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Abstract We study existence and stability for solutions of $-Lu + g(x,u) = \omega$ where L is a second order elliptic operator, g a Caratheodory function and ω a measure in $\overline{\Omega}$. We present a unified theory of the Dirichlet problem and the Poisson equation. We prove the stability of the problem with respect to weak convergence of the data.

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1 Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, L a uniformly elliptic second order differential operator in divergence form with Lipschitz continuous coefficients and g is a real valued Caratheodory function defined in $\Omega \times \mathbb{R}$. If ω is a Radon measure on $\overline{\Omega}$, we study existence and stability of solutions of the generalized equation

$$-Lu + g(x,u) = \omega \tag{1.1}$$

in $\overline{\Omega}$. Precise assumptions are made on the coefficients of L so that uniqueness holds. A fundamental contribution is made by Benilan and Brezis [6], [3] who study the case where $L = \Delta$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is nondecreasing and positive on $\mathbb{R}_+$: if μ is a bounded measure in Ω and g satisfies the *subcriticality assumption*

$$\int_1^\infty (g(s) - g(-s))\, s^{-2\frac{N-1}{N-2}} ds < \infty, \tag{1.2}$$

then there exists a unique function $u \in L^1(\Omega)$ such that $g \circ u \in L^1(\Omega)$ (where $g \circ u(x) = g(x, u(x))$) satisfying

$$\int_\Omega (-u\Delta\zeta + g \circ u\, \zeta)\, dx = \int_\Omega \zeta d\mu, \tag{1.3}$$

for any $\zeta \in C_0^2(\Omega)$.

The boundary value problem with measures is first investigated by Gmira and Véron [7]. By adapting the method introduced by Benilan and Brezis they obtain the existence and uniqueness of a weak solution of

$$\begin{aligned} -\Delta u + g(u) &= 0 && \text{in } \Omega \\ u &= \lambda && \text{in } \partial\Omega \end{aligned} \tag{1.4}$$

when λ is a Radon measure. They assume that g, always nondecreasing, satisfies the *boundary subcriticality assumption*

$$\int_1^\infty (g(s) - g(-s))\, s^{-\frac{2N}{N-2}} ds < \infty, \tag{1.5}$$

and prove the existence and uniqueness of a weak solution to (1.4). For this problem, in the integral identity (1.3) the right hand-side is replaced by $-\int_{\partial\Omega}\zeta_{\mathbf{n}}d\lambda$ (where $\zeta_{\mathbf{n}} = \nabla u.\mathbf{n}$ is the outward normal derivative on $\partial\Omega$).

In [13] Véron extends Benilan-Brezis results in replacing Δ by a general uniformly elliptic second order differential operator with smooth coefficients. If g is nondecreasing and satisfies, for some $\alpha \in [0,1]$, the *α-subcriticality assumption*,

$$\int_1^\infty (g(s) - g(-s))\, s^{-2\frac{N+\alpha-1}{N+\alpha-2}} ds < \infty, \tag{1.6}$$

then if μ belongs to $\mathfrak{M}_{\rho^\alpha}(\Omega)$, which means

$$\|\mu\|_{\mathfrak{M}_{\rho^\alpha}} := \int_{\Omega} \rho^\alpha d\,|\mu| < \infty, \tag{1.7}$$

where $\rho(x) := \mathrm{dist}\,(x, \partial\Omega)$, there exists a unique $u \in L^1(\Omega)$ such that $g(u) \in L^1_\rho(\Omega)$ satisfying

$$\int_\Omega (-uL^*\zeta + g(u)\zeta)\, dx = \int_\Omega \zeta d\mu \qquad \forall \zeta \in C_c^{1,L^*}(\overline{\Omega}). \tag{1.8}$$

where

$$C_c^{1,L^*}(\overline{\Omega}) = \{\zeta \in C^1(\overline{\Omega}) : \zeta = 0 \text{ on } \partial\Omega,\ L^*\zeta \in L^\infty(\Omega)\}, \tag{1.9}$$

where L^* is the adjoint operator to L. Furthermore he proves the weak stability of the problem. it means that if u_n is a set of solutions of

$$\begin{aligned} -Lu_n + g(u_n) &= \mu_n \qquad &&\text{in } \Omega \\ u_n &= 0 &&\text{in } \partial\Omega \end{aligned} \tag{1.10}$$

for a sequence of measure $\{\mu_n\}$ such that

$$\lim_{n\to\infty} \int_\Omega \zeta d\mu_n = \int_\Omega \zeta d\mu \tag{1.11}$$

for all $\zeta \in C(\overline{\Omega})$ verifying $\sup_\Omega \rho^{-\alpha}|\zeta| < \infty$, then $u_n \to u$ where u satisfies (1.1). However, a careful observation of the existence and stability statements proved in [13, Th 3.7, Cor 3.8] shows that the result is slightly stronger than the one stated since it implies the following:

Let $\alpha \in [0,1]$ and $g : \mathbb{R} \mapsto \mathbb{R}$ be continuous function which satisfies the α-subcriticality assumption (1.6)*. If $\{\mu_n\}$ is a sequence of Radon measures in $\overline{\Omega}$ such that*

$$\int_{\overline{\Omega}} \rho^\alpha d\,|\mu_n| \leq M \tag{1.12}$$

for some $M > 0$ and (1.11) *holds for ζ such that $\rho^{-\alpha}\zeta \in C(\overline{\Omega})$, then the corresponding solution u_n of* (1.10) *converges to the solution u of* (1.1)*. In particular, if $\alpha = 1$, it contains the case where there exists a Radon measure λ on $\partial\Omega$ such that*

$$\lim_{n\to\infty} \int_\Omega \zeta d\mu_n = -\int_{\partial\Omega} \zeta_{\mathbf{n}} d\lambda \qquad \forall \zeta \in C_c^1(\overline{\Omega}). \tag{1.13}$$

The case where the nonlinearity g depends on the $\rho(x)$ variable has investigated by Marcus [8]. If $g(x,r)\mathrm{sign}\,\mathrm{r} \leq \rho(\mathrm{x})^\beta\tilde{\mathrm{g}}(|\mathrm{r}|)\mathrm{sign}\,\mathrm{r}$ for some $\beta > -2$ and $\tilde{g}$ satisfying a subcriticality assumption

$$\int_1^\infty (\tilde{g}(s) - \tilde{g}(-s))\, s^{-\frac{2N+\beta-1}{N-1}} ds < \infty, \tag{1.14}$$

then there exists a weak solution to problem (1.4) for any Radon measure λ. Furthermore stability holds.

The subcriticality is a key hypothesis in all the previous results: essentially it means that the problem can be solved for any measure if it can be solved for a Dirac measure. The different integral assumptions are just the transcription that the fact that g of the fundamental solution of the associated linear equation is integrable for a suitable measure associated to the distance function ρ.

The aim of this article is twofold: 1- to unify the problems for measures in Ω and on $\partial\Omega$; 2- to present under the form of an integrability condition a classical sufficient condition of solvability which has the advantage of being a natural extension to the supercritical case the previous subcriticality assumptions and to provide new results results of existence and stability for (1.1) in the spirit of [13]. A function $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ belongs to the class $G_{h,\Psi}$ if it is a Caratheodory function and there exist a continuous and nondecreasing function $\tilde{g} : \mathbb{R} \mapsto \mathbb{R}$ vanishing at 0, a locally integrable nonnegative function h defined in Ω and a nonnegative continuous nonincreasing function $\Psi : [0,\infty) \mapsto [0,\infty)$, such that

$$|g(x,r)| \leq h(x)\, |\tilde{g}(r)| \qquad \forall (x,r) \in \Omega \times \mathbb{R}, \tag{1.15}$$

and the Ψ-*integrability condition* holds, i.e.

$$-\int_0^\infty (\tilde{g}(s) - \tilde{g}(-s))\, d\Psi(t) ds < \infty. \tag{1.16}$$

Let G and K be respectively the Green and Poisson kernels corresponding to the operator L in Ω and $\mathbb{G}[.]$ and $\mathbb{K}[.]$ the corresponding potential operators. The natural subcritical assumptions in the framework of Marcus's results (with h instead of ρ^β) for solving

$$\begin{aligned} -Lu + g(x,u) &= \mu && \text{in } \Omega \\ u &= \lambda && \text{in } \partial\Omega \end{aligned} \tag{1.17}$$

would be

$$\int_1^\infty (\mathbb{G}[|\mu|] + \mathbb{K}[|\lambda|])\, h(x)\rho(x) dx < \infty. \tag{1.18}$$

However this type of condition is not satisfactory since it may not hold if μ and λ are merely integrable functions since the problem admits always weak solutions. More generally it does not define a clear class of measures for which we can solve problem (1.17). We introduce new classes of Radon measures whose Green and Poisson potentials belong to a weighted Marcinkiewicz space-type space. Let Ψ be a continuous nonincreasing and nonnegative function defined on $[0,\infty)$ and m is a bounded positive Borel measure in Ω and denote

$$M_m^{\Psi}(\Omega) := \left\{ f \in \mathcal{B}(\Omega) : \exists\, C > 0 \ \text{ s.t. } \int_{\lambda_f(t)} dm(x) \leq C\Psi(t)\,,\ \forall t > 0 \right\} \tag{1.19}$$

where $\mathcal{B}(\Omega)$ denotes the space of Borel functions in Ω and $\lambda_f(t) = \{x \in \Omega : |f(x)| > t\}$. The main results of this article are the two next statements:

Theorem A *Let g be an element of the class $G_{h,\Psi}$ with $\rho h \in L^1(\Omega)$. Then for any $\mu \in \mathfrak{M}_\rho(\Omega)$ and $\lambda \in \mathfrak{M}(\partial\Omega)$ such that $\mathbb{G}[|\mu|]$ and $\mathbb{K}[|\lambda|]$ belong to $M^{\Psi}_{\rho h}(\Omega)$, there exists a solution to problem* (1.17)*. If $r \mapsto g(x,r)$ is nondecreasing for a.e. $x \in \Omega$, this solution is unique.*

Actually we shall introduce a unique formulation for the data (μ,λ) as a *unique* measure ω on $\overline{\Omega}$ which allows to replace (1.17) by (1.1), and a unique assumption on the *extended* Green operator $\overline{\mathbb{G}}[|\omega|]$. We prove in particular the following:

Theorem B *Assume the assumptions on h, Ψ and g of Theorem A are satisfied and $r \mapsto g(x,r)$ is nondecreasing. If $\{(\omega_n\}$ is a sequence of measures in $\mathfrak{M}_\rho(\overline{\Omega})$ which converges to $\omega \in \mathfrak{M}_\rho(\overline{\Omega})$ in the sense that*

$$\int_{\overline{\Omega}} \zeta d\omega_n \to \int_{\overline{\Omega}} \zeta d\omega \tag{1.20}$$

for any ζ such that $\rho^{-1}\zeta \in C(\overline{\Omega})$ and if the $\overline{\mathbb{G}}[|\omega_n|]$ are bounded in $M^{\Psi}_{\rho h}(\Omega)$, then the corresponding solutions u_{ω_n} of problem (1.10) *converges to the solution u_ω of problem* (1.1)*. If g satisfies the $\Delta_2$conditions, the convergence remains valid if only the $\overline{\mathbb{G}}[|\omega_{sn}|]$ are bounded in $M^{\Psi}_{\rho h}(\Omega)$, where ω_{sn} denotes the singular parts of ω_n.*

2 Linear equations and measures

Since $\partial\Omega$ is C^2, there exists $\delta_0 > 0$ such that, If $x \in \Omega$ is such that $\rho(x) \leq \delta_0$, there exists a unique $\sigma := \sigma(x) \in \partial\Omega$ such that $|x - \rho(x)| = \rho(x)$. For $\delta > 0$ we denote

$$\Omega_\delta := \{x \in \Omega : \rho(x) > \delta\}\,,\ \Omega'_\delta := \{x \in \Omega : \rho(x) < \delta\}\,,\ \Sigma_\delta := \{x \in \Omega : \rho(x) = \delta\}\,,\ \Sigma := \Sigma_0 = \partial\Omega.$$

The mapping $x \mapsto (\rho(x), \sigma(x))$ is a C^1 diffeomorphism from $\overline{\Omega'_{\delta_0}}$ onto $[0,\delta_0] \times \Sigma$.

2.1 Weighted measures on $\overline{\Omega}$

We denote by $\mathfrak{M}(\Omega)$ the set of Radon measures in Ω. If $\alpha \in [0,1]$, we denote by $\mathfrak{M}_{\rho^\alpha}(\Omega)$ the subset of $\mathfrak{M}(\Omega)$ of measures such that

$$\|\mu\|_{\mathfrak{M}_{\rho^\alpha}} := \int_\Omega \rho^\alpha d\,|\mu| < \infty. \tag{2.21}$$

We also set

$$C_\alpha(\overline{\Omega}) := \{\zeta \in C(\Omega) : \rho^{-\alpha}\zeta \in C(\overline{\Omega})\}\}, \tag{2.22}$$

with norm

$$\|\zeta\|_{C_\alpha} := \sup_{x\in\overline{\Omega}} \rho^{-\alpha}(x)\,|\zeta(x)|\,. \tag{2.23}$$

Thus, if $\mu \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ and $\zeta \in C_\alpha(\overline{\Omega})$, there holds

$$\left|\int_\Omega \zeta d\mu\right| \leq \|\mu\|_{\mathfrak{M}_{\rho^\alpha}} \|\zeta\|_{C_\alpha}\,. \tag{2.24}$$

Furthermore, since

$$\int_{\Omega_{\delta_0}} \rho^\alpha d\,|\mu| + \sum_{n=1}^{\infty} \int_{\{2^{-n}\delta_0<\rho\leq 2^{1-n}\delta_0\}} \rho^\alpha d\,|\mu| = \int_\Omega \rho^\alpha d\,|\mu| < \infty,$$

there holds

$$\lim_{\delta\to 0} \int_{\Omega'_\delta} \rho^\alpha d\,|\mu| = 0. \tag{2.25}$$

We say that a sequence $\{\mu_n\} \subset \mathfrak{M}_{\rho^\alpha}(\Omega)$ converges weakly to $\mu \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ if, for any $\zeta \in C_\alpha(\overline{\Omega})$, there holds

$$\lim_{n\to\infty} \int_\Omega \zeta d\mu_n = \int_\Omega \zeta d\mu. \tag{2.26}$$

However, the left-hand side expression of (2.26) may exist but not being a Radon measure in Ω. Therefore we define a more general set of linear functionals on C_α

Definition 2.1 *We denote by $\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$ the set of continuous linear functionals ω on $C_\alpha(\overline{\Omega})$ such that there exists a sequence $\{\mu_n\} \subset \mathfrak{M}_{\rho^\alpha}(\Omega)$ which converges weakly to ω.*

The natural norm in $\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$ is

$$\|\omega\|_{\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})} = \sup\{|\omega(\zeta)| : \zeta \in C_\alpha(\overline{\Omega}), \|\zeta\|_{C_\alpha} \leq 1\}. \tag{2.27}$$

Proposition 2.2 *If $\omega \in \mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$, its restriction to $C_c(\Omega)$ is a Radon measure, denoted by μ, which belongs to $\mathfrak{M}_{\rho^\alpha}(\Omega)$. Furthermore, there exists a Radon measure λ on $\partial\Omega$ such that*

$$\omega(\zeta) - \int_\Omega \zeta d\mu = \int_{\partial\Omega} \psi\lfloor_{\partial\Omega} d\lambda \qquad \forall \zeta \in C_\alpha(\overline{\Omega}) \text{ and } \psi = \rho^{-\alpha}\zeta \in C(\overline{\Omega}). \tag{2.28}$$

Proof. Since ω is continuous, there exists $C > 0$ such that

$$|\omega(\zeta)| \leq C\,\|\zeta\|_{C_\alpha} \qquad \forall \zeta \in C_\alpha(\overline{\Omega}). \tag{2.29}$$

This holds in particular if $\zeta \in C_c(\Omega)$ and proves that the restriction of ω to $C_c(\Omega)$ is a Radon measure that we denote by μ (as well as the associated Borel measure in Ω) and

$$\omega(\zeta) = \int_\Omega \zeta d\mu \qquad \forall \zeta \in C_c(\Omega).$$

Let $\{\mu_n\} \subset \mathfrak{M}_{\rho^\alpha}(\Omega)$ such that

$$\lim_{n\to\infty} \int_\Omega \zeta d\mu_n = \omega(\zeta) \qquad \forall \zeta \in C_\alpha(\overline{\Omega}).$$

By the Banach-Steinhaus theorem there exists $C > 0$ such that $\|\mu_n\|_{\mathfrak{M}_{\rho^\alpha}} \leq C$ for all $n \in \mathbb{N}$. Since for $\zeta \in C_c(\Omega)$,

$$\omega(\zeta) - \int_\Omega \zeta d\mu = \lim_{n\to\infty} \int_\Omega \zeta d(\mu_n - \mu)$$

and

$$\left| \int_\Omega \zeta d(\mu_n - \mu) \right| \leq 2C \|\zeta\|_{C_\alpha},$$

it follows that $\{\lambda_n\} := \{\rho^\alpha(\mu_n - \mu)\}$ is a sequence of Radon measures on Ω, bounded in $\mathfrak{M}_{\rho^\alpha}(\Omega)$ and such that

$$\lim_{n\to\infty} \int_\Omega \zeta d\lambda_n = 0 \qquad \forall \zeta \in C_c(\Omega).$$

Therefore there exists a Radon measure λ with support in $\partial\Omega$ and a subsequence λ_{n_k} such that

$$\lim_{n\to\infty} \int_\Omega \psi d\lambda_{n_k} = \int_{\partial\Omega} \psi\lfloor_{\partial\Omega} d\lambda,$$

which implies (2.28). □

Corollary 2.3 *The mapping* $T : \mathfrak{M}_{\rho^\alpha}(\Omega) \times \mathfrak{M}(\partial\Omega) \mapsto \mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$ *defined by*

$$T[\mu,\lambda](\zeta) = \int_\Omega \zeta d\mu + \int_{\partial\Omega} \psi\lfloor_{\partial\Omega} d\lambda \qquad \forall \zeta \in C_\alpha(\overline{\Omega}) \text{ and } \psi = \rho^{-\alpha}\zeta \in C(\overline{\Omega}). \tag{2.30}$$

is one to one. Furthermore

$$\max\{\|\mu\|_{\mathfrak{M}_{\rho^\alpha}(\Omega)}, \|\lambda\|_{\mathfrak{M}(\partial\Omega)}\} \leq \|T[\mu,\lambda]\|_{\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})} \leq \|\mu\|_{\mathfrak{M}_{\rho^\alpha}(\Omega)} + \|\lambda\|_{\mathfrak{M}(\partial\Omega)}. \tag{2.31}$$

Proof. The mapping T is onto from Proposition 2.2. The mapping T is one to one since if $T[\mu,\lambda] = 0$, then $\mu = 0$ and $\int_{\partial\Omega} \psi\lfloor_{\partial\Omega} d\lambda = 0$ for any $\psi \in C(\overline{\Omega})$. This implies $\lambda = 0$. The right-hand side inequality (2.31) is clear since $\sup|\psi\lfloor_{\partial\Omega}| \leq \|\zeta\|_{C_\alpha}$. Because of (2.25)

$$\int_\Omega \rho^\alpha d|\mu| = \sup\left\{ \int_\Omega \zeta d\mu : \zeta \in C_c(\Omega), \|\zeta\|_{C_\alpha} \leq 1 \right\}$$

This implies

$$\|\mu\|_{\mathfrak{M}_{\rho^\alpha}(\Omega)} \leq \|T[\mu,\lambda]\|_{\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})}$$

If $\phi \in C(\partial\Omega)$ is such that $|\phi| \leq 1$ and Φ is its harmonic lifting in Ω, the function $\zeta = \rho^\alpha\Phi$ belongs to $C_\alpha(\overline{\Omega})$ and satisfies $\|\zeta\|_{C^\alpha} \leq 1$. Let $\{\eta_n\} \subset C^\infty(\mathbb{R}^N)$ such that $0 \leq \eta_n \leq 1$, $\eta_n(x) = 0$ if $\rho(x) \geq 2/n$, $\eta_n(x) = 1$ if $\rho(x) \leq 1/n$. Then $\zeta_n = \eta_n\rho^\alpha\Phi$ belongs also to $C_\alpha(\overline{\Omega})$ and $\|\zeta_n\|_{C^\alpha} \leq 1$. Since

$$T[\mu,\lambda](\zeta_n) = \int_\Omega \zeta_n d\mu + \int_{\partial\Omega} \phi d\lambda$$

and $\int_\Omega \zeta_n d\mu \to 0$ as $n \to \infty$, we derive

$$\|T[\mu,\lambda]\|_{\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})} \geq \int_{\partial\Omega} \phi d\lambda.$$

This ends to proof. □

Remark. If λ is a Radon measure on $\partial\Omega$ and we can define its δ^α-lifting $\Lambda_{\delta^\alpha}[\lambda] \in \mathfrak{M}(\Omega)$ by

$$\int_\Omega \zeta d\lambda_{\delta^\alpha} = \delta^{-\alpha} \int_\Omega \zeta(\delta,\sigma) d\lambda(\sigma).$$

Clearly $\lambda_{\delta^\alpha} \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ and if $\zeta \in C_\alpha(\overline{\Omega})$ and $\ell_\alpha(\zeta) = -\lim_{\rho\to 0} \rho^{-\alpha}\zeta$, then $\ell_\alpha(\zeta) \in C(\partial\Omega)$, there holds

$$\lim_{\delta\to 0} \int_\Omega \zeta d\lambda_{\delta^\alpha} = \int_\Sigma \ell_\alpha(\zeta) d\lambda. \tag{2.32}$$

In the particular case where $\alpha = 1$ $\ell_\alpha(\zeta) = \zeta_{\mathbf{n}} := \lim_{\rho\to 0} \rho^{-1}\zeta$, and

$$\lim_{\delta\to 0} \int_\Omega \zeta d\lambda_\delta = -\int_\Sigma \zeta_{\mathbf{n}} d\lambda. \tag{2.33}$$

2.2 The linear operator

Let $x = (x_1, ..., x_N)$ the coordinates in $\mathbb{R}^N$ and Ω a bounded domain in $\mathbb{R}^N$. We consider the operator L in divergence form defined by

$$Lu := -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^N b_i \frac{\partial u}{\partial x_i} - \sum_{i=1}^N \frac{\partial}{\partial x_i}(c_i u) + du \tag{2.34}$$

where the a_{ij}, b_i and c_i are Lipschitz continuous and d is bounded and measurable in Ω. We assume that the ellipticity condition

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq a \sum_{i1}^N \xi_i^2 \qquad \forall \xi \in \mathbb{R}^N \tag{2.35}$$

holds for almost x in Ω, for some $a > 0$. We also assume the positivity condition

$$\int_\Omega \left(dv + \frac{1}{2}\sum_{i=1}^N (b_i + c_i)\frac{\partial v}{\partial x_i}\right) dx \geq 0 \qquad \forall v \in C_c^1(\Omega),\ v \geq 0 \tag{2.36}$$

Under these assumptions, the bilinear form

$$(u,v) \mapsto A_L(u,v) = \int_\Omega \left(\sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum_{i=1}^N \left(b_i \frac{\partial u}{\partial x_i} v + c_i \frac{\partial v}{\partial x_i} u\right) + duv\right) dx \tag{2.37}$$

is continuous and coercive on $W^{1,2}(\Omega)$. We define the adjoint operator L^* by

$$L^*u := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} c_i \frac{\partial u}{\partial x_i} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(b_i u) + du \tag{2.38}$$

We denote by $G = G_L$ and $K = K_L$ the Green and Poisson kernels corresponding to the operator L in Ω. We recall the following equivalence statement [10], [2]

Proposition 2.4 *Assume Ω has a C^2 boundary and (2.36) holds. Then there exists a positive constant C such that*

$$CG_{-\Delta} \leq G \leq C^{-1}G_{-\Delta} \qquad \text{in } \Omega \times \Omega \setminus D_\Omega \tag{2.39}$$

where $D_\Omega = x \in \Omega \times \Omega : x :\neq y$ and

$$CK_{-\Delta} \leq K \leq C^{-1}K_{-\Delta} \qquad \text{in } \Omega \times \partial\Omega. \tag{2.40}$$

2.3 Linear equation with measure data

If $m \in \mathfrak{M}_+(\Omega)$ is a bounded Borel measure and $\Psi : [0,\infty) \mapsto [0,\infty)$ is continuous and nonincreasing, we define the subset $M_m^\Psi(\Omega)$ of the set $\mathcal{B}(\Omega)$ of Borel mesurable functions by

$$M_m^\Psi(\Omega) := \left\{ f \in \mathcal{B}(\Omega) : \exists C > 0 \text{ s.t. } \int_{\lambda_f(t)} dm(x) \leq C\Psi(t)\,,\ \forall t > 0 \right\} \tag{2.41}$$

where

$$\lambda_f(t) = \{x \in \Omega : |f(x)| > t\}. \tag{2.42}$$

Notice that $\Psi(t) \leq m(\Omega)$ for $t \geq 0$. Denote

$$\bar{\lambda}_f(t) = \{x \in \Omega : |f(x)| \geq t\}. \tag{2.43}$$

Since Ψ is continuous, (2.41) implies

$$\int_{\bar{\lambda}_f(t)} dm(x) \leq C\Psi(t)\,,\ \forall t > 0.$$

If we modify Ψ in order to impose $\Psi(0) = m(\Omega)$, (2.41) is equivalent to

$$M_m^\Psi(\Omega) := \left\{ f \in \mathcal{B}(\Omega) : \exists C > 0 \text{ s.t. } \int_{\bar{\lambda}_f(t)} dm(x) \leq C\Psi(t)\,,\ \forall t \geq 0 \right\} \tag{2.44}$$

We denote by $C_m^\Psi(f)$ the smallest constant C such that (2.41) holds. If $t \mapsto \Psi(t)/\Psi(2t)$ remains bounded on $[0,\infty)$, $M_m^\Psi(\Omega)$ is a vector space $f \mapsto C_m^\Psi(f)$ is a quasi-norm on the quotient space $M_m^\Psi(\Omega)/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation $f_1\mathcal{R}f_2 \iff f_1 - f_2 = 0$ m-a.e. in Ω. In general $M_m^\Psi(\Omega)$ is not a vector space

When $\Psi(t) = t^{-p}$ with $p \geq 1$ and $m(x) = \rho(x)^\alpha$, with $\alpha \in [0,1]$, we denote by $M_{\rho^\alpha}^p(\Omega)$ the corresponding Marcinkiewicz space. The following results proved in [5] with $L = -\Delta$ are valid for a general operator L

Proposition 2.5 *Let $\alpha \in [0,1]$, $N \geq 2$. If $\mu \in \mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$ and $N+\alpha-2>0$,*

$$\|\mathbb{G}[\mu]\|_{M_{\rho^\alpha}^{(N+\alpha)/(N+\alpha-2)}} \leq C\,\|\mu\|_{\mathfrak{M}_{\rho^\alpha}}\,, \tag{2.45}$$

$$\|\nabla\mathbb{G}[\mu]\|_{M_{\rho^\alpha}^{(N+\alpha)/(N+\alpha-1)}} \leq C\,\|\mu\|_{\mathfrak{M}_{\rho^\alpha}}\,. \tag{2.46}$$

Furthermore, for any $\gamma \in [0,1]$ and $\lambda \in \mathfrak{M}(\partial\Omega)$,

$$\|\mathbb{K}[\lambda]\|_{M_{\rho^\gamma}^{(N+\gamma)/(N-1)}} \leq C\,\|\lambda\|_{\mathfrak{M}}\,. \tag{2.47}$$

We recall the following result proved in [13, Th 2.9]

Theorem 2.6 *Let $\alpha \in [0,1]$. For every $\mu \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ and $\lambda \in \mathfrak{M}(\partial\Omega)$, there exists a unique $u := u_{\mu,\lambda} \in L^1(\Omega)$ satisfying*

$$\begin{aligned} -Lu &= \mu \qquad && \text{in } \Omega \\ u &= \lambda \qquad && \text{in } \partial\Omega, \end{aligned} \tag{2.48}$$

in the following weak sense

$$-\int_\Omega uL^*\zeta dx = \int_\Omega \zeta d\mu - \int_{\partial\Omega} \zeta_{\mathbf{n}} d\lambda \qquad \forall \zeta \in C_{c^{1,L}}(\overline{\Omega}). \tag{2.49}$$

Furthermore, if $\{(\mu_{n,\lambda_n})\}$ is bounded in $\mathfrak{M}_{\rho^\alpha}(\Omega) \times \mathfrak{M}(\partial\Omega)$ and converges weakly with respect to $C_\alpha(\overline{\Omega}) \times C(\partial\Omega)$ to $(\mu,\lambda) \in \mathfrak{M}_{\rho^\alpha}(\Omega) \times \mathfrak{M}(\partial\Omega)$, then u_{μ_n,λ_n} converges to $u_{\mu,\lambda}$.

Remark. If we define the measure $\omega \in \mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$ by $\omega = T[\mu,\lambda]$ (see (2.30)), then it can also be expressed by

$$\int_{\overline{\Omega}} \zeta d\omega := \int_\Omega \zeta d\mu - \int_{\partial\Omega} \zeta_{\mathbf{n}} d\lambda \quad \forall \zeta \in C_1(\overline{\Omega}), \tag{2.50}$$

since $\zeta \in C_1(\overline{\Omega})$ implies that $\zeta_{\mathbf{n}}$ exists on $\partial\Omega$ and is continuous. We define the *global* Green operator on $\overline{\Omega}$ by

$$\overline{\mathbb{G}}[\omega] := \mathbb{G}[\mu]) + \mathbb{P}_L[\lambda]. \tag{2.51}$$

and (2.48) is replaced by the unique equation

$$-Lu = \omega \qquad \text{in } \overline{\Omega}. \tag{2.52}$$

Then (2.45)-(2.47) with $\alpha = 1$ are equivalent to

$$\left\|\overline{\mathbb{G}}[\omega]\right\|_{M_\rho^{(N+1)/(N-1)}} \leq C\,\|\omega\|_{\mathfrak{M}_\rho}\,. \tag{2.53}$$

Furthermore, we say that $u \in L^1(\Omega)$ is a subsolution of (2.52) in $\overline{\Omega}$, if

$$-\int_\Omega uL^*\zeta dx \leq \int_{\overline{\Omega}} \zeta d\omega := \int_\Omega \zeta d\mu - \int_{\partial\Omega} \zeta_{\mathbf{n}} d\lambda \qquad \forall \zeta \in C_c^{1,L^*}(\overline{\Omega}),\ \zeta \geq 0. \tag{2.54}$$

Comparison principle applies, thus $u \leq \overline{\mathbb{G}}[\omega]$. A supersolution is defined similarly.

Remark. If $\omega = T[\mu,\lambda] \in \mathfrak{M}_\alpha^+(\overline{\Omega})$ its Lebesgue decomposition is $\omega_r + \omega_s = T[\mu_r,\lambda_r] + T[\mu_s,\lambda_s]$ where μ_r and λ_r are the absolutely continuous part with respect to the Hausdorff measures $d\mathcal{H}^N$ and $d\mathcal{H}^{N-1}$ and μ_s and λ_s the respective singular parts. Similarly if $\omega = T[\mu,\lambda]$, then $\omega = \omega^+ - \omega^-$ where $\omega^+ = T[\mu^+,\lambda^+]$ and $\omega^- = T[\mu^-,\lambda^-]$.

2.4 Regularity results

We define the class of measures $B^p_h(\overline{\Omega})$ by

$$B^\Psi_h(\overline{\Omega}) := \{\omega \in \mathfrak{M}_\rho(\overline{\Omega}) : \overline{\mathbb{G}}[|\omega|] \in M^\Psi_{\rho h}\Omega)\}. \tag{2.55}$$

By Proposition 2.4, this class remains unchanged if we replace $-\Delta$ by L and the Green operator for L by the one of $-\Delta$. If $\Psi(t) = t^{-p}$ and $h = 1$, the corresponding class of measures is larger that the usual

$$\tilde{B}^p(\overline{\Omega}) := \{\omega \in \mathfrak{M}_\rho(\overline{\Omega}) : \overline{\mathbb{G}}[|\omega|] \in L^p_\rho(\Omega)\} \tag{2.56}$$

which corresponds to negative Besov spaces: if $\omega = T[\mu, \lambda]$, then the regularity results for harmonic functions [9] and solution of Laplace equation [1] yields to

$$\tilde{B}^p(\overline{\Omega}) = B^{-\frac{2}{p},p}(\Omega). \tag{2.57}$$

Example 1 If $h(x) = (\rho(x))^\beta$, with $\beta > -2$. Then $\omega = T[0, \lambda] \in B^p_{\rho^\beta}(\overline{\Omega})$ if and only if $\overline{\mathbb{G}}[|\omega|] \in M_{\rho^{\beta+1}}(\Omega)$. This means that $\lambda \in B^{-s,p}_\infty(\partial\Omega)$ with $s = (\beta + 2)/p$ (see [11] for the definition of $B^{\alpha,p}_q$.

3 The main results

Definition 3.1 *We say that a Caratheodory function $g : \Omega \times \mathbb{R}$ belongs to the class $G_{h,\Psi}$ if there exist a nonnegative function $h \in L^1_\rho(\Omega)$, a continuous nondecreasing function $\tilde{g}$ defined on $\mathbb{R}_+$ and vanishing at $r = 0$ such that $0 \leq g(x, r)\mathrm{sign\,r} \leq \mathrm{h(x)\tilde{g}(|r|)}$ in $\Omega \times \mathbb{R}$ and a continuous nonincreasing function $\Psi : [0, \infty) \mapsto [0, \infty)$ with the property that*

$$-\int_1^\infty \tilde{g}(s)d\Psi(s) < \infty. \tag{3.58}$$

Lemma 3.2 *Let μ be a nonnegative measure in $\mathfrak{M}(\Omega)$ and $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ a Caratheodory function such that $0 \leq g(x, r)\mathrm{sign\,r} \leq \mathrm{h(x)\tilde{g}(|r|)}$ where $h \in L^1_\rho(\Omega)$ and $\tilde{g}$ is a continuous and nondecreasing function $\tilde{g}$ defined on $\mathbb{R}_+$ and vanishing at $r = 0$. Then*
(i) If $g \in G_{h,\Psi}$ and $\mu \in B^\Psi_h(\overline{\Omega})$, then $\tilde{g} \circ \overline{\mathbb{G}}[\mu] \in L^1_{\rho h}(\Omega)$.
(ii) if $\tilde{g} \circ \overline{\mathbb{G}}[\mu] \in L^1_{\rho h}(\Omega)$ and , then $\mu \in B^\Psi_h(\overline{\Omega})$ and $g \in G_{h,\Psi}$ with $\Psi(s) = \theta_{\lambda_{\overline{\mathbb{G}}[\mu]}}(s)$, where $\lambda_{\overline{\mathbb{G}}[\mu]}(s)$ is defined by (2.42) with f replaced by $\overline{\mathbb{G}}[\mu]$ and $\theta_{\lambda_{\overline{\mathbb{G}}[\mu]}}(s) = \int_{\lambda_{\overline{\mathbb{G}}[\mu]}(s)} d(\rho h)$.

Proof. This due to the fact that

$$\int_\Omega \tilde{g}(\overline{\mathbb{G}}[\mu])\rho h dx = -\int_0^\infty \tilde{g}(s)d\theta_{\lambda_{\overline{\mathbb{G}}[\mu]}}(s). \tag{3.59}$$

Therefore, if $\theta_{\lambda_{\overline{\mathbb{G}}[\mu]}}(s) \leq \Psi(s)$, it proves (i). Conversely, if $\Psi(s) = \theta_{\lambda_{\overline{\mathbb{G}}[\mu]}}(s)$, then $\mu \in B^\Psi_h(\overline{\Omega})$ and $g \in G_{h,\Psi}$. □

The following existence result extends to one in [13]

Theorem 3.3 *Assume g belongs to the class $G_{h,\Psi}$. Then for any $\omega \in B_h^{\Psi}(\overline{\Omega})$ there exists a function $u \in L^1(\Omega)$ such that $g \circ u \in L^1(\Omega)$ satisfying*

$$\int_{\Omega} (-uL^*\zeta + g \circ u\,\zeta)\,dx = \int_{\overline{\Omega}} \zeta d\omega \qquad \forall \zeta \in C_c^{1,L^*}(\overline{\Omega}). \tag{3.60}$$

Furthermore u is unique if $r \mapsto g(x,r)$ is nondecreasing for a.e. $x \in \Omega$.

Proof. It is essentially [13, Theorem 3.7]. Since $0 \le g(x,r)\mathrm{sign}\,\mathrm{r} \le \mathrm{h(x)}\tilde{\mathrm{g}}(|\mathrm{r}|)$, we define the following truncation $g_k(.,r)$ for any $k > 0$.

$$g_k(x,r) = g(x,r)\chi_{\Theta_k} \tag{3.61}$$

where $\Theta_k = \{x \in \Omega : h(x) \le k\}$. Then $0 \le g(x,r)\mathrm{sign}\,\mathrm{r} \le \mathrm{k}\tilde{\mathrm{g}}(|\mathrm{r}|)$ and there exists a solution u_k to

$$-Lu_k + g_k \circ u_k = \omega \qquad \text{in } \overline{\Omega}\ . \tag{3.62}$$

Actually, in [13, Theorem 3.7] the proof is done with $\mu \in \mathfrak{M}_{\rho^\alpha}(\Omega)$ for any $\alpha \in [0,1]$, but due to our definition of measures in $\mathfrak{M}_{\rho^\alpha}(\overline{\Omega})$, it is also valid in this case.

Step 2: Convergence when $k \to \infty$. By Brezis'estimates (see e.g. [13, Th 2.4]), for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, $\zeta \ge 0$, one has

$$\int_{\Omega} (-|u_k|\,L^*\zeta + \mathrm{sign(u_k)}g_k(x,u_k)\zeta)\,dx \le \int_{\overline{\Omega}} \zeta d\,|\omega|\,. \tag{3.63}$$

and

$$\|u_k\|_{L^1} + \|\rho g_k(.,u_k)\|_{L^1_\rho} \le C_1\,\|\omega\|_{\mathfrak{M}_\rho}\,. \tag{3.64}$$

Furthermore, by estimates of Proposition 2.5 and since $|u_k| \le \overline{\mathbb{G}}[|\omega|]$, there holds,

$$\|u_k\|_{M_\rho^{(N+1)/N}} + \|\nabla u_k\|_{M_\rho^{(N+1)/N}} \le C\,\|\omega\|_{\mathfrak{M}_\rho}\,. \tag{3.65}$$

Since the right-hand side of (3.65) is bounded independently of k fixed, there exist a subsequence $\{u_{k_j}\}$ and a function $u \in W^{1,q}_{loc}(\Omega)$, for any $1 \le q < (N+1)/N$, such that $u_{k_j} \to u$ a.e. in Ω - and thus $g_{k_j} \circ u_{k_j} \to g \circ u$ a.e. - and weakly in $W^{1,q}_{loc}(\Omega)$ when $k_j \to \infty$. Let $R > 0$ and $E \subset \Omega$ be a Borel subset, then

$$\begin{aligned}\int_E \left|g_{k_j} \circ u_{k_j}\right| \rho dx &\le \int_{E\cap\{\left|u_{k_j}\right|\le R\}} \tilde{g}(\left|u_{k_j}\right|)\rho h dx + \int_{E\cap\{\left|u_{k_j}\right|> R\}} \tilde{g}(\left|u_{k_j}\right|)\rho h dx \\ &\le \tilde{g}(R)\int_E \rho h dx - \int_R^\infty \tilde{g}(s) d\theta_{u_{k_j}}(s),\end{aligned} \tag{3.66}$$

where, we recall it,

$$\theta_{u_{k_j}}(s) := \int_{\lambda_{u_{k_j}}(s)} d(\rho h).$$

Since $\left|u_{k_j}\right| \leq \overline{\mathbb{G}}[|\omega|]$, $\theta_{u_{k_j}}(s) \leq \theta_{\overline{\mathbb{G}}[|\omega|]}(s)$. By assumption,

$$\theta_{\overline{\mathbb{G}}[|\omega_n|]}(s) \leq C\Psi(s) \qquad \forall s > 0,$$

with

$$C = C^{\Psi}_{\rho h}(\overline{\mathbb{G}}[|\omega|]).$$

Furthermore, by a standard integration by parts in Stieltjes integrals and for a.e. R,

$$\begin{aligned}
-\int_R^\infty \tilde{g}(s)d\theta_{u_{k_j}}(s) &= \tilde{g}(R)\theta_{u_{k_j}}(R) + \int_R^\infty \theta_{u_{k_j}}(s)d\tilde{g}(s)) \\
&\leq \tilde{g}(R)\theta_{u_{k_j}}(R) + C\int_R^\infty \Psi(s)d\tilde{g}(s) \\
&\leq \tilde{g}(R)\theta_{u_{k_j}}(R) - C\tilde{g}(R)\Psi(R) - C\int_R^\infty \tilde{g}(s)d\Psi(s) \\
&\leq -C\int_R^\infty \tilde{g}(s)d\Psi(s).
\end{aligned} \tag{3.67}$$

Since condition (3.58) holds, it follows

$$\lim_{R\to\infty}\int_R^\infty \tilde{g}(s)d\Psi(s) = 0. \tag{3.68}$$

Given $\epsilon > 0$, we first choose $R > 0$ such that

$$-C\int_R^\infty \tilde{g}(s)d\Psi(s) \leq \epsilon/2.$$

Then we put $\delta = \epsilon/(2(1 + \tilde{g}(R))$ and derive

$$\int_E \rho dx \leq \delta \Longrightarrow \int_E \left|g_{k_j}(u_{k_j})\right| \rho h dx \leq \epsilon.$$

Therefore $\{g_{k_j} \circ u_{k_j}\}$ is uniformly integrable in $L^1_\rho(\Omega)$. It follows by Vitali's convergence theorem

$$\lim_{k\to\infty} g_{k_j} \circ u_{k_j} = g \circ u \qquad \text{in } L^1_\rho(\Omega). \tag{3.69}$$

Let $\zeta \in C^{1,L}_c(\overline{\Omega})$. If we let $k_j \to \infty$ in the equality

$$\int_\Omega \left(-u_{k_j}L^*\zeta + g_{k_j} \circ u_{k_j}\zeta\right) dx = \int_{\overline{\Omega}} \zeta d\omega, \tag{3.70}$$

we derive

$$\int_\Omega \left(-uL^*\zeta + g \circ u\zeta\right) dx = \int_{\overline{\Omega}} \zeta d\omega. \tag{3.71}$$

Uniqueness follows classicaly if $g(x, .)$ is nonndecreasing. □

The following extension of the previous result is an adaptation of [13, Th. 3.20]

Theorem 3.4 *Assume g belongs to the class $G_{h,\Psi}$ and satisfies the following Δ_2-condition*

$$\left|g(x,r+r')\right| \leq \theta\left(\left|g(x,r)\right| + \left|g(x,r')\right|\right) + \ell(x) \qquad \forall x \in \Omega,\, \forall (r,r') \in \mathbb{R}\times\mathbb{R}, \tag{3.72}$$

for some nonnegative $\ell \in L^1_\rho(\Omega)$. Suppose also that $r \mapsto g(x,r)$ is nondeacreasing. If $\omega \in \mathfrak{M}_\rho(\overline{\Omega})$ has Lebesgue decomposition $\omega = \omega_r + \omega_s$ with regular part with respect to the Lebesgues measures ω_r and singular part ω_s, and if ω_s belongs to $B^\Psi_h(\overline{\Omega})$, then there exists a unique solution u to (3.60).

Proof. If g satisfies (3.72), g_k defined by (3.61) shares the same property with the same ℓ. Therefore, by [13, Th 3.12], there exists a solution u_k to (3.62). Actually, in this result it is only assume that ℓ in (3.72) is a constant, but the proof is valid if it is a nonnegative function in $L^1_\rho(\Omega)$. Let v_k and v'_k be weak solutions in $\overline{\Omega}$ of $-Lv_k + g_k \circ v_k = \omega^+_r$ and $-Lv'_k - g_k \circ (-v'_k) = \omega^-_r$ respectively. Set $w_k = v_k + \overline{\mathbb{G}}(\omega^+_s)$ and $w'_k = v'_k + \overline{\mathbb{G}}(\omega^-_s)$. Then $-Lw_k + g_k \circ w_k \geq \omega^+$ and $-Lw'_k - g_k \circ (-w'_k) \geq \omega^-$ in $\overline{\Omega}$. By monotonicity $-w'_k \leq u_k \leq w_k$, thus $g_k(-w'_k) \leq g_k(u_k) \leq g_k(w_k)$. The estimates (3.64) and (3.65) are satisfied, therefore there exist a function $u \in L^1(\Omega)$ and a subsequence u_{k_j} which converges to u a.e. in Ω. Furthermore

$$\begin{aligned} g_k(x,u_k) &\leq \theta\left(g_k(x,v_k) + g_k(x,\overline{\mathbb{G}}(\omega^+_s))\right) + \ell \\ &\leq \theta\left(g_k(x,v_k) + g(x,\overline{\mathbb{G}}(\omega^+_s))\right) + \ell \end{aligned} \tag{3.73}$$

Since the sequence $\{|g_k|\}$ increases, $\{v_k\}$ and $\{v'_k\}$ decrease. Therefore $v_k \downarrow v$ and $v'_k \downarrow v'$ which satisfy $-Lv + g \circ v = \omega^+_r$ and $-Lv' - g_k \circ (-v') = \omega^-_r$ respectively in $\overline{\Omega}$. Therefore $g_k \circ v_k \to g \circ v$ and $g_k \circ v'_k \to -g \circ (-v')$ in $L^1_\rho(\Omega)$ respectively. Since

$$g_k \circ \overline{\mathbb{G}}(\omega^+_s) \leq g \circ \overline{\mathbb{G}}(\omega^+_s)$$

and $\omega_s \in B^\Psi_h(\overline{\Omega})$, $g \circ \overline{\mathbb{G}}(\omega^+_s$ by Lemma 3.2, the right-hand side term of inequality (3.73) is uniformly integrable in $L^1_\rho(\Omega)$. Similarly

$$g_k(x,u_k) \geq \theta\left(g_k(x,-v'_k) + g(x,-\overline{\mathbb{G}}(\omega^-_s))\right) - \ell \tag{3.74}$$

and the right-hand side of (3.74) is also uniformly integrable in $L^1_\rho(\Omega)$. We conclude as in Theorem 3.3 . $\square$

4 Stability

Lemma 4.1 *Let $\{\omega_n\} \subset B^\Psi_h(\overline{\Omega})$ be a sequence of measures such that $C^\Psi_\rho(\mathbb{G}[|\omega_n|])$ is bounded independently of n. Then $\{\omega_n\}$ remains bounded in $\mathfrak{M}_\rho(\overline{\Omega})$. If $\omega_n \to \omega$ weakly in $\mathfrak{M}_\rho(\overline{\Omega})$, then $\omega \in B^\Psi_h(\overline{\Omega})$.*

Proof. Since $C^\Psi_\rho(\overline{\mathbb{G}}[|\omega_n|])$ is uniformly bounded, the sequence $\{g \circ \overline{\mathbb{G}}[|\omega_n|])\}$ is bounded in $L^1_\rho(\Omega)$ by Lemma 3.2. Since $\omega_n \to \omega$ weakly in $\mathfrak{M}_\rho(\overline{\Omega})$, $\overline{\mathbb{G}}[\omega_n] \to \overline{\mathbb{G}}[\omega]$ in $L^1_\rho(\Omega)$ and, up to a subsequence, a.e. in Ω. Therefore, and up to sets of zero Lebesgue measure,

$$\lambda_{\overline{\mathbb{G}}[\omega]}(t) \subset \bigcap_{n\geq 0}\left(\bigcup_{p\geq n}\lambda_{\overline{\mathbb{G}}[\omega_p]}(t)\right) \subset \bigcap_{n\geq 0}\left(\bigcup_{p\geq n}\overline{\lambda}_{\overline{\mathbb{G}}[\omega_p]}(t)\right) \subset \overline{\lambda}_{\overline{\mathbb{G}}[\omega]}(t). \tag{4.75}$$

Therefore

$$\limsup_{n\to\infty}\theta_{\lambda_{\overline{\mathbb{G}}[\omega_n]}(t)} \leq \theta_{\overline{\lambda}_{\overline{\mathbb{G}}[\omega]}(t)}. \tag{4.76}$$

Conversely, for any $x \in \lambda_{\overline{\mathbb{G}}[\omega]}(t)$, i.e. such that $\overline{\mathbb{G}}[\omega](x) > t$, there exists n_x such that $x \in \lambda_{\overline{\mathbb{G}}[\omega_n]}(t)$ if $n \geq n_x$. This implies

$$\lim_{n\to\infty}\chi_{\lambda_{\overline{\mathbb{G}}[\omega_n]}(t)}\chi_{\lambda_{\overline{\mathbb{G}}[\omega]}(t)} = \chi_{\lambda_{\overline{\mathbb{G}}[\omega]}(t)},$$

and

$$\liminf_{n\to\infty}\theta_{\lambda_{\overline{\mathbb{G}}[\omega_n]}(t)} \geq \theta_{\lambda_{\overline{\mathbb{G}}[\omega]}(t)}. \tag{4.77}$$

Since $\theta_{\lambda_{\overline{\mathbb{G}}[\omega_n]}(t)} \leq C_\rho^\Psi(\overline{\mathbb{G}}[|\omega_n|])\Psi(t)$ and the $C_\rho^\Psi(\overline{\mathbb{G}}[|\omega_n|])$ are bounded, it follows that ω belongs to $B_h^\Psi(\overline{\Omega})$. □

Theorem 4.2 *Assume g belongs to the class $G_{h,\Psi}$ and $r \mapsto g(x,r)$ is nondecreasing for a.e. $x \in \Omega$. Let $\{\omega_n\} \subset B_h^\Psi(\overline{\Omega})$ be a sequence of measures such that $C_\rho^\Psi(\mathbb{G}[|\omega_n|])$ is bounded independently of n which converges to ω weakly with respect to $C_1(\overline{\Omega})$. Then the solution u_n of*

$$-Lu_n + g\circ u_n = \omega_n \qquad \text{in } \overline{\Omega} \tag{4.78}$$

converges to the solution u of

$$-Lu + g\circ u = \omega \qquad \text{in } \overline{\Omega} \tag{4.79}$$

Proof. Since u_n satisfies the Brezis estimates (3.64) and (3.65), there exists a subsequence $\{u_{n_j}\}$ and $u \in L^1(\Omega)$ such that $u_{n_j} \to u$ a.e. in Ω and in $L^1(\Omega)$. As in the proof of Theorem 3.3, the problem is to prove the convergence of the $g\circ u_{n_j}$ in $L^1_\rho(\Omega)$. But this is a clearly obtained by the uniform integrability, as in the proof of Theorem 3.3-Step 2, using the fact that, in (3.67), the $\theta_{u_{n_j}}$ are bounded by $sup_n C^\Psi_{\rho h}(\overline{\mathbb{G}}[\omega_n])\Psi$. □

Theorem 4.3 *Assume g belongs to the class $G_{h,\Psi}$, satisfies the Δ_2-condition* (3.72) *and $r \mapsto g(x,r)$ is nondeacreasing. Let $\{\omega_n\} \subset \mathfrak{M}_\rho(\overline{\Omega})$ has Lebesgue decomposition $\omega_n = \omega_{n\,r} + \omega_{n\,s}$ if $\{\omega_{n\,s}\} \subset B_h^\Psi(\overline{\Omega})$ are such that the $C^\Psi_{\rho h}(\overline{\mathbb{G}}[\omega_{n\,s}])$ are uniformly bounded, then the solutions u_n of* (4.78) *converges in $L^1(\Omega)$ to the solution u of* (4.79).

Proof. The argument follows the one of Theorem 3.4. Let v_n and v'_n be weak solutions in $\overline{\Omega}$ of $-Lv_n + g\circ v_n = \omega^+_{n\,r}$ and $-Lv'_n - g\circ(-v'_n) = \omega^-_{n\,r}$ respectively. Set $w_n = v_n + \overline{\mathbb{G}}(\omega^+_{n\,s})$ and $w'_k = v'_k + \overline{\mathbb{G}}(\omega^-_{n\,s})$. Then $-Lw_n + g\circ w_n \geq \omega^+_n$ and $-Lw'_n - g\circ(-w'_n) \geq \omega^-_n$. By monotonicity $-w'_n \leq u_n \leq w_n$, thus $g(-w'_n) \leq g(u_n) \leq g(w_n)$. The estimates (3.64) and (3.65) are satisfied therefore there exist a function $u \in L^1(\Omega)$ and a subsequence u_{n_j} which converges to u a.e. in Ω and in $L^1(\Omega)$. Furthermore

$$\begin{aligned} g(x,u_n) &\leq \theta\left(g(x,v_n) + g(x,\overline{\mathbb{G}}(\omega^+_{n\,s})\right) + \ell \\ &\leq \theta\left(g(x,v_n) + g(x,\overline{\mathbb{G}}(\omega^+_{n\,s})\right) + \ell. \end{aligned} \tag{4.80}$$

Classicaly $v_n \to v$ $v'_n \to v'$ in $L^1(\Omega)$ which satisfy $-Lv + g \circ v = \omega_r^+$ and $-Lv' - g_k \circ (-v') = \omega_r^-$ respectively. Therefore $g \circ v_n \to g \circ v$ and $g \circ v' \to -g \circ (-v')$ in $L^1_\rho(\Omega)$ respectively. Since $C^\Psi_{\rho h}(\overline{\mathbb{G}}[\omega_{n\,s}])$ is uniformly bounded the $g \circ \overline{\mathbb{G}}[\omega_{n\,s}]$ are uniformly integrable in $L^1_\rho(\Omega)$ by Lemma 3.2. Therefore the $(g \circ u_n)^+$ are uniformly integrable in $L^1_\rho(\Omega)$. Similarly

$$g(x, u_n) \geq \theta\left(g(x, -v'_k) + g(x, -\overline{\mathbb{G}}(\omega_s^-))\right) - \ell \tag{4.81}$$

and the $(g \circ u_n)^-$ are also uniformly integrable in $L^1_\rho(\Omega)$. The conclusion follows in the same way as in Theorem 3.4. □

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