Self-similar solutions of the p-Laplace heat equation: the case when p>2

Marie-Françoise Bidaut-Véron

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Marie Françoise Bidaut-Véron*

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Abstract

We study the self-similar solutions of the equation

$$u_t - div(|\nabla u|^{p-2} \nabla u) = 0,$$

in $\mathbb{R}^N$, when $p > 2$. We make a complete study of the existence and possible uniqueness of solutions of the form

$$u(x,t) = (\pm t)^{-\alpha/\beta} w((\pm t)^{-1/\beta} |x|)$$

of any sign, regular or singular at $x = 0$. Among them we find solutions with an expanding compact support or a shrinking hole (for $t > 0$), or a spreading compact support or a focussing hole (for $t < 0$). When $t < 0$, we show the existence of positive solutions oscillating around the particular solution $U(x,t) = C_{N,p}(|x|^p / (-t))^{1/(p-2)}$.

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*Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Faculté des Sciences, Parc Grandmont, 37200 Tours, France. e-mail:veronmf@univ-tours.fr

1 Introduction and main results

Here we consider the self-similar solutions of the degenerate heat equation involving the p-Laplace operator

$$u_t - div(|\nabla u|^{p-2}\nabla u) = 0. \qquad (\mathbf{E}_u)$$

in $\mathbb{R}^N$, with $p > 2$. This study is the continuation of the work started in [4], relative to the case $p < 2$. It can be read independently. We set

$$\gamma = \frac{p}{p-2}, \qquad \eta = \frac{N-p}{p-1}, \qquad (1.1)$$

thus $\gamma > 1$, $\eta < N$,

$$\frac{N+\gamma}{p-1} = \eta + \gamma = \frac{N-\eta}{p-2}. \qquad (1.2)$$

If u is a solution, then for any $\alpha, \beta \in \mathbb{R}$, $u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^\beta t)$ is a solution of $(\mathbf{E}_u)$ if and only if

$$\beta = \alpha(p-2) + p = (p-2)(\alpha+\gamma); \qquad (1.3)$$

notice that $\beta > 0 \Longleftrightarrow \alpha > -\gamma$. Given $\alpha \in \mathbb{R}$ such that $\alpha \neq -\gamma$, we search self-similar solutions, radially symmetric in x, of the form:

$$u = u(x,t) = (\varepsilon\beta t)^{-\alpha/\beta} w(r), \qquad r = (\varepsilon\beta t)^{-1/\beta}|x|, \qquad (1.4)$$

where $\varepsilon = \pm 1$. By translation, for any real T, we obtain solutions defined for any $t > T$ when $\varepsilon\beta > 0$, or $t < T$ when $\varepsilon\beta < 0$. We are lead to the equation

$$\left(|w'|^{p-2}w'\right)' + \frac{N-1}{r}|w'|^{p-2}w' + \varepsilon(rw' + \alpha w) = 0 \qquad \text{in } (0,\infty). \qquad (\mathbf{E}_w)$$

Our purpose is to give a complete description of all the solutions, with constant or changing sign. Equation $(\mathbf{E}_w)$ is very interesting, because it is singular at any zero of w', since $p > 2$, implying a nonuniqueness phenomena.

For example, concerning the constant sign solutions near the origin, it can happen that

$$\lim_{r\to 0} w = a \neq 0, \qquad \lim_{r\to 0} w' = 0,$$

we will say that w is *regular*, or

$$\lim_{r\to 0} w = \lim_{r\to 0} w' = 0,$$

we say that w is *flat*. Or different kinds of singularities may occur, either at the level of w :

$$\lim_{r\to 0} w = \infty,$$

or at the level of the gradient:

$$\begin{aligned}
&\lim_{r\to 0} w = a \in \mathbb{R}, \qquad \lim_{r\to 0} w' = \pm\infty, \qquad \text{when } p > N > 1,\\
&\lim_{r\to 0} w = a \in \mathbb{R}, \qquad \lim_{r\to 0} w' = b \neq 0 \qquad \text{when } p > N = 1.
\end{aligned}$$

We first show that any local solution w of $(\mathbf{E}_w)$ can be defined on $(0,\infty)$, thus any solution u of equation $(\mathbf{E}_u)$ associated to w by (1.4) is defined on $\mathbb{R}^N\backslash\{0\}\times(0,\pm\infty)$. Then we prove the existence of regular solutions, flat ones, and of all singular solutions mentioned above.

Moreover, for $\varepsilon = 1$, there exist solutions w with a compact support $(0,\bar{r})$; then $u \equiv 0$ on the set

$$D = \left\{(x,t) : x \in \mathbb{R}^N, \quad \beta t > 0, \quad |x| > (\beta t)^{1/\beta}\bar{r}\right\}.$$

For $\varepsilon = -1$, there exist solutions with a hole: $w(r) = 0 \Longleftrightarrow r \in (0,\bar{r})$. Then $u \equiv 0$ on the set

$$H = \left\{(x,t) : x \in \mathbb{R}^N, \quad \beta t < 0, \quad |x| < (-\beta t)^{1/\beta}\bar{r}\right\}.$$

The free boundary is of parabolic type for $\beta > 0$, of hyperbolic type for $\beta < 0$. This leads to four types of solutions, and we prove their existence:

• If $t > 0$, with $\varepsilon = 1, \beta > 0$, we say that u has an *expanding support*; the support increases from $\{0\}$ as t increases from 0.

• If $t > 0$, with $\varepsilon = -1, \beta < 0$, we say that u has a *shrinking hole*: the hole decreases from infinity as t increases from 0;

• If $t < 0$, with $\varepsilon = 1, \beta < 0$, we say that u has a *spreading support*: the support increases to be infinite as t increases to 0.

• If $t < 0$, with $\varepsilon = -1, \beta > 0$, we say that u has a *focussing hole*: the hole disappears as t increases to 0.

Up to our knowledge, some of them seem completely new, as for example the solutions with a shrinking hole or a spreading support. In particular we find again and improve some results of [8] concerning the existence of focussing type solutions.

Finally for $t < 0$ we also show the existence of positive solutions turning around the fundamental solution U given at (1.8) with a kind of periodicity, and also the existence of changing sign solutions doubly oscillating in $|x|$ near 0 and infinity.

As in [4] we reduce the problem to dynamical systems.

When $\varepsilon = -1$, a critical negative value of α is involved:

$$\alpha^* = -\gamma + \frac{\gamma(N+\gamma)}{(p-1)(N+2\gamma)}. \tag{1.5}$$

1.1 Explicit solutions

Obviously if w is a solution of $(\mathbf{E}_w)$, $-w$ is also a solution. Some particular solutions are well-known.

The solution U. For any α such that $\varepsilon(\alpha+\gamma) < 0$, that means $\varepsilon\beta < 0$, there exist flat solutions of $(\mathbf{E}_w)$, given by

$$w(r) = \pm \ell r^{\gamma}, \tag{1.6}$$

where

$$\ell = \left(\frac{|\alpha+\gamma|}{\gamma^{p-1}(\gamma+N)}\right)^{1/(p-2)} > 0. \tag{1.7}$$

They correspond to a unique solution of $(\mathbf{E}_u)$ called U, defined for $t < 0$, such that $U(0,t) = 0$, flat, blowing up at $t = 0$ for fixed $x \neq 0$:

$$U(x,t) = C\left(\frac{|x|^p}{-t}\right)^{1/(p-2)}, \qquad C = ((p-2)\gamma^{p-1}(\gamma+N))^{1/(2-p)}. \tag{1.8}$$

The case $\alpha = N$. Then $\beta = \beta_N = N(p-2)+p > 0$, and the equation has a first integral

$$w + \varepsilon r^{-1}\left|w'\right|^{p-2} w' = Cr^{-N}. \tag{1.9}$$

All the solutions corresponding to $C = 0$ are given by

$$\begin{aligned} w = w_{K,\varepsilon}(r) &= \pm\left(K - \varepsilon\gamma^{-1} r^{p'}\right)_+^{(p-1)/(p-2)}, \qquad K \in \mathbb{R}, \\ u = \pm u_{K,\varepsilon}(x,t) &= \pm(\varepsilon\beta_N t)^{-N/\beta_N}\left(K - \varepsilon\gamma^{-1}(\varepsilon\beta_N t)^{-p'/\beta_N}|x|^{p'}\right)_+^{(p-1)/(p-2)}. \end{aligned} \tag{1.10}$$

For $\varepsilon = 1$, $K > 0$, they are defined for $t > 0$, called *Barenblatt solutions,* regular with a compact support. Given $c > 0$, the function $u_{K,1}$, defined on $\mathbb{R}^N \times (0,\infty)$, is the unique solution of equation $(\mathbf{E}_u)$ with initial data $u(0) = c\delta_0$, where δ_0 is the Dirac mass at 0, and K being linked by $\int_{\mathbb{R}^N} u_K(x,t)dt = c$. The $u_{K,1}$ are the only nonnegative solutions defined on $\mathbb{R}^N \times (0,\infty)$, such that $u(x,0) = 0$ for any $x \neq 0$. For $\varepsilon = -1$, the $u_{K,-1}$ are defined for $t < 0$; for $K > 0$, w does not vanish on $(0,\infty)$; for $K<0$, w is flat with a hole near 0. For $K=0$, we find again the function w given at (1.6).

The case $\alpha=\eta\neq 0$. We exhibit a family of solutions of $(\mathbf{E}_w)$:

$$w(r)=Cr^{-\eta}, \qquad u(t,x)=C\left|x\right|^{-\eta}, \qquad C\neq 0. \tag{1.11}$$

The solutions u, independent of t, are p-harmonic in $\mathbb{R}^N$; they are fundamental solutions when $p<N$. When $p>N$, w satisfies $\lim_{r\to 0} w=0$, and $\lim_{r\to 0} w'=\infty$ for $N>1$, $\lim_{r\to 0} w'=b$ for $N=1$.

The case $\alpha=-p'$. Equation $(\mathbf{E}_w)$ admits regular solutions of the form

$$w(r)=\pm K\left(N(Kp')^{p-2}+\varepsilon r^{p'}\right), \qquad u(x,t)=\pm K\left(N(Kp')^{p-2}t+|x|^{p'}\right), \qquad K>0. \tag{1.12}$$

Here $\beta>0$; in the two cases $\varepsilon=1, t>0$ and $\varepsilon=-1, t<0$, u is defined for any $t\in\mathbb{R}$ and of the form $\psi(t)+\Phi(|x|)$ with Φ nonconstant, and $u(.,t)$ has a constant sign for $t>0$ and changing sign for $t<0$.

The case $\alpha=0$. Equation $(\mathbf{E}_w)$ can be explicitely solved: either $w'\equiv 0$, thus $w\equiv a\in\mathbb{R}$, u is a constant solution of $(\mathbf{E}_u)$, or there exists $K\in\mathbb{R}$ such that

$$\left|w'\right|=r^{-(\eta+1)}\left(K-\frac{\varepsilon}{\gamma+N}r^{N-\eta}\right)_+^{1/(p-2)}; \tag{1.13}$$

and w follows by integration, up to a constant, and then $u(x,t)=w(|x|/(\varepsilon pt)^{1/p})$. If $\varepsilon=1$, then $t>0$, $K>0$ and w' has a compact support; up to a constant, u has a compact support. If $\varepsilon=-1$, then $t<0$; for $K>0$, w is strictly monotone; for $K<0$, w is flat, constant near 0; for $K=0$, we find again (1.6). For $\varepsilon=\pm 1, K>0$, observe that $\lim_{r\to 0} w=\pm\infty$ if $p\leqq N$; and $\lim_{r\to 0} w=a\in\mathbb{R}$, $\lim_{r\to 0} w'=\pm\infty$ if $p>N>1$; and $\lim_{r\to 0} w=a\in\mathbb{R}$, $\lim_{r\to 0} w'=K$ if $p>N=1$. In particular we find solutions such that $w=cr^{|\eta|}(1+o(1))$ near 0, with $c>0$.

(v) Case $N=1$ **and** $\alpha=-(p-1)/(p-2)<0$. Here $\beta=1$, and we find the solutions

$$w(r)=\pm\left(Kr+\varepsilon\left|\alpha\right|^{p-1}\left|K\right|^p\right)_+^{(p-1)/(p-2)}, \quad u(x,t)=\pm\left(K\left|x\right|+\left|\alpha\right|^{p-1}\left|K\right|^p t\right)_+^{(p-1)/(p-2)}, \tag{1.14}$$

If $\varepsilon=1, t>0$, then w has a singularity at the level of the gradient, and either $K>0$, $w>0$, or $K<0$ and w has a compact support. If $\varepsilon=-1, t<0$ then $K>0$, w has a hole.

1.2 Main results

In the next sections we provide an exhaustive study of equation $(\mathbf{E}_w)$. Here we give the main results relative to the function u. Let us show how to return from w to u. Suppose that the behaviour of w is given by

$$\lim_{r\to 0} r^\lambda w(r)=c\neq 0, \qquad \lim_{r\to\infty} r^\mu w(r)=c'\neq 0, \qquad \text{where } \lambda,\mu\in\mathbb{R}.$$

(i) Then for fixed $t \neq 0$, the function u has a behaviour in $|x|^{-\lambda}$ near $x = 0$, and a behaviour in $|x|^{-\mu}$ for large $|x|$.

If $\lambda = 0$, then u is defined on $\mathbb{R}^N \times (0, \pm\infty)$. Either w is regular, then $u(.,t) \in C^1\left(\mathbb{R}^N \times (0,\infty)\right)$; we will say that u is **regular**; nevertheless the regular solutions u presents a singularity at time $t = 0$ if and only if $\alpha < -\gamma$ or $\alpha > 0$. Or a singularity can appear for u at the level of the gradient.

If $\lambda < 0$, thus u is defined on $\mathbb{R}^N \times (0, \pm\infty)$ and $u(0,t) = 0$; either w is flat, we also say that u is **flat**, or a singularity appears at the level of the gradient.

If $0 < \lambda < N$, then $u(.,t) \in L^1_{loc}\left(\mathbb{R}^N\right)$ for $t \neq 0$, we say that $x = 0$ is a **weak** *singularity.* We will show that there exist no stronger singularity.

If $\lambda < N < \mu$; then $u(.,t) \in L^1\left(\mathbb{R}^N\right)$.

(ii) For fixed $x \neq 0$, the behaviour of u near $t = 0$, depends on the sign of β:

$$\lim_{t\to 0} |x|^{\mu} |t|^{(\alpha-\mu)/\beta} u(x,t) = C \neq 0 \quad \text{if} \quad \alpha > -\gamma,$$
$$\lim_{t\to 0} |x|^{\lambda} |t|^{(\alpha-\lambda)/\beta} u(x,t) = C \neq 0 \quad \text{if} \quad \alpha < -\gamma.$$

If $\mu < 0, \alpha > -\gamma$ or $\lambda < 0, \alpha < -\gamma$, then $\lim_{t\to 0} u(x,t) = 0$.

1.2.1 Solutions defined for $t > 0$

Here we look for solutions u of $(\mathbf{E}_u)$ of the form (1.4) defined on $\mathbb{R}^N \backslash \{0\} \times (0,\infty)$. That means $\varepsilon\beta > 0$ or equivalently $\varepsilon = 1$, $-\gamma < \alpha$ (see Section 6) or $\varepsilon = -1, \alpha < -\gamma$ see (Section 7). We begin by the case $\varepsilon = 1$, treated at Theorem 6.1.

Theorem 1.1 *Assume $\varepsilon = 1$, and $-\gamma < \alpha$.*

(1) Let $\alpha < N$.

All regular solutions on $\mathbb{R}^N \backslash \{0\} \times (0,\infty)$ have a strict constant sign, in $|x|^{-\alpha}$ near ∞ for fixed t, with initial data $L|x|^{-\alpha}$ $(L \neq 0)$ in $\mathbb{R}^N$; thus $u(.,t) \notin L^1\left(\mathbb{R}^N\right)$, and u is unbounded when $\alpha < 0$.
There exist nonnegative solutions such that near $x = 0$,

$$\left.\begin{array}{lll} \text{for } p < N, & u \text{ has a weak singularity in } |x|^{-\eta}, & \\ \text{for } p = N, & u \text{ has a weak singularity in } \ln|x|, & \\ \text{for } p > N, & u \in C^0(\mathbb{R}^N \times (0,\infty)), \quad u(0,t) = a > 0, & \text{with a singular gradient,} \end{array}\right\} \qquad (1.15)$$

and u has an ***expanding compact support*** *for any $t > 0$, with initial data $L|x|^{-\alpha}$ in $\mathbb{R}^N \backslash \{0\}$.*
There exist positive solutions with the same behaviour as $x \to 0$, in $|x|^{-\alpha}$ near ∞ for fixed t; and also solutions such that u has one zero for fixed $t \neq 0$, and the same behaviour.

If $p > N$, there exist positive solutions satisfying (1.15), and also positive solutions such that

$$u \in C^0(\mathbb{R}^N \times (0,\infty)\,, \quad u(0,t) = 0, \text{ in } |x|^{|\eta|} \text{ near } 0, \text{ with a singular gradient,} \tag{1.16}$$

in $|x|^{-\alpha}$ near ∞ for fixed t, with and initial data $L\,|x|^{-\alpha}$ in $\mathbb{R}^N \backslash \{0\}$.

(2) Let $\alpha = N$.

All ***regular (Barenblatt)*** *solutions are nonnegative, have a* ***compact support*** *for any $t > 0$. If $p \leqq N$, all the other solutions have one zero for fixed t, satisfy (1.15) or (1.16) and have the same behaviour at ∞.*

(3) Let $N < \alpha$.

All regular solutions u have a finite number $m \geqq 1$ of simple zeros for fixed t, and $u(.,t) \in L^1\left(\mathbb{R}^N\right)$. Either they are in $|x|^{-\alpha}$ near ∞ for fixed t, then there exist solutions with m zeros, compact support, satisfying (1.15); or they have a compact support. All the solutions have m or $m+1$ zeros. There exist solutions satisfying (1.15) with $m+1$ zeros, and in $|x|^{-\alpha}$ near ∞. If $p > N$, there exist solutions satisfying (1.15) with m zeros; there exist also solutions with m zeros, $u(0,t) = 0$, and a singular gradient, in $|x|^{-\alpha}$ near ∞.

Next we come to the case $\varepsilon = -1$, which is the subject of Theorem 7.1.

Theorem 1.2 *Assume $\varepsilon = -1$ and $\alpha < -\gamma$.*

All the solutions u on $\mathbb{R}^N \backslash \{0\} \times (0,\infty)$, in particular the regular ones, are ***oscillating around*** *0 for fixed $t > 0$ and large $|x|$, and $r^{-\gamma} w$ is asymptotically periodic in $\ln r$. Moreover there exist*

solutions such that $r^{-\gamma} w$ is ***periodic*** *in $\ln r$, in particular $C_1 t^{-|\alpha/\beta|} \leqq |u| \leqq C_2 t^{-|\alpha/\beta|}$ for some $C_1, C_2 > 0$;*

solutions $u \in C^1(\mathbb{R}^N \times [0,\infty))$, $u(x,0) \equiv 0$, with a ***shrinking hole****;*

flat *solutions $u \in C^1(\mathbb{R}^N \times [0,\infty))$, in $|x|^{|\alpha|}$ near 0, with initial data $L\,|x|^{|\alpha|}$ $(L \neq 0)$;*

solutions satisfying (1.15) near $x = 0$, and if $p > N$, solutions satisfying (1.16) near 0.

1.2.2 Solutions defined for $t < 0$

We look for solutions u of $(\mathbf{E}_u)$ of the form (1.4) defined on $\mathbb{R}^N \backslash \{0\} \times (-\infty, 0)$. That means $\varepsilon\beta < 0$ or equivalently $\varepsilon = 1$, $\alpha < -\gamma$ (see Section 8, Theorem 8.1) or $\varepsilon = -1, \alpha > -\gamma$ (see Section 9). In the case $\varepsilon = 1$, we get the following:

Theorem 1.3 *Assume $\varepsilon = 1$, and $\alpha < -\gamma$.*

The function $U(x,t) = C\left(\frac{|x|^p}{-t}\right)^{1/(p-2)}$ is a positive ***flat*** *solution on $\mathbb{R}^N \backslash \{0\} \times (-\infty, 0)$.*

All regular solutions have a constant sign, are unbounded in $|x|^\gamma$ near ∞ for fixed t, and blow up at $t = 0$ like $(-t)^{-|\alpha|/|\beta|}$ for fixed $x \neq 0$.

*There exist **flat positive** solutions $u \in C^1(\mathbb{R}^N \times (-\infty, 0])$, in $|x|^\gamma$ near ∞ for fixed t, with **final data** $L\,|x|^{|\alpha|}$ $(L > 0)$.*

*There exist **nonnegative** solutions satisfying (1.15) near 0, with a **spreading compact support**, blowing up near $t = 0$ (like $|t|^{-(\eta+|\alpha|)/|\beta|}$ for $p < N$, or $|t|^{-|\alpha|/|\beta|}\ln|t|$ for $p = N$, or $(-t)^{-|\alpha|/|\beta|}$for $p > N$).*

There exist positive solutions with the same behaviour near 0, in $|x|^\gamma$ near ∞, blowing up as above at $t = 0$, and solutions with one zero for fixed t, and the same behaviour. If $p > N$, there exist positive solutions satisfying (1.15) (resp. (1.16)) near 0, in $|x|^\gamma$ near ∞ for fixed t, blowing up at $t = 0$ like $|t|^{-|\alpha|/|\beta|}$ (resp. $|t|^{(|\eta|-|\alpha|)/|\beta|}$) for fixed x.

Up to a symmetry, all the solutions are described.

The most interesting case is $\varepsilon = -1, -\gamma < \alpha$. For simplicity we will assume that $p < N$. The case $p \geqq N$ is much more delicate, and the complete results can be read in terms of w at Theorems 9.4, 9.6, 9.9, 9.10, 9.11 and 9.12. We discuss according to the position of α with respect to $-p'$ and α^* defined at (1.5). Notice that $\alpha^* < -p'$.

Theorem 1.4 *Assume $\varepsilon = -1$, and $-p' \leqq \alpha \neq 0$. The function U is still a flat solution on $\mathbb{R}^N \backslash \{0\} \times (-\infty, 0)$.*

(1) Let $0 < \alpha$.

All regular solutions have a strict constant sign, in $|x|^\gamma$ near ∞ for fixed t, blowing up at $t = 0$ like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$.

*There exist nonnegative solutions with a **focussing hole:** $u(x,t) \equiv 0$ for $|x| \leqq C\,|t|^{1/\beta}$, $t > 0$, in $|x|^\gamma$ near ∞ for fixed t, blowing up at $t = 0$ like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$.*

*There exist positive solutions u with a **(weak) singularity** in $|x|^{-\eta}$at $x = 0$, in $|x|^{-\alpha}$ near ∞ for fixed t, with $u(.,t) \in L^1(\mathbb{R}^N)$ if $\alpha > N$, with final data $L\,|x|^{-\alpha}$ $(L > 0)$ in $\mathbb{R}^N \backslash \{0\}$.*

There exist positive solutions u in $|x|^{-\eta}$at $x = 0$, in $|x|^\gamma$ near ∞ for fixed t, blowing up at $t = 0$ like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$; solutions with one zero and the same behaviour.

(2) Let $-p' < \alpha < 0$.

*All regular solutions have **one zero** for fixed t, and the same behaviour. There exist solutions with one zero, in $|x|^{-\eta}$at $x = 0$, in $|x|^{|\alpha|}$ near ∞ for fixed t, with final data $L\,|x|^{-\alpha}$ $(L > 0)$ in $\mathbb{R}^N \backslash \{0\}$. There exist solutions with one zero, u in $|x|^{-\eta}$at $x = 0$, in $|x|^\gamma$ near ∞ for fixed t, blowing up at $t = 0$ like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$; solutions with two zeros and the same behaviour.*

3) Let $\alpha = -p'$.

*All regular solutions have **one zero** and are in $|x|^{|\alpha|}$ near ∞ for fixed t, and with **final data** $L\,|x|^{|\alpha|}$ $(L > 0)$. The other solutions have one or two zeros, are in $|x|^{-\eta}$at $x = 0$, in $|x|^\gamma$ near ∞ for fixed t.*

In any case, up to a symmetry, all the solutions are described.

Theorem 1.5 *Assume* $\varepsilon = -1, -\gamma < \alpha < -p'$*. Then* U *is still a flat solution on* $\mathbb{R}^N \backslash \{0\} \times (-\infty, 0)$.

(1)Let $\alpha \leqq \alpha^*$.

Then there exist ***positive flat solutions****, in* $|x|^\gamma$ *near 0, in* $|x|^{|\alpha|}$ *near* ∞ *for fixed* t*, with* ***final data*** $L|x|^{-\alpha}$ $(L > 0)$ *in* $\mathbb{R}^N$.

All the other solutions, among them the ***regular ones****, have an* ***infinity of zeros****:* $u(t,.)$ *is oscillating around* 0 *for large* $|x|$*. There exist solutions with a focussing hole, and solutions with a singularity in* $|x|^{-\eta}$ *at* $x = 0$*. There exist solutions* ***oscillating also for small*** $|x|$*, such that* $r^{-\gamma}w$ *is periodic in* $\ln r$.

(2) There exist a ***critical unique value*** $\alpha_c \in (\max(\alpha^*, -p')$ *such that for* $\alpha = \alpha_c$*, there exists nonnegative solutions with a* ***focussing hole*** *near 0, in* $|x|^{|\alpha|}$ *near* ∞ *for fixed* t*, with* ***final data*** $L|x|^{-\alpha}$ $(L > 0)$ *in* $\mathbb{R}^N$*. And* $\alpha_c > -(p-1)/(p-2)$.
There exist positive flat solutions, such that $|x|^{-\gamma} u$ *is bounded on* $\mathbb{R}^N$ *for fixed* t*, blowing up at* $t = 0$ *like* $(-t)^{-1/(p-2)}$ *for fixed* $x \neq 0$*. The regular solutions are oscillating around* 0 *as above. There exist solutions* ***oscillating around*** 0*, such that* $r^{-\gamma}w$ *is* ***periodic*** *in* $\ln r$*. There are solutions with a weak singularity in* $|x|^{-\eta}$ *at* $x = 0$*, and oscillating around 0 for large* $|x|$.

(3) Let $\alpha^* < \alpha < \alpha_c$.

The regular solutions are as above. There exist solutions of the same types as above. Moreover there exist ***positive*** *solutions, such that* $r^{-\gamma}w$ *is* ***periodic*** *in* $\ln r$*, thus there exist* $C_1, C_2 > 0$ *such that*

$$C_1 \left(\frac{|x|^p}{|t|}\right)^{1/(p-2)} \leqq u \leqq C_2 \left(\frac{|x|^p}{|t|}\right)^{1/(p-2)}$$

There exist ***positive*** *solutions, such that* $r^{-\gamma}w$ *is asymptotically periodic in* $\ln r$ *near* 0 *and in* $|x|^\gamma$ *near* ∞ *for fixed* t*; and also, solutions with a hole, and oscillating around 0 for large* $|x|$*. There exist solutions positive near* 0*, oscillating near* ∞*, and* $r^{-\gamma}w$ *is* ***doubly asymptotically periodic*** *in* $\ln r$.

4) Let $\alpha_c < \alpha < -p'$.

There exist nonnegative solutions with a focussing hole near 0, in $|x|^\gamma$ *near* ∞ *for fixed* t*, blowing up at* $t = 0$ *like* $(-t)^{-1/(p-2)}$ *for fixed* $x \neq 0$*. Either the regular solutions have an* ***infinity*** *of zeros for fixed* t*, then the same is true for all the other solutions. Or they have a* ***finite*** *number* $m \geqq 2$ *of zeros, and can be in* $|x|^\gamma$ *or* $|x|^{|\alpha|}$ *near* ∞ *(in that case they have a final data* $L|x|^{|\alpha|}$*); all the other solutions have* m *or* $m+1$ *zeros.*

In the case $\alpha = \alpha_c$, we find again the existence and uniqueness of the focussing solutions introduced in [8].

2 Different formulations of the problem

In all the sequel we assume

$$\alpha \neq 0,$$

recalling that the solutions w are given explicitely by (1.13) when $\alpha = 0$. Defining

$$J_N(r) = r^N \left(w + \varepsilon r^{-1} \left| w' \right|^{p-2} w' \right), \qquad J_\alpha(r) = r^{\alpha - N} J_N(r), \tag{2.1}$$

equation ($\mathbf{E}_w$) can be written in an equivalent way under the forms

$$J_N'(r) = r^{N-1}(N-\alpha)w, \qquad J_\alpha'(r) = -\varepsilon(N-\alpha) r^{\alpha-2} \left| w' \right|^{p-2} w'. \tag{2.2}$$

If $\alpha = N$, then J_N is constant, so we find again (1.9).

We mainly use logarithmic substitutions; given $d \in \mathbb{R}$, setting

$$w(r) = r^{-d} y_d(\tau), \qquad Y_d = -r^{(d+1)(p-1)} \left| w' \right|^{p-2} w', \qquad \tau = \ln r, \tag{2.3}$$

we obtain the equivalent system:

$$\left.\begin{aligned} y_d' &= d y_d - |Y_d|^{(2-p)/(p-1)} Y_d, \\ Y_d' &= (p-1)(d-\eta) Y_d + \varepsilon e^{(p+(p-2)d)\tau} (\alpha y_d - |Y_d|^{(2-p)/(p-1)} Y_d). \end{aligned}\right\} \tag{2.4}$$

At any point τ where $w'(\tau) \neq 0$, the functions y_d, Y_d satisfy the equations

$$y_d'' + (\eta - 2d) y_d' - d(\eta - d) y_d + \frac{\varepsilon}{p-1} e^{((p-2)d+p)\tau} \left| d y_d - y_d' \right|^{2-p} \left(y_d' + (\alpha - d) y_d \right) = 0, \tag{2.5}$$

$$\begin{aligned} Y_d'' + (p-1)(\eta - 2d - p') Y_d' + \varepsilon e^{((p-2)d+p)\tau} |Y_d|^{(2-p)/(p-1)} \left(Y_d'/(p-1) + (\alpha - d) Y_d \right) \\ -(p-1)^2 (\eta - d)(p' + d) Y_d = 0, \end{aligned} \tag{2.6}$$

The main case is $d = -\gamma$: setting $y = y_{-\gamma}$,

$$w(r) = r^\gamma y(\tau), \qquad Y = -r^{(-\gamma+1)(p-1)} \left| w' \right|^{p-2} w', \qquad \tau = \ln r, \tag{2.7}$$

we are lead to the *autonomous* system

$$\left.\begin{aligned} y' &= -\gamma y - |Y|^{(2-p)/(p-1)} Y, \\ Y' &= -(\gamma + N) Y + \varepsilon (\alpha y - |Y|^{(2-p)/(p-1)} Y). \end{aligned}\right\} \tag{\textbf{S}}$$

Its study is fundamental: its phase portrait allows to study all the *signed* solutions of equation $(\mathbf{E}_w)$. Equation (2.5) takes the form

$$(p-1)y''+(N+\gamma p)y'+\gamma(\gamma+N)y+\varepsilon\left|\gamma y+y'\right|^{2-p}\left(y'+(\alpha+\gamma)y\right)=0, \qquad (\mathbf{E}_y)$$

Notice that $J_N(r) = r^{N+\gamma}(y(\tau) - \varepsilon Y(\tau))$.

Remark 2.1 *Since $(\boldsymbol{S})$ is autonomous, for any solution w of $(\boldsymbol{E}_w)$ of the problem, all the functions $w_\xi(r) = \xi^{-\gamma}w(\xi r), \xi > 0$, are also solutions.*

Notation 2.2 *In the sequel we set $\varepsilon\infty := +\infty$ if $\varepsilon = 1$, $\varepsilon\infty := -\infty$ if $\varepsilon = -1$.*

2.1 The phase plane of system (S)

In the phase plane (y, Y) we denote the four quadrants by

$$\mathcal{Q}_1 = (0,\infty)\times(0,\infty), \quad \mathcal{Q}_2 = (-\infty,0)\times(0,\infty), \quad \mathcal{Q}_3 = -\mathcal{Q}_1, \quad \mathcal{Q}_4 = -\mathcal{Q}_2.$$

Remark 2.3 *The vector field at any point $(0,\xi), \xi > 0$ satisfies $y' = -\xi^{1/(p-1)} < 0$, thus points to $\mathcal{Q}_2$; moreover $Y' < 0$ if $\varepsilon = 1$. The field at any point $(\varphi, 0), \varphi > 0$ satisfies $Y' = \varepsilon\alpha\varphi$, thus points to $\mathcal{Q}_1$ if $\varepsilon\alpha > 0$ and to $\mathcal{Q}_4$ if $\varepsilon\alpha < 0$; moreover $y' = -\gamma\varphi < 0$.*

If $\varepsilon(\gamma+\alpha) \geqq 0$, system (**S**) has a unique stationary point $(0,0)$. If $\varepsilon(\gamma+\alpha) < 0$, it admits three stationary points:

$$(0,0), \qquad M_\ell = (\ell, -(\gamma\ell)^{p-1}) \in \mathcal{Q}_4, \qquad M'_\ell = -M_\ell \in \mathcal{Q}_2, \tag{2.8}$$

where ℓ is defined at (1.7). The point $(0,0)$ is singular because $p > 2$; its study concern in particular the solutions w with a *double zero.* When $\varepsilon(\gamma+\alpha) < 0$, the point M_ℓ is associated to the solution $w \equiv \ell r^\gamma$ of equation $(\mathbf{E}_w)$ given at (1.1).

Linearization around M_ℓ. Near the point M_ℓ, setting

$$y = \ell + \overline{y}, \qquad Y = -(\gamma\ell)^{p-1} + \overline{Y}, \tag{2.9}$$

system (**S**) is equivalent in $\mathcal{Q}_4$ to

$$\overline{y}' = -\gamma\overline{y} - \varepsilon\nu(\alpha)\overline{Y} + \Psi(\overline{Y}), \qquad \overline{Y}' = \varepsilon\alpha\overline{y} - (\gamma+N+\nu(\alpha))\overline{Y} + \varepsilon\Psi(\overline{Y}), \tag{2.10}$$

where

$$\nu(\alpha) = -\frac{\gamma(N+\gamma)}{(p-1)(\gamma+\alpha)}, \text{ and } \Psi(\vartheta) = ((\gamma\ell)^{p-1}-\vartheta)^{1/(p-1)} - \gamma\ell + \frac{(\gamma\ell)^{2-p}}{p-1}\vartheta, \qquad \vartheta < (\gamma\ell)^{p-1}, \tag{2.11}$$

thus $\varepsilon\nu(\alpha) > 0$. The linearized problem is given by

$$\overline{y}' = -\gamma\overline{y} - \varepsilon\nu(\alpha)\overline{Y}, \qquad \overline{Y}' = \varepsilon\alpha\overline{y} - (\gamma + N + \nu(\alpha))\overline{Y}.$$

Its eigenvalues $\lambda_1 \leqq \lambda_2$ are the solutions of equation

$$\lambda^2 + (2\gamma + N + \nu(\alpha))\lambda + p'(N + \gamma) = 0 \tag{2.12}$$

The discriminant Δ of the equation (2.12) is given by

$$\Delta = (2\gamma + N + \nu(\alpha))^2 - 4p'(N + \gamma) = (N + \nu(\alpha))^2 - 4\nu(\alpha)\alpha. \tag{2.13}$$

For $\varepsilon = 1$, M_ℓ is a *sink*, and a node point, since $\nu(\alpha) > 0$, and $\alpha < 0$, thus $\Delta > 0$. For $\varepsilon = -1$, we have $\nu(\alpha) < 0$; the nature of M_ℓ depends on the critical value α^* defined at (1.5); indeed

$$\alpha = \alpha^* \Longleftrightarrow \lambda_1 + \lambda_2 = 0.$$

Then M_ℓ is a *sink* when $\alpha > \alpha^*$ and a *source* when $\alpha < \alpha^*$. Moreover α^* corresponds to a spiral point, and M_ℓ is a node point when $\Delta \geqq 0$, that means $\alpha \leqq \alpha_1$, or $\gamma > N/2 + \sqrt{p'(N+\gamma)}$ and $\alpha_2 \leqq \alpha$, where

$$\alpha_1 = -\gamma + \frac{\gamma(N+\gamma)}{(p-1)(2\gamma + N + 2(p'(N+\gamma))^{1/2})}, \qquad \alpha_2 = -\gamma + \frac{\gamma(N+\gamma)}{(p-1)(2\gamma + N - 2(p'(N+\gamma))^{1/2})}. \tag{2.14}$$

When $\Delta > 0$, and $\lambda_1 < \lambda_2$, one can choose a basis of eigenvectors

$$e_1 = (-\varepsilon\nu(\alpha), \lambda_1 + \gamma) \quad \text{and} \quad e_2 = (\varepsilon\nu(\alpha), -\gamma - \lambda_2). \tag{2.15}$$

Remark 2.4 *One verifies that $\alpha^* < -1$; and $\alpha^* < -(p-1)/(p-2)$ if and only if $p > N$. Also $\alpha_2 \leqq 0$, and $\alpha_2 = 0 \Longleftrightarrow N = p/((p-2)^2$; and $\alpha_2 > -p' \Longleftrightarrow \gamma^2 - 7\gamma - 8N < 0$, which is not always true.*

As in [4, Theorem 2.16] we prove that the Hopf bifurcation point is not degenerate, which implies the existence of small cycles near α^*.

Proposition 2.5 *Let $\varepsilon = -1$, and $\alpha = \alpha^* > -\gamma$. Then M_ℓ is a weak source. If $\alpha > \alpha^*$ and $\alpha - \alpha^*$ is small enough, there exists a unique limit cycle in $\mathcal{Q}_4$, attracting at $-\infty$.*

2.2 Other systems for positive solutions

When w has a constant sign, we define two functions associated to (y, Y) :

$$\zeta(\tau) = \frac{|Y|^{(2-p)/(p-1)} Y}{y}(\tau) = -\frac{rw'(r)}{w(r)}, \qquad \sigma(\tau) = \frac{Y}{y}(\tau) = -\frac{|w'(r)|^{p-2} w'(r)}{rw(r)}. \tag{2.16}$$

Thus ζ describes the behaviour of w'/w and σ is the slope in the phase plane (y, Y). They satisfy the system

$$\left.\begin{aligned} \zeta' &= \zeta(\zeta-\eta) + \varepsilon\,|\zeta y|^{2-p}(\alpha-\zeta)/(p-1) = \zeta(\zeta-\eta+\varepsilon(\alpha-\zeta)/(p-1)\sigma), \\ \sigma' &= \varepsilon(\alpha-N) + \left(|\sigma y|^{(2-p)/(p-1)}\,\sigma - N\right)(\sigma-\varepsilon) = \varepsilon(\alpha-\zeta) + (\zeta-N)\,\sigma. \end{aligned}\right\} \tag{$\mathbf{Q}$}$$

In particular, System ($\mathbf{Q}$) provides a short proof of the local existence and uniqueness of the *regular* solutions: they correspond to its stationary point $(0, \varepsilon\alpha/N)$, see Section 3.1.

Moreover, if w and w' have a strict constant sign, that means in any quadrant $\mathcal{Q}_i$, we can define

$$\psi = \frac{1}{\sigma} = \frac{y}{Y} \tag{2.17}$$

We obtain a new system relative to (ζ, ψ) :

$$\left.\begin{aligned} \zeta' &= \zeta(\zeta-\eta+\varepsilon(\alpha-\zeta)\psi/(p-1)), \\ \psi' &= \psi(N-\zeta+\varepsilon\,(\zeta-\alpha)\psi)\,. \end{aligned}\right\} \tag{$\mathbf{P}$}$$

We are reduced to a polynomial system, thus with no singularity. System ($\mathbf{P}$) gives the existence of singular solutions when $p > N$, corresponding to its stationary point $(\eta, 0)$, see Section 5.

We will also consider another system in any $\mathcal{Q}_i$: setting

$$\zeta = -1/g, \qquad \sigma = -s, \qquad d\tau = gsd\nu = |Y|^{(p-2)/(p-1)}\,d\nu, \tag{2.18}$$

we find

$$\left.\begin{aligned} dg/d\nu &= g(s(1+\eta g) + \varepsilon(1+\alpha g)/(p-1)), \\ ds/d\nu &= -s(\varepsilon(1+\alpha g) + (1+Ng)s). \end{aligned}\right\} \tag{$\mathbf{R}$}$$

System ($\mathbf{R}$) allows to get the existence of solutions w with a hole or a compact support, and other solutions, corresponding to its stationary points $(0, -\varepsilon)$ and $(-1/\alpha, 0)$; it provides a complete study of the singular point $(0, 0)$ of system ($\mathbf{S}$), see Sections 3.3, 5; and of the focussing solutions, see Section 9.

Remark 2.6 *The particular solutions can be found again in the different phase planes, where their trajectories are lines:*

For $\alpha = N$, the solutions (1.10) correspond to $Y \equiv \varepsilon y$, that means $\sigma \equiv \varepsilon$.

For $\alpha = \eta \neq 0$ the solutions (1.11) correspond to $\zeta \equiv \eta$.

For $\alpha = -p'$, the solutions (1.12) are given by $\zeta + \varepsilon N\sigma \equiv \alpha$.

For $N = 1$, $\alpha = -(p-2)/(p-1)$, the solutions (1.14) satisfy $\alpha g + \varepsilon s \equiv -1$.

3 Global existence

3.1 Local existence and uniqueness

Proposition 3.1 *Let $r_1 > 0$ and $a, b \in \mathbb{R}$. If $(a,b) \neq (0,0)$, there exists a unique solution w of equation ($\boldsymbol{E}_w$) in a neighborhood $\mathcal{V}$ of r_1, such that w and $|w'|^{p-2} w' \in C^1(\mathcal{V})$ and $w(r_1) = a$, $w'(r_1) = b$. It extends on a maximal interval I where $(w(r), w'(r)) \neq (0,0)$.*

Proof. If $b \neq 0$, the Cauchy theorem directly applies to system (**S**). If $b = 0$ the system is a priori singular on the line $\{Y = 0\}$ since $p > 2$. In fact it is only singular at $(0,0)$. Indeed near any point $(\xi, 0)$ with $\xi \neq 0$, one can take Y as a variable, and

$$\frac{dy}{dY} = F(Y,y), \qquad F(Y,y) := \frac{\gamma y + |Y|^{(2-p)/(p-1)} Y}{(\gamma + N)Y + \varepsilon(|Y|^{(2-p)/(p-1)} Y - \alpha y)},$$

where F is continuous in Y and C^1 in y, hence local existence and uniqueness hold. ∎

Notation 3.2 *For any point $P_0 = (y_0, Y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the unique trajectory in the phase plane (y, Y) of system ($\boldsymbol{S}$) going through P_0 is denoted by $\mathcal{T}_{[P_0]}$. By symmetry, $\mathcal{T}_{[-P_0]} = -\mathcal{T}_{[P_0]}$.*

Next we show the existence of regular solutions. Our proof is short, based on phase plane portrait, and not on a fixed point method, rather delicate because $p > 2$, see [3].

Theorem 3.3 *For any $a \in \mathbb{R}$, $a \neq 0$, there exists a unique solution $w = w(.,a)$ of equation ($\boldsymbol{E}_w$) in an interval $[0, r_0)$, such that w and $|w'|^{p-2} w' \in C^1([0, r_0))$ and*

$$w(0) = a, \qquad w'(0) = 0; \tag{3.1}$$

and then $\lim_{r \to 0} |w'|^{p-2} w'/rw = -\varepsilon\alpha/N$. In other words in the phase plane (y, Y) there exists a unique trajectory $\mathcal{T}_r$ such that $\lim_{\tau \to -\infty} y = \infty$, and $\lim_{\tau \to -\infty} Y/y = \varepsilon\alpha/N$.

Proof. We have assumed $\alpha \neq 0$ (when $\alpha = 0, w \equiv a$ from (1.13)). If such a solution w exists, then from (2.1) and (2.2), $J_N'(r) = r^{N-1}(N-\alpha)a(1+o(1))$ near 0. Thus $J_N(r) = r^{N-1}(1 - \alpha/N)a(1+o(1))$, hence $\lim_{r \to 0} |w'|^{p-2} w'/rw = -\varepsilon\alpha/N$; in other words, $\lim_{\tau \to -\infty} \sigma = \varepsilon\alpha/N$. And

$\lim_{\tau\to-\infty} y = \infty$, thus $\lim_{\tau\to-\infty} \zeta = 0$, and $\varepsilon\alpha\zeta > 0$ near $-\infty$. Reciprocally consider system (**Q**). The point $(0, \varepsilon\alpha/N)$ is stationary. Setting $\sigma = \varepsilon\alpha/N + \bar{\sigma}$, the linearized system near this point is given by

$$\zeta' = p'\zeta, \qquad \bar{\sigma}' = \varepsilon\zeta(\alpha - N)/N - N\bar{\sigma}.$$

One finds is a saddle point, with eigenvalues $-N$ and p'. Then there exists a unique trajectory $\mathcal{T}_r'$ in the phase-plane (ζ, σ) starting at $-\infty$ from $(0, \varepsilon\alpha/N)$ with the slope $\varepsilon(\alpha - N)/N(N + p') \neq 0$ and $\varepsilon\alpha\zeta > 0$. It corresponds to a unique trajectory $\mathcal{T}_r$ in the phase plane (y, Y), and $\lim_{\tau\to-\infty} y = \infty$, since $y = |\sigma| |\zeta|^{1-p})^{1/(p-2)}$. For any solution (ζ, σ) describing $\mathcal{T}_r'$, the function $w(r) = r^{\gamma}(|\sigma| |\zeta|^{1-p}(\tau))^{1/(p-2)}$ satisfies $\lim_{r\to 0} |w'|^{p-2} w'/rw = -\varepsilon\alpha/N$. As a consequence, $w^{(p-2)/(p-1)}$ has a finite nonzero limit, and $\lim_{r\to 0} w' = 0$; thus w is regular. Local existence and uniqueness follows for any $a \neq 0$, by Remark 2.1. ∎

Definition 3.4 *The trajectory $\mathcal{T}_r$ in the plane (y, Y) and its opposite $-\mathcal{T}_r$ will be called regular trajectories. We shall say that y is regular. Observe that $\mathcal{T}_r$ starts in $\mathcal{Q}_1$ if $\varepsilon\alpha > 0$, and in $\mathcal{Q}_4$ if $\varepsilon\alpha < 0$.*

Remark 3.5 *From Theorem 3.3 and Remark 2.1, all regular solutions are obtained from one one of them: $w(r, a) = aw(a^{-1/\gamma}r, 1)$. Thus they have the same behaviour near ∞.*

3.2 Sign properties

Next we give informations on the zeros of w or w', by using the monotonicity properties of the functions y_d, Y_d, in particular y, Y, and ζ and σ. At any extremal point τ, they satisfy respectively

$$y_d''(\tau) = y_d(\tau)\left(d(\eta - d) + \frac{\varepsilon(d-\alpha)}{p-1} e^{((p-2)d+p)\tau} |dy_d(\tau)|^{2-p}\right), \tag{3.2}$$

$$Y_d''(\tau) = Y_d(\tau)\left((p-1)^2(\eta - d)(p' + d) + \varepsilon(d-\alpha)e^{((p-2)d+p)\tau} |Y_d(\tau)|^{(2-p)/(p-1)}\right), \tag{3.3}$$

$$(p-1)y''(\tau) = \gamma^{2-p}y(\tau)\left(-\gamma^{p-1}(N+\gamma) - \varepsilon(\gamma+\alpha)|y(\tau)|^{2-p}\right) = -|Y(\tau)|^{(2-p)/(p-1)} Y'(\tau), \tag{3.4}$$

$$Y''(\tau) = Y(\tau)\left(-\gamma(N+\gamma) - \varepsilon(\gamma+\alpha)|Y(\tau)|^{(2-p)/(p-1)}\right) = \varepsilon\alpha y'(\tau), \tag{3.5}$$

$$(p-1)\zeta''(\tau) = -\varepsilon(p-2)((\alpha-\zeta)|\zeta|^{2-p}|y|^{-p} yy')(\tau) = \varepsilon(p-2)((\alpha-\zeta)(\gamma+\zeta)|\zeta y|^{2-p})(\tau), \tag{3.6}$$

$$(p-1)\sigma''(\tau) = -(p-2)((\sigma-\varepsilon)|\sigma|^{(2-p)/(p-1)} Y |y|^{(4-3p)/(p-1)} y')(\tau) = \zeta'(\tau)(\sigma(\tau) - \varepsilon). \tag{3.7}$$

Proposition 3.6 *Let $w \not\equiv 0$ be any solution of ($\boldsymbol{E}_w$) on an interval I.*

(i) If $\varepsilon = 1$ and $\alpha \leqq N$, then w has at most one simple zero; if $\alpha < N$ and w is regular, it has no zero. If $\alpha = N$ it has no simple zero and a compact support. If $\alpha > N$ and w is regular, it has at least one simple zero.

(ii) If $\varepsilon = -1$ and $\alpha \geqq \min(0,\eta)$, then w has at most one simple zero. If $w \not\equiv 0$ has a double zero, then it has no simple zero. If $\alpha > 0$ and w is regular, it has no zero.

(iii) If $\varepsilon = -1$ and $-p' \leqq \alpha < \min(0,\eta)$, then w' has at most one simple zero, consequently w has at most two simple zeros, and at most one if w is regular. If $\alpha < -p'$, the regular solutions have at least two zeros.

Proof. (i) Let $\varepsilon = 1$. Consider two consecutive simple zeros $\rho_0 < \rho_1$ of w, with $w > 0$ on (ρ_0, ρ_1) ; hence $w'(\rho_1) < 0 < w'(\rho_0)$. If $\alpha \leqq N$, we find from (2.1),

$$J_N(\rho_1) - J_N(\rho_0) = -\rho_1^{N-1}\left|w'(\rho_1)\right|^{p-2} - \rho_0^{N-1}w'(\rho_0)^{p-1} = (N-\alpha)\int_{\rho_0}^{\rho_1} s^{N-1}wds,$$

which is contradictory; thus w has at most one simple zero. The contradiction holds as soon as ρ_0 is simple, even if ρ_1 is not. If w is regular with $w(0) > 0$, and ρ_1 is a first zero, and $\alpha < N$,

$$J_N(\rho_1) = -\rho_1^{N-1}\left|w'(\rho_1)\right|^{p-1} = (N-\alpha)\int_0^{\rho_1} s^{N-1}wds > 0,$$

which is still impossible. If $\alpha = N$, the (Barenblatt) solutions are given by (1.10). Next suppose $\alpha > N$ and w regular. If $w > 0$, then $J_N < 0$, thus $w^{-1/(p-1)}w' + r^{1/(p-1)} < 0$. Then the function $r \mapsto r^{p'} + \gamma w^{(p-2)/(p-1)}$ is non increasing and we reach a contradiction for large r. Thus w has a first zero ρ_1, and $J_N(\rho_1) < 0$, thus $w'(\rho_1) \neq 0$.

(ii) Let $\varepsilon = -1$ and $\alpha \geqq \min(\eta, 0)$. Here we use the substitution (2.3) from some $d \neq 0$. If y_d has a maximal point, where it is positive, and is not constant, then (3.2) holds. Taking $d \in (0, \min(\alpha,\eta))$ if $\eta > 0$, $d = \eta$ if $\eta \leqq 0$, we reach a contradiction. Hence y_d has at most a simple zero, and no simple zero if it has a double one. Suppose w regular and $\alpha > 0$. Then $w' > 0$ near 0, from Theorem 3.3. As long as w stays positive, any extremal point r is a strict minimum, from $(\mathbf{E}_w)$, thus in fact w' stays positive.

(iii) Let $\varepsilon = -1$ and $-p' \leqq \alpha < \min(0,\eta)$. Suppose that w' and has two consecutive zeros $\rho_0 < \rho_1$, and one of them is simple, and use again (2.3) with $d = \alpha$. Then the function Y_α has an extremal point τ, where it is positive and is not constant; from (3.3),

$$Y_\alpha''(\tau) = (p-1)^2(\eta-\alpha)(p'+\alpha)Y_\alpha(\tau), \tag{3.8}$$

thus $Y_\alpha''(\tau) \geqq 0$, which is contradictory. Next consider the regular solutions. They satisfy $Y_\alpha(\tau) = e^{(\alpha(p-1)+p)\tau}(\left|\alpha\right| a/N)(1+o(1)$ near $-\infty$, from Theorem 3.3 and (2.3), thus $\lim_{\tau\to-\infty} Y_\alpha = 0$. As above Y_α cannot have any extremal point, then Y_α is positive and increasing. In turn $w' < 0$ from (2.3), hence w has at most one zero. ■

Proposition 3.7 *Let $w \not\equiv 0$ be any solution of $(\boldsymbol{E}_w)$ on an interval I. If $\varepsilon = 1$, then w has a finite number of isolated zeros. If $\varepsilon = -1$, it has a finite number of isolated zeros in any interval $[m, M] \cap I$ with $0 < m < M < \infty$.*

Proof. Let Z be the set of isolated zeros on I. If w has two consecutive isolated zeros $\rho_1 < \rho_2$, and $\tau \in (e^{\rho_1}, e^{\rho_2})$ is a maximal point of $|y_d|$, from (3.2), it follows that

$$\varepsilon e^{((p-2)d+p)\tau} |dy_d(\tau)|^{2-p} (d-\alpha) \leqq (p-1)d(d-\eta). \tag{3.9}$$

That means with $\rho = e^\tau \in (\rho_1, \rho_2)$,

$$\varepsilon \rho^p |w(\rho)|^{2-p} (d-\alpha) \leqq (p-1)d^{p-1}(d-\eta). \tag{3.10}$$

First suppose $\varepsilon = 1$ and fix $d > \alpha$. Consider the energy function

$$E(r) = \frac{1}{p'} |w'|^p + \frac{\alpha}{2} w^2.$$

It is nonincreasing since $E'(r) = -(N-1)r^{-1} |w'|^p - rw'^2$, thus bounded on $I \cap [\rho_1, \infty)$. Then w is bounded, ρ_2 is bounded, Z is a bounded set. If Z is infinite, there exists a sequence of zeros (r_n) converging to some point $\overline{r} \in [0, \infty)$, and a sequence (τ_n) of maximal points of $|y_d|$ converging to $\overline{\tau} = \ln \overline{r}$. If $\overline{r} > 0$, then $w(\overline{r}) = w'(\overline{r}) = 0$; we get a contradiction by taking $\rho = \rho_n = e^{\tau_n}$ in (3.10), because the left-hand side tends to ∞. If $\overline{r} = 0$, fixing now $d < \eta$, there exists a sequence (τ_n) of maximal points of $|y_d|$ converging to $-\infty$. Then $w(\rho_n) = O(\rho_n^{p/(p-2)})$, and $w'(\rho_n) = -d\rho_n^{-1} w(\rho_n) = O(\rho_n^{2/(p-2)})$, thus $E(\rho_n) = o(1)$. Since E is monotone, it implies $\lim_{r \to 0} E(r) = 0$, hence $E \equiv 0$, and $w \equiv 0$, which is contradictory. Next suppose $\varepsilon = -1$ and fix $d < \alpha$. If $Z \cap [m, M]$ is infinite, we construct a sequence converging vers some $\overline{r} > 0$ and reach a contradiction as above. ■

Proposition 3.8 *Let y be any non constant solution of ($\boldsymbol{E}_y$), on a maximal interval I where $(y, Y) \neq (0, 0)$, and s be an extremity of I.*

(i) If y has a constant sign near s, then the same is true for Y.

(ii) If $y > 0$ is strictly monotone near s, then Y, ζ, σ are monotone near s.

(iii) If $y > 0$ is not strictly monotone near s, then $s = \pm\infty$, $\varepsilon(\gamma + \alpha) < 0$ and y oscillates around ℓ.

(iv) If y is oscillating around 0 near s, then $\varepsilon = -1, s = \pm\infty, \alpha < -p'$; if $\alpha > -\gamma$, then $|y| > \ell$ at the extremal points.

Proof. (i) The function w has at most one extremal point on I : at such a point, it satisfies $(|w'|^{p-2} w')' = -\varepsilon \alpha w$ with $\alpha \neq 0$. From (2.7), Y has a constant sign near s.

(ii) Suppose y strictly monotone near s. At any extremal point τ of Y, we find $Y''(\tau) = \varepsilon \alpha y'(\tau)$ from (3.5). Then $y'(\tau) \neq 0$, $Y''(\tau)$ has a constant sign. Thus τ is unique, and Y is strictly monotone near s. Next consider ζ. If there exists τ_0 such that $\zeta(\tau_0) = \alpha$, then $\zeta'(\tau_0) = \alpha(\alpha - \eta)$, from system **(Q)**. If $\alpha \neq \eta$, then τ_0 is unique, thus $\alpha - \zeta$ has a constant sign near s. Then $\zeta''(\tau)$ has a constant sign at any extremal point τ of ζ, from (3.6), thus ζ is strictly monotone near s. If $\alpha = \eta$, then $\zeta \equiv \alpha$. At last consider σ. If there exists τ_0 such that $\sigma(\tau_0) = \varepsilon$, then $\sigma'(\tau_0) = \varepsilon(\alpha - N)$ from System **(Q).** If $\alpha \neq N$, then τ_0 is unique, and $\sigma - \varepsilon$ has a constant sign near s. Thus $\sigma''(\tau)$ has a constant sign at any extremal point τ of σ, from (3.7) and assertion (i). If $\alpha = N$, then $\sigma \equiv \varepsilon$.

(iii) Let y be positive and not strictly monotone near s. There exists a sequence (τ_n) strictly monotone, converging to $\pm\infty$, such that $y'(\tau_n) = 0$, $y''(\tau_{2n}) > 0 > y''(\tau_{2n+1})$. Since $y(\tau_n) = \gamma^{-1} |Y|^{(2-p)/(p-1)} Y(\tau_n)$, we deduce $Y < 0$ near s, from (i). From (3.5),

$$-\varepsilon(\gamma+\alpha)y(\tau_{2n+1})^{2-p} \leqq \gamma^{p-1}(N+\gamma) \leqq -\varepsilon(\gamma+\alpha))y(\tau_{2n})^{2-p}, \tag{3.11}$$

thus $\varepsilon(\gamma+\alpha) < 0$ and $y(\tau_{2n}) < \ell < y(\tau_{2n+1})$, and $Y(\tau_{2n+1}) < -(\gamma\ell)^{p-1} < Y(\tau_{2n})$. If s is finite, then $y(s) = y'(s) = 0$, which is impossible; thus $s = \pm\infty$.

(iv) If y is changing sign, then $\varepsilon = -1$ and $\alpha < -p'$, from Propositions 3.6 and 3.7. At any extremal point τ,

$$(\alpha+\gamma)\,|y(\tau)|^{2-p} \leqq \gamma^{p-1}(N+\gamma)$$

from (3.4); if $\alpha > -\gamma$ it means $|y| > \ell$. ■

3.3 Double zeros and global existence

Theorem 3.9 *For any $\overline{r} > 0$, there exists a unique solution w of ($\boldsymbol{E}_w$) defined in a interval $[\overline{r}, \overline{r} \pm h)$ such that*

$$w > 0 \quad \text{on } (\overline{r}, \overline{r} \pm h) \quad \text{and} \quad w(\overline{r}) = w'(\overline{r}) = 0.$$

Moreover $\varepsilon h < 0$ and

$$\lim_{r \to \overline{r}} |(\overline{r} - r)|^{(p-1)/(2-p)}\, \overline{r}^{1/(2-p)} w(r) = \pm((p-2)/(p-1))^{(p-1)/(p-2)}. \tag{3.12}$$

In other words in the phase plane (y, Y) there exists a unique trajectory $\mathcal{T}_\varepsilon$ converging to $(0,0)$ at $\varepsilon\infty$. It has the slope ε and converges in finite time; it depends locally continuously of α.

Proof. Suppose that a solution $w \not\equiv 0$ exists on $[\overline{r}, \overline{r} \pm h)$ with $w(\overline{r}) = w'(\overline{r}) = 0$. From Propositions 3.7 and 3.8, up to a symmetry, $y > 0$, $|Y| > 0$ near $\overline{\tau} = \ln \overline{r}$, and $\lim_{\tau \to \ln \overline{r}} y = 0$, and σ, ζ are monotone near $\ln r$. Let μ and λ be their limits. If $|\mu| = \infty$, then $|\lambda| = \infty$, because $\zeta = |Y|^{(2-p)/(p-1)} \sigma$, $|\zeta|^{p-2} \zeta = \sigma y^{2-p}$; then $f = 1/\zeta$ tends to 0; but

$$f' = -1 + \eta f + \varepsilon \frac{1 - \alpha f}{(p-1)\sigma}, \tag{3.13}$$

thus f' tends to -1, which is impossible. Thus μ is finite. If λ is finite, then $\mu = 0$, thus $\lambda = \alpha$, from system (**Q**), $\ln w$ is integrable at $\overline{r}$, which is not true. Then $\lambda = \varepsilon\infty$, hence

$$\mu = \lim_{\tau \to \ln \overline{r}} \sigma = \varepsilon,$$

from system ($\mathbf{Q}$). Then $\varepsilon Y > 0$ near $\bar{\tau}$, then $\varepsilon w' < 0$ near $\bar{r}$, thus $\varepsilon h < 0$. Consider system ($\mathbf{R}$): as τ tends to $\bar{\tau}$, ν tends to $\pm\infty$, and (g, s) converges to the stationary point $(0, -\varepsilon)$.

Reciprocally, setting $s = -\varepsilon/\beta + h$, the linearized system of system ($\mathbf{R}$) at this point is given by

$$\frac{dg}{d\nu} = -\varepsilon\frac{p-2}{p-1}g, \qquad \frac{dh}{d\nu} = (\alpha - N)g + \varepsilon h.$$

The eigenvalues are $-\varepsilon(p-2)/(p-1)$ and ε, thus we find a saddle point. There are two trajectories converging to $(0, -\varepsilon)$. The first one satisfies $g \equiv 0$, it does not correspond to a solution of the initial problem. Then there exists a unique trajectory converging to $(0, -\varepsilon)$, as ν tends to $\varepsilon\infty$, with $g > 0$ near $\varepsilon\infty$. It is associated to the eigenvalue $-\varepsilon(p-2)/(p-1)$ and the eigenvector $((2p-3)/(p-1), \varepsilon(N-\alpha))$. It satisfies $dg/d\nu = -\varepsilon((p-2)/(p-1))g(1+o(1))$, thus $dg/d\tau = ((p-2)/(p-1))(1+o(1))$. Then τ has a finite limit $\bar{\tau}$, and τ increases to $\bar{\tau}$ if $\varepsilon = 1$ and decreases to $\bar{\tau}$ if $\varepsilon = -1$. In turn $|Y|^{(p-2)/(p-1)} = gs$ tends to 0, and s tends to ε, thus (y, Y) tends to $(0, 0)$ as τ tends to $\bar{\tau}$. Then w and w' converges to 0 at $\bar{r} = e^{\bar{\tau}}$. And $w'w^{-1/(p-1)} + (\varepsilon + o(1))r^{1/(p-1)} = 0$, which implies (3.12).

Corollary 3.10 *Let $r_1 > 0$, and $a, b \in \mathbb{R}$ and w be any local solution such that $w(r_1) = a$, $w'(r_1) = b$.*

(i) If $(a, b) = (0, 0)$, then w has a unique extension by 0 on (r_1, ∞) if $\varepsilon = 1$, on $(0, r_1)$ if $\varepsilon = -1$.

(ii) If $(a, b) \neq (0, 0)$, w has a unique extension to $(0, \infty)$.

■

Proof. (i) Assume $a = b = 0$, the function $w \equiv 0$ is a solution. Let w be any local solution near r_1, defined in an interval $(r_1 - h_1, r_1 + h_1)$ with $w(r_1) = w'(r_1) = 0$. Suppose that there exists $h_2 \in (0, h_1)$ such that $w(r_1 + \varepsilon h_1) \neq 0$. Let $\bar{h} = \inf\{h \in (0, h_1) : w(r_1 + \varepsilon h) \neq 0\}$, and $\bar{r} = r_1 + \varepsilon\bar{h}$, thus $w(\bar{r}) = w'(\bar{r}) = 0$, and for example $w > 0$ on some interval $(\bar{r}, \bar{r} + \varepsilon k))$ with $k > 0$. This contradicts theorem 3.9. Thus $w \equiv 0$ on $(r_1, r_1 + \varepsilon h_1)$.

(ii) From Theorems 3.9 and 3.3, w has no double zero for $\varepsilon(r - r_1) < 0$, and has a unique extension to a maximal interval with no double zero. From (i) it has a unique extension to $(0, \infty)$. In particular any local regular solution is defined on $[0, \infty)$. ■

4 Asymptotic behaviour

Next the function y is supposed to be monotone, thus w has a constant sign near 0 or ∞, we can assume that $w > 0$.

Proposition 4.1 *Let y be any solution of ($\boldsymbol{E}_y$) strictly monotone and positive near $s = \pm\infty$.*

(1) Then (ζ,σ) *has a limit* (λ,μ) *near* s*, given by is some of the values*

$$A_\gamma = \left(-\gamma, \varepsilon\frac{\alpha+\gamma}{N+\gamma}\right), \quad A_r = (0, \varepsilon\alpha/N), \quad A_\alpha = (\alpha, 0),$$
$$L_\eta = \eta\,(1,\infty)\,(\textit{if } p \neq N), \quad L_+ = (0,\infty)\,(\textit{if } p \geqq N), \quad L_- = (0,-\infty)\,(\textit{if } p > N). \tag{4.1}$$

(2) More precisely,

(i) Either $\varepsilon(\gamma+\alpha) < 0$ *and* (y,Y) *converges to* $\pm M_\ell$*. Then* $(\lambda,\mu) = A_\gamma$ *and* $(\varepsilon = 1, s = \infty)$ *or* $(\varepsilon = -1, s$=-∞ *for* $\alpha \leqq \alpha^*$, $s = \infty$ *for* $\alpha > \alpha^*)$*.*

(ii) Or (y,Y) *converges to* $(0,0)$*. Then* $(s = \infty$ *and* $-\gamma < \alpha)$ *or* $(s = -\infty$ *and* $\alpha < -\gamma)$*, or* $(s = \varepsilon\infty$ *and* $\alpha = -\gamma)$ *and* $(\lambda,\mu) = A_\alpha$*.*

(iii) Or $\lim_{\tau\to s} y = \infty$*. Then* $s = -\infty$*. If* $p < N$*, then* $(\lambda,\mu) = A_r$ *or* L_η*. If* $p = N$*, then* $(\lambda,\mu) = A_r$ *or* L_+*. If* $p > N$*, then* $(\lambda,\mu) = A_r, L_\eta, L_+$ *or* L_-*.*

Proof. (1) The functions Y,σ,ζ are also monotone, and by definition $\zeta\sigma > 0$. Thus ζ has a limit $\lambda \in [-\infty,\infty]$ and σ has a limit $\mu \in [-\infty,\infty]$, and $\lambda\mu \geqq 0$.

(i) λ is finite. Indeed if $\lambda = \pm\infty$, then $f = 1/\zeta$ tends to 0. From (3.13), either $\mu = \pm\infty$, then f' tends to -1, which is imposible; or μ is finite, thus $\mu = \varepsilon$ from system (**Q**), then f' tends to $(2-p)/(p-1)$, which is still contradictory.

(ii) Either μ is finite, thus (λ,μ) is a stationary point of system **(Q)**, equal to A_γ, A_r or A_α.

(iii) 0r $\mu = \pm\infty$ and $(\lambda,0)$ is a stationary point of system (**P**).

- If $p \neq N$, either $\lambda = \eta \neq 0$ and $(\lambda,\mu) = L_\eta$; or $\lambda = 0$ and $(\lambda,\mu) = L_+$ or L_-. In the last case (ζ,ψ) converges to $(0,0)$, and $\zeta'/\psi' = -(\eta\zeta/N\psi)(1+o(1))$, thus $\eta < 0$, that means $p > N$.

- If $p = N$, then again (ζ,ψ) converges to $(0,0)$, thus $\mu = \pm\infty$, and $\psi' = N\psi(1+o(1))$, and necessarily $s = -\infty$. We make the substitution (2.4) with $d = 0$. Then $y_0(\tau) = w(r)$, and y_0 satisfies

$$y_0' = -\left|Y_0\right|^{(2-p)/(p-1)} Y_0 = -\zeta y_0 = o(y_0), \qquad Y_0' = \varepsilon e^{p\tau} y_0(\alpha-\zeta) = \varepsilon e^{p\tau} y_0\alpha(1+o(1).$$

Thus for any $\upsilon > 0$, we get $y_0 = O(e^{-\upsilon\tau})$ and $1/y_0 = O(e^{\upsilon\tau})$. Then Y_0' is integrable, and Y_0 has a finite limit $|k|^{p-2}k$. Suppose that $k = 0$. Then $Y_0 = O(e^{(p-\upsilon)\tau})$, and y_0 has a finite limit $a \geqq 0$. If $a \neq 0$, then $Y_0' = \varepsilon\alpha a e^{p\tau}(1+o(1))$; in turn $Y_0 = p^{-1}\varepsilon\alpha a e^{p\tau}(1+o(1))$, and $\psi = e^{p\tau}y_0/Y_0$ does not tend to 0. If $a = 0$, then $y_0 = O(e^{p'\tau})$, which contradicts the estimate of $1/y_0$. Thus $k > 0$ and

$$y_0 = -k\tau(1+o(1), \qquad Y_0 = k^{p-1}(1+o(1)); \tag{4.2}$$

hence $(\lambda,\mu) = L_+$.

(2) Since y is monotone, we encounter one of the three following cases:

(i) (y,Y) converges to $\pm M_\ell$. Then $(\lambda,\mu)=A_\gamma$ and M_ℓ is a source (or a weak source) for $\alpha \leqq \alpha^*$, a sink for $\alpha>\alpha^*$.

(ii) y tends to 0. Since λ is finite, (y,Y) converges to $(0,0)$. And $|\sigma|=|\zeta|^{p-1}y^{p-2}$ tends to 0, thus $(\lambda,\mu)=A_\alpha$. If $-\gamma<\alpha$, seeing that $y'=-y(\gamma+\zeta)<0$ we find $s=\infty$. If $\alpha<-\gamma$, then $s=-\infty$. If $\alpha=-\gamma<0$, then $\varepsilon(\gamma+\zeta)>0$, from the first equation of **(Q),** thus $\varepsilon y'<0$, hence $s=\varepsilon\infty$.

(iii) y tends to ∞. Either $\lambda\neq 0$, thus $|\sigma|=|\zeta|^{p-1}y^{p-2}$ tends to ∞, and $\lambda=\eta$ from system (**Q**), thus $p\neq N$, $(\lambda,\mu)=L_\eta$. Or $\lambda=0$ and μ is finite, thus $\mu=\varepsilon\alpha/N$, $(\lambda,\mu)=A_r$. Or $(\lambda,\mu)=L_0$; then either $p=N$, $L_0=L_\eta$, or $p>N$. In any case, $y'=-y(\gamma+\zeta)<0$, from (1.2), hence $s=-\infty$. ■

Next we apply these results to the functions w :

Proposition 4.2 *We keep the assumptions of Proposition 4.1. Let w be the solution of ($\boldsymbol{E}_w$) associated to y by (2.7).*

(i) If $(\lambda,\mu)=A_\gamma$ (near 0 or ∞), then

$$\lim r^{-\gamma}w=\ell. \tag{4.3}$$

(ii) If $(\lambda,\mu)=A_\alpha$ (near 0 or ∞), then

$$\lim r^{\alpha}w=L>0 \qquad \text{if } \alpha\neq-\gamma, \tag{4.4}$$

$$\lim r^{-\gamma}(\ln r)^{1/(p-2)}w=((p-2)\gamma^{p-1}(N+\gamma))^{-1/(p-2)} \qquad \text{if } \alpha=-\gamma. \tag{4.5}$$

(iii) If $p<N$ and $(\lambda,\mu)=L_\eta$, then

$$\lim_{r\to0} r^{\eta}w=c>0. \tag{4.6}$$

(iv) If $p>N$ and $(\lambda,\mu)=L_\eta$, then

$$\lim_{r\to0} r^{-|\eta|}w=c>0. \tag{4.7}$$

(v) If $p=N$ and $(\lambda,\mu)=L_+$, then

$$\lim_{r\to0}|\ln r|^{-1}w=k>0, \qquad \lim_{r\to0} rw'=-k \qquad \text{if } p=N. \tag{4.8}$$

(vi) If $p>N$ and $(\lambda,\mu)=L_+$, or L_-, then

$$\lim_{r\to0} w=a>0, \qquad \lim_{r\to0}(-r^{(N-1)/(p-1)}w')=c>0, \tag{4.9}$$

or

$$\lim_{r\to0} w=a>0, \qquad \lim_{r\to0}(-r^{(N-1)/(p-1)}w')=c<0. \tag{4.10}$$

Proof. (i) This follows directly from (2.7).

(ii) From (2.16), $rw'(r) = -\alpha w(r)(1+o(1)$. We are lead to three cases.

• Either $-\gamma < \alpha$, and $s = \infty$. For any $\upsilon > 0$, we find $w = O(r^{-\alpha+\upsilon})$ and $1/w = O(r^{\alpha+\upsilon})$ near ∞ and $w' = O(r^{-\alpha-1+\upsilon})$. Then $J'_\alpha(r) = O(r^{\alpha(2-p)-p-1+\upsilon})$, hence J'_α is integrable, J_α has a limit L. And $\lim r^\alpha w = L$, seeing that $J_\alpha(r) = r^\alpha w(1+o(1))$. If $L = 0$, then $r^\alpha w = O(r^{\alpha(2-p)-p+\upsilon})$, which contradicts the estimate of $1/w = O(r^{\alpha+\upsilon})$ for υ small enough. Thus $L > 0$.

• Or $\alpha < -\gamma$ and $s = -\infty$. For any $\upsilon > 0$, we find $w = O(r^{-\alpha-\upsilon})$ and $1/w = O(r^{\alpha+\upsilon})$ near 0 and $w' = O(r^{-\alpha-1-\upsilon})$. Then $J'_\alpha(r) = O(r^{\alpha(2-p)-p-1-\upsilon})$, and J'_α is still integrable, J_α has a limit L, and $\lim r^\alpha w = L$. If $L = 0$, then $r^\alpha w = O(r^{\alpha(2-p)-p-\upsilon})$, which contradicts the estimate of $1/w$. Thus again $L > 0$.

• Or $\alpha = -\gamma$ and $s = \varepsilon\infty$. Then $Y = -\gamma^{p-1}y^{p-1}(1+o(1))$, and $\mu = 0$, thus $y - \varepsilon Y = y(1+o(1))$. From System (**S**),

$$(y-\varepsilon Y)' = \varepsilon(N+\gamma)Y = -\varepsilon(N+\gamma)\gamma^{p-1}\,(y-\varepsilon Y)^{p-1}\,(1+o(1)).$$

Then $y = (N+\gamma)\gamma^{p-1}(p-2)\,|\tau|)^{-1/(p-2)}(1+o(1))$, which is equivalent to (4.5).

(iii) From (2.16), we get $rw'(r) = -\eta w(r)(1+o(1)$. We use (2.3) with $d = \eta$, thus $y_\eta = r^\eta w$. We find $y_\eta = O(e^{-\upsilon\tau})$, $1/y_\eta = O(e^{-\upsilon\tau})$, in turn $Y_\eta = O(e^{-\upsilon\tau})$. From (2.4), $Y'_\eta = O(e^{(p+(p-2)\eta-\upsilon)\tau})$, thus Y'_η is integrable, hence Y_η has a finite limit. Now $(e^{-\eta\tau}y_\eta)' = -e^{-\eta\tau}Y_\eta^{1/(p-1)}$, and $\eta > 0$, thus y_η has a limit c. If $c = 0$, then $Y_\eta = O(e^{(p+(p-2)\eta-\upsilon)\tau})$, $y_\eta = O(e^{((p+(p-2)\eta)/(p-1)-\upsilon)\tau})$, which contradicts $1/y_\eta = O(e^{-\upsilon\tau})$ for υ small enough. Then (4.6) holds.

(iv) As above, Y_η has a finite limit. In turn $r^{-|\eta|+1}w' = |Y_\eta|^{(2-p)/(p-1)}\,Y_\eta$ has a limit $c\,|\eta|$ and w has a limit $a \geqq 0$. From (2.16), $rw' = |\eta|\,w(1+o(1)$, hence $a = 0$. Then $c \geqq 0$; if $b = 0$, then $Y < 0$, the function $v = -e^{(\gamma+N)\tau}Y > 0$ tends to 0 and

$$v' = -e^{(\gamma+N)\tau}\varepsilon(\alpha-\eta)y(1+o(1)) = -\varepsilon(\alpha-\eta)\,|\eta|\,e^{-(\gamma+N)(p-2)/(p-1)\tau}v^{1/(p-1)};$$

we reach again a contradiction.Thus $a = 0$ and $c > 0$, and (4.7) holds.

(v) Assertion (4.8) follows from (4.2).

(vi) Here $rw' = o(w)$, thus $w + |w'| = O(r^{-k})$ for any $k > 0$. Then J'_N is integrable, J_N has a limit at 0, and $\lim_{r\to 0} r^N w = 0$. Thus $\lim_{r\to 0} r^{(N-1)/(p-1)}w' = -c \in \mathbb{R}$, $\lim_{r\to 0} J_N = -\varepsilon\,|c|^{p-2}\,c$, $\lim_{r\to 0} w = a \geq 0$. If $c = 0$, then $J_N(r) = \int_0^r J'_N(s)ds$, implying that $\lim_{r\to 0} w' = 0$. Either $a > 0$ and then w is regular, then $\lim_{\tau\to-\infty}\sigma = \varepsilon$; or $a = 0$, then $w' > 0$ and $(w')^{p-1} = O(rw)$; in both cases we get a contradiction. Thus $c \neq 0$. If $a = 0$, we find $\lim_{\tau\to-\infty}\zeta = \eta$, which is not true, hence $a > 0$. In any case (4.9) or (4.10) holds. ■

Now we study the cases where y is not monotone, and eventually changing sign.

Proposition 4.3 *Suppose $\varepsilon = -1$. Let $w \not\equiv 0$ be any solution of $(\boldsymbol{E}_w)$.*

(i) If $\alpha \leqq -\gamma$, then w is oscillating near 0 at ∞.

(ii) If $\alpha < 0$, then y and Y are bounded at ∞.

Proof. (i) Suppose by contradiction that $w \geqq 0$ for large r, then $y \geqq 0$ for large τ. If $y > 0$ near ∞, then from Proposition 3.8, either y is constant, which is impossible since $(0, 0)$ is the unique stationary point; or y is strictly monotone, which contradicts Proposition 4.1. Then there exists a sequence (τ_n) tending to ∞ such that $y(\tau_n) = y'(\tau_n) = 0$; from Theorem 3.10, $y \equiv 0$ on $(-\infty, \tau_n)$, thus $y \equiv 0$.

(ii) Consider the function

$$\tau \mapsto R(\tau) = \frac{y^2}{2} + \frac{|Y|^{p'}}{p' |\alpha|};$$

it satisfies

$$R'(\tau) = -\gamma y^2 + \frac{1}{|\alpha|} |Y|^{2/(p-1)} - \frac{N+\gamma}{|\alpha|} |Y|^{p'}.$$

From the Young inequality,

$$|\alpha| (R'(\tau) + \gamma R(\tau)) = |Y|^{2/(p-1)} - (N + \frac{1}{p-2}) |Y|^{p'} \leqq (\frac{2}{Np+\gamma})^{(p-2)/2} \leqq 1$$

thus $R(\tau)$ is bounded for large τ, at least by $1/|\alpha| \gamma$. ■

Proof.

Proposition 4.4 *(i) Assume $\varepsilon = 1$, or $\varepsilon = -1$, $\alpha \notin (\alpha_2, \alpha_1)$. Then for any trajectory of system $(\boldsymbol{S})$ in $\mathcal{Q}_4$ near $\pm\infty$, y is strictly monotone near $\pm\infty$.*

(ii) Assume $\varepsilon = 1$, and $\alpha \leqq \alpha^*$ or $-p' \leqq \alpha$. Then system $(\boldsymbol{S})$ admits no cycle in $\mathcal{Q}_4$ (or $\mathcal{Q}_2$).

■

Proof. (i) In any case M_ℓ is a node point. Following [4, Theorem 2.24], we use the linearization defined by (2.9). Consider the line L given by the equation $A\overline{y} + \overline{Y} = 0$, where A is a real parameter. The points of L are in $\mathcal{Q}_4$ whenever $\overline{Y} < (\gamma\ell)^{p-1}$ and $-\ell < \overline{y}$. We get

$$A\overline{y}' + \overline{Y}' = \left(\varepsilon\nu(\alpha)A^2 + (N + \nu(\alpha))A + \varepsilon\alpha\right) \overline{y} + (A + \varepsilon)\Psi(\overline{Y}).$$

From (2.13), apart from the case $\varepsilon = 1, \alpha = N$, we can find an A such that

$$\varepsilon\nu(\alpha)A^2 + (N + \nu(\alpha))A + \varepsilon\alpha = 0,$$

and $A + \varepsilon \neq 0$. Moreover $\Psi(\overline{Y}) \leqq 0$ on $L \cap \mathcal{Q}_4$. Indeed $(p-1)\Psi'(t) = -((\gamma\ell)^{p-1} - t)^{(2-p)/(p-1)} + (\gamma\ell)^{2-p}$, thus Ψ has a maximum 0 on $\left(-\infty, (\delta\ell)^{p-1}\right)$ at point 0. Then the orientation of the vector field does not change along $L \cap \mathcal{Q}_4$. In particular y cannot oscillate around ℓ, thus y is monotone, from Proposition 3.8. If $\varepsilon = 1, \alpha = N$, then $Y \equiv y \in (\ell, \infty)$ defines the trajectory $\mathcal{T}_r$, corresponding to the solutions given by (1.10) with $K > 0$. No solution y can oscillate around ℓ, since the trajectory cannot meet $\mathcal{T}_r$.

(ii) Suppose that there exists a cycle in $\mathcal{Q}_4$.

• Assume $\alpha \leqq \alpha^*$. Here M_ℓ is a source, or a weak source, from Proposition 2.5. Any trajectory starting from M_ℓ at $-\infty$ has a limit cycle in $\mathcal{Q}_1$, which is attracting at ∞. Writing System (**S**) under the form $y' = f_1(y, Y), Y' = f_2(y, Y)$, the mean value of the Floquet integral on the period $[0, \mathcal{P}]$ is given by

$$I = \oint (\frac{\partial f_1}{\partial y}(y, Y) + \frac{\partial f_2}{\partial Y}(y, Y))d\tau = \oint (\frac{|Y|^{(2-p)/(p-1)}}{p-1} - 2\gamma - N)d\tau. \tag{4.11}$$

Such a cycle is not unstable, thus $I \leqq 0$. Now

$$\oint (\alpha y' - \gamma Y')d\tau = 0 = (\alpha + \gamma) \oint |Y|^{1/(p-1)} d\tau - \gamma(\gamma + N) \oint |Y| d\tau.$$

From the Jensen and Hölder inequalities, since $1/(p-1) < 1$,

$$\gamma(\gamma + N)(\oint |Y|^{1/(p-1)} d\tau)^{p-2} \leqq \alpha + \gamma,$$

$$1 \leqq \left(\oint |Y|^{(2-p)/(p-1)})d\tau \right) \left(\oint |Y|^{1/(p-1)} d\tau \right)^{p-2} \leqq \frac{(p-1)(2\gamma + N)}{\gamma(\gamma + N)}(\alpha + \gamma),$$

then $\alpha^* < \alpha$, which is contradictory.

• Assume $-p' \leqq \alpha < 0$. Consider the functions $y_\alpha = e^{(\alpha+\gamma)\tau} y$ and $Y_\alpha = e^{(\alpha+\gamma)(p-1)\tau} Y$ defined by (2.3) with $d = \alpha$. They vary respectively from 0 to ∞ and from 0 to $-\infty$. They have no extremal point. Indeed at such a point, from (3.2) and (3.3) y''_α or Y''_α have a strict constant sign for $\alpha \neq \eta, p'$, which is contradictory. If $\alpha = \eta$ or p', from uniqueness y_α or Y_α is constant, thus y or Y is monotone, which is impossible. In any case $y'_\alpha > 0 > Y'_\alpha$ on $(-\infty, \infty)$. Next, from (2.5) and (2.6),

$$\frac{y''_\alpha}{y'_\alpha} + \eta - 2\alpha - \frac{1}{p-1} Y^{(2-p)/(p-1)} = \alpha(\eta - \alpha)\frac{y_\alpha}{y'_\alpha}, \tag{4.12}$$

$$\frac{Y''_\alpha}{Y'_\alpha} + (p-1)(\eta - 2\alpha - p') - \frac{1}{p-1} Y^{(2-p)/(p-1)} = (p-1)^2(\eta - \alpha)(p' + \alpha)\frac{Y_\alpha}{Y'_\alpha}. \tag{4.13}$$

Let us integrate on the period $\mathcal{P}$. If $\eta \leqq \alpha < 0$, then $\eta - N - 2(\alpha + \gamma) \geqq 0$ from (4.12), which is contradictory. If $-p' \leqq \alpha < \eta$, then $-2(\alpha + p' + \gamma) > 0$ from (4.13), still contradictory. ■

5 New local existence results

At Proposition 4.1 we gave all the *possible* behaviours of the positive solutions near $\pm\infty$. Next we prove their existence, and uniqueness or multiplicity. The case $p > N$ is very delicate.

Theorem 5.1 *(i) Suppose $p < N$. In the phase plane (y, Y) of system $(\boldsymbol{S})$ there exist an infinity of trajectories $\mathcal{T}_\eta$ such that* $\lim_{\tau\to-\infty}(\zeta,\sigma) = L_\eta$; *the corresponding w satisfy (4.6).*

(ii) Suppose $p > N$. There exist a unique trajectory $\mathcal{T}_u$ such that $\lim_{\tau\to-\infty}(\zeta,\sigma) = L_\eta$; *in other words for any* $c \neq 0$, *there exists a unique solution w of equation $(\boldsymbol{E}_w)$ such that (4.7) holds.*

Proof. Suppose that such a trajectory exists in the plane (y, Y). In the phase plane (ζ, ψ) of System $(\mathbf{P})$, ζ and ψ keep a strict constant sign, because the two axes $\zeta = 0$ and $\psi = 0$ contain particular trajectories, and (ζ, ψ) converges to $(\eta, 0)$ at $-\infty$. Reciprocally, setting $\zeta = \eta + \bar{\zeta}$, the linearized problem at point $(\eta, 0)$

$$\bar{\zeta}' = \eta\bar{\zeta} + \eta(\alpha - \eta)\varepsilon\psi/(p-1), \qquad \psi' = (N-\eta)\psi,$$

admits the eigenvalues η and $N - \eta$. The trajectories linked to the eigenvalue η are tangent to the line $\psi = 0$.

(i) Case $p < N$. Then $\eta > 0$, and $(\eta, 0)$ is a source. In the plane (ζ, ψ) there exist an infinity of trajectories, starting from this point at $-\infty$, such that $\psi > 0$, and $\lim_{\tau\to-\infty}\zeta = \eta$, thus $\zeta > 0$. In the phase plane (y, Y), setting $y = (\psi\,|\zeta|^{p-2}\,\zeta)^{2-p}$ and $Y = y/\psi$, they correspond to an infinity of trajectories in the plane (y, Y) such that $\lim_{\tau\to-\infty}(\zeta,\sigma) = L_\eta$, and (4.6) holds from Proposition (4.2).

(ii) Case $p > N$. Then $\eta < 0$, and $(\eta, 0)$ is a saddle point. In the plane (ζ, ψ), there exists a unique trajectory starting from $(\eta, 0)$, tangentially to the vector $(\eta(\alpha-\eta)\varepsilon/(p-1), N-\eta)$, with $\psi < 0$; it defines a unique trajectory $\mathcal{T}_u$ in the plane (y, Y), and (4.7) holds. From Remark 2.1, we get a solution for any $c \neq 0$. ■

Theorem 5.2 *(i) Suppose $p = N$. In the phase plane (y, Y), there exists an infinity of trajectories $\mathcal{T}_+$ such that* $\lim_{\tau\to-\infty}(\zeta,\sigma) = L_+$; *then w satisfies (4.8).*

(ii) Suppose $p > N$. Then there exist an infinity of trajectories $\mathcal{T}_+$ (resp. $\mathcal{T}_-$) such $\lim_{\tau\to-\infty}(\zeta,\sigma) = L_+$ *(resp. L_-); then the corresponding solutions w of $(\boldsymbol{E}_w)$ satisfy (4.9) (resp. (4.10).*

More precisely for any $k > 0$ (for $p = N$) or any $a > 0$ and $c \neq 0$ (for $p > N$) there exists a unique function w satisfying those conditions.

Proof. If $\lim_{\tau\to-\infty}(\zeta,\sigma) = L_\pm$, then $\lim_{\tau\to-\infty}(\zeta,\psi) = (0,0)$, with $\zeta\psi > 0$ in case of L_+, $\zeta\psi < 0$ in case of L_-. The linearization of System $(\mathbf{P})$ near $(0,0)$ is given by

$$\zeta' = |\eta|\,\zeta, \qquad \psi' = N\psi.$$

(i) Case $p=N$. The phase plane study is delicate because 0 is a center, thus we use a fixed method. Suppose that such a trajectory exists, and consider the substitution (2.3) with $d=0$. From (4.2), there exists $k>0$ such that $\zeta=|Y_0|^{(2-p)/(p-1)}/y_0=-\tau^{-1}(1+o(1))>0$, and $\psi=-k^{2-p}\tau e^{N\tau}(1+o(1))>0$. Then $\zeta'=\tau^{-2}(1+o(1))$ from System (**P**). The function

$$V=\psi e^{-N/\zeta}\zeta$$

satisfies $\lim_{\tau\to-\infty}V=k^{2-p}$, and

$$V'=\frac{Ve^{N/\zeta}}{(N-1)\zeta^2}(\varepsilon\,(\alpha-\zeta)\,(N-(N-2)\zeta)V+2N(N-1)\zeta^2e^{-N/\zeta}).$$

Thus $\varepsilon\alpha(V-k^{2-p})>0$ near $-\infty$. Moreover $\lim_{\tau\to-\infty}\zeta'/V'=0$, so that ζ can be considered as a function of V near k^{2-p}, with $\lim_{V\to k^{2-p}}\zeta=0$ and

$$\frac{d\zeta}{dV}=K(V,\zeta),\qquad K(V,\zeta):=\frac{\zeta^2}{V}\frac{\varepsilon\,(\alpha-\zeta)\,V+(N-1)\zeta^2e^{-N/\zeta}}{\varepsilon\,(\alpha-\zeta)\,(N-(N-2)\zeta)V+2N(N-1)\zeta^2e^{-N/\zeta}}.$$

Reciprocally, extending the function $\zeta^2e^{-N/\zeta}$ by 0 for $\zeta\leqq 0$, the function K is of class C^1 near $(k^{2-p},0)$. For any $k>0$, there exists a unique local solution $V\mapsto\zeta(V)$ on a interval $\mathcal{V}$ where $\varepsilon\alpha(V-k^{2-p})>0$, such that $\zeta(k^{2-p})=0$. And $d\zeta/dV=(\zeta^2/Nk^{2-p})(1+o(1))$ near 0, thus $\zeta>0$. In the plane (ζ,ψ), taking one point P on the curve $\mathcal{C}=\left\{(\zeta(V),V\zeta(V)e^{N/\zeta(V)}):v\in\mathcal{V}\right\}$, there exists a unique solution of System (**P**) issued from P at time 0. Its trajectory is on $\mathcal{C}$, thus it converges to $(0,0)$, with $\zeta,\psi>0$. It corresponds to a unique trajectory $\mathcal{T}_+$ in the plane (y,Y), and (ζ,σ) converges to L_+, as τ tends to $-\infty$, from Proposition 4.1. The corresponding functions w satisfy (4.8) from Proposition (4.2).

(ii) Case $p>N$. Here $(0,0)$ is a source for System (**P**). The lines $\zeta=0$ and $\psi=0$ contain trajectories. There exists an infinity of trajectories converging to $(0,0)$, with $\zeta\psi\neq 0$; moreover, if $N\geqq 2$, then $|\eta|<N$, thus $\lim_{\tau\to-\infty}(\psi/\zeta)=0$. Our claim is more precise. Given $a>0$ and $c\neq 0$, we look for a solution w of ($\mathbf{E}_w$) such that $\lim_{r\to 0}w=a$, $\lim_{r\to 0}r^{\eta+1}w'=-c$. By scaling we can assume $a=1$. If w_1 is a such a solution, then ζ and ψ have the sign of c near 0, and $\zeta\,(\tau)=ce^{|\eta|\tau}(1+o(1))$ and $|c|^{p-2}\,c\psi\,(\tau)=e^{N\tau}(1+o(1))$. The function

$$v=c(|c|^{p-2}\,c\psi)^{1/\kappa}/\zeta,\qquad\text{with }\kappa=N/\left|\eta\right|>1,$$

satisfies $\lim_{\tau\to-\infty}v=1$, and can be expressed locally as a function of ζ, and

$$\frac{dv}{d\zeta}=H(\zeta,v),\qquad H(\zeta,v):=-\frac{v}{\kappa}\frac{(p-1)(\kappa+1)+\varepsilon(\kappa-p+1)\,|c|^{1-p-\kappa}\,(\zeta-\alpha)\,|\zeta|^{\kappa-1}\,v^\kappa}{(p-1)(\zeta-\eta)+\varepsilon\,|c|^{1-p-\kappa}\,(\alpha-\zeta)\,|\zeta|^{\kappa-1}\,\zeta v^\kappa}.$$

Reciprocally, there exists a unique solution $\zeta\mapsto v(\zeta)$ of this equation on a small interval $[0,hc)$, with $h>0$, such that $v(0)=1$. Indeed H is locally continuous in ξ and C^1 in v. Taking one point P on the curve $\mathcal{C}' = \left\{ (\zeta, |c|^{1-p-\kappa} |\zeta|^{\kappa-1} \zeta v(\zeta)) : \zeta \in [0, hc) \right\}$, there exists a unique solution of System (**P**) issued from P at time 0. Its trajectory is on $\mathcal{C}'$, thus converges to $(0,0)$ with $\zeta\psi > 0$. It corresponds to a solution (y, Y) of System (**S**), such that (ζ, σ) converges to L_+, as τ tends to $-\infty$, from Proposition 4.1. The corresponding function, called w_2, satisfies $\lim_{r\to 0} r^{\eta+1} w_2^{\gamma^{-1}|\eta|-1} w_2' = -c$; thus w_2 has a limit a_2, and $\lim_{r\to 0} r^{\eta-1} w_2' = a_2^{1-s} b$. Moreover $a_2 \neq 0$, because $a_2 = 0$ implies that $r^{-\gamma} w_2$ has a nonzero limit, thus (ζ, σ) converges to A_γ. The function $w(r) = a_2^{-1} w_2(a_2^{1/\gamma} r)$ satisfies $\lim_{r\to 0} w = 1$, and $\lim_{r\to 0} r^{\eta-1} w' = -c$, and the proof is done.

Theorem 5.3 *(i) In the phase plane (y, Y), for any $\alpha \neq 0$ there exists at least a trajectory $\mathcal{T}_\alpha$ converging to $(0,0)$ with $y > 0$, and $\lim(\zeta, \sigma) = A_\alpha$. The convergence holds at ∞ if $-\gamma < \alpha$, or $-\infty$ if $\alpha < -\gamma$, or $\varepsilon\infty$ if $\alpha = -\gamma$.*

(ii) If $\varepsilon(\gamma + \alpha) < 0$, $\mathcal{T}_\alpha$ is unique, it is the unique trajectory converging to $(0,0)$ at $-\varepsilon\infty$ with $y > 0$, and it depends locally continuously of α.

■

Proof. (i) Suppose that such a trajectory exists. Then τ tends to ∞ if $-\gamma < \alpha$, or $-\infty$ if $\alpha < -\gamma$, or $\varepsilon\infty$ if $\alpha = -\gamma$, from Proposition 4.1. Consider System (**R**), where g, s and ν are defined by (2.18). Then (g, s) converges to $(-1/\alpha, 0)$, with $gs > 0$, and ν tends to the same limits as τ, since Y converges to 0. Reciprocally, in the plane (g, s), let us show the existence of a trajectory converging to $(-1/\alpha, 0)$, different from the line $s = 0$. Setting $g = -1/\alpha + \bar{g}$, the linearized system at this point is

$$\frac{d\bar{g}}{d\nu} = -\frac{\varepsilon}{p-1}\bar{g} + \frac{\eta - \alpha}{\alpha^2} s, \qquad \frac{ds}{d\nu} = 0,$$

thus we find a center: the eigenvalues are 0 and $\lambda = \varepsilon/(p-1)$. Since the system is polynomial, it is known that System (**R**) admits a trajectory, depending locally continuously of α, such that $sg > 0$, and tangent to the eigenvector $((p-1)(\eta - \alpha), \varepsilon\alpha^2)$. It satisfies $ds/d\nu = (p-2)(\alpha+\gamma)s^2(1+o(1))$. Then $ds/d\tau = -(p-2)\alpha(\alpha+\gamma)s(1+o(1))$, thus τ tends to $\pm\infty$. And $|y|^{p-2} = |s| \, |g|^{1/(p-1)}$, then y tends to 0, (y, Y) converges to $(0,0)$, and $\lim(\zeta, \sigma) = A_\alpha$.

(ii) Suppose $\varepsilon(\gamma + \alpha) < 0$. Consider two trajectories $\mathcal{T}_1, \mathcal{T}_2$ in the plane (y, Y), converging to $(0,0)$ at $-\varepsilon\infty$, with $y > 0$. They are different from $\mathcal{T}_\varepsilon$ which converges at $\varepsilon\infty$, thus $\lim(\zeta_i, \sigma_i) = (\alpha, 0)$ from Proposition 4.1. Then ζ_1, ζ_2 can locally be expressed as a function of y, and

$$y\frac{d(\zeta_1 - \zeta_2)^2}{dy} = 2(F(\zeta_1, y) - F(\zeta_2, y))\,(\zeta_1 - \zeta_2)$$

near 0, where

$$F(\zeta, y) = \frac{1}{\gamma + \zeta}(-\zeta(\zeta - \eta) + \frac{\varepsilon}{p-1}|\zeta y|^{2-p}(\zeta - \alpha)).$$

Then $(\zeta_1 - \zeta_2)^2$ is nonincreasing, seeing that $\partial F/\partial\zeta(\zeta, y) = -((p-1)\varepsilon(\gamma+\alpha))^{-1} |\alpha y|^{2-p} (1 + o(1))$. Hence $\zeta_1 \equiv \zeta_2$ near 0, and $\mathcal{T}_1 \equiv \mathcal{T}_2$. ■

6 The case $\varepsilon = 1,\ -\gamma \leqq \alpha$

In that Section and in Sections 7, 8 and 9 we describe the solutions of $(\mathbf{E}_w)$. When we give a *uniqueness* result, we mean that w is unique, *up to a scaling,* from Remark 2.1.

Theorem 6.1 *Assume* $\varepsilon = 1,\ -\gamma \leqq \alpha\ (\alpha \neq 0)$.

Any solution w *of* $(\boldsymbol{E}_w)$ *has a finite number of simple zeros, and satisfies (4.4) or (4.5) near* ∞ *or has a compact support. Either* w *is regular, or* $|w|$ *satisfies (4.6),(4.8), (4.7),(4.9) or (4.10) near* 0*, and there exist solutions of each type.*

(1) Case $\alpha < N$*. All regular solutions have a strict constant sign, and satisfy (4.4) or (4.5) near* ∞*. Moreover there exist (and exhaustively, up to a symmetry)*

(i) a unique nonnegative solution with (4.6)or (4.8) or (4.9)) near 0*, and compact support;*
(ii) positive solutions with the same behaviour at 0 *and (4.4) or (4.5) near* ∞;
(iii) solutions with one simple zero, and $|w|$ *has the same behaviour at* 0 *and* ∞;
(iv) for $p > N$*, a unique positive solution with (4.7) near* 0*, and (4.4) or (4.5) near* ∞*;*
(v) for $p > N$*, positive solutions with (4.10) near* 0*, and (4.4) or (4.5) near* ∞.

(2) Case $\alpha = N$*. Then the regular (Barenblatt) solutions have a constant sign with compact support. If* $p \leqq N$*, all the other solutions are of type (iii). If* $p > N$*, there exist also solutions of type (iv) and (v).*

(3) Case $\alpha > N$*.*

Either the regular solutions have m *simple zeros and satisfy satisfies (4.4) near* ∞*. Then there exist*
(vi) a unique solution with m *simple zeros,* $|w|$ *satisfies (4.6), (4.8) or(4.9) near* 0*, with compact support;*
(vii) solutions with $m+1$ *simple zeros,* $|w|$ *satisfies (4.6), (4.8) or (4.9) near* 0*, and (4.4) or (4.5) near* ∞;
(viii) for $p > N$*, solutions with* m *simple zeros,* $|w|$ *satisfies (4.9),(4.7) or (4.10) near* 0*, and (4.4) or (4.5) near* ∞.

Or the regular solutions have m *simple zeros and a compact support. Then the other solutions are of type (vii) or (viii).*

th 6.1,fig1: $\varepsilon = 1, N = 2, p = 3, \alpha = -2$ th 6.1,fig2: $\varepsilon = 1, N = 2, p = 3, \alpha = 1$

th 6.1,fig3: $\varepsilon = 1, N = 2, p = 3, \alpha = 2$ th 6.1,fig4: $\varepsilon = 1, N = 2, p = 3, \alpha = 50$

Proof. All the solutions w have a finite number of simple zeros, from Proposition 3.7 and Theorem 3.9. Either they have a compact support. Or y has a strict constant sign and is monotone near ∞, and converge to $(0,0)$ at ∞, and (4.4) or (4.5) holds, from Propositions 3.8, 4.1.

In the phase plane (y, Y), system (**S**) admits only one stationary point $(0,0)$. The trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_4$ when $\alpha < 0$, in $\mathcal{Q}_1$ when $\alpha > 0$, and $\lim_{\tau \to -\infty} y = \infty$, with an asymptotical direction of slope α/N. From Propositions 4.1 and 4.2 all the nonregular solutions $\pm w$ satisfy (4.6), (4.8), (4.7), (4.9) or (4.10) near $-\infty$. The existence of solutions of any kind is proved at Theorems 5.1 and 5.2. When $p \leqq N$, they correspond to trajectories $\pm\mathcal{T}_\eta$ such that $\mathcal{T}_\eta$ starts in $\mathcal{Q}_1$ with an infinite slope, in any case above $\mathcal{T}_r$. When $p > N$, there is a unique trajectory $\mathcal{T}_u$ satisfying (4.7), starting in $\mathcal{Q}_4$, under $\mathcal{T}_r$; the trajectories $\mathcal{T}_+$ start from $\mathcal{Q}_1$, above $\mathcal{T}_r$; the trajectories $\mathcal{T}_-$ start in $\mathcal{Q}_4$ under $\mathcal{T}_r$. From Theorem 3.9, there exists a unique trajectory $\mathcal{T}_\varepsilon$ converging to $(0,0)$ in $\mathcal{Q}_1$ at ∞, with the slope 1.

(1) Case $\alpha < N$. From Proposition 3.6, all the solutions w have at most one simple zero.

The regular solutions stay positive, and $\mathcal{T}_r$ stays in its quadrant, $\mathcal{Q}_4$ or $\mathcal{Q}_1$, from Remark 2.3 (see figures 1 and 2). Then $\mathcal{T}_\varepsilon$ stays in $\mathcal{Q}_1$, because it cannot meet $\mathcal{T}_r$ for $\alpha > 0$, or the line $\{Y = 0\}$ for $\alpha < 0$, from Remark 2.3; and the corresponding w is of type (i).

Consider any trajectory $\mathcal{T}_{[P]}$ with $P \in \mathcal{Q}_1$ above $\mathcal{T}_\varepsilon$. It cannot stay in $\mathcal{Q}_1$ because it does not meet $\mathcal{T}_\varepsilon$ and converges to $(0,0)$ with a slope 0. Thus it enters $\mathcal{Q}_2$ from Remark 2.3. Then y has a unique zero, and $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_1$ before P, and in $\mathcal{Q}_2 \cup \mathcal{Q}_3$ after P. Since $\mathcal{T}_{[P]}$ cannot meet $\pm\mathcal{T}_\varepsilon$, and $\lim_{\tau\to\infty} \zeta = \alpha$, $\mathcal{T}_{[P]}$ ends up in $\mathcal{Q}_3$ if $\alpha > 0$, in $\mathcal{Q}_2$ if $\alpha < 0$. It has the same behaviour as $\mathcal{T}_\varepsilon$ at $-\infty$, and w is of type (iii).

Next consider $\mathcal{T}_{[P]}$ for any $P \in \mathcal{Q}_1 \cup\ \mathcal{Q}_4$ between $\mathcal{T}_\varepsilon$ and $\mathcal{T}_r$. Then y stays positive, and $\mathcal{T}_{[P]}$ necessarily starts from $\mathcal{Q}_1$, and w is of type (ii).

At least take any $P \in \mathcal{Q}_1 \cup \mathcal{Q}_4$ under $\mathcal{T}_r$. If $p \leqq N$, $\mathcal{T}_{[P]}$ starts from $\mathcal{Q}_3$ and y has a unique zero, and $-w$ is of type (iii). If $p > N$, either $-w$ is of type (iii), or $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_4$. From Theorems 5.1, 5.2, either $\mathcal{T}_{[P]}$ coincides with $\mathcal{T}_u$, and w is of type (iv), or with one of the trajectories $\mathcal{T}_-$, thus w is of type (v).

(2) Case $\alpha = N$. All the solutions are given by (1.9), which is equivalent to $J_N \equiv C$, where J_N is defined by (2.1). For $C = 0$, the regular (Barenblatt) solutions, given by (1.10), are nonnegative, with a compact support. In other words the trajectory $\mathcal{T}_\varepsilon$ given by Theorem 5.3 coincides with $\mathcal{T}_r$, it is given by $y \equiv Y$, $y > 0$ (see figure 3). The only change in the phase plane is the nonexistence of solutions of type (ii).

(3) Case $\alpha > N$.

The regular solutions have a number $m \geqq 1$ of simple zeros, from Proposition 3.6 (see figure 4). As above, $\mathcal{T}_r$ starts from $\mathcal{Q}_1$ with a finite slope α/N.

Either $\mathcal{T}_r \neq \mathcal{T}_\varepsilon$. Then the regular solutions satisfy $\lim_{r\to\infty} r^\alpha w = L \neq 0$. Since $\mathcal{T}_\varepsilon$ cannot meet $\mathcal{T}_r$, $\mathcal{T}_\varepsilon$ also cuts the line $\{y = 0\}$ at m points, and the corresponding w is of type (vi). For any $P \in \mathcal{Q}_1$ above $\mathcal{T}_r$, the trajectory $\mathcal{T}_{[P]}$ cuts the line $\{y = 0\}$ at $m + 1$ points and w is of type (vii). If $p > N$, there exist trajectories starting from $\mathcal{Q}_1$ between $\mathcal{T}_\varepsilon$ and $\mathcal{T}_r$, with (4.9), such that w has m simple zeros, and trajectories with (4.7) or (4.10), m zeros, and $\lim_{r\to\infty} r^\alpha w = L \neq 0$.

Or $\mathcal{T}_r = \mathcal{T}_\varepsilon$, the regular solutions have a compact support, and we only find solutions of type (vii), (viii). ■

Remark 6.2 *In the case $\alpha = \eta < 0$, the solutions (iv) are given by (1.11). In the case $N = 1$, $\alpha = -(p-1)/(p-2)$, the solutions of types (i) and (v) are given by (1.14).*

Remark 6.3 *We conjecture that there exists an increasing sequence* $(\bar{\alpha}_m)$ *, with* $\bar{\alpha}_0 = N$ *such that the regular solutions* w *have* m *simple zeros for* $\alpha \in (\bar{\alpha}_{m-1}, \bar{\alpha}_m)$ *, with* $\lim_{r\to\infty} r^{\alpha} w = L \neq 0$*, and* m *simple zeros and a compact support for* $\alpha = \bar{\alpha}_m$*, in which case* $\mathcal{T}_r = \mathcal{T}_{\varepsilon}$*.*

7 The case $\varepsilon = -1, \alpha \leqq -\gamma$

Theorem 7.1 *Assume* $\varepsilon = -1$*,* $\alpha \leqq -\gamma$*. Then all the solutions* w *of* $(\boldsymbol{E}_w)$*, among them the regular ones, are ocillating near* ∞ *and* $r^{-\gamma} w$ *is asymptotically periodic in* $\ln r$*. There exist*

(i) solutions such that $r^{-\gamma} w$ *is periodic in* $\ln r$*;*
(ii) a unique solution with a hole;
(iii) flat solutions w *with (4.4) or (4.5) near* 0*;*
(iv) solutions with (4.6) or(4.8) or (4.9) or also (4.10) near 0*;*
(v) for $p > N$*, a unique solution with (4.7) near* 0*.*

th 7.1,fig5: $\varepsilon = -1, N = 1, p = 3, \alpha = -4$

Proof. Here again, $(0, 0)$ is the unique stationary point in the plane (y, Y). Any solution y of $(\mathbf{E}_y)$ oscillates near ∞, and (y, Y) is bounded from Proposition 4.3. From the strong form of the Poincaré-Bendixon theorem, see [7, p.239], all the trajectories have a limit cycle or are periodic. In particular $\mathcal{T}_r$ starts in $\mathcal{Q}_1$, since $\varepsilon\alpha > 0$, with the asymptotical direction $\varepsilon\alpha/N$. and it has a limit cycle $\mathcal{O}$. There exists a periodic trajectory of orbit $\mathcal{O}$, thus w is of type (i) (see figure 5).

From Theorem 5.2 there exists a unique trajectory $\mathcal{T}_{\varepsilon}$ starting from $(0, 0)$ with the slope -1, $y > 0$; it has a limit cycle $\mathcal{O}_{\varepsilon} \subset \mathcal{O}$, and w is of type (ii). For any P in the bounded domain delimitated by $\mathcal{O}_{\varepsilon}$, not located on $\mathcal{T}_{\varepsilon}$, the trajectory $\mathcal{T}_{[P]}$ does not meet $\mathcal{T}_{\varepsilon}$, and admits $\mathcal{O}_{\varepsilon}$ as limit cycle; near $-\infty$, y has a constant sign, is monotone and converges to $(0,0)$ from Propositions 3.8 and 4.1, and $\lim_{\tau\to-\infty}\zeta=\alpha$. This show again the existence of such trajectories, proved at Theorem 5.1, and there is an infinity of them; and w is if type (iii).

From Theorems 5.1 and 5.2, there exist trajectories starting from infinity, with $\mathcal{O}$ as limit cycle, and w is of type (iv) or (v). If $\mathcal{O}=\mathcal{O}_\varepsilon$, all the solutions are described. ■

8 Case $\varepsilon=1,\alpha<-\gamma$.

Theorem 8.1 *Assume $\varepsilon=1$, $\alpha<-\gamma$. Then $w\equiv\pm\ell r^\gamma$ is a solution of ($\mathbf{E}_w$). All regular solutions have a strict constant sign, and satisfy (4.3) near ∞. Moreover there exist (exhaustively, up to a symmetry)*

(i) a unique positive flat solution with (4.4) near 0 and (4.3) near ∞;
(ii) a unique nonnegative solution with (4.6) or (4.8) or (4.9) near 0, and compact support;
(iii) positive solutions with the same behaviour near 0 and (4.3) near ∞;
(iv) solutions with one zero and the same behaviour near 0, and $|w|$ satisfies (4.3) near ∞;
(v) for $p>N$, positive solutions with (4.7) near 0 and (4.3) near ∞;
(vi) for $p>N$, positive solutions with (4.10) near 0 and (4.3) near ∞.

th 8.1, fig6: $\varepsilon=1,N=2,p=3,\alpha=-6$

Proof. Here system (**S**) admits three stationary points in the plane (y,Y), given at (2.8), thus $w\equiv\pm\ell r^\gamma$ is a solution; and M_ℓ is a sink (see figure 6). Any solution y of ($\mathbf{E}_y$) has at most one zero, and is strictly monotone near $\pm\infty$, from Propositions 3.6 and 3.8.

From Theorems 3.9 and 5.3, there exists a unique trajectory $\mathcal{T}_\varepsilon$ converging to $(0,0)$ in $\mathcal{Q}_1$ at ∞, and a unique trajectory $\mathcal{T}_\alpha$ converging to $(0,0)$ in $\mathcal{Q}_4$ at $-\infty$. The trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_4$ with the asymptotical direction $-|\alpha|/N$. From Remark 2.3, $\mathcal{Q}_4$ is positively invariant, and $\mathcal{Q}_1$ negatively invariant. Then $\mathcal{T}_\varepsilon$ stays in $\mathcal{Q}_1$, and $\mathcal{T}_\alpha$ and $\mathcal{T}_r$ in $\mathcal{Q}_4$. From Proposition 4.1, all the trajectories, apart from $\pm\mathcal{T}_\varepsilon$, converge to $\pm M_\ell$ at ∞. Then $\mathcal{T}_r$ converges to M_ℓ, and w satisfies (4.3) near ∞. And $\mathcal{T}_\alpha$ also converges to M_ℓ, and w is of type (i).

From Propositions 4.1, Theorems 5.1 and 5.2, all the nonregular solutions which are positive near $-\infty$ satisfy (4.6), (4.8), (4.9), (4.10) or (4.7), and there exist such solutions. For $p < N$ (resp. $p = N$), they correspond to trajectories $\mathcal{T}_\eta$ (resp. $\mathcal{T}_+$) starting in $\mathcal{Q}_1$. For $p > N$, there is a unique trajectory $\mathcal{T}_u$ satisfying (4.7), starting in $\mathcal{Q}_4$ under $\mathcal{T}_r$; and the trajectories $\mathcal{T}_+$ satisfying (4.9) start from $\mathcal{Q}_1$; the trajectories $\mathcal{T}_-$ satisfying (4.10) and the unique trajectory $\mathcal{T}_u$ satisfying (4.7) start from $\mathcal{Q}_4$, under $\mathcal{T}_r$. Since $\mathcal{T}_\varepsilon$ stays in $\mathcal{Q}_1$, it defines solutions w of type (ii).

Consider the basis of eigenvectors (e_1, e_2) defined at (2.15), where $\nu(\alpha) > 0$, associated to the eigenvalues $\lambda_1 < \lambda_2$. One verifies that $\lambda_1 < -\gamma < \lambda_2$; thus e_1 points towards $\mathcal{Q}_3$ and e_2 points towards $\mathcal{Q}_4$. There exist unique trajectories $\mathcal{T}_{e_1}$ and $\mathcal{T}_{-e_1}$ converging to M_ℓ, tangentially to e_1 and $-e_1$. All the other trajectories converging to M_ℓ at ∞ are tangent to $\pm e_2$. Let

$$\mathcal{M} = \left\{ |Y|^{(2-p)/(p-1)} Y = -\gamma y \right\}, \qquad \mathcal{N} = \left\{ (N+\gamma)Y + \varepsilon |Y|^{(2-p)/(p-1)} Y = \varepsilon \alpha y \right\}$$

be the sets of extremal points of y and Y.

The trajectory $\mathcal{T}_r$ starts above the curves $\mathcal{M}$ and $\mathcal{N}$, thus $y' < 0$ and $Y' > 0$ near $-\infty$. And $\mathcal{T}_r$ converges to M_ℓ at ∞, tangentially to e_2. Indeed if $\mathcal{T}_r = \mathcal{T}_{e_1}$, then y has a minimal point such that $y < \ell$ and $Y < -(\gamma\ell)^{p-1}$, then (y, Y) cannot be on $\mathcal{M}$. If $\mathcal{T}_r = \mathcal{T}_{-e_1}$, then Y has a maximal point such that $y > \ell$ and $Y < -(\gamma\ell)^{p-1}$, then also (y, Y) cannot be on $\mathcal{N}$. Finally $\mathcal{T}_r$ cannot end up tangentially to $-e_2$, it would intersect $\mathcal{T}_{e_1}$ or $\mathcal{T}_{-e_1}$.

The trajectory $\mathcal{T}_\alpha$ converge to M_ℓ tangentially to $-e_2$. Indeed if $\mathcal{T}_\alpha = \mathcal{T}_{e_1}$, then Y has a maximal point such that $y < \ell$ and $Y < -(\gamma\ell)^{p-1}$; if $\mathcal{T}_\alpha = \mathcal{T}_{-e_1}$, then y has a maximal point such that $y > \ell$ and $Y > -(\gamma\ell)^{p-1}$. In any case we reach a contradiction. Moreover $\mathcal{T}_{e_1}$ does not stay in $\mathcal{Q}_4$: y would have a minimal point such that $y < \ell$ and $Y < -(\gamma\ell)^{p-1}$, which is impossible; thus $\mathcal{T}_{e_1}$ starts in $\mathcal{Q}_3$, and enters $\mathcal{Q}_4$ at some point $(\xi_1, 0)$ with $\xi_1 < 0$. And $-w$ is of type (iv).

Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of $\mathcal{Q}_1 \cup \mathcal{Q}_4$ delimitated by $\mathcal{T}_r, \mathcal{T}_\alpha$ and $\mathcal{T}_\varepsilon$, comes from $\mathcal{Q}_1$, and converges to M_ℓ in $\mathcal{Q}_4$, in particular $\mathcal{T}_{-e_1}$; the corresponding w are of type (iii).

Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of $\mathcal{Q}_3 \cup \mathcal{Q}_4$ delimitated by $\mathcal{T}_{e_1}, \mathcal{T}_\alpha$ and $-\mathcal{T}_\varepsilon$, goes from $\mathcal{Q}_3$ to $\mathcal{Q}_4$, and $\mathcal{T}_{[P]}$ converges to M_ℓ at ∞, and $-w$ is of type (iv). For any $\xi < \xi_1$, the trajectory $\mathcal{T}_{[(0,\xi)]}$ is of the same type. If $p \leqq N$, any trajectory in the domain under $\mathcal{T}_r$, and $\mathcal{T}_{e_1}$ is of the same type.

If $p > N$, moreover in this domain there exists a the unique trajectory $\mathcal{T}_u$ and trajectories of the type $\mathcal{T}_-$ corresponding to solutions w of type (v) and (vi), from Theorems 5.1 and 5.2. Up to a symmetry, all the solutions are described, and all of them do exist. ■

9 Case $\varepsilon = -1, -\gamma < \alpha$

Here again System (S) admits the three stationary points (2.8), thus $w \equiv \pm \ell r^\gamma$ is a solution of ($\mathbf{E}_w$). The behaviour is very rich: it depends on the position of α with respect to α^* defined at (1.5), and 0, $-p'$, and η (in case $p > N$), and also α_1, α_2 defined at (2.14). We start from some general remarks.

Remark 9.1 *(i) There exists a unique trajectory $\mathcal{T}_\varepsilon$ starting from $(0,0)$ in $\mathcal{Q}_4$ with the slope -1, from Theorem 3.9.*
(ii) There exists a unique trajectory $\mathcal{T}_\alpha$ converging to $(0,0)$ at ∞, in $\mathcal{Q}_1$ if $\alpha > 0$, in $\mathcal{Q}_4$ if $\alpha < 0$, with a slope 0 at $(0,0)$, and $\lim_{\tau\to\infty} \zeta = \alpha$, from Theorem 5.3.
(iii) From Remark 2.3, if $\alpha > 0$, $\mathcal{Q}_4$ is positively invariant and $\mathcal{Q}_1$ negatively invariant. If $\alpha < 0$, at any point $(0,\xi), \xi < 0$, the vector field points to $\mathcal{Q}_4$, and at any point $(\varphi, 0), \varphi > 0$, it points to $\mathcal{Q}_1$. Thus if $\mathcal{T}_\varepsilon$ does not stay in $\mathcal{Q}_1$, then $\mathcal{T}_\alpha$ stays in the bounded domain delimitated by $\mathcal{Q}_4 \cap \mathcal{T}_\varepsilon$. If $\mathcal{T}_\alpha$ does not stay in $\mathcal{Q}_4$, then $\mathcal{T}_\varepsilon$ stays in the bounded domain delimitated by $\mathcal{Q}_4 \cap \mathcal{T}_\alpha$. If $\mathcal{T}_\varepsilon$ is homoclinic, in other words $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$, it stays in $\mathcal{Q}_4$.

Remark 9.2 *From Propositions 4.1, Theorems 5.1 and 5.2, all the nonregular solutions positive near $-\infty$ satisfy (4.6) for $p < N$, (4.8) for $p = N$, corresponding to trajectories $\mathcal{T}_\eta, \mathcal{T}_+$ starting from$\mathcal{Q}_1$; and (4.9), (4.10) or (4.7) for $p > N$, corresponding to trajectories $\mathcal{T}_+$ starting from $\mathcal{Q}_1$, and $\mathcal{T}_-, \mathcal{T}_u$ starting from $\mathcal{Q}_4$.*

Remark 9.3 *Any trajectory $\mathcal{T}$ is bounded near ∞ from Proposition 4.3. From the strong form of the Poincaré-Bendixon theorem, any trajectory $\mathcal{T}$ bounded at $\pm\infty$ converges to $(0,0)$ or $\pm M_\ell$, or its limit set $\Gamma_\pm$ at $\pm\infty$ is a cycle, or it is homoclinic, namely $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$. If there exists a limit cycle surrounding $(0,0)$, it also surrounds the points $\pm M_\ell$, from Proposition 3.8.*

The simplest case is $\alpha > 0$.

Theorem 9.4 *Assume $\varepsilon = -1$, $\alpha > 0$.*

Then $w \equiv \ell r^\gamma$ is a solution w of ($\mathbf{E}_w$). All regular solutions have a strict constant sign; and satisfy (4.3) near ∞. There exist (exhaustively, up to a symmetry)

(i) a unique nonnegative solution with a hole, and (4.3) near ∞;
(ii) a unique positive solution with (4.6), or (4.8) or (4.9), and (4.4) near ∞;
(iii) positive solutions with the same behaviour near 0, and (4.3) near ∞;

(iv) solutions with one zero, the same behaviour near 0, *and* $|w|$ *satisfies (4.3) near* ∞;
(v) for $p > N$, *a unique positive solution with (4.7) near* 0, *and (4.3) near* ∞;
(vi) for $p > N$, *positive solutions with (4.10) near* 0, *and (4.3) near* ∞.

th 9.4, fig7: $\varepsilon = -1, N = 1, p = 3, \alpha = 0.7$ th 9.4, fig8: $\varepsilon = -1, N = 1, p = 3, \alpha = 1$

Proof. Any solution y of $(\mathbf{E}_y)$ has at most one zero, and y is strictly monotone near ∞, from Propositions 3.6 and 4.4. The point M_ℓ is a sink and a node point, since $\alpha > 0 \geqq \alpha_2$ (see figure 7). Consider the basis eigenvectors $(e_1,\, e_2)$, defined at (2.15), where $\nu(\alpha) < 0$, associated to the eigenvalues $\lambda_1 < \lambda_2 < 0$. One verifies that $\lambda_1 < -\gamma < \lambda_2$, thus e_1 points towards $\mathcal{Q}_3$ and e_2 points towards $\mathcal{Q}_4$. There exist unique trajectories $\mathcal{T}_{e_1}$ and $\mathcal{T}_{-e_1}$ tangent to e_1 and $-e_1$ at ∞. All the other trajectories which converge to M_ℓ end up tangentially to $\pm e_1$.

The trajectory $\mathcal{T}_\alpha$ stays in $\mathcal{Q}_1$ from Remark 9.1; near $-\infty$ it is of type $\mathcal{T}_\eta$ for $p < N$, and $\mathcal{T}_+$ for $p \geqq N$; it defines the solution of type (ii). Since $\mathcal{T}_\alpha$ is the unique trajectory converging to $(0, 0)$ at ∞, all the trajectories, apart from $\pm\mathcal{T}_\alpha$, converge to $\pm M_\ell$ at ∞, from Propositions 3.8 and 4.1.

The trajectories $\mathcal{T}_r$ and $\mathcal{T}_\varepsilon$ start in $\mathcal{Q}_4$, and stay in it from Remark 9.1, and both converge to M_ℓ at ∞, then w satisfies (4.3); and $\mathcal{T}_r$ starts with the asymptotical direction $-\alpha/N$. And $\mathcal{T}_\varepsilon$ defines the solution of type (i).

As in the proof of Theorem 8.1, $\mathcal{T}_r$ ends up tangentially to e_2, and $\mathcal{T}_\varepsilon$ tangentially to $-e_2$. Moreover $\mathcal{T}_{e_1}$ does not stay in $\mathcal{Q}_4$, it starts in $\mathcal{Q}_3$, and converges to M_ℓ in $\mathcal{Q}_4$, and $-w$ is of type (iv). Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of $\mathcal{Q}_4$ between $\mathcal{T}_{e_1}$, $\mathcal{T}_\varepsilon$, starts from $\mathcal{Q}_3$, enters $\mathcal{Q}_4$ at some point $(0, \xi), \xi > \xi_1$, and has the same type as $\mathcal{T}_{e_1}$. Any trajectory $\mathcal{T}_{[(0,\xi)]}$ with $\xi < \xi_1$ is of the same type.

Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of $\mathcal{Q}_1 \cup \mathcal{Q}_4$ above $\mathcal{T}_r \cup \mathcal{T}_\varepsilon$, starts from $\mathcal{Q}_1$, and converges to M_ℓ in $\mathcal{Q}_4$, in particular $\mathcal{T}_{-e_1}$; the corresponding w are of type (iii). If $p \leqq N$, all the solutions are described. If $p > N$, moreover there exist trajectories staying in $\mathcal{Q}_4 : \mathcal{T}_u$ and the $\mathcal{T}_-$, starting under $\mathcal{T}_r$, corresponding to types (v) and (vi). ■

Remark 9.5 *For $\alpha = N$, $\mathcal{T}_r$ and $\mathcal{T}_\varepsilon$ are given by (1.10), respectively with $K > 0$ and $K < 0$. The trajectory $\mathcal{T}_\varepsilon$ describes the portion $0M_\ell$ of the line $\{Y = -y\}$, and $\mathcal{T}_r$ the complementary half-line in $\mathcal{Q}_4$ (see figure 8).*

Next we assume $-p' \leqq \alpha < 0$. The case $p > N$ is delicate: indeed the special value $\alpha = \eta$ is involved, because $\eta < 0$.

Theorem 9.6 *Assume $\varepsilon = -1, p \leqq N$, and $-p' \leqq \alpha < 0$. Then $w \equiv \ell r^\gamma$ is a solution w of ($\boldsymbol{E}_w$).*

There exist a unique nonnegative solution with a hole, satisfying (4.3) at ∞.

(1) If $\alpha \neq -p'$, all regular solutions have one zero, and $|w|$ satisfies (4.3) near ∞. There exist (exhaustively, up to a symmetry)

- *for $p \leqq N$,*

  (i) a unique solution with one zero, with (4.6) or (4.8) near 0,and (4.4) near ∞;
  (ii) solutions with one zero, with (4.6) or (4.8) near 0, and $|w|$ satisfies (4.3) near ∞;
  (iii) solutions with two zeros, with (4.6) or (4.8) near 0, and (4.3) near ∞;

- *for $p > N$, $\eta < \alpha$,*

  (iv) a unique positive solution, with (4.10) near 0, and (4.4) near ∞;
  (v) a unique positive solution, with (4.7) near 0, and (4.3) near ∞;
  (vi) positive solutions, with (4.10) near 0, and (4.3) near ∞;
  (vii) solutions with one zero with (4.10) or (4.9) near 0, and (4.3) near ∞;

- *for $p > N, \alpha < \eta$,*

  (viii) a unique solution with one zero, with (4.9) near 0, and (4.4) near ∞;
  (ix) a unique solution with one zero, with (4.7) near 0, and $|w|$ satisfies (4.3) near ∞;
  (x) solutions with one zero, with (4.9) or (4.9) near 0, and $|w|$ satisfies (4.3) near ∞;
  (xi) solutions with two zeros, with (4.9) near 0, and (4.3) near ∞.

- *for $p > N, \alpha = \eta$, solutions of the form $w = cr^{|\eta|}$ $(c > 0)$. The other solutions are of type (vii).*

(2) If $\alpha = -p'$, all regular solutions have one zero and satisfy (4.4) near ∞. The solutions without hole are of types (ii), (iii) for $p \leqq N$, (ix), (x), (xi) for $p > N$.

th 9.6, fig9: $\varepsilon = -1, N = 1, p = 3, \alpha = -0.7$ th 9.6, fig10: $\varepsilon = -1, N = 1, p = 3, \alpha = -1.49$

th 9.6, fig11: $\varepsilon = -1, N = 1, p = 3, \alpha = -3/2$

Proof. Here again M_ℓ is a sink; but it is a node point only if $\alpha \geqq \alpha_{2.}$. The phase plane (y, Y) does not contain any cycle, from Proposition 4.4. From Proposition 3.6, any solution y has at most two zeros, and Y at most one.

The unique trajectory $\mathcal{T}_\alpha$ ends up in $\mathcal{Q}_4$ with the slope 0. From the uniqueness of $\mathcal{T}_\alpha$ and $\mathcal{T}_\varepsilon$, all the trajectories, apart from $\pm\mathcal{T}_\alpha$, converge to $\pm M_\ell$ at ∞, from Proposition 4.1 and Remark 9.3. Since $\varepsilon\alpha > 0$, the trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_1$, and y has at most one zero. Then $\mathcal{T}_r$ converges to $-M_\ell$ in $\mathcal{Q}_2$, or $\mathcal{T}_r = -\mathcal{T}_\alpha$.

The trajectory $\mathcal{T}_\varepsilon$ starts in $\mathcal{Q}_4$ with the slope -1, satisfies $y \geqq 0$ from Proposition 3.6. If $\mathcal{T}_\varepsilon$ converge to $(0,0)$, then $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$, thus it is homoclinic. Then M_ℓ is in the bounded component defined by $\mathcal{T}_\varepsilon$, and $\mathcal{T}_\varepsilon$ meets $\mathcal{T}_r$, which is impossible. Hence $\mathcal{T}_\varepsilon$ converges to M_ℓ in $\mathcal{Q}_4$, and w is nonnegative with a hole and satisfies (4.3) near ∞.

If $\alpha \neq -p'$, we claim that $\mathcal{T}_r \neq -\mathcal{T}_\alpha$. Indeed suppose $\mathcal{T}_r = -\mathcal{T}_\alpha$. Consider the functions y_α, Y_α, defined by (2.3) with $d = \alpha$. Then Y_α stays positive, and $Y_\alpha = O(e^{(\alpha(p-1)+p)\tau})$ at ∞, thus

$$\lim_{\tau\to\infty} Y_\alpha = 0, \quad \lim_{\tau\to\infty} Y_\alpha = c > 0, \quad \lim_{\tau\to-\infty} y_\alpha = \infty, \quad \lim_{\tau\to\infty} y_\alpha = L < 0.$$

Moreover y_α, Y_α have no extremal point: at such a point, from (3.2), (3.3) the second derivatives have a strict constant sign; then $Y'_\alpha > 0 > y'_\alpha$. If $\alpha < \eta$ (in particular if $p \leqq N$), from (4.13), near ∞,

$$(p-1)Y''_\alpha/Y'_\alpha \geqq |Y|^{(2-p)/(p-1)}\,(1+o(1)),$$

thus $Y''_\alpha > 0$ near ∞, which is contradictory; if $\alpha > \eta$, from (4.12)

$$(p-1)y''_\alpha/y'_\alpha \geqq |Y|^{(2-p)/(p-1)}\,(1+o(1)),$$

thus $y''_\alpha < 0$ near ∞, still contradictory. If $\alpha = \eta$, $\mathcal{T}_\alpha = \mathcal{T}_u$ from (1.11), thus again $\mathcal{T}_r \neq -\mathcal{T}_\alpha$.

If $p > N$ and $\alpha \neq \eta$, we claim that $\mathcal{T}_\alpha \neq \mathcal{T}_u$. Indeed suppose $\mathcal{T}_\alpha = \mathcal{T}_u$. This trajectory stays $\mathcal{Q}_4$, the function ζ stays negative, and $\lim_{\tau\to-\infty}\zeta = \eta$, $\lim_{\tau\to\infty}\zeta = \alpha$. If ζ has an extremal point ϑ, then $\vartheta \in (\alpha,\eta)$ from System (**Q**), and ζ'' has a constant sign, the sign of $\alpha - \zeta$; it is impossible. Thus ζ is monotone; then $(\alpha-\eta)\zeta' > 0$, which contradicts System (**Q**).

(1) Case $\alpha \neq -p'$. Since $\mathcal{T}_r \neq -\mathcal{T}_\alpha$, $\mathcal{T}_r$ converges to $-M_\ell$, and y has one zero, and $|w|$ satisfies (4.3).

• Case $p \leqq N$. All the other trajectories start in $\mathcal{Q}_3$ or $\mathcal{Q}_1$, from Remarks 9.1 and 9.2. For any $\varphi > 0$, the trajectory $\mathcal{T}_{[(\varphi,0)]}$ goes from $\mathcal{Q}_4$ into $\mathcal{Q}_1$, and converges to $-M_\ell$ in $\mathcal{Q}_2$, since it cannot meet $\mathcal{T}_r$ and $-\mathcal{T}_\varepsilon$; thus y has two zeros, and w is of type (iii). The trajectory $\mathcal{T}_\alpha$ cannot meet $\mathcal{T}_{[(\varphi,0)]}$, thus y has one zero, and it has the same behaviour at $-\infty$, and w is of type (i). All the trajectories $\mathcal{T}_{[P]}$ with P in the interior domain of $\mathcal{Q}_1$ delimitated by $-\mathcal{T}_\varepsilon$ and $\mathcal{T}_r$ start from $\mathcal{Q}_1$ and converge to $-M_\ell$, y has precisely one zero, and has the same behaviour at $-\infty$, and w is of type (ii).

• Case $p > N$, $\eta < \alpha$ (see figure 9). Any solution y has at most one simple zero. The trajectory $\mathcal{T}_\alpha$ stays in $\mathcal{Q}_4$. Indeed if it started in $\mathcal{Q}_3$, then for any trajectory $\mathcal{T}_{[(0,\xi)]}$ with $(0,\xi)$ above $-\mathcal{T}_\alpha$, the function y would have two zeros. Since $\mathcal{T}_\alpha \neq \mathcal{T}_u$, we have $\mathcal{T}_\alpha \in \mathcal{T}_-$, and w is of type (iv). The trajectory $\mathcal{T}_u$ necessarily stays in $\mathcal{Q}_4$ and converges to M_ℓ, and w is of type (v). The trajectories $\mathcal{T}_{[P]}$, with P in the domain delimitated by $\mathcal{T}_u, \mathcal{T}_\alpha$ and $\mathcal{T}_\varepsilon$, are of type $\mathcal{T}_-$ and converge in $\mathcal{Q}_4$ to M_ℓ, and w is of type (vi). The trajectories $\mathcal{T}_{[P]}$, with P in the domain delimitated by $\mathcal{T}_r, \mathcal{T}_\alpha$ and $-\mathcal{T}_\varepsilon$, are of type $\mathcal{T}_-$, and converge to $-M_\ell$, and y has one zero. The trajectories $\mathcal{T}_{[P]}$, with P in the domain delimitated by $\mathcal{T}_r$ and $-\mathcal{T}_u$, are of type $\mathcal{T}_+$,converge to $-M_\ell$, and y has one zero. Both define solutions w of type (vii).

• Case $p > N, \alpha < \eta$ (see figure 10). We have seen that $\mathcal{T}_r \neq -\mathcal{T}_\alpha$. If $\mathcal{T}_\alpha \in \mathcal{T}_+$, then ζ decreases from 0 to α, which contradicts System (**Q**) at ∞. Then $\mathcal{T}_\alpha$ does not stay in $\mathcal{Q}_4$, it starts in $\mathcal{Q}_3$ and $-\mathcal{T}_\alpha \in \mathcal{T}_-$, hence y has a zero, and w is of type (viii). Then $\mathcal{T}_u$ and the trajectories $\mathcal{T}_-$ converge to $-M_\ell$, and y has one zero. The trajectories $\mathcal{T}_{[P]}$, with P in the domain delimitated by $\mathcal{T}_r, -\mathcal{T}_\alpha$ and $-\mathcal{T}_\varepsilon$, are of type $\mathcal{T}_+$ and converge to $-M_\ell$, y has one zero. They correspond to w is of type (ix) or (x). The trajectories $\mathcal{T}_{[P]}$, with P in $\mathcal{Q}_4$ above $\mathcal{T}_r$, cut the line $\{y = 0\}$ twice, and converge to M_ℓ, and w is of type (xi).

• Case $p > N, \alpha = \eta$. Then $\mathcal{T}_\alpha = \mathcal{T}_u$, the functions $w = cr^{-\eta}$ $(c > 0)$ are particular solutions. The phase plane study is the same, and gives only solutions of type (vii).

(2) Case $\alpha = -p'$ (see figure 11). Here $\mathcal{T}_r = -\mathcal{T}_\alpha$, since the regular solutions are given by (1.12). Thus there exist no more solutions of type (ii) or (viii). ■

Next we study the behaviour of all the solutions when $\alpha < -p'$. In particular we prove the existence and uniqueness of an α_c for which there exists an homoclinic trajectory. Thus we find again some results obtained in [8], with new detailed proofs. We also improve the bounds for α_c, in particular $\alpha^* < \alpha_c$.

Lemma 9.7 *Let*

$$\alpha_p := -(p-1)/(p-2).$$

If $N = 1$, for $\alpha = \alpha_p$, then there exists an homoclinic trajectory in the phase plane (y, Y). If $N \geqq 2$, for $\alpha = \alpha_p$, there is no homoclinic trajectory, moreover $\mathcal{T}_\alpha$ converges to M_ℓ at $-\infty$ or has a limit cycle in $\mathcal{Q}_4$.

Proof. In the case $N = 1$, $\alpha = \alpha_p$, the explicit solutions (1.14) define an homoclinic trajectory in the phase plane (y, Y), namely $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$. In the phase plane (g, s) of System **(R)**, from Remark 2.6, they correspond to the line $s \equiv 1 + \alpha g$, joining the stationary points $(0, 1)$ and $(-1/\alpha, 0)$.

Next assume $N \geqq 2$ and consider the trajectory $\mathcal{T}_\alpha$ in the plane (y, Y). In the plane (g, s) of System (**R**), the corresponding trajectory $\mathcal{T}'_\alpha$ ends up at $(-1/\alpha, 0)$, as ν tends to ∞ from (2.18), with the slope $-k_p$. If $\mathcal{T}_\alpha$ is homoclinic, then $\mathcal{T}'_\alpha$ converges to $(0, 1)$ as ν tends to $-\infty$. Consider the segment

$$T = \{(g, -k(g + 1/\alpha_p) : g \in [0, 1/|\alpha_p|]\}, \qquad \text{with} \qquad k = p'\alpha_p^2/(N + 2/(p-2)) > k_p.$$

Its extremity $(0, k/|\alpha_p|)$ is strictly under $(0, 1)$. The domain $\mathcal{R}$ delimitated by the axes, which are particular orbits, and T, is negatively invariant: indeed, at any point of T, we find

$$k\frac{dg}{d\nu} + \frac{ds}{d\nu} = (N-1)p'ks(g - \frac{1}{\gamma})^2.$$

The trajectory $\mathcal{T}'_\alpha$ ends up in $\mathcal{R}$, thus it stays in it, hence $\mathcal{T}'_\alpha$ cannot join $(0,1)$. In the phase plane (y,Y), $\mathcal{T}_\alpha$ is not homoclinic, and $\mathcal{T}_\alpha$ stays in $\mathcal{Q}_4$, and Remark 9.3 applies. ∎

Remark 9.8 *Notice that* $\alpha^* \leqq \alpha_p \Leftrightarrow N \leqq p$.

Theorem 9.9 *Assume* $\varepsilon = -1$, *and* $\alpha < -p'$. *There exists a unique* $\alpha_c < 0$ *such that there exists an homoclinic trajectory in the plane* (y,Y); *in other words* $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$. *If* $N = 1$, *then* $\alpha_c = \alpha_p$. *If* $N \geqq 2$, *then*

$$max(\alpha^*, \alpha_p) < \alpha_c < \min(\alpha_2, -p'). \tag{9.1}$$

Proof. In order to prove the existence of an homoclinic orbit for System (**S**), we could consider a Poincaré application as in [4], but it does not give uniqueness. Thus we consider the system $(\mathbf{R}_\beta)$ obtained from $(\mathbf{R})$ by setting $s = \beta S$:

$$\left.\begin{array}{ll} \frac{dg}{d\nu} = gF(g,S), & F(g,S) := \beta S(1+\eta g) - \frac{1}{p-1}(1+\alpha g), \\ \frac{dS}{d\nu} = SG(g,S), & G(g,S) := 1+\alpha g - \beta(1+Ng)S. \end{array}\right\} \qquad (\mathbf{R}_\beta)$$

Its stationary points are

$$(0,0), \qquad A' = (1/|\alpha|, 0), \qquad B' = (0, 1/\beta), \qquad M' = (1/\gamma, 1/(N+\gamma)(p-2)),$$

where M' corresponds to M_ℓ. The existence of homoclinic trajectory for System (**S**) resumes to the existence of a trajectory for System $(\mathbf{R}_\beta)$ in the plane (g,S), starting from B' and ending at A'.

(i) *Existence.* We can assume that $\alpha \in (\alpha_1, \min(\alpha_2, -p'))$, from Proposition 4.4. In the plane (g,S), consider the trajectories $\mathcal{T}'_\varepsilon$ and $\mathcal{T}'_\alpha$ corresponding to $\mathcal{T}_\varepsilon \cap \mathcal{Q}_4$ and $\mathcal{T}_\alpha \cap \mathcal{Q}_4$ in the plane (y,Y). Then $\mathcal{T}'_\varepsilon$ starts from B' and $\mathcal{T}'_\alpha$ ends up at A'. From Remark 9.1, for any $\alpha \in (\alpha_1, \alpha_2)$, with $\alpha \leqq -p'$, we have three possibilities:

• $\mathcal{T}'_\varepsilon$ is converging to M' as ν tends to ∞ and turns around this point, since α is a spiral point, or it has a limit cycle in $\mathcal{Q}_1$ around M'. And $\mathcal{T}'_\alpha$ admits the line $g = 0$ as an asymptote as ν tends to $-\infty$, which means that $\mathcal{T}_\alpha$ does not stay in $\mathcal{Q}_4$ in the plane (y,Y). Then $\mathcal{T}'_\varepsilon$ meats the line

$$L := \{g = 1/\gamma\}$$

at a first point $(1/\gamma, S_0(\alpha))$. And $\mathcal{T}'_\alpha$ meats L at a last point $(1/\gamma, S_1(\alpha))$, such that $S_0(\alpha) - S_1(\alpha) < 0$;

• $\mathcal{T}'_\alpha$ is converging to M' at $-\infty$ or it has a limit cycle in $\mathcal{Q}_1$ around M'. And $\mathcal{T}'_\varepsilon$ admits the line $S = 0$ as an asymptote at ∞, which means that $\mathcal{T}_\varepsilon$ does not stay in $\mathcal{Q}_4$. Then with the same notations, $S_0(\alpha) - S_1(\alpha) > 0$.

• $\mathcal{T}'_\varepsilon = \mathcal{T}'_\alpha$, equivalently $S_0(\alpha) - S_1(\alpha) = 0$.

The function $\alpha \mapsto \varphi(\alpha) = S_0(\alpha) - S_1(\alpha)$ is continuous, from Theorems 3.9 and 5.3. If $-p' < \alpha_2$, then $\varphi(-p')$ is well defined and $\varphi(-p') < 0$; indeed $\mathcal{T}_\alpha = -\mathcal{T}_r$, thus $\mathcal{T}_\alpha$ does not stay in $\mathcal{Q}_4$ from Theorem 9.6. If $\alpha_2 \leqq -p'$, in the plane (y, Y), the trajectory $\mathcal{T}_{\alpha_2}$ leaves $\mathcal{Q}_4$, from Proposition 4.4, because α_2 is a sink, and transversally from Remark 9.1. The same happens for $\mathcal{T}_{\alpha_{2-\upsilon}}$ for $\upsilon > 0$ small enough, by continuity, thus $\varphi(\alpha_2 - \upsilon) < 0$. From Lemma 9.7, $\varphi(\alpha_p) > 0$ if $N \geqq 2$, and $\varphi(\alpha_p) = 0$ if $N = 1$. In any case there exists at least an α_c satisfying (9.1), such that $\varphi(\alpha_c) = 0$.

(ii) *Uniqueness*. First observe that $1 + \eta g > 0$; indeed $1 + \eta / |\alpha| > (p' + \eta)/|\alpha| > 0$. Now

$$(p-1)F + G = p\beta S(1/\gamma - g) = (p-2)\beta S(1 - \gamma g),$$

hence the curves $\{F = 0\}$ and $\{G = 0\}$ intersect at M' and A', $\{G = 0\}$ contains B' and is above $\{F = 0\}$ for $g \in (0, 1/\gamma)$ and under it for $g \in (1/\gamma, 1/|\alpha|)$. Moreover $\mathcal{T}'_\varepsilon$ has a negative slope at B', thus $F > 0 > G$ near 0 from $(\mathbf{R}_\beta)$. And $\mathcal{T}'_\varepsilon$ cannot meet $\{G = 0\}$ for $(0, 1/\gamma)$, because on this curve the vector field is $(gF, 0)$ and $F > 0$. Thus $\mathcal{T}'_\varepsilon$ satisfies $F > 0 > G$ on $(0, 1/\gamma)$. In the same way $\mathcal{T}'_\alpha$ has a negative slope $-\theta\alpha^2/(p-1)(\eta + |\alpha|) < 0$ at $1/|\alpha|$, thus $F > 0 > G$ near $1/|\alpha|$. And $\mathcal{T}'_\alpha$ cannot meet $\{F = 0\}$, because the vector field on this curve is $(0, SG)$ and $G < 0$. Thus $\mathcal{T}'_\alpha$ satisfies $F > 0 > G$ on $(1/\gamma, 1/|\alpha|)$.

Let $\alpha < \bar{\alpha}$. Then $\mathcal{T}'_\varepsilon$ is above $\bar{\mathcal{T}}'_\varepsilon$ near $g = 0$, and $\mathcal{T}'_\alpha$ is at the left of $\mathcal{T}'_{\bar{\alpha}}$ near $S = 0$. We show that $\varphi(\alpha) > \varphi(\bar{\alpha})$. First suppose that $\mathcal{T}'_\varepsilon$ and $\bar{\mathcal{T}}'_\varepsilon$ (or $\mathcal{T}'_\alpha$ and $\bar{\mathcal{T}}'_{\bar{\alpha}}$) intersect at a first point P_1 (or a last point) such $g \neq 1/\gamma$. Then at this point

$$\frac{1}{p-1}\frac{g}{S}\frac{dS}{dg} + 1 = \frac{(p-2)(1-\gamma g)S}{(p-1)S(1+\eta g) - \beta^{-1}(1+\alpha g)} = \frac{(p-2)(1-\gamma g)S}{h_S(g) - \beta^{-1}(1-\gamma g)} \tag{9.2}$$

with $h_S(g) = (p-1)S(1+\eta g) - g/(p-2)$. Thus the denominator, which is positive, is increasing in α on $(0, 1/\gamma)$, decreasing on $(1/\gamma, 1/|\alpha|)$; in any case $dS/dg > d\overline{S}/dg$ at P_1, which is contradictory. Next suppose that there is an intersection on L. At such a point $P_1 = (1/\gamma, S_1) = (1/\gamma, \overline{S}_1)$ the derivatives are equal from (9.2), and P_1 is above M', because $F > 0$. At any points $(g, S(g)) \in \mathcal{T}'_\varepsilon$ (or $\mathcal{T}'_\alpha$), $(g, \overline{S}(g)) \in \bar{\mathcal{T}}'_\varepsilon$ (or $\bar{\mathcal{T}}'_{\bar{\alpha}}$), setting $g = 1/\gamma + u$,

$$\Phi(u) = \left(\frac{1}{p-1}\frac{g}{S}\frac{dS}{dg} + 1\right)\frac{1}{(p-2)S} = -\frac{\gamma}{h_S(1/\gamma)}u + \frac{1}{h_S^2(1/\gamma)}\left(\frac{\gamma}{\beta} + h'_S(1/\gamma)\right)u^2(1 + o(1)),$$

$$\bar{\Phi}(u) = \left(\frac{1}{p-1}\frac{g}{\overline{S}}\frac{d\overline{S}}{dg} + 1\right)\frac{1}{(p-2)\overline{S}} = -\frac{\gamma}{h_{\overline{S}}(1/\gamma)}u + \frac{1}{h_{\overline{S}}^2(1/\gamma)}\left(\frac{\gamma}{\beta} + h'_{\overline{S}}(1/\gamma)\right)u^2(1 + o(1)),$$

And $h_S(1/\gamma) = h_{\overline{S}}(1/\gamma) > 0$, and $h'_S(1/\gamma) = h'_{\overline{S}}(1/\gamma)$, then

$$(\Phi - \bar{\Phi})(u) = \frac{\gamma u^2(1/\beta - 1/\bar{\beta})}{h(1/\gamma)}(1 + o(1)).$$

This implies $d^2(S-\overline{S})/dg^2 = 0$ and $d^3(S-\overline{S})/dg^3 = 2S_1\gamma^2(p-1)(p-2)(1/\beta - 1/\bar{\beta}) > 0$, which is a contradiction. Then $\mathcal{T}'_\varepsilon$ and $\bar{\mathcal{T}}'_\varepsilon$ cannot intersect on this line, similarly for $\mathcal{T}'_\alpha$ and $\bar{\mathcal{T}}'_{\bar{\alpha}}$. Hence $\varphi(\alpha) > \varphi(\bar{\alpha})$, which proves the uniqueness.

As a consequence, for $\alpha < \alpha_c$, $\varphi(\alpha) > 0$, in the plane (y, Y), $\mathcal{T}_\varepsilon$ does not stay in $\mathcal{Q}_4$; for $\alpha > \alpha_c$, $\varphi(\alpha) < 0$, $\mathcal{T}_\alpha$ does not stay in $\mathcal{Q}_4$. From Lemma 9.7, it follows that $\alpha_p < \alpha_c$ if $N \geqq 2$. Moreover $\alpha^* < \alpha_c$. Indeed α^* is a weak source from Proposition 2.5, thus for $\alpha > \alpha^*$ small enough, there exists a unique cycle $\mathcal{O}$ around M_ℓ, which is unstable. For such an α, $\mathcal{T}_\varepsilon$ cannot stay in $\mathcal{Q}_4$: it would have $\mathcal{O}$ as a limit cycle at ∞, which contradicts the unstability. ■

Next we discuss according to the position of α with respect to α^* and α_c.

Theorem 9.10 *Assume* $\varepsilon = -1$, *and* $\alpha \leqq \alpha^*$. *Then*

(i) there exist a unique flat positive solution w *of* $(\boldsymbol{E}_w)$ *with (4.3) near* 0, *and (4.4) near* ∞;
(ii) All the other solutions are oscillating at ∞, *among them the regular ones, and* $r^{-\gamma}w$ *is asymptotically periodic in* $\ln r$. *There exist solutions with a hole, also with (4.3), (4.6) or (4.9) or (4.9) or (4.7) near* 0. *There exist solutions such that* $r^{-\gamma}w$ *is periodic in* $\ln r$.

th 9.10,fig 12: $\varepsilon = -1, N = 1, p = 3, \alpha = -2.53$ th 9.10, fig 13: $\varepsilon = -1, N = 1, p = 3, \alpha = -2.2$

Proof. Here $\alpha < \alpha_c$, from Theorem 9.9, and the trajectory $\mathcal{T}_\alpha$ stays in $\mathcal{Q}_4$. From Proposition 4.4, it converges at $-\infty$ to M_ℓ, and w is of type (i).

The trajectory $\mathcal{T}_\varepsilon$ leaves $\mathcal{Q}_4$, and cannot converge either to $(0,0)$ since $\mathcal{T}_\varepsilon \neq \mathcal{T}_\alpha$, or to $\pm M_\ell$, because this point is a source, or a weak source. Recall that M_ℓ is a node point for $\alpha \leqq \alpha_1$ (see figure 12,, where $\alpha_1 \cong -2.50$), or a spiral point (see figure 13). And $\mathcal{T}_\varepsilon$ is bounded at ∞ from Proposition 4.3. Then it has a limit cycle $\mathcal{O}_\varepsilon$ surrounding $(0,0)$ from Proposition 4.4, and $\pm M_\ell$ from Remark 9.3. Thus w is oscillating around 0 near ∞, $r^{-\gamma}w$ is asymptotically periodic in $\ln r$.

The solutions w corresponding to $\mathcal{O}_\varepsilon$ are oscillating and $r^{-\gamma}w$ is periodic in $\ln r$. Any trajectory $\mathcal{T}_{[P]}$ with P in the interior domain delimitated by $\mathcal{O}_\varepsilon$ converges to M_ℓ at $-\infty$ and has the same limit cycle at ∞. The trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_1$, with $\lim_{\tau\to-\infty} y = \infty$ and cannot converge to any stationary point at ∞. It is bounded, thus has a limit cycle $\mathcal{O}_r$ surrounding $\mathcal{O}_0$. For any $P \notin \mathcal{T}_r$ in the exterior domain to $\mathcal{O}_r$, the trajectory $\mathcal{T}_{[P]}$ admits $\mathcal{O}_r$ as a limit cycle at ∞, and y is necessarily monotone at $-\infty$, thus (4.6) or (4.9) or (4.9) or (4.7) near 0; all those solutions exist. The question of the uniqueness of the cycle ($\mathcal{O}_r = \mathcal{O}_\varepsilon$) is open.

Theorem 9.11 *Let α_c be defined by Theorem 9.9.*

(1) Let $\alpha^* < \alpha < \alpha_c$. Then all regular solutions w of $(\boldsymbol{E}_w)$ are oscillating around 0 near ∞, and $r^{-\gamma}w$ is asymptotically periodic in $\ln r$. There exist

(i) ***positive*** *solutions, such that $r^{-\gamma}w$ is periodic in $\ln r$;*
(ii) a unique positive solution such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0, with (4.4) near ∞;
(iii) positive solutions such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0, with (4.3) near ∞;
(iv) solutions oscillating around 0 such that $r^{-\gamma}w$ is periodic in $\ln r$;
(v) solutions with a hole, oscillating near ∞, such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$;
(vi) solutions satisfying (4.6) or (4.9) or (4.9) or (4.7) near 0, oscillating around 0 near ∞, such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$;
(vii) solutions positive near 0, oscillating near ∞, such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0 and ∞.

(2) Let $\alpha = \alpha_c$.

(viii) There exist a ***unique nonnegative*** *solution* ***with a hole****, with (4.4) near ∞.*
The regular solutions are as above. There exist solutions of types (iv), (vi), and
(ix) positive solutions such that $r^{-\gamma}w$ is bounded from above near 0, with (4.3) near ∞.

■

th 9.11,fig 14: $\varepsilon = -1, N = 1, p = 3, \alpha = -2.1$ th 9.11,fig 15: $\varepsilon = -1, N = 1, p = 3, \alpha = -2$

Proof. (1) Let $\alpha^* < \alpha < \alpha_c$ (see figure 14). Then $\mathcal{T}_\alpha$ stays in $\mathcal{Q}_4$, but cannot converge neither to M_ℓ which is a sink, nor to $(0,0)$ since $\mathcal{T}_\alpha \neq \mathcal{T}_\varepsilon$. It has a limit cycle $\mathcal{O}_\alpha$ in $\mathcal{Q}_4$ at $-\infty$, surrounding M_ℓ, and w is of type (ii). The orbit $\mathcal{O}_\alpha$ corresponds to solutions of type (i). There exist positive solutions converging to M_ℓ at ∞, with a limit cycle $\mathcal{O}_\ell$ at $-\infty$ surrounded by $\mathcal{O}_\alpha$, and w is of type (iii). This cycle is unique ($\mathcal{O}_\ell = \mathcal{O}_\alpha$) for $\alpha - \alpha^*$ small enough, from Proposition 2.5. The trajectory $\mathcal{T}_\varepsilon$ still cannot stay in $\mathcal{Q}_4$. As in the case $\alpha \leqq \alpha^*$, $\mathcal{T}_\varepsilon$ has a limit cycle $\mathcal{O}_\varepsilon$ surrounding the three stationary points, w is of type (v), and $\mathcal{T}_r$ is oscillating around 0, and there exist solutions of type (vi). Any trajectory $\mathcal{T}_{[P]}$ with $P \notin \mathcal{T}_\varepsilon$ in $\mathcal{Q}_4$ in the domain delimitated by $\mathcal{O}_\alpha$ and $\mathcal{O}_\varepsilon$ admits $\mathcal{O}_\alpha$ as a limit cycle at $-\infty$ and $\mathcal{O}_\varepsilon$ at ∞, and w is of type (vii).

(2) Let $\alpha = \alpha_c$ (see figure 15). The homoclinic trajectory $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$ corresponds to the solution w of type (viii). The trajectory $\mathcal{T}_r$ has a limit cycle $\mathcal{O}_r$ surrounding the three points. Thus there exist solutions of types (iv) or (vi). Any trajectory ending up at M_ℓ at ∞ is bounded, contained in the domain delimitated by $\mathcal{T}_\varepsilon$, and its limit set at $-\infty$ is the homoclinic trajectory $\mathcal{T}_\varepsilon$, or a cycle around M_ℓ, and w is of type (ix).

Theorem 9.12 *Assume* $\varepsilon = -1$, *and* $\alpha_c < \alpha < -p'$.

There exist a unique nonnegative solution w *of* ($\boldsymbol{E}_w$) *with a hole, with* $r^{-\gamma}w$ *bounded from above and below at* ∞. *The regular solutions have at least two zeros.*

(1) Either there exist oscillating solutions such that $r^{-\gamma}w$ *is periodic in* $\ln r$. *Then the regular solutions have an infinity of zeros, and* $r^{-\gamma}w$ *is asymptotically periodic in* $\ln r$. *There exist*

(i) solutions satisfying (4.6) or (4.9) or (4.9) or (4.7) near 0*, oscillating near* ∞*, such that* $r^{-\gamma}w$ *is asymptotically periodic in* $\ln r$;
(ii) a unique solution oscillating near 0, such that $r^{-\gamma}w$ *is asymptotically periodic in* $\ln r$*, and with (4.4) near* ∞;
(iii) solutions positive near 0*, with* $r^{-\gamma}w$ *bounded, and oscillating near* ∞*, such that* $r^{-\gamma}w$ *is asymptotically periodic in* $\ln r$*.*

(2) Or all the solutions have a finite number of zeros, and at least two. Two cases may occur:

• *Either regular solutions have* m *zeros and* $r^{-\gamma}w$ *bounded from above and below at* ∞*. Then there exist*

(iv) solutions with m *zeros, with (4.6) or (4.9), with (4.4) near* ∞;
(v) solutions with m *zeros with (4.6) or (4.9) and* $r^{-\gamma}w$ *bounded from above and below at* ∞;
(vi) solutions with $m+1$ *zeros with (4.6) or (4.9) and* $r^{-\gamma}w$ *bounded from above and below at* ∞;
(vii) (for $p > N$*) a unique solution with* m *zeros,with (4.7) or (4.10) and* $r^{-\gamma}w$ *bounded from above and below at* ∞*.*

• *Or regular solutions have* m *zeros and (4.4) holds near* ∞*. Then there exist solutions of type (vi) or (vii).*

■

th 9.12,fig 16: $\varepsilon = -1, N = 1, p = 3, \alpha = -1.98$ th 9.12, fig 17: $\varepsilon = -1, N = 1, p = 3, \alpha = -1.90$

Proof. Here $\mathcal{T}_\varepsilon$ stays in $\mathcal{Q}_4$, converges to M_ℓ or has a limit cycle around M_ℓ, thus w has a hole and $r^{-\gamma}w$ bounded from above and below at ∞. If $\alpha \geqq \alpha_2$, there is no cycle in $\mathcal{Q}_4$, from Proposition 4.4, thus $\mathcal{T}_\varepsilon$ converges to M_ℓ.

(1) Either there exists a cycle surrounding $(0,0)$ and $\pm M_\ell$, thus solutions w oscillating around 0, such that $r^{-\gamma}w$ is periodic in $\ln r$. Then $\mathcal{T}_r$ has such a limit cycle $\mathcal{O}_r$, and w is oscillating around 0. The trajectory $\mathcal{T}_\alpha$ has a limit cycle at $-\infty$ of the same type $\mathcal{O}_\alpha \subset \mathcal{O}_r$, and w is of type (ii). For any $P \notin \mathcal{T}_\varepsilon$ in the interior domain in $\mathcal{O}_\alpha$, $\mathcal{T}_{[P]}$ admits $\mathcal{O}_\alpha$ as a limit cycle at $-\infty$ and converges to M_ℓ at ∞, or has a limit cycle in $\mathcal{Q}_4$; and w is of type (iii). For any $P \notin \mathcal{T}_r$, in the domain exterior to $\mathcal{O}_r, \mathcal{T}_{[P]}$ has $\mathcal{O}_\alpha$ as limit cycle at ∞, and w is of type (i).

(2) Or no such cycle exists. Then any trajectory converges at ∞, any trajectory, apart from $\pm\mathcal{T}_\alpha$, converges to $\pm M_\ell$ or has a limit cycle in $\mathcal{Q}_1$. All the trajectories end up in $\mathcal{Q}_2$ or $\mathcal{Q}_4$. Since $\mathcal{T}_r$ starts in $\mathcal{Q}_1$, y has at least one zero. Suppose that it is unique. Then $\mathcal{T}_r$ converges to $-M_\ell$, thus Y stays positive. Consider the function $Y_\alpha = e^{(\alpha+\gamma)(p-1)\tau}Y$ defined by (2.3) with $d = \alpha$. From Theorem 3.3, $Y_\alpha = (a\,|\alpha|\,/N)e^{(\alpha(p-1)+p)\tau}(1+o(1))$ near $-\infty$; thus Y_α tends to ∞, since $\alpha < p'$. And $Y_\alpha = (\gamma\ell)^{p-1}e^{(\alpha+\gamma)(p-1)\tau}$ near ∞, thus also Y_α tends to ∞; then it has a minimum point τ, and from (2.6), $Y_\alpha''(\tau) = (p-1)^2(\eta-\alpha)(p'+\alpha)Y_\alpha < 0$, which is contradictory. Thus y has a number $m \geqq 2$ of zeros.

Either $\mathcal{T}_r \neq \mathcal{T}_\alpha$. Since the slope of $\mathcal{T}_\alpha$ near $-\infty$ is infinite and the slope of $\mathcal{T}_r$ is finite, $\mathcal{T}_\alpha$ cuts the line $\{y=0\}$ at m points, starts from $\mathcal{Q}_1$, and w is of type (iv). For any P in the domain of $\mathcal{Q}_1$ between $\mathcal{T}_r$ and $\mathcal{T}_\alpha$, $\mathcal{T}_{[P]}$ cuts $\{y=0\}$ at $m+1$ points, and w is of type (v). For any P in the domain of $\mathcal{Q}_1$ above $\mathcal{T}_r$, $\mathcal{T}_{[P]}$ cuts the line $\{y=0\}$ at $m+1$ points, and w is of type (vi). If $p > N$, the trajectories $\mathcal{T}_-$ and $\mathcal{T}_u$ cut the line $\{y=0\}$ at m points, and w is of type (vii).

Or $\mathcal{T}_r = \mathcal{T}_\alpha$, and then we find only trajectories with w of type (vi) or (vii). ■

Remark 9.13 *Consider the regular solutions in the range $\alpha_c < \alpha < -p'$. We conjecture that there exists a decreasing sequence $(\bar{\alpha}_n)$, with $\bar{\alpha}_0 = -p'$ and $\alpha_c < \bar{\alpha}_n$ such that for $\alpha \in (\bar{\alpha}_m, \bar{\alpha}_{m-1})$, y has m zeros and converges to $\pm M_\ell$; and for $\alpha = \bar{\alpha}_m$, y has $m+1$ zeros and converges to $(0,0)$, thus $\mathcal{T}_r = \mathcal{T}_\alpha$. We presume that $(\bar{\alpha}_m)$ has a limit $\bar{\alpha} > \alpha_c$. And for $\alpha < \bar{\alpha}$, y has an infinity of zeros, in other words there exists a cycle $\mathcal{O}_r$ surrounding $\{0\}$ and $\pm M_\ell$.*

Numerically, for $\alpha = \alpha_c$, the cycle $\mathcal{O}_r$ seems to be the unique cycle surrounding the three points. But for $\alpha > \alpha_c$ and $\alpha - \alpha_c$ small enough, there exist ***two different cycles*** *$\mathcal{O}_\alpha \subset \mathcal{O}_r$ (see figure 15). As α increases, we observe the coalescence of those cycles; they disappear after some value $\bar{\alpha}$ (see figure 16).*

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