Quasilinear Elliptic Hamilton-Jacobi Equations on Complete Manifolds

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Partial Differential Equations/Differential Geometry

QUASILINEAR ELLIPTIC HAMILTON-JACOBI EQUATIONS ON COMPLETE MANIFOLDS

Marie-Françoise Bidaut-Véron [1], Marta Garcia-Huidobro [2], Laurent Véron [3]

Résumé. Let (M^n, g) be a n-dimensional complete, non-compact and connected Riemannian manifold, with Ricci tensor $Ricc_g$ and sectional curvature Sec_g. Assume $Ricc_g \geq (1-n)B^2$, and either $p > 2$ and $Sec_g(x) = o(dist^2(x,a))$ when $dist^2(x,a) \to \infty$ for $a \in M$, or $1 < p < 2$ and $Sec_g(x) \leq 0$. If $q > p-1 > 0$, any C^1 solution of (E) $-\Delta_p u + |\nabla u|^q = 0$ on M satisfies $|\nabla u(x)| \leq c_{n,p,q} B^{\frac{1}{q+1-p}}$ for some constant $c_{n,p,q} > 0$. As a consequence there exists $c_{n,p} > 0$ such that any positive p-harmonic function v on M satisfies $v(a)e^{-c_{n,p}B\text{dist}\,(x,a)} \leq v(x) \leq v(a)e^{c_{n,p}B\text{dist}\,(x,a)}$ for any $(a,x) \in M \times M$.

Equations de Hamilton-Jacobi quasilinéaires sur une variété complète

Résumé. Soit (M^n, g) une variété riemannienne n-dimensionnelle complète, non compacte et connexe de courbures de Ricci $Ricc_g$ et sectionnelle Sec_g. On suppose $Ricc_g \geq (1-n)B^2$ et $Sec_g(x) = o(dist^2(x,a))$ si $dist^2(x,a) \to \infty$ pour $a \in M$ si $p > 2$, ou $Sec_g(x) \leq 0$ si $1 < p < 2$. Si $q > p-1 > 0$, toute solution de classe C^1 de (E) $-\Delta_p u + |\nabla u|^q = 0$ sur M satisfait à $|\nabla u(x)| \leq c_{n,p,q} B^{\frac{1}{q+1-p}}$ où $c_{n,p,q} > 0$ est une constante. On en déduit qu'il existe $c_{n,p} > 0$ tel que toute fonction p-harmonique positive v sur M satisfait à l'encadrement suivant, $v(a)e^{-c_{n,p}B\text{dist}\,(x,a)} \leq v(x) \leq v(a)e^{c_{n,p}B\text{dist}\,(x,a)}$ pour tout $(a,x) \in M \times M$.

Version française abrégée. Soit (M^n, g) une variété riemannienne complète, non-compacte et connexe de courbure de Ricci $Ricc_g$ et courbure sectionnelle Sec_g. Pour tout $p > 1$, on dénote par $u \mapsto \Delta_p u := div\left(|\nabla u|^{p-2}\nabla u\right)$ le p-Laplacien sur M pour la métrique g. Notre résultat principal est le suivant

Theorème 1. *Soit $B \geq 0$ tel que $Ricc_g \geq (1-n)B^2$ et $q > p-1 > 0$. On suppose*

$$\lim_{dist(x,a)\to\infty} \frac{Sec_g(x)}{(dist(x,a))^2} = 0 \tag{1}$$

pour tout $a \in M$ si $p > 2$, ou $Sec_g \leq 0$ si $1 < p < 2$. Il existe alors $c_{n,p,q} > 0$ telle que toute solution $u \in C^1(M)$ de

$$-\Delta_p u + |\nabla u|^q = 0 \qquad \text{sur } M \tag{2}$$

vérifie

$$|\nabla u(x)| \leq c_{n,p,q} B^{\frac{1}{q+1-p}} \qquad \forall x \in M. \tag{3}$$

Une des conséquences est un théorème de type Liouville.

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Corollaire 2. *Supposons que* $Ricc_g \geq 0$*,* $q > p-1 > 0$ *et que les hypothèses du Théorème 1 portant sur la courbure sectionnelle soient vérifiées si* $p \neq 2$*. Alors toute solution* $u \in C^1(M)$ *de (2) est constante.*

Si v est une fonction p-harmonique positive sur M, la fonction $u := -(p-1)\ln v$ vérifie

$$-\Delta_p u + |\nabla u|^p = 0 \qquad \text{sur } M. \tag{4}$$

En utilisant le résultat du théorème 1, on en déduit

Théorème 3. *Supposons que* $p > 1$ *et que les hypothèses du Théorème 1 portant sur la courbure soient vérifiées. Il existe alors une constante* $c_{n,p} > 0$ *telle que toute fonction p-harmonique et positive* v *sur* M *vérifie*

$$v(a)e^{-c(n,p)B\mathrm{dist}\,(x,a)} \leq v(x) \leq v(a)e^{c(n,p)B\mathrm{dist}\,(x,a)} \qquad \forall (a,x) \in M \times M. \tag{5}$$

Quand $p = 2$ Cheng et Yau [1] ont montré que toute fonction harmonique positive sur une variété riemannienne complète à courbure de Ricci positive est une constante. Dans le cas des fonctions p-harmoniques positives et sous l'hypothèse de minoration uniforme de la courbure sectionnelle, $Sec_g \geq -B^2$, Kotschwar et Ni [4] montrent que toute fonction p-harmonique positive v sur M vérifie l'estimation suivante,

$$\frac{|\nabla v|}{v} \leq (p-1)B. \tag{6}$$

Notons que leur hypothèse implique $Ricc_g \geq (1-n)B^2$.

Let (M^n, g) be a complete, connected and non compact Riemannian manifold with Ricci curvature $Ricc_g$ and sectionnal curvature Sec_g. For $p > 1$ we denote by Δ_p the p-Laplacian defined in the metric g by

$$\Delta_p u := div\left(|\nabla u|^{p-2}\,\nabla u\right),$$

and thus Δ_2 is the Laplace-Beltrami operator on M. If $p = 2$ a classical result due to Cheng and Yau [1] asserts that if $Ricc_g$ is nonnegative, any nonnegative harmonic function v is a constant. In [4], Kotschwar et Ni obtained sharper results dealing with positive p-harmonic functions under the assumption that $Sec_g \geq -B^2$. They proved that if v is such a function, it satisfies

$$\frac{|\nabla v|}{v} \leq (p-1)B. \tag{1}$$

Their assumption on Sec_g implies $Ricc_g \geq (1-n)B^2$. They also noticed that if $p = 2$ their estimate holds under the previous lower estimate on the Ricci curvature. In this note we give an extension of their result in imbedding it the more general class of quasilinear Hamilton-Jacobi type equations

$$-\Delta_p u + |\nabla u|^q = 0 \qquad \text{on } M. \tag{2}$$

Our main result is the following

Theorem 1. *Let $B \geq 0$ such that $Ricc_g \geq (1-n)B^2$. If $p > 2$ we assume that for any $a \in M$*

$$\lim_{dist(x,a)\to\infty} \frac{Sec_g(x)}{(dist(x,a))^2} = 0, \tag{3}$$

and if $1 < p < 2$ that $Sec_g \leq 0$. Then there exists $c_{n,p,q} > 0$ such that any solution $u \in C^1(M)$ of (2) satisfies

$$|\nabla u(x)| \leq c_{n,p,q} B^{\frac{1}{q+1-p}} \qquad \forall x \in M. \tag{4}$$

A clear consequence of (3) is the following Liouville theorem

Corollary 2. *Assume $Ricc_g \geq 0$ and that the assumptions of Theorem 1 concerning Sec_g hold if $p \neq 2$. Then any solution $u \in C^1(M)$ of (2) is constant.*

If v is a positive p-harmonic function on M, then $u := -(p-1)\ln v$ satisfies

$$-\Delta_p u + |\nabla u|^p = 0 \qquad \text{on } M. \tag{5}$$

Therefore estimate (4) yields to the following result

Theorem 3. *Assume that $p > 1$ and the curvature assumptions of Theorem 1 are fulfilled. Then there exists a constant $c_{n,p} > 0$ such that any positive p-harmonic function v on M satisfies*

$$v(a)e^{-c(n,p)B\text{dist}\,(x,a)} \leq v(x) \leq v(a)e^{c(n,p)B\text{dist}\,(x,a)} \qquad \forall (a,x) \in M \times M. \tag{6}$$

Proof of Theorem 1. Let $M_+ := \{x \in M : |\nabla u(x)| > 0\}$. Then M_+ is open and $u \in C^3(M_+)$ since the equation is no longer degenerate. The proof is based upon the fact that $z = |\nabla u|^2$ is a subsolution of an elliptic differential inequality with a superlinear absorption term (see [5] for other applications). We denote by TM the tangent bundle of M and by $\langle .,. \rangle$ the scalar product induced by the metric g. We recall that any C^3-function u verifies the Böchner-Weitzenböck formula; combined with Schwarz inequality it yields to

$$\begin{aligned} \frac{1}{2}\Delta_2 |\nabla u|^2 &= \left|D^2 u\right|^2 + \langle \nabla \Delta_2 u, \nabla u\rangle + Ricc_g(\nabla u, \nabla u) \\ &\geq \frac{1}{n}|\Delta_2 u|^2 + \langle \nabla \Delta_2 u, \nabla u\rangle + Ricc_g(\nabla u, \nabla u), \end{aligned} \tag{7}$$

where $D^2 u$ is the Hessian. If u is a C^1 solution of (2), then $z = |\nabla u|^2$ satisfies

$$-\Delta_2 u - \frac{p-2}{2}\frac{\langle \nabla z, \nabla u\rangle}{z} + z^{\frac{q+2-p}{2}} = 0 \tag{8}$$

on M_+. Replacing $\Delta_2 u$ in (7) it follows that, for any $a > 0$,

$$\begin{aligned} \Delta_2 z + (p-2)\frac{\langle D^2 z(\nabla u), \nabla u\rangle}{z} &\geq \frac{2a^2}{N} z^{q+2-p} - \frac{1}{Na^2}\frac{\langle \nabla z, \nabla u\rangle^2}{z^2} - \frac{(p-2)}{2}\frac{|\nabla z|^2}{z} \\ &+ (p-2)\frac{\langle \nabla z, \nabla u\rangle^2}{z^2} + (q+2-p)z^{\frac{q-p}{2}}\langle \nabla z, \nabla u\rangle - (N-1)B^2 z. \end{aligned} \tag{9}$$

Since $z^{\frac{q-p}{2}}|\langle \nabla z, \nabla u\rangle| \leq z^{\frac{q+1-p}{2}}\dfrac{|\nabla z|}{\sqrt{z}}$, we can take $a = a(p,q) > 0$ large enough so that the right-hand side of (9) is bounded from below by $Cz^{q+2-p} - D\dfrac{|\nabla z|^2}{z}$ for some C, $D > 0$ which depend only on p and q. We set

$$\mathcal{A}(v) := -\Delta_2 v - (p-2)\frac{\langle D^2 v(\nabla u), \nabla u\rangle}{|\nabla u|^2} = -\sum_{i,j=1}^{N} a_{ij} v_{x_i x_j}$$

where the a_{ij} depend on ∇u and satisfy

$$\theta\,|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \leq \Theta\,|\xi|^2 \qquad \forall\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n,$$

where $\theta = \min\{1, p-1\}$ and $\Theta = \max\{1, p-1\}$. Then

$$\mathcal{L}^*(z) := \mathcal{A}(z) + Cz^{q+2-p} - D\frac{|\nabla z|^2}{z} - (n-1)B^2 z \leq 0 \qquad \text{in } M_+. \tag{10}$$

The next lemma is a local estimate.

Lemma 1. *Let $B_R(a) \subset M^n(g)$ be the geodesic ball of radius $R > 0$ and center a. Assume that $Ricc_g \geq -(n-1)B^2$ and either $Sec_g \geq -S^2$ for some $S^2 := S_R^2$ in $B_R(a)$ if $p > 2$, or $Sec_g \leq 0$ if $1 < p < 2$. Then there exists $c = c(n, p, q) > 0$ such that the function*

$$w(x) = \lambda\left(R^2 - r^2(x)\right)^{-\frac{2}{q+1-p}} + \mu \quad \text{with } r = r(x) = d(x, a), \tag{11}$$

satisfies $\mathcal{L}^*(w) \geq 0$ in $B_R(a)$, provided

$$\lambda = c\max\left\{(R^4B^2)^{\frac{1}{q+1-p}}, ((1 + B + (p-2)_+S)R^3)^{\frac{1}{q+1-p}}\right\} \tag{12}$$

and

$$\mu = ((n-1)B^2)^{\frac{1}{q+1-p}}. \tag{13}$$

Proof. We recall that $\Delta_2 w = w'' + w'\Delta_2 r$ and by [6, Lemma 1]

$$\Delta_2 r \leq (n-1)B\coth(Br) \leq \frac{n-1}{r}(1 + Br).$$

Then

$$\Delta_2 w \leq \frac{4}{q+1-p}(R^2 - r^2)^{-\frac{2(q+2-p)}{q+1-p}}\left(\frac{2r^2(q+3-p)}{q+1-p} + (R^2 - r^2)(1 + (n-1)(1 + Br)\right). \tag{14}$$

Moverover from [3, Chap 2, p. 23]

$$D^2 w = w'' dr \otimes dr + w' D^2 r. \tag{15}$$

If $0 \geq Sec_g(x) \geq -S^2$, there holds

$$0 \leq D^2 r \leq S\coth(Sr)g \leq \frac{S}{r}(1 + Sr)\,g. \tag{16}$$

Therefore, if $p \geq 2$ and $Sec_g \geq -S^2$, we get

$$\frac{\langle D^2 w(\nabla u), \nabla u\rangle}{|\nabla u|^2} \leq \frac{4}{q+1-p}(R^2 - r^2)^{-\frac{2(q+2-p)}{q+1-p}}\left(\frac{2r^2(q+3-p)}{q+1-p} + (R^2 - r^2)(2 + Sr)\right), \tag{17}$$

while, if $p \leq 2$ and $Sec_g \leq 0$,

(18)
$$\frac{\langle D^2 w(\nabla u), \nabla u\rangle}{|\nabla u|^2} \leq \frac{4}{q+1-p}(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}}\left(\frac{2r^2(q+3-p)}{q+1-p}+2(R^2-r^2)\right).$$

As a consequence

$$\begin{aligned}\mathcal{A}(w) &= -\Delta w - (p-2)\frac{\langle D^2 w(\nabla u), \nabla u\rangle}{|\nabla u|^2} \\ &\geq -k\lambda(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}}(R^2+(R^2-r^2)B_p r)\end{aligned} \tag{19}$$

for some $k = k(n,p,q)$, where $B_p = B + (p-2)_+ S$. Since

$$w^{q+2-p} \geq \lambda^{q+2-p}\left(R^2-r^2\right)^{-\frac{2(q+1-p)}{q+1-p}} + \mu^{q+2-p},$$

we have

(20)
$$\mathcal{L}^*(w) \geq \lambda(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}}\left(-k(R^2+(R^2-r^2)B_p r) - D\frac{16}{(q+1-p)^2}r^2 + C\lambda^{q+1-p}\right)$$
$$+\mu^{q+2-p} - (n-1)B^2\lambda\left(R^2-r^2\right)^{-\frac{2}{q+1-p}} - (n-1)B^2\mu.$$

We first take

$$\mu = ((n-1)B^2)^{\frac{1}{q+1-p}}. \tag{21}$$

Next we choose λ in order to have, uniformly for $0 \leq r < R$,

$$2^{-1}C\lambda^{q+1-p} \geq k\left(R^2+(R^2-r^2)B_p r\right) + \frac{16Dr^2}{(q+1-p)^2}$$

and

$$2^{-1}C\lambda^{q+2-p}(R^2-r^2)^{-\frac{2(q+2-p)}{q+1-p}} \geq (n-1)B^2\lambda\left(R^2-r^2\right)^{-\frac{2}{q+1-p}}.$$

There exists $c = c(n,p,q)$ such that, if

$$\lambda = c\max\left\{(R^4B^2)^{\frac{1}{q+1-p}}, ((1+B_p)R^3)^{\frac{1}{q+1-p}}\right\}, \tag{22}$$

then $\mathcal{L}^*(w) \geq 0$ holds.

Lemma 2. *Under the assumptions of Lemma 1, any C^1 solution of (2) in M satisfies*

(23)
$$|\nabla u(x)| \leq c_{n,p,q}\max\left\{B^{\frac{1}{q+1-p}}, (1+B_p)^{\frac{1}{2(q+1-p)}}(d(x,\partial\Omega))^{-\frac{1}{2(q+1-p)}}\right\} \qquad \forall x \in \Omega,$$

for every domain $\Omega \subset M$, where $B_p = B + (p-2)_+ S$ and $S = S_{d(x,\partial\Omega)}$.

Proof. Assume $a \in \Omega$, with $R < d(a,\partial\Omega)$. Let w be as in Lemma 1, then in any connected component G of $\{x \in B_R(a) : z(x) - w(x) > 0\}$ we find

(24)
$$\mathcal{A}(z-w) + C\left(z^{q+2-p} - w^{q+2-p}\right) - (n-1)B^2(z-w) - D\left(\frac{|\nabla z|^2}{z} - \frac{|\nabla w|^2}{w}\right) \leq 0.$$

By the mean value theorem and since $w(a)$ is the minimum of w, there holds

$$C(z^{q+2-p} - w^{q+2-p}) - (n-1)B^2(z-w) > 0, \tag{25}$$

provided $C(q+2-p)(w(a))^{q+1-p} > (n-1)B^2$. Since $w(a) > \mu = \left((n-1)B^2\right)^{\frac{1}{q+1-p}}$ and $q+2-p > 1$, this condition is fulfilled, up to replacing μ by $A\mu$ for some $A = A(p,q) > 1$. If $x_0 \in G$ is such that $z - w$ is maximal at x_0, we derive that

$$\mathcal{A}(z-w) + C\left(z^{q+2-p} - w^{q+2-p}\right) - (N-1)B^2(z-w) - D\left(\frac{|\nabla z|^2}{z} - \frac{|\nabla w|^2}{w}\right) \leq 0$$

if $x = x_0$, which is a contradiction. Thus $G = \emptyset$, $z \leq w$ and (23) follows.

The proof of Theorem 1 and Corollary 2 follows by taking $\Omega = B_R(x)$ and letting $R \to \infty$.

Proof of Theorem 3. We take $q = p$ and assume that v is p-harmonic and positive. If we write $v = e^{-\frac{u}{p-1}}$, then u satisfies

$$-\Delta_p u + |\nabla u|^p = 0.$$

If $Ricc_g(x) \geq 0$, u is constant by Corollary 2, and so is v. If $\inf\{Ricc_g(x) : x \in M\} = (1-n)B^2 < 0$ we apply (23) to ∇u. If γ is a minimizing geodesic from a to x, then $|\gamma'(t)| = 1$ and

$$u(x) - u(a) = \int_0^{d(x,a)} \frac{d}{dt} u \circ \gamma(t) dt = \int_0^{d(x,a)} \langle \nabla u \circ \gamma(t), \gamma'(t) \rangle dt.$$

Since

$$|\langle \nabla u \circ \gamma(t), \gamma'(t)| \leq |\nabla u \circ \gamma(t)| \leq c_{n,p,p} B,$$

we obtain

$$u(a) - c_{n,p,p} B \operatorname{dist}(x,a) \leq u(x) \leq u(a) + c_{n,p,p} B \operatorname{dist}(x,a) \qquad \forall x \in M. \tag{26}$$

Then (6) follows since $u = (1-p)\ln v$.

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