ON RECURRENCE OF REFLECTED RANDOM WALK ON THE HALF-LINE

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WITH AN APPENDIX ON RESULTS OF MARTIN BENDA

ABSTRACT. Let (Y_n) be a sequence of i.i.d. real valued random variables. Reflected random walk (X_n) is defined recursively by $X_0 = x \geq 0$, $X_{n+1} = |X_n - Y_{n+1}|$. In this note, we study recurrence of this process, extending a previous criterion. This is obtained by determining an invariant measure of the embedded process of reflections.

1. INTRODUCTION

Reflected random walk was described and studied by FELLER [10]; apparently, it was first considered by VON SCHELLING [16] in the context of telephone networks.

Let $(Y_n)_{n \geq 0}$ be a sequence of i.i.d. real valued random variables, and let $S_n = Y_1 + \ldots + Y_n$ be the classical associated random walk. Reflected random walk is obtained by considering a non-negative initial random variable X_0 independent of the Y_n and considering $X_0 - S_n$, $n = 0, 1, \ldots$, as long as this is non-negative. When it becomes negative, we change sign and continue from the new (reflected) point by subtracting $Y_{n+1}, Y_{n+2}, \ldots$, until the next reflection, and so on. Thus, we consider the Markov chain X_n given by $X_{n+1} = |X_n - Y_{n+1}|$. We are interested in recurrence of this process on its essential (i.e., maximal irreducible) classes.

We start by considering the situation when $Y_n \geq 0$ (of course excluding the trivial case $Y_n \equiv 0$), so that the increments of (X_n) are non-positive except possibly at the moments of reflection. In this case, FELLER [10] and KNIGHT [13] have computed an invariant measure for the process when the Y_n are non-lattice random variables, while BOUDIBA [5], [6] has provided such a measure when the Y_n are lattice variables. LEGUESDRON [14], BOUDIBA [6] and BENDA [2] have also studied its uniqueness (up to constant factors). When that invariant measure has finite total mass – which holds if and only if $\mathsf{E}(Y_1) < \infty$ – the process is (topologically) recurrent: with probability 1, it returns infinitely often to each open set that is charged by the invariant measure.

Our main result is that reflected random walk is still recurrent when $Y_n \geq 0$ and $\int_0^\infty \Pr[Y_1 \geq t]^2 \, dt < \infty$; see §3 for the case when the Y_n are lattice random variables, and §4 for the non-lattice case. The result is based on considering the process of reflections, that is, reflected random walk observed at the instances of reflection, see §2. We determine an invariant measure for the latter. The above "quadratic tail" condition holds if and only if that measure is finite. This holds, in particular, when $\mathsf{E}(Y_1^{1/2}) < \infty$.

Subsequently, in §5, we also consider the case when the Y_n may assume negative as well as positive values. Reflected random walk is of interest when $\limsup_n S_n = \infty$ almost surely. Let $Y_1 = Y_1^+ - Y_1^-$ be the decomposition into positive and negative part. If $\mathsf{E}(Y_1^-) < \mathsf{E}(Y_1^+)$ then the situation is similar to the case when $Y_1 \geq 0$ a.s., and we get recurrence when $\mathsf{E}\big(\sqrt{Y_1^+}\big) < \infty$. If the Y_n are centered, that is, $0 < \mathsf{E}(Y_1^-) = \mathsf{E}(Y_1^+)$, then we get recurrence under the moment condition $\mathsf{E}\Big(\sqrt{Y_1^+}^3\Big) < \infty$, which turns out to be almost sharp.

Our methods are based on interesting and useful work of M. Benda in his PhD thesis [2] (in German) and the two subsequent preprints [3], [4] which have remained unpublished. For this reason, we outline those results in the Appendix (§6).

2. The process of reflections

In this and the next two sections, we suppose always that (Y_n) is a sequence of i.i.d, non-constant, non-negative random variables. Let μ be the (common) distribution of Y_n, a non-degenerate probability measure on $[0\,,\infty)$, and $F(x) = F_\mu(x) = \mu([0\,,x])$ the associated distribution function $(x \geq 0)$. Denote by $\mu^{(n)}$ its n-th convolution power, the distribution of S_n, with $\mu^{(0)} = \delta_0$. Since $S_n \to \infty$ almost surely, the potential

$$(2.1) \qquad\qquad \mathcal{U} = \sum_{n=0}^{\infty} \mu^{(n)}$$

defines a Radon measure on $[0\,,\infty)$, that is, $\mathcal{U}(B) < \infty$ if B is a bounded Borel set.

Now consider the sequence of stopping times $(\mathbf{r}(k))_{k \geq 0}$, where $\mathbf{r}(0) = 0$, and $\mathbf{r}(k)$ $(k > 0)$ is the time of the k-th reflection:

$$(2.2) \qquad \begin{aligned} \mathbf{r}(k+1) &= \inf\big\{n > \mathbf{r}(k) : X_n = -(X_{n-1} - Y_n)\big\} \\ &= \inf\big\{n > \mathbf{r}(k) : (Y_{\mathbf{r}(k)+1} + \ldots + Y_{n-1} + Y_n) \geq X_{\mathbf{r}(k)}\big\}\,. \end{aligned}$$

Once more because $S_n \to \infty$, each $\mathbf{r}(k)$ is finite almost surely. We call the embedded process $R_k = X_{\mathbf{r}(k)}$ the *process of reflections*.

(2.3) Lemma. *The process of reflections is a Markov chain with transition probabilities given as follows: if $B \subset [0\,,\infty)$ is a Borel set, then*

$$q(0, B) = \mu(B) \quad and \quad q(x, B) = \int_{[0\,,x)} \mu(B + x - w)\,\mathcal{U}(dw)\,, \quad if\ x > 0\,.$$

Proof. It is clear that (R_k) is a (time-homogeneous) Markov chain. We compute

$$q(x, B) = \mathsf{Pr}[R_1 \in B \mid R_0 = x] = \sum_{n=1}^{\infty} \mathsf{Pr}[\mathbf{r}(1) = n\,,\ S_n - x \in B]$$

$$= \sum_{n=1}^{\infty} \mathsf{Pr}[S_{n-1} < x\,,\ S_n - x \in B] = \sum_{n=1}^{\infty} \int_{[0\,,x)} \mathsf{Pr}[Y_n + w - x \in B]\,\mu^{(n-1)}(dw)$$

$$= \int_{[0\,,x)} \mu(B + x - w)\,\mathcal{U}(dw)\,,$$

as proposed. $\square$

It is an instructive exercise, relying on the fact that $\mathsf{supp}(\mu) \subset [0\,,\,\infty)$, to show directly that $q(\cdot,\cdot)$ is stochastic.

Now the idea is the following: if the embedded process of reflections is recurrent, then also the original reflected Markov chain must be recurrent.

3. The lattice case

We start with the discrete case, which is instructive and has to be treated separately anyway. Here we suppose that there is $\kappa > 0$ such that $\mathsf{supp}(\mu) \subset \kappa \cdot \mathbb{N}_0$, and we may assume without loss of generality that $\kappa = 1$. (By $\mathbb{N}_0$ we denote the non-negative integers.)

The one-step transition probabilities of (X_n) are

$$(3.1) \qquad p(x,y) = \begin{cases} \mu(x)\,, & \text{if } y = 0\,, \\ \mu(x+y)\,, & \text{if } x < y\,, \\ \mu(x-y) + \mu(x+y)\,, & \text{if } x \geq y > 0\,. \end{cases}$$

We write $p^{(n)}(x,y) = \Pr[X_n = y \mid X_0 = x]$ for the n-step transition probabilities. Set

$$d = \gcd \mathsf{supp}(\mu) \quad \text{and} \quad N = \sup \mathsf{supp}(\mu)\,.$$

If the reflected Markov chain starts in a deterministic point $X_0 = x_0 \in [0\,,\,\infty)$, then (X_n) evolves within the state space

$$\mathsf{S}(x_0) = \{kd \pm x_0 : k \in \mathbb{Z}\} \cap [0\,,\,\infty)\,.$$

Recall that an essential class of a denumberable Markov chain is a subset C of the state space which is irreducible and absorbing: if $x \in \mathsf{C}$ then $p^{(n)}(x,y) > 0$ for some n if and only if $y \in \mathsf{C}$. The next lemma follows from [6] when the starting point x_0 is rational, and when it is irrational, it is immediately seen to be true as well.

(3.2) Lemma. *The reflected random walk (X_n) starting at x_0 is absorbed after finitely many steps by the essential class $\mathsf{C}(x_0) = \mathsf{S}(x_0) \cap [0\,,\,N]$.*

When we speak of recurrence of (X_n) with starting point x_0 then we mean recurrence on $\mathsf{C}(x_0)$. This is known to be independent of x_0 [6].

If $N = \infty$ then $\mathsf{C}(x_0) = \mathsf{S}(x_0)$. Also, if $\mathsf{supp}(\mu)$ is finite then $\mathsf{C}(x_0)$ is finite and carries a unique invariant probability measure. An invariant measure ν (not necessarily with finite total mass) exists always. Its formula is due to [5], where only $x_0 \in \mathbb{Z}$ is considered, but it can be adapted to the present situation with arbitrary starting point as follows. Set

$$(3.3) \qquad \nu(0) = \frac{1 - \mu(0)}{2} \quad \text{and} \quad \nu(x) = \frac{\mu(x)}{2} + \mu\big((x\,,\,\infty)\big)\,, \quad \text{if } x > 0\,.$$

Here, we mean of course $\mu(x) = \mu(\{x\})$, so that $\mu(x) = 0$ when $x \in [0\,,\,\infty) \setminus \mathbb{N}_0$. Then the invariant measure ν_{x_0} on $\mathsf{C}(x_0)$ is given by the restriction of ν to that essential class: if $B \subset \mathsf{C}(x_0)$ then $\nu_{x_0}(B) = \sum_{x \in B} \nu(x)$.

(3.4) Corollary. *The reflected random walk starting at x_0 is positive recurrent on $\mathsf{C}(x_0)$ if and only if the first moment $\sum_n n\,\mu(n)$ of Y_k is finite.*

If the reflected random walk is (positive or null) recurrent on $\mathsf{C}(x_0)$, then it follows of course from the basic theory of denumerable Markov chains that ν_{x_0} is the unique invariant measure (up to multiplication with constants).

We now consider the process of reflections.

(3.5) Lemma. *The set $\mathsf{C}(x_0)$ is also the unique essential class for (R_k) starting at x_0.*

Proof. Since $\mathsf{C}(x_0)$ is the only essential class for (X_n), we only need to verify that it is an irreducible class for (R_k). We have to show that for $x, y \in \mathsf{C}(x_0)$, it occurs with positive probability that (X_n), starting at x, reaches y at some reflection time $\mathbf{r}(k)$.

There is $m \in \mathsf{supp}\,\mu$ such that $m \geq y$. Then also $m - y \in \mathsf{C}(x_0)$, and there is n such that $p^{(n)}(x, m - y) > 0$. But from $m - y$, the reflected random walk can reach y (the reflection of $-y$) in a single step with positive probability $\mu(m)$, and this occurs at a reflection time. $\qquad\square$

Our simple new contribution is the following.

(3.6) Theorem. *Set* $\quad \rho(0) = \dfrac{1 - \mu(0)}{2} \quad$ *and*

$$\rho(x) = \sum_{k=1}^{\infty} \left(\frac{\mu(x)}{2} + \mu\big((x\,,\,x+k)\big) + \frac{\mu(x+k)}{2} \right) \mu(k), \quad \text{if } x > 0\,.$$

Then the restriction ρ_{x_0} of ρ to $\mathsf{C}(x_0)$ is an invariant measure for the process of reflections (R_k) on $\mathsf{C}(x_0)$. It is unique (up to multiplication by a constant), if ν_{x_0} is the unique invariant measure (up to multiplication by a constant) for the reflected random walk (X_n) on $\mathsf{C}(x_0)$.

Proof. We first show that ρ_{x_0} is invariant. The index x_0 will be ommitted whenever this does not obscure the arguments. Also, note that by its definition, $\rho \equiv 0$ on $\mathsf{S}(x_0) \setminus \mathsf{C}(x_0)$, so that we can think of ρ_{x_0} as a measure on the whole of $\mathsf{S}(x_0)$ with no mass outside $\mathsf{C}(x_0)$.

Consider the signed measure $\mathcal{A}$ defined by $\mathcal{A}(x) = \delta_0(x) - \mu(x)$ for $x \geq 0$. Then we have the convolution formula $\mathcal{A} * \mathcal{U} = \mathcal{U} * \mathcal{A} = \delta_0$, that is

$$(3.7) \qquad\qquad \sum_{j=0}^{n} \mathcal{A}(j)\,\mathcal{U}(n - j) = \delta_0(n)\,.$$

Now we verify that for each real $x \in (0\,,\,N]$,

$$(3.8) \qquad\qquad \rho(x) = \sum_{k=0}^{\infty} \mathcal{A}(k)\,\nu(x + k)\,.$$

Indeed, the last sum is equal to

$$\big(1 - \mu(0)\big)\nu(x) - \sum_{k=1}^{\infty} \mu(k)\,\nu(x + k) = \sum_{k=1}^{\infty} \mu(k)\,\big(\nu(x) - \nu(x + k)\big),$$

which is equal to $\rho(x)$. We remark here that the sum in (3.8) is absolutely convergent, since $\nu(\cdot) \leq 1$. Combining (3.8) with the inversion formula (3.7), we get

$$\sum_{k=0}^{\infty} \mathcal{U}(k)\, \rho(x+k) = \sum_{k=0}^{\infty} \mathcal{U}(k) \sum_{l=0}^{\infty} \mathcal{A}(l)\, \nu(x+k+l)$$

$$= \sum_{n=0}^{\infty} \nu(x+n) \sum_{k=0}^{n} \mathcal{U}(k)\, \mathcal{A}(n-k)$$

$$= \sum_{n=0}^{\infty} \nu(x+n)\, \delta_0(n)\,,$$

that is,

$$(3.9) \qquad \nu(x) = \sum_{k=0}^{\infty} \mathcal{U}(k)\, \rho(x+k)\,, \quad x > 0\,.$$

If σ is any measure on $\mathsf{C}(x_0)$ then we write

$$\mathsf{E}_\sigma(\cdot) = \sum_{w \in \mathsf{C}(x_0)} \sigma(w)\mathsf{E}_w(\cdot)\,,$$

where $\mathsf{E}_w(\cdot)$ denotes expectation when the starting point is $X_0 = w$. We claim that

$$(3.10) \qquad \nu(x) = \mathsf{E}_{\rho_{x_0}}\left(\sum_{j=0}^{\mathbf{r}(1)-1} \mathbf{1}_x(X_j) \right), \quad \text{if } x \in \mathsf{C}(x_0)\,.$$

Indeed, if $x = 0$ then the right hand side of (3.10) is $\rho(0) = \nu(0)$, since the reflected random walk can reach the state 0 before the first reflection only when it starts at 0, in which case $\mathbf{r}(1) = 1$. If $x > 0$, $x \in \mathsf{C}(x_0)$ then the reflected walk starting from $w \in \mathsf{C}(x_0)$ can reach x before the first reflection only if $w = x + k$ for some $k \in \mathbb{N}_0$ such that $k = S_j$ for some $j \geq 0$. We compute

$$\mathsf{E}_{x+k}\left(\sum_{j=0}^{\mathbf{r}(1)-1} \mathbf{1}_x(X_j) \right) = \mathsf{E}_{x+k}\left(\sum_{n=1}^{\infty} \mathbf{1}_n\big(\mathbf{r}(1)\big) \sum_{j=0}^{n-1} \mathbf{1}_x(X_j) \right)$$

$$= \sum_{j=0}^{\infty} \mathsf{Pr}[X_j = x, \mathbf{r}(1) > j \mid X_0 = x + k]$$

$$= \sum_{j=0}^{\infty} \mathsf{Pr}[x = x + k - S_j] = \mathcal{U}(k)\,.$$

Therefore

$$\mathsf{E}_{\rho_{x_0}}\left(\sum_{j=0}^{\mathbf{r}(1)-1} \mathbf{1}_x(X_j) \right) = \sum_{k=0}^{\infty} \rho(x+k)\mathcal{U}(k) = \nu(x)\,, \quad \text{if } x > 0\,,$$

as proposed. From (3.10), we infer that

$$\sum_w \nu(w)\, p(w,x) = \mathsf{E}_{\rho_{x_0}}\left(\sum_{j=1}^{\mathbf{r}(1)} \mathbf{1}_x(X_j)\right).$$

Now ν satisfies $\sum_w \nu(w)\, p(w,x) = \nu(x)$, and applying (3.10) once more, we obtain

$$\mathsf{E}_{\rho_{x_0}}\left(\mathbf{1}_x(X_0)\right) = \mathsf{E}_{\rho_{x_0}}\left(\mathbf{1}_x(X_{\mathbf{r}})\right).$$

The left hand side is $\rho(x)$, while the right hand side is $\sum_w \rho(w)q(w,x)$, where $q(\cdot,\cdot)$ is the transition kernel of the process of reflections. Thus, ρ_{x_0} is invariant for (R_k) on the state space $\mathsf{C}(x_0)$.

We now prove uniqueness. In view of Lemma 3.5, this is of course obvious by the basic theory of denumerable Markov chains, when $\rho_{x_0}(\mathsf{C}(x_0)) < \infty$, but this is not supposed in our statement.

So let $\bar\rho$ be another invariant measure for (R_k) on $\mathsf{C}(x_0)$, again considered on $\mathsf{S}(x_0)$ with zero mass outside $\mathsf{C}(x_0)$. Using the formula of Lemma 2.3 for the transition probabilities of (R_k), we get for $y \in \mathsf{C}(x_0)$

$$\bar\rho(y) = \sum_{w\in\mathsf{C}(x_0)} \bar\rho(w) \sum_{k\in\mathbb{N}_0:0\leq k<w} \mathcal{U}(k)\,\mu(w+y-k) = \sum_{k=0}^{\infty}\sum_{w\in\mathsf{C}(x_0):w>k} \mathcal{U}(k)\,\bar\rho(w)\,\mu(w+y-k)$$

To have a non-zero contribution in the last double sum, $w + y$ has to be integer, d must divide both k and $w + y$, and $x = w - k \in \mathsf{C}(x_0)$. Therefore we can rewrite

$$\bar\rho(y) = \sum_{k=0}^{\infty}\sum_{x\in\mathsf{C}(x_0):x>0} \mathcal{U}(k)\,\bar\rho(x+k)\,\mu(x+y).$$

Now let $x \in \mathsf{C}(x_0)$, $x > 0$. Again, there is $m \in \mathsf{supp}(\mu)$ with $x \leq m$, and $y = m - x \in \mathsf{C}(x_0)$. Therefore

$$\sum_{k=0}^{\infty}\mathcal{U}(k)\,\bar\rho(x+k) \leq \frac{\bar\rho(y)}{\mu(m)} < \infty$$

for each $x \in \mathsf{C}(x_0)$ with $x > 0$. This allows us to define a new measure $\bar\nu$ on $\mathsf{C}(x_0)$ by $\bar\nu(0) = \bar\rho(0)$, if $0 \in \mathsf{C}(x_0)$, and

$$\bar\nu(x) = \sum_{k=0}^{\infty}\mathcal{U}(k)\,\bar\rho(x+k), \quad \text{if } x > 0,$$

and a straightforward exercise shows that it is legitimate to apply the inversion formula (3.7) to deduce that

$$\bar\rho(x) = \sum_{k=0}^{\infty}\mathcal{A}(k)\,\bar\nu(x+k), \quad \text{if } x > 0,$$

The same computations as that lead to (3.9) and (3.10) show that

$$\bar\nu(x) = \mathsf{E}_{\bar\rho_{x_0}}\left(\sum_{j=0}^{\mathbf{r}(1)-1} \mathbf{1}_x(X_j)\right)$$

is an invariant measure for (X_n) on $\mathsf{C}(x_0)$. By uniqueness of the latter, $\bar{\nu} = c \cdot \nu_{x_0}$ for some $c > 0$. Therefore $\bar{\rho} = c \cdot \rho_{x_0}$. $\quad\square$

(3.11) Corollary. *The total mass of ρ_{x_0} is finite for some (equivalently, every) starting point x_0 if and only if*

$$(3.12) \qquad\qquad \sum_{k=0}^{\infty} \big(1 - F_\mu(k)\big)^2 < \infty \, .$$

Proof. We write $H(x) = 1 - F_\mu(x)$. For real $\alpha \geq 0$, let $\Sigma(\alpha) = \sum_{k=0}^{\infty} \rho(\alpha + kd)$. Let α_0 be the unique number in $(0\,,\,d]$ such that $x_0 - \alpha_0$ is an integer multiple of d. If $\alpha_0 = d$ or $\alpha_0 = d/2$ we have $\rho\big(\mathsf{C}(x_0)\big) = \rho(0)\delta_0\big(\mathsf{C}(x_0)\big) + \Sigma(\alpha_0)$, while otherwise $\rho(\mathsf{C}(x_0)) = \Sigma(\alpha_0) + \Sigma(d - \alpha_0)$. Thus, we prove that for any $\alpha \in (0\,,\,d]$, we have $\Sigma(\alpha) < \infty$ if and only if (3.12) holds. Recalling that $\mu(x) = 0$ if x is not a multiple of d, we compute $\Sigma(\alpha) = \Sigma_0(\alpha) + \Sigma_1(\alpha)$, where

$$\Sigma_0(\alpha) = \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{\mu(\alpha + kd) - \mu(\alpha + kd + md)}{2} \, \mu(md)$$

is always finite, and

$$\begin{aligned}
\Sigma_1(\alpha) &= \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \Big(H(\alpha + kd) - H(\alpha + kd + md) \Big) \mu(md) \\
&= \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} H(\alpha + kd) \, \mu(md) \\
&= \sum_{k=0}^{\infty} H(\alpha + kd) \sum_{m=k+1}^{\infty} \mu(md) = \sum_{k=0}^{\infty} H(\alpha + kd) \, H(kd) \, .
\end{aligned}$$

Since $H(\cdot)$ is decreasing, on one hand

$$\Sigma_1(\alpha) \leq \sum_{k=0}^{\infty} H(kd)^2 = \frac{1}{d} \sum_{k=0}^{\infty} \big(1 - F_\mu(k)\big)^2 \, ,$$

and on the other hand

$$\Sigma_1(\alpha) \geq \sum_{k=0}^{\infty} H\big((k+1)d\big)^2 = \frac{1}{d} \sum_{k=0}^{\infty} \big(1 - F_\mu(k)\big)^2 - H(0)^2 \, .$$

Thus, $\Sigma_1(\alpha)$ and the sum in (3.12) are finite, resp. infinite, simultaneously. $\quad\square$

The following is now immediate.

(3.13) Theorem. *Suppose that the "quadratic tail" condition (3.12) holds. Then the process of reflections (R_k) is positive recurrent on $\mathsf{C}(x_0)$ for each starting point $x_0 \geq 0$. If in addition $\mathsf{E}(Y_1) = \sum_{k \geq 0} k \, \mu(k) < \infty$, then the reflected random walk (X_n) is also positive recurrent on C_{x_0}, while it is null recurrent when $\mathsf{E}(Y_1) = \infty$.*

Finally, it is easy to relate the "quadratic tail" condition with a moment condition.

(3.14) Lemma. *If $\mathsf{E}\big(\sqrt{Y_1}\big) = \sum_{k \geq 0} \sqrt{k} \, \mu(k) < \infty$, then (3.12) holds.*

Proof. We use the Cauchy-Schwarz inequality:

$$\left(\sum_{k=n+1}^{\infty}\mu(k)\right)^2 \leq \left(\sum_{k=n+1}^{\infty}\mu(k)\sqrt{k}\right)\left(\sum_{k=n+1}^{\infty}\mu(k)/\sqrt{k}\right) \leq \mathsf{E}\left(\sqrt{Y_1}\right)\sum_{k=n+1}^{\infty}\mu(k)/\sqrt{k}\,.$$

Therefore,

$$\sum_{n=0}^{\infty}\bigl(1-F_\mu(n)\bigr)^2 \leq \mathsf{E}\left(\sqrt{Y_1}\right)\sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\mu(k)/\sqrt{k} = \left(\mathsf{E}\left(\sqrt{Y_1}\right)\right)^2,$$

which is finite. □

4. THE NON-LATTICE CASE

We now consider the case when $\mathsf{supp}(\mu) \subset [0\,,\infty)$, but there is no $\kappa > 0$ such that $\mathsf{supp}(\mu) \subset \kappa \cdot \mathbb{N}_0$. Again, denote $N = \sup\mathsf{supp}(\mu)$, and set $\mathsf{C} = [0\,,N]$ if $N < \infty$, resp. $\mathsf{C} = [0\,,\infty)$, if $N = \infty$. The transition probabilities of the reflected random walk are

$$p(x, B) = \mu(\{y : |x - y| \in B\})\,,$$

where $B \subset [0\,,\infty)$ is a Borel set. For the following, we need to specify in more detail the probability space on which we are working. This is the product space $(\Omega, \mathsf{Pr}) = (\mathsf{C}^{\mathbb{N}}, \mu^{\mathbb{N}})$, where Y_n is the n-th projection. It will be convenient to write X_n^x for the reflected walk starting at $x \geq 0$, so that $X_0^x = x$ and $X_{n+1}^x = |X_n^x - Y_{n+1}|$ as in the Introduction. We also write $X_{k,n}^x$ $(n \geq k)$ for the reflected walk starting at time k at x, so that $X_n^x = X_{0,n}^x$. Note that we always have

(4.1) $$|X_{k,n+1}^x - X_{k,n+1}^y| \leq |X_{k,n}^x - X_{k,n}^y|\,.$$

The following is due to [10], [13] and [14].

(4.2) Lemma. (a) *The reflected random walk with any starting point is absorbed after finitely many steps by the interval* C.

(b) *It is topologically irreducible on* C, *that is, for every* $x \in \mathsf{C}$ *and open set* $B \subset \mathsf{C}$, *there is* n *such that* $p^{(n)}(x, B) = \mathsf{Pr}[X_n \in B \mid X_0 = x] > 0\,.$

(c) *The measure* ν *on* C *given by*

$$\nu(dx) = \bigl(1 - F_\mu(x)\bigr)\,dx\,,$$

where dx *is Lebesgue measure, is an invariant measure for* $p(\cdot,\cdot)$.

From (4.1), one deduces the following.

(4.3) Lemma. $$\mathsf{Pr}[X_n^x \to \infty] \in \{0\,,1\}\,,$$
and the value is the same for each starting point x.

Proof. By (4.1), the event $[X_n^x \to \infty]$ is in the tail σ-algebra of the (Y_n). □

If $\mathsf{Pr}[X_n^x \to \infty] = 1$, then we call the reflected random walk *transient*.

We now state two important results that were proved in [14] in the case when $\mathsf{E}(Y_1) < \infty$, and in the general case in [2].

(4.4) Proposition. *In the non-lattice case, reflected random walk is* locally contractive: *for every bounded interval $I \subset \mathsf{C}$ and all $x, y \in \mathsf{C}$,*

$$\lim_{n \to \infty} \mathbf{1}_I(X_n^x) \, |X_n^x - X_n^y| = 0 \quad \text{almost surely.}$$

If $\Pr[X_n^x \to \infty] = 0$, then one even has

$$\lim_{n \to \infty} |X_n^x - X_n^y| = 0 \quad \text{almost surely.}$$

Of course, also reflected random walk started at time k is locally contractive for each $k \geq 0$. The proof of Proposition 4.4 is outlined in the Appendix.

For $\omega \in \Omega$, let $\mathsf{L}^x(\omega)$ be the set of (finite) accumulation points of the sequence $X_n^x(\omega)$. In the transient case, $\mathsf{L}^x(\cdot)$ is almost surely empty. Otherwise, contractivity implies that there is a set $\mathsf{L} \subset \mathsf{C}$, the *attractor* of the process, such that

(4.5) $$\Pr[\mathsf{L}^x(\cdot) = \mathsf{L} \ \text{ for all } \ x \in \mathsf{C}] = 1 \,.$$

Thus, for any $x \in \mathsf{C}$, every open set that interesects L is visited infinitely often by (X_n^x) with probability 1. In other words, the attractor L is *topologically recurrent*, so that it is justified to call the random walk *recurrent* when $\Pr[X_n^x \to \infty] = 0$.

Proposition 4.4 has the following important consequence, see the Appendix.

(4.6) Proposition. *In the recurrent case, $\mathsf{L} = \mathsf{C}$, and the measure ν defined in Lemma 4.2.c is the unique invariant Radon measure for $p(\cdot, \cdot)$ up to multiplication with constants.*

Thus, we have topological recurrence on the whole of C. Now, ν is invariant even in the transient case. If $\mathsf{E}(Y_1) < \infty$ then $\nu(\mathsf{C}) < \infty$, and we have recurrence by [14]. As in the lattice case, we want to extend this recurrence criterion. Here is the continuous analogue of Theorem 3.6 regarding the process of reflections of §2, with a rather similar proof.

(4.7) Theorem. *In the non-lattice case, the measure ρ on C, given by*

$$\rho(dx) = \left(\int_{[0,\infty)} \mu\big((x, \, x + y]\big) \, \mu(dy) \right) dx$$

is an invariant measure for the process of reflections (R_k). It is unique (up to multiplication with constants), if the measure ν is the unique invariant measure for the reflected random walk (up to multiplication with constants).

Proof. We use once more the convolution identity $\mathcal{A} * \mathcal{U} = \mathcal{U} * \mathcal{A} = \delta_0$, where $\mathcal{A} = \delta_0 - \mu$. For any Radon measure $\mathcal{M}$ on $\mathbb{R}$, we denote by $\check{\mathcal{M}}$ its reflection: $\check{\mathcal{M}}(B) = \mathcal{M}(-B)$ for Borel sets $B \subset \mathbb{R}$. We write again $H(x) = 1 - F_\mu(x)$ for the density of ν with respect to Lebesgue measure, and $h(x)$ for the density of ρ. Then

$$h(x) = \int_{[0,\infty)} \big(H(x) - H(x + y)\big) \, \mu(dy) = H(x) - \check{\mu} * H(x) = \check{\mathcal{A}} * H(x),$$

that is, $\rho = \check{A} * \nu$. Therefore with the same computations as above,

$$\nu(B) = \check{\mathcal{U}} * \rho(B) = \mathsf{E}_\rho \left(\sum_{j=0}^{\mathbf{r}(1)-1} \mathbf{1}_B(X_j) \right),$$

where of course we intend $\mathsf{E}_\rho = \int \mathsf{E}_w(\cdot)\, \rho(dw)$. Now invariance of ν for the reflected random walk implies invariance of ρ for the process of reflections precisely as in the proof of Theorem 3.6.

For proving uniqueness, let $\bar{\rho}$ be an invariant (Radon) measure for (R_k). Once we can prove that the convolution $\bar{\nu} = \check{\mathcal{U}} * \bar{\rho}$, restricted to $[0, \infty)$, defines a Radon measure (i.e., is finite on compact sets), we can proceed as before: $\bar{\nu}$ is invariant for (X_n), whence $\bar{\nu} = c \cdot \nu$ for some $c > 0$, and $\bar{\rho} = \check{A} * \bar{\nu} = c \cdot \check{A} * \nu = c \cdot \rho$.

If $N < \infty$ then $\bar{\rho}$ has finite mass, since it must be concentrated on $[0, N]$ by Lemma 4.2. Let $\mathcal{U}_N$ be the restriction of $\mathcal{U}$ to $[0, N]$. It is also a finite measure, and on $[0, N]$, we have $\check{\mathcal{U}} * \bar{\rho} = \check{\mathcal{U}}_N * \bar{\rho}$, which is again finite.

Now suppose that $N = \infty$. Let $a > 0$. Then $\mathsf{supp}(\mu)$ contains an element $M > a$. Choose b such that $a < b < M$. Now let f be a compactly supported, continuous function on $\mathbb{R}$, supported within $[0, \infty)$, such that $f \equiv 1$ on $[M-b, M+b]$. Then the convolution

$$\mu * \check{f}(x) = \int_{[0,\infty)} f(v - x)\, \mu(dv)$$

defines a continuous function.

If $x \in [0, a]$ then $f(v-x) = 1$ for all $v \in [M-b+x, M+b+x] \supset [M-b+a, M+b]$. Therefore

$$\mu * \check{f}(x) \geq \mu\big([M-b+a, M+b]\big) > 0 \quad \text{for each} \quad x \in [0, a].$$

Using this, the invariance of $\bar{\rho}$ for (R_k), the formula of Lemma 2.3, and Fubini's theorem, we now compute the finite number

$$\int_{[0,\infty)} f(x)\, \bar{\rho}(dx) = \int_{[0,\infty)} \int_{[0,\infty)} f(y)\, q(x, dy)\, \bar{\rho}(dx)$$

$$= \int_{[0,\infty)} \int_{[0,\infty)} \int_{[0,x)} f(y+w-x)\, \mathcal{U}(dw)\, \mu(dy)\, \bar{\rho}(dx)$$

$$= \int_{[0,\infty)} \int_{[0,\infty)} \int_{[w,\infty)} f(y+w-x)\, \bar{\rho}(dx)\, \mathcal{U}(dw)\, \mu(dy)$$

$$= \int_{[0,\infty)} \int_{[0,\infty)} \int_{[0,\infty)} f(y-x)\, [\delta_{-w} * \bar{\rho}](dx)\, \mathcal{U}(dw)\, \mu(dy)$$

$$= \int_{[0,\infty)} \int_{[0,\infty)} \mu * \check{f}(x)\, [\delta_{-w} * \bar{\rho}](dx)\, \mathcal{U}(dw)$$

$$= \int_{[0,\infty)} \mu * \check{f}(x)\, [\check{\mathcal{U}} * \bar{\rho}](dx)$$

$$\geq \mu\big([M-b+a, M+b]\big)\, \check{\mathcal{U}} * \bar{\rho}\big([0, a]\big).$$

Therefore $\check{\mathcal{U}} * \bar{\rho}\big([0, a]\big)$ is finite for each $a > 0$. $\qquad\qquad\square$

The following is now obtained precisely as in the lattice case.

(4.8) Corollary. *The invariant measure ρ of the process of reflections has finite mass if and only if*

$$(4.9) \qquad \int_{[0,\infty)} \big(1 - F_\mu(x)\big)^2 \, dx < \infty \,.$$

This holds, in particular, when $\mathsf{E}\big(\sqrt{Y_1}\big) = \int_{[0,\infty)} \sqrt{x}\,\mu(dx) < \infty \,.$

We now want to deduce recurrence of reflected random walk. This is not as straightforward as in the case of Markov chains with a denumerable state space.

(4.10) Proposition. *Let $J = (a,b) \subset \mathsf{C}$ be a bounded, open interval. Then, setting $J(\varepsilon) = (a + \varepsilon\,, b - \varepsilon)$,*

$$\mathsf{Pr}\left[\exists\ \varepsilon > 0\ :\ \sum_{n=0}^\infty \mathbf{1}_{J(\varepsilon)}(X_n^x) = \infty\right] \in \{0,1\}\,.$$

Proof. Each of the countably many events

$$\left[\lim_{n\to\infty} \mathbf{1}_{[0,m]}(X_{k,n}^{\bar{x}})\,|X_{k,n}^{\bar{x}} - X_{k,n}^{\bar{y}}| = 0\right] \subset \Omega\,,$$

where $\bar{x}, \bar{y} \in \mathsf{C}$ are rational and $k, m \in \mathbb{N}_0$, has probability 1. Let Ω_0 be their intersection, so that $\mathsf{Pr}(\Omega_0) = 1$. Consider the event

$$A_J^x = \Omega_0 \cap \bigcup_{0 < \varepsilon < (b-a)/2} B_{J(\varepsilon)}^x\,, \quad \text{where} \quad B_{J(\varepsilon)}^x = \left[\sum_{n=0}^\infty \mathbf{1}_{J(\varepsilon)}(X_n^x) = \infty\right]\,.$$

We claim that A_J^x does not depend on x. Let $y \in \mathsf{C}$. If $\omega \in A_J^x$ then there is $\varepsilon \in (0\,, \frac{b-a}{2})$ such that $\omega \in B_{J(\varepsilon)}^x$. There are rational numbers $\bar{x}, \bar{y} \in \mathsf{C}$ such that $|x - \bar{x}| < \varepsilon/4$ and $|y - \bar{y}| < \varepsilon/4$. Since $\omega \in \Omega_0$, we have

$$\mathbf{1}_J\big(X_n^{\bar{x}}(\omega)\big)\,\big|X_n^{\bar{x}}(\omega) - X_n^{\bar{y}}(\omega)\big| < \varepsilon/4$$

for all sufficiently large n. Since $|X_n^x - X_n^{\bar{x}}| \le |x - \bar{x}|$ and $|X_n^y - X_n^{\bar{y}}| \le |y - \bar{y}|$, we get that $X_n^y(\omega) \in J(\varepsilon/4)$ whenever $X_n^x(\omega) \in J(\varepsilon)$. Therefore, $A_J^x \subset A_J^y$, and exchanging the role of x and y, we see that $A_J = A_J^x$ is the same for all x.

Now, we claim that A_J is in the tail σ-algebra of the $(Y_n)_{n\ge 1}$. Let $\omega \in A_J$ and $\bar{\omega} \in \Omega$ such that for some $k \in \mathbb{N}$, $Y_n(\bar{\omega}) = Y_n(\omega)$ for $n > k$. Then clearly $\bar{\omega} \in \Omega_0$. Set $u = Y_k(\omega)$ and $v = Y_k(\bar{\omega})$. Then we have $X_n^x(\omega) = X_{k,n}^u(\omega)$ and $X_n^x(\bar{\omega}) = X_{k,n}^v(\omega)$ for all $n \ge k$. Now the same "$\varepsilon/4$"-argument as above implies that $\bar{\omega} \in A_J$.

Therefore $\mathsf{Pr}(A_J) \in \{0\,, 1\}$ by the 0-1 law of Kolmogorov. $\qquad\square$

(4.11) Theorem. *Suppose that the "quadratic tail" condition (4.9) holds. Then, for every starting point $x > 0$, the reflected random walk (X_n^x) is topologically recurrent: for every bounded, open interval $J \subset \mathsf{C}$,*

$$\mathsf{Pr}\left[\sum_{n=0}^\infty \mathbf{1}_J(X_n^x) = \infty\right] = 1\,.$$

If in addition $\mathsf{E}(Y_1) = \int_{[0,\infty)} x\,\mu(dx) < \infty$, *then* (X_n^x) *is positive recurrent, while it is null recurrent when* $\mathsf{E}(Y_1) = \infty$.

Proof. We write (R_n^x) for the process of reflections starting at $x \in \mathsf{C}$, and define

$$M_n^x = \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{J(\varepsilon)}(R_k^x) \quad \text{and} \quad M^x = \limsup_{n\to\infty} M_n^x,$$

where $\varepsilon > 0$ is chosen such that $J(\varepsilon)$ is non-empty. The measure ρ of Theorem 4.7 is supported by the whole of C, and $\rho(\mathsf{C}) < \infty$ by assumption. We have

$$\int_{[0,\infty)} \int_\Omega M_n^x \, d\mathsf{Pr}\, \rho(dx) = \rho\big(J(\varepsilon)\big)$$

Since $\rho(\mathsf{C}) < \infty$ by assumption and $0 \le M_n \le 1$, we may apply the "lim sup"-variant of the Lemma of Fatou to obtain

$$\int_{[0,\infty)} \int_\Omega M^x \, d\mathsf{Pr}\, \rho(dx) \ge \rho\big(J(\varepsilon)\big)\,.$$

Therefore there must be $x \in \mathsf{C}$ such that

$$\int_\Omega M^x \, d\mathsf{Pr} \ge \frac{3\rho\big(J(\varepsilon)\big)}{4\rho(\mathsf{C})}\,.$$

Consequently,

$$0 < \mathsf{Pr}[M_x \ge c] \le \mathsf{Pr}\left[\sum_{n=0}^\infty \mathbf{1}_{J(\varepsilon)}(X_n^x) = \infty\right], \quad \text{where} \quad c = \frac{\rho\big(J(\varepsilon)\big)}{2\rho(\mathsf{C})} > 0\,.$$

Proposition 4.10 now yields that

$$\mathsf{Pr}\left[\exists\, \varepsilon > 0 \,:\, \sum_{n=0}^\infty \mathbf{1}_{J(\varepsilon)}(X_n^x) = \infty\right] = 1\,,$$

and the result follows. $\square$

Note that we should be careful in stating that the process of reflections itself is topologically recurrent on C when it has a finite invariant Radon measure. Indeed, it is by no means clear that it inherits local contractivity, or even the property to be Fellerian, from reflected random walk.

5. General reflected random walk

In this section, we drop the restriction that the random variables Y_n are non-negative. Thus, the "ordinary" random walk $S_n = Y_1 + \cdots + Y_n$ may visit the positive as well as the negative half-axis. Again, μ will denote the distribution of each of the Y_n. In the lattice case, we suppose without loss of generality that $\mathsf{supp}(\mu) \subset \mathbb{Z}$, and that the group generated by $\mathsf{supp}(\mu)$ is the whole of $\mathbb{Z}$. In the non-lattice case, the closed group generated by $\mathsf{supp}(\mu)$ is $\mathbb{R}$.

We start with a simple observation ([4] has a more complicated proof).

(5.1) Lemma. *If μ is symmetric, then reflected random walk is (topologically) recurrent if and only if the random walk S_n is recurrent.*

Proof. If μ is symmetric, then also $|S_n|$ is a Markov chain. Indeed, for a Borel set $B \subset [0, \infty)$,

$$\Pr[\, |S_{n+1}| \in B \mid S_n = x] = \mu(-x + B) + \mu(-x - B) - \mu(-x)\, \delta_0(B)$$
$$= \Pr[\, |S_{n+1}| \in B \mid S_n = -x]\,,$$

and we see that $|S_n|$ has the same transition probabilities as the reflected random walk governed by μ. $\qquad\square$

Recall the classical result that when $\mathsf{E}(|Y_1|) < \infty$ and $\mathsf{E}(Y_1) = 0$ then S_n is recurrent; see CHUNG AND FUCHS [9]. So if μ is symmetric and has finite first moment then reflected random walk is recurrent.

In general, we should exclude that $S_n \to -\infty$, since in that case there are only finitely many reflections, and reflected random walk tends to $+\infty$ almost surely.

Let $Y_n^+ = \max\{Y_n, 0\}$ and $Y_n^- = \max\{-Y_n, 0\}$, so that $Y_n = Y_n^+ - Y_n^-$. The following is well-known.

(5.2) Lemma. *If* (a) $\mathsf{E}(Y_1^-) < \mathsf{E}(Y_1^+) \leq \infty$, *or if* (b) $0 < \mathsf{E}(Y_1^-) = \mathsf{E}(Y_1^+) < \infty$, *then* $\limsup S_n = \infty$ *almost surely, so that there are infinitely many reflections.*

We note that Proposition 4.4 is also valid here, since its proof (see the Appendix) does not require non-negativity of Y_n. Also, when the Y_n may assume both positive and negative values with positive probability, then the essential class, resp. classes, on which reflected random walk evolves is/are unbounded. In the non-lattice case this is $\mathsf{C} = [0, \infty)$, and X_n^x is locally contractive.

In the sequel, we assume that $\limsup S_n = \infty$ almost surely. Then the (non-strictly) ascending *ladder epochs*

$$\boldsymbol{\ell}(0) = 0\,, \quad \boldsymbol{\ell}(k+1) = \inf\{n > \boldsymbol{\ell}(k) : S_n \geq S_{\boldsymbol{\ell}(k)}\}$$

are all almost surely finite, and the random variables $\boldsymbol{\ell}(k+1) - \boldsymbol{\ell}(k)$ are i.i.d. We can consider the *embedded random walk* $S_{\boldsymbol{\ell}(k)}$, $k \geq 0$, which tends to ∞ almost surely. Its increments $\overline{Y}_k = S_{\boldsymbol{\ell}(k)} - S_{\boldsymbol{\ell}(k-1)}$, $k \geq 1$, are i.i.d. non-negative random variables with distribution denoted $\overline{\mu}$. Furthermore, if $\overline{X}_k^x$ denotes the reflected random walk associated with the sequence $(\overline{Y}_k)$, while X_n^x is our original reflected random walk associated with (Y_n), then

$$\overline{X}_k^x = X_{\boldsymbol{\ell}(k)}^x\,,$$

since no reflection can occur between times $\boldsymbol{\ell}(k)$ and $\boldsymbol{\ell}(k+1)$.

(5.3) Lemma. [2] *The embedded reflected random walk $\overline{X}_k^x$ is recurrent if and only the original reflected random walk is recurrent.*

Proof. Since both processes are locally contractive, each of the two processes is transient if and only if it tends to $+\infty$ almost surely: in the lattice case this is clear, and in the non-lattice case it follows from local contractivity. If $\lim_n X_n^x = \infty$ then clearly also $\lim_k X_{\boldsymbol{\ell}(k)}^x = \infty$ a.s. Conversely, suppose that $\lim_k \overline{X}_k^x \to \infty$ a.s. If $\boldsymbol{\ell}(k) \leq n < \boldsymbol{\ell}(k+1)$ then $X_n^x \geq X_{\boldsymbol{\ell}(k)}^x$. (Here, k is random, depending on n and $\omega \in \Omega$, and when $n \to \infty$ then $k \to \infty$ a.s.) Therefore, also $\lim_n X_n^x = \infty$ a.s. $\qquad\square$

As long as $\limsup S_n = \infty$, we can consider the reflection times as in (2.2) for the case of non-negative Y_n. The observation that there is no reflection between times $\boldsymbol{\ell}(k)$ and $\boldsymbol{\ell}(k+1)$ yields the following.

(5.4) Lemma. *The reflection times for (X_n^x) and $(\overline{X}_k^x)$ are the same, so that reflected random walk and embedded reflected random walk have the same process of reflections. In particular, if the latter has a finite invariant measure, resp., if it is non-transient, then (X_n^x) is (topologically) recurrent on its essential class(es).*

We can now deduce the following.

(5.5) Theorem. *Reflected random walk (X_n^x) is (topologically) recurrent on its essential class(es), if*

(a) $\mathsf{E}(Y_1^-) < \mathsf{E}(Y_1^+)$ *and* $\mathsf{E}\left(\sqrt{Y_1^+}\right) < \infty$, *or if*

(b) $0 < \mathsf{E}(Y_1^-) = \mathsf{E}(Y_1^+)$ *and* $\mathsf{E}\left(\sqrt{Y_1^+}^3\right) < \infty$.

Proof. We show that in each case the assumptions imply that $\mathsf{E}\left(\sqrt{\overline{Y}_1}\right) < \infty$. Then we can apply Lemma 3.14, resp. Corollary 4.8 to deduce recurrence of $(\overline{X}_k^x)$. This in turn yields recurrence of (X_n^x) by Lemma 5.4.

(a) Under the first set of assumptions,

$$\mathsf{E}\left(\sqrt{\overline{Y}_1}\right) = \mathsf{E}\left(\sqrt{Y_1 + \ldots + Y_{\boldsymbol{\ell}(1)}}\right) \leq \mathsf{E}\left(\sqrt{Y_1^+ + \ldots + Y_{\boldsymbol{\ell}(1)}^+}\right)$$
$$\leq \mathsf{E}\left(\sqrt{Y_1^+} + \ldots + \sqrt{Y_{\boldsymbol{\ell}(1)}^+}\right) = \mathsf{E}\left(\sqrt{Y_1^+}\right) \cdot \mathsf{E}(\boldsymbol{\ell}(1))$$

by Wald's identity. Thus, we now are left with proving $\mathsf{E}(\boldsymbol{\ell}(1)) < \infty$. If $\mathsf{E}(Y_1^+) < \infty$, then $\mathsf{E}(|Y_1|) < \infty$ and $\mathsf{E}(Y_1) > 0$ by assumption, and in this case it is well known that $\mathsf{E}(\boldsymbol{\ell}(1)) < \infty$; see e.g. [10, Thm. 2 in §XII.2, p. 396-397]. If $\mathsf{E}(Y_1^+) = \infty$ then there is $M > 0$ such that $Y_n^{(M)} = \min\{Y_n, M\}$ (which has finite first moment) satisfies $\mathsf{E}(Y_n^{(M)}) = \mathsf{E}(Y_1^{(M)}) > 0$. The first increasing ladder epoch $\boldsymbol{\ell}^{(M)}(1)$ associated with $S_n^{(M)} = Y_1^{(M)} + \ldots + Y_n^{(M)}$ has finite expectation by what we just said, and $\boldsymbol{\ell}(1) \leq \boldsymbol{\ell}^{(M)}(1)$. Thus, $\boldsymbol{\ell}(1)$ is integrable.

(b) If the Y_n are centered, non-zero and $\mathsf{E}\left((Y_1^+)^{1+a}\right) < \infty$, where $a > 0$, then $\mathsf{E}\left((\overline{Y}_1)^a\right) < \infty$, as was shown by CHOW AND LAI [8]. In our case, $a = 1/2$. $\qquad\square$

In conclusion, we discuss sharpness of the sufficient recurrence conditions $\mathsf{E}\left(\sqrt{Y_1}^3\right) < \infty$ in the centered case, resp $\mathsf{E}(\sqrt{Y_1}) < \infty$ in the case when $Y_1 \geq 0$.

(5.6) Example. Define a symmetric probability measure μ on $\mathbb{Z}$ by

$$\mu(0) = 0, \quad \mu(k) = \mu(-k) = c/k^{1+a} \quad (k \neq 0),$$

where $a > 0$ and c is the proper normalizing constant. Then it is known that the associated symmetric random walk S_n on $\mathbb{Z}$ is recurrent if and only if $a \geq 1$, see SPITZER [17, p. 87]. By Lemma 5.1, the associated reflected random walk is also recurrent, but when $1 \leq a \leq 3/2$ then condition (b) of Theorem 5.5 does not hold.

Nevertheless, we can also show that in general, the sufficient condition $\mathsf{E}\left(\sqrt{\overline{Y}_1}\right) < \infty$ for recurrence of reflected random walk with non-negative increments $\overline{Y}_n$ is very close to being sharp. (We write $\overline{Y}_n$ because we shall represent this as an embedded random walk in the next example.)

(5.7) Proposition. *Let μ_0 be a probability measure on $\mathbb{N}_0$ such that $\mu_0(n) \geq \mu_0(n+1)$ for all $n \geq 0$ and*
$$\mu_0(n) \sim c \, (\log n)^b / n^{3/2} \, , \quad as \; n \to \infty \, ,$$
where $b > 1/2$ and $c > 0$. Then the associated reflected random walk on $\mathbb{N}_0$ is transient.

Note that μ_0 has finite moment of order $\frac{1}{2} - \varepsilon$ for every $\varepsilon > 0$, while the moment of order $\frac{1}{2}$ is infinite.

The proof needs some preparation. Let (Y_n) be i.i.d. random variables with values in $\mathbb{Z}$ that have finite first moment and are non-constant and centered, and let μ be their common distribution. The first *strictly ascending* and *strictly descending* ladder epochs of the random walk $S_n = Y_1 + \ldots + Y_n$ are
$$\mathbf{t}_+(1) = \inf\{n > 0 : S_n > 0\} \quad \text{and} \quad \mathbf{t}_-(1) = \inf\{n > 0 : S_n < 0\} \, ,$$
respectively. They are almost surely finite. Let μ_+ be the distribution of $S_{\mathbf{t}_+(1)}$ and μ_- the distribution of $S_{\mathbf{t}_-(1)}$, and – as above – $\overline{\mu}$ the distribution of $\overline{Y}_1 = S_{\ell(1)}$. We denote the characteristic function associated with any probability measure σ on $\mathbb{R}$ by $\widehat{\sigma}(t)$, $t \in \mathbb{R}$. Then, following FELLER [10, (3.11) in §XII.3], *Wiener-Hopf-factorization* tells us that

(5.8)
$$\mu = \overline{\mu} + \mu_- - \overline{\mu} * \mu_- \quad \text{and} \quad \overline{\mu} = u \cdot \delta_0 + (1 - u) \cdot \mu_+ \, ,$$
$$\text{where} \quad u = \overline{\mu}(0) = \sum_{n=1}^{\infty} \mathsf{Pr}[S_1 < 0 \, , \ldots, S_{n-1} < 0 \, , \; S_n = 0] < 1 \, .$$

(Recall that $*$ is convolution.)

(5.9) Lemma. *Let μ_0 be a probability measure on $\mathbb{N}_0$ such that $\mu_0(n) \geq \mu_0(n+1)$ for all $n \geq 0$. Then there is a symmetric probability measure μ on $\mathbb{Z}$ such that that the associated first (non-strictly) ascending ladder random variable has distribution μ_0.*

Proof. We decompose $\mu_0 = \mu_0(0) \cdot \delta_0 + \big(1 - \mu_0(0)\big) \cdot \mu_\times$, where $\mu_\times$ is supported by $\mathbb{N}$ (i.e., $\mu_\times(0) = 0$). If μ_0 is the law of the first strictly ascending ladder random variable associated with some symmetric measure μ, then by (5.8) we must have $\mu_- = \check{\mu}_\times$, the reflection of $\mu_\times$ at 0, and

(5.10) $\mu = \mu_0 + \check{\mu}_\times - \mu_0 * \check{\mu}_\times = \mu_0(0) \cdot \delta_0 + \big(1 - \mu_0(0)\big) \cdot (\mu_\times + \check{\mu}_\times - \mu_\times * \check{\mu}_\times) \, .$

We *define* μ in this way. The monotonicity assumption on μ_0 implies that μ is a probability measure: indeed, it is straightforward to show that $\mu(k) \geq 0$ for each $k \in \mathbb{Z}$.

The measure μ of (5.10) is non-degenerate and symmetric. If it induces a recurrent random walk (S_n), then the ascending and descending ladder epochs are a.s. finite. If (S_n) is transient, then $|S_n| \to \infty$ almost surely, but it cannot be $\mathsf{Pr}[S_n \to \infty] > 0$ since in that case this probaility had to be 1 Kolmogorov's 0-1-law, while symmetry would yield $\mathsf{Pr}[S_n \to -\infty] = \mathsf{Pr}[S_n \to \infty] \leq 1/2$. Therefore $\liminf S_n = -\infty$ and $\limsup S_n = +\infty$

almost surely, a well-known fact, see e.g. [10, Thm. 1 in §XII.2, p. 395]. Consequently, the ascending and descending ladder epochs are again a.s. finite. Therefore the probability measures μ_+ and $\mu_- = \check{\mu}_+$ (the laws of $S_{\mathbf{t}_{\pm}(1)}$) are well defined. By the uniqueness theorem of Wiener-Hopf-factorization [10, Thm. 1 in §XII.3, p. 401], it follows that $\mu_- = \check{\mu}_\times$ and that the distribution of the first (non-strictly) ascending ladder random variable is $\overline{\mu} = \mu_0$. □

Proof of Proposition 5.7. Let μ be the symmetric measure associated with μ_0 according to (5.10) in Lemma 5.9. Then its characteristic function $\widehat{\mu}(t)$, given by (5.8), is non-negative real. A well-known criterion says that the random walk S_n associated with μ is transient if and only if (the real part of) $1/\big(1 - \widehat{\mu}(t)\big)$ is integrable in a neighbourhood of 0. Returning to $\overline{\mu}$, it is a standard exercise (see [10, Ex. 12 in Ch. XVII, Section 12]) to show that there is $A \in \mathbb{C}$, $A \neq 0$ such that its characteristic function satisfies

$$\widehat{\overline{\mu}}(t) = 1 + A\sqrt{t}\,(\log t)^b\,\big(1 + o(t)\big) \quad \text{as } t \to 0\,.$$

By (5.8),

$$1 - \widehat{\mu}(t) = (1 - u)\big(1 - \widehat{\mu_+}(t)\big)\big(1 - \widehat{\mu_-}(t)\big)\,.$$

We deduce

$$\widehat{\mu}(t) = 1 + \big(1 - \mu_0(0)\big)\,|A|^2\,t\,(\log t)^{2b}\,\big(1 + o(t)\big) \quad \text{as } t \to 0\,.$$

The function $1/\big(1 - \widehat{\mu}(t)\big)$ is integrable near 0. By Lemma 5.1, the associated reflected random walk is transient. But then also the embedded reflected random walk associated with $S_{\boldsymbol{\ell}(n)}$ is transient by Lemma 5.3. This is the reflected random walk governed by $\overline{\mu}$. □

6. Appendix: local contractivity

Here, we come back to propositions 4.4 and 4.6. They arise as special cases of two main results in the PhD thesis of Benda [2] and of the contents of the two papers [3] and [4], which were accepted for publication but remained unpublished. For this reason, we give an outline, resp. published references for their proofs. In [3], this is placed in the following more general context. Let (X, d) be a proper metric space (i.e., closed balls are compact), and let $\mathfrak{G}$ be the monoid of all continuous mappings $\mathsf{X} \to \mathsf{X}$. It carries the topology of uniform convergence on compact sets. Now let $\widetilde{\mu}$ be a regular probability measure on $\mathfrak{G}$, and let $(F_n)_{n \geq 1}$ be a sequence of i.i.d. $\mathfrak{G}$-valued random variables (functions) with common distribution $\widetilde{\mu}$. The measure $\widetilde{\mu}$ gives rise to the stochastic iterated function system (SFS) X_n^x defined by

$$(6.1) \qquad X_0^x = x \in \mathsf{X}, \quad \text{and} \quad X_n^x = F_n(X_{n-1}^x), \quad n \geq 1\,.$$

In the setting of the above Sections 2–4, we have $\mathsf{X} = [0, \infty)$ with the standard distance, and $F_n(x) = |x - Y_n|$, so that the measure $\widetilde{\mu}$ is the image of the distribution μ of the Y_n in §2 under the mapping $[0, \infty) \to \mathfrak{G}$, $y \mapsto g_y$, where $g_y(x) = |x - y|$.

(6.2) Definition. The SFS is called *locally contractive,* if for all $x \in \mathsf{X}$ and every compact $K \subset \mathsf{X}$,

$$\mathbf{1}_K(X_n^x) \cdot \sup_{y \in K} d(X_n^x, X_n^y) \to 0 \quad \text{almost surely, as } n \to \infty\,.$$

This notion was first introduced by BABILLOT, BOUGEROL AND ELIE [1] and was later exploited systematically by BENDA, who (in personal comunication) also gives credit to unpublished work of his late PhD advisor KELLERER, compare with the posthumous publication [12].

Using Kolomogorov's 0-1 law (and properness of X), one gets a general variant of Lemma 4.3.

(6.3) Lemma. *For a locally contractive SFS of contractions,*

$$either \quad \mathsf{Pr}[d(X_n^x, x) \to \infty] = 0 \quad for \ all \ x \in \mathsf{X},$$
$$or \quad \mathsf{Pr}[d(X_n^x, x) \to \infty] = 1 \quad for \ all \ x \in \mathsf{X}.$$

Proof. Let $B(r)$, $r \in \mathbb{N}$ be the open balls in X with radius r and fixed center $o \in \mathsf{X}$. It has compact closure by properness of X. Consider

(6.4) $$X_{m,n}^x = F_n \circ F_{n-1} \circ \ldots \circ F_{m+1}(x)$$

for $n > m$, so that $X_n^x = X_{0,n}^x$. Then local contractivity implies that for each $x \in \mathsf{X}$, we have $\mathsf{Pr}(\Omega_0) = 1$ for the event Ω_0 consisting of all $\omega \in \Omega$ with

(6.5) $$\lim_{n \to \infty} \mathbf{1}_{B(r)}\big(X_n^x(\omega)\big) \cdot \sup_{y \in B(r)} d\big(X_{m,n}^x(\omega), X_{m,n}^y(\omega)\big) = 0 \quad \text{for each } r \in \mathbb{N}, \ m \in \mathbb{N}_0.$$

Clearly, Ω_0 is invariant with respect to the shift of the sequence (F_n).

Now let $\omega \in \Omega_0$ be such that the sequence $\big(X_n^x(\omega)\big)_{n \geq 0}$ accumulates at some $w \in \mathsf{X}$. Fix m and set $v = X_m^x(\omega)$. Then also $\big(X_{m,n}^v(\omega)\big)_{n \geq m}$ accumulates at w. Now let $y \in \mathsf{X}$ be arbitrary. Then there is r such that $v, w, y \in B(r)$. Therefore also $\big(X_{m,n}^y(\omega)\big)_{n \geq m}$ accumulates at w. In particular, the fact that $\big(X_n^x(\omega)\big)_{n \geq 0}$ accumulates at some point does not depend on the initial trajectory, i.e., on the specific realization of $F_1, \ldots, F_m$. We infer that the set

$$\big\{\omega \in \Omega_0 : \big(X_n^x(\omega)\big)_{n \geq 0} \text{ accumulates in } \mathsf{X}\big\}$$

is a tail event of $(F_n)_{n \geq 1}$. On its complement in Ω_0, we have $d(X_n^x, x) \to \infty$. $\qquad \square$

If $d(X_n^x, x) \to \infty$ almost surely, then we call the SFS *transient*. What has been said about the attractor in (4.5) for reflected random walk is true in general. For $\omega \in \Omega$, let $\mathsf{L}^x(\omega)$ be the set of accumulation points of $\big(X_n^x(\omega)\big)$ in X. A straightforward extension of the argument used in the last proof (using again properness of X) yields the following.

(6.6) Lemma. *For any non-transient, locally contractive SFS, there is a set* $\mathsf{L} \subset \mathsf{X}$ *– the attractor – such that*

$$\mathsf{Pr}[\mathsf{L}^x(\cdot) = \mathsf{L} \ \text{for all} \ x \in \mathsf{C}] = 1,$$

Thus, (X_n^x) is *(topologically) recurrent* on L when $\mathsf{Pr}[d(X_n^x, x) \to \infty] = 0$.

(6.7) Proposition. *For a recurrent locally contractive SFS, there is a unique invariant Radon measure ν on X up to multiplication with constants, and* $\mathsf{supp}(\nu) = \mathsf{L}$.

This is contained in [2] and [3]. The proof of the existence of such a measure supported by L is rather straightforward, compare with the old survey by Foguel [11]. (One first

constructs an excessive measure supported by L via a ratio limit argument, an then uses recurrence to obtain that it has to be invariant.) For a proof of uniqueness that is available in print, see BROFFERIO [7, Thm. 3], who considers only SFS of affine mappings, but the argument carries over to general locally contractive SFS without changes.

Let us now consider a more specific class of SFS: within $\mathfrak{G}$, we consider the closed submonoid $\mathfrak{L}$ of all *contractions* of X, i.e., mappings $f : \mathsf{X} \to \mathsf{X}$ with Lipschitz constant $L(f) \le 1$. We suppose that the probability measure $\widetilde{\mu}$ that governs the SFS is supported by $\mathfrak{L}$, that is, each random function F_n of (6.1) satisfies $L(F_n) \le 1$. In this case, one does not need local contractivity in order to obtain Lemma 6.3; this follows directly from properness of X and the inequality

$$d(X_n^x, X_n^y) \le d(x, y).$$

Let $\mathfrak{S}(\widetilde{\mu})$ be the closed sub-semigroup of $\mathfrak{L}$ generated by $\mathsf{supp}(\widetilde{\mu})$. The following key result of [2] is inspired by [13, Thm. 2.2], where reflected random walk with $\mathsf{E}(Y_n) < \infty$ is studied.

(6.8) Proposition. *If* (i) *the SFS of contractions is non-transient, and* (ii) *the semigroup* $\mathfrak{S}(\widetilde{\mu}) \subset \mathfrak{L}$ *contains a constant function, then*

$$D_n(x, y) = d(X_n^x, X_n^y) \to 0 \quad \text{almost surely, as } n \to \infty.$$

Proof. Since $D_{n+1}(x, y) \le D_n(x, y)$, the limit $D_\infty(x, y) = \lim_n D_n(x, y)$ exists and is between 0 and $d(x, y)$. We set $w(x, y) = \mathsf{E}\big(D_\infty(x, y)\big)$. First of all, we claim that

(6.9) $$\lim_{m \to \infty} w(X_m^x, X_m^y) = D_\infty(x, y) \quad \text{almost surely.}$$

To see this, consider $X_{m,n}^x$ as in (6.4). Then $D_{m,\infty}(x, y) = \lim_n d(X_{m,n}^x, X_{m,n}^y)$ has the same distribution as $D_\infty(x, y)$, whence $\mathsf{E}\big(D_{m,\infty}(x, y)\big) = w(x, y)$. Therefore, we also have

$$\mathsf{E}\big(D_{m,\infty}(X_m^x, X_m^y) \mid F_1, \dots, F_m\big) = w(X_m^x, X_m^y).$$

On the other hand, $D_{m,\infty}(X_m^x, X_m^y) = D_\infty(x, y)$, and the bounded martingale

$$\Big(\mathsf{E}\big(D_\infty(x, y) | F_1, \dots, F_m\big)\Big)_{m \ge 1}$$

converges almost surely to $D_\infty(x, y)$. The proposed statement (6.9) follows.

Now let $\varepsilon > 0$ be arbitrary, and fix $x, y \in \mathsf{X}$. We have to show that the event $A = [D_\infty(x, y) \ge \varepsilon]$ has probability 0.
(i) By non-transience,

$$\Pr\left(\bigcup_{r \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcup_{n \ge m} [X_n^x, \, X_n^y \in \mathsf{B}(r)]\right) = 1.$$

On A, we have $D_n(x, y) \ge \varepsilon$ for all n. Therefore we need to show that $\Pr(A_r) = 0$ for each $r \in \mathbb{N}$, where

$$A_r = \bigcap_{m \in \mathbb{N}} \bigcup_{n \ge m} [X_n^x, \, X_n^y \in \mathsf{B}(r), \, D_n(x, y) \ge \varepsilon].$$

(ii) By the second hypothesis, there is $x_0 \in X$ which can be approximated uniformly on compact sets by functions of the form $f_k \circ \ldots \circ f_1$, where $f_j \in \mathsf{supp}(\widetilde{\mu})$. Therefore, given r there is $k \in \mathbb{N}$ such that

$$\Pr(C_{k,r}) > 0, \quad \text{where} \quad C_{k,r} = \left[\sup_{u \in \mathsf{B}(r)} d(X_k^u, x_0) \leq \varepsilon/4 \right].$$

On $C_{k,r}$ we have $D_\infty(u,v) \leq D_k(u,v) \leq \varepsilon/2$ for all $u,v \in \mathsf{B}(r)$. Therefore, setting $\delta = \Pr(C_{k,r}) \cdot (\varepsilon/2)$, we have for all $u,v \in \mathsf{B}(r)$ with $d(u,v) \geq \varepsilon$ that

$$\begin{aligned} w(u,v) &= \mathsf{E}\big(\mathbf{1}_{C_{k,r}} D_\infty(u,v)\big) + \mathsf{E}\big(\mathbf{1}_{\mathsf{X} \setminus C_{k,r}} D_\infty(u,v)\big) \\ &\leq \Pr(C_{k,r}) \cdot (\varepsilon/2) + \big(1 - \Pr(C_{k,r})\big) \cdot d(u,v) \leq d(u,v) - \delta. \end{aligned}$$

We conclude that on A_r, there is a (random) sequence (n_ℓ) such that

$$w(X_{n_\ell}^x, X_{n_\ell}^y) \leq D_{n_\ell}(x,y) - \delta.$$

Passing to the limit on both sides, we see that (6.9) is violated on A_r, since $\delta > 0$. Therefore $\Pr(A_r) = 0$ for each r. $\qquad\square$

(6.10) Corollary. *If the semigroup $\mathfrak{S}(\widetilde{\mu}) \subset \mathfrak{L}$ contains a constant function, then the SFS is locally contractive.*

Proof. In the transient case, X_n^x can visit any compact K only finitely often, whence $\mathbf{1}_K(X_n^x) \to 0$ a.s. In the non-transient case, we use the fact that by properness, X has a dense, countable subset Y. Proposition 6.8 implies that with probability 1, we have $\lim_n D_n(x,w) = 0$ for all $w \in Y$. If $K \subset \mathsf{X}$ is compact and $\varepsilon > 0$ then there is a finite $W \subset Y$ such that $d(y,W) < \varepsilon$ for every $y \in K$. Therefore

$$\sup_{y \in K} D_n(x,y) \leq \underbrace{\max_{w \in W} D_n(x,w)}_{\to 0 \text{ a.s.}} + \varepsilon,$$

since $D_n(x,y) \leq D_n(x,w) + D_n(w,y) \leq D_n(x,w) + d(w,y)$. $\qquad\square$

Proof of Proposition 4.4. Reflected random walk is an SFS of contractions, since $L(g_y) = 1$ for the function $g_y(x) = |x - y|$. [14, Prop. 2] shows that the constant function $x \mapsto 0$ is contained in the semigroup $\mathfrak{S}(\widetilde{\mu})$, where μ is the law of the increments Y_n and $\widetilde{\mu}$ its image in the semigroup $\mathfrak{L}$ of contractions of $\mathsf{X} = [0, \infty)$ under the mapping $y \mapsto g_y$, $g_y(x) = |x - y|$. Note that this statement and its proof in [14] are completely deterministic, regarding topological properties of the set $\mathsf{supp}(\mu) \subset [0, \infty)$, and do not rely on any moment condition. $\qquad\square$

Proof of Proposition 4.6. If reflected random walk is recurrent, then we know from Proposition 6.7 that there is a unique invariant Radon measure up to multiplication with constants, and its support is the attractor L. On the other hand, we already have the invariant measure ν given in Lemma 4.2.c, and its support is C. $\qquad\square$

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