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1. Introduction

We study the relation between the exponential growth rate of volume in a pinched negatively curved manifold and the critical exponent of its lattices. These objects have a long and interesting story and are closely related to the geometry and the dynamical properties of the geodesic flow of the manifold (see e.g. [**4**], [**9**],[**20**] and references therein).

Throughout this paper, X will denote a complete and simply connected Riemannian manifold of dimension $N \geq 2$ and we will assume that X has *pinched negative curvature*, that is its sectional curvature K_X is bounded between two negative constants $-b^2 \leq -a^2 < 0$. A *Kleinian group* of X is a torsion free and discrete subgroup Γ of $Is(X)$; then, Γ operates freely and properly discontinuously on X and the quotient manifold $M := X/\Gamma$ has a fundamental group which can be identified with Γ. The group Γ is called a *lattice* when the volume of M is finite ; the lattice is said to be *uniform* if M is compact.

Recall that the exponential growth rate of X, also known as the *volume entropy* of X, is defined as

$$\omega(X) = \limsup_{R \to +\infty} \frac{1}{R} \ln v_X(\mathbf{x}, R)$$

where $v_X(\mathbf{x}, R)$ is the volume of the open ball $B_X(\mathbf{x}, R)$ of X, centered at the point $\mathbf{x}$ and with radius R. By the triangular inequality, this quantity does not depend on the base point $\mathbf{x}$; furthermore, under our pinching assumption, Bishop-Gunther's comparison theorem (see [**14**]) implies

$$(N-1)a \leq \omega(X) \leq (N-1)b. \tag{1}$$

The invariant $\omega(X)$ has been intensively studied when $Is(X)$ admits a *uniform* lattice Γ. It turns out that, in this case, $\omega(X)$ is a true limit and equals the topological entropy of the geodesic flow of the compact manifold M

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(see [**17**]). Furthermore, with a suitable normalization on the volume of M, it is a complete invariant of locally symmetric metrics on M (see [**4**]).

The second object of our interest in this paper is the *Poincaré series* $P_\Gamma(s,\mathbf{x})$ of a Kleinian group Γ, defined by

$$P_\Gamma(s,\mathbf{x}) = \sum_{\gamma\in\Gamma} e^{-sd(\mathbf{x},\gamma\mathbf{x})},$$

for $\mathbf{x}\in X$ and $s\in\mathbb{R}$. Its abscissa of convergence, called the *critical exponent of* Γ, is equal to

$$\delta(\Gamma) = \limsup_{R\to\infty} \frac{1}{R}\ln v_\Gamma(\mathbf{x},R),$$

where $v_\Gamma(\mathbf{x},R)$ is the cardinality of the "ball" $B_\Gamma(\mathbf{x},R) := \{\gamma\in\Gamma/d(\mathbf{x},\gamma\mathbf{x})\leq R\}$; again, by the triangular inequality, $\delta(\Gamma)$ does not depend on $\mathbf{x}$.

A way to understand the dynamic significance of the volume entropy $\omega(X)$ and its relation with $\delta(\Gamma)$ is to consider the Laplace transform of the Γ-invariant volume form dv_X on X, namely

$$I_X(s) = \int_0^{+\infty} e^{-sr} v_X(\mathbf{x},r)\mathrm{d}r.$$

The abscissa of convergence of $I_X(s)$ coincides with $\omega(X)$.

By a Fubini type argument, we also have $I_X(s) = \dfrac{1}{s}\displaystyle\int_X e^{-sd(\mathbf{x},y)}dv_X(y)$. If D is a Borel fundamental domain for the action of Γ on X, we get, by invariance of dv_X :

$$sI_X(s) = \sum_{\gamma\in\Gamma}\int_{\gamma D} e^{-sd(\mathbf{x},y)}dv_X(y) = \sum_{\gamma\in\Gamma}\int_D e^{-sd(\gamma^{-1}\mathbf{x},y)}dv_X(y)$$

which, in turns, yields :

$$P_\Gamma(s,\mathbf{x})\int_D e^{-sd(\mathbf{x},y)}dv_X(y) \leq sI_X(s) \leq P_\Gamma(s,\mathbf{x})\int_D e^{sd(\mathbf{x},y)}dv_X(y) \tag{2}$$

From the left-hand side of (2) it immediately follows that we always have

$$\delta(\Gamma)\leq\omega(X). \tag{3}$$

Moreover, from the right-hand side of (2), we have $\delta(\Gamma)=\omega(X)$ when Γ is a uniform lattice.

In this paper we shall investigate the case where X *admits a non-uniform lattice* Γ. Let us emphasize that, under this assumption, if X also admits a uniform lattice Γ_0 then X is a symmetric space of non compact type (and rank 1). Actually, as the curvature does not vanish, the manifold X is not a Riemannian product ; then (by [**11**], Corollary 9.2.2), either X is symmetric or the isometry group of X is discrete. But, in this last case, Γ_0 would have finite index in $Is(X)$ (see [**11**] 1.9.34) and, if φ is a parabolic isometry of X, then φ^n would belong to Γ_0 for some $n\geq 1$, which contradicts the fact that a uniform lattice contains only axial elements.

Somewhat surprisingly, the equality $\delta(\Gamma)=\omega(X)$ may fail for a non uniform lattice Γ ; actually, in the last section of this paper, we shall prove

THEOREM 1.1. *There exists a complete and simply connected Riemannian surface X with pinched negative curvature which admits a non uniform lattice Γ such that*

$$\delta(\Gamma) < \omega(X).$$

Our construction extends to any dimension. To explain it, recall that to each cuspidal end of the quotient manifold X/Γ corresponds a maximal parabolic subgroup $\mathcal{P} \subset \Gamma$, which has a *lower critical exponent* :

$$\delta^-(\mathcal{P}) = \liminf_{R\to\infty} \frac{1}{R} \ln v_{\mathcal{P}}(\mathbf{x}, R).$$

In strictly negative curvature, this exponent is nonzero, despite the fact that $\mathcal{P}$ is virtually nilpotent (see [**6**]). The key point is that, in the variable curvature setting, $\delta^-(\mathcal{P})$ may be distinct from $\delta(\mathcal{P})$, as was suggested a long time ago to the second author by B. Bowditch ; in contrast, it is well known that the critical exponent of any non elementary Kleinian group always is a true limit [**19**]. We shall show in Section 5 that the inequality $\omega(X) > \delta(\Gamma)$ may appear as soon as $\delta^-(\mathcal{P}) < \delta(\mathcal{P})/2$.

On the other hand, our example induces us to introduce a notion of pinching for non uniform lattices which ensures that $\omega(X) = \delta(\Gamma)$. Namely, we say that Γ is *parabolically* $1/2$*-pinched* if for any maximal parabolic subgroup $\mathcal{P} \subset \Gamma$, we have

$$\frac{\delta(\mathcal{P})}{\delta^-(\mathcal{P})} \leq 2 \tag{4}$$

We will prove

THEOREM 1.2. *Let X be a complete, simply connected Riemannian manifold with pinched negative curvature. Then for any lattice $\Gamma \subset Is(X)$ which is parabolically $1/2$-pinched, we have $\delta(\Gamma) = \omega(X)$.*

Moreover, we notice that, under the assumptions of this theorem, the invariant $\omega(X)$ is a true limit ; this follows from Corollary 4.5, combined with the fact that $\delta(\Gamma)$ is a limit.

We shall see that Theorem 1.2 covers the case of lattices in any 1/4-pinched negatively curved manifold (i.e. $\frac{b^2}{a^2} \leq 4$). As far as we know, even in the classical case of Riemannian negatively curved symmetric spaces of rank one (which are 1/4-pinched, cp. [**15**]), there does not exist an elementary proof of this result. Nevertheless, for those spaces, the equality $\omega(X) = \delta(\Gamma)$ can be easily deduced from a general and deep result of A. Eskin and C. McMullen in [**13**] on lattices of affine symmetric spaces, obtained by algebraic methods. In contrast, the context of variable negative curvature forces us to use only elementary geometric arguments.

The equality $\omega(X) = \delta(\Gamma)$ actually holds under a milder geometric assumption than 1/4-pinched curvature. Namely, we will say that a manifold $M = X/\Gamma$ has *asymptotically* $1/4$*-pinched curvature* when, for any $\epsilon > 0$, there exists a compact set $C_\epsilon \subset M$, such that the metric is $(\frac{1}{4+\epsilon})$-pinched on $M \setminus C_\epsilon$. A direct consequence of Theorem 1.2 is

Corollary 1.3. *Let X be a complete, simply connected Riemannian manifold with pinched negative curvature and let Γ be a lattice of X. If $M := X/\Gamma$ has asymptotically $1/4$-pinched curvature, then $\delta(\Gamma) = \omega(X)$.*

We remark that the pinching constant $\frac{1}{4}$ is optimal because, for every $\epsilon > 0$, the example we construct in Theorem 1.1 can be chosen so that the curvature is $\frac{1}{4+\epsilon}$-pinched.

The paper is organized as follows. Section 2 deals with elementary geometrical estimates inside horoballs. In Section 3, we relate the volume growth of balls inside a horoball $\mathcal{H}$ with the critical exponent of ample parabolic subgroups preserving $\mathcal{H}$. In section 4, we first give an elementary proof of the equality $\omega(X) = \delta(\Gamma)$ for $\frac{1}{4}$-pinched manifolds; this is of interest since the main idea about the behavior of a ball intersecting a horoball appears clearly in the proof. The proofs of Theorem 1.2 and Corollary 1.3 will follow. Section 5 is devoted to the construction of the example of Theorem 1.1; this relies on pretty technical results about convex functions, postponed to the Appendix.

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We fix here once and for all some notation about asymptotic behavior of functions :

Notations. We shall write $f \overset{c}{\preceq} g$ *(or simply $f \preceq g$) when $f(R) \leq cg(R)$ for some constant $c > 0$ and R large enough. The notation $f \overset{c}{\asymp} g$ (or simply $f \asymp g$) means $f \overset{c}{\preceq} g \overset{c}{\preceq} f$.*

Analogously, we whall write $f \overset{c}{\sim} g$ (or simply $f \sim g$) when $|f(R) - g(R)| \leq c$ for some constant $c > 0$ and R large enough.

The upper and lower exponential growth rates of a function f are denoted by $\omega^+(f)$ (or simpler $\omega(f)$) and $\omega^-(f)$ respectively; namely we have

$$\omega^-(f) := \liminf_{R\to+\infty} \frac{\ln f(R)}{R} \quad \textit{and} \quad \omega^+(f) = \omega(f) := \limsup_{R\to+\infty} \frac{\ln f(R)}{R}.$$

Finally, if f and g are two real functions, we denote by $f * g$ the discrete convolution of f with g, defined by $f * g(R) = \sum_{n=0}^{[R]} f(n)g(R-n)$ for any $R \geq 0$.

2. Radial flow and geometry of horoballs

As the curvature is bounded from above by $-a^2 < 0$, we have the following classical inequality :

Lemma 2.1. *Let T be a geodesic triangle with different vertices $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ and angle at $\mathbf{y}$ greater than $\alpha > 0$. Then there is a constant $D = D(\alpha, a)$ such that*

$$d(\mathbf{x}, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) - D.$$

Proof. See [**8**].

□

Let $X(\infty)$ be the boundary at infinity of X. Fix a point ξ in $X(\infty)$ and consider its associated *radial semi-flow*, $(\psi_{\xi,t})_{t\geq 0}$ defined as follows : for any $\mathbf{x}\in X$, the point $\psi_{\xi,t}(\mathbf{x})$ lies on the geodesic ray $[\mathbf{x},\xi)$ at distance t from $\mathbf{x}$. For any horosphere $\partial\mathcal{H}$ centered at ξ, we set $\partial\mathcal{H}(t)=\psi_{\xi,t}(\partial\mathcal{H})$, and we let d_t be the distance induced by d on the horosphere $\partial\mathcal{H}(t)$. For any points $x,y\in\partial\mathcal{H}(t)$, we have (see [**16**])

$$\frac{2}{a}\sinh\Big(\frac{a}{2}d(x,y)\Big)\leq d_t(x,y)\leq\frac{2}{b}\sinh\Big(\frac{b}{2}d(x,y)\Big). \tag{5}$$

By [**16**], the differential of the map $\psi_{\xi,t}:\partial\mathcal{H}\to\partial\mathcal{H}(t)$ satisfies, for any vector $v\in T(\partial\mathcal{H})$ and any $t\geq 0$

$$e^{-bt}||v||\leq||d\psi_{\xi,t}(v)||\leq e^{-at}||v||. \tag{6}$$

This readily implies the estimates

$$e^{-b(N-1)t}\leq|Jac(\psi_{\xi,t})|\leq e^{-a(N-1)t}. \tag{7}$$

In particular, if μ_t is the Riemannian measure induced on $\partial\mathcal{H}(t)$ by the metric on X, we have, for any Borel set $A\subset\partial\mathcal{H}$

$$e^{-b(N-1)t}\mu_0(A)\leq\mu_t(\psi_{\xi,t}(A))=\int_A|Jac(\psi_{\xi,t})|(x)d\mu_0(x)\leq e^{-a(N-1)t}\mu_0(A). \tag{8}$$

If the points $\mathbf{x},\mathbf{y}$ belong to the horosphere $\partial\mathcal{H}$, we set

$$t_{\mathbf{x},\mathbf{y}}=\inf\{t\geq 0/d_t(\psi_{\xi,t}(\mathbf{x}),\psi_{\xi,t}(\mathbf{y}))\leq 1\}.$$

The next lemma, which precises Lemma 4 in [**9**], will be of major importance in the following.

Lemma 2.2. *There exists a constant $c=c(a,b)>0$, only depending on the bounds on the curvature, such that, for any horosphere $\partial\mathcal{H}$ and any $\mathbf{x},\mathbf{y}\in\partial\mathcal{H}$, the arc $\gamma_{\mathbf{x},\mathbf{y}}$ which is the ordered union of the three geodesic segments $[\mathbf{x},\psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{x})]$, $[\psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{x}),\psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{y})]$ and $[\psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{y}),\mathbf{y}]$ is a $(1,c)$-quasigeodesic. Furthermore, for any $s,t\geq 0$, we have*

$$d(\psi_{\xi,s}(\mathbf{x}),\psi_{\xi,t}(\mathbf{x}))\overset{c}{\sim}\varphi(s,t)$$

where φ is the function defined on $\mathbb{R}_+\times\mathbb{R}_+$ by

$$\varphi(s,t)=\begin{cases}2t_{\mathbf{x},\mathbf{y}}-s-t & \text{when}\quad s,t\leq t_{\mathbf{x},\mathbf{y}}\\ |s-t| & \text{otherwise.}\end{cases}$$

In particular, we have $d(\mathbf{x},\mathbf{y})\overset{c}{\sim}2t_{\mathbf{x},\mathbf{y}}$.

Proof. If $d_0(\mathbf{x},\mathbf{y})\leq 1$, the arc $\gamma_{\mathbf{x},\mathbf{y}}$ is the geodesic segment $[\mathbf{x},\mathbf{y}]$ and the lemma is obvious in this case. We now assume $d_0(\mathbf{x},\mathbf{y})>1$. Let $x=\psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{x})$ and $y=\psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{y})$. From the right hand side of (5), the distance $d(x,y)$ is bounded from below by $b':=\frac{2}{b}\sinh^{-1}\frac{b}{2}$.

Let us now fix a point ξ' on the boundary at infinity of the space $\mathbb{H}^N_a$ of constant curvature $-a^2$, and two points x',y' on the same horosphere centered at ξ', and at distance b' each from the other on this space; comparing the triangles $x\ y\ \xi$ and $x'\ y'\ \xi'$ we deduce that $\widehat{x\ y\ \xi}\leq\widehat{x'\ y'\ \xi'}\leq\frac{\pi}{2}-\theta$, for some constant $\theta>0$ depending only on a and b. Since $\widehat{\mathbf{x}\ x\ y}\geq\pi/2$, we have $\widehat{x\ y\ \mathbf{x}}\leq\pi/2$ and so $\widehat{\mathbf{x}\ y\ \mathbf{y}}\geq\theta$. Applying Lemma 2.1 successively to the triangles $\mathbf{x}\ x\ y$

(with $\alpha \geq \pi/2$) and $\mathbf{x}\ y\ \mathbf{y}$ (with $\alpha \geq \theta$) we obtain $d(\mathbf{x}, \mathbf{y}) \sim d(\mathbf{x}, x) + d(y, \mathbf{y})$. The second point follows from the first one, computing the distance between $\psi_{\xi,s}(\mathbf{x})$ and $\psi_{\xi,t}(\mathbf{y})$ along $\gamma_{\mathbf{x},\mathbf{y}}$. □

Applying this lemma, we obtain the

PROPOSITION 2.3. *There exists a constant $c = c(a, b) > 0$ such that for any point ξ in $X(\infty)$, any horoball $\mathcal{H}$ centered at ξ and any $\mathbf{x} \in \partial\mathcal{H}$ and $R > 0$ we have*

$$B_X(\psi_{\xi,R/2}(\mathbf{x}), R/2) \subset B_X(\mathbf{x}, R) \cap \mathcal{H} \subset B_X(\psi_{\xi,R/2}(\mathbf{x}), R/2 + c).$$

Proof. We need only to prove the second inclusion, the first one being obvious. For $\mathbf{z} \in B_X(\mathbf{x}, R) \cap \mathcal{H}$, denote by $\mathbf{y}$ the projection of $\mathbf{z}$ on $\partial\mathcal{H}$ and by $\mathbf{z}_0$ the intersection of the horosphere centered at ξ and containing $\mathbf{z}$ with the geodesic ray $[\mathbf{x}, \xi)$.

Assume first $t_{\mathbf{x},\mathbf{y}} \leq \max\{R/2, d(\mathbf{y}, \mathbf{z})\}$; setting $s = R/2$ and $t = d(\mathbf{y}, \mathbf{z})$ in the previous lemma, we get $d(\psi_{\xi,R/2}(\mathbf{x}), \mathbf{z}) \sim |s - t| = d(\psi_{\xi,R/2}(\mathbf{x}), \mathbf{z}_0) \leq R/2$ (the last inequality following from the fact that $d(\mathbf{x}, \mathbf{z}_0) \leq d(\mathbf{x}, \mathbf{z}) \leq R$).

Assume now $t_{\mathbf{x},\mathbf{y}} \geq \max\{R/2, d(\mathbf{y}, \mathbf{z})\}$; applying twice the previous lemma, we get in this case

$$\begin{cases} d(\mathbf{x}, \mathbf{z}) \sim 2t_{\mathbf{x},\mathbf{y}} - d(\mathbf{z}, \mathbf{y}) & \text{(setting } s = 0 \text{ and } t = d(\mathbf{y}, \mathbf{z})) \\ d(\psi_{\xi,R/2}(\mathbf{x}), \mathbf{z}) \sim 2t_{\mathbf{x},\mathbf{y}} - d(\mathbf{z}, \mathbf{y}) - R/2 & \text{(setting } s = R/2 \text{ and } t = d(\mathbf{y}, \mathbf{z})). \end{cases}$$

Since $\mathbf{z} \in B_X(\mathbf{x}, R)$ there, thus exists $c > 0$ such that $d(\psi_{\xi,R/2}(\mathbf{x}), \mathbf{z}) \leq R/2 + c$. □

In the next section, we will consider discrete parabolic subgroups of $Is(X)$; any such group fixes one point $\xi \in X(\infty)$ and preserves any horoball $\mathcal{H}$ centered at ξ. We shall investigate the relation between the critical exponent of $\mathcal{P}$ and the volume growth of X. Here we shall limit ourselves to remark :

COROLLARY 2.4. *If X is homogeneous, then for any discrete parabolic subgroup $\mathcal{P}$ of $Is(X)$, we have*

$$\delta(\mathcal{P}) \leq \omega(X)/2.$$

This fact is well known when X is a rank one symmetric space ; Proposition 2.3 allows to understand the geometrical reason of this inequality. Actually, let $\mathcal{H}$ be an horoball preserved by $\mathcal{P}$ and let $\mathbf{x} \in \partial\mathcal{H}$. As $\mathcal{P}$ is discrete, we have $d := \frac{1}{2} \inf_{p \in \mathcal{P}} d(\mathbf{x}, p\mathbf{x}) > 0$, then

$$\bigsqcup_{p / d(\mathbf{x}, p\mathbf{x}) \leq R} B_X(p\mathbf{x}, d) \times [0, 1] \ \subset \ B_X(\mathbf{x}, R + d + 1) \cap \mathcal{H}.$$

By Proposition 2.3, we deduce $v_{\mathcal{P}}(\mathbf{x}, R) \preceq \sup_{\mathbf{y} \in \mathcal{H}} v_X\Big(\mathbf{y}, \dfrac{R + d + 1}{2} + c\Big)$. As X is homogeneous, for any $\epsilon > 0$, we have $v_X(\mathbf{y}, r) \preceq e^{(\omega(X) + \epsilon)r}$ uniformly in $\mathbf{y}$. The Corollary follows. □

3. Growth of ample parabolic subgroups

Let be $\mathcal{P}$ a parabolic subgroup of $Is(X)$ fixing $\xi \in X(\infty)$. We shall say that $\mathcal{P}$ is **ample** if it acts cocompactly on every horoball $\partial\mathcal{H}$ centered at ξ. This holds in particular when $\mathcal{P}$ is a maximal parabolic subgroup of a non uniform lattice of $Is(X)$.

We then fix a (relatively compact) Borel fundamental domain $\mathcal{C} \subset \partial\mathcal{H}$ for the action of $\mathcal{P}$ on $\partial\mathcal{H}$. For any $t \geq 0$, the set $\mathcal{C}_t := \psi_{\xi,t}(\mathcal{C})$ is a fundamental domain for the action of $\mathcal{P}$ on $\partial\mathcal{H}(t)$; in the same way, the set $\mathcal{E} := \cup_{t\geq 0}\mathcal{C}_t$, which is canonically homeomorphic to $\mathcal{C} \times \mathbb{R}^+$, is a fundamental domain for the action of $\mathcal{P}$ on the horoball $\mathcal{H}$.

We now associate to any ample parabolic group $\mathcal{P}$ a function $\mathcal{A}_\mathcal{P}$ which will play a crucial role in this paper :

Definition 3.1. The **horospherical area** of $\mathcal{P}$ is the function $\mathcal{A}_\mathcal{P}(\mathbf{x}, t)$ defined by

$$\forall\, \mathbf{x} \in \partial\mathcal{H}, \forall\, t \geq 0 \qquad \mathcal{A}_\mathcal{P}(\mathbf{x}, t) := \mu_t(\psi_{\xi,t}(\mathcal{C})).$$

The function $t \mapsto \mathcal{A}_\mathcal{P}(\mathbf{x}, t)$ is decreasing and does not depend on the choice of the fundamental domain $\mathcal{C}$; furthermore, by inequalities (8), for any R and $R_0 > 0$, we have

$$e^{-(N-1)bR_0} \leq \frac{\mathcal{A}_\mathcal{P}(\mathbf{x}, R+R_0)}{\mathcal{A}_\mathcal{P}(\mathbf{x}, R)} \leq e^{-(N-1)aR_0}. \tag{9}$$

The following proposition stresses the relation between the function $\mathcal{A}_\mathcal{P}$ and the orbital counting function $v_\mathcal{P}(\mathbf{x}, R)$ of $\mathcal{P}$.

Proposition 3.2. *There exists a constant $c = c(a, b, diam(\mathcal{C})) > 0$ such that for any $\mathbf{x} \in X$*

$$v_\mathcal{P}(\mathbf{x}, R) \stackrel{c}{\asymp} \frac{1}{\mathcal{A}_\mathcal{P}(\mathbf{x}, \frac{R}{2})}.$$

In particular, we have

$$\delta(\mathcal{P}) = \omega\Big(\frac{1}{\mathcal{A}_\mathcal{P}(\mathbf{x}, \frac{R}{2})}\Big) \quad \text{and} \quad \delta^-(\mathcal{P}) = \omega^-\Big(\frac{1}{\mathcal{A}_\mathcal{P}(\mathbf{x}, \frac{R}{2})}\Big). \tag{10}$$

Proof. We recall that d_t denotes the horospherical distance on the horosphere $\partial\mathcal{H}(t)$. We let c be the constant of Lemma 2.2 such that $d(\mathbf{x}, \mathbf{y}) \stackrel{c}{\sim} 2t_{\mathbf{x},\mathbf{y}}$ for $\mathbf{x}, \mathbf{y}$ on $\partial\mathcal{H}$. If $d(\mathbf{x}, \mathbf{y}) = R$, as $t_{\mathbf{x},\mathbf{y}} \stackrel{c/2}{\sim} \frac{R}{2}$, we deduce

$$d_{\frac{R+c}{2}}\Big(\mathbf{x}(\frac{R+c}{2}), \mathbf{y}(\frac{R+c}{2})\Big) \leq 1 \quad \text{and} \quad d_{\frac{R-c}{2}}\Big(\mathbf{x}(\frac{R-c}{2}), \mathbf{y}(\frac{R-c}{2})\Big) \geq 1.$$

This implies that $\psi_{\frac{R+c}{2}}(B_X(\mathbf{x}, R) \cap \partial\mathcal{H}) \subset B_1$ and $\psi_{\frac{R-c}{2}}(B_X(\mathbf{x}, R) \cap \partial\mathcal{H}) \subset B_2$ with

$$B_1 := B_{\partial\mathcal{H}(\frac{R+c}{2})}\Big(\mathbf{x}(\frac{R+c}{2}), 1\Big) \quad \text{and} \quad B_2 := B_{\partial\mathcal{H}(\frac{R-c}{2})}\Big(\mathbf{x}(\frac{R-c}{2}), 1\Big).$$

Gauss equation implies that the sectional curvature of all horospheres for the induced metric is in between $a^2 - b^2$ and $2b(b-a)$ (see ([**7**], section 1.4, example (*iii*)). Therefore, there exist positive constants $v^- = v^-(a, b, \mathbf{x})$ and $v^+ = v^+(a, b, \mathbf{x})$ such that $v^- \leq vol(B_i) \leq v^+$ for the induced volume form on the horospheres and $i = 1, 2$.

Now, there are at most $v_{\mathcal{P}}(\mathbf{x}, R)$ distinct fundamental domains $p(\mathcal{C})$ included in $B_X(\mathbf{x}, R) \cap \partial\mathcal{H}$ and since the radial semi-flow $(\psi_{\xi,t})_{t\geq 0}$ is equivariant with respect to the action of $\mathcal{P}$ on the horospheres $\partial\mathcal{H}(t)$, there are also at most $v_{\mathcal{P}}(\mathbf{x}, R)$ distinct fundamental domains $p(\mathcal{C}(\frac{R+c}{2}))$ included in $\psi_{\frac{R+c}{2}}(B_X(\mathbf{x}, R) \cap \partial\mathcal{H})$. Therefore, we have $v_{\mathcal{P}}(\mathbf{x}, R) \leq \dfrac{v^+}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R+c}{2})}$ and by (9), this leads to

$$v_{\mathcal{P}}(\mathbf{x}, R) \preceq \frac{1}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R}{2})}.$$

On the other hand, we can cover the set $B_X(\mathbf{x}, R) \cap \partial\mathcal{H}$ with $v_{\mathcal{P}}(\mathbf{x}, R + d)$ distinct fundamental domains $p(\mathcal{C})$; by the equivariance of $(\psi_{\xi,t})_t$ we deduce again that $\psi_{\frac{R-c}{2}}(B_X(\mathbf{x}, R) \cap \mathcal{H})$ can be covered by $v_{\mathcal{P}}(\mathbf{x}, R + d))$ fundamental domains as well. Therefore, using (9) again

$$v_{\mathcal{P}}(\mathbf{x}, R) \geq \frac{v^-}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R-c-d}{2})} \succeq \frac{1}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R}{2})}.$$

□

We now estimate the volume of a ball of radius R, inside the horoball $\mathcal{H}$. We have

PROPOSITION 3.3. *There exists a constant $c = c(a, b, diam(\mathcal{C})) > 0$ such that*

$$vol(B_X(\mathbf{x}, R) \cap \mathcal{H}) \overset{c}{\asymp} \int_0^R \frac{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, t)}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{t+R}{2})} dt.$$

To get this result, we need the following refinement of Proposition 2.3.

LEMMA 3.4. *There exists a constant $\Delta = \Delta(a, b, diam(\mathcal{C}))$ such that*

$$p(\mathcal{C}) \times \Big[(2t_p - R + \Delta)^+, (R - \Delta)^+\Big[\subset \Big(p(\mathcal{E}) \cap B_X(\mathbf{x}, R)\Big) \subset p(\mathcal{C}) \times \Big[(2t_p - R - \Delta)^+, R\Big].$$

Proof. Let $\Delta = c + diam(\mathcal{C})$, where c is the constant of Lemma 2.2. We first prove the right hand side inclusion. Let $\mathbf{z} = (\mathbf{z}_0, t) \in p(\mathcal{C}) \times \mathbb{R}^+$ and assume that this point belongs to $B_X(\mathbf{x}, R)$. Clearly $t \leq R$ as $t = B_\xi(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{z}) \leq R$. If $t_p \leq \frac{R+\Delta}{2}$ there is nothing left to prove; on the other hand, if $t_p > \frac{R+\Delta}{2}$, then $2t_p - t \overset{c}{\sim} d(\mathbf{x}, \mathbf{z}) < R$ hence $t \in [(2t_p - R + \Delta)^+, R]$. Let us now consider the case where $\mathbf{z} \in p(\mathcal{C}) \times [(2t_p - R + \Delta)^+, (R - \Delta)^+]$. We may assume that $R \geq \Delta$ and $t_p \leq R - \Delta$, otherwise there is nothing to prove. If $t \geq t_p$ we have $d(\mathbf{x}, \mathbf{z}) \overset{c}{\sim} t \leq R - \Delta$, otherwise we have $d(\mathbf{x}, \mathbf{z}) \overset{c}{\sim} 2t_p - t \leq 2t_p - (2t_p - R + \Delta)^+$; therefore, in both cases $\mathbf{z} \in B_X(\mathbf{x}, R)$. □

Proof of Proposition 3.3 . We simply write $\mathcal{A}(R) = \mathcal{A}_{\mathcal{P}}(\mathbf{x}, R)$. Recall that

$$B_X(\mathbf{x}, R) \cap \mathcal{H} = \bigsqcup_{p \in \mathcal{P}} B_X(\mathbf{x}, R) \cap p(\mathcal{E}).$$

By Lemma 3.4, we have $B_X(\mathbf{x},R)\cap p(\mathcal{E})\subset p(\mathcal{C})\times[(2t_p-R-\Delta)^+,R]$. Then, we find

$$\begin{aligned}\sum_{p\in\mathcal{P}} vol\Big(B_X(\mathbf{x},R)\cap p(\mathcal{E})\Big) &= \sum_{t_p\le R+\frac{\Delta}{2}}\int_{(2t_p-R-\Delta)^+}^{R}\mathcal{A}(t)dt\\ &= \sum_{t_p\le R+\frac{\Delta}{2}}\int_0^R \mathcal{A}(t)1_{[(2t_p-R-\Delta)^+,+\infty[}(t)dt\end{aligned}$$

Now, as $d(\mathbf{x},p\mathbf{x})\overset{c}{\sim}2t_p\le c\le\Delta$, for every fixed $t\in[0,R]$ we have

$$\begin{aligned}\sharp\Big\{p\in\mathcal{P}/t_p\le R+\frac{\Delta}{2}\text{ and }2t_p-R-\Delta\le t\Big\} &\le v_{\mathcal{P}}\Big(\mathbf{x},\frac{t+R+\Delta}{2}+\Delta\Big)\\ &\le \frac{v^+}{\mathcal{A}(\frac{t+R+3\Delta}{2})}\\ &\preceq \frac{1}{\mathcal{A}(\frac{R+t}{2})},\end{aligned}$$

where we have successively used Proposition 3.2 and (9). This yields

$$vol(B_X(\mathbf{x},R)\cap\mathcal{H})\preceq\int_0^R\frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})}dt.$$

We now prove the converse inequality. Again, by Proposition 3.4, we deduce

$$B_X(\mathbf{x},R)\cap p(\mathcal{E})\supset p(\mathcal{C})\times[(2t_p-R+\Delta)^+,R-\Delta].$$

We only consider those p's such that $\frac{R-\Delta}{2}\le t_p\le R-\Delta$; summing over these p's, we find

$$\begin{aligned}\sum_{\frac{R-\Delta}{2}\le t_p\le R-\Delta} vol\Big(B_X(\mathbf{x},R)\cap p(\mathcal{E})\Big) &= \sum_{\frac{R-\Delta}{2}\le t_p\le R-\Delta}\int_{2t_p-R-\Delta}^{R-\Delta}\mathcal{A}(t)dt\\ &\ge \sum_{\frac{R-\Delta}{2}\le t_p\le R-\Delta}\int_{R_0}^{R-\Delta}\mathcal{A}(t)1_{[2t_p-R+\Delta,R-\Delta]}(t)dt\end{aligned}$$

for any $R_0\ge 0$. Now, for every fixed $t\in[R_0,R-\Delta]$, we have

$$\begin{aligned}\sharp\Big\{p\in\mathcal{P}/\frac{R-\Delta}{2}\le t_p\le R-\Delta\text{ and }2t_p-R+\Delta\le t\Big\} &\ge v_{\mathcal{P}}\Big(\mathbf{x},t+R-2\Delta\Big)-v_{\mathcal{P}}\Big(\mathbf{x},R\Big)\\ &\ge \frac{v^-}{\mathcal{A}(\frac{t+R-2\Delta}{2})}-\frac{v^+}{\mathcal{A}(\frac{R}{2})}\\ &\ge \frac{1}{\mathcal{A}(\frac{t+R}{2})}\Big(v^-\frac{\mathcal{A}(\frac{t+R}{2})}{\mathcal{A}(\frac{t+R-2\Delta}{2})}-v^+\frac{\mathcal{A}(\frac{t+R}{2})}{\mathcal{A}(\frac{R}{2})}\Big)\\ &\ge \frac{1}{\mathcal{A}(\frac{t+R}{2})}\Big(v^-e^{-b(N-1)\Delta}-v^+e^{-a(N-1)R_0/2}\Big)\end{aligned}$$

by Proposition 3.2 and (9). Therefore, if R_0 is large enough, we find

$$vol\Big(B_X(\mathbf{x},R)\cap\mathcal{H}\Big)\succeq\int_{R_0}^{R-\Delta}\frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})}dt.$$

We can replace this last integral by $\displaystyle\int_0^R \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})}dt$ since, $\displaystyle\int_{R-\Delta}^R \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})}dt$ is bounded in terms of a, b and Δ and for R large enough

$$\int_{R_0}^{R-\Delta} \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})}dt \geq \int_{R_0}^{2R_0} \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})}dt \asymp \frac{1}{\mathcal{A}(R/2)} \asymp \int_0^{R_0} \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})}dt.$$

□

As a direct consequence of Propositions 3.2 and 3.3, we obtain

Corollary 3.5. *For any $\epsilon > 0$ and $\mathbf{x} \in \partial\mathcal{H}$, we have*
i) if $\delta(\mathcal{P}) \geq 2\delta^-(\mathcal{P})$ then

$$e^{(\delta^-(\mathcal{P})-\epsilon)R} \preceq vol\Big(B_X(\mathbf{x}, R) \cap \mathcal{H}\Big) \preceq e^{2\left(\delta(\mathcal{P})-\delta^-(\mathcal{P})+\epsilon\right)R}$$

ii) if $\delta(\mathcal{P}) < 2\delta^-(\mathcal{P})$ then

$$e^{(\delta^-(\mathcal{P})-\epsilon)R} \preceq vol\Big(B_X(\mathbf{x}, R) \cap \mathcal{H}\Big) \preceq e^{2\left(\delta(\mathcal{P})+\epsilon\right)R}.$$

4. Growth of nonuniform lattices

We suppose now that the manifold X admits a nonuniform lattice Γ. Let us recall some well known geometrical properties of Γ proved in the general context of geometrically finite groups in ([**5**]). Since the volume of $M = X/\Gamma$ is finite, the *limit set* of Γ equals $X(\infty)$ and is the disjoint union of its *radial* subset and of finitely many orbits $\Gamma\xi_1, \ldots, \Gamma\xi_l$ of points, called *bounded parabolic fixed points*. By definition, a point ξ_i corresponds to a end of the manifold M and is fixed by a parabolic subgroup of Γ. Denote $\mathcal{P}_i$ the maximal parabolic subgroup fixing the point ξ_i. This group preserves any horoball $\mathcal{H}$ centered at ξ_i and acts cocompactly on the horosphere $\partial\mathcal{H}$. By Margulis' lemma (see [**20**]), there exist closed horoballs $\mathcal{H}_{\xi_1}, \ldots, \mathcal{H}_{\xi_l}$ centered respectively at $\xi_1, \ldots, \xi_l$, such that all the horoballs $\gamma.\mathcal{H}_{\xi_i}$, for $1 \leq i \leq l$ and $\gamma \in \Gamma$, are disjoint or coincide. We fix an origin $\mathbf{o} \in X$ and a convex Borel fundamental domain $\mathcal{D}$ in X for the action of Γ, containing the geodesic rays $[\mathbf{o}, \xi_1)$..., $[\mathbf{o}, \xi_l)$. For each $1 \leq i \leq l$, we set $\mathcal{E}_i = \mathcal{D} \cap \mathcal{H}_{\xi_i}$ and $\mathcal{C}_i = \mathcal{D} \cap \partial\mathcal{H}_{\xi_i}$. Those both sets are fundamental domains for the action of the group $\mathcal{P}_i$ respectively on $\mathcal{H}_{\xi_i}$ and $\partial\mathcal{H}_{\xi_i}$. Moreover, the set $\mathcal{C}_0 = \mathcal{D} \setminus \left(\cup_{i=1}^l \mathcal{E}_i\right)$, and hence each $\mathcal{C}_i$, is relatively compact. We may assume that $\mathbf{o}$ belongs to the interior of $\mathcal{C}_0$.

The quotient manifold M is therefore decomposed into the disjoint union of a relatively compact set C_0 and finitely many ends of finite volume $E_i = \mathcal{H}_{\xi_i}/\mathcal{P}_i$, which are the projections on M of the domains $\mathcal{C}_0$ and $\mathcal{E}_i$ respectively.

We first precise some bounds on the critical exponent $\delta(\Gamma)$ in terms of bounds on the curvature of X.

Lemma 4.1. *We have $(N-1)a \leq \delta(\Gamma) \leq (N-1)b$.*

In particular, when X is the real hyperbolic space $\mathbb{H}_a^N$ of constant curvature $-a^2$, we have $\delta(\Gamma) = (N-1)a$ and hence $\delta(\Gamma) = \omega(\mathbb{H}_a^N)$.

Proof. The inequality $\delta(\Gamma) \leq (N-1)b$ follows from (3) and (1). It remains to prove the left hand side inequality of the Lemma

If $\delta(\Gamma) = \omega(X)$, the inequality follows from(1). Assume now $\delta(\Gamma) < \omega(X)$ and consider $s \in]\delta(\Gamma), \omega(X)[$. Inequality (2) implies

$$\int_{\mathcal{D}} e^{sd(\mathbf{o},\mathbf{x})} dv_X(\mathbf{x}) = +\infty$$

which, by the decomposition $\mathcal{D} = \mathcal{C}_0 \cup \left(\bigcup_{i=1}^{l} \mathcal{E}_i \right)$, is equivalent to

$$\max_{i \in \{1,\cdots,l\}} \int_{\mathcal{E}_i} e^{sd(\mathbf{o},\mathbf{x})} dv_X(\mathbf{x}) = +\infty. \tag{11}$$

Note now that for $\mathbf{x} \in \mathcal{E}_i$, we have $B_{\xi_i}(\mathbf{o},\mathbf{x}) \leq d(\mathbf{o},\mathbf{x}) \leq B_{\xi_i}(\mathbf{o},\mathbf{x}) + diam(\mathcal{C}_i)$ where $B_{\xi_i}(.,.)$ denotes the Busemann function centered at ξ_i. Therefore the integrals $\int_{\mathcal{E}_i} e^{sd(\mathbf{o},\mathbf{x})} dv_X(\mathbf{x})$ and $\int_{\mathcal{E}_i} e^{sB_{\xi_i}(\mathbf{o},\mathbf{x})} dv_X(\mathbf{x})$ are of the same nature.

By (8), we have

$$\int_{\mathcal{E}_i} e^{sB_{\xi_i}(\mathbf{o},\mathbf{x})} dv_X(\mathbf{x}) = \int_{d(\mathbf{o},\mathcal{C}_i)}^{+\infty} e^{st} \mu_t(\psi_{\xi_i,t}(\mathcal{C}_i)) dt \leq \mu_0(\mathcal{C}_i) \int_0^{+\infty} e^{t[s-(N-1)a]} \mathrm{d}t$$

and the last integral must be divergent for all $s \in]\delta(\Gamma), \omega(X)[$, so $\delta(\Gamma) \geq (N-1)a$. □

Recall that $v_X(\mathbf{o}, R)$ denotes the volume of the open ball $B_X(\mathbf{o}, R)$ and that $v_\Gamma(\mathbf{o}, R)$ represents the cardinality of the intersection of this ball with $\Gamma(\mathbf{o})$. The following estimate will be used to obtain a upper bound for $\delta(\Gamma)$.

Proposition 4.2. *There exists a constant $\Delta = \Delta(a, b, diam(\mathcal{C}_0)) > 0$ such that, for all $R > 0$, we have*

$$v_X(\mathbf{o}, R-\Delta) \preceq v_\Gamma(\mathbf{o}, R) + \sum_{i=1}^{l} \sum_{n=0}^{[R]} v_\Gamma(\mathbf{o}, n+1) \times vol\Big(B_X(\mathbf{x}_i, R-n+\Delta) \cap \mathcal{H}_{\xi_i} \Big) \tag{12}$$

where $\mathbf{x}_i$ denotes the intersection of the geodesic ray $[\mathbf{o}, \xi_\mathbf{i})$ with the horosphere $\partial\mathcal{H}_{\xi_i}$.

Proof. Set $d_0 = diam(\mathcal{C}_0)$.We have

$$B_X(\mathbf{o}, R) = \Big(B_X(\mathbf{o}, R) \cap \Gamma.\mathcal{C}_0 \Big) \bigcup \left(\bigcup_{1 \leq i \leq l} \Big(B_X(\mathbf{o}, R) \cap \Gamma.\mathcal{H}_{\xi_i} \Big) \right) \tag{13}$$

whence

$$B_X(\mathbf{o}, R) \cap \Gamma.\mathcal{C}_0 \quad \subset \bigcup_{\gamma \in B_\Gamma(\mathbf{o}, R+d_0)} \gamma(\mathcal{C}_0)$$

and

$$vol\Big(B_X(\mathbf{o}, R) \cap \Gamma.\mathcal{C}_0 \Big) \preceq v_\Gamma(R + d_0).$$

Now, for each $i \in \{1, ..., l\}$ we define a map on Γ as follows : for any $\gamma \in \Gamma$, let $x_{\gamma,i}$ be the intersection of the ray $[\mathbf{o}, \gamma(\xi_i))$ with the horosphere $\gamma(\partial\mathcal{H}_{\xi_i})$. Since $\mathcal{C}_i$ is a fundamental domain for the action of $\mathcal{P}_i$ on $\partial\mathcal{H}_{\xi_i}$ there exist a finite number of elements $\bar{\gamma}$ in $\gamma\mathcal{P}_i$ such that $x_{\gamma,i} \in \bar{\gamma}(\mathcal{C}_i)$. Choose one of those elements and denote it by $\bar{\gamma}_i$. Let $\bar{\Gamma}_i$ be the set of all $\bar{\gamma}_i$ for γ in Γ. Since $d(x_{\gamma,i}, \bar{\gamma}_i\mathbf{o}) \le d_0$, and since the angle at $x_{\gamma,i}$ between the geodesic segments $[x_{\gamma,i}, \mathbf{o}]$ and $[x_{\gamma,i}, x]$ is greater than $\pi/2$, by lemma 2.1 there exists a constant $d_1 > 0$ such that for every $\gamma \in \Gamma$ and $x \in \gamma\mathcal{H}_{\xi_i} \cap B_X(\mathbf{o}, R)$, we have :

$$d(\mathbf{o}, \bar{\gamma}_i\mathbf{o}) + d(\bar{\gamma}_i\mathbf{o}, x) - d_1 \le d(\mathbf{o}, x).$$

We have by (13)

$$B_X(\mathbf{o}, R) \cap \Gamma.\mathcal{H}_{\xi_i} \subset \left(\bigcup_{0 \le n \le [R+d_0]} \quad \bigcup_{\substack{\overline{\gamma} \in \overline{\Gamma}_i \\ n \le d(\mathbf{o}, \overline{\gamma}\mathbf{o}) < n+1}} B_X(\overline{\gamma}\mathbf{o}, R - n + d_1) \cap \overline{\gamma}.\mathcal{H}_{\xi_i} \right).$$

For each i denote $\mathbf{x}_i$ the intersection of the geodesic ray $[\mathbf{o}, \xi_\mathbf{i})$ with the horosphere $\partial\mathcal{H}_{\xi_i}$. One has

$$vol\Big(B_X(\overline{\gamma}\mathbf{o}, R - n + d_1) \cap \overline{\gamma}.\mathcal{H}_{\xi_i} \Big) \le vol\Big(B_X(\mathbf{x}_i, R - n + d_1 + d_0) \cap \mathcal{H}_{\xi_i} \Big),$$

while

$$\sharp\{\bar{\gamma} \in \bar{\Gamma}_i / n \le d(\mathbf{o}, \bar{\gamma}.\mathbf{o}) < n+1\} \le v_\Gamma(\mathbf{o}, n+1),$$

so

$$v_X(\mathbf{o}, R - d_0) \preceq v_\Gamma(\mathbf{o}, R) + \sum_{i=1}^{l} \sum_{n=0}^{[R]} v_\Gamma(\mathbf{o}, n+1) \times vol\Big(B_X(\mathbf{x}_i, R - n + d_1) \cap \mathcal{H}_{\xi_i} \Big).$$

The lemma follows with $\Delta \ge \max(d_0, d_1)$. □

Proposition 4.2 is crucial to establish Theorem 1.2 ; we first give an elementary proof of this result, in the case where X is 1/4-pinched.

4.1. Proof of Theorem 1.2 : the $\frac{1}{4}$−pinched curvature case. We prove here that if (X, g) is a complete, simply connected Riemannian manifold with 1/4-pinched negative curvature, then for any lattice $\Gamma \subset Is(X)$, we have $\delta(\Gamma) = \omega(X)$.

We use the notations of Proposition 4.2.

By (3), we need only to show that $\omega(X) \le \delta(\Gamma)$. By Proposition 2.3, we know that for $r > 0$ the set $B_X(\mathbf{x}_i, r) \cap \mathcal{H}_{\xi_i}$ is included in the ball of radius $r/2 + c$ centered at the point $\psi_{\xi_i, r/2}(\mathbf{x}_i)$. Then, (12) leads to the following inequality

(14)

$$v_X(\mathbf{o}, R - \Delta) \preceq v_\Gamma(\mathbf{o}, R) + \sum_{n=0}^{[R]} v_\Gamma(\mathbf{o}, n+1) \times \sup_{\mathbf{x} / B_X(\mathbf{x}, \frac{R-n+\Delta}{2}) \subset \Theta} vol\Big(B_X\Big(\mathbf{x}, \frac{R - n + \Delta}{2}\Big)\Big).$$

From Bishop Gunther's theorem and the fact that $b^2 \le 4a^2$, we have

$$vol\Big(B_X(\mathbf{x}, r) \cap \Theta \Big) \le v_X(\mathbf{x}, r) \preceq e^{b(N-1)r} \preceq e^{2a(N-1)r},$$

for any $\mathbf{x} \in X$ and $r > 0$. We conclude that $\omega(X) \le (N-1)a \le \delta(\Gamma)$ using Lemma 4.1. □

Remark - The above proof uses in a crucial way Lemma 4.1 and it still works if we relax the pinching assumption as follows :

For any $\epsilon > 0$*, there exists a compact set* $C_\epsilon \subset M$ *such that the curvature on* $M \setminus C_\epsilon$ *belongs to* $[-(4+\epsilon)a^2, -a^2]$.

However, this condition is much stronger than the $\left(\frac{1}{4+\epsilon}\right)$-pinching assumption and the proof of Corollary 1.3 requires the more precise estimates of the volume of balls obtained in the previous section.

4.2. Proof of Theorem 1.2 : the general case. We fix here a non uniform lattice $\Gamma \subset Is(X)$ and apply the results of Section 3 to each maximal parabolic subgroup $\mathcal{P}_i$ of Γ. We first set the

Definition 4.3. Let $M = X/\Gamma$ be a complete Riemannian manifold of finite volume with $-b^2 \leq K_X \leq -a^2 < 0$ and with ends $E_1, ..., E_l$. For $1 \leq i \leq l$, the **cuspidal function** $\mathcal{F}_i$ associated with E_i is defined by

$$\forall \mathbf{x} \in X, \forall R > 0 \qquad \mathcal{F}_i(\mathbf{x}, R) = \int_0^R \frac{\mathcal{A}_i(\mathbf{x}, t)}{\mathcal{A}_i\left(\mathbf{x}, \frac{t+R}{2}\right)} dt$$

where $\mathcal{A}_i(\mathbf{x}, t)$ is the horospherical area function associated with E_i.

By (9), the growth rates $\omega^\pm(\mathcal{F}_i(\mathbf{x}, .))$ depend only on the ends E_i of M as for any points $\mathbf{x}, \mathbf{y} \in X$ and any $R_0 > 0$ fixed, we have $\mathcal{F}_i(\mathbf{x}, R) \asymp \mathcal{F}_i(\mathbf{y}, R)$. Those functions are of major importance in order to estimate $v_X(\mathbf{x}, R)$; namely, we have the

Proposition 4.4. *There exists* $\Delta = \Delta(a, b, diam(\mathcal{C}_0)) > 0$ *such that*

(i) $\quad v_X(\bullet, R+\Delta) \succeq v_\Gamma(\bullet, R) + \sum_{i=1}^{l} \mathcal{F}_i(\bullet, R)$

(ii) $\quad v_X(\bullet, R+\Delta) \preceq v_\Gamma(\bullet, R) + \sum_{i=1}^{l} v_\Gamma(\bullet, .) * \mathcal{F}_i(\bullet, .)(R)$

which leads to the

Corollary 4.5. *We have* $\omega^\pm(X) = \max\Big(\delta(\Gamma), \omega^\pm(\mathcal{F}_1), ..., \omega^\pm(\mathcal{F}_l)\Big)$.

Proof of Proposition 4.4. *Part (i).* We have

$$B_X(\mathbf{o}, R) \supset \bigsqcup_{\gamma \in B_\Gamma(\mathbf{o}, R-d_0)} \gamma(\mathcal{C}_0) \cup \bigcup_{i=1}^{l} (B_X(\mathbf{o}, R) \cap \mathcal{H}_i)\,.$$

On the other hand $B_X(\mathbf{o}, R) \cap \mathcal{H}_i \supset B_X(\mathbf{x}_i, R-d_0) \cap \mathcal{H}_i$, and by Proposition 3.3, we have

$$v_X(\mathbf{o}, R) \succeq v_\Gamma(\mathbf{o}, R-d_0) + \sum_{i=1}^{l} \mathcal{F}_i(\mathbf{x}_i, R)$$

with $\mathcal{F}_i(\mathbf{x}_i, R) \asymp \mathcal{F}_i(\mathbf{o}, R)$; the first inequality follows.

Part (ii) follows by plugging Proposition 3.3 in (12). $\square$

Proof of Theorem 1.2 By Corollary 4.5, it is enough to show that $\omega(\mathcal{F}_i) \leq \delta(\Gamma)$ for $1 \leq i \leq l$. By Proposition 3.2, we have, for any $\epsilon > 0$:

$$\mathcal{A}_i(t) \preceq e^{-(2\delta^-(\mathcal{P})-\epsilon)t} \quad \text{and} \quad \mathcal{A}_i\Big(\frac{t+R}{2}\Big) \succeq e^{-(\delta(\mathcal{P})+\epsilon)(t+R)}.$$

So, we obtain $\mathcal{F}_i(t) \preceq e^{(\delta(\mathcal{P})+\epsilon)R} \displaystyle\int_0^R e^{(\delta(\mathcal{P})-2\delta^-(\mathcal{P})+2\epsilon)t}dt \preceq e^{(\delta(\mathcal{P})+3\epsilon)R}$ as $\delta(\mathcal{P}) - 2\delta^-(\mathcal{P}) \leq 0$, therefore $\omega(\mathcal{F}_i) \leq \delta(\mathcal{P}) \leq \delta(\Gamma)$. $\square$.

Proof of Corollary 1.3 Assume that $M = X/\Gamma$ is asymptotically $\frac{1}{4}$-pinched. Then, for any fixed $\epsilon > 0$ we know that outside a compact subset C_ϵ the curvature of M is between $-\beta^2$ and $-\alpha^2$, with $\beta^2 \leq (4+\epsilon)\alpha^2$. Therefore we have

$$e^{-\beta(N-1)t} \preceq \mathcal{A}_i(t) \preceq e^{-\alpha(N-1)t}$$

hence, by Proposition 3.2, we deduce that

$$\frac{\delta(\mathcal{P})}{\delta^-(\mathcal{P})} \leq \frac{\beta}{\alpha} \leq 2+\epsilon$$

for every maximal parabolic subgroup of Γ. As ϵ is arbitrary, we deduce that M is parabolically $\frac{1}{4}$-pinched, and we conclude by Theorem 1.2. $\square$

Remark. We have seen that, under the assumptions of Theorem 1.2, we have $\omega(\mathcal{F}_i) \leq \delta(\Gamma)$ for all $1 \leq i \leq l$; in particular, $\omega(X)$ is a limit in this case.

5. An end with the leading role

We shall construct in this section a pinched, negatively curved surface $S = X/\Gamma$ of finite volume such that $\omega(X) > \delta(\Gamma)$. The surface we exhibit is homeomorphic to a 3-punctured sphere, and we shall deform a hyperbolic metric on one end E of S.
Our construction rests on two main ideas :

i) we can deform the metric in the end E varying the sectional curvature from α^2 to β^2 on different bands of E, in order that the function $\mathcal{F}$ associated to E satisfies $\omega(\mathcal{F}) > \delta(\mathcal{P})$.

ii) we set $\epsilon := \omega(\mathcal{F}) - \delta(\mathcal{P})$ and we show that the above deformation of the metric can be performed in such a way that $\delta(\Gamma) < \delta(\mathcal{P}) + \epsilon$ also.

By Corollary 4.5 we conclude that $\omega(X) > \delta(\Gamma)$.

Fix positive real numbers α and β such that $\beta > 2\alpha$. We can construct sequences of disjoint intervals $[p_n, q_n]$, $[r_n, s_n]$ included in $[\Delta^{n-1}, \Delta^n]$ (for some $\Delta > 1$), and a C^2 convex, decreasing function $\mathcal{A}(t)$ on $[\Delta, +\infty[$ whose restrictions to $[p_n, q_n]$ and $[r_n, s_n]$ coincide respectively with $e^{-\alpha t}$ and $e^{-\beta t}$. More precisely, we can arrange the points p_n, q_n, r_n and s_n in order that $q_n \geq p_n + 1$ and $t \in [p_n, q_n] \Leftrightarrow \frac{t+\Delta^n}{2} \in [r_n, s_n]$, and we can choose $\mathcal{A}$ such that $e^{-\beta t} \leq \mathcal{A}(t) \leq e^{-\beta t}$ and $\frac{\mathcal{A}''(t)}{\mathcal{A}(t)} \in [\alpha^2 - \eta, \beta^2 + \eta]$ for all $t \in [\Delta, +\infty[$ and some $\eta > 0$. The existence of such intervals and of the function $\mathcal{A}$ is rather technical and we postponed the details of proof to the Appendix (Section 6).

By construction, the function $\mathcal{F}(R) := \displaystyle\int_0^R \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})}\mathrm{d}t$ satisfies :

$$\omega(\mathcal{F}) \geq \limsup_{n\to+\infty} \frac{1}{\Delta^n} \ln \int_{p_n}^{q_n} \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+\Delta^n}{2})}\mathrm{d}t > \beta/2.$$

We can now construct the surface of Theorem 1.1. Start from a 3-punctured sphere S with a metric g_0 of finite volume and constant curvature $-\alpha^2$. Let $\Gamma = \pi_1(S)$ and let $\mathcal{P}$ be the maximal parabolic subgroup associated with the end E of S. Consider the horospherical parametrization $\sigma : [0,1[\times\mathbb{R}^+ \to \mathcal{E}$ of $\mathcal{E}$; with respect to these coordinates, the hyperbolic metric writes $g = e^{-2\alpha t}dx^2 + dt^2$. We now perturb g on $E_n = \sigma([0,1[\times[p_n,+\infty[)$ to obtain a new C^2-metric g_n such that $g_n = \mathcal{A}^2(t)dx^2 + dt^2$ on E_n, for $\mathcal{A}$ defined above. We shall denote by d and d_n the distances on X associated respectively to g and g_n and we let $\delta_n(\Gamma)$, $\delta_n(\mathcal{P})$ be the critical exponents of Γ and $\mathcal{P}$ relatively to the new metric g_n. Notice that $K_X = -\dfrac{\mathcal{A}''}{\mathcal{A}}$ is pinched between $-\beta^2 - \eta$ and $-\alpha^2 + \eta$; furthermore $\mathcal{A}(R)$ is precisely the horospherical area (length) function of $\mathcal{P}$, with respect to g_n, so $\delta_n(\mathcal{P}) = \beta/2$ for all n, by Proposition 3.2 (while $\delta_n^-(\mathcal{P}) \leq \alpha/2$). Since we know that $\omega(\mathcal{F}) = \beta/2 + \epsilon$ for some $\epsilon > 0$, it will be enough to show that :

PROPOSITION 5.1. *For n large enough, we have $\delta_n(\Gamma) \in]\delta_n(\mathcal{P}), \delta_n(\mathcal{P}) + \epsilon[$.*

Proof. Let p be a generator of $\mathcal{P}$ and choose another parabolic element $q \in \Gamma$ such that Γ is the free non abelian group over p and q. Fix $N \geq 2$; each element $\gamma \in \Gamma \setminus \{id\}$ can be written in a unique way as

$$\gamma = p^{l_1} q^{m_1} \cdots p^{l_k} q^{m_k}, \tag{15}$$

where $l_i, m_i \in \mathbb{Z}^*$ except for l_1 and m_k which may be zero. Given this decomposition, we select those l_i such that $|l_i| \geq N$, say $l_{i_1}, \cdots l_{i_r}$, and write

$$\gamma = Q_1 p^{l_{i_1}} Q_2 \cdots p^{l_{i_r}} Q_r \tag{16}$$

where each Q_i is a subword of the expression (15), containing powers of q and powers of p not exceeding N in absolute value. Note that decomposition (16) is still unique. We denote by $\mathcal{Q}_N$ the subset of elements $\gamma \in \Gamma$ which write simply $\gamma = Q_1$ in (16).

Now let $\mathbf{o} \in X$ and $\mathcal{D}$ be the Dirichlet domain for the action of Γ, centered at $\mathbf{o}$. Roughly speaking, the union of the geodesic segments

$$[\mathbf{o}, Q_1(\mathbf{o})], [Q_1(\mathbf{o}), Q_1 p^{l_{i_1}}(\mathbf{o})], \ \cdots \ , \ [Q_1 ... p^{l_{i_r}}(\mathbf{o}), \gamma(\mathbf{o})]$$

represents a quasigeodesic which stays close to $[\mathbf{o}, \gamma(\mathbf{o})]$ and each of its subsegments corresponds to the excursion of the geodesic loop γ alternatively outside or inside the cusp E. We now precise this argument.

As $K_X \leq -\alpha^2 + \eta$, there exits a minimal angle $\theta_0 > 0$ such that for all $\mathbf{x} \in p^{\pm 2}(\mathcal{D})$ and all $\mathbf{y} \in q^{\pm 1}(\mathcal{D})$, we have $\widehat{\mathbf{x}\,\mathbf{o}\,\mathbf{y}} \geq \theta_0$. Then, when $Q_1 \neq id$ in (16), by a ping-pong argument we deduce that $\angle_{\mathbf{o}}(Q_1^{-1}\mathbf{o}, p^{l_{i_1}} Q_2 \cdots Q_r \mathbf{o}) \geq \theta_0$, as $l_{i_1} \geq N \geq 2$. Therefore, by Lemma 2.1, there exists a constant $d = d(\alpha, \theta_0) > 0$ such that

$$d_n(\mathbf{o}, \gamma(\mathbf{o})) \geq d_n(\mathbf{o}, Q_1(\mathbf{o})) + d_n(\mathbf{o}, p^{l_{i_1}} Q_2 \cdots p^{l_{i_{r-1}}} Q_r(\mathbf{o})) - d$$

Repeating this argument yields

$$d_n(\mathbf{o},\gamma(\mathbf{o})) \geq \sum_{i=0}^{r} d_n(\mathbf{o},Q_i(\mathbf{o})) + \sum_{j=1}^{r-1} d_n(\mathbf{o},p^{l_{i_j}}(\mathbf{o})) - 2rd.$$

Consequently
(17)

$$\sum_{\gamma\in\Gamma} e^{-sd_n(\mathbf{o},\gamma(\mathbf{o}))} \leq \sum_{\gamma\in\mathcal{Q}_N} e^{-sd_n(\mathbf{o},\gamma(\mathbf{o}))} + \sum_{r\geq 1}\left(e^{2sd}\sum_{|k|\geq N} e^{-sd_n(\mathbf{o},p^k(\mathbf{o}))}\sum_{\gamma\in\mathcal{Q}_N} e^{-sd_n(\mathbf{o},\gamma(\mathbf{o}))}\right)^r$$

If n is large enough with respect to N, every element of $\mathcal{Q}_N$ correspond to a geodesic loop staying in the part of S where the curvature is constant equal to $-\alpha^2$. For that choice of n anf for $s = \frac{\beta+\epsilon}{2}$, we have

$$\sum_{\gamma\in\mathcal{Q}_N} e^{-sd_n(\mathbf{o},\gamma\mathbf{o})} \leq \sum_{\gamma\in\Gamma} e^{-sd(\mathbf{o},\gamma\mathbf{o})} := A.$$

The latter series converges because the value of the critical exponent of any lattice in the space of constant curvature case $-\alpha^2$ is α and $\alpha < s$.
Furthermore

$$\begin{aligned}
\sum_{|k|\geq N} e^{-sd(\mathbf{o},p^k(\mathbf{o}))} &\preceq \sum_{m\geq d(\mathbf{o},p^N(\mathbf{o}))} v_{\mathcal{P}}(\mathbf{o},m)e^{-sm}\\
&\preceq \sum_{m\geq d(\mathbf{o},p^N(\mathbf{o}))} \frac{e^{-sm}}{\mathcal{A}(\frac{m}{2})}\\
&\preceq \sum_{m\geq d(\mathbf{o},p^N(\mathbf{o}))} e^{-(s-\frac{\beta}{2})m} = \sum_{m\geq d(\mathbf{o},p^N(\mathbf{o}))} e^{-\epsilon m/2}
\end{aligned}$$

so that $\sum_{|k|\geq N} e^{-sd(\mathbf{o},p^k(\mathbf{o}))} \to 0$ when $N \to +\infty$. Then, we can choose N and n such that

$$\sum_{\gamma\in\mathcal{Q}_N} e^{-sd(\mathbf{o},\gamma(\mathbf{o}))} \leq A < +\infty \qquad \text{and} \qquad \left(e^{2sd}\sum_{|k|\geq N} e^{-sd(\mathbf{o},p^k(\mathbf{o}))}A\right) < 1.$$

For that choice, (17) implies that the Poincaré series associated with Γ converges at s and consequently : $\delta(\Gamma) \leq s < \delta(\Gamma)+\epsilon$.

Remark. Notice that the curvature of S is not asymptotically $\frac{1}{4}$-pinched as $\beta > 2\alpha$; but, letting $\alpha \to \beta/2$ and $\eta \to 0$, the metric can be choosen so that K_S is asymptotically $(\frac{1}{4+\epsilon})$-pinched, for any $\epsilon > 0$.

□

6. Appendix

Let t_0, t_1, t_2, t_3 be four real numbers satisfying $t_0 < t_1 < t_2 < t_3$. Denote by φ_1 a C^2 convex and decreasing function on $[t_0,t_1]$ and φ_2 a C^2 convex and decreasing function on $[t_2, t_3]$. A straightforward geometric argument on epigraphs of φ_1 and φ_2 shows that the following inequalities :

$$\varphi_1'(t_1)(t_2 - t_1) \underset{(a)}{<} \varphi_2(t_2) - \varphi_1(t_1) \underset{(b)}{<} \varphi_2'(t_2)(t_2 - t_1) \tag{18}$$

are necessary and sufficient for the existence of a C^2 convex decreasing function ψ on $[t_0, t_3]$ such that $\psi_{|[t_0,t_1]} \equiv \varphi_1$ and $\psi_{|[t_2,t_3]} \equiv \varphi_2$.

Lemma 6.1. *Let α, β two positive reals such that $\alpha < \beta$.*
(I) *Inequalities (18) are satisfied for $\varphi_1(t) = e^{-\alpha t}$ and $\varphi_2(t) = e^{-\beta t}$ when $t_2 - t_1 > \frac{1}{\alpha}$.*
(II) *Inequalities (18) are satisfied for $\varphi_1(t) = e^{-\beta t}$ and $\varphi_2(t) = e^{-\alpha t}$ when $t_2 > (\frac{\beta}{\alpha} + \epsilon)t_1$ for any $\epsilon > 0$.*

Proof. Case **(I)** :

$$(a) \Leftrightarrow e^{-\beta t_2 + \alpha t_1} + \alpha(t_2 - t_1) > 1$$

and the second inequality is satisfied when $t_2 - t_1 > \frac{1}{\alpha}$. Note that this condition is optimal if we want such an inequality to be satisfied for arbitrary large t_1 because with $u = t_2 - t_1$, this inequality becomes

$$e^{(\alpha-\beta)t_1 - \beta u} + \alpha u > 1$$

and this inequality cant be satisfied for small u when t_1 is too large.
With the previous notations,

$$(b) \Leftrightarrow e^{\beta u} e^{(\beta-\alpha)t_1} - \beta u - 1 > 0$$

and the latter inequality is always satisfied because $e^x - x - 1 > 0$ for all $x > 0$.

Case **(II)** :

$$(a) \Leftrightarrow e^{-\alpha t_2 + \beta t_1} + \beta(t_2 - t_1) > 1$$

and this second inequality is satisfied when $t_2 - t_1 > \frac{1}{\beta}$. The same remark as in the case (I).
With the previous notations too,

$$(b) \Leftrightarrow e^{\alpha u} e^{(\alpha-\beta)t_1} - \alpha u - 1 > 0$$

with $u = t_2 - t_1$. If we set $t_2 = (1 + x)t_1 + f(t_1)$ and substitute in the last term, a necessary condition in order to realise (b) is $(x + 1) \geq \frac{\beta}{\alpha}$ and if we set $(x + 1) = \frac{\beta}{\alpha}$ and replace, we get $e^{\alpha f(t_1)} - (\beta - \alpha)t_1 - f(t_1) - 1 > 0$. The conclusion follows.

□

Lemma 6.2. *Let $t_0 < t_1 < t_2 < t_3$ and $\eta > 0$. There exists $A = A(\eta, \alpha, \beta) > 0$ and $B = B(\alpha, \beta) > 0$ such that if $t_2 > A.t_1$ and $t_0 > B$,*
(I) *There exists a C^2 convex and decreasing function ψ on $[t_0, t_3]$ satisfying :*

$$(\mathbf{C_1}) \begin{cases} \forall t \in [t_0, t_1], & \psi(t) = e^{-\alpha t} \\ \forall t \in [t_2, t_3], & \psi(t) = e^{-\beta t} \\ \forall t \in [t_0, t_3], & \alpha^2 - \eta \leq \frac{\psi''(t)}{\psi(t)} \leq \beta^2 - \eta \quad \textit{and} \quad \psi(t) \geq e^{-\beta t} \end{cases}$$

(II) *There exists a C^2 convex and decreasing function ψ on $[t_0, t_3]$ such that we have*

$$(\mathbf{C_2})\begin{cases} \forall\, t \in [t_0, t_1], & \psi(t) = e^{-\beta t} \\ \forall\, t \in [t_2, t_3], & \psi(t) = e^{-\alpha t} \\ \forall\, t \in [t_0, t_3], & \alpha^2 - \eta \leq \frac{\psi''(t)}{\psi(t)} \leq \beta^2 + \eta \quad \textit{and} \quad \psi(t) \geq e^{-\beta t} \end{cases}$$

Proof. By the previous remark, if we choose $A > \frac{\beta}{\alpha}$ and $B > \frac{1}{\beta - \alpha}$, inequalities (18) are satisfied. In both cases, set

$$\psi(t) = e^{-t\varphi(t)} \quad t \in [t_0, t_3]$$

where φ is constant on $[t_0, t_1]$ and $[t_2, t_3]$ (depending in an obvious way on case I or II). Consider a C^2 function $\phi : [0, 1] \to [\alpha, \beta]$; set $s = \lambda(t - t_1)$ where $\lambda = \frac{1}{t_2 - t_1}$ and put $\varphi(t) = \phi(s)$ for $t \in [t_1, t_2]$. A straightforward calculus gives, for $s \in [0, 1]$:

$$\begin{aligned} \frac{\psi''(t)}{\psi(t)} &= ((s\phi(s))' + \lambda t_1 \phi'(s))^2 - \lambda\,(2\phi'(s) + (s + \lambda t_1)\phi''(s)) \\ &= (k'(s))^2 + \lambda t_1 (2k'(s)\phi'(s) + \lambda(t_1(\phi'(s))^2 - \phi''(s))) - \lambda(2\phi'(s) + s\phi''(s)) \\ &= (k'(s))^2 + \theta(\lambda) \end{aligned}$$

where $k(s) := s\phi(s)$ and θ is a function such that $\theta(\lambda) \to 0$ when $\lambda \to 0$.
Set $M_i = \sup_{s \in [0,1]} |\phi^{(i)}(s)|$ for $i = 1, 2$ (which depend only on (α, β)) and $C = \frac{1}{8(\beta+1)(M_1 + M_2 + \beta)}$. The previous equalities implie

$$(k'(s))^2 - \frac{\eta}{2} \leq \frac{\psi''(t)}{\psi(t)} \leq (k'(s))^2 + \frac{\eta}{2} \tag{19}$$

when $\lambda t_1 < C.\eta$ i.e. for $t_2 > (1 + \frac{1}{C.\eta})t_1 := A.t_1$. We show in both cases that we can choose a C^2 function ϕ with values in $[\alpha, \beta]$ such that for all $s \in [0, 1]$:

$$\alpha - \frac{\eta}{4} \leq k'(s) \leq \beta + \frac{\eta}{4}. \tag{20}$$

Case **(I)** : choose $\phi : [0, 1] \to [\alpha, \beta]$ non decreasing satisfying $\phi(0) = \alpha$, $\phi(1) = \beta$ and $\phi'(0) = \phi'(1) = \phi''(0) = \phi''(1) = 0$. Then, the function φ can be extend on $[t_0, t_3]$ in a C^2 manner and on $[0, 1]$, we have $k'(s) = (s\phi(s))' = \phi(s) + s\phi'(s) \geq \alpha$ and $\phi(s) \leq \beta$ so that $\psi''/\psi \geq \alpha^2 - \eta$ and $\psi(t) \geq e^{-\beta t}$ are both satisfied on $[t_0, t_3]$. It implies in particular that the function ψ constructed is convex on $[t_0, t_3]$. Note that in this case, the inequality $\lambda.t_1 < C\eta$ must be satisfied, for, in the second expression of $\frac{\psi''}{\psi}$, the term $(t_1(\phi'(s))^2 - \phi''(s))$ is negative in the neighborhood of $s_0 = \inf\{s\,;\, \phi'(s) = 0\}$.
It is left to show that ϕ or equivalently k can be choosen so that $k'(s) = \phi(s) + s\phi'(s) \leq \beta + \frac{\eta}{4}$. The boundary conditions for ϕ up to the first order translate to $k(0) = 0$, $k(1) = \beta$, $k'(0) = \alpha$ and $k'(1) = \beta$. For $\epsilon_1 \in]0, 1[$, consider the C^0-piecewise affine function $\bar{k}$ defined on $[0, \epsilon_1]$ by $\bar{k}(t) = \alpha.t$, on $[1 - \epsilon_1, 1]$ by $\bar{k}(t) = \beta.t$ and affine on $[\epsilon_1, 1 - \epsilon_1]$. If we choose ϵ_1 small enough (depending on η and α), we can smooth $\bar{k}$ to obtain a C^2 function k on $[0, 1]$ in such a way that the dérivative k' satisfies

$$\begin{cases} k'(s) = \alpha & s \in [0, \epsilon_1/2] \\ k'(s) \leq \beta + \eta/(4\beta) & s \in [\epsilon_1/2, 1 - \epsilon_1/2] \\ k'(s) = \beta & s \in [1 - \epsilon_1/2, 1] \end{cases}$$

so that $(k'(s))^2 \leq \beta^2 - \eta/2$.

Case **(II)** : this case is similar. We choose $\phi\,:\,[0,1] \to [\alpha,\beta]$ non increasing satisfying $\phi(0)=\beta$, $\phi(1)=\alpha$ and $\phi'(0)=\phi'(1)=\phi''(0)=\phi''(1)=0$, or equivalently (up to the first order), we choose $k(s)=s\phi(s)$ satisfying $k(0)=0$, $k(1)=\int_0^1 k'(s)ds=\alpha$, $k'(0)=\beta$ and $k'(1)=\alpha$. The construction is symmetric to the previous one. In both cases, the desired inequalities : (20), (19) and $e^{-\alpha t} \leq \psi(t) \leq e^{-\alpha t}$ are satisfied. □

Let us now construct the sequences of intervals $[p_n,q_n]$, $[r_n,s_n]$ and the function $\mathcal{A}$ we used in Section 4. Let $A>1$ and $B>0$ given by Lemma 6.2. We set

$$\left\{ \begin{array}{l} p_n=(1-\lambda_0)\Delta^{n-1}+\lambda_0\Delta^n \quad \textit{and} \quad r_n=\frac{p_n+\Delta^n}{2} \\ q_n=(1-\mu_0)\Delta^{n-1}+\mu_0\Delta^n \quad \textit{and} \quad s_n=\frac{q_n+\Delta^n}{2} \end{array} \right.$$

for Δ, λ_0 and μ_0 to be defined.

Fix (λ_0,μ_0) in the (nonempty) set $(]0,1[^2 \cap \{(\lambda,\mu)\ ;\ 1+\lambda-2A\mu>0 \wedge \mu>\lambda\})$. The polynomial function $P(x)=2\lambda_0x^2+((2-A)-2\lambda_0-A\mu_0)x-A(1-\mu_0)$ tends to infinity as $x\to+\infty$; thus, we can choose a positive real number Δ such that both inequalities

$$\Delta > \frac{2A-1+\lambda_0-2A\mu_0}{1+\lambda_0-2A\mu_0} \tag{21}$$

$$P(q_0)>0 \tag{22}$$

are satisfied.

Inequality (21) insures that $r_n > Aq_n$ and inequality (22) insures that $p_{n+1} > As_n$. By Lemma 6.2, there exists $n_0 \in \mathbb{N}^*$ and a C^2-convex and decreasing function $\mathcal{A}$ on $[\Delta^{n_0-1},+\infty[$ satisfying $\frac{\mathcal{A}''(t)}{\mathcal{A}(t)} \geq \alpha^2-\eta$ and $\mathcal{A}(t)\geq e^{-\beta t}$ for all $t\in[\Delta^{n_0-1},+\infty[$, and such that for $n\geq n_0$, we have :

$$\left\{ \begin{array}{ll} \mathcal{A}(t)=e^{-\alpha t} & \forall\, t\in[p_n,q_n] \\ \mathcal{A}(t)=e^{-\beta t} & \forall\, t\in[r_n,s_n]. \end{array} \right.$$

Note that by construction $t\in[p_n,q_n] \Leftrightarrow \frac{t+R_n}{2}\in[r_n,s_n]$.

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