Some existence results to the Dirichlet problem for the minimal hypersurface equation on non mean convex domains of a Riemannian manifold

Ari Aiolfi, Jaime Ripoll, Marc Soret

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1   Introduction

As it is well known, the Dirichlet problem for the minimal hypersurface equation

$$\begin{cases} \mathcal{M}\left[u\right] := \operatorname{div} \dfrac{\operatorname{grad} u}{\sqrt{1 + \left|\operatorname{grad} u\right|^2}} = 0 \text{ in } \Omega, \ u \in C^{2,\alpha}\left(\overline{\Omega}\right) \\ \\ u|_{\partial\Omega} = \varphi, \end{cases} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ is solvable for an arbitrary continuous boundary data φ only if the domain is *mean convex* (Theorem 1 of [6]). This result (the existence part) has been extended and generalized to Riemannian manifolds (more generally to constant mean curvature graphs in warped products) in [2].

In the paper [6] H. Jenkins and J. Serrin noted that a condition involving $\operatorname{osc}\left(\varphi\right) := \sup\varphi - \inf\varphi$, $\left|D\varphi\right|$ and $\left|D^2\varphi\right|$ should be enough to ensure the solvability of (1) in arbitrary bounded domains. In fact, they proved that if $\operatorname{osc}\left(\varphi\right) \leq \mathcal{B}\left(\left|D\varphi\right|, \left|D^2\varphi\right|, \Omega\right)$ then the (1) is solvable (Theorem 2 of [6]). The function $\mathcal{B}$ has an explicit form (Section 3, p. 179 of [6]) and is infinity at the points where the domain is mean convex. Theorem 2 of [6] then extends Theorem 1 of [6].

In the present paper we first obtain an extension of Theorem 2 of Jenkins-Serrin [6] to the minimal hypersurface PDE on a domain Ω in an arbitrary

complete Riemannian manifold M. In the next result grad and div are the gradient and divergence operators in M. Then, u is a solution of (1) if and only if the graph of u in $M \times \mathbb{R}$ is a minimal surface. We prove

Theorem 1 *Let M^n be a complete $n-$dimensional Riemannian manifold M, $n \geq 2$. Given a bounded domain Ω in M - whose boundary $\partial\Omega$ has mean curvature H w.r.t. the inward unit normal vector - let $\varphi \in C^{2,\alpha}(\partial\Omega)$ be such that $\operatorname{osc}(\varphi) \leq \mathcal{C}(|D\varphi|, |D^2\varphi|, |A|, \operatorname{Ric}_M)$. Then the Dirichlet problem (1) for the minimal hypersurface equation is solvable. Moreover, $\mathcal{C}$ is given explicitly by (16) and (9) and $\mathcal{C} = +\infty$ at the mean convex points of $\partial\Omega$. It follows that if $\partial\Omega$ is mean convex (that is, $H \geq 0$) then (1) is solvable for any continuous boundary data.*

Next we apply Theorem 1 to the exterior Dirichlet problem for the minimal hypersurface equation when M is complete and noncompact. This problem consists in proving existence, uniqueness and describing the asymptotic behavior of a solution of (1) where Ω is an exterior open subset of M, that is, $M \backslash \Omega$ is compact.

It seems that the first mathematician to take up with the exterior Dirichlet problem was J. C. C. Nitsche, who proved ([10], §760) that any solution u of (1), in the case $M = \mathbb{R}^2$, has at most linear growth and has a well defined Gauss map at infinity. This problem has been investigated further by R. Krust [7], E. Kuwert [9], Kutev and Tomi in [8] and, more recently in [11].

An investigation of the exterior Dirichlet problem for the minimal surface equation in the Riemannian setting was initiated in [3]. There the authors considered only the special case of vanishing boundary data assuming, in the case $K_M \geq 0$, a condition on the decay of the sectional curvature of M and, in the case that $K_M \leq -k^2$, $k > 0$, that $M \backslash \Omega$ is convex and M simply connected.

In this paper we continue the investigation of [3] in the case $K_M \leq -k^2$, $k > 0$. First, we allow Ω to be any exterior domain and the boundary data not need be zero. Moreover, the asymptotic behaviour of the solution will be prescribed by any given continuous function defined on the asymptotic boundary $\partial_\infty M$ of M. For this last part we use the ideas of [12], as follows.

Recall that if M is a Hadamard manifold (complete, simply connected, $K_M < 0$) the asymptotic boundary $\partial_\infty M$ and the cone compactifaction $\overline{M}$ of M are well defined by using the so called cone topology (see [1]). According to [12], we say that M satisfies the *strict convexity condition (SC condition)*

if, given $x \in \partial_\infty M$ and a relatively open subset $W \subset \partial_\infty M$ containing x, there exists a C^2 open subset $\Omega \subset \overline{M}$ such that $x \in \text{Int}\,(\partial_\infty \Omega) \subset W$, where $\text{Int}\,(\partial_\infty \Omega)$ denotes the interior of $\partial_\infty \Omega$ in $\partial_\infty M$, and $M \setminus \Omega$ is convex. We prove

Theorem 2 *Let M be a Hadamard manifold satisfying the SC condition and assume that $K_M \leq -k^2$, $k > 0$. Let Ω be an exterior $C^{2,\alpha}$ domain. Given $\varphi \in C^{2,\alpha}\,(\partial \Omega)$ such that $\text{osc}\,(\varphi) \leq \mathcal{C}\,(|D\varphi|, |D^2\varphi|, |A|, Ric_M)$ and $\psi \in C^0\,(\partial_\infty M)$ there is an unique solution $u \in C^{2,\alpha}\,(\overline{\Omega})$ of $\mathcal{M}\,[u] = 0$ in Ω such that $u|_{\partial \Omega} = \varphi$. Moreover, u extends continuously to $\partial_\infty M$ and $u|_{\partial_\infty M} = \psi$.*

We mention that under the hypothesis $K_M \leq -k^2 < 0$, any $2-$dimensional Hadamard manifold satisfies the SC convexity condition, since any two distinct points of $\partial_\infty M$ can be connected by a minimizing geodesic. It is also proved in [12] that this condition is also satisfied if the metric of M is rotationally symmetric or if the sectional curvature of M has at most exponential decay, precisely, if $\inf_{B_R} K_M \geq -Ce^{(k-\varepsilon)R}$ for $R \geq R_0$ and for some $\varepsilon > 0$. In [5] it is proved that if the SC condition is not required then there are examples of $3-$dimensional complete manifolds with $K_M \leq -k^2 < 0$ in which only the constant functions are bounded solutions of the minimal PDE on M that extends continuously to $\partial_\infty M$. In such manifolds, if $u \in C^\infty\,(M) \cap C^0\,(\overline{M})$ is solution of an exterior Dirichlet problem for the minimal hypersurface equation then $u|_{\partial_\infty M}$ is constant.

2   An extension of a Theorem of Jenkins and Serrin

We begin with some preliminary facts.

2.1   Normal coordinates for the inner halftube of $\partial \Omega$

Let $\varphi \in C^2\,(\partial \Omega)$ be given. Let d be the Riemannian distance in M and set $\rho(z) := d(z, \partial \Omega)$, $z \in \Omega$. For ρ_0 sufficiently small, the normal exponential map

$$\exp_{\partial \Omega} : \partial \Omega \times [0, \rho_0) \longrightarrow U_{\rho_0} = \{z \in \Omega; \rho(z) < \rho_0\} \subset M,$$

is a diffeomorphism.

Let $\{T_1(x), ..., T_{n-1}(x)\}_{x \in V_{r_0}}$ be the orthonormal frame defined on an neighborhood $V_{r_0} \subset \partial\Omega$ of a point $p \in \partial\Omega$ by parallel transport of a given orthonormal frame at p (by construction $\nabla_{T_i} T_j(p) = 0, i, j = 1, \cdots, n-1$). For each $i \in \{1, ..., n-1\}$, extend also $T_i|_{\partial\Omega}$ by parallel transport along the normal geodesic $t \to \exp_{\partial\Omega} t\eta$, where η is the inward unitary normal field to $\partial\Omega$ and $t \leq \rho_0$, and denote this extension again by T_i. Then, for each $(x, t) \in V_{r_0} \times [0, \rho_0]$, $\{T_1(x,t), ..., T_{n-1}(x,t)\}$ is a orthonormal frame on the equidistant hypersurface $\rho(z) = t$. We complete the orthonormal frame on U_{ρ_0} by setting $T_n(x, t) = \nabla\rho(x, t)$ for all $(x, t) \in V_{r_0} \times [0, \rho_0)$. We extend also φ to U_{ρ_0} by setting $\varphi(\exp_{\partial\Omega}(x, t)) = \varphi(x)$, $(x, t) \in \partial\Omega \times [0, \rho_0)$.

By construction $\nabla_{T_n} T_i(x, t) = 0$ where $(x, t) \in V_{r_0} \times [0, \rho_0)$, $i \in \{1, ..., n-1\}$ and $T_n(\varphi) = \nabla_{T_n} T_n = 0$. Define

$$\omega(z) = \varphi(z) + \psi(\rho(z)), \tag{2}$$

where $z \in U_{\rho_0}$ and $\psi \in C^2([0, \infty))$ is to be determined later. Let $\mathcal{M}$ denote, as above, the minimal hypersurface equation operator. We have

$$W^3 \mathcal{M}(\omega) = -\frac{1}{2} \nabla_{\nabla\omega}(|\nabla\omega|^2) + W^2 \operatorname{div}(\nabla\omega) \tag{3}$$

with $W^2 := (1 + |\nabla\omega|^2)$. Then $\mathcal{M}(\omega) \leq 0$ if only if

$$-\sum_{i=1}^{n} \langle \nabla_{T_i} \nabla\omega, \nabla\omega \rangle T_i(\omega) + W^2 \sum_{i=1}^{n} \langle \nabla_{T_i} \nabla\omega, T_i \rangle \leq 0. \tag{4}$$

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Lemma 3 *The following equalities hold for ω :*

$$\begin{cases} \langle \nabla_{T_n} \nabla \omega, \nabla \omega \rangle = \sum_{i,k=1}^{n-1} II(T_i, T_k) T_i(\varphi) T_k(\varphi) + \psi' \psi'' \\ \langle \nabla_{T_n} \nabla \omega, T_n \rangle = \psi'' \\ \sum_{i=1}^{n-1} \langle \nabla_{T_i} \nabla \omega, \nabla \omega \rangle T_i(\varphi) = \sum_{i,j=1}^{n-1} T_i\left(T_j(\varphi)\right) T_j(\varphi) T_i(\varphi) \\ \sum_{i=1}^{n-1} \langle \nabla_{T_i} \nabla \omega, T_i \rangle = \sum_{i=1}^{n-1} T_i\left(T_i(\varphi)\right) + \sum_{i=1}^{n-1} T_i(\varphi) \operatorname{div}_{\Omega(\rho)} T_i - \psi'(n-1)H. \end{cases}$$

Proof. We will use throughout the proof that $\nabla_{T_n} T_i = 0$, $i = 1, \cdots, n$. To prove equality one and two we need to compute $\nabla_{T_n} \nabla \omega$. Since

$$\nabla \omega = \sum_{i=1}^{n-1} T_i(\varphi) T_i + \psi' T_n \tag{5}$$

then

$$\nabla_{T_n} \nabla \omega = \sum_{i=1}^{n-1} T_n(T_i(\varphi)) T_i + T_n(\psi') T_n. \tag{6}$$

We obtain for the first terms of (6)

$$T_n\left(T_i(\varphi)\right) = [T_n, T_i]\varphi = (\nabla_{T_n} T_i - \nabla_{T_i} T_n)\varphi = -\nabla_{T_i} T_n(\varphi)$$

$$= -\sum_{k=1}^{n-1} \langle \nabla_{T_i} T_n, T_k \rangle T_k(\varphi) = \sum_k II(T_i, T_k) T_k(\phi) \tag{7}$$

Moreover

$$T_n\left(\psi'(\rho)\right) T_n = \psi''(\rho) T_n(\rho) T_n = \psi'' T_n \tag{8}$$

Proof of i**) and** ii**):** From (5), (6), (7) and (8) we obtain

$$\langle \nabla_{T_n}(\nabla \omega), \nabla \omega \rangle = \sum_{i=1}^{n-1} T_i(\varphi) T_n(T_i(\varphi)) + \psi' T_n(\psi') = \sum_{i,k=1}^{n-1} II(T_i, T_k) T_i(\varphi) T_k(\varphi) + \psi' \psi''$$

and $\langle \nabla_{T_n} \nabla \omega, T_n \rangle = \psi''$.

Proof of iii**):** Note that $\nabla \varphi^T$ - the projection of $\nabla \varphi$ on hypersurfaces

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parallel to $\partial\Omega$ - is $\nabla\varphi$ since φ is independent of ρ. Furthermore $\nabla_{T_i}\psi = 0, i = 1, \cdots n - 1$; hence we have

$$\frac{1}{2}(\nabla\varphi)^T(|\nabla\omega|^2) = \sum_{i=1}^{n-1} T_i(\varphi)\nabla_{\nabla\varphi^T}(T_i(\varphi)) = \sum_{i,j=1}^{n-1} T_i\left(T_j\left(\varphi\right)\right) T_j\left(\varphi\right) T_i\left(\varphi\right).$$

Proof of iv): Using (5) we have

$$\sum_{i=1}^{n-1} \langle\nabla_{T_i}\nabla\omega, T_i\rangle = \sum_{i=1}^{n-1} T_i(T_i(\varphi)) - (n-1)\psi'H + \sum_{i=1}^{n-1} T_i\left(\varphi\right)\mathrm{div}_{\Omega(\rho)}(T_i).$$

■

2.1.1 Barriers for the Dirichlet problem on Ω for the minimal surface equation

Lemma 4 *Let H and A respectively the mean curvature and the shape operator of $\partial\Omega$ w.r.t. to the inner orientation and set $H_{\inf} := \inf_{\partial\Omega} H$. Let R be an upperbound of the Ricci curvature of M. The function $\omega(z) := \varphi(z) + a\ln(1 + bt)$ is superharmonic w.r.t. $\mathcal{M}$ on U_ε, where:*
i) if $H_{\inf} < 0$, $a = b^{-1}$, $0 < \varepsilon < \min\{\frac{1}{2b}, \rho_0\}$ being the constant b given by

$$\begin{aligned} b/3 &= ||D\varphi||_{\partial\Omega}^2\left(||D^2\varphi||_{\partial\Omega} + ||A||_{\partial\Omega}\right) \\ &+ (2 + ||D\varphi||_{\partial\Omega}^2)(||D^2\varphi||_{\partial\Omega} + (n-1)^2\rho_0\,||D\varphi||_{\partial\Omega}\,R - (n-1)H_{\inf}); \end{aligned} \quad (9)$$

ii) if $H_{\inf} \geq 0$, $b > a^{-1}$, $0 < \varepsilon < \min\{a - b^{-1}, \rho_0\}$, being the constant a given by

$$\begin{aligned} a^{-1} &= ||D\varphi||_{\partial\Omega}^2\left(||D^2\varphi||_{\partial\Omega} + ||A||_{\partial\Omega}\right) \\ &+ (2 + ||D\varphi||_{\partial\Omega}^2)(||D^2\varphi||_{\partial\Omega} + (n-1)^2\rho_0\,||D\varphi||_{\partial\Omega}\,R). \end{aligned}$$

Proof. Let us introduce the following notations for terms containing only φ and its derivatives:

$$\alpha := \sum_{i=1}^{n-1} [T_i\left(\varphi\right)]^2, \beta = \sum_{i=1}^{n-1} T_i\left(T_i\left(\varphi\right)\right), \mu := \sum_{j=1}^{n-1} T_j\left(\varphi\right)\mathrm{div}\,T_j$$

$$\lambda := \sum_{i,j=1}^{n-1} T_i\left(T_j\left(\varphi\right)\right) T_j\left(\varphi\right) T_i\left(\varphi\right), \theta := \sum_{i,k=1}^{n-1} II(T_i, T_k)T_i(\varphi)T_k\left(\varphi\right).$$

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From Lemma 3, we plug in inequality (4) the preceeding terms and obtain a differential inequality for ψ:

$$-\lambda + \left(1 + \alpha + [\psi']^2\right)(\sigma - \psi'(n-1)H) - (\theta + \psi'\psi')\psi' + \left(1 + \alpha + [\psi']^2\right)\psi'' \leq 0.$$

where $\sigma := \beta + \mu$.
Set

$$H_{\mathrm{inf}} := \inf_{\partial\Omega} H|_{\partial\Omega} \tag{10}$$

and define $\psi(t) = a\ln(1 + bt)$, where $a > 0$ and $b > 0$ are constant to be determined. Since $\psi' > 0$, we can replace the function H in the inequality (2.1.1) by H_{inf} :

$$-\lambda + (1 + \alpha)\sigma + \sigma[\psi']^2 - \theta\psi' + (1 + \alpha)\psi'' \quad +$$
$$- (n-1)H_{\mathrm{inf}}[\psi']^3 - (n-1)(1+\alpha)H_{\mathrm{inf}}\psi' \quad \leq 0.$$

Setting $\delta := -\lambda + \sigma(1 + \alpha)$, $c := (n-1)(1+\alpha) > 0$, then last inequality becomes

$$\delta + \sigma[\psi']^2 - \theta\psi' + (1 + \alpha)\psi'' \quad +$$
$$- (n-1)H_{\mathrm{inf}}[\psi']^3 - cH_{\mathrm{inf}}\psi' \quad \leq 0. \tag{11}$$

We first suppose $H_{\mathrm{inf}} < 0$. Set $ab = 1$; then replacing $\psi' = (1 + bt)^{-1}$ and $\psi'' = -[\psi']^2 b$,

$$\delta(1 + bt)^3 + \sigma(1 + bt) - \theta(1 + bt)^2 - (1 + \alpha)b(1 + bt)$$
$$- (n-1)H_{\mathrm{inf}} - cH_{\mathrm{inf}}(1 + bt)^2 \leq 0.$$

Taking absolute values, dividing by $1 + bt$ and expanding w.r.t. t, (2.1.1) is true if

$$|\delta|b^2 t^2 + (2|\delta| + |\theta| - cH_{\mathrm{inf}})bt + |\delta| + |\theta| + |\sigma| - b(\alpha + 1) \quad +$$
$$-(c + n - 1)H_{\mathrm{inf}} \quad \leq 0 \tag{12}$$

It is clear that a sufficient condition for inequality (12) is that :

$$\begin{cases} t \leq \frac{1}{\sqrt{3|\delta|b}} \\ t \leq \frac{1}{3[2|\delta| + |\theta| - cH_{\mathrm{inf}}]} \\ b/3 \geq |\delta| + |\sigma| + |\theta| - (c + n - 1)H_{\mathrm{inf}} \end{cases}. \tag{13}$$

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Notice that these inequalities are *a fortiori* satisfied if we replace in these expressions the functions α, $|\beta|$, $|\mu|$, $|\lambda|$ and $|\theta|$ by their supremum on $\partial\Omega$. We obtain

$$\alpha := ||D\varphi||^2, \beta = ||D^2\varphi||, \ \lambda := ||D^2\varphi||||D\varphi||^2, \ \theta := ||A||||D\varphi||^2.$$

To estimate μ we need to bound $\text{div}(T_j)$. We derivate the equation $\nabla_{T_n} T_j = 0$ with respect to $T_i, i = 1, \cdots, n-1$. We obtain an evolution equation for $\text{div }T_j$ along the normal geodesic : $\nabla_{T_n} \text{div}(T_j) = (n-1)\text{Ric}(T_n, T_j)$, with the initial condition $\text{div}(T_j(p)) = 0$. This yields

$$\mu := (n-1)^2||D\varphi||\rho_0 \sup_{y \in V_{r_0} \times [0,\rho_0]} \text{Ric}(y).$$

Let us fix b such that third inequality in (13) is an equality. We then obtain expression of b in Lemma 4. Replacing b by its expression (9) the first two inequalities of (13) hold if

$$t \leq t_0 := \min\left(\frac{1}{2b}, \rho_0\right).$$

This conclude the proof of i).

Now, suppose $H_{\inf} \geq 0$. In this case, inequality (11) is satisfied if $\delta + \sigma(\psi')^2 - \theta\psi' + (1 + \alpha)\psi'' \leq 0$ which, after replacing $\psi' = ab(1 + bt)^{-1}$ and $\psi'' = -[\psi']^2 a^{-1}$, become

$$\delta(1 + bt)^2 + \sigma a^2 b^2 - \theta ab(1 + bt) - (1 + \alpha)ab^2 \leq 0. \tag{14}$$

Since $\alpha \geq 0$, (14) is satisfied if

$$|\delta|(1 + bt)^2 + |\sigma|a^2 b^2 + |\theta|ab(1 + bt) \leq ab^2. \tag{15}$$

Notice that for $1 + bt \leq ab$, equation (15) is true if we replace $1 + bt$ by ab, obtaining

$$a(|\delta| + |\sigma| + |\theta|) \leq 1.$$

Fix $a = 1/(|\delta| + |\sigma| + |\theta|)$ where we already assume for $|\delta|, |\sigma|$ and $|\theta|$ their supremum on $\partial\Omega$. Then, for all $b > 1/a$ and $t \leq \min\{a - 1/b, \rho_0\}$ we have (15) satisfied and this conclude the proof of ii). ∎

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2.2 Proof of Theorem 1.

Let

$$\mathcal{C} = \frac{1}{b}\ln(1 + b\varepsilon), \tag{16}$$

where b and ε are defined in i) of Lemma 4, if $H_{\inf} < 0$ ($\mathcal{C} = +\infty$ if $H_{\inf} \geq 0$ - according with ii) of Lemma 4). For the first part where $\varphi \in C^{2,\alpha}(\partial\Omega)$ one uses the continuity method by setting

$$V = \left\{t \in [0,1] \mid \exists u_t \in C^{2,\alpha}(\overline{\Omega}) \text{ such that } \mathcal{M}[u_t] = 0,\ u_t|_{\partial\Omega} = t\varphi\right\}.$$

Clearly $V \neq \varnothing$ since $t = 0 \in V$. Moreover, V is open by the implicit function theorem.

Let $t_n \in V$ be a sequence converging to $t \in [0,1]$ and $u_n := u_{t_n} \in C^{2,\alpha}(\overline{\Omega})$ be the solutions such that $u_n|_{\partial\Omega} = t_n\varphi$. By the maximum principle the sequence u_n has uniformly bounded C^0 norm. Moreover, $\varphi - \psi|_{\partial\Omega} \leq u_n|_{\partial\Omega} \leq \omega|_{\partial\Omega}$ where the functions φ and ψ are defined in equation (2) and Lemma 4. It follows that

$$\max_{\partial\Omega} |\mathrm{grad}\, u_n| \leq \max\left\{\max_{\partial\Omega}|\mathrm{grad}\,\sigma|, \max_{\partial\Omega}|\mathrm{grad}\,\omega|\right\} < \infty.$$

By Section 5 of [2] there is $C > 0$ such that $\max_\Omega |\mathrm{grad}\, u_n| \leq C$ so that $|u_n|_1 \leq D < \infty$ with D not depending on n. Hölder estimates and PDE linear elliptic theory ([4]) guarantees that u_n is equicontinous in the $C^{2,\beta}$ norm for some $\beta > 0$ and hence contains a subsequence converging uniformly on the C^2 norm to a solution $u \in C^2(\overline{\Omega})$. Regularity theory of linear elliptic PDE ([4]) implies that $u \in C^{2,\alpha}(\overline{\Omega})$. This proves the first part of the theorem.

Assume now that Ω is mean convex and let $\varphi \in C^0(\partial\Omega)$ be given. Let $\varphi_n^\pm \in C^{2,\alpha}(\partial\Omega)$ be a monotonic sequence of functions converging from above and from below to φ in the C^0 norm. It follows by what we have proved above the existence of solutions $u_n^\pm \in C^{2,\alpha}(\overline{\Omega})$ of $\mathcal{M} = 0$ in Ω such that $u_n^\pm|_{\partial\Omega} = \psi_n^\pm$. The sequence $u_n^\pm$ is uniformly bounded in the C^0 norm by the maximum principle. Therefore, by Theorem 1.1 of [13] and linear elliptic PDE theory the sequence $u_n^\pm$ contains a subsequence $v_n \in C^{2,\alpha}(\overline{\Omega})$ converging uniformly on the C^2 norm on compacts subsets of Ω to a solution $u \in C^2(\Omega)$ of $\mathcal{M} = 0$. Since

$$\varphi_n^- \leq u_n^- \leq u_n^+ \leq \varphi_n^+$$

and $\varphi_n^\pm$ converges to φ it follows by the maximum principle that u extends continously to $\overline{\Omega}$ and $u|_{\partial\Omega} = \varphi$. This concludes the proof of Theorem 1.

3 The exterior Dirichlet problem for the minimal hypersurface PDE with prescribed asymptotic boundary

For the proof of Theorem 2 we shall make use of the following definition given in [12]. We first recall that a function $\Sigma \in C^0(M)$ is a supersolution for $\mathcal{M}$ if, given a bounded domain $U \subset M$ and if $u \in C^0\left(\overline{U}\right)$ is a solution of $\mathcal{M} = 0$ in U, then $u|_{\partial U} \leq \Sigma|_{\partial U}$ implies that $u \leq \Sigma|_U$.

Given $x \in \partial_\infty M$ and an open subset $\Omega \subset M$ such that $x \in \partial_\infty \Omega$, an *upper barrier for $\mathcal{M}$ relative to x and Ω with height C* is a function $\Sigma \in C^0(M)$ such that

(i) Σ is a supersolution for $\mathcal{M}$;

(ii) $\Sigma \geq 0$ and $\lim_{p \in M, p \to x} \Sigma(p) = 0$, w.r.t. the cone topology and according to [1];

(iii) $\Sigma_{M \setminus \Omega} \geq C$.

Similarly, we define subsolutions and lower barriers.

We say that M is *regular at infinity with respect to $\mathcal{M}$* if, given $C > 0$, $x \in \partial_\infty M$ and an open subset $W \subset \partial_\infty M$ with $x \in W$, there exist an open set $\Omega \subset M$ such that $x \in \text{Int}\, \partial_\infty \Omega \subset W$ and upper and lower barriers $\Sigma, \sigma : M \to \mathbb{R}$ relatives to x and Ω, with height C.

3.1 Proof of Theorem 2

Consider a continuous extension Ψ of $\psi \in C^0(\partial_\infty M)$ which is C^∞ in M. That is, $\Psi \in C^\infty(M) \cap C^0\left(\overline{M}\right)$, $M := M \cup \partial_\infty M$ and $\Psi|_{\partial_\infty M} = \psi$. Let $o \in M$ be a fixed point in M and let $N > 0$ be such that the open geodesic ball B_n centered at o with radius n contains the boundary $\partial \Omega$ of the exterior domain Ω ($n \in \mathbb{N}$ and $n \geq N$). Set $\Omega_n = \Omega \cap B_n$. It follows from the Hessian comparison theorem that $\partial \Omega_n \setminus \partial \Omega$ is convex, in particular mean convex. From this fact and from the hypothesis on the oscillation of φ, it follows from Theorem 1 the existence of a solution $u_n \in C^{2,\alpha}\left(\overline{\Omega}_n\right)$ of $\mathcal{M} = 0$ in Ω_n such that $u_n|_{\partial \Omega} = \varphi$ and $u_n|_{\partial \Omega_n \setminus \partial \Omega} = \Psi|_{\partial \Omega_n \setminus \partial \Omega}$.

By the maximum principle u_n is uniformly bounded. It follows from Theorem 1.1 of [13] and the diagonal method that u_n contains a subsequence

converging uniformly on the C^2 norm on compact subsets of Ω to a solution $u \in C^\infty(\Omega)$ of $\mathcal{M} = 0$. By Lemma **??** and regularity theory $u \in C^{2,\alpha}\left(\overline{\Omega}\right)$ and $u|_{\partial\Omega} = \varphi$.

Since M satisfies the SC condition- given in the Introduction- it follows from Theorem 10 of [12] that M is regular at infinity with respect to the minimal hypersurface PDE. From Theorem 4 of [12] it follows that u extends continuously to $\partial_\infty M$ and satisfies the boundary condition $u|_{\partial_\infty M} = \psi$. This concludes the proof of Theorem 2.

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