On the L^2-critical nonlinear Schrödinger Equation with a nonlinear damping.

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ON THE L^2-CRITICAL NONLINEAR SCHRÖDINGER EQUATION WITH A NONLINEAR DAMPING.

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ABSTRACT. We consider the Cauchy problem for the L^2-critical nonlinear Schrödinger equation with a nonlinear damping. According to the power of the damping term, we prove the global existence or the existence of finite time blowup dynamics with the log-log blow-up speed for $\|\nabla u(t)\|_{L^2}$.

1. INTRODUCTION

In this paper, we study the blowup and the global existence of solutions for the focusing NLS equation with a nonlinear damping (NLS_{ap}):

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u + ia|u|^p u = 0, (t,x) \in [0,\infty[\times\mathbb{R}^d, d = 1,2,3,4. \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \quad (1.1)$$

with initial data $u(0) = u_0 \in H^1(\mathbb{R}^d)$ where $a > 0$ is the coefficient of friction and $p \geq 1$. Note that if we replace $+|u|^{\frac{4}{d}}u$ by $-|u|^{\frac{4}{d}}u$, (1.1) becomes the defocusing NLS equation.

Equation (1.1) arises in various areas of nonlinear optics, plasma physics and fluid mechanics. Fibich [7] noted that in the nonlinear optics context, the origin of the nonlinear damping is multiphoton absorption. For example, in the case of solids the number p corresponds to the number of photons it takes to make a transition from the valence band to the conduction band. Similar behavior can occur with free atoms, in this case p corresponds to the number of photons needed to make a transition from the ground state to some excited state or to the continuum.

The Cauchy problem for (1.1) was studied by Kato [11] and Cazenave [3] and it is known that if $p < \frac{4}{d-2}$, then the problem is locally well-posed in $H^1(\mathbb{R}^d)$: For any $u_0 \in H^1(\mathbb{R}^d)$, there exist $T \in (0,\infty]$ and a unique solution $u(t)$ of (1.1) with $u(0) = u_0$ such that $u \in C([[0,T]);H^1(\mathbb{R}^d))$. Moreover, T is the maximal existence time of the solution $u(t)$ in the sense that if $T < \infty$ then $\lim_{t\to T} \|u(t)\|_{H^1(\mathbb{R}^d)} = \infty$.

Let us notice that for $a = 0$ (1.1) becomes the L^2-critical nonlinear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \quad (1.2)$$

For $u_0 \in H^1$, a sharp criterion for global existence for (1.2) has been exhibited by Weinstein [25]: Let Q be the unique positive solution to

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \tag{1.3}$$

For $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the solution of (1.2) is global in H^1. This follows from the conservation of the energy and the L^2 norm and the sharp Gagliardo-Nirenberg inequality:

$$\forall u \in H^1, E(u) \geq \frac{1}{2}(\int |\nabla u|^2)\left(1 - (\frac{\int |u|^2}{\int |Q|^2})^{\frac{2}{d}}\right).$$

On the other hand, there exists explicit solutions with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ that blow up in finite time in the regime $\frac{1}{T-t}$.

In the series of papers [15, 23], Merle and Raphael have studied the blowup for (1.2) with $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \delta$, δ small and have proven the existence of the blowup regime corresponding to the log-log law:

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \left(\frac{\log |\log(T-t)|}{T-t}\right)^{\frac{1}{2}}. \tag{1.4}$$

In [6], Darwich has proved in case of the linear damping ($p = 0$), the global existence in H^1 for $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, and has showed that the log-log regime is stable by such perturbations (i.e. there exist solutions blows up in finite time with the log-log law).

Numerical observations suggest that this finite time blowup phenomena persists in the case of the nonlinear damping for $p < \frac{4}{d}$ (see Fibich [7] and [21]). Passot and Sulem [21] have proved that the solutions are global in $H^1(\mathbb{R}^2)$ in the case where the power of the damping term is strictly greater to the focusing nonlinearity. The case where the power of the damping term is equal to the focusing nonlinearity, "small damping prevents blow-up ? "was an open question for Sparber and Antonelli in their paper [1] and for Fibich and Klein in their paper [8]. Our results can gives an answer for their open problem, at least for the L^2-critical case. In fact, our aim in this paper is study for each value of (d, p), the existence of blow-up solutions as well as global existence criteria. And know if the regime log-log still stable by such perturbations.

Let us now our results:

Theorem 1.1. *Let u_0 in $H^1(\mathbb{R}^d)$ with $d = 1, 2, 3, 4$:*

(1) *if $\frac{4}{d-2} > p \geq \frac{4}{d}$, then the solution of (1.1) is global in H^1.*

(2) *if $1 \leq p < \frac{4}{d}$ and $1 \leq p \leq 2$, then there exists $0 < \alpha < \|Q\|_{L^2}$ such that for any $u_0 \in H^1$ with $\|u_0\|_{L^2} < \alpha$, the emanating solution is global in H^1.*

(3) *if $1 \leq p < \frac{4}{d}$, then there exists $\delta_0 > 0$ such that $\forall a > 0$ and $\forall \delta \in]0, \delta_0[$, there exists $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$, such that the solution of (1.1) blows up in finite time in the log-log regime.*

(4) *if $1 \leq p < \frac{4}{d}$, then there does not exists an intial data u_0 with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ such that the solution u of (1.1) blow up in finite*

time with this law:

$$\frac{1}{(T-t)^{\beta-\epsilon}} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T-t)^{\beta+\epsilon}},$$

for any $\beta \in]0, \frac{2}{pd}[$ and $0 < \epsilon < \frac{2-\beta pd}{8+pd}$.

Remark 1.1. *Note that, part (4) of Theorem 1.1 prove in particular that we dont have the blowup in the regime log-log for any $p \in [1, \frac{4}{d}[$ and in the regime $\frac{1}{t}$ for $d = 1$ and $1 \leq p < 2$, for initial data with critical or subcritical mass.*

In the "critical" case $p = \frac{4}{d}$, we have more precisely :

Theorem 1.2. *Let $p = \frac{4}{d}$, then the initial-value probem (1.1) is globally well posed in $H^s(R^d)$, $s \geq 0$. Moreover, there exist unique u_+ in L^2 such that*

$$\|u(.,t) - e^{it\Delta}u_+\|_{L^2} \longrightarrow 0, \ t \longrightarrow +\infty, \tag{1.5}$$

where $e^{it\Delta}$ is the free evolution.

Remark 1.2. *Theorem 1.2 and part (1) and (2) of Theorem 1.1 still hold in the defocusing case.*

Remark 1.3. *Note that if $u(t, .)$ is a solution of $(NLS_{a,p})$ then $\overline{u(-t, .)}$ is a solution of $(NLS_{-a,p})$, then we dont have the scattering in $-\infty$, because this changes the sign of the coefficient of friction.*

2. Proof of part (4) of Theorem 1.1

Special solutions play a fundamental role for the description of the dynamics of (NLS). They are the solitary waves of the form $u(t, x) = \exp(it)Q(x)$, where Q solves:

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \tag{2.1}$$

The pseudo-conformal transformation applied to the stationary solution $e^{it}Q(x)$ yields an explicit solution for (NLS)

$$S(t, x) = \frac{1}{|t|^{\frac{d}{2}}} Q(\frac{x}{t}) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}$$

which blows up at $T = 0$.
Note that

$$\|S(t)\|_{L^2} = \|Q\|_{L^2} \text{ and } \|\nabla S(t)\|_{L^2} \sim \frac{1}{t} \tag{2.2}$$

It turns out that $S(t)$ is the unique minimal mass blow-up solution in H^1 up to the symmetries of the equation (see [14]).

A known lower bound (see [19]) on the blow-up rate for (NLS) is

$$\|\nabla u(t)\|_{L^2} \geq \frac{C(u_0)}{\sqrt{T-t}}. \tag{2.3}$$

Note that this blow-up rate is the one of $S(t)$ given by (2.2) and log-log given by (1.4). Now, we will prove the part (4) of Theorem 1.1.

For this we need the following Theorem (see [10]) :

Theorem 2.1. *Let $(v_n)_n$ be a bounded family of $H^1(\mathbb{R}^d)$, such that:*

$$\limsup_{n \to +\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^d)} \leq M \quad and \quad \limsup_{n \to +\infty} \|v_n\|_{L^{\frac{4}{d}+2}} \geq m. \qquad (2.4)$$

Then, there exists $(x_n)_n \subset \mathbb{R}^d$ such that:

$$v_n(\cdot + x_n) \rightharpoonup V \quad weakly,$$

with $\|V\|_{L^2(\mathbb{R}^d)} \geq (\frac{d}{d+4})^{\frac{d}{4}} \frac{m^{\frac{d}{2}+1}+1}{M^{\frac{d}{2}}} \|Q\|_{L^2(\mathbb{R}^d)}$.

Let us recall the following quantities:

L^2-norm : $\|u(t,x)\|_{L^2} = \int |u(t,x)|^2 dx.$

Energy : $E(u(t,x)) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{d}{4+2d}\|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$

Kinetic momentum : $P(u(t)) = Im(\int \nabla u \overline{u}(t,x)).$

Remark 2.1. *It is easy to prove that if u is a solution of (1.1) on $[0,T[$, then for all $t \in [0,T[$ it holds*

$$\frac{d}{dt}\|u(t)\|_{L^2} = -2a \int |u|^{p+2}, \qquad (2.5)$$

$$\frac{d}{dt}E(u(t)) = -a(\|u^{\frac{p}{2}}\nabla u\|_{L^2}^2 - C_p\|u\|_{L^{\frac{4}{d}+2+p}}^{\frac{4}{d}+2+p}) \qquad (2.6)$$

and

$$\frac{d}{dt}P(u(t)) = -2aIm \int \overline{u}|u|^p \nabla u. \qquad (2.7)$$

where $C_p = \frac{4+2d+pd}{4+2d}$.

Now we are ready to prove part (4) of Theorem 1.1:
Suppose that there exist an initial data u_0 with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, such that the corresponding solution $u(t)$ blows up with the following law

$$\frac{1}{(T-t)^{\beta-\epsilon}} \lesssim \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{(T-t)^{\beta+\epsilon}},$$

where β; $0 < \beta < \beta(p,d) = \frac{2}{pd}$ and $0 < \epsilon < \frac{2-\beta pd}{8+pd}$.
Recall that

$$E(u(t)) = E(u_0) - a \int_0^t K(u(\tau))d\tau, \quad t \in [0,T[, \qquad (2.8)$$

where $K(u(t)) = (\|u^{\frac{p}{2}}\nabla u\|_{L^2}^2 - C_p\|u\|_{L^{\frac{4}{d}+2+p}}^{\frac{4}{d}+2+p})$.

By Gagliardo-Nirenberg inequality and (2.5), we have:

$$E(u(t)) \lesssim E(u_0) + \int_0^t \|u\|_{L^2}^{\frac{4}{d}+p-\frac{p}{2}} \|\nabla u\|_{L^2}^{2+\frac{pd}{2}} \lesssim E(u_0) + \int_0^t \|\nabla u\|_{L^2}^{2+\frac{pd}{2}}$$

Since the choise of ϵ, we obtain that

$$0 \leqslant \lim_{t\to T} \frac{\int_0^t \|\nabla u(\tau)\|_{L^2}^{2+\frac{pd}{2}} d\tau}{\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2} \lesssim \lim_{t\to T} (T-t)^{\frac{1}{2}(2-\beta pd-\epsilon(8+pd))} = 0, \qquad (2.9)$$

let

$$\rho(t) = \frac{\|\nabla Q\|_{L^2(\mathbb{R}^d)}}{\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}} \quad \text{and} \quad v(t,x) = \rho^{\frac{d}{2}} u(t,\rho x)$$

$(t_k)_k$ a sequence such that $t_k \to T$ and $\rho_k = \rho(t_k), v_k = v(t_k, .)$. The family $(v_k)_k$ satisfies

$$\|v_k\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^2(\mathbb{R}^d)} \leq \|Q\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|\nabla v_k\|_{L^2(\mathbb{R}^d)} = \|\nabla Q\|_{L^2(\mathbb{R}^d)}.$$

Remark that $\lim_{k\longrightarrow+\infty} E(v_k) = 0$, because:

$$0 \leq \frac{1}{2}\left(\int |\nabla v_k|^2\right)\left(1 - \left(\frac{\int |v_k|^2}{\int |Q|^2}\right)^2\right) \leq E(v_k) = \rho_k^2 E(u_0) - a\rho_k^2 \int_0^{t_k} K(u(\tau))d\tau$$

$$\leq \rho_k^2 E(u_0) + \frac{1}{\|\nabla u(t_k)\|_{L^2(\mathbb{R}^d)}^2} \int_0^{t_k} \|\nabla u(\tau)\|_{L^2}^{2+\frac{pd}{2}} d\tau$$

then using (2.9), the energy of v_k tends to 0. Which yields

$$\|v_k\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \to \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2. \qquad (2.10)$$

The family $(v_k)_k$ satisfies the hypotheses of Theorem 2.1 with

$$m^{\frac{4}{d}+2} = \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 \quad \text{and} \quad M = \|\nabla Q\|_{L^2(\mathbb{R}^d)},$$

thus there exists a family $(x_k)_k \subset \mathbb{R}$ and a profile $V \in H^1(\mathbb{R})$ with $\|V\|_{L^2(\mathbb{R}^d)} \geq \|Q\|_{L^2(\mathbb{R}^d)}$, such that,

$$\rho_k^{\frac{d}{2}} u(t_k, \rho_k \cdot +x_k) \rightharpoonup V \in H^1 \quad \text{weakly}. \qquad (2.11)$$

Using (2.11), $\forall A \geq 0$

$$\liminf_{n\to+\infty} \int_{B(0,A)} \rho_n^d |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{B(0,A)} |V|^2 dx.$$

But $\lim_{n\to+\infty} \frac{1}{\rho_n} = +\infty$ thus $\frac{1}{\rho_n} > A$, $\rho_n A < 1$. This gives immediately:

$$\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leq 1} |u(t_n,x)|^2 dx \geq \liminf_{n\to+\infty} \int_{|x-x_n|\leq \rho_n A} |u(t_n,x)|^2 dx \geq \int_{|x|\leq A} |V|^2 dx.$$

This it is true for all $A > 0$ thus :

$$\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leq 1} |u(t_n,x)|^2 dx \geq \int Q^2, \tag{2.12}$$

then

$$\|u_0\|_{L^2} > \liminf_{n\to+\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leq 1} |u(t_n,x)|^2 dx \geqslant \|Q\|_{L^2}.$$

This gives the proof, the fact that $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$.

3. Global existence.

In this section, we prove assertion (1) and (2) of Theorem 1.1 and Theorem 1.2. To prove part (1), we will prove that the H^1-norm of u is bounded for any time. To prove part (2), we use generalised Gagliardo-Nirenberg inequalities to show that the energy is non increasing. Finally to prove Theorem 1.2, we etablish an a priori estimate on the critical Strichartz norm.

Theorem 3.1. *Let $p \geq 1$ for $d = 1,2$ or $1 \leq p \leq \frac{4}{d-2}$ for $d \geq 3$, then the initial-value probem (1.1) is locally well posed in $H^1(\mathbb{R}^d)$(If $p < \frac{4}{d-2}$ the minimal time of the existence depends on $\|u_0\|_{H^1}$.) .*

Proof: See [3] page 93 Theorem 4.4.1.

To prove the following proposition, we will proceed in the same way as in the section 3.1 in [21].

Proposition 3.1. *Let u be a solution of (1.1) and $\frac{4}{d-2} \geqslant p > \frac{4}{d}$ then*

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \leq \|\nabla u(0)\|_{L^2(\mathbb{R}^d)} e^{a^{(\frac{-4t}{pd-4})}}.$$

Proof: Multiply Eq. (1.1) by $\Delta\overline{u}$, integrate and take the imaginary part, this gives

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 dx + a\int |u|^p|\nabla u|^2 + a\Re\int u\nabla|u|^p\nabla\overline{u}dx = -\frac{4}{d}\Im\int u\nabla\overline{u}\Re(u\nabla u)|u|^{\frac{4}{d}-2}. \tag{3.1}$$

In the l.h.s, a simple calculation shows that the third term rewrites in the form $\frac{p}{4}\int |u|^{p-2}(\nabla|u|^2)^2$.Equation (3.1) becomes:

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 dx + a\int |u|^p|\nabla u|^2 + a\frac{p}{4}\int |u|^{p-2}(\nabla|u|^2)^2 \leq \frac{2}{d}\int |u|^{\frac{4}{d}}|\nabla u|^2. \tag{3.2}$$

To estimate the r.h.s of (3.2), we rewrite it as ($p > \frac{4}{d}$)

$$\int |u|^{\frac{4}{d}}|\nabla u|^2 = \int |u|^{\frac{4}{d}}|\nabla u|^{\frac{8}{pd}}|\nabla u|^{2-\frac{8}{pd}}.$$

Now by Hölder inequality we obtain that

$$\int |u|^{\frac{4}{d}}|\nabla u|^2 \leq (\int |u|^p|\nabla u|^2)^{\frac{4}{pd}}(\int |\nabla u|^2)^{1-\frac{4}{pd}}.$$

Then inequality (3.2) takes the form:

$$\frac{d}{dt}w(t) + 2av(t) \leq \frac{4}{d}v(t)^{\frac{4}{pd}}w(t)^{1-\frac{4}{pd}}.$$

where $w(t) = \int |\nabla u|^2$ and $v(t) = \int |u|^p |\nabla u|^2$.

Using Young's inequality $ab \leq \epsilon a^q + C\epsilon^{-\frac{1}{q-1}} b^{q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$, with $q = \frac{pd}{4}$ and $\epsilon = \frac{ad}{2}$ we obtain :

$$\frac{d}{dt} w(t) \leq a^{-\frac{1}{\frac{pd}{4}-1}} w(t).$$

This ensures that:

$$w(t) \leq w(0) e^{a^{\left(-\frac{4t}{pd-4}\right)}}. \square$$

This show that the H^1-norm of u is bounded for any time and gives directly the proof of part one of Theorem 1.1 in the case $p > 4/d$.

Now we will prove the global existence for small data, for this we will use the following generalized Gagliardo-Niremberg inequalities (see for instance [9]):

Lemma 3.1. Let q, r be any real numbers satisying $1 \leq q$, $r \leq \infty$, and let j, m be any integers satisfying $0 \leq j < m$. If u is any functions in $C_0^m(\mathbb{R}^d)$, then

$$\|D^j u\|_{L^s} \leq C \|D^m u\|_r^a \|u\|_q^{1-a}$$

where

$$\frac{1}{s} = \frac{j}{d} + a(\frac{1}{r} - \frac{m}{d}) + (1-a)\frac{1}{q},$$

for all a in the interval

$$\frac{j}{m} \leq a \leq 1,$$

where C is a constant depending only on d, m, j,q,r and a.

As a direct consequence we get :

Lemma 3.2. Let $1 \leq p \leq 2$ and $v \in C_0^\infty(R^d)$ then:

$$\int |v|^{\frac{4}{d}+2+p} \leq C(\int |\nabla(|v|^{\frac{p+2}{2}})|^2) \times (\int |v|^2)^{\frac{2}{d}}.$$

where $c > 0$ depending only on d and p.

Proof: Take $s = \frac{\frac{4}{d}+2+p}{1+\frac{p}{2}}$, $q = \frac{2}{1+\frac{p}{2}}$ $r = 2$, $j = 0$ and $m = 1$,then by Lemma 3.1 we obtain that:

$$|u|_{L^{\frac{\frac{4}{d}+2+p}{1+\frac{p}{2}}}} \leq C|\nabla u|_{L^2}^{\frac{4+2p}{\frac{8}{d}+4+2p}} |u|_{L^{\frac{2}{1+\frac{p}{2}}}}^{\frac{\frac{8}{d}}{\frac{8}{d}+4+2p}}.$$

Taking $u = |v|^{1+\frac{p}{2}}$, we obtain our lemma. $\square$

Now we can prove the following proposition:

Proposition 3.2. Let $1 \leq p \leq 2$. There exists $0 < \alpha = \alpha(p,d) < \|Q\|_{L^2}$, such that for any $u_0 \in H^1$ with $\|u_0\|_{L^2} < \alpha$, it holds

$$\frac{d}{dt} E(u(t)) \leq 0, \ \forall t > 0.$$

Proof: We can whrite that:

$$\frac{d}{dt}E(u(t)) = a\Big(C_p \int |u|^{\frac{4}{d}+p+2} - \frac{4}{(p+2)^2} \int |\nabla(|u|^{\frac{p+2}{2}})|^2\Big),$$

then by Lemma 3.2 we obtain that:

$$\frac{d}{dt}E(u(t)) \le a(\int |\nabla(|u|^{\frac{p+2}{2}})|^2)(C_p C(\int |u|^2)^{\frac{2}{d}} - \frac{4}{(p+2)^2})$$

Choosing $\alpha^{\frac{2}{d}} < \frac{4}{(p+2)^2}\frac{1}{C_p C}$, and using that $\|u(t)\|_{L^2} \le \|u_0\|_{L^2}$ for all $t \ge 0$ (see 2.5 below) we get the result.   □

Now the proof of part (2) of Theorem 1.1 follows from the sharp Gagliardo-Nirenberg inequality :

$$\forall u \in H^1, E(u) \ge \frac{1}{2}\Big(\int |\nabla u|^2\Big)\Big(1 - (\frac{\int |u|^2}{\int |Q|^2})^{\frac{2}{d}}\Big).$$

Proposition 3.2 together with the above inequality ensure that the H^1-norm of u is uniformly bounded in time. This leads to the global existence result for small initial data when $1 \le p \le 2$.

3.1. Critical case ($p = \frac{4}{d}$).
Now we will treat the critical case and prove Theorem 1.2. First let us prove that, if the solution blows up in finite time T, then $\|u\|_{L^{\frac{4}{d}+2}([0,T];L^{\frac{4}{d}+2}(R^d))} = +\infty$.

Proposition 3.3. *Let u be the unique maximal solution of (1.1) in $[0, T^*)$; if $T^* < \infty$, then $\|u\|_{L^\sigma([0,T],L^\sigma)} = \infty$ where $\sigma = \frac{4}{d} + 2$.*

To prove this claim, denoting by $S(\cdot)$ the free evolution of the linear Schrödinger equation and defining the notion of admissible pair in the following way : An ordered pair (q, r) is called admissible if $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, 2 < q \le \infty$, we will use the following proposition:

Proposition 3.4. *There exists $\delta > 0$ with the following property. If $u_0 \in L^2(R^d)$ and $T \in (0, \infty]$ are such that $\|S(.)u_0\|_{L^\sigma([0,T],L^\sigma)} < \delta$, there exists a unique solution $u \in C([0,T], L^2(R^d)) \cap L^\sigma([0,T], L^\sigma(R^n))$ of (1.1). In addition, $u \in L^q([0,T], L^r(R^d))$ for every admissible pair (q, r); for $t \in [0,T]$. Finally, u depends continuously in $C([0,T], L^2(R^d)) \cap L^\sigma([0,T], L^\sigma(R^d))$ on $u_0 \in L^2(R^n)$. If $u_0 \in H^1(R^d)$, then $u \in C([0,T], H^1(R^d))$.*

See [4] for the proof.

We need also the following lemma (see [4]):

Lemma 3.3. *Let $T \in (0, \infty]$, let $\sigma = \frac{4}{d} + 2$, and let (q, r) be an admissible pair. Then, whenever $u \in L^\sigma([0,T], L^\sigma(\mathbb{R}^d))$, it follows that $F(u) = -i\int_0^t S(t-s)(|u|^{\frac{4}{d}}u + ia|u|^p u)ds \in C([0,T], H^{-1}(\mathbb{R}^d)) \cap L^q(0,T, L^r(\mathbb{R}^d))$. Furthermore, there exists K, independent of T, such that*

$$\|Fv - Fu\|_{L^q(]0,T[,L^r)} < K(\|u\|_{L^\sigma(]0,T[,L^\sigma(\mathbb{R}^d))}^{\frac{4}{d}} + \|v\|_{L^\sigma(]0,T[,L^\sigma(\mathbb{R}^d))}^{\frac{4}{d}})\|u-v\|_{L^\sigma(]0,T[,L^\sigma(\mathbb{R}^d))}$$

(3.3)

for every $u, v \in L^\sigma(]0,T[, L^\sigma(\mathbb{R}^d))$.

Proof of Proposition 3.3:

Let $u_0 \in L^2(\mathbb{R}^d)$. Observe that $\|S(.)u_0\|_{L^\sigma(0,T,L^\sigma)} \longrightarrow 0$ as $T \longrightarrow 0$. Thus for sufficiently small T, the hypotheses of Proposition 3.4 are satisfied. Applying iteratively this proposition, we can construct the maximal solution $u \in C([0,T^*), L^2(\mathbb{R}^d))) \cap L^\sigma([0,T^*), L^\sigma(\mathbb{R}^d))$ of (1.1). We proceed by contradiction, assuming that $T^* < \infty$, and $\|u\|_{L^\sigma(]0,T[,L^\sigma)} < \infty$. Let $t \in [0,T^*)$. For every $s \in [0,T^*-t)$ we have

$$S(s)u(t) = u(t+s) - F(u(t+\cdot))(s).$$

From (3.3), we thus obtain

$$\|S(.)u(t)\|_{L^\sigma([0,T^*-t),L^\sigma(\mathbb{R}^d))} \leq \|u\|_{L^\sigma(]t,T^*[,L^\sigma)} + K(\|u\|_{L^\sigma(]t,T^*[,L^\sigma)})^{\frac{4}{d}+1}$$

Therefore, for t fixed close enough to T^*, it follows that

$$\|S(.)u(t)\|_{L^\sigma([0,T^*-t),L^\sigma(\mathbb{R}^d))} \leq \delta.$$

Applying Proposition 3.4, we find that u can be extended after T^*, which contradicts the maximality.

Corollary 3.1. *For $p = \frac{4}{d}$, the solution of (1.1) is global.*

Proof Multiply equation (1.1) by $\overline{u}$, and take the imaginary part to obtain:

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2a\|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d+2}} = 0.$$

Hence $\forall t \in \mathbb{R}_+$

$$\|u\|_{L^{\frac{4}{d}+2}[0,t[L^{\frac{4}{d}+2}(R^d)} \leq \frac{1}{2a}\|u_0\|_{L^2}^2.$$

The global existence follows then directly from Propostion 3.3. Now to finish the proof of Theorem 1.2, we will prove the scattering:
Let $v(t) = e^{-it\Delta}u(t) := S(-t)u(t)$ then

$$v(t) = u_0 + i \int_0^t S(-s)(|u(s)|^{\frac{4}{d}}u(s) + ia|u(s)|^{\frac{4}{d}}u)ds.$$

Therefore for $0 < t < \tau$,

$$v(t) - v(\tau) = i \int_\tau^t S(-s)(|u(s)|^{\frac{4}{d}}u(s) + ia|u(s)|^{\frac{4}{d}}u)ds.$$

It follows from Strichartz's estimates (see the proof of Lemma 4.2) that:

$$\|v(t)-v(\tau)\|_{L^2} = \|i \int_\tau^t S(-s)(|u(s)|^{\frac{4}{d}}u(s)+ia|u(s)|^{\frac{4}{d}}u)ds\|_{L^2} \leq C \parallel u \parallel_{L^{\frac{4}{d}+2}([t,\tau]\times\mathbb{R}^d)}^{\frac{4}{d}+1}.$$

The right hand side goes to zero when $t, \tau \longrightarrow +\infty$, then scattering follows from the Cauchy criterion.
This completes the proof of Theorem 1.2.

4. Blow up solution.

In this section, we will prove the existence of the explosive solutions in the case $1 \leq p < 4/d$.

Theorem 4.1. *Let $1 \leq p < 4/d$. There exist a set of initial data Ω in H^1, such that for any $0 < a < a_0$ with $a_0 = a_0(p)$ small enough, the emanating solution $u(t)$ to (1.1) blows up in finite time in the log-log regime.*

The set of initial data Ω is the set described in [15] in order to initialize the log-log regime. It is open in H^1. Using the continuity with regard to the initial data and the parameters, we easily obtain the following corollary:

Corollary 4.1. *Let $1 \leq p < 4/d$ and $u_0 \in H^1$ be an initial data such that the corresponding solution $u(t)$ of (1.2) blows up in the loglog regime. There exist $\beta_0 > 0$ and $a_0 > 0$ such that if $v_0 = u_0 + h_0$, $\|h_0\|_{H^1} \leq \beta_0$ and $a \leq a_0$, the solution $v(t)$ for (1.1) with the initial data v_0 blows up in finite time.*

Assertion (3) of Theorem 1.1 now follows directly from this corollary together the results of [15] on the L^2-critical NLS equation and a scaling argument in order to drop the smallness condition on the damped coefficient $a > 0$.
Now to prove Theorem 4.1, we look for a solution of (1.1) such that for t close enough to blowup time, we shall have the following decomposition:

$$u(t,x) = \frac{1}{\lambda^{\frac{d}{2}}(t)}(Q_{b(t)} + \epsilon)(t, \frac{x - x(t)}{\lambda(t)})e^{i\gamma(t)}, \qquad (4.1)$$

for some geometrical parameters $(b(t), \lambda(t), x(t), \gamma(t)) \in (0,\infty) \times (0,\infty) \times \mathbb{R}^d \times \mathbb{R}$, here $\lambda(t) \sim \frac{1}{\|\nabla u(t)\|_{L^2}}$, and the profiles Q_b are suitable deformations of Q related to some extra degeneracy of the problem.

Note that we will abbreviated our proof because it is very very close to the case of linear damping ($p = 0$ see Darwich [6]). Actually, as noticed in [22], we only need to prove that in the log-log regime the L^2 norm does not grow, and the growth of the energy(resp the momentum) is below $\frac{1}{\lambda^2}$ (resp $\frac{1}{\lambda}$) . In this paper, we will prove that in the log-log regime, the growth of the energy and the momentum are bounded by:

$$E(u(t)) \lesssim log(\lambda(t))\lambda(t)^{-\frac{pd}{2}}, \ P(u(t)) \lesssim log(\lambda(t))\lambda(t)^{1-\frac{pd}{4}}.$$

Let us recall that a fonction u :$[0,T] \longmapsto H^1$ follows the log-log regime if the following uniform controls on the decomposition (4.1) hold on $[0,T]$:

- Control of $b(t)$

$$b(t) > 0, \ b(t) < 10b(0). \qquad (4.2)$$

- Control of λ:

$$\lambda(t) \leq e^{-e^{\frac{\pi}{100b(t)}}} \qquad (4.3)$$

and the monotonicity of λ:

$$\lambda(t_2) \leq \frac{3}{2}\lambda(t_1), \forall \ 0 \leq t_1 \leq t_2 \leq T. \qquad (4.4)$$

Let $k_0 \leq k_+$ be integers and $T^+ \in [0, T]$ such that

$$\frac{1}{2^{k_0}} \leq \lambda(0) \leq \frac{1}{2^{k_0 - 1}}, \frac{1}{2^{k_+}} \leq \lambda(T^+) \leq \frac{1}{2^{k_+ - 1}} \tag{4.5}$$

and for $k_0 \leq k \leq k_+$, let t_k be a time such that

$$\lambda(t_k) = \frac{1}{2^k}, \tag{4.6}$$

then we assume the control of the doubling time interval:

$$t_{k+1} - t_k \leq k\lambda^2(t_k). \tag{4.7}$$

• control of the excess of mass:

$$\int |\nabla \epsilon(t)|^2 + \int |\epsilon(t)|^2 \, e^{-|y|} \leq \Gamma_{b(t)}^{\frac{1}{4}}. \tag{4.8}$$

4.1. Control of the energy and the kinetic momentum in the log-log regime.
We recall the Strichartz estimates. An ordered pair (q, r) is called admissible if $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $2 < q \leq \infty$. We define the Strichartz norm of functions $u : [0, T] \times \mathbb{R}^d \longmapsto C$ by:

$$\|u\|_{S^0([0,T] \times \mathbb{R}^d)} = \sup_{(q,r) admissible} \|u\|_{L_t^q L_x^r([0,T] \times \mathbb{R}^d)} \tag{4.9}$$

and

$$\|u\|_{S^1([0,T] \times \mathbb{R}^d)} = \sup_{(q,r) admissible} \|\nabla u\|_{L_t^q L_x^r([0,T] \times \mathbb{R}^d)} \tag{4.10}$$

We will sometimes abbreviate $S^i([0, T] \times \mathbb{R}^2)$ with S_T^i or $S^i[0, T]$, $i = 1, 2$. Let us denote the Hölder dual exponent of q by q' so that $\frac{1}{q} + \frac{1}{q'} = 1$. The Strichartz estimates may be expressed as:

$$\|u\|_{S_T^0} \leq \|u_0\|_{L^2} + \|(i\partial_t + \Delta)u\|_{L_t^{q'} L_x^{r'}} \tag{4.11}$$

where (q, r) is any admissible pair. Now we will derive an estimate on the energy, to check that it remains small with respect to λ^{-2}:

Lemma 4.1. *Assuming that (4.2)-(4.8) hold, then the energy and kinetic momentum of the solution u to (1.1) are controlled on $[0, T]$ by:*

$$|E(u(t))| \leq C(log(\lambda(t))\lambda(t)^{-\frac{pd}{4}}), \tag{4.12}$$

$$|P(u(t))| \leq C(log(\lambda(t))\lambda(t)^{1-\frac{pd}{4}}). \tag{4.13}$$

To prove this lemma, we shall need the following one:

Lemma 4.2. *Let u be a solution of (1.1) emanating for u_0 in H^1. Then $u \in C([0, \Delta T], H^1)$ where $\Delta T = \|u_0\|_{L^2}^{\frac{d-4}{d}} \|u_0\|_{H^1}^{-2}$, and we have the following control*

$$\|u\|_{S^0[t,t+\Delta T]} \leq 2 \|u_0\|_{L^2} \; , \; \|u\|_{S^1[t,t+\Delta T]} \leq 2 \|u_0\|_{H^1(\mathbb{R}^d)} \, .$$

Proof of **Lemma 4.1:** According to (4.7) each interval $[t_k, t_{k+1}]$, can be divided into k intervals, $[\tau_k^j, \tau_k^{j+1}]$ such that the estimates of the previous lemma are true. From (2.6) and the Gagliardo-Nirenberg inequality, we obtain that:

$$\frac{d}{dt}E(u(t)) \lesssim \|u\|_{L^2}^{\frac{4}{d}+p-\frac{pd}{2}} \|\nabla u\|_{L^2}^{2+\frac{pd}{2}}$$

Using (2.5) this gives

$$\int_{\tau_k^j}^{\tau_{kj+1}} \frac{d}{dt}E(u(t))dt \leq C \int_{\tau_{kj}}^{\tau_k^{j+1}} \|\nabla u(t)\|_{L^2}^{2+\frac{pd}{2}},$$

then by Lemma 4.2, we obtain that:

$$\int_{\tau_k^j}^{\tau_{kj+1}} \frac{d}{dt}E(u(t))dt \leq C(\tau_{kj+1} - \tau_{kj})\lambda^{-2-\frac{pd}{2}}(\tau_{kj})$$

Note that $\tau_k^{j+1} - \tau_k^j \sim \lambda^2(\tau_k^j) \sim \lambda^2(t_k)$, then

$$\int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt}E(u(t))dt \leq C\lambda^{-\frac{pd}{2}}(t_k)$$

Summing from $j = 1$ to $J_k \leq CK$, we obtain that:

$$\sum_{j=1}^{J_k} \int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt}E(u(t))dt \leq Ck\lambda^{-\frac{pd}{2}}(t_k)$$

Now taking $T^+ = T$ and summing from K_0 to K^+, we obtain:

$$\int_0^{T^+} \frac{d}{dt}E(u(t))dt \leq CK^+\lambda^{-\frac{pd}{2}}(T^+) \lesssim Clog(\lambda(T))\lambda^{-\frac{pd}{2}}(T).$$

Note that $\log(\lambda(T))\lambda^{-\frac{pd}{2}}(T)$ is small with to respect $\frac{1}{\lambda^2}$ because $p < \frac{4}{d}$.

Now we prove **(4.13)**: From (2.7) we have :

$$|\frac{d}{dt}P(u(t))| \leq \int |\overline{u}||\nabla u||u|^p$$

By Gagliardo-Nirenberg inequality we have:

$$\|u\|_{L^{2p+2}}^{2p+2} \leq \|u\|_{L^2}^{2p+2-dp} \|\nabla u\|_{L^2}^{dp}$$

then

$$\frac{d}{dt}P(u(t)) \leq (\int |u|^{2(p+1)})^{\frac{1}{2}} \|\nabla u\|_{L^2} \leq C \|\nabla u\|_{L^2}^{1+\frac{pd}{2}}.$$

Then:

$$\int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt}P(u(t)) \leq C(\tau_k^{j+1} - \tau_k^j) \left\|\nabla u(\tau_k^j)\right\|_{L^2}^{1+\frac{pd}{2}} \leq C \|\nabla u(t_k)\|_{L^2}^{-1+\frac{pd}{2}}$$

Summing successively into j and k we obtain that:

$$\int_0^{T^+} \frac{d}{dt}P(u(t)) \lesssim \log(\lambda(T^+))\lambda^{1-\frac{pd}{2}}(T^+).$$

Remark that this quantity is small with to respect $\frac{1}{\lambda}$ because $p < \frac{4}{d}$. $\qquad\square$

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