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Huyuan Chen, Laurent Veron

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Huyuan Chen[1] **Laurent Véron**[2]
Laboratoire de Mathématiques et Physique Théorique
CNRS UMR 7350
Université François Rabelais, Tours, France

Abstract

We study the existence of weak solutions to (E) $(-\Delta)^\alpha u+g(u)=\nu$ in a bounded regular domain Ω in $\mathbb{R}^N (N\geq 2)$ which vanish in $\mathbb{R}^N\setminus\Omega$, where $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha\in(0,1)$, ν is a Radon measure and g is a nondecreasing function satisfying some extra hypotheses. When g satisfies a subcritical integrability condition, we prove the existence and uniqueness of a weak solution for problem (E) for any measure. In the case where ν is Dirac measure, we characterize the asymptotic behavior of the solution. When $g(r)=|r|^{k-1}r$ with k supercritical, we show that a condition of absolute continuity of the measure with respect to some Bessel capacity is a necessary and sufficient condition in order (E) to be solved.

Contents

Key words: Fractional Laplacian, Radon measure, Dirac measure, Green kernel, Bessel capacities.

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[1]chenhuyuan@yeah.net
[2]Laurent.Veron@lmpt.univ-tours.fr

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded C^2 domain and $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. We are concerned with the existence of weak solutions to the semilinear fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + g(u) = \nu \quad & \text{in} \quad \Omega, \\ u = 0 \quad & \text{in} \quad \Omega^c, \end{aligned} \tag{1.1}$$

where $\alpha \in (0, 1)$, ν is a Radon measure such that $\int_\Omega \rho^\beta d|\nu| < \infty$ for some $\beta \in [0, \alpha]$ and $\rho(x) = dist(x, \Omega^c)$. The fractional Laplacian $(-\Delta)^\alpha$ is defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \to 0^+} (-\Delta)^\alpha_\epsilon u(x),$$

where for $\epsilon > 0$,

$$(-\Delta)^\alpha_\epsilon u(x) = -\int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz \tag{1.2}$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if} \quad \mathrm{t} \in [0, \epsilon], \\ 1, & \text{if} \quad \mathrm{t} > \epsilon. \end{cases}$$

When $\alpha = 1$, the semilinear elliptic problem

$$\begin{aligned} -\Delta u + g(u) = \nu \quad & \text{in} \quad \Omega, \\ u = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned} \tag{1.3}$$

has been extensively studied by numerous authors in the last 30 years. A fundamental contribution is due to Brezis [7], Benilan and Brezis [2], where ν is a bounded measure in Ω and the function $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, positive on $(0, +\infty)$ and satisfies the subcritical assumption:

$$\int_1^{+\infty} (g(s) - g(-s)) s^{-2\frac{N-1}{N-2}} ds < +\infty.$$

They proved the existence and uniqueness of the solution for problem (1.3). Baras and Pierre [1] studied (1.3) when $g(u) = |u|^{p-1}u$ for $p > 1$ and ν is absolutely continuous with respect to the Bessel capacity $C_{2,\frac{p}{p-1}}$, to obtain a solution. In [37] Véron extended Benilan and Brezis results in replacing the Laplacian by a general uniformly elliptic second order differential operator with Lipschitz continuous coefficients; he obtained existence and uniqueness results for solutions, when $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 1]$ where $\mathfrak{M}(\Omega, \rho^\beta)$ denotes the space of Radon measures in Ω satisfying

$$\int_\Omega \rho^\beta d|\nu| < +\infty, \tag{1.4}$$

$\mathfrak{M}(\Omega, \rho^0) = \mathfrak{M}^b(\Omega)$ is the set of bounded Radon measures and g is nondecreasing and satisfies the β-subcritical assumption:

$$\int_1^{+\infty} (g(s) - g(-s))s^{-2\frac{N+\beta-1}{N+\beta-2}} ds < +\infty.$$

The study of general semilinear elliptic equations with measure data have been investigated, such as the equations involving measures boundary data which was initiated by Gmira and Véron [20] who adapted the method introduced by Benilan and Brezis to obtain the existence and uniqueness of solution. This subject has been vastly expanded in recent years, see the papers of Marcus and Véron [25, 26, 27, 28], Bidaut-Véron and Vivier [5], Bidaut-Véron, Hung and Véron [4].

Recently, great attention has been devoted to non-linear equations involving fractional Laplacian or more general integro-differential operators and we mention the reference [8, 9, 10, 14, 15, 24, 30, 32]. In particular, the authors in [23] used the duality approach to study the equations of

$$(-\Delta)^\alpha v = \mu \quad \text{in} \quad \mathbb{R}^N,$$

where μ is a Radon measure with compact support. In [14] the authors obtained the existence of large solutions to equation

$$(-\Delta)^\alpha u + g(u) = f \quad \text{in} \quad \Omega, \tag{1.5}$$

where Ω is a bounded regular domain. In [13] we considered the properties of possibly singular solutions of (1.5) in punctured domain . It is a well-known fact [36] that for $\alpha = 1$ the weak singular solutions of (1.5) in punctured domain are classified according the type of singularities they admits: either weak singularities with Dirac mass, or strong singularities which are the upper limit of solutions with weak singularities. One of our interests is to extend these properties to any $\alpha \in (0, 1)$ and furthermore to consider general Radon measures.

In this paper we study the existence and uniqueness of solutions of (1.1) in a measure framework. Before stating our main theorem we make precise the notion of weak solution used in this article.

Definition 1.1 *We say that u is a weak solution of (1.1), if $u \in L^1(\Omega)$, $g(u) \in L^1(\Omega, \rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + g(u)\xi] dx = \int_\Omega \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha, \tag{1.6}$$

where $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ is the space of functions ξ satisfying:
(i) $\mathrm{supp}(\xi) \subset \bar{\Omega}$,
(ii) $(-\Delta)^\alpha \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^\alpha \xi(x)| \le C$ for some $C > 0$,
(iii) there exist $\varphi \in L^1(\Omega, \rho^\alpha dx)$ and $\epsilon_0 > 0$ such that $|(-\Delta)^\alpha_\epsilon \xi| \le \varphi$ a.e. in Ω, for all $\epsilon \in (0, \epsilon_0]$.

We notice that for $\alpha = 1$, the test space $\mathbb{X}_\alpha$ is used as $C_0^{1,L}(\Omega)$, which has similar properties like (i) and (ii). The counter part for the Laplacian of assumption (iii) would be that the difference quotient $\nabla_{x_j,h}[u](.) := h^{-1}[\partial_{x_j}u(.+h\mathbf{e}_j)-\partial_{x_j}u(.)]$ is bounded by an L^1-function, which is true since

$$\nabla_{x_j,h}[u](x) = h^{-1}\int_0^h \partial^2_{x_j,x_j}u(x+s\mathbf{e}_j)ds.$$

We denote by G the Green kernel of $(-\Delta)^\alpha$ in Ω and by $\mathbb{G}[.]$ the Green operator defined by

$$\mathbb{G}[f](x) = \int_\Omega G(x,y)f(y)dy, \qquad \forall f\in L^1(\Omega,\rho^\alpha dx). \tag{1.7}$$

For $N\ge 2$, $0<\alpha<1$ and $\beta\in[0,\alpha]$, we define the critical exponent

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if} \quad \beta\in[0,\frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if} \quad \beta\in(\frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha,\alpha]. \end{cases} \tag{1.8}$$

Our main result is the following:

Theorem 1.1 *Assume $\Omega\subset\mathbb{R}^N$ $(N\ge 2)$ is an open bounded C^2 domain, $\alpha\in(0,1)$, $\beta\in[0,\alpha]$ and $k_{\alpha,\beta}$ is defined by (1.8). Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous, nondecreasing function, satisfying*

$$g(r)r\ge 0, \quad \forall r\in\mathbb{R} \quad \text{and} \quad \int_1^\infty (g(s)-g(-s))s^{-1-k_{\alpha,\beta}}ds<\infty. \tag{1.9}$$

Then for any $\nu\in\mathfrak{M}(\Omega,\rho^\beta)$ problem (1.1) admits a unique weak solution u. Furthermore, the mapping: $\nu\mapsto u$ is increasing and

$$-\mathbb{G}(\nu_-)\le u\le \mathbb{G}(\nu_+) \quad \text{a.e. in } \Omega \tag{1.10}$$

where ν_+ and ν_- are respectively the positive and negative part in the Jordan decomposition of ν.

We note that for $\alpha=1$ and $\beta\in[0,1)$, we have

$$k_{1,\beta} > \frac{N+\beta}{N-2+\beta}, \tag{1.11}$$

where $k_{1,\beta}$ is given in (1.8) and the number in right hand side of (1.11) is from Theorem 3.7 in [37]. Inspired by [20, 37], the existence of solution could be extended in assuming that $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies *the (N,α,β)-weak-singularity assumption*, that is, there exists $r_0>0$ such that

$$g(x,r)r\ge 0, \quad \forall (x,r)\in\Omega\times(\mathbb{R}\setminus(-r_0,r_0)),$$

and

$$|g(x,r)| \leq \tilde{g}(|r|), \quad \forall (x,r) \in \Omega \times \mathbb{R},$$

where $\tilde{g} : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and satisfies

$$\int_1^\infty \tilde{g}(s)s^{-1-k_{\alpha,\beta}}ds < \infty.$$

We also give a stability result which shows that problem (1.1) is weakly closed in the space of measures $\mathfrak{M}(\Omega,\rho^\beta)$. In the last section we characterize the behaviour of the solution u of (1.1) when $\nu = \delta_a$ for some $a \in \Omega$. We also study the case where $g(r) = |r|^{k-1}r$ when $k \geq k_{\alpha,\beta}$, which doesn't satisfy (1.9). *We show that a necessary and sufficient condition in order a weak solution to problem*

$$\begin{aligned} (-\Delta)^\alpha u + |u|^{k-1}u = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{1.12}$$

to exist where ν is a positive bounded measure is that ν vanishes on compact subsets K of Ω with zero $C_{2\alpha,k'}$ Bessel-capacity.

The paper is organized as follows. In Section 2 we give some properties of Marcinkiewicz spaces and obtain the optimal index k for which there holds

$$\|\mathbb{G}(\nu)\|_{M^k(\Omega,\rho^\gamma dx)} \leq C\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}. \tag{1.13}$$

We also gives some integration by parts formulas and prove a Kato's type inequalities. In Section 3, we prove Theorem 1.1. It Section 4 we give applications the cases where the measure is a Dirac mass and where the nonlinearity is a power function.

2 Linear estimates

2.1 The Marcinkiewicz spaces

We recall the definition and basic properties of the Marcinkiewicz spaces.

Definition 2.1 *Let $\Omega \subset \mathbb{R}^N$ be an open domain and μ be a positive Borel measure in Ω. For $\kappa > 1$, $\kappa' = \kappa/(\kappa-1)$ and $u \in L^1_{loc}(\Omega, d\mu)$, we set*

$$\|u\|_{M^\kappa(\Omega,d\mu)} = \inf\{c \in [0,\infty] : \int_E |u|d\mu \leq c\left(\int_E d\mu\right)^{\frac{1}{\kappa'}}, \ \forall E \subset \Omega \text{ Borel set}\} \tag{2.1}$$

and

$$M^\kappa(\Omega,d\mu) = \{u \in L^1_{loc}(\Omega,d\mu) : \|u\|_{M^\kappa(\Omega,d\mu)} < \infty\}. \tag{2.2}$$

$M^\kappa(\Omega,d\mu)$ is called the Marcinkiewicz space of exponent κ or weak L^κ space and $\|.\|_{M^\kappa(\Omega,d\mu)}$ is a quasi-norm. The following property holds.

Proposition 2.1 *[3, 16] Assume* $1 \leq q < \kappa < \infty$ *and* $u \in L^1_{loc}(\Omega, d\mu)$. *Then there exists* $C(q,\kappa) > 0$ *such that*

$$\int_E |u|^q d\mu \leq C(q,\kappa)\|u\|_{M^\kappa(\Omega,d\mu)} \left(\int_E d\mu\right)^{1-q/\kappa},$$

for any Borel set E *of* Ω.

For $\alpha \in (0,1)$ and $\beta, \gamma \in [0,\alpha]$ we set

$$k_1(t) = \frac{\gamma}{\alpha} + \frac{N-(N-2\alpha)\frac{\gamma}{\alpha}}{N-2\alpha+t}, \quad k_2(t) = \gamma + \frac{N-(N-2\alpha)\frac{\gamma}{\alpha}}{N-2\alpha+t}t \tag{2.3}$$

and

$$t_{\alpha,\beta,\gamma} = \min\{t \in [0,\alpha] : \ \frac{k_2(t)}{k_1(t)} \geq \beta\}. \tag{2.4}$$

Remark 2.1 *The quantity* $t_{\alpha,\beta,\gamma}$ *is well defined, since*

$$\frac{k_2(\alpha)}{k_1(\alpha)} = \frac{\gamma + \alpha\frac{N-(N-2\alpha)\frac{\gamma}{\alpha}}{N-\alpha}}{\frac{\gamma}{\alpha} + \frac{N-(N-2\alpha)\frac{\gamma}{\alpha}}{N-\alpha}} = \alpha \geq \beta.$$

Remark 2.2 *The function* $t \mapsto k_1(t)$ *is decreasing in* $[0,\alpha]$ *with the following bounds*

$$k_1(0) = \frac{N}{N-2\alpha} \quad \textit{and} \quad k_1(\alpha) = \frac{N+\gamma}{N-\alpha} > 1.$$

Remark 2.3 *The function* $t \mapsto \frac{k_2(t)}{k_1(t)}$ *is increasing in* $[0,\alpha]$, *since*

$$\left(\frac{k_2(t)}{k_1(t)}\right)' = \frac{[N-(N-2\alpha)\frac{\gamma}{\alpha}](N+\gamma)}{k_1^2(t)} > 0.$$

As a consequence (2.4) is equivalent to

$$t_{\alpha,\beta,\gamma} = \max\{0, t_\beta\}, \tag{2.5}$$

where

$$t_\beta = \frac{\beta N - (N-2\alpha)\gamma}{N-(N-3\alpha+\beta)\frac{\gamma}{\alpha}}. \tag{2.6}$$

is the solution of $\frac{k_2(t)}{k_1(t)} = \beta$.

Proposition 2.2 *Let* $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ *be an open bounded* C^2 *domain and* $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ *with* $\beta \in [0,\alpha]$. *Then*

$$\|\mathbb{G}[\nu]\|_{M^{k_{\alpha,\beta,\gamma}}(\Omega,\rho^\gamma dx)} \leq C\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \tag{2.7}$$

where $\gamma \in [0,\alpha]$, $\mathbb{G}[\nu](x) = \int_\Omega G(x,y)d\nu(y)$ *where* G *is Green's kernel of* $(-\Delta)^\alpha$ *and*

$$k_{\alpha,\beta,\gamma} = \begin{cases} \frac{N+\gamma}{N-2\alpha+\beta}, & \text{if } \gamma \leq \frac{\mathrm{N}\beta}{\mathrm{N}-2\alpha}, \\ \frac{N}{N-2\alpha}, & \text{if not.} \end{cases} \tag{2.8}$$

Proof. For $\lambda > 0$ and $y \in \Omega$, we denote

$$A_\lambda(y) = \{x \in \Omega \setminus \{y\} : G(x,y) > \lambda\} \quad \text{and} \quad m_\lambda(y) = \int_{A_\lambda(y)} \rho^\gamma(x)dx.$$

From [11], there exists $C > 0$ such that for any $(x,y) \in \Omega \times \Omega$, $x \neq y$,

$$G(x,y) \leq C \min\left\{\frac{1}{|x-y|^{N-2\alpha}}, \frac{\rho^\alpha(x)}{|x-y|^{N-\alpha}}, \frac{\rho^\alpha(y)}{|x-y|^{N-\alpha}}\right\} \tag{2.9}$$

and

$$G(x,y) \leq C\frac{\rho^\alpha(y)}{\rho^\alpha(x)|x-y|^{N-2\alpha}}. \tag{2.10}$$

Therefore, if $\gamma \in [0,\alpha]$ and $x \in A_\lambda(y)$, there holds

$$\rho^\gamma(x) \leq \frac{C\rho^\gamma(y)}{\lambda^{\frac{\gamma}{\alpha}}|x-y|^{(N-2\alpha)\frac{\gamma}{\alpha}}}. \tag{2.11}$$

Let $t \in [0,\alpha]$ be such that $\frac{k_2(t)}{k_1(t)} \geq \beta$, where $k_1(t)$ and $k_2(t)$ are given in (2.3), then

$$G(x,y) \leq \left(\frac{C}{|x-y|^{N-2\alpha}}\right)^{1-\frac{t}{\alpha}} \left(\frac{C\rho^\alpha(y)}{|x-y|^{N-\alpha}}\right)^{\frac{t}{\alpha}} = \frac{C\rho^t(y)}{|x-y|^{N-2\alpha+t}}.$$

We observe that

$$A_\lambda(y) \subset \left\{x \in \Omega \setminus \{y\} : \frac{C\rho(y)^t}{|x-y|^{N-2\alpha+t}} > \lambda\right\} \subset D_\lambda(y)$$

where $D_\lambda(y) := \left\{x \in \Omega : |x-y| < (\frac{C\rho^t(y)}{\lambda})^{\frac{1}{N-2\alpha+t}}\right\}$; together with (2.11), this implies

$$m_\lambda(y) \leq \int_{D_\lambda(y)} \frac{C\rho^\gamma(y)}{\lambda^{\frac{\gamma}{\alpha}}|x-y|^{(N-2\alpha)\frac{\gamma}{\alpha}}}dx \leq C\rho(y)^{k_2(t)}\lambda^{-k_1(t)}.$$

For any Borel set E of Ω, we have

$$\int_E G(x,y)\rho^\gamma(x)dx \leq \int_{A_\lambda(y)} G(x,y)\rho^\gamma(x)dx + \lambda\int_E \rho^\gamma(x)dx$$

and

$$\begin{aligned}\int_{A_\lambda(y)} G(x,y)\rho^\gamma(x)dx &= -\int_\lambda^\infty s dm_s(y) \\ &= \lambda m_\lambda(y) + \int_\lambda^\infty m_s(y)ds \\ &\leq C\rho(y)^{k_2(t)}\lambda^{1-k_1(t)}.\end{aligned}$$

Thus,

$$\int_E G(x,y)\rho^\gamma(x)dx \le C\rho(y)^{k_2(t)}\lambda^{1-k_1(t)} + \lambda\int_E \rho^\gamma(x)dx.$$

By choosing $\lambda = [\rho(y)^{-k_2(t)}\int_E \rho^\gamma(x)dx]^{-\frac{1}{k_1(t)}}$, we have

$$\int_E G(x,y)\rho^\gamma(x)dx \le C\rho(y)^{\frac{k_2(t)}{k_1(t)}}(\int_E \rho^\gamma(x)dx)^{\frac{k_1(t)-1}{k_1(t)}}.$$

Therefore,

$$\begin{aligned}
\int_E \mathbb{G}(|\nu|)(x)\rho^\gamma(x)dx &= \int_\Omega\int_E G(x,y)\rho^\gamma(x)dxd|\nu(y)| \\
&\le C\int_\Omega \rho(y)^{\frac{k_2(t)}{k_1(t)}}d|\nu(y)|\left(\int_E \rho^\gamma(x)dx\right)^{\frac{k_1(t)-1}{k_1(t)}} \\
&\le C\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}\left(\int_E \rho^\gamma(x)dx\right)^{\frac{k_1(t)-1}{k_1(t)}},
\end{aligned}$$

since by our choice of t, $\frac{k_2(t)}{k_1(t)} \ge \beta$, which guarantees that

$$\int_\Omega \rho(y)^{\frac{k_2(t)}{k_1(t)}}d|\nu(y)| \le \max_\Omega \rho^{\frac{k_2(t)}{k_1(t)}-\beta}\int_\Omega \rho(y)^\beta d|\nu(y)|.$$

As a consequence,

$$\|\mathbb{G}(\nu)\|_{M^{k_1(t)}(\Omega,\rho^\gamma dx)} \le C\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}.$$

Therefore,

$$k_{\alpha,\beta,\gamma} := \max\{k_1(t) : t\in[0,\alpha]\} = k_1(t_{\alpha,\beta,\gamma}),$$

where $t_{\alpha,\beta,\gamma}$ is defined by (2.4) and $k_{\alpha,\beta,\gamma}$ is given by (2.8). We complete the proof. $\square$

We choose the parameter γ in order to make $k_{\alpha,\beta,\gamma}$ the largest possible, and denote

$$k_{\alpha,\beta} = \max_{\gamma\in[0,\alpha]} k_{\alpha,\beta,\gamma}. \tag{2.12}$$

Since $\gamma \mapsto k_{\alpha,\beta,\gamma}$ is increasing, the following statement holds.

Proposition 2.3 *Let $N \ge 2$ and $k_{\alpha,\beta}$ be defined by (2.12), then*

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if} \quad \beta\in[0, \frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if} \quad \beta\in(\frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha, \alpha]. \end{cases} \tag{2.13}$$

2.2 Non-homogeneous problem

In this subsection, we study some properties of the solution of the linear non-homogeneous, which will play a key role in the sequel. We assume that $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is an open bounded domain with a C^2 boundary.

Lemma 2.1 (i) *There exists $C > 0$ such that for any $\xi \in \mathbb{X}_\alpha$ there holds*

$$\|\xi\|_{C^\alpha(\bar{\Omega})} \leq C\|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)} \tag{2.14}$$

and

$$\|\rho^{-\alpha}\xi\|_{C^\theta(\bar{\Omega})} \leq C\|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)}. \tag{2.15}$$

where $0 < \theta < \min\{\alpha, 1-\alpha\}$. In particular, for $x \in \Omega$

$$|\xi(x)| \leq C\|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)}\rho^\alpha(x). \tag{2.16}$$

(ii) *Let u be the solution of*

$$\begin{aligned} (-\Delta)^\alpha u = f \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{2.17}$$

where $f \in C^\gamma(\bar{\Omega})$ for $\gamma > 0$. Then $u \in \mathbb{X}_\alpha$.

Proof. (i). Estimates (2.14) and (2.15) are consequences of [30, Prop 1.1] and [30, Th 1.2] respectively. Furthermore, if η_1 is the solution of (2.17) with $f \equiv 1$ in Ω, then $\eta_1 > 0$ in Ω and by follows [30, Th 1.2], there exists $C > 0$ such that

$$C^{-1} \leq \frac{\eta_1}{\rho^\alpha} \leq C \quad \text{in} \quad \Omega. \tag{2.18}$$

In this expression the right-side follows [30, Th 1.2] and the left-hand side inequality follows from the maximum principle and [11, Th 1.2]. Since

$$-\|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)} \leq (-\Delta)^\alpha \xi \leq \|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)} \quad \text{in } \Omega,$$

it follows by the comparison principle,

$$-\|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)}\eta_1(x) \leq \xi(x) \leq \|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)}\eta_1(x).$$

which, together with (2.18), implies (2.16).
(ii) For $r > 0$, we denote

$$\Omega_r = \{z \in \Omega : \ dist(z, \partial\Omega) > r\}.$$

Since $f \in C^\gamma(\bar{\Omega})$, then by Corollary 1.6 part (i) and Proposition 1.1 in [30], for $\theta \in [0, \min\{\alpha, 1-\alpha, \gamma\})$, there exists $C > 0$ such that for any $r > 0$, we have

$$\|u\|_{C^{2\alpha+\theta}(\Omega_r)} \leq Cr^{-\alpha-\theta}$$

and

$$\|u\|_{C^\alpha(\mathbb{R}^N)} \le C.$$

Then for $x \in \Omega$, letting $r = \rho(x)/2$,

$$|\delta(u, x, y)| \le Cr^{-\alpha-\theta}|y|^{2\alpha+\theta}, \quad \forall y \in B_r(0) \tag{2.19}$$

and

$$|\delta(u, x, y)| \le C|y|^\alpha, \quad \forall y \in \mathbb{R}^N,$$

where $\delta(u, x, y) = u(x + y) + u(x - y) - 2u(x)$. Thus,

$$\begin{aligned} |(-\Delta)^\alpha_\epsilon u(x)| &\le \frac{1}{2}\int_{\mathbb{R}^N} \frac{|\delta(u, x, y)|}{|y|^{N+2\alpha}}\chi_\epsilon(|y|)dy \\ &\le \frac{1}{2}\int_{B_r(0)} \frac{|\delta(u, x, y)|}{|y|^{N+2\alpha}}dy + \frac{1}{2}\int_{B_r^c(0)} \frac{|\delta(u, x, y)|}{|y|^{N+2\alpha}}dy \\ &\le \frac{Cr^{-\alpha-\theta}}{2}\int_{B_r(0)} \frac{1}{|y|^{N-\theta}}dy + \frac{C}{2}\int_{B_r^c(0)} \frac{1}{|y|^{N+\alpha}}dy \\ &\le C\rho(x)^{-\alpha}, \quad x \in \Omega, \end{aligned}$$

for some $C > 0$ independent of ϵ. Moreover, $\rho^{-\alpha}$ is in $L^1(\Omega, \rho^\alpha dx)$. Finally, we prove $(-\Delta)^\alpha_\epsilon u \to (-\Delta)^\alpha u$ as $\epsilon \to 0^+$ pointwise. For $x \in \Omega$, choosing $\epsilon \in (0, \rho(x)/2)$, then by (2.19),

$$\begin{aligned} |(-\Delta)^\alpha u(x) - (-\Delta)^\alpha_\epsilon u(x)| &\le \frac{1}{2}\int_{B_\epsilon(0)} \frac{|\delta(u, x, y)|}{|y|^{N+2\alpha}}dy \\ &\le C\rho(x)^{-\alpha-\theta}\epsilon^\theta \\ &\to 0, \quad \epsilon \to 0^+. \end{aligned}$$

The proof is complete. □

The following Proposition is the Kato's type estimate for proving the uniqueness of the solution of (1.1).

Proposition 2.4 *If $\nu \in L^1(\Omega, \rho^\alpha dx)$, there exists a unique weak solution u of the problem*

$$\begin{aligned} (-\Delta)^\alpha u = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c. \end{aligned} \tag{2.20}$$

For any $\xi \in \mathbb{X}_\alpha$, $\xi \ge 0$, we have

$$\int_\Omega |u|(-\Delta)^\alpha \xi dx \le \int_\Omega \xi \text{sign}(u)\nu dx \tag{2.21}$$

and

$$\int_\Omega u_+(-\Delta)^\alpha \xi dx \le \int_\Omega \xi \text{sign}_+(u)\nu dx, \tag{2.22}$$

We note here that for $\alpha = 1$, the proof of Proposition 2.4 could be seen in [37, Th 2.4]. For $\alpha \in (0,1)$, we first prove some integration by parts formula.

Lemma 2.2 *Assume $u, \xi \in \mathbb{X}_\alpha$, then*

$$\int_\Omega u(-\Delta)^\alpha \xi dx = \int_\Omega \xi(-\Delta)^\alpha u dx. \tag{2.23}$$

Proof. Denote

$$(-\Delta)^\alpha_{\Omega,\epsilon} u(x) = -\int_\Omega \frac{u(z)-u(x)}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|)dz. \tag{2.24}$$

By the definition of $(-\Delta)^\alpha_\epsilon$, we have

$$\begin{aligned}
(-\Delta)^\alpha_\epsilon u(x) &= -\int_\Omega \frac{u(z)-u(x)}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|)dz + u(x)\int_{\Omega^c} \frac{\chi_\epsilon(|x-z|)}{|z-x|^{N+2\alpha}} dz \\
&= (-\Delta)^\alpha_{\Omega,\epsilon} u(x) + u(x)\int_{\Omega^c} \frac{\chi_\epsilon(|x-z|)}{|z-x|^{N+2\alpha}} dz.
\end{aligned}$$

We claim that

$$\int_\Omega \xi(x)(-\Delta)^\alpha_{\Omega,\epsilon} u(x)dx = \int_\Omega u(x)(-\Delta)^\alpha_{\Omega,\epsilon}\xi(x)dx, \quad \text{for } u, \xi \in \mathbb{X}_\alpha. \tag{2.25}$$

By using the fact of

$$\int_\Omega\int_\Omega \frac{[u(z)-u(x)]\xi(x)}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|)dzdx = \int_\Omega\int_\Omega \frac{[u(x)-u(z)]\xi(z)}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|)dzdx,$$

we have

$$\begin{aligned}
&\int_\Omega \xi(x)(-\Delta)^\alpha_{\Omega,\epsilon} u(x)dx \\
&= -\frac{1}{2}\int_\Omega\int_\Omega [\frac{(u(z)-u(x))\xi(x)}{|z-x|^{N+2\alpha}} + \frac{(u(x)-u(z))\xi(z)}{|z-x|^{N+2\alpha}}]\chi_\epsilon(|x-z|)dzdx \\
&= \frac{1}{2}\int_\Omega\int_\Omega \frac{[u(z)-u(x)][\xi(z)-\xi(x)]}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|)dzdx.
\end{aligned}$$

Similarly, by the fact that $u \in \mathbb{X}_\alpha$,

$$\int_\Omega u(x)(-\Delta)^\alpha_{\Omega,\epsilon}\xi(x)dx = \frac{1}{2}\int_\Omega\int_\Omega \frac{[u(z)-u(x)][\xi(z)-\xi(x)]}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|)dzdx.$$

Then (2.25) holds. In order to prove (2.23), we first notice that by (2.25),

$$\begin{aligned}
&\int_\Omega \xi(x)(-\Delta)^\alpha_\epsilon u(x)dx \\
&= \int_\Omega \xi(x)(-\Delta)^\alpha_{\Omega,\epsilon} u(x)dx + \int_\Omega u(x)\xi(x)\int_{\Omega^c} \frac{\chi_\epsilon(|x-z|)}{|z-x|^{N+2\alpha}} dzdx \\
&= \int_\Omega u(x)(-\Delta)^\alpha_{\Omega,\epsilon}\xi(x)dx + \int_\Omega u(x)\xi(x)\int_{\Omega^c} \frac{\chi_\epsilon(|x-z|)}{|z-x|^{N+2\alpha}} dzdx \\
&= \int_\Omega u(x)(-\Delta)^\alpha_\epsilon \xi(x)dx.
\end{aligned} \tag{2.26}$$

Since u and ξ belongs to $\mathbb{X}_\alpha$, $(-\Delta)^\alpha_\epsilon \xi \to (-\Delta)^\alpha \xi$ and $(-\Delta)^\alpha_\epsilon u \to (-\Delta)^\alpha u$ and $|u(-\Delta)^\alpha_\epsilon \xi| + |\xi(-\Delta)^\alpha_\epsilon u| \le C\varphi$ for some $C > 0$ and $\varphi \in L^1(\Omega, \rho^\alpha dx)$. It follows by the Dominated Convergence Theorem

$$\lim_{\epsilon \to 0^+} \int_\Omega \xi(x)(-\Delta)^\alpha_\epsilon u(x)dx = \int_\Omega \xi(x)(-\Delta)^\alpha u(x)dx$$

and

$$\lim_{\epsilon \to 0^+} \int_\Omega (-\Delta)^\alpha_\epsilon \xi(x)u(x)dx = \int_\Omega (-\Delta)^\alpha \xi(x)u(x)dx.$$

Letting $\epsilon \to 0^+$ of (2.26) we conclude that (2.23) holds. □

For $1 \le p < \infty$ and $0 < s < 1$, $W^{s,p}(\Omega)$ is the set of $\xi \in L^p(\Omega)$ such that

$$\int_\Omega \int_\Omega \frac{|\xi(x) - \xi(y)|^p}{|x-y|^{N+sp}} dydx < \infty. \tag{2.27}$$

This space is endowed with the norm

$$\|\xi\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |\xi(x)|^p dx + \int_\Omega \int_\Omega \frac{|\xi(x) - \xi(y)|^p}{|x-y|^{N+sp}} dydx \right)^{\frac{1}{p}}. \tag{2.28}$$

Furthermore, if Ω is bounded, the following Poincaré inequality holds [35, p 134].

$$\left(\int_\Omega |\xi(x)|^p dx \right)^{\frac{1}{p}} \le C \left(\int_\Omega \int_\Omega \frac{|\xi(x) - \xi(y)|^p}{|x-y|^{N+sp}} dydx \right)^{\frac{1}{p}}, \quad \forall \xi \in C^\infty_c(\Omega). \tag{2.29}$$

Lemma 2.3 *Let $u \in \mathbb{X}_\alpha$ and γ be C^2 in the interval $u(\bar{\Omega})$ and satisfy $\gamma(0) = 0$, then $u \in W^{\alpha,2}(\Omega)$, $\gamma \circ u \in \mathbb{X}_\alpha$ and for all $x \in \Omega$, there exists $z_x \in \bar{\Omega}$ such that*

$$(-\Delta)^\alpha (\gamma \circ u)(x) = (\gamma' \circ u)(x)(-\Delta)^\alpha u(x) - \frac{\gamma'' \circ u(z_x)}{2} \int_\Omega \frac{(u(y) - u(x))^2}{|y-x|^{N+2\alpha}} dy. \tag{2.30}$$

Proof. Since $u \in C(\bar{\Omega})$ vanishes in Ω^c, $\gamma \circ u$ shares the same properties. By (2.14), for any x and y in Ω

$$(u(x) - u(y))^2 \le C|x-y|^{2\alpha} \|(-\Delta)^\alpha u\|^2_{L^\infty(\Omega)}.$$

Then $u \in W^{\alpha,2}(\Omega)$. Similarly $\gamma \circ u \in W^{\alpha,2}(\Omega)$. Furthermore

$$(\gamma \circ u)(y) - (\gamma \circ u)(x) = (\gamma' \circ u)(x) \left(u(y) - u(x)\right) + \int_{u(x)}^{u(y)} (u(y) - t)\gamma''(t)dt.$$

By the mean value theorem, there exists some $\tau \in [0,1]$ such that

$$\int_{u(x)}^{u(y)}(u(y)-t)\gamma''(t)dt = \frac{\gamma''(\tau u(y)+(1-\tau)u(x))}{2}(u(y)-u(x))^2.$$

Since γ'' is continuous and u is continuous in $\bar{\Omega}$,

$$\left|\int_{u(x)}^{u(y)}(u(y)-t)\gamma''(t)dt\right| \leq \frac{\|\gamma''\circ u\|_{L^\infty(\bar{\Omega})}}{2}(u(y)-u(x))^2$$

and by (2.14),

$$\left|\int_{|y-x|>\epsilon}\int_{u(x)}^{u(y)}(u(y)-t)\gamma''(t)dt\frac{dy}{|y-x|^{N+2\alpha}}\right| \\ \leq \frac{\|\gamma''\circ u\|_{L^\infty}}{2}\int_\Omega (u(y)-u(x))^2\frac{dy}{|y-x|^{N+2\alpha}}.$$

Notice also that $\tau u(y)+(1-\tau)u(x) \in u(\bar{\Omega}) := I$, therefore

$$\min_{t\in I}\gamma''(t) \leq \gamma''(\tau u(y)+(1-\tau)u(x)) \leq \max_{t\in I}\gamma''(t),$$

thus

$$\frac{\min_{t\in I}\gamma''(t)}{2}\int_\Omega \frac{(u(y)-u(x))^2}{|y-x|^{N+2\alpha}}dy \\ \leq \int_\Omega\int_{u(x)}^{u(y)}(u(y)-t)\gamma''(t)dt\frac{dy}{|y-x|^{N+2\alpha}} \\ \leq \frac{\max_{t\in I}\gamma''(t)}{2}\int_\Omega \frac{(u(y)-u(x))^2}{|y-x|^{N+2\alpha}}dy.$$

Since γ'' is continuous, there exists $t_0 \in I$ such that

$$\int_\Omega\int_{u(x)}^{u(y)}(u(y)-t)\gamma''(t)dt\frac{dy}{|y-x|^{N+2\alpha}} = \frac{\gamma''(t_0)}{2}\int_\Omega \frac{(u(y)-u(x))^2}{|y-x|^{N+2\alpha}}dy$$

and since u is continuous in $\mathbb{R}^N$ and vanishes in Ω^c, there exists $z_x \in \bar{\Omega}$ such that $t_0 = u(z_x)$, which ends the proof. $\square$

Proof of Proposition 2.4. *Uniqueness.* Let w be a weak solution of

$$\begin{aligned}(-\Delta)^\alpha w &= 0 \quad \text{in} \quad \Omega \\ w &= 0 \quad \text{in} \quad \Omega^c.\end{aligned} \tag{2.31}$$

If ω is a Borel subset of Ω and $\eta_{\omega,n}$ the solution of

$$\begin{aligned}(-\Delta)^\alpha \eta_{\omega,n} &= \zeta_n \quad \text{in} \quad \Omega \\ \eta_{\omega,n} &= 0 \quad \text{in} \quad \Omega^c,\end{aligned} \tag{2.32}$$

where $\zeta_n : \bar{\Omega} \mapsto [0,1]$ is a $C^1(\bar{\Omega})$ function such that

$$\zeta_n \to \chi_\omega \quad \text{in } L^\infty(\bar{\Omega}) \quad \text{as } n \to \infty.$$

Then by Lemma 2.1 part (ii), $\eta_{\omega,n} \in \mathbb{X}_\alpha$ and

$$\int_\Omega w\zeta_n dx = 0.$$

Then passing the limit of $n \to \infty$, we have

$$\int_\omega w dx = 0.$$

This implies $w = 0$.

Existence and estimate (2.21). For $\delta > 0$ we define an even convex function ϕ_δ by

$$\phi_\delta(t) = \begin{cases} |t| - \frac{\delta}{2}, & \text{if} \quad |\mathrm{t}| \geq \delta, \\ \frac{t^2}{2\delta}, & \text{if} \quad |\mathrm{t}| < \delta/2. \end{cases} \tag{2.33}$$

Then for any $t, s \in \mathbb{R}$, $|\phi_\delta'(t)| \leq 1$, $\phi_\delta(t) \to |t|$ and $\phi_\delta'(t) \to \mathrm{sign(t)}$ when $\delta \to 0^+$. Moreover

$$\phi_\delta(s) - \phi_\delta(t) \geq \phi_\delta'(t)(s - t). \tag{2.34}$$

Let $\{\nu_n\}$ be a sequence functions in $C^1(\bar{\Omega})$ such that

$$\lim_{n\to\infty} \int_\Omega |\nu_n - \nu|\rho^\alpha dx = 0.$$

Let u_n be the corresponding solution to (2.20) with right-hand side ν_n, then by Lemma 2.1, $u_n \in \mathbb{X}_\alpha$ and by Lemmas 2.2, 2.3, for any $\delta > 0$ and $\xi \in \mathbb{X}_\alpha$, $\xi \geq 0$,

$$\begin{aligned} \int_\Omega \phi_\delta(u_n)(-\Delta)^\alpha \xi dx &= \int_\Omega \xi(-\Delta)^\alpha \phi_\delta(u_n) dx \\ &\leq \int_\Omega \xi\phi_\delta'(u_n)(-\Delta)^\alpha u_n dx \\ &= \int_\Omega \xi\phi_\delta'(u_n)\nu_n dx. \end{aligned} \tag{2.35}$$

Letting $\delta \to 0$, we obtain

$$\int_\Omega |u_n|(-\Delta)^\alpha \xi dx \leq \int_\Omega \xi \mathrm{sign}(u_n)\nu_n dx \leq \int_\Omega \xi|\nu_n| dx. \tag{2.36}$$

If we take $\xi = \eta_1$, we derive from Lemma 2.1

$$\int_\Omega |u_n| dx \leq C \int_\Omega |\nu_n|\rho^\alpha dx. \tag{2.37}$$

Similarly

$$\int_\Omega |u_n - u_m| dx \leq C \int_\Omega |\nu_n - \nu_m| \rho^\alpha dx. \tag{2.38}$$

Therefore $\{u_n\}$ is a Cauchy sequence in L^1 and its limit u is a weak solution of (2.20). Letting $n \to \infty$ in (2.36) we obtain (2.21). Inequality (2.22) is proved by replacing ϕ_δ by $\tilde{\phi_\delta}$ which is zero on $(-\infty, 0]$ and ϕ_δ on $[0, \infty)$. $\square$

The next result is a higher order regularity result

Proposition 2.5 *Let the assumptions of Proposition 2.2 be fulfilled and* $0 \leq \beta \leq \alpha$. *Then for* $p \in (1, \frac{N}{N+\beta-2\alpha})$ *there exists* $c_p > 0$ *such that for any* $\nu \in L^1(\Omega, \rho^\beta dx)$

$$\|\mathbb{G}[\nu]\|_{W^{2\alpha-\gamma,p}(\Omega)} \leq c_p \|\nu\|_{L^1(\Omega,\rho^\beta dx)} \tag{2.39}$$

where $\gamma = \beta + \frac{N}{p'}$ *if* $\beta > 0$ *and* $\gamma > \frac{N}{p'}$ *if* $\beta = 0$.

Proof. We use Stampacchia's duality method [33] and put $u = \mathbb{G}[\nu]$. If $\psi \in C_c^\infty(\bar{\Omega})$, then

$$\begin{aligned}\left| \int_\Omega \psi(-\Delta)^\alpha u dx \right| &\leq \int_\Omega |\nu||\psi| dx \\ &\leq \sup_\Omega |\rho^{-\beta}\psi| \int_\Omega |\nu| \rho^\beta dx \\ &\leq \|\psi\|_{C^\beta(\bar{\Omega})} \|\nu\|_{L^1(\Omega,\rho^\beta dx)}.\end{aligned} \tag{2.40}$$

By Sobolev-Morrey embedding type theorem (see e.g. [29, Th 8.2]), for any $p \in (1, \frac{N}{N+\beta-2\alpha})$ and $p' = \frac{p}{p-1}$,

$$\|\psi\|_{C^\beta(\bar{\Omega})} \leq C \|\psi\|_{W^{\gamma,p'}(\Omega)}$$

with $\gamma = \beta + \frac{N}{p'}$ if $\beta > 0$ and $\gamma > \frac{N}{p'}$ if $\beta = 0$. Therefore,

$$\left| \int_\Omega \psi(-\Delta)^\alpha u dx \right| \leq C \|\psi\|_{W^{\gamma,p'}(\Omega)} \|\nu\|_{L^1(\Omega,\rho^\beta dx)}, \tag{2.41}$$

which implies that the mapping $\psi \mapsto \int_\Omega \psi(-\Delta)^\alpha u dx$ is continuous on $W^{\gamma,p'}(\Omega)$ and thus

$$\|(-\Delta)^\alpha u\|_{W^{-\gamma,p}(\Omega)} \leq C \|\nu\|_{L^1(\Omega,\rho^\beta dx)}. \tag{2.42}$$

Since $(-\Delta)^{-\alpha}$ is an isomorphism from $W^{-\gamma,p}(\Omega)$ into $W^{2\alpha-\gamma,p}(\Omega)$, it follows that

$$\|u\|_{W^{2\alpha-\gamma,p}(\Omega)} \leq C \|\nu\|_{L^1(\Omega,\rho^\beta dx)}. \tag{2.43}$$

$\square$

Proposition 2.6 *Under the assumptions of Proposition 2.5 the mapping* $\nu \mapsto \mathbb{G}[\nu]$ *is compact from* $L^1(\Omega, \rho^\beta dx)$ *into* $L^q(\Omega)$ *for any* $q \in [1, \frac{N}{N+\beta-2\alpha})$.

Proof. By [29, Th 6.5] the embedding of $W^{2\alpha-\gamma,p}(\Omega)$ into $L^q(\Omega)$ is compact, this ends the proof. $\square$

3 Proof of Theorem 1.1

Before proving the main we give a general existence result in $L^1(\Omega, \rho^\alpha dx)$.

Proposition 3.1 *Suppose that* Ω *is an open bounded* C^2 *domain of* $\mathbb{R}^N$ $(N \geq 2)$, $\alpha \in (0,1)$ *and the function* $g : \mathbb{R} \to \mathbb{R}$ *is continuous, nondecreasing and* $rg(r) \geq 0$ *for all* $r \in \mathbb{R}$. *Then for any* $f \in L^1(\Omega, \rho^\alpha dx)$ *there exists a unique weak solution* u *of (1.1) with* $\nu = f$. *Moreover the mapping* $f \mapsto u$ *is increasing.*

Proof. *Step 1: Variational solutions.* If $w \in L^2(\Omega)$, we denote by $\underline{w}$ its extension by 0 in Ω^c and by $W_c^{\alpha,2}(\Omega)$ the set of function in $L^2(\Omega)$ such that

$$\|w\|^2_{W_c^{\alpha,2}(\Omega)} := \int_{\mathbb{R}^N} |\hat{\underline{w}}|^2(1+|x|^\alpha)dx < \infty,$$

where $\hat{\underline{w}}$ is the Fourier transform of $\underline{w}$. For $\epsilon > 0$ we set

$$J(w) = \frac{1}{2}\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{\alpha}{2}}\underline{w}\right)^2 dx + \int_\Omega (j(w) + \epsilon w^2)dx,$$

with domain $D(J) = \{w \in W_c^{\alpha,2}(\mathbb{R}^N) \text{ s.t. } j(w) \in L^1(\Omega)\}$ and $j(s) = \int_0^s g(t)dt$. Furthermore since there holds $J(w) \geq \sigma\|w\|^2_{W_c^{\alpha,2}}$ for some $\sigma > 0$, the subdifferential ∂J of J is a maximal monotone in the sense of Browder-Minty (see [6] and the references therein) which satisfies $R(\partial J) = L^2(\Omega)$. Then for any $f \in L^2(\Omega)$ there exists a unique u_ϵ in the domain $D(\partial J)$ such that $\partial J(u_\epsilon) = f$. Since for any $\psi \in W_c^{\alpha,2}(\Omega)$

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}}\underline{w}\,(-\Delta)^{\frac{\alpha}{2}}\underline{\psi}dx = (4\pi)^\alpha \int_{\mathbb{R}^N} \hat{\underline{w}}\,\hat{\underline{\psi}}|x|^{2\alpha}dx = \int_\Omega \psi(-\Delta)^\alpha \underline{w}dx,$$

$$\partial J(u_\epsilon) = (-\Delta)^\alpha u_\epsilon + g(u_\epsilon) + 2\epsilon u = f,$$

with $u_\epsilon \in W_c^{2\alpha,2}(\Omega)$ such that $g(u_\epsilon) \in L^2(\Omega)$. This is also a consequence of [6, Cor 2.11]. If f is assumed to be bounded, then $u \in C^\alpha(\overline{\Omega})$ by [30, Prop 1.1]. Note that more delicate variational formulations can be found in [21], [22].

Step 2: L^1 *solutions.* For $n \in \mathbb{N}^*$ we denote by $u_{n,\epsilon}$ the solution of

$$\begin{array}{rll} (-\Delta)^\alpha u_{n,\epsilon} + g(u_{n,\epsilon}) + 2\epsilon u_{n,\epsilon} = f_n & \text{in } \Omega \\ u_{n,\epsilon} = 0 & \text{in } \Omega^c \end{array} \tag{3.1}$$

where $f_n = \text{sgn}(f)\min\{n, |f|\}$. By (2.36) with $\xi = \eta_1$,

$$\int_\Omega (|u_{n,\epsilon}| + (2\epsilon|u_{n,\epsilon}| + |g(u_{n,\epsilon})|)\eta_1)\,dx \leq \int_\Omega |f_n|\eta_1 dx \leq \int_\Omega |f|\eta_1 dx, \tag{3.2}$$

and, for $\epsilon' > 0$ and $m \in \mathbb{N}^*$,

$$\int_\Omega \left(|u_{n,\epsilon} - u_{m,\epsilon'}| + |g(u_{n,\epsilon}) - g(u_{m,\epsilon'})|\eta_1\right) dx \\ \leq \int_\Omega \left(|f_n - f_m| + 2\epsilon|u_{n,\epsilon}| + 2\epsilon'|u_{m,\epsilon'}|\right) \eta_1 dx. \tag{3.3}$$

Since $f_n \to f$ in $L^1(\Omega, \rho^\alpha dx)$, $\{u_{n,\epsilon}\}$ and $\{g \circ u_{n,\epsilon}\}$ are Cauchy filters in $L^1(\Omega)$ and $L^1(\Omega, \rho^\alpha dx)$ respectively. Set $u = \lim_{n\to\infty, \epsilon\to 0} u_{n,\epsilon}$, we derive from the following identity valid for any $\xi \in \mathbb{X}_\alpha$

$$\int_\Omega (u_{n,\epsilon}(-\Delta)^\alpha \xi + g(u_{n,\epsilon})\xi)\, dx = \int_\Omega (f_n - \epsilon u_{n,\epsilon})\, \xi dx$$

that u is a solution of (1.1). Uniqueness follows from (2.36)-(3.3), since for any f and f' in $L^1(\Omega, \rho^\alpha dx)$, the any couple (u, u') of weak solutions with respective right-hand side f and f' satisfies

$$\int_\Omega \left(|u - u'| + |g(u) - g(u')|\eta_1\right) dx \leq \int_\Omega |f - f'|\eta_1 dx. \tag{3.4}$$

Finally, the monotonicity of the mapping $f \mapsto u$ follows from (2.22) thanks to which (3.4) is transformed into

$$\int_\Omega \left((u - u')_+ + (g(u) - g(u'))_+\eta_1\right) dx \leq \int_\Omega (f - f')_+\eta_1 dx. \tag{3.5}$$

□

Proof of Theorem 1.1. Uniqueness follows from (3.4). For existence we define

$$C_\beta(\bar\Omega) = \{\zeta \in C(\bar\Omega) : \rho^{-\beta}\zeta \in C(\bar\Omega)\}$$

endowed with the norm

$$\|\zeta\|_{C_\beta(\bar\Omega)} = \|\rho^{-\beta}\zeta\|_{C(\bar\Omega)}.$$

We consider a sequence $\{\nu_n\} \subset C^1(\bar\Omega)$ such that $\nu_{n,\pm} \to \nu_\pm$ in the duality sense with $C_\beta(\bar\Omega)$, which means

$$\lim_{n\to\infty} \int_{\bar\Omega} \zeta \nu_{n,\pm} dx = \int_{\bar\Omega} \zeta d\nu_\pm$$

for all $\zeta \in C_\beta(\bar\Omega)$. It follows from the Banach-Steinhaus theorem that $\|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)}$ is bounded independently of n, therefore

$$\int_\Omega (|u_n| + |g(u_n)|\eta_1)\, dx \leq \int_\Omega |\nu_n|\eta_1 dx \leq C. \tag{3.6}$$

Therefore $\|g(u_n)\|_{\mathfrak{M}(\Omega,\rho^\alpha)}$ is bounded independently of n. For $\epsilon > 0$, set $\xi_\epsilon = (\eta_1+\epsilon)^{\frac{\beta}{\alpha}} - \epsilon^{\frac{\beta}{\alpha}}$, which is concave in the interval $\eta(\bar{\omega})$. Then, by Lemma 2.3 part (ii),

$$
\begin{aligned}
(-\Delta)^\alpha \xi_\epsilon &= \frac{\beta}{\alpha}(\eta_1+\epsilon)^{\frac{\beta-\alpha}{\alpha}}(-\Delta)^\alpha \eta_1 - \frac{\beta(\beta-\alpha)}{\alpha^2}(\eta_1+\epsilon)^{\frac{\beta-2\alpha}{\alpha}} \int_\Omega \frac{(\eta_1(y)-\eta_1(x))^2}{|y-x|^{N+2\alpha}} dy \\
&\geq \frac{\beta}{\alpha}(\eta_1+\epsilon)^{\frac{\beta-\alpha}{\alpha}},
\end{aligned}
$$

and $\xi_\epsilon \in \mathbb{X}_\alpha$. Since

$$
\int_\Omega (|u_n|(-\Delta)^\alpha \xi_\epsilon + |g(u_n)|\xi_\epsilon)\, dx \leq \int_\Omega \xi_\epsilon d|\nu_n|,
$$

we obtain

$$
\int_\Omega \left(|u_n| \frac{\beta}{\alpha}(\eta_1+\epsilon)^{\frac{\beta-\alpha}{\alpha}} + |g(u_n)|\xi_\epsilon \right) dx \leq \int_\Omega \xi_\epsilon d|\nu_n|.
$$

If we let $\epsilon \to 0$, we obtain

$$
\int_\Omega \left(|u_n| \frac{\beta}{\alpha}\eta_1^{\frac{\beta-\alpha}{\alpha}} + |g(u_n)|\eta_1^{\frac{\beta}{\alpha}} \right) dx \leq \int_\Omega \eta_1^{\frac{\beta}{\alpha}} d|\nu_n|.
$$

By Lemma 2.3, we derive the estimate

$$
\int_\Omega \left(|u_n|\rho^{\beta-\alpha} + |g(u_n)|\rho^\beta \right) dx \leq C\|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)} \leq C'. \tag{3.7}
$$

Since $u_n = \mathbb{G}[\nu_n - g(u_n)]$, it follows by (2.7), that

$$
\|u_n\|_{M^{k_{\alpha,\beta}}(\Omega,\rho^\beta dx)} \leq \|\nu_n - g(u_n)\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \tag{3.8}
$$

where $k_{\alpha,\beta}$ is defined by (2.13). By Corollary 2.6 the sequence $\{u_n\}$ is relatively compact in the $L^q(\Omega)$ for $1 \leq q < \frac{N}{N+\beta-2\alpha}$. Therefore there exist a sub-sequence $\{u_{n_k}\}$ and some $u \in L^1(\Omega) \cap L^q(\Omega)$ such that $u_{n_k} \to u$ in $L^q(\Omega)$ and almost every where in Ω. Furthermore $g(u_{n_k}) \to g(u)$ almost every where. Put $\tilde{g}(r) = g(|r|) - g(-|r|)$ and we note that $|g(r)| \leq \tilde{g}(|r|)$ for $r \in \mathbb{R}$ and $\tilde{g}$ is nondecreasing. For $\lambda > 0$, we set $S_\lambda = \{x \in \Omega : |u_{n_k}(x)| > \lambda\}$ and $\omega(\lambda) = \int_{S_\lambda} \rho^\beta dx$. Then for any Borel set $E \subset \Omega$, we have

$$
\begin{aligned}
\int_E |g(u_{n_k})|\rho^\beta dx &= \int_{E\cap S_\lambda^c} |g(u_{n_k})|\rho^\beta dx + \int_{E\cap S_\lambda} |g(u_{n_k})|\rho^\beta dx \\
&\leq \tilde{g}(\lambda)\int_E \rho^\beta dx + \int_{S_\lambda} \tilde{g}(|u_{n_k}|)\rho^\beta dx \\
&\leq \tilde{g}(\lambda)\int_E \rho^\beta dx - \int_\lambda^\infty \tilde{g}(s) d\omega(s).
\end{aligned}
$$

But

$$\int_\lambda^\infty \tilde{g}(s)d\omega(s) = \lim_{T\to\infty}\int_\lambda^T \tilde{g}(s)d\omega(s).$$

Since $u_{n_k} \in M^{k_{\alpha,\beta}}(\Omega, \rho^\beta dx)$, $\omega(s) \le cs^{-k_{\alpha,\beta}}$ and

$$\begin{aligned}
-\int_\lambda^T \tilde{g}(s)d\omega(s) &= -\Big[\tilde{g}(s)\omega(s)\Big]_{s=\lambda}^{s=T} + \int_\lambda^T \omega(s)d\tilde{g}(s)\\
&\le \tilde{g}(\lambda)\omega(\lambda) - \tilde{g}(T)\omega(T) + c\int_\lambda^T s^{-k_{\alpha,\beta}}d\tilde{g}(s)\\
&\le \tilde{g}(\lambda)\omega(\lambda) - \tilde{g}(T)\omega(T) + c\left(T^{-k_{\alpha,\beta}}\tilde{g}(T) - \lambda^{-k_{\alpha,\beta}}\tilde{g}(\lambda)\right)\\
&\qquad\qquad + \frac{c}{k_{\alpha,\beta}+1}\int_\lambda^T s^{-1-k_{\alpha,\beta}}\tilde{g}(s)ds.
\end{aligned}$$

By assumption (1.9) there exists $\{T_n\} \to \infty$ such that $T_n^{-k_{\alpha,\beta}}\tilde{g}(T_n) \to 0$ when $n \to \infty$. Furthermore $\tilde{g}(\lambda)\omega(\lambda) \le c\lambda^{-k_{\alpha,\beta}}\tilde{g}(\lambda)$, therefore

$$-\int_\lambda^\infty \tilde{g}(s)d\omega(s) \le \frac{c}{k_{\alpha,\beta}+1}\int_\lambda^\infty s^{-1-k_{\alpha,\beta}}\tilde{g}(s)ds.$$

Notice that the above quantity on the right-hand side tends to 0 when $\lambda \to \infty$. The conclusion follows: for any $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\frac{c}{k_{\alpha,\beta}+1}\int_\lambda^\infty s^{-1-k_{\alpha,\beta}}\tilde{g}(s)ds \le \frac{\epsilon}{2}$$

and $\delta > 0$ such that

$$\int_E \rho^\beta dx \le \delta \Longrightarrow \tilde{g}(\lambda)\int_E \rho^\beta dx \le \frac{\epsilon}{2}.$$

This proves that $\{g\circ u_{n_k}\}$ is uniformly integrable in $L^1(\Omega, \rho^\beta dx)$. Then $g\circ u_{n_k} \to g\circ u$ in $L^1(\Omega, \rho^\beta dx)$ by Vitali convergence theorem. Letting $n_k \to \infty$ in the identity

$$\int_\Omega \left(u_{n_k}(-\Delta)^\alpha\xi + \xi g\circ u_{n_k}\right)dx = \int_\Omega \nu_{n_k}\xi dx$$

where $\xi \in \mathbb{X}_\alpha$, it infers that u is a weak solution of (1.1).

The right-hand side of estimate (1.9) follows from the fact that $v_{n,+} := \mathbb{G}[\nu_{n,+}]$ satisfies

$$(-\Delta)^\alpha v_{n,+} + g(v_{n,+}) = \nu_{n,+} + g(v_{n,+}) \ge \nu_n$$

Therefore $v_{n,+} \ge u_n$ by Proposition 3.1. Letting $n \to \infty$ yields to (1.10). The left-hand side is proved similarly.

To prove the mapping $\nu \mapsto u$ is increasing. Let $\nu_1, \nu_2 \in \mathfrak{M}(\Omega, \rho^\beta)$ and $\nu_1 \geq \nu_2$, then there exist two sequences $\{\nu_{1,n}\}$ and $\{\nu_{2,n}\}$ in $C^\infty(\bar{\Omega})$ such that $\nu_{1,n} \geq \nu_{2,n}$ and

$$\nu_{i,n} \to \nu_i \quad \text{as n} \to \infty, \quad \text{i} = 1, 2.$$

Let $u_{i,n}$ be the unique solution of (1.1) with $\nu_{i,n}$ and u_i be the unique solution of (1.1) with ν_i where $i = 1, 2$. Then $u_{1,n} \geq u_{2,n}$. Moveover, by uniqueness $u_{i,n}$ convergence to u_i in $L^1(\Omega)$ for $i = 1$ and $i = 2$. Then we have $u_1 \geq u_2$. $\Box$

Corollary 3.1 *Under the hypotheses of Theorem 1.1, we further assume that $\{\nu_n\}$ is a sequence of measures in $\mathfrak{M}(\Omega, \rho^\beta)$ and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ such that for any $\xi \in C_\beta(\bar{\Omega})$,*

$$\int_\Omega \xi d\nu_n \to \int_\Omega \xi d\nu \quad \text{as} \quad n \to \infty.$$

Then the sequence $\{u_n\}$ of weak solutions to

$$\begin{aligned} (-\Delta)^\alpha u_n + g \circ u_n = \nu_n \quad &\text{in} \quad \Omega, \\ u_n = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{3.9}$$

converges to the solution u of (1.1) in $L^q(\Omega)$ for $1 \leq q < \frac{N}{N+\beta-2\alpha}$ and $\{g \circ u_n\}$ converges to $g \circ u$ in $L^1(\Omega, \rho^\beta dx)$.

Proof. The method is an adaptation of [38]. Since $\nu_n \to \nu$ in the duality sense of $C_\beta(\overline{\Omega})$, there exists $M > 0$ such that

$$\|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)} \leq M, \qquad \forall n \in \mathbb{N}.$$

Therefore (3.7) and (3.8) hold (but with u_n solution of (3.9)). The above proof shows that $\{g \circ u_n\}$ is uniformly integrable in $L^1(\Omega, \rho^\beta dx)$ and $\{u_n\}$ relatively compact in $L^q(\Omega)$ for $1 \leq q < \frac{N}{N+\beta-2\alpha}$. Thus, up to a subsequence $\{u_{n_k}\} \subset \{u_n\}$, $u_{n_k} \to u$, and u is the weak solution of (1.1). Since u is unique, $u_n \to u$ as $n \to \infty$. $\Box$

Remark 3.1 *Under the hypotheses of Theorem 1.1, we assume $\nu \geq 0$, then*

$$\mathbb{G}(\nu) - \mathbb{G}(g(\mathbb{G}(\nu))) \leq u \leq \mathbb{G}(\nu). \tag{3.10}$$

Indeed, since g is nondecreasing and $u \leq \mathbb{G}(\nu)$, then

$$\begin{aligned} u &= \mathbb{G}(\nu) - \mathbb{G}(g(u)) \\ &\geq \mathbb{G}(\nu) - \mathbb{G}(g(\mathbb{G}(\nu))). \end{aligned}$$

4 Applications

4.1 The case of a Dirac mass

In this subsection we characterize the asymptotic behavior of a solution near a singularity created by a Dirac mass.

Theorem 4.1 *Assume that Ω is an open, bounded and C^2 domain of $\mathbb{R}^N$ ($N \geq 2$) with $0 \in \Omega$, $\alpha \in (0,1)$, $\nu = \delta_0$ and the function $g : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and (1.9) holds for*

$$k_{\alpha,0} = \frac{N}{N-2\alpha}. \tag{4.1}$$

Then problem (1.1) admits a unique positive weak solution u such that

$$\lim_{x\to 0} u(x)|x|^{N-2\alpha} = C, \tag{4.2}$$

for some $C > 0$.

Remark 4.1 *We note here that a weak solution u of (1.1) with $\nu = \delta_0$ satisfies*

$$\begin{aligned} (-\Delta)^\alpha u + g(u) = 0 \quad &\text{in} \quad \Omega \setminus \{0\}, \\ u = 0 \quad &\text{in} \quad \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{4.3}$$

The asymptotic behavior (4.2) is one of the possible singular behaviors of solutions of (4.3) given in [13].

Before proving Theorem 4.1, we give an auxiliary lemma.

Lemma 4.1 *Assume that $g : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and (1.9) holds with $k_{\alpha,\beta} > 1$. Then*

$$\lim_{s\to\infty} g(s)s^{-k_{\alpha,\beta}} = 0.$$

Proof. Since

$$\int_s^{2s} g(t)t^{-1-k_{\alpha,\beta}}dt \geq g(s)(2s)^{-1-k_{\alpha,\beta}} \int_s^{2s} dt = 2^{-1-k_{\alpha,\beta}} g(s)s^{-k_{\alpha,\beta}}$$

and by (1.9),

$$\lim_{s\to\infty} \int_s^{2s} g(t)t^{-1-k_{\alpha,\beta}}dt = 0.$$

Then

$$\lim_{s\to\infty} g(s)s^{-k_{\alpha,\beta}} = 0.$$

The proof is complete. $\square$

Proof of Theorem 4.1. Existence, uniqueness and positiveness follow from Theorem 1.1 with $\beta = 0$. For (4.2), we shall use (1.10). From [12] there holds,

$$0 < \frac{C}{|x|^{N-2\alpha}} - G(x,0) < \frac{C}{\rho(0)^{N-2\alpha}}, \quad x \in \Omega \setminus \{0\}. \tag{4.4}$$

for some $C > 0$ dependent of N and α. Since

$$\mathbb{G}(\delta_0)(x) = G(x,0) < \frac{C}{|x|^{N-2\alpha}}, \quad x \in \Omega \setminus \{0\},$$

then

$$\begin{aligned}
0 &\leq \mathbb{G}(g(\mathbb{G}(\delta_0)))(x)|x|^{N-2\alpha} \\
&\leq \int_\Omega \frac{1}{|x-y|^{N-2\alpha}} g(\frac{C}{|y|^{N-2\alpha}})dy|x|^{N-2\alpha} \\
&\leq \int_\Omega \frac{1}{|e_x-y|^{N-2\alpha}} g(\frac{C}{(|x||z|)^{N-2\alpha}})dz|x|^{N} \\
&= |x|^N \int_{\Omega\cap B_{1/2}(e_x)} \frac{1}{|e_x-y|^{N-2\alpha}} g(\frac{C}{(|x||z|)^{N-2\alpha}})dz \\
&\quad +|x|^N \int_{\Omega\cap B^c_{1/2}(e_x)} \frac{1}{|e_x-y|^{N-2\alpha}} g(\frac{C}{(|x||z|)^{N-2\alpha}})dz \\
&:= A_1(x) + A_2(x), \quad x \in \Omega \setminus \{0\},
\end{aligned}$$

where $e_x = x/|x|$. By Lemma 4.1,

$$\begin{aligned}
A_1(x) &\leq |x|^N g(\frac{2^{N-2\alpha}C}{|x|^{N-2\alpha}}) \int_{B_{1/2}(e_x)} \frac{1}{|e_x-y|^{N-2\alpha}}dz \\
&\to 0 \quad \text{as} \quad |x| \to 0,
\end{aligned}$$

and by (1.9),

$$\begin{aligned}
A_2(x) &\leq \bar{C}|x|^N \int_{B_R(0)} g(\frac{C}{(|x||z|)^{N-2\alpha}})dz \\
&\leq \bar{C} \int_{\frac{R^{1/(N-2\alpha)}}{|x|}}^{\infty} g(Cs)s^{-1-\frac{N}{N-2\alpha}}ds \\
&\to 0 \quad \text{as} \quad |x| \to 0,
\end{aligned}$$

where $R > 0$ such that $B_R(0) \supset \Omega$. That is

$$\lim_{|x|\to 0} \mathbb{G}(g(\mathbb{G}(\delta_0)))(x)|x|^{N-2\alpha} = 0. \tag{4.5}$$

We plug (4.4) and (4.5) into (3.10), then (4.2) holds. $\square$

4.2 The power case

If $g(s)=|s|^{k-1}s$ with $k\geq 1$, then (1.9) is satisfied if $1\leq k<k_{\alpha,\beta}$ where $k_{\alpha,\beta}$ defined by (2.13) is called the *critical exponent* with limit values $k_{\alpha,0}=\frac{N}{N-2\alpha}$ and $k_{\alpha,\alpha}=\frac{N+\alpha}{N-\alpha}$. If we consider the problem

$$\begin{aligned}(-\Delta)^\alpha u+|u|^{k-1}u=\nu \qquad &\text{in} \quad \Omega,\\ u=0 \qquad &\text{in} \quad \Omega^c,\end{aligned} \tag{4.6}$$

then if $1<k<k_{\alpha,\beta}$ it is solvable for any $\nu\in\mathfrak{M}(\Omega,\rho^\beta)$, but it may not be the case if $k\geq k_{\alpha,\beta}$. As in the case $\alpha=1$, the sharp solvability of (4.6) is associated to a concentration property of the measure ν and this concentration is expressed by the mean of Bessel capacities. If $k>1$ and $k'=\frac{k}{k-1}$, we define for any compact set $K\subset\Omega$,

$$C^{\Omega}_{2\alpha,k'}(K)=\inf\{\|\phi\|^{k'}_{W^{2\alpha,k'}(\Omega)}:\phi\in C^\infty_c(\Omega),0\leq\phi\leq 1,\phi\equiv 1 \text{ on } K\}. \tag{4.7}$$

Then $C_{2\alpha,k'}$ is an outer measure or capacity in Ω extended to Borel sets by standard processes. Our result is the following in the case of bounded measures

Theorem 4.2 *Assume Ω is an open bounded C^2 domain in $\mathbb{R}^N$ and $k>1$. Then problem* (4.6) *can be solved with a nonnegative bounded measure ν if and only if ν satisfies on compact subsets $K\subset\Omega$*

$$C^{\Omega}_{2\alpha,k'}(K)=0\Longrightarrow\nu(K)=0. \tag{4.8}$$

Proof. *1-The condition is necessary.* Assume u is a weak solution and let $K\subset\Omega$ be compact. Let $\phi\in C^\infty_c(\mathbb{R}^N)$ such that $0\leq\phi\leq 1$ and $\phi(x)=1$ for all $x\in K$, and set $\xi=\phi^{k'}$, then $\xi\in\mathbb{X}_\alpha$ and

$$\int_\Omega\left(u(-\Delta)^\alpha\xi+u^k\xi\right)dx=\int_\Omega\xi d\nu.$$

Since $\xi\geq\chi_K$ it follows from (2.30) that

$$\int_\Omega\left(k'\phi^{k'-1}u(-\Delta)^\alpha\phi+\phi^{k'}u^k\right)dx\geq\nu(K). \tag{4.9}$$

By Hölder's inequality

$$\left|\int_\Omega\phi^{k'-1}u(-\Delta)^\alpha\phi dx\right|\leq\left(\int_\Omega\phi^{k'}u^kdx\right)^{\frac{1}{k}}\left(\int_\Omega|(-\Delta)^\alpha\phi|^{k'}dx\right)^{\frac{1}{k'}} \tag{4.10}$$

By [29, Th 5.4], there exists $\tilde\phi\in W^{2\alpha,k'}(\mathbb{R}^N)$ such that $\tilde\phi\lfloor_\Omega=\phi$ and

$$\|\tilde\phi\|_{W^{2\alpha,k'}(\mathbb{R}^N)}\leq C\|\phi\|_{W^{2\alpha,k'}(\Omega)}$$

Then, by standard regularity result on the Riesz potential $(-\Delta)^{-\alpha}$ in $\mathbb{R}^N$,

$$\begin{aligned}\left|\int_\Omega \phi^{k'-1}u(-\Delta)^\alpha\phi dx\right| &\leq \left(\int_\Omega \phi^{k'}u^k dx\right)^{\frac{1}{k}}\left(\int_{\mathbb{R}^N}|(-\Delta)^\alpha\phi|^{k'}dx\right)^{\frac{1}{k'}}\\ &\leq C'\left(\int_\Omega \phi^{k'}u^k dx\right)^{\frac{1}{k'}}\|\tilde{\phi}\|_{W^{2\alpha,k'}(\mathbb{R}^N)} \qquad (4.11)\\ &\leq C'\left(\int_\Omega \phi^{k'}u^k dx\right)^{\frac{1}{k'}}\|\phi\|_{W^{2\alpha,k'}(\Omega)}.\end{aligned}$$

Therefore (4.11), yields to

$$C\|\phi\|_{W^{2\alpha,k'}(\Omega)}\left(\int_\Omega \phi^{k'}u^k dx\right)^{\frac{1}{k}}+\int_\Omega \phi^{k'}u^k dx\geq \nu(K). \tag{4.12}$$

If $C^{\Omega}_{2\alpha,k'}(K)=0$, there exists a sequence $\{\phi_n\}\subset C_c^\infty(\Omega)$ such that $0\leq\phi_n\leq 1$ and $\phi_n=1$ on K and $\|\phi_n\|_{W^{2\alpha,k'}(\Omega)}\to 0$ as $n\to\infty$. Furthermore K has zero Lebesgue measure and $\phi_n\to 0$ almost everywhere. If we replace ϕ by ϕ_n in (4.12) and let $n\to\infty$ we obtain $\nu(K)=0$.

2-The condition is sufficient. We first assume that $\nu\in W^{-2\alpha,k}(\Omega)\cap\mathfrak{M}^b_+(\Omega)$; for $n\in\mathbb{N}$, we denote by u_n the solution of

$$\begin{aligned}(-\Delta)^\alpha u+|T_n(u)|^{k-1}T_n(u)&=\nu \qquad \text{in} \quad \Omega\\ u&=0 \qquad \text{in} \quad \Omega^c\end{aligned} \tag{4.13}$$

where $T_n(r)=\operatorname{sign}(r)\min\{n,|r|\}$. Such a solution exists by Theorem 1.1, is nonnegative and the sequence $\{u_n\}$ is decreasing and converges to some nonnegative u since $\{T_n(r)\}$ is increasing on $\mathbb{R}_+$. Furthermore

$$0\leq u_n\leq\mathbb{G}[\nu],$$

by (1.10). This implies that the convergence holds in $L^1(\Omega)$. Since $\nu\in W^{-2\alpha,k}(\Omega)$, $G[\nu]\in L^k(\Omega)$, it infers that

$$|T_n(u_n)|^{k-1}T_n(u_n)=(T_n(u_n))^k\leq(\mathbb{G}[\nu])^k.$$

Since for any $\xi\in\mathbb{X}_\alpha$ there holds

$$\int_\Omega\left(u_n(-\Delta)^\alpha\xi+(T_n(u_n))^k\xi\right)dx=\int_\Omega\xi d\nu \tag{4.14}$$

we can let $n\to\infty$ and conclude that u is a solution of (4.6), unique by (3.4). Next we assume that (4.8) holds. By a result of Feyel and de la Pradelle [19] (see also [17]), there exists an increasing sequence $\{\nu_n\}\subset$

$W^{-2\alpha,k}(\Omega) \cap \mathfrak{M}^b_+(\Omega)$ which converges to ν in the weak sense of measures. This implies that the sequence $\{u_n\}$ of weak solutions of

$$\begin{aligned} (-\Delta)^\alpha u_n + u_n^k &= \nu_n \qquad &\text{in} \quad \Omega \\ u_n &= 0 \qquad &\text{in} \quad \Omega^c \end{aligned} \tag{4.15}$$

is increasing with limit u. Taking $\eta_1 := \mathbb{G}[1]$ as a test function in the weak formulation, we have

$$\int_\Omega \left(u_n + u_n^k \eta_1 \right) dx = \int_\Omega \eta_1 d\nu_n \leq \int_\Omega \eta_1 d\nu.$$

Therefore $u_n \to u$ in $L^1(\Omega) \cap L^k(\Omega, \rho^\alpha dx)$. Letting $n \to \infty$ we deduce that u satisfies (4.6). $\square$

Remark 4.2 *If ν is a signed bounded measure a sufficient condition for solving* (4.6) *is*

$$C^\Omega_{2\alpha,k'}(K) = 0 \Longrightarrow |\nu|(K) = 0. \tag{4.16}$$

This can be obtained by using the fact that the solutions of (4.6) *with right-hand side ν_+ and $-\nu_-$ are respectively a supersolution and a subsolution of* (4.6)*. It is not clear whether it is also a necessary condition.*

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