Powers of sequences and convergence of ergodic averages

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POWERS OF SEQUENCES AND CONVERGENCE OF ERGODIC AVERAGES

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ABSTRACT. A sequence (s_n) of integers is good for the mean ergodic theorem if for each invertible measure preserving system $(X, \mathcal{B}, \mu, T)$ and any bounded measurable function f, the averages $\frac{1}{N}\sum_{n=1}^{N} f(T^{s_n}x)$ converge in the $L^2(\mu)$ norm. We construct a sequence (s_n) that is good for the mean ergodic theorem, but the sequence (s_n^2) is not. Furthermore, we show that for any set of bad exponents B, there is a sequence (s_n) where (s_n^k) is good for the mean ergodic theorem exactly when k is not in B. We then extend this result to multiple ergodic averages of the form $\frac{1}{N}\sum_{n=1}^{N} f_1(T^{s_n}x)f_2(T^{2s_n}x)\ldots f_\ell(T^{\ell s_n}x)$. We also prove a similar result for pointwise convergence of single ergodic averages.

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1. INTRODUCTION

1.1. **Main result.** It is well known that for any fixed positive integer k the sequence $1^k, 2^k, 3^k \ldots$ is good for the mean ergodic theorem. This means that for every measure preserving system and function f in $L^2(\mu)$, the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^{n^k}x) \tag{1.1}$$

converge in the $L^2(\mu)$ norm as $N \to \infty$. Using the spectral theorem for unitary operators, this is equivalent to the convergence of the averages

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i n^k \alpha} \tag{1.2}$$

as $N \to \infty$ for any real number α.

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An illustrative question for our paper is the following. Is there a sequence $s_1, s_2, s_3 \dots$ of positive integers so that the averages

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i s_n \alpha} \tag{1.3}$$

converge for any real number α, but for some α, the averages

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i s_n^2 \alpha} \tag{1.4}$$

do not converge as $N \to \infty$? In other words, we ask if there is a sequence $s_1, s_2, s_3 \dots$ of positive integers which is good for the mean ergodic theorem, but the sequence of squares $s_1^2, s_2^2, s_3^2 \dots$ of the sequence is not good for the mean ergodic theorem?

Similarly, we can ask: is there a sequence $s_1, s_2, s_3 \dots$ of positive integers which is *not* good for the mean ergodic theorem, but the sequence of squares $s_1^2, s_2^2, s_3^2 \dots$ of the sequence *is* good for the mean ergodic theorem?

Perhaps surprisingly, the answer to *both* questions is *yes*, indicating that the convergence properties of positive powers of a sequence are independent of those of the original sequence. In fact, in this paper we prove the following result showing the total independence of powers of a sequence for the mean ergodic theorem.

Theorem A. *Let B be an arbitrary set of positive integers. Then there exists an increasing sequence $s_1, s_2, s_3 \dots$ of positive integers such that*

- *The sequence $s_1^g, s_2^g, s_3^g \dots$ is good for the mean ergodic theorem for any "good" exponent $g \in \mathbb{N} \setminus B$.*
- *The sequence $s_1^b, s_2^b, s_3^b \dots$ is not good for the mean ergodic theorem for any "bad" exponent $b \in B$.*

Using the spectral theorem for unitary operators, we get the following equivalent formulation of our theorem:

Theorem A′. *Let B be an arbitrary set of positive integers. Then there exists an increasing sequence $s_1, s_2, s_3 \dots$ of positive integers such that*

- *For $g \in \mathbb{N} \setminus B$, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i s_n^g \alpha} \tag{1.5}$$

  converge as $N \to \infty$ for any real number α.
- *For $b \in B$, there exists a real number α such that the averages*

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i s_n^b \alpha} \tag{1.6}$$

  do not converge as $N \to \infty$.

Similar results related to issues of recurrence were proved in [FrLW2]. The original motivation to search for results which express the independence of powers of a sequence for various properties, comes from the papers of Deshouillers, Erdős, Sárközy ([DES]) and Deshouillers, Fouvry ([DFo]). In these papers, the authors prove results analogous to ours but for bases of the positive integers.

In our paper, we generalize Theorem A to multiple ergodic averages and prove a version for pointwise convergence of single ergodic averages. We state these generalizations in the next subsection, where we also give precise definitions of the concepts used throughout the paper.

1.2. **Definitions and generalizations.** All along the article we shall use the word *system*, or the term *measure preserving system*, to designate a quadruple $(X, \mathcal{B}, \mu, T)$, where T is an *invertible* measure preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. By insisting that our system is invertible, we make sure that T^k is well defined for negative integers k.

Definition 1.1. Let ℓ be a positive integer, (s_n) be a sequence of integers, and $(X, \mathcal{B}, \mu, T)$ be a system.

We say that the sequence of integers (s_n) is *good for ℓ-convergence for the system* $(X, \mathcal{B}, \mu, T)$ if for any bounded, measurable functions $f_1, f_2, \ldots, f_\ell$, the averages

$$\frac{1}{N}\sum_{n=1}^{N} f_1(T^{s_n}x)\cdot f_2(T^{2s_n}x)\cdot \ldots \cdot f_\ell(T^{\ell s_n}x)$$

converge in the $L^2(\mu)$ norm as $N \to \infty$.

We say the sequence (s_n) is *universally good for ℓ-convergence* if it is good for ℓ-convergence for any system $(X, \mathcal{B}, \mu, T)$. We often abbreviate this by saying that "(s_n) is good for ℓ-convergence" and refer to the case $\ell = 1$ as *single mean convergence.*

We refer the reader to [RW] for examples of sequences that are good for single mean convergence. Examples of sequences that are good for ℓ-convergence for every $\ell \in \mathbb{N}$, include $s_n = n$, shown by Host and Kra ([HKr1]) and later by Ziegler ([Z]), as well as $s_n = p(n)$ where $p(n)$ is an integer polynomial, as shown by Host and Kra ([HKr2]) and Leibman ([Lei]).

We give a strengthening of Theorem A that related to problems of ℓ-convergence:

Theorem 1.2. *Let B be an arbitrary set of positive integers and $\ell \in \mathbb{N}$. Then there exists an increasing sequence (s_n) of positive integers such that*

- *For every $g \in \mathbb{N} \setminus B$, the sequence (s_n^g) is good for ℓ-convergence.*
- *For every $b \in B$, the sequence (s_n^b) is not good for ℓ-convergence.*

Next we introduce a notion related to pointwise convergence of ergodic averages.

Definition 1.3. Let (s_n) be a sequence of integers, and let $(X, \mathcal{B}, \mu, T)$ be a system.

We say that the sequence (s_n) is *good for the pointwise ergodic theorem for the system* $(X, \mathcal{B}, \mu, T)$ if for any $f \in L^2(\mu)$, the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^{s_n}x)$$

converge as $N \to \infty$ for almost every x.

We say the sequence (s_n) is *universally good for the pointwise ergodic theorem* if it is good for the pointwise ergodic theorem for any system $(X, \mathcal{B}, \mu, T)$. We often abbreviate this by saying that "(s_n) is good for the pointwise ergodic theorem."

Remark. The preceding definition is given for the class of functions in $L^2(\mu)$. It is known that similar definitions for the class of functions in $L^p(\mu)$ gives rise to different notions for different values of $p \in [1, +\infty]$ (see [RW, Chapter VII]). However, for the sequences that we construct in the present paper (and which have positive density) the properties of being good for the class of functions in $L^p(\mu)$ are equivalent for all $p > 1$. For $p = 1$, our arguments do not apply, the reason being that there is no strong maximal inequality along the sequence (n^k) in $L^1(\mu)$.

Notice that any sequence that is good for the pointwise ergodic theorem is also good for the mean ergodic theorem. Various examples of sequences that are good for the pointwise ergodic theorem are known (see e.g. [RW]). A particular example that will be used later is the case of a sequence $(p(n))$ where p is any polynomial with integer coefficients. This case was treated by Bourgain in [Bou].

We give a strengthening of Theorem A related to problems of pointwise convergence:

Theorem 1.4. *Let B be an arbitrary set of positive integers. Then there exists an increasing sequence (s_n) of positive integers such that*

- *For every $g \in \mathbb{N} \setminus B$, the sequence (s_n^g) is good for the pointwise ergodic theorem.*
- *For every $b \in B$, the sequence (s_n^b) is not good for the mean ergodic theorem.*

1.3. **Further remarks and conjectures.** If for a given $\ell \in \mathbb{N}$, the sequence (s_n) is good for ℓ-convergence, then, as Theorem 1.2 shows, we cannot assert in general that any of its powers (s_n^k), where $k \geq 2$, is good for ℓ-convergence. In contrast with this, we expect the following result to be true:

Conjecture 1. *Suppose that the sequence of integers (s_n) is good for ℓ-convergence for every $\ell \in \mathbb{N}$. Then for every $k \in \mathbb{N}$, the sequence (s_n^k) is good for ℓ-convergence for every $\ell \in \mathbb{N}$.*

To support this conjecture, let us mention that if the sequence (s_n) is good for 2-convergence then the sequence (s_n^2) is good for single mean convergence (see Lemma 3.1 below). In fact, Conjecture 1 would be true if the following more general statement holds:

Conjecture 2. *Suppose that the sequence of integers (s_n) is good for mean $(k\ell)$-convergence. Then the sequence (s_n^k) is good for ℓ-convergence.*

The conjecture holds for $\ell = 1$. This is shown in [FrLW1] and a key ingredient of the proof is the spectral theorem for unitary operators which gives convenient necessary and sufficient conditions for single mean convergence. We currently do not have such a convenient characterization for ℓ-convergence when $\ell \geq 2$. The following conjecture would fill this gap if true:

Conjecture 3. *Let (s_n) be a sequence of integers. The following three statements are equivalent:*

- *The sequence (s_n) is good for ℓ-convergence.*
- *The sequence (s_n) is good for ℓ-convergence for every ℓ-step nilsystem.*[1]
- *For every ℓ-step nilmanifold $X = G/\Gamma$, every $a \in G$, and $f \in C(X)$, the sequence $\left(\frac{1}{N}\sum_{n=1}^{N} f(a^{s_n}\Gamma)\right)$ converges as $N \to +\infty$.*

[1] If G is an ℓ-step nilpotent Lie group and Γ is a discrete cocompact subgroup, then the homogeneous space $X = G/\Gamma$ is called an *ℓ-step nilmanifold*. If $T_a(g\Gamma) = (ag)\Gamma$ for some $a \in G$, $\mathcal{X}$ is the Borel σ-algebra of X, and m is the Haar measure on X, then the system $(X, \mathcal{X}, m, T_a)$ is called an ℓ-step nilsystem.

We are mainly interested in knowing if the third (or second) condition implies the first. In the case that the set $S = \{s_n, n \in \mathbb{N}\}$ has positive upper density, this implication follows immediately from the nilsequence decomposition result of Bergelson, Host, and Kra (Theorem 1.9 in [BeHKr]).

For general sequences (s_n), using the spectral theorem for unitary operators, we can verify Conjecture 3 for $\ell = 1$. It is possible to see that if Conjecture 3 is true then Conjecture 2 is also true (and hence Conjecture 1).

Notation: The following notation will be used throughout the article: $Tf = f \circ T$, $e(t) = e^{2\pi i t}$.

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2. Single mean convergence

In this section we shall prove Theorem A. This will help us illustrate the main ideas behind the more complicated arguments we shall use in the proof of Theorem 1.2 and of Theorem 1.4 (which both extend Theorem A).

We shall prove our theorem in its equivalent formulation of Theorem A′.

Let B be a fixed set of positive integers (possibly empty), and let α be a fixed irrational number. It is sufficient to find a sequence (s_n) satisfying the following two conditions:

(s1) For every $b \in B$, the sequence $\left(\frac{1}{N}\sum_{n=1}^{N} e(s_n^b\alpha)\right)$ diverges.

(s2) For every $g \in \mathbb{N} \setminus B$ and every $\beta \in \mathbb{R}$, the sequence $\left(\frac{1}{N}\sum_{n=1}^{N} e(s_n^g\beta)\right)$ converges.

The rest of this section will be devoted to the construction of a sequence (s_n) that satisfies conditions (s1) and (s2).

2.1. Definition of the sequence (s_n).

We denote

(2.1)
$$I_+ = \left\{x \in \mathbb{T} \mid \cos(2\pi x) \geq \sqrt{2}/2\right\} \quad \text{and} \quad I_- = \left\{x \in \mathbb{T} \mid \cos(2\pi x) \leq -\sqrt{2}/2\right\}.$$

These are intervals of length 1/4 on the torus. The sequence (s_n) consists of the elements of a set S, taken in increasing order, that is defined as follows:

(2.2)
$$S = \bigcup_{j\geq 1} \left\{n \in \mathbb{N} \mid [2^{2j-1} \leq n < 2^{2j},\ n^{b_j}\alpha \in I_+] \text{ or } [2^{2j} \leq n < 2^{2j+1},\ n^{b_j}\alpha \in I_-]\right\}$$

for some appropriately chosen sequence (b_j) of elements of B. We shall construct a sequence (b_j) so that:

• Every element of B appears infinitely often in the sequence (b_j). This guarantees that condition (s1) holds.

• The first appearance of elements of B in the sequence (b_j) happens late enough (with respect to j) to guarantee that certain equidistribution properties are satisfied. All unspecified elements b_j will be set to be equal to some fixed element of B. This will enable us to verify condition (s2).

Let us now state explicitly the properties that the sequence (b_j) is going to satisfy:

(b1) $b_j \in B$, and all elements of B appear infinitely often in the sequence (b_j).

(b2) We have

$$\lim_{j\to\infty} \sup_{N>2^{2^{j-1}}} \left| \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{I_\pm}(n^{b_j}\alpha) - \frac{1}{4} \right| = 0.$$

(b3) For every nonzero $l \in \mathbb{Z}$, $\beta \in \mathbb{R}$, and $g \in G$, we have

$$\lim_{j\to\infty} \sup_{N\geq 2^{2^{j-1}}} \left| \frac{1}{N}\sum_{n=1}^{N} e(ln^{b_j}\alpha + n^g\beta) \right| = 0.$$

In the next subsection we construct such a sequence (b_j).

2.2. **Construction of the sequence** (b_j)**.** The following lemma will be essential for our construction:

Lemma 2.1. *Let $b, g \in \mathbb{N}$ with $b > g$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{I_\pm}(n^b\alpha) = \frac{1}{4}, \text{ and}$$

$$\lim_{N\to\infty} \sup_{\beta\in\mathbb{R}} \left| \frac{1}{N}\sum_{n=1}^{N} e(n^b\alpha + n^g\beta) \right| = 0.$$

Proof. The first part follows from Weyl's equidistribution theorem. The second part is a direct consequence of van der Corput's classical inequality (see e.g. [KN]). Applying it g times, we can estimate the trigonometric sums by a quantity that converges to 0 as $N \to \infty$ uniformly in β. This completes the proof. □

Note that, for each fixed b only finitely many g's are involved in the lemma. Therefore, the convergence is also uniform in g.

Proposition 2.2. *There exists a sequence (b_j) that satisfies conditions (b1), (b2), and (b3) of Section 2.1.*

Proof. Using Lemma 2.1, we get that for every $b \in \mathbb{N}$, $l \in \mathbb{N}$, and $\varepsilon > 0$, there exists $J = J(b, l, \varepsilon) \geq 1$ that satisfies

$$\sup_{N\geq 2^{2^{J-1}}} \left| \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{I_\pm}(n^b\alpha) - \frac{1}{4} \right| \leq \varepsilon; \tag{2.3}$$

and such that

$$\sup_{N\geq 2^{2^{J-1}}} \sup_{\beta\in\mathbb{R}} \sup_{g\in\mathbb{N},\, g<b} \left| \frac{1}{N}\sum_{n=1}^{N} e(ln^b\alpha + n^g\beta) \right| \leq \varepsilon. \tag{2.4}$$

Furthermore, we can assume that $J = J(b, l, \varepsilon)$ is increasing with respect to the variables b, l, and decreasing with respect to the variable ε.

We write $B = \{a_t, t \in \mathbb{N}\}$ where $a_1 < a_2 < \ldots$. We construct a sequence (b_j) that satisfies the following conditions: (i) every integer a_t appears infinitely often in the range of (b_j), (ii) for $t \geq 2$, the first appearance of a_t in (b_j) happens at a time j that is greater than $J(a_t, t, 1/t)$, and (iii) all values of b_j that are left unspecified are set to be equal to a_1. Notice that condition (ii) guarantees that (2.3) and (2.4) hold for $b = b_j$ and $J = j$.

The explicit construction goes as follows:

Define $J_1 = J(a_1, 1, 1)$ and $b_{J_1} = a_1$.

Define $J_2 = \max\{J(a_2, 2, 1/2), J_1 + 2\}$ and $b_{J_2} = a_2$, $b_{J_2+1} = a_1$.

Inductively, we define $J_t = \max\{J(a_t, t, 1/t), J_{t-1} + t\}$

and $b_{J_t} = a_t$, $b_{J_t+1} = a_{t-1}, \ldots, b_{J_t+t-1} = a_1$.

We claim that this sequence (b_j) has the advertised properties. First, we note that every integer a_t appears infinitely many times in the range of (b_j), so condition (b1) holds. Now let $\beta \in \mathbb{R}$, $g \in G$, and $l \in \mathbb{N}$. Choose j large enough so that $J_t \leq j < J_{t+1}$ for some $t \geq l$. Then, by construction, we have $b_j \in \{a_1, a_2, \ldots, a_t\}$. If $N \geq 2^{2j-1}$ then $N \geq 2^{2J_t-1}$, and $J_t \geq J(a_k, l, 1/t)$ for all k between 1 and t. It follows from (2.3) that

$$\sup_{N \geq 2^{2j-1}} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{I_\pm}(n^{b_j}\alpha) - \frac{1}{4} \right| \leq \frac{1}{t},$$

and so condition (b2) holds. Furthermore, it follows from (2.4) that for every b_j greater than g we have

$$\sup_{N \geq 2^{2j-1}} \left| \frac{1}{N} \sum_{n=1}^{N} e(ln^{b_j}\alpha + n^g\beta) \right| \leq \frac{1}{t}. \tag{2.5}$$

It remains to deal with those b_j that are less than g. Since there are only finitely many values of the sequence (b_j) that are less than g, by Weyl's equidistribution theorem we have

$$\lim_{N\to\infty} \sup_{j\in\mathbb{N}, b_j < g} \left| \frac{1}{N} \sum_{n=1}^{N} e(ln^{b_j}\alpha + n^g\beta) \right| = 0.$$

Combining this with (2.5) gives that condition (b3) also holds, completing the proof. □

2.3. **The sequence (s_n) satisfies conditions (s1) and (s2).** The goal of this section is to complete the proof of Theorem A′ by proving the following proposition:

Proposition 2.3. *Suppose that the sequence (b_j) satisfies conditions (b1),(b2), and (b3) of Section 2.2. Then the sequence (s_n) defined by* (2.2) *satisfies conditions (s1) and (s2).*

First we show that the set S in (2.2) has positive density.

Lemma 2.4. *Suppose that the sequence (b_j) satisfies condition (b2) of Section 2.1. Then the set S in* (2.2) *has density* $1/4$.

Proof. This is a direct consequence of Lemma 5.3 in the Appendix. □

Since S has positive density, condition (s1) is equivalent to

$$\text{For every } b \in B, \text{ the sequence } \left(\frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_S(n) e(n^b\alpha) \right) \text{ diverges,} \tag{2.6}$$

and condition (s2) is equivalent to

(2.7)

$$\text{For every } g\in\mathbb{N}\setminus B \text{ and } \beta\in\mathbb{R}, \text{ the sequence } \Big(\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_S(n)e(n^g\beta)\Big) \text{ converges.}$$

We first show that the conditions imposed on the sequence (b_j) guarantee that condition (2.6) (and as a result condition (s1)) is satisfied by the set S.

Proposition 2.5. *Suppose that the sequence (b_j) satisfies conditions (b1) and (b2) of Section 2.1. Then the sequence (s_n) defined by* (2.2) *satisfies condition (s1).*

Proof. Fix $b\in B$. By condition $(b1)$ there are arbitrarily large values of j for which $b_j=b$. For any such j we have

$$\frac{1}{2^{2j}}\sum_{n=1}^{2^{2j}}\mathbb{1}_S(n)\cos(2\pi n^b\alpha)-\frac{1}{2^{2j+1}}\sum_{n=1}^{2^{2j+1}}\mathbb{1}_S(n)\cos(2\pi n^b\alpha)=$$

$$\frac{1}{2^{2j+1}}\sum_{n=1}^{2^{2j}}\mathbb{1}_S(n)\cos(2\pi n^b\alpha)-\frac{1}{2^{2j+1}}\sum_{n=2^{2j}+1}^{2^{2j+1}}\mathbb{1}_S(n)\cos(2\pi n^b\alpha)\geq$$

$$\frac{1}{2^{2j+1}}\sum_{n=1}^{2^{2j-1}}\mathbb{1}_S(n)(-1)+\frac{1}{2^{2j+1}}\sum_{n=2^{2j-1}+1}^{2^{2j}}\mathbb{1}_S(n)(\sqrt{2}/2)-\frac{1}{2^{2j+1}}\sum_{n=2^{2j}+1}^{2^{2j+1}}\mathbb{1}_S(n)(-\sqrt{2}/2).$$

Using condition $(b2)$ and Lemma 2.4, we see that, for large j, the last quantity is near $-\frac{1}{16}+\frac{\sqrt{2}}{32}+\frac{\sqrt{2}}{16}$, which is positive. This shows that (2.6) holds. Since the set S has positive density (Lemma 2.4), condition (s1) is also satisfied. □

Next we show that the conditions imposed on the sequence (b_j) guarantee that condition (2.7) (and hence condition (s2)) is satisfied. We first need two lemmas.

Lemma 2.6. *Let l,m be two nonzero integers, $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, and suppose that the sequence (b_j) satisfies condition (b3) of Section 2.1. Let us define a sequence (e_n) by*

$$e_n=\begin{cases}e(ln^{b_j}\alpha), & n\in[2^{2j-1},2^{2j});\\ e(mn^{b_j}\alpha), & n\in[2^{2j},2^{2j+1}).\end{cases}$$

Then for every $\beta\in\mathbb{R}$ and $g\in\mathbb{N}\setminus B$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e_n\,e(n^g\beta)=0.$$

Proof. Fix $\beta\in\mathbb{R}$ and $g\in\mathbb{N}\setminus B$. By condition (b3) we have for every nonzero integer k that

$$\lim_{j\to\infty}\sup_{N>2^{2j-1}}\left|\frac{1}{N}\sum_{n=1}^{N}e(kn^{b_j}\alpha+n^g\beta)\right|=0.$$

Applying this for $k=l$, and $k=m$, and using Lemma 5.3 in the Appendix we get the advertised result. □

Given a sequence (b_j) of positive integers and functions $\phi, \psi \colon \mathbb{T} \to \mathbb{C}$, define the sequence

$$f_n(\phi,\psi) = \begin{cases} \phi(n^{b_j}\alpha), & n \in [2^{2j-1}, 2^{2j}); \\ \psi(n^{b_j}\alpha), & n \in [2^{2j}, 2^{2j+1}). \end{cases} \tag{2.8}$$

Lemma 2.7. *Let $\phi_0 = \mathbf{1}_{I_+} - 1/4, \psi_0 = \mathbf{1}_{I_-} - 1/4$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and suppose that the sequence (b_j) satisfies condition (b3) of Section 2.1. Then for every $\beta \in \mathbb{R}$ and $g \in \mathbb{N} \setminus B$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n(\phi_0,\psi_0)\, e(n^g\beta) = 0.$$

Proof. By Lemma 2.6, the result is true if in place of ϕ_0 and ψ_0 we use trigonometric polynomials. To complete the proof, we use a standard approximation argument. We give the details for the convenience of the reader.

It suffices to show that for every $\varepsilon > 0$ there exist trigonometric polynomials ϕ and ψ with zero integral such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |f_n\left(\phi_0,\psi_0\right) - f_n(\phi,\psi)| \le \varepsilon. \tag{2.9}$$

We can approximate in $L^1(\mathbb{T})$ the function ϕ_0 by a trigonometric polynomial. After composing with a translation this trigonometric polynomial will approximate the function ψ_0 as well. So there exist two trigonometric polynomials ϕ and ψ, with zero integral, such that $\int_{\mathbb{T}} |\phi - \phi_0| = \int_{\mathbb{T}} |\psi - \psi_0| = \theta \le \varepsilon$. Notice that

$$|f_n\left(\phi_0,\psi_0\right) - f_n(\phi,\psi)| = f_n\left(|\phi-\phi_0|, |\psi-\psi_0|\right). \tag{2.10}$$

So in order to establish (2.9) it suffices to show that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n(|\phi-\phi_0|, |\psi-\psi_0|) \le \varepsilon. \tag{2.11}$$

Let $\Phi_0 = |\phi - \phi_0|$ and $\Psi_0 = |\psi - \psi_0|$. Since both Φ_0 and Ψ_0 are Riemann integrable and have integral θ, the following holds: For every $\delta > 0$ there exist four continuous functions $\phi_1, \phi_2, \psi_1, \psi_2$, with zero integral, such that

$$\phi_1 + \theta - \delta \le \Phi_0 \le \phi_2 + \theta + \delta \quad \text{and} \quad \psi_1 + \theta - \delta \le \Psi_0 \le \psi_2 + \theta + \delta.$$

It follows that

$$f_n(\phi_1,\psi_1) + \theta - \delta \le f_n(\Phi_0,\Psi_0) \le f_n(\phi_2,\psi_2) + \theta + \delta. \tag{2.12}$$

Moreover, since

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n(\phi,\psi) = 0 \tag{2.13}$$

for every trigonometric polynomials ϕ and ψ with zero integral, by uniform approximation, this remains true if ϕ and ψ are continuous functions on the torus, with zero integral. Thus, we deduce from (2.12) and (2.13) (applied to $\phi = \phi_i$, $\psi = \psi_i$ for $i = 1, 2$), and the fact that δ was arbitrary, that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n(\Phi_0,\Psi_0) = \theta \le \varepsilon.$$

Hence, (2.11) is established, completing the proof. □

Proposition 2.8. *Suppose that the sequence (b_j) satisfies conditions (b2) and (b3) of Section 2.1. Then the sequence (s_n) defined by* (2.2) *satisfies condition (s2).*

Proof. We apply Lemma 2.7. We get for every $\beta \in \mathbb{R}$ and $g \in \mathbb{N} \setminus B$ that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (\mathbb{1}_S(n) - 1/4)\, e(n^g \beta) = 0.$$

Hence, condition (2.7) is satisfied. Since the set S has positive density (Lemma 2.4), condition (s2) is also satisfied. □

Combining Propositions 2.5 and 2.8 we deduce Proposition 2.3, completing the proof of Theorem A'.

3. Multiple mean convergence

In this section we shall prove Theorem 1.2. The argument is similar to the one used to prove Theorem A, but there are some extra complications since our analysis relies on some more intricate multiple convergence results. To avoid repetition, we do not give details of proofs that can be immediately extracted using arguments of the previous section.

3.1. **A reduction.** We need one preliminary result that was proved in [FrLW1] in the special case where the polynomial p is a monomial. A very similar argument gives the following more general result:

Lemma 3.1. *Suppose that the sequence (s_n) is good for ℓ-convergence. Then for every polynomial $p \in \mathbb{R}[t]$ with $\deg p \leq \ell$, the sequence $\left(\frac{1}{N}\sum_{n=1}^{N} e(p(s_n))\right)$ converges.*

Example 1. Let us illustrate how one proves Lemma 3.1 in the case where $\ell = 2$. Suppose that the sequence (s_n) is good for 2-convergence. Let $p(t) = 2\alpha t^2 + \beta t + \gamma$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. We define the transformation $R\colon \mathbb{T}^3 \to \mathbb{T}^3$ by

$$R(t_1, t_2, t_3) = (t_1 + \alpha, t_2 + 2t_1 + \alpha, t_3 + \beta),$$

and for $k \in \mathbb{Z}$ the functions

$$f_1(t_1, t_2, t_3) = e(k(-2t_2 + t_3)), \quad f_2(t_1, t_2, t_3) = e(kt_2).$$

Then

$$R^n(t_1, t_2, t_3) = (t_1 + n\alpha, t_2 + 2nt_1 + n^2\alpha, t_3 + n\beta).$$

As a consequence, the averages

$$\frac{1}{N}\sum_{n=1}^{N} R^{s_n} f_1 \cdot R^{2s_n} f_2 = e(k(t_3 - t_2)) \cdot \frac{1}{N}\sum_{n=1}^{N} e(k(2\alpha s_n^2 + \beta s_n))$$

converge as $N \to \infty$.

Using the previous lemma, we can deduce Theorem 1.2 from the following result that we shall prove next:

Proposition 3.2. *Let B be an arbitrary set of positive integers, $\ell \in \mathbb{N}$, and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then there exists an increasing sequence (s_n) of integers such that*

(s1) *For every $b \in B$, the sequence $\left(\frac{1}{N}\sum_{n=1}^{N} e((s_n^{\ell b} + s_n^b)\alpha)\right)$ diverges.*

(s2) *For every $g \in \mathbb{N} \setminus B$, system $(X, \mathcal{B}, \mu, T)$, and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, the sequence $\left(\frac{1}{N}\sum_{n=1}^{N} T^{s_n^g} f_1 \cdot T^{2s_n^g} f_2 \cdot \ldots \cdot T^{\ell s_n^g} f_\ell\right)$ converges in $L^2(\mu)$.*

The rest of this section will be devoted to the construction of a sequence (s_n) that satisfies conditions (s1) and (s2).

3.2. **Definition of the sequence (s_n).** Let α be any irrational number and I_+ and I_- be the intervals defined in (2.1). The sequence (s_n) consists of the elements of a set S, taken in increasing order, that is defined as follows:

$$S = \bigcup_{j\geq 1} \big\{ n \in \mathbb{N} \mid [2^{2j-1} \leq n < 2^{2j},\ (n^{\ell b_j} + n^{b_j})\alpha \in I_+] \ \text{ or } \ [2^{2j} \leq n < 2^{2j+1},\ (n^{\ell b_j} + n^{b_j})\alpha \in I_-] \big\} \tag{3.1}$$

where the sequence (b_j) satisfies the following conditions:

(b1) $b_j \in B$, and all elements of B appear infinitely often in the sequence (b_j).

(b2) We have
$$\lim_{j\to\infty} \sup_{N > 2^{2j-1}} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{I_\pm}(n^{b_j}\alpha) - \frac{1}{4} \right| = 0.$$

(b3) For every system $(X, \mathcal{B}, \mu, T)$, functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, and nonzero $k \in \mathbb{Z}$, we have
$$\lim_{j\to\infty} \sup_{N \geq 2^{2j-1}} \left\| \frac{1}{N} \sum_{n=1}^{N} e(k(n^{\ell b_j} + n^{b_j})\alpha) \cdot T^{n^g} f_1 \cdot T^{2n^g} f_2 \cdot \ldots \cdot T^{\ell n^g} f_\ell \right\|_{L^2(\mu)} = 0.$$

In the next subsection we construct such a sequence (b_j).

3.3. **Construction of the sequence (b_j).** We need two preliminary results. The first was proved in [FrLW2] using the machinery of nil-factors:

Lemma 3.3. *Let $(X, \mathcal{B}, \mu, T)$ be a system, $f_1, \ldots, f_\ell \in L^\infty(\mu)$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. If b, g are distinct positive integers then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e\big((n^{\ell b} + n^b)\alpha\big) \cdot T^{n^g} f_1 \cdot T^{2n^g} f_2 \cdot \ldots \cdot T^{\ell n^g} f_\ell = 0 \tag{3.2}$$

where the convergence takes place in $L^2(\mu)$.

The second is the following result:

Lemma 3.4. *Let $\ell, g \in \mathbb{N}$. Then there exists $d(\ell, g) \in \mathbb{N}$ such that for every $d \geq d(\ell, g)$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we have*

$$\lim_{N\to\infty} \sup_{S_1} \left\| \frac{1}{N} \sum_{n=1}^{N} e\big((n^{\ell d} + n^d)\alpha\big) \cdot T^{n^g} f_1 \cdot T^{2n^g} f_2 \cdot \ldots \cdot T^{\ell n^g} f_\ell \right\|_{L^2(\mu)} = 0,$$

where S_1 is the collection of all systems $(X, \mathcal{B}, \mu, T)$ and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$ with $\|f_i\|_{L^\infty(\mu)} \leq 1$ for $i = 1, \ldots, \ell$.

Proof. The main idea is to apply a Hilbert space version of van der Corput's classical inequality (see Section 5.1) several times in order to get an upper bound for the expression

$$\left\|\frac{1}{N}\sum_{n=1}^{N} e\big((n^{\ell d}+n^d)\alpha\big)\cdot T^{n^g}f_1\cdot T^{2n^g}f_2\cdot\ldots\cdot T^{\ell n^g}f_\ell\right\|_{L^2(\mu)}^{2^{\ell d-1}} \tag{3.3}$$

that does not depend on the transformation T or the functions $f_1,\ldots,f_\ell$. Such an estimate can be obtained using a rather standard argument, very much along the lines of the polynomial exhaustion technique introduced by Bergelson in [Be1]. To do this we shall use Lemma 5.6 and Proposition 5.7 in the Appendix. We get that for d large enough (depending on ℓ and g only), the quantity in (3.3) is bounded by a constant multiple of

$$\frac{1}{H_1\cdots H_{\ell d-1}}\sum_{1\leq h_i\leq H_i}\left|\frac{1}{N}\sum_{n=1}^{N} e\big(\Delta_{h_1,\ldots,h_{\ell d-1}}(n^{\ell d}+n^d)\alpha\big)\right| + o_{N,H_i,H_i\prec N}(1), \tag{3.4}$$

where $\Delta_h(a_n)=a_{n+h}-a_n$, $\Delta_{h_1,\ldots,h_r}(a_n)=\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_r}(a_n)$, and $o_{N,H_i,H_i\prec N}(1)$ denotes a quantity that goes to zero as $N,H_i\to\infty$ in a way that $H_i/N\to 0$. Notice that the sequence $\Delta_{h_1,\ldots,h_{\ell d-1}}(n^{\ell d}+n^d)$ is linear in n. Since α is irrational, letting $N\to+\infty$ and then $H_i\to+\infty$, we get that the quantity (3.4) converges to 0. This completes the proof. □

Using that $\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{I_\pm}((n^{\ell b}+n^b)\alpha)\to\frac{1}{4}$ for $b\in\mathbb{N}$, and Lemmas 3.3 and 3.4, the next result is proved in a similar fashion as Proposition 2.2.

Proposition 3.5. *There exists a sequence (b_j) that satisfies conditions (b1), (b2), and (b3) of Section 3.2.*

3.4. **The sequence (s_n) satisfies (s1) and (s2).** The next result is proved in essentially the same way as Proposition 2.3 and allows us to immediately deduce Theorem 1.2.

Proposition 3.6. *Suppose that the sequence (b_j) satisfies conditions (b1), (b2), and (b3) of Section 3.2. Then the sequence (s_n) defined by* (3.1) *satisfies conditions (s1) and (s2) of Proposition 3.2.*

4. Pointwise convergence

We shall prove Theorem 1.4. The argument is similar to the one used to prove Theorem A. However, extra complications arise since, as is typical for pointwise results, we need to establish quantitative estimates for some trigonometric sums.

Throughout this section we shall assume that the irrational α is badly approximable, that means, there exists a positive real number c such that for every $p\in\mathbb{Z}$ and $q\in\mathbb{N}$ we have $|\alpha-p/q|\geq c/q^2$. In fact, for convenience, we shall fix α to be the *golden mean* $(\sqrt{5}+1)/2$, in which case the previous estimate holds with $c=1/3$.

4.1. **A reduction.** Theorem 1.4 is a direct consequence of the following result:

Theorem 4.1. *Let B be an arbitrary set of positive integers and α be the golden mean. Then, there exists an increasing sequence (s_n) of integers such that*

(s1) *For every $b\in B$ the sequence $\big(\frac{1}{N}\sum_{n=1}^{N} e(s_n^b\alpha)\big)$ diverges.*

(s2) *For every $g \in \mathbb{N} \setminus B$ the sequence (s_n^g) is good for the pointwise ergodic theorem.*

Our first goal is to find a more convenient condition to replace (s2). To do this we are going to use the following lemma:

Lemma 4.2. *Let $(w_n)_{n\in\mathbb{N}}$ be a bounded sequence of complex numbers, and $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive integers, such that for all $\gamma > 1$ we have*

$$\sum_{k=1}^{\infty} \sup_{\beta\in\mathbb{R}} \left| \frac{1}{[\gamma^k]} \sum_{n=1}^{[\gamma^k]} w_n\, e(a_n\beta) \right|^2 < +\infty.$$

Then for every system $(X, \mathcal{B}, \mu, T)$ and $f \in L^2(\mu)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} w_n \cdot T^{a_n} f = 0 \qquad \mu\text{-almost everywhere.}$$

Proof. Using the spectral theorem for unitary operators we get that

$$\Big\| \frac{1}{N} \sum_{n=1}^{N} w_n \cdot T^{a_n} f \Big\|_2^2 = \int \Big| \frac{1}{N} \sum_{n=1}^{N} w_n\, e(a_n t) \Big|^2 \, d\sigma_f(t)$$

holds for every $N \in \mathbb{N}$, where σ_f denotes the spectral measure of the function f. As a consequence

$$\Big\| \frac{1}{N} \sum_{n=1}^{N} w_n \cdot T^{a_n} f \Big\|_2^2 \leq \|f\|_2^2 \cdot \sup_{t\in\mathbb{R}} \Big| \frac{1}{N} \sum_{n=1}^{N} w_n\, e(a_n t) \Big|^2.$$

Combining this with our hypothesis, we get that if $\gamma > 1$, then

$$\lim_{k\to+\infty} \frac{1}{[\gamma^k]} \sum_{n=1}^{[\gamma^k]} w_n \cdot T^{a_n} f = 0 \qquad \mu\text{-almost everywhere for } f \in L^2(\mu).$$

(We used that $\sum_{k=1}^{\infty} \|f_k\|_2 < \infty$ implies $f_k \to 0$ pointwise.) The announced result now follows from Lemma 1.5 in [RW]. □

Let I_+, I_- be the intervals defined by (2.1). Given a sequence of positive integers (b_j) and functions $\phi, \psi \colon \mathbb{T} \to \mathbb{T}$ let $(f_n(\phi, \psi))$ be the sequence defined by (2.8). As in Section 2, we define a sequence (s_n) by taking the elements of the set S given by

$$\mathbb{1}_S(n) = f_n(\mathbb{1}_{I_+}, \mathbb{1}_{I_-})$$

in increasing order.

Proposition 4.3. *Let B be an arbitrary set of positive integers, $\alpha \in \mathbb{R}$, and let the sequences $(f_n(\phi, \psi))$ and (s_n) be as above. Suppose that*

(s1') *For every $b \in B$, the sequence $\left(\frac{1}{N}\sum_{n=1}^{N} e(s_n^b \alpha)\right)$ diverges.*

(s2') *For all trigonometric polynomials ϕ and ψ with zero integral, $\gamma > 1$, and $g \in \mathbb{N} \setminus B$, we have*

$$\sum_{k\geq 0} \sup_{\beta\in\mathbb{R}} \left| \frac{1}{[\gamma^k]} \sum_{n=1}^{[\gamma^k]} f_n(\phi, \psi)\, e(n^g \beta) \right| < +\infty.$$

Then the sequence (s_n) satisfies conditions (s1) and (s2) of Theorem 4.1.

Proof. It is clear that if (s1') holds then also (s1) holds.

It remains to show that condition (s2') implies that for $g \in \mathbb{N} \setminus B$ the sequence (s_n^g) is good for the pointwise ergodic theorem. We start by observing that condition (s2'), combined with Lemma 4.2, guarantees that for every system $(X, \mathcal{B}, \mu, T)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n(\phi, \psi) \cdot T^{n^g} f = 0 \quad \mu\text{-almost everywhere for } f \in L^\infty(\mu). \tag{4.1}$$

Using the approximation argument of Lemma 2.7 and the fact $\frac{1}{N}\sum_{n=1}^{N} f_n(\phi,\psi) \to 0$ (this follows from (4.1)), we conclude that (4.1) remains true if we replace $f_n(\phi,\psi)$ by $f_n(\mathbb{1}_{I_+}, \mathbb{1}_{I_-}) - 1/4 = \mathbb{1}_S(n) - 1/4$. As a consequence, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (\mathbb{1}_S(n) - 1/4) \cdot T^{n^g} f = 0 \quad \mu\text{-almost everywhere for } f \in L^\infty(\mu). \tag{4.2}$$

Using Bourgain's maximal inequality ([Bou]) for the ergodic averages $\frac{1}{N}\sum_{n=1}^{N} T^{n^g} f$, we can to replace $L^\infty(\mu)$ by $L^2(\mu)$ in the preceding statement. Finally, using Bourgain's pointwise ergodic theorem ([Bou], or see the Appendix B of [Be2] for a simpler proof) for the ergodic averages along g-th powers, we get that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_S(n) \cdot T^{n^g} f \quad \text{exists } \mu\text{-almost everywhere for } f \in L^\infty(\mu).$$

Since the set S has positive density (this follows by setting $f = 1$ in (4.2)), we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{s_n^g} f \quad \text{exists } \mu\text{-almost everywhere for } f \in L^2(\mu).$$

Therefore, the sequence (s_n^g) is good for the pointwise ergodic theorem. This completes the proof. □

The rest of this section will be devoted to the construction of a sequence (s_n) that satisfies conditions (s1') and (s2').

4.2. **Definition of the sequence** (s_n)**.** We remind the reader the set of integers B is given and the irrational number α is the golden mean. Let I_+ and I_- be the intervals defined in (2.1). The sequence (s_n) consists of the elements of a set S, taken in increasing order, that is defined as follows:

(4.3)
$$S = \bigcup_{j\geq 1} \left\{ n \in \mathbb{N} \mid [2^{2j-1} \leq n < 2^{2j},\ n^{b_j}\alpha \in I_+] \text{ or } [2^{2j} \leq n < 2^{2j+1},\ n^{b_j}\alpha \in I_-] \right\}$$

where the sequence of integers (b_j) satisfies the following conditions:

(b1) $b_j \in B$, and all elements of B appear infinitely often in the sequence (b_j).

(b2) We have

$$\lim_{j\to\infty} \sup_{N > 2^{2j-1}} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{I_\pm}(n^{b_j}\alpha) - \frac{1}{4} \right| = 0.$$

(b3) For every nonzero $m \in \mathbb{Z}$, and $g \in \mathbb{N} \setminus B$, we have

$$\lim_{j\to\infty} \sup_{N\geq 2^{2j-1}} \sup_{\beta\in\mathbb{R}} \left| \frac{1}{N^{1-\eta(N)}} \sum_{n=1}^{N} e(mn^{b_j}\alpha + n^g\beta) \right| < +\infty$$

where $\eta(N) = (\log_2(N))^{-1/2}$.

In the next subsection we construct such a sequence (b_j).

4.3. **Construction of the sequence** (b_j)**.** The key ingredient in the construction is the following exponential sum estimate:

Proposition 4.4. *Let α be the golden mean, and $(\eta(N))$ be a sequence of real numbers which tends to zero at infinity. For every $b, g \in \mathbb{N}$ with $b \neq g$, and nonzero $m \in \mathbb{Z}$, there exists $N_0 = N_0(b, g, m)$, such that if $N > N_0$ then*

$$\sup_{\beta\in\mathbb{R}} \left| \sum_{n=1}^{N} e(mn^b\alpha + n^g\beta) \right| \leq N^{1-\eta(N)}.$$

Proof. This is an immediate consequence of Lemmas 5.4 and 5.5 in the Appendix. □

Using that $\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{I_\pm}(n^b\alpha) \to \frac{1}{4}$ for $b \in \mathbb{N}$, and Proposition 4.4, the next result is proved in essentially the same way as Proposition 2.2 (the argument is actually somewhat simpler in this case, since we have already combined the estimates dealing with $b < g$ and $b > g$ into a single estimate).

Proposition 4.5. *There exists a sequence (b_j) that satisfies conditions (b1), (b2), and (b3) of Section 4.2.*

4.4. **The sequence** (s_n) **satisfies conditions (s1) and (s2).** The goal of this section is to prove the following proposition, that allows us to immediately deduce Theorem 1.4.

Proposition 4.6. *Suppose that the sequence (b_j) satisfies conditions (b1), (b2), and (b3) of Section 4.2. Then the sequence (s_n) defined by* (4.3) *satisfies conditions (s1) and (s2) of Theorem 4.1.*

Before starting the proof of Proposition 4.6, let us gather some useful properties that the sequence $\eta(N) = (\log_2(N))^{-1/2}$ satisfies:

(η1) The sequence $\left(N^{-\eta(N)}\right)$ is decreasing.

(η2) For every $\gamma > 1$, we have $\displaystyle\sum_{k\geq 0} \left[\gamma^k\right]^{-\eta([\gamma^k])} < +\infty$.

(η3) If we define

$$\rho(N) = \frac{1}{N} \sum_{i=1}^{[\log_2 N]} 2^{i(1-\eta(2^i))},$$

then for every $\gamma > 1$, we have $\displaystyle\sum_{k} \rho\left(\left[\gamma^k\right]\right) < +\infty$.

The first property is obvious.

To check the second property notice that $\eta\left(\left[\gamma^k\right]\right) \sim c/\sqrt{k}$, so $\left[\gamma^k\right]^{-\eta([\gamma^k])} = O\left(\gamma^{-c\sqrt{k}}\right)$ for some constant $c > 0$.

To check the third property notice that

$$\sum_{i=1}^{l} 2^{i(1-\eta(2^i))} = \sum_{i=1}^{l} 2^{i-\sqrt{i}} = \sum_{i=1}^{[l/2]-1} 2^{i-\sqrt{i}} + \sum_{i=[l/2]}^{l} 2^{i-\sqrt{i}} \leq 2^{l/2} + 2^{-\sqrt{l/2}}2^{l+1},$$

hence

$$\sum_{i=1}^{l} 2^{i(1-\eta(2^i))} \leq 2^{l-\sqrt{l/2}+2}.$$

It follows that

$$\rho\left(\left[\gamma^k\right]\right) \leq \frac{1}{[\gamma^k]} 2^{\log_2\left([\gamma^k]\right)-\sqrt{\log_2([\gamma^k])/2}+2} = O\left(2^{-\sqrt{k(\log_2\gamma)/2}}\right),$$

which implies ($\eta3$).

Lemma 4.7. *Suppose that the sequence* (b_j) *satisfies condition (b3) of Section 4.2. If* ϕ *and* ψ *are two trigonometric polynomials with zero integral, let* $(f_n(\phi,\psi))$ *be the sequence defined by* (2.8). *Then for every* $\gamma > 1$ *and for every* $g \in \mathbb{N}\setminus B$, *we have*

$$\sum_{k\geq 0} \sup_{\beta\in\mathbb{R}} \left| \frac{1}{[\gamma^k]} \sum_{n=1}^{[\gamma^k]} f_n(\phi,\psi)\, e(n^g\beta) \right| < +\infty.$$

Proof. For brevity we shall write f_n instead of $f_n(\phi,\psi)$. Since (b_j) satisfies condition (b3), there exists a positive constant $C = C(\phi,\psi,g)$ such that for every large enough j and $N \geq 2^{2j-1}$, we have

$$\sup_{\beta\in\mathbb{R}} \left| \sum_{n=1}^{N} \xi(n^{b_j}\alpha)\, e\left(n^g\beta\right) \right| \leq CN^{1-\eta(N)}, \tag{4.4}$$

where ξ is either ϕ or ψ. We shall use this estimate to find an upper bound for the averages

$$\frac{1}{N}\sum_{n=1}^{N} f_n\, e(n^g\beta).$$

We start by noticing that

$$\frac{1}{2^{2j}} \sum_{n=2^{2j}+1}^{2^{2j+1}} f_n\, e(n^g\beta) = 2\frac{1}{2^{2j+1}} \sum_{n=1}^{2^{2j+1}} \psi(n^{b_j}\alpha)\, e(n^g\beta) - \frac{1}{2^{2j}} \sum_{n=1}^{2^{2j}} \psi(n^{b_j}\alpha)\, e(n^g\beta).$$

We also get a similar estimate with $2j+1$ is in place of $2j$ and ϕ in place of ψ. It follows from (4.4) that there exists $j_0 \geq 0$ (depending on ϕ, ψ and g), such that for every $j \geq j_0$, we have

$$\sup_{\beta\in\mathbb{R}} \left| \frac{1}{2^j} \sum_{n=2^j+1}^{2^{j+1}} f_n\, e(n^g\beta) \right| \leq C\,(2^j)^{-\eta(2^j)}. \tag{4.5}$$

Now consider a large enough integer N, then $N \in (2^j, 2^{j+1}]$ for some $j \geq j_0$. We split the sum between 1 and N into several pieces

$$\sum_{n=1}^{N} \cdot = \sum_{n=1}^{2^{j_0}} \cdot + \sum_{i=j_0}^{j-1} \sum_{n=2^i+1}^{2^{i+1}} \cdot + \sum_{n=2^j+1}^{N} \cdot,$$

and we get the following estimate
(4.6)
$$\left|\frac{1}{N}\sum_{n=1}^{N} f_n\, e(n^g\beta)\right| \leq C\frac{2^{2j_0}}{N} + \frac{1}{N}\sum_{i=j_0}^{j-1}\left|\sum_{n=2^i+1}^{2^{i+1}} f_n\, e(n^g\beta)\right| + \left|\frac{1}{N}\sum_{n=2^j+1}^{N} f_n\, e(n^g\beta)\right|.$$

By (4.5) we have

$$\frac{1}{N}\sum_{i=j_0}^{j-1}\left|\sum_{n=2^i+1}^{2^{i+1}} f_n\, e(n^g\beta)\right| \leq C\frac{1}{N}\sum_{i=1}^{[\log_2 N]} 2^{i(1-\eta(2^i))}. \tag{4.7}$$

Moreover, since

$$\left|\frac{1}{N}\sum_{n=2^j+1}^{N} f_n\, e(n^g\beta)\right| \leq \left|\frac{1}{N}\sum_{n=1}^{N}\xi(n^{b_j}\alpha)\, e(n^g\beta)\right| + \left|\frac{1}{2^j}\sum_{n=1}^{2^j}\xi(n^{b_j}\alpha)\, e(n^g\beta)\right|,$$

where ξ is either ϕ or ψ, we get using (4.4) and property (η1) that

$$\left|\frac{1}{N}\sum_{n=2^j+1}^{N} f_n\, e(n^g\beta)\right| \leq C\left(N^{-\eta(N)} + (2^j)^{-\eta(2^j)}\right) \leq 2C\,(N/2)^{-\eta(N/2)}. \tag{4.8}$$

Combining equations (4.6), (4.7), and (4.8), and using properties (η1), (η2), and (η3), we get the advertised result. □

Proof of Proposition 4.6. Let (s_n) be the sequence defined by (4.3). By Proposition 4.3, it suffices to verify properties (s1') and (s2') mentioned there. Using Proposition 2.5, we see that properties (b1) and (b2) give property (s1'). Also, using Lemma 4.7, we see that property (b3) gives property (s2'). This completes the proof. □

5. Appendix

We prove some results that were used in the main part of the article.

5.1. **Van der Corput's lemma.** The following is a Hilbert space version of a classical elementary estimate of van der Corput. It appears in a form similar to the one stated below in [Be1].

Lemma 5.1. *Let $v_1,\ldots,v_N$ be vectors of a Hilbert space with $\|v_i\| \leq 1$ for $i = 1,\ldots,N$. Then for every integer H between 1 and N we have*

$$\left\|\frac{1}{N}\sum_{n=1}^{N} v_n\right\|^2 \leq \frac{2}{H} + \frac{4}{H}\sum_{h=1}^{H-1}\Big|\frac{1}{N}\sum_{n=1}^{N-h}\langle v_{n+h}, v_n\rangle\Big|.$$

An immediate corollary of the preceding lemma is the following:

Corollary 5.2. *Let $v_1,\ldots,v_N$ be vectors of a Hilbert space with $\|v_i\| \leq 1$ for $i = 1,\ldots,N$. Then for every integer H between 1 and N we have*

$$\left\|\frac{1}{N}\sum_{n=1}^{N} v_n\right\|^2 \leq \frac{4}{H}\sum_{h=1}^{H}\Big|\frac{1}{N}\sum_{n=1}^{N}\langle v_{n+h}, v_n\rangle\Big| + o_{N,H,H\prec N}(1),$$

where $o_{N,HH\prec N}(1)$ denotes a quantity that goes to zero as $N, H \to \infty$ in a way that $H/N \to 0$.

5.2. **Dyadic intervals.** The next lemma allows us, under suitable assumptions, to concatenate dyadic pieces of sequences and create a new sequence with average zero.

Lemma 5.3. *Let* $(u_{n,j})_{n,j\in\mathbb{N}}$ *be a family of complex numbers that satisfy*

$$\lim_{j\to\infty}\sup_{N>2^j}\left|\frac{1}{N}\sum_{n=1}^{N}u_{n,j}\right|=0.$$

Define the sequence (u_n) *by*

$$u_n=u_{n,j}\quad \text{if}\;\; 2^j\le n<2^{j+1}.$$

Then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}u_n=0.$$

Proof. Let $\varepsilon>0$. We define

$$\varepsilon(j)=\sup_{N>2^j}\left|\frac{1}{N}\sum_{n=1}^{N}u_{n,j}\right|$$

and

$$\varepsilon'(j)=\sup_{k\ge j}\varepsilon(k).$$

By our assumption, there exists j_0 such that $\varepsilon'(j)<\varepsilon$ for every $j\ge j_0$.

We start by noticing that

$$\frac{1}{2^j}\sum_{n=2^j+1}^{2^{j+1}}u_n=2\,\frac{1}{2^{j+1}}\sum_{n=1}^{2^{j+1}}u_{n,j}-\frac{1}{2^j}\sum_{n=1}^{2^j}u_{n,j}.$$

Therefore, for every $j\in\mathbb{N}$ we have

$$\left|\frac{1}{2^j}\sum_{n=2^j+1}^{2^{j+1}}u_n\right|\le 3\,\varepsilon(j). \tag{5.1}$$

Now suppose that $N>2^{j_0}$, then $N\in(2^j,2^{j+1}]$ for some $j\ge j_0$. We split the sum between 1 and N into several pieces

$$\sum_{n=1}^{N}\cdot=\sum_{n=1}^{2^{j_0}}\cdot+\sum_{i=j_0}^{j-1}\sum_{n=2^i+1}^{2^{i+1}}\cdot+\sum_{n=2^j+1}^{N}\cdot,$$

in order to get the following upper bound

$$\left|\frac{1}{N}\sum_{n=1}^{N}u_n\right|\le\left|\frac{1}{N}\sum_{n=1}^{2^{j_0}}u_n\right|+\frac{1}{N}\sum_{i=j_0}^{j-1}\left|\sum_{n=2^i+1}^{2^{i+1}}u_n\right|+\left|\frac{1}{N}\sum_{n=2^j+1}^{N}u_n\right|.$$

Using (5.1), we get

$$\frac{1}{N}\sum_{i=j_0}^{j-1}\left|\sum_{n=2^i+1}^{2^{i+1}}u_n\right|\le\frac{1}{N}\sum_{i=j_0}^{j-1}3\,\varepsilon(i)\,2^i\le 3\,\varepsilon'(j_0)\frac{1}{N}\sum_{i=j_0}^{j-1}2^i\le 3\varepsilon'(j_0).$$

We also have

$$\frac{1}{N}\sum_{n=2^j+1}^{N} u_n = \frac{1}{N}\sum_{n=1}^{N} u_{n,j} - \frac{1}{N}\sum_{n=1}^{2^j} u_{n,j},$$

which gives

$$\left|\frac{1}{N}\sum_{n=2^j+1}^{N} u_n\right| \leq \left|\frac{1}{N}\sum_{n=1}^{N} u_{n,j}\right| + \left|\frac{1}{2^j}\sum_{n=1}^{2^j} u_{n,j}\right| \leq 2\,\varepsilon'(j).$$

Putting these estimates together we get

$$\left|\frac{1}{N}\sum_{n=1}^{N} u_n\right| \leq \left|\frac{1}{N}\sum_{n=1}^{2^{j_0}} u_n\right| + 5\varepsilon'(j_0).$$

By taking N large enough we can make sure that the right hand side becomes less than $6\,\varepsilon$. This completes the proof. □

5.3. **Exponential sum estimates.** We shall establish two exponential sum estimates. The first gives non-trivial power type savings when one deals with exponential sums involving polynomials with leading coefficient an integer multiple of the golden mean. Let us recall the "bad approximation property" of the golden mean α

$$\text{For all non-zero integers } q \text{ we have } d(q\alpha,\mathbb{Z}) \geq 1/(3q). \tag{5.2}$$

Lemma 5.4. *Let α be the golden mean and $b \in \mathbb{N}$. There exists $C = C(b) > 0$ such that for every $m, N \in \mathbb{N}$ we have*

$$\sup_{P\in\mathbb{R}[X],\deg(P)<b}\left|\sum_{n=1}^{N} e(mn^b\alpha + P(n))\right| \leq C\, m^{2^{1-b}} N^{1-4^{1-b}}. \tag{5.3}$$

Proof. We use an induction on b . For $b = 1$, we have

$$\left|\sum_{n=1}^{N} e(mn\alpha)\right| \leq \frac{2}{|e(m\alpha)-1|} = \frac{1}{|\sin(\pi m\alpha)|} \leq \frac{1}{2\mathrm{d}(m\alpha,\mathbb{Z})} \leq \frac{3m}{2},$$

by the bad approximation property (5.2).

Suppose that the estimate (5.3) holds for the integer b. We are going to show that it also holds for the integer $b+1$. Let us define

$$S(m,N,b) = \sup_{P\in\mathbb{R}[X],\deg(P)<b}\left|\sum_{n=1}^{N} e(mn^b\alpha + P(n))\right|.$$

From van der Corput's inequality (Lemma 5.1), we deduce that for every integer H between 1 and N, we have

$$S(m,N,b+1)^2 \leq \frac{2N^2}{H} + \frac{4N}{H}\sum_{h=1}^{H-1} S(mh(b+1), N-h, b). \tag{5.4}$$

The induction hypothesis gives that for some constant $C = C_b$ we have

$$S(mh(b+1), N-h, b) \leq C\,(mh)^{2^{1-b}}(N-h)^{1-4^{1-b}},$$

and so for h between 1 and $H-1$ we have

$$S(mh(b+1), N-h, b) \leq C\, m^{2^{1-b}}(H-1)^{2^{1-b}} N^{1-4^{1-b}}. \tag{5.5}$$

Combining (5.4) and (5.5) we get

$$S(m,N,b+1)^2 \le 4\,C\,m^{2^{1-b}}\left(N^2H^{-1}+N^{2-4^{1-b}}(H-1)^{2^{1-b}}\right). \tag{5.6}$$

Choosing $H=\left[N^{2^{1-2b}}\right]+1$ gives

$$N^2H^{-1}\le N^{2(1-4^{-b})}, \qquad N^{2-4^{1-b}}(H-1)^{2^{1-b}}\le N^{2(1-2\cdot 4^{-b}+2\cdot 8^{-b})}\le N^{2(1-4^{-b})}.$$

Using this together with (5.6) we find that

$$S(m,N,b+1)^2\le 4\,C\,m^{2^{1-b}}N^{2(1-4^{-b})}.$$

Taking square roots establishes (5.3) for $b+1$. This completes the induction and the proof. □

The second lemma gives non-trivial power type savings for exponential sums involving polynomials that have an integer multiple of the golden mean as a non-leading (non-constant) coefficient. Its proof is a simplification of an argument that appears in [BosKoQW].

Lemma 5.5. *Let α be the golden mean and $g\in\mathbb{N}$. For every $\varepsilon>0$, there exists $C=C(\varepsilon,g)<+\infty$ such that, for every $\beta\in\mathbb{R}$, $b\in\mathbb{N}$ with $b<g$, $N\in\mathbb{N}$, and nonzero $m\in\mathbb{Z}$ with $|m|\le N^{2^{-g-2}}$, we have*

$$\left|\sum_{n=1}^{N} e(mn^b\alpha+n^g\beta)\right|\le C\,N^{1+\varepsilon-2^{-2g-1}}. \tag{5.7}$$

Proof. The proof proceeds as follows: If β is not well approximable by rationals in a way to be made precise below, then classical estimates of Weyl immediately give the advertised estimate. If β is well approximated by rationals, using partial summation we can replace β by a rational (up to a small error), and reduce the problem to studying an exponential sum involving a polynomial that has an integer multiple of α as leading coefficient. In this case, again the classical estimates of Weyl give the advertised result.

So let us first recall Weyl's classical estimate (see e.g. [V]). For every $k\in\mathbb{N}$ and $\varepsilon>0$, there exists a constant C satisfying the following property: for every $N\in\mathbb{N}$, $\beta\in\mathbb{R}$, and relatively prime $r,s\in\mathbb{N}$ with $|\beta-r/s|<1/s^2$, and for every real polynomial $P(x)$ with leading coefficient βx^k, we have

$$\left|\sum_{n=1}^{N} e(P(n))\right|\le CN^{1+\varepsilon}\left(\frac{1}{s}+\frac{1}{N}+\frac{s}{N^k}\right)^{1/2^{k-1}}.$$

We fix $\beta\in\mathbb{R}$, $b\in\mathbb{N}$ with $b<g$, $N\in\mathbb{N}$, nonzero $m\in\mathbb{Z}$ with $|m|\le N^{2^{-g-2}}$, and we define $\gamma=2^{-g-2}$.

By Dirichlet's principle, there exist $r,s\in\mathbb{N}$, relatively prime, such that $s\le N^{g-\gamma}$ and

$$\left|\beta-\frac{r}{s}\right|\le\frac{1}{N^{g-\gamma}s}.$$

We distinguish two cases: either $s>N^\gamma$ (bad approximation) or $s\le N^\gamma$ (good approximation).

Case 1. Suppose that $s>N^\gamma$. By Weyl's estimate, we have

$$\left|\sum_{n=1}^{N} e(mn^b\alpha+n^g\beta)\right|\le C(\varepsilon,g)N^{1+\varepsilon}(N^{-\gamma}+N^{-1}+N^{-\gamma})^{2^{1-g}},$$

which implies the estimate (5.7), since $\gamma = 2^{-g-2}$.

Case 2. Suppose now that $s \le N^\gamma$. By Dirichlet's principle, there exist $t, u \in \mathbb{N}$, relatively prime, such that $u \le N^{b-1/2}$ and

$$\left| ms^b\alpha - \frac{t}{u} \right| \le \frac{1}{N^{b-1/2}u}. \tag{5.8}$$

The bad approximation property of α mentioned in (5.2) gives that $mus^b \ge \frac{1}{3}N^{b-1/2}$. Since $s \le N^\gamma$ and $|m| \le N^\gamma$ we have $u \ge \frac{1}{3}N^{b-1/2-\gamma b-\gamma}$.

Consider now an integer M between $N^{1-\gamma}$ and N. We are going to compare the sums $\sum_{n\le M} e(mn^b\alpha + n^g\beta)$ with the sums $\sum_{n\le M} e(mn^b\alpha + \frac{r}{s}n^g)$ that are easier to estimate. Let us first estimate the second sum. We have

$$\sum_{n\le M} e\left(mn^b\alpha + \frac{r}{s}n^g\right) = \sum_{j=1}^{s} \sum_{i\ge 0,\, si+j\le M} e\left(m(si+j)^b\alpha + \frac{r}{s}j^g\right),$$

hence

$$\left|\sum_{n\le M} e\left(mn^b\alpha + \frac{r}{s}n^g\right)\right| \le \sum_{j=1}^{s} \left|\sum_{i\ge 0,\, si+j\le M} e\left(m(si+j)^b\alpha\right)\right|. \tag{5.9}$$

By Weyl's estimate and (5.8), we have

$$\left|\sum_{i\ge 0,\, si+j\le M} e\left(m(si+j)^b\alpha\right)\right| \le C(\varepsilon,b)\left(\frac{M}{s}\right)^{1+\epsilon}\left(\frac{1}{u} + \frac{s}{M} + u\left(\frac{s}{M}\right)^b\right)^{2^{1-b}}.$$

Using that

$$N^{1-\gamma} \le M \le N\ , \quad 1 \le s \le N^\gamma, \quad \text{and} \quad \frac{1}{3}N^{b-1/2-\gamma b-\gamma} \le u \le N^{b-1/2},$$

we obtain

$$\left|\sum_{i\ge 0,\, si+j\le M} e\left(m(si+j)^b\alpha\right)\right| \le$$
$$C(\varepsilon,b)\frac{N^{1+\varepsilon}}{s}\left(N^{-b+1/2+\gamma b+\gamma} + N^{-1+2\gamma} + N^{-1/2+2\gamma b}\right)^{2^{1-b}}.$$

The term $N^{-1/2+2\gamma b}$ is dominant, and is bounded by $N^{-1/4}$. It follows that

$$\left|\sum_{i\ge 0,\, si+j\le M} e\left(m(si+j)^b\alpha\right)\right| \le C(\varepsilon,b)\frac{N^{1+\varepsilon-2^{-b-1}}}{s}.$$

Since the integer g is fixed and $b < g$ we have

$$\left|\sum_{i\ge 0,\, si+j\le M} e\left(m(si+j)^b\alpha\right)\right| \le C(\varepsilon,g)\frac{N^{1+\varepsilon-2^{-g-1}}}{s}.$$

In conjunction with (5.9) this gives

$$\left|\sum_{n\le M} e\left(mn^b\alpha + \frac{r}{s}n^g\right)\right| \le C(\varepsilon,g)N^{1+\varepsilon-2^{-g-1}}. \tag{5.10}$$

We come back to our main goal of estimating the sums $\sum_{n\leq M} e(mn^b\alpha + n^g\beta)$. We are going to use summation by parts. We set $S(M) = \sum_{n\leq M} e\left(mn^b\alpha + \frac{r}{s}n^g\right)$ and notice that

$$\left|\sum_{n\leq N} e(mn^b\alpha + n^g\beta)\right| \leq N^{1-\gamma} + \left|\sum_{N^{1-\gamma}<n\leq N} e(mn^b\alpha + n^g\beta)\right|$$
$$\leq N^{1-\gamma} + \left|\sum_{N^{1-\gamma}<n\leq N} (S(n) - S(n-1))\, e\left(n^g(\beta - \frac{r}{s})\right)\right|.$$

We have

$$\left|e\left(n^g(\beta - \frac{r}{s})\right) - e\left((n+1)^g(\beta - \frac{r}{s})\right)\right| \leq Cn^{g-1}\left|\beta - \frac{r}{s}\right|,$$

where the constant C does not depend on β because $\beta - r/s$ is uniformly bounded. We know that for $n \leq N$ we have $n^{g-1}|\beta - r/s| \leq N^{-1+\gamma}$. So using partial summation, we obtain

$$\left|\sum_{n\leq N} e(mn^b\alpha + n^g\beta)\right| \leq$$
$$N^{1-\gamma} + \left|S\left([N^{1-\gamma}]+1\right)\right| + |S(N)| + C\sum_{N^{1-\gamma}<n\leq N} |S(n)|N^{-1+\gamma}.$$

Using (5.10) we conclude that

$$\left|\sum_{n\leq N} e(mn^b\alpha + n^g\beta)\right| \leq C(\varepsilon, g)N^{1+\varepsilon-2^{-g-1}+\gamma}.$$

Recalling that $\gamma = 2^{-g-2}$, we derive an estimate stronger than (5.7). This completes the proof. □

5.4. **The PET induction argument.** We give the details needed to complete the proof of Lemma 3.4. The next result follows immediately from Corollary 5.2:

Lemma 5.6. *Consider a family of integer polynomials $\{p_1, \ldots, p_k\}$ and an integer polynomial p, all of them having zero constant term. Let $\{q_1, \ldots, q_{k'}\}$ be the family of distinct integer polynomials that is defined using the following operation: we start with the family of polynomials*

$$p_1(n+h) - p_1(h) - p(n), \ldots, p_k(n+h) - p_k(h) - p(n), p_1(n) - p(n), \ldots, p_k(n) - p(n),$$

and we remove polynomials that are identically zero and repetitions of polynomials. Then for every system $(X, \mathcal{B}, \mu, T)$, and sequence of complex numbers (u_n) with $\|u_n\|_\infty \leq 1$, we have

$$\sup_{f_1,\ldots,f_k} \left\|\frac{1}{N}\sum_{n=1}^{N} u_n\, T^{p_1(n)}f_1 \cdot \ldots \cdot T^{p_k(n)}f_k\right\|_2^2 \leq$$
$$\frac{4}{H}\sum_{h=1}^{H} \sup_{f_1,\ldots,f_{k'}} \left\|\frac{1}{N}\sum_{n=1}^{N} u_{n+h}\bar{u}_n\, T^{q_1(n)}f_1 \cdot \ldots \cdot T^{q_{k'}(n)}f_{k'}\right\|_2 + o_{N,H,H\prec N}(1),$$

where the supremums are taken over families of functions in $L^\infty(\mu)$ bounded by 1.

Let $\mathcal{P}$ be a family of non-constant integer polynomials with zero constant term. The maximum degree of the polynomials is called the *degree* of the polynomial family and we denote it by d. Let $\mathcal{P}_i$ be the subfamily of polynomials of degree i in $\mathcal{P}$. We let w_i denote the number of distinct leading coefficients that appear in the family $\mathcal{P}_i$. The vector $(d, w_d, \ldots, w_1)$ is called the *type* of the polynomial family $\mathcal{P}$. We use an induction scheme, often called PET induction (Polynomial Exhaustion Technique), on types of polynomial families that was introduced by Bergelson in [Be1]. We order the set of all possible types lexicographically, this means, $(d, w_d, \ldots, w_1) > (d', w'_{d'}, \ldots, w'_1)$ if and only if in the first instance where the two vectors disagree the coordinate of the first vector is greater than the coordinate of the second vector.

Proposition 5.7 (**Bergelson's PET** [Be1]). *Let $\{p_1, \ldots, p_k\}$ be a family of non-constant integer polynomials with zero constant term. After applying finitely many times the operation defined in Lemma 5.6 (for good choices of the auxiliary polynomial p at each step) it is possible to obtain the empty family of polynomials.*

Proof. Let p_{min} be any member of the family $\{p_1, \ldots, p_k\}$ that has minimal degree. Notice that after applying the operation defined in Lemma 5.6 for $p = p_{min}$, we obtain a new family of polynomials that has type strictly less than the type of the family $\{p_1, \ldots, p_k\}$. The result now follows using induction on the type of the polynomial family. □

The following two examples illustrate how a typical PET induction argument works:

Example 2. Suppose that we start with the family of polynomials $\mathcal{P}_0 = \{n, 2n\}$ that has type $(1, 2)$. Applying the operation defined in Lemma 5.6 with $p(n) = n$ we obtain the family $\mathcal{P}_1 = \{n\}$ that has type $(1, 1)$. After one more application of the operation we obtain an empty family of polynomials.

Example 3. Suppose that we start with the family of polynomials $\mathcal{P}_0 = \{n^2, 2n^2\}$ that has type $(2, 2, 0)$. Applying successively the operation defined in Lemma 5.6 we obtain the following families of polynomials: Using $p(n) = n^2$ we get the family

$$\mathcal{P}_1 = \{2nh_1, n^2 + 4nh_1, n^2\}$$

that has type $(2, 1, 1)$. Using $p(n) = 2nh_1$ we get the family

$$\mathcal{P}_2 = \{n^2 + 2n(h_1 + h_2), n^2 + 2n(h_2 - h_1), n^2 + 2nh_1, n^2 - 2nh_1\}$$

that has type $(2, 1, 0)$. Using $p(n) = n^2$ we get the family

$$\mathcal{P}_3 = \{2n(h_1 + h_2 + h_3), 2n(h_3 + h_2 - h_1), 2n(h_1 + h_3), \\ 2n(h_3 - h_1), 2n(h_1 + h_2), 2n(h_2 - h_1), 2nh_1, -2nh_1\}$$

that has type at most $(1, 8)$ (actually equal to $(1, 8)$ for most values of h_1, h_2, h_3). The last family consists of linear polynomials and can be dealt as in Example 2. After 8 more operations we arrive to an empty family of polynomials.

References

[Be1] V. Bergelson. Weakly mixing PET. *Ergodic Theory Dynam. Systems* **7** (1987), no. 3, 337–349.

[Be2] V. Bergelson. Combinatorial and Diophantine applications of ergodic theory. Appendix A by A. Leibman and Appendix B by A. Quas and M. Wierdl. *Handbook of dynamical systems*, Vol. 1B, 745–869, Elsevier B. V., Amsterdam, 2006.

[BeHKr] V. Bergelson, B. Host and B. Kra. Multiple recurrence and nilsequences. *Inventiones Math.* **160** (2005), 261–303.

[BosKoQW] M. Boshernitzan, G. Kolesnik, A. Quas, M. Wierdl. Ergodic averaging sequences. *J. Anal. Math.* **95** (2005), 63–103.

[Bou] J. Bourgain. On the maximal ergodic theorem for certain subsets of the integers. *Israel J. Math.* **6**1 (1988), 39–72.

[DES] J. Deshouillers, P. Erdös, A. Sárközy. On additive bases. *Acta Arith.* **30** (1976), no. 2, 121–132.

[DFo] J. Deshouillers, E. Fouvry. On additive bases II. *J. London Math. Soc. (2)* **14** (1976), no. 3, 413–422.

[FrLW1] N. Frantzikinakis, E. Lesigne, M. Wierdl. Sets of k-recurrence but not $(k+1)$-recurrence. *Annales de l'Institut Fourier* **56** (2006), no. 4, 839–849.

[FrLW2] N. Frantzikinakis, E. Lesigne, M. Wierdl. Powers of sequences and recurrence. *Proc. Lond. Math. Soc.* (3) **98** (2009), no. 2, 504–530.

[HKr1] B. Host, B. Kra. Nonconventional ergodic averages and nilmanifolds. *Annals of Math.* **161** (2005), 397–488.

[HKr2] B. Host, B. Kra. Convergence of polynomial ergodic averages. *Isr. J. Math.* **149** (2005), 1–19.

[HKr3] B. Host, B. Kra. Uniformity seminorms on l^∞ and spplications. To appear *J. Anal. Math.* Available at arXiv:0711.3637.

[KN] L. Kuipers, H. Niederreiter. Uniform distribution of sequences. Pure and Applied Mathematics. *Wiley-Interscience*, New York-London-Sydney, (1974).

[Lei] A. Leibman. Convergence of multiple ergodic averages along polynomials of several variables. *Isr. J. Math.* **146** (2005), 303–316.

[RW] J. Rosenblatt, M. Wierdl. Pointwise theorems via harmonic analysis. Ergodic theory and its connections with harmonic analysis (Alexandria, 1993). *London Math. Soc. Lecture Note Ser.* **205**, 3–151.

[V] R. Vaughan. The Hardy-Littlewood method. Second edition, *Cambridge Tracts in Mathematics* **125**, Cambridge University Press, Cambridge, 1997.

[Z] T. Ziegler. Universal characteristic factors and Furstenberg averages. *J. Amer. Math. Soc.* **20** (2007), 53–97.

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