
\end{aligned}$$

which is exactly the new equation for $\tilde{w}$ in the case (56). It completes the proof of the lemma. □

Proof of Lemma 3.4. For simplicity, we fix R and set $\varphi = \varphi_R$ for simplicity. Therefore φ_r denotes the derivative of φ wrt the space variable. We compute

$$\Phi_t = C\varphi_t, \quad D\Phi = \varphi_r Dh,$$
$$D^2\Phi = \varphi_r D^2 h + \varphi_{rr} Dh \otimes Dh,$$

with

$$h = \overline{C}(1+|x|^p), \quad Dh = p\overline{C}|x|^{p-2}x, \quad D^2h = p\overline{C}(|x|^{p-2}Id + (p-2)|x|^{p-4}x\otimes x).$$

For all $(x,t)\in I\!R^N\times(0,T]$,

$$\begin{aligned} & \mathcal{L}(\Phi(x,t)) \\ = \;& C\varphi_t - \left(\mathrm{Trace}(\sigma\sigma^T D^2h) + C_b(1+|x|)|Dh|\right)\varphi_r - \mathrm{Trace}(\sigma\sigma^T Dh\otimes Dh)\varphi_{rr} \\ & +(1-\mu)\frac{L}{4}h - (1-\mu)\mathrm{e}^{-Lt} f\left(x,t,0,\mathrm{e}^{Lt}s\left(\frac{\varphi_r}{\mu-1}+1\right)Dh\right). \end{aligned} \tag{67}$$

Using **(C)**(ii) and the fact that $p'(p-1)=p$, we have the following estimates:

$$\begin{aligned} & |Dh| \le p\overline{C}|x|^{p-1}, \quad |D^2h| \le p(p-1)\overline{C}|x|^{p-2}, \\ & C_b(1+|x|)|Dh| \le pC_b h, \\ & |\mathrm{Trace}(\sigma\sigma^T D^2h)| \le p(p-1)NC_\sigma^2 h, \\ & 0 \le \mathrm{Trace}(\sigma\sigma^T Dh\otimes Dh) \le C_\sigma^2(1+|x|^2)p^2\overline{C}^2|x|^{2(p-1)} \le p^2C_\sigma^2h^2. \end{aligned} \tag{68}$$

Now, the assumption **(B)**(i) on the growth of f plays a crucial role:

$$\begin{aligned} & f\left(x,t,0,\mathrm{e}^{Lt}s\left(\frac{\varphi_r}{\mu-1}+1\right)Dh\right) \\ \le\;& C_f\left(1+|x|^p+\left|\mathrm{e}^{Lt}s\left(\frac{\varphi_r}{\mu-1}+1\right)Dh\right|^{p'}\right) \\ \le\;& C_f\left(\frac{1}{\overline{C}} + p^{p'}C_s^{p'}\overline{C}^{p'-1}\mathrm{e}^{Lp't}\left(\frac{\mathrm{e}^T}{1-\mu}+1\right)^{p'}\right)h(x) \end{aligned}$$

since $\varphi_r \le \mathrm{e}^T$ (Lemma 3.3) and $|Dh|^{p'} \le p^{p'}\overline{C}^{p'-1}h(x)$ (because $p'(p-1)=p$). It follows from (67),

$$\begin{aligned} & \mathcal{L}(\Phi(x,t)) \\ \ge\;& C\varphi_t - (p(p-1)NC_\sigma^2 + pC_b)\varphi_r - p^2C_\sigma^2h^2\varphi_{rr} \\ & +(1-\mu)\left(\frac{L}{4} - \frac{C_f\mathrm{e}^{-Lt}}{\overline{C}} - p^{p'}C_s^{p'}\overline{C}^{p'-1}\mathrm{e}^{Lp't}\left(\frac{\mathrm{e}^T}{1-\mu}+1\right)^{p'}\right)h(x). \end{aligned}$$

We take

$$\begin{aligned} C &> \max\left\{p(p-1)NC_\sigma^2+pC_b,\, p^2C_\sigma^2\right\}, \\ L &> \frac{4C_f}{\overline{C}} + 4p^{p'}C_s^{p'}\overline{C}^{p'-1}\mathrm{e}^{p'}\left(\frac{\mathrm{e}^T}{1-\mu}+1\right)^{p'} + 1 \quad \text{and (45) holds}, \\ \tau &= \frac{1}{L}. \end{aligned}$$

For this choice of parameters, for all $(x,t)\in I\!R^N\times(0,\tau]$, we have

$$\mathcal{L}(\Phi(x,t)) \ge C\left(\varphi_t(h,Ct) - h\varphi_r(h,Ct) - h^2\varphi_{rr}(h,Ct)\right) + (1-\mu)h > 0$$

since φ is a solution of (48) and $h>0$. This proves the lemma. □

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