Keller-Osserman estimates for some quasilinear elliptic systems

Marie-Françoise Bidaut-Véron, Marta Garcia-Huidobro, Cecilia Yarur

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Marie-Françoise BIDAUT-VERON* Marta GARCÍA-HUIDOBRO† Cecilia YARUR‡

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Abstract

In this article we study quasilinear systems of two types, in a domain Ω of $\mathbb{R}^N$: with absorption terms, or mixed terms:

$$(A)\begin{cases} \mathcal{A}_p u = v^\delta, \\ \mathcal{A}_q v = u^\mu, \end{cases} \qquad (M)\begin{cases} \mathcal{A}_p u = v^\delta, \\ -\mathcal{A}_q v = u^\mu, \end{cases}$$

where $\delta, \mu > 0$ and $1 < p, q < N$, and $D = \delta\mu - (p-1)(q-1) > 0$; the model case is $\mathcal{A}_p = \Delta_p, \mathcal{A}_q = \Delta_q$. Despite of the lack of comparison principle, we prove a priori estimates of Keller-Osserman type:

$$u(x) \leq Cd(x, \partial\Omega)^{-\frac{p(q-1)+q\delta}{D}}, \qquad v(x) \leq Cd(x, \partial\Omega)^{-\frac{q(p-1)+p\mu}{D}}.$$

Concerning system (M), we show that v always satisfies Harnack inequality. In the case $\Omega = B(0,1)\backslash\{0\}$, we also study the behaviour near 0 of the solutions of more general weighted systems, giving a priori estimates and removability results. Finally we prove the sharpness of the results.

Keywords. Quasilinear elliptic systems, a priori estimates, large solutions, asymptotic behaviour, Harnack inequality.

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*Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Faculté des SCiences, 37200 Tours France. E-mail address: veronmf@univ-tours.fr

†Departamento de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago de Chile. E-mail address: mgarcia@mat.puc.cl

‡Departamento de Matemática y C.C., Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago de Chile. E-mail address: cecilia.yarur@usach.cl

1 Introduction

In this article we study the nonnegative solutions of quasilinear systems in a domain Ω of $\mathbb{R}^N$, either with absorption terms, or mixed terms, that is,

$$(A) \left\{ \begin{array}{l} \mathcal{A}_p u = v^\delta, \\ \mathcal{A}_q v = u^\mu, \end{array} \right. \qquad (M) \left\{ \begin{array}{l} \mathcal{A}_p u = v^\delta, \\ -\mathcal{A}_q v = u^\mu, \end{array} \right. \tag{1.1}$$

where

$$\delta, \mu > 0 \quad \text{and} \quad 1 < p, q < N.$$

The operators are given in divergence form by

$$\mathcal{A}_p u := div\left[\mathrm{A}_p(x, u, \nabla u)\right], \qquad \mathcal{A}_q v := div\left[\mathrm{A}_q(x, v, \nabla v)\right],$$

where A_p and A_q are Carathéodory functions. In our main results, we suppose that $\mathcal{A}_p$ *is S-p-C (strongly-p-coercive)*, that means (see [8])

$$\mathrm{A}_p(x, u, \eta).\eta \geq K_{1,p} |\eta|^p \geq K_{2,p} |\mathrm{A}_p(x, u, \eta)|^{p'}, \qquad \forall (x, u, \eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N.$$

for some $K_{1,p}, K_{2,p} > 0$, and similarly for $\mathcal{A}_q$. The model type for $\mathcal{A}_p$ is the p-Laplace operator

$$u \longmapsto \Delta_p u = div(|\nabla u|^{p-2} \nabla u).$$

We prove *a priori estimates of Keller-Osserman type* for such operators, under a natural condition of "superlinearity":

$$D = \delta\mu - (p-1)(q-1) > 0, \tag{1.2}$$

and we deduce Liouville type results of nonexistence of entire solutions. We also study the behaviour near 0 of nonnegative solutions of possibly weighted systems of the form

$$(A_w) \left\{ \begin{array}{l} \mathcal{A}_p u = |x|^a v^\delta, \\ \mathcal{A}_q v = |x|^b u^\mu, \end{array} \right. \qquad (M_w) \left\{ \begin{array}{l} \mathcal{A}_p u = |x|^a v^\delta, \\ -\mathcal{A}_q v = |x|^b u^\mu, \end{array} \right.$$

in $\Omega \backslash \{0\}$, where

$$a, b \in \mathbb{R}, \qquad a > -p, \quad b > -q.$$

In particular we discuss about the *Harnack inequality* for u or v.

Recall some classical results in the scalar case. For the model equation with an absorption term

$$\Delta_p u = u^Q, \tag{1.3}$$

in Ω, with $Q > p-1$, the first estimate was obtained by Keller [19] and Osserman [24] for $p = 2$, and extended to the case $p \neq 2$ in [29]: any nonnegative solution $u \in C^2(\Omega)$ satisfies

$$u(x) \leq C d(x, \partial\Omega)^{-p/(Q-p+1)}, \tag{1.4}$$

where $d(x, \partial\Omega)$ is the distance to the boundary, and $C = C(N, p, Q)$. For the equation with a source term

$$-\Delta_p u = u^Q,$$

up to now estimate (1.4), valid for any $Q > p-1$ in the radial case, has been obtained only for $Q < Q^*$, where $Q^* = \frac{N(p-1)+p}{N-p}$ is the Sobolev exponent, with difficult proofs, see [18], [9] in the case $p = 2$ and [27] in the general case $p > 1$. For $p = 2$, the estimate, with a universal constant, is not true for $Q = \frac{N+1}{N-3}$, and the problem is open between Q^* and $\frac{N+1}{N-3}$.

Up to our knowledge all the known estimates for systems are related with systems for which some comparison properties hold, of competitive type, see [16], or of cooperative type, see [11]; or with quasilinear operators in [17], [32]. Problems (A) and (M) have been the object of very few works because such properties do not hold. The main ones concern systems (A_w) and (M_w) in the linear case $p = q = 2$, see [5] and [6]; the proofs rely on the inequalities satisfied by the mean values $\overline{u}$ and $\overline{v}$ on spheres of radius r, they cannot be extended to the quasilinear case. A radial study of system (A) was introduced in [15], and recently in [7].

The problem with two source terms

$$(S)\begin{cases} -\mathcal{A}_p u = |x|^a v^\delta, \\ -\mathcal{A}_q v = |x|^b u^\mu, \end{cases}$$

was analyzed in [8]. The results are based on integral estimates, still valid under weaker assumptions: from [8], $\mathcal{A}_p$ is called *W-p-C (weakly-p-coercive)* if

$$\mathrm{A}_p(x,u,\eta).\eta \geq K_p \ |\mathrm{A}_p(x,u,\eta)|^{p'}, \qquad \forall (x,u,\eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \tag{1.5}$$

for some $K_p > 0$; similarly for $\mathcal{A}_q$. When $\delta, \mu < Q_1$, where $Q_1 = \frac{N(p-1)}{N-p}$, punctual estimates were deduced for S-p-C, S-q-C operators and it was shown that u and v satisfy the Harnack inequality.

In **Section 2**, we give our main tools for obtaining a priori estimates. First we show that the technique of integral estimates if fundamental, and can be used also for systems (A) and (M). In Proposition 2.1 we consider both equations with absorption or source terms

$$-\mathcal{A}_p u + f = 0, \quad \text{or} \quad -\mathcal{A}_p u = f, \tag{1.6}$$

in a domain Ω, where $f \in L^1_{loc}(\Omega)$, $f \geq 0$, and obtain local integral estimates of f with respect to u in a ball $B(x_0,\rho)$. When $\mathcal{A}_p$ is S-p-C, they imply minorizations by the Wölf potential of f in the ball

$$W^f_{1,p}(B(x_0,\rho)) = \int_0^\rho \left(t^p \fint_{B(x_0,t)} f \right)^{\frac{1}{p-1}} \frac{dt}{t}, \tag{1.7}$$

extending the first results of [20], [21]. The second tool is the well known weak Harnack inequalities for solutions of (1.6) in case of S-p-C operators, and a more general version in case of equation with absorption, which appears to be very useful. The third one is a boostrap argument given in [5] which remains essential.

In **Section 3** we study both systems (A) and (M). When $\mathcal{A}_p = \Delta_p$ and $\mathcal{A}_q = \Delta_q$, they admit particular radial solutions

$$u^*(x) = A^* \, |x|^{-\gamma} \,, v^*(r) = B^* \, |x|^{-\xi} \,,$$

where

$$\gamma = \frac{p(q-1)+q\delta}{D}, \qquad \xi = \frac{q(p-1)+p\mu}{D}, \tag{1.8}$$

whenever

$$\gamma > \frac{N-p}{p-1} \quad \text{and} \quad \xi > \frac{N-q}{q-1} \qquad \text{for system } (A),$$
$$\gamma > \frac{N-p}{p-1} \quad \text{and} \quad \xi < \frac{N-q}{q-1} \qquad \text{for system } (M).$$

Our main result for the system with absorption term (A) extends precisely the Osserman-Keller estimate of the scalar case (1.3):

Theorem 1.1 *Assume that*

$$\mathcal{A}_p \text{ is S-}p\text{-C}, \quad \mathcal{A}_q \text{ is S-}q\text{-C}, \tag{1.9}$$

and (1.2) holds. Let $u \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$, $v \in W^{1,q}_{loc}(\Omega) \cap C(\Omega)$ *be nonnegative solutions of*

$$\begin{cases} -\mathcal{A}_p u + v^\delta \leq 0, \\ -\mathcal{A}_q v + u^\mu \leq 0, \end{cases} \qquad \text{in } \Omega.$$

Then for any $x \in \Omega$

$$u(x) \leq Cd(x,\partial\Omega)^{-\gamma}, \qquad v(x) \leq Cd(x,\partial\Omega)^{-\xi}, \tag{1.10}$$

with $C = C(N,p,q,\delta,\mu,K_{1,p},K_{2,p},K_{1,q},K_{2,q})$.

Our second result shows that the mixed system (M) also satisfies the Osserman-Keller estimate, *without any restriction on* δ *and* μ, and moreover the second function v *always satisfies Harnack inequality:*

Theorem 1.2 *Assume (1.2),(1.9). Let* $u \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$, $v \in W^{1,q}_{loc}(\Omega) \cap C(\Omega)$ *be nonnegative solutions of*

$$\begin{cases} -\mathcal{A}_p u + v^\delta \leq 0, \\ -\mathcal{A}_q v \geqq u^\mu, \end{cases} \qquad \text{in } \Omega.$$

Then (1.10) still holds for any $x \in \Omega$.

Moreover, if u, v *are any nonnegative solution of system* (M), *then* v *satisfies Harnack inequality in* Ω, *and there exists another* $C > 0$ *as above, such that the punctual inequality holds*

$$u^\mu(x) \leq Cv^{q-1}(x)d(x,\partial\Omega)^{-q}. \tag{1.11}$$

Notice that the results are new even for $p = q = 2$. As a consequence we deduce Liouville properties:

Corollary 1.3 *Assume (1.2),(1.9). Then there exist no entire nonnegative solutions of systems* (A) *or* (M).

Section 4 concerns the behaviour near 0 of systems with possible weights (A_w) and (M_w), where γ, ξ are replaced by

$$\gamma_{a,b} = \frac{(p+a)(q-1)+(q+b)\delta}{D}, \qquad \xi_{a,b} = \frac{(q+b)(p-1)+(p+a)\mu}{D}, \tag{1.12}$$

in other terms $\delta\xi_{a,b} = (p-1)\gamma_{a,b} + p + a$, $\mu\gamma_{a,b} = (q-1)\xi_{a,b} + q + b$. We set $B_r = B(0,r)$ and $B'_r = B_r \backslash \{0\}$ for any $r > 0$. Our results extend and simplify the results of [5], [6] in a significant way:

Theorem 1.4 *Assume (1.2),(1.9). Let* $u \in W^{1,p}_{loc}(B'_1) \cap C(B'_1)$, $v \in W^{1,q}_{loc}(B'_1) \cap C(B'_1)$ *be nonnegative solutions of*

$$\begin{cases} -\mathcal{A}_p u + |x|^a v^\delta \leq 0, \\ -\mathcal{A}_q v + |x|^b u^\mu \leq 0, \end{cases} \quad \text{in } B'_1. \tag{1.13}$$

Then there exists $C = C(N,p,q,a,b,\delta,\mu,K_{1,p},K_{2,p},K_{1,q},K_{2,q}) > 0$ *such that*

$$u(x) \leq C\,|x|^{-\gamma_{a,b}}, \qquad v(x) \leq C\,|x|^{-\xi_{a,b}} \qquad \text{in } B'_{\frac{1}{2}}. \tag{1.14}$$

Theorem 1.5 *Assume (1.2),(1.9). Let* $u \in W^{1,p}_{loc}(B'_1) \cap C(B'_1)$, $v \in W^{1,q}_{loc}(B'_1) \cap C(B'_1)$ *be nonnegative solutions of*

$$\begin{cases} -\mathcal{A}_p u + |x|^a v^\delta \leq 0, \\ -\mathcal{A}_q v \geq |x|^b u^\mu, \end{cases} \quad \text{in } B'_1. \tag{1.15}$$

in B'_1. *Then there exists* $C > 0$ *as in theorem 1.4 such that*

$$u(x) \leq C\,|x|^{-\gamma_{a,b}}, \qquad v(x) \leq C\min(|x|^{-\xi_{a,b}}, |x|^{-\frac{N-q}{q-1}}), \qquad \text{in } B'_{\frac{1}{2}}. \tag{1.16}$$

Moreover if (u,v) *is any nonnegative solution of* (M_w), *then* v *satisfies Harnack inequality in* $B'_{\frac{1}{2}}$, *and there exist another* $C > 0$ *as above, such that*

$$|x|^{b+q} u^\mu(x) \leq C v^{q-1}(x), \qquad \text{in } B'_{\frac{1}{2}}. \tag{1.17}$$

Moreover we give removability results for the two systems (A_w) and (M_w), see Theorems 4.1, 4.2, whenever $\mathcal{A}_p$ and $\mathcal{A}_q$ satisfy monotonicity and homogeneity properties, extending to the quasilinear case [5, Corollary 1.2] and [6, Theorem 1.1].

In **Section 5** we show that our results on Harnack inequality are optimal, even in the radial case. And we prove the sharpness of the removability conditions.

2 Main tools

For any $x \in \mathbb{R}^N$ and $r > 0$, we set $B(x,r) = \left\{y \in \mathbb{R}^N / \ |y-x| < r\right\}$ and $B_r = B(0,r)$.
For any function $w \in L^1(\Omega)$, and for any weight function $\varphi \in L^\infty(\Omega)$ such that $\varphi \geq 0$, $\varphi \neq 0$, we denote by

$$\oint_\varphi w = \frac{1}{\int_\Omega \varphi} \int_\Omega w\varphi$$

the mean value of w with respect to φ and by

$$\oint_{\Omega} w = \frac{1}{|\Omega|}\int_{\Omega} w = \oint_{1} w.$$

For any function $g \in L^1_{loc}(\Omega)$, we say that a function $u \in W^{1,p}_{loc}(\Omega)$ satisfies

$$-\mathcal{A}_p u \geqq g \qquad \text{in } \Omega, \qquad (\text{resp. } \leqq, \text{ resp. } =)$$

if $\mathrm{A}_p(x,u,\nabla u) \in L^{p'}_{loc}(\Omega)$ and

$$-\int_{\Omega} \mathrm{A}_p(x,u,\nabla u).\nabla\phi \geqq \int_{\Omega} g\phi, \qquad (\text{resp. } \leqq, \text{ resp. } =) \tag{2.1}$$

for any nonnegative $\phi \in W^{1,\infty}(\Omega)$ with compact support in Ω.

2.1 Integral estimates under weak conditions

Next we prove integral inequalities on the second member f of equations (1.6) in terms of the function u, for either with source or with absorption terms, obtained by multiplication by u^{α} with $\alpha < 0$ for the source case, $\alpha > 0$ for the absorption case. The method is now classical, initiated by Serrin [26] and Trudinger [28], leading to Harnack inequalities for S-p-C operators. These estimates were developped for the p-Laplace operator in [20]. Under weak conditions on the operator, this technique of multiplication by u^{α} was used with specific f for obtaining Liouville results in [23]. It was developped for general f in [8, Proposition 2.1] where the notion of W-p-C operator was introduced. More recent Liouville results were given in [10, Theorem 2.1], and in [14] for the case of absorption terms.

Proposition 2.1 *Let $\mathcal{A}_p$ be W-p-C. Let $f \in L^1_{loc}(\Omega)$, $f \geq 0$ and let $u \in W^{1,p}_{loc}(\Omega)$ be any nonnegative solution of inequality*

$$-\mathcal{A}_p u \geqq f, \qquad \text{in } \Omega, \tag{2.2}$$

or of inequality

$$-\mathcal{A}_p u + f \leqq 0, \qquad \text{in } \Omega. \tag{2.3}$$

Let $\xi \in \mathcal{D}(\Omega)$, with values in $[0,1]$, and $\varphi = \xi^{\lambda}$, $\lambda > 0$, and $S_{\xi} = \mathrm{supp}|\nabla\xi|$.

Then for any $\ell > p-1$, there exists $\lambda(p,\ell)$ such that for $\lambda \geq \lambda(p,\ell)$, there exists $C = C(N,p,K_p,\ell,\lambda) > 0$ such that

$$\int_{\Omega} f\varphi \leqq C\,|S_{\xi}|\max_{\Omega}|\nabla\xi|^p \left(\oint_{S_{\xi}} u^{\ell}\varphi\right)^{\frac{p-1}{\ell}}. \tag{2.4}$$

Proof. (i) First assume that $\ell > p-1+\alpha$, with $\alpha \in (1-p,0)$ in case of equation (2.2), $\alpha \in (0,1)$ (any $\alpha > 0$ if $u \in L^{\infty}_{loc}(\Omega)$) in case of equation (2.3). We claim that there exists $\lambda(p,\alpha,\ell)$ such that for any $\lambda \geq \lambda(p,\alpha,\ell)$

$$\int_{\Omega} f u^{\alpha}\varphi \leqq C\,|S_{\xi}|\max_{\Omega}|\nabla\xi|^p \left(\oint_{S_{\xi}} u^{\ell}\varphi\right)^{\frac{p-1+\alpha}{\ell}}, \tag{2.5}$$

for some $C = C(N, p, K_p, \alpha, \ell, \lambda)$. For proving (2.5), one can assume that $u^\ell \in L^1(B(x_0, \rho))$. Let $\varphi = \xi^\lambda$, where $\lambda > 0$ will be chosen after. Let $\delta > 0, k \geq 1$, and (η_n) be a sequence of mollifiers; we set $u_\delta = u + \delta$, $u_{\delta,k} = \min(u, k) + \delta$ and approximate u by $u_{\delta,k,n} = u_{\delta,k} * \eta_n$, and we take $\phi = u^\alpha_{\delta,k,n}\varphi$ as a test function. Then in any case, from (1.5) and Hölder inequality,

$$\begin{aligned}
&|\alpha| \int_\Omega u^{\alpha-1}_{\delta,k,n} \varphi \mathrm{A}_p(x, u, \nabla u).\nabla u_{\delta,k,n} + \int_\Omega f u^\alpha_{\delta,k,n}\varphi \\
&\leq \lambda \int_{S_\xi} u^\alpha_{\delta,k,n} \xi^{\lambda-1} |\mathrm{A}_p(x, u, \nabla u)||\nabla\xi| \\
&\leq \lambda K_p^{-1/p'} \int_{S_\xi} u^\alpha_{\delta,k,n} \xi^{\lambda-1} (\mathrm{A}_p(x, u, \nabla u).\nabla u)^{1/p'} |\nabla\xi| \\
&\leq \lambda K_p^{-1/p'} \left(\int_{S_\xi} u^{\alpha-1}_{\delta,k,n} \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u \right)^{\frac{1}{p'}} \left(\int_{S_\xi} u^{\alpha+p-1}_{\delta,k,n} \xi^{\lambda-p} |\nabla\xi|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Otherwise $(\nabla u_{\delta,k,n})$ tends to $\chi_{\{u\leq k\}}\nabla u$ in $L^p_{loc}(\Omega)$, and up to subsequence *a.e.* in Ω, and $\mathrm{A}_p(x, u, \nabla u) \in L^{p'}_{loc}(\Omega)$. By letting $n \to \infty$, we obtain

$$\begin{aligned}
&|\alpha| \int_{\{u\leq k\}} u^{\alpha-1}_{\delta,k} \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u + \int_\Omega f u^\alpha_{\delta,k}\xi^\lambda \\
&\leq \lambda K_p^{-1/p'} \left(\int_{S_\xi} u^{\alpha-1}_{\delta,k} \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u \right)^{\frac{1}{p'}} \left(\int_{S_\xi} u^{\alpha+p-1}_{\delta,k} \xi^{\lambda-p} |\nabla\xi|^p \right)^{\frac{1}{p}} \\
&\leq \frac{|\alpha|}{2} \int_{S_\xi} u^{\alpha-1}_{\delta,k} \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u + C \int_{S_\xi} u^{\alpha+p-1}_{\delta,k} \xi^{\lambda-p} |\nabla\xi|^p,
\end{aligned}$$

with $C = C(\alpha, K_p, p, \lambda)$; otherwise, for $\alpha < 1$ (or $u \in L^\infty_{loc}(\Omega)$ and taking $k \geq \sup_{S_\xi} u$)

$$\begin{aligned}
\int_\Omega u^{\alpha-1}_{\delta,k} \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u &= \int_{\{u\leq k\}} u^{\alpha-1}_{\delta,k} \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u + \int_{\{u>k\}} u^{\alpha-1}_{\delta,k} \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u \\
&\leq \int_{\{u\leq k\}} u^{\alpha-1}_{\delta,k} \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u + Mk^{\alpha-1}
\end{aligned}$$

where $M = \int_\Omega \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u$ (or $M = 0$) is independent of k and δ. Then, for any $\theta > 1$,

$$\begin{aligned}
&\frac{|\alpha|}{2} \int_{\{u\leq k\}} u^{\alpha-1}_{\delta,k} \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u + \int_\Omega f u^\alpha_{\delta,k}\xi^\lambda \leq C \int_{S_\xi} u^{\alpha+p-1}_{\delta,k} \xi^{\lambda-p} |\nabla\xi|^p + M\,|\alpha|\,k^{\alpha-1} \\
&\leq C \left(\int_{S_\xi} u^{(\alpha+p-1)\theta}_{\delta,k} \xi^\lambda \right)^{\frac{1}{\theta}} \left(\int_{S_\xi} \xi^{\lambda-p\theta'} |\nabla\xi|^{p\theta'} \right)^{\frac{1}{\theta'}} + M\,|\alpha|\,k^{\alpha-1}.
\end{aligned}$$

Choosing $\theta = \ell/(\alpha + p - 1) > 1$, and $\lambda \geq \lambda(p, \alpha, \ell) = p\theta'$, we find

$$\begin{aligned}
&\frac{|\alpha|}{2} \int_{\{u \leq k\}} u_{\delta,k}^{\alpha-1} \xi^\lambda \mathrm{A}_p(x, u, \nabla u).\nabla u + \int_\Omega f u_{\delta,k}^\alpha \xi^\lambda \\
&\leq C \left(\int_{S_\xi} u_{\delta,k}^\ell \varphi \right)^{\frac{\alpha+p-1}{\ell}} \left(\int_{S_\xi} |\nabla \xi|^{p\theta'} \right)^{\frac{1}{\theta'}} + M\,|\alpha|\,k^{\alpha-1} \\
&\leq C\,|S_\xi|^{\frac{1}{\theta'}} \max_\Omega |\nabla \xi|^p \left(\int_{S_\xi} u_\delta^\ell \varphi \right)^{\frac{\alpha+p-1}{\ell}} + M\,|\alpha|\,k^{\alpha-1},
\end{aligned}$$

with a new constant $C = C(N, p, K, \alpha, \ell)$. As $k \to \infty$, we deduce

$$\frac{|\alpha|}{2} \int_\Omega u_\delta^{\alpha-1} \varphi \mathrm{A}_p(x, u, \nabla u).\nabla u + \int_\Omega f u_\delta^\alpha \varphi \leq C\,|S_\xi|^{\frac{1}{\theta'}} \max_\Omega |\nabla \xi|^p \left(\int_{S_\xi} u_\delta^\ell \varphi \right)^{\frac{\alpha+p-1}{\ell}}. \tag{2.6}$$

Finally as $\delta \to 0$ we get (2.5) with a new constant C. Moreover we deduce an estimate of the gradient terms:

$$\frac{|\alpha|}{2} \int_\Omega u^{\alpha-1} \varphi \mathrm{A}_p(x, u, \nabla u).\nabla u \leq C\,|S_\xi|^{\frac{1}{\theta'}} \max_\Omega |\nabla \xi|^p \left(\int_\Omega u^\ell \varphi \right)^{\frac{\alpha+p-1}{\ell}}. \tag{2.7}$$

(ii) Next we only assume that $\ell > p - 1$, $u^\ell \in L^1(B(x_0, \rho))$. Let φ as above, and fix some $\alpha = \alpha(p, \ell)$ such that $\alpha \in (1 - p, 0)$ and $(1 - \alpha)(p - 1) < \ell$ for (2.2), $\alpha \in (0, 1)$ and $\alpha + p - 1 < \ell$ for (2.3). In any case $\tau = \ell/(1 - \alpha)(p - 1) > 1$, and $1/\theta p' + 1/p\tau = (p - 1)/\ell$. Let $\lambda \geq \lambda(p, \alpha(p, \ell), \ell) \geq p\tau'$. We take φ as a test function and from (2.6) we deduce successively, with new constants C,

$$\begin{aligned}
\int_\Omega f\varphi &\leq \lambda \int_\Omega \xi^{\lambda-1} |\mathrm{A}_p(x, u, \nabla u)|\,|\nabla \xi| \leq C \int_\Omega \xi^{\lambda-1} |\mathrm{A}_p(x, u, \nabla u)|\,|\nabla \xi|\, u_\delta^{\frac{\alpha-1}{p'}} u_\delta^{\frac{1-\alpha}{p'}} \\
&\leq C \left(\int_{S_\xi} u_\delta^{\alpha-1} |\mathrm{A}_p(x, u, \nabla u)|^{p'} \varphi \right)^{\frac{1}{p'}} \left(\int_{S_\xi} u_\delta^{(1-\alpha)(p-1)} \xi^{\lambda-p} |\nabla \xi|^p \right)^{\frac{1}{p}} \\
&\leq C \left(\int_{S_\xi} u_\delta^{\alpha-1} \varphi \mathrm{A}_p(x, u, \nabla u).\nabla u \right)^{\frac{1}{p'}} \left(\int_{S_\xi} u_\delta^\ell \varphi \right)^{\frac{1}{p\tau}} \left(\int_{S_\xi} \xi^{\lambda - p\tau'} |\nabla \xi|^{p\tau'} \right)^{\frac{1}{p\tau'}} \\
&\leq C\,|S_\xi|^{\frac{1}{\theta' p'} + \frac{1}{p\tau'}} \max_\Omega |\nabla \xi|^p \left(\int_{S_\xi} u_\delta^\ell \varphi \right)^{\frac{1}{p'\theta} + \frac{1}{p\tau}} \\
&\leqq C\,|S_\xi|^{1 - \frac{p-1}{\ell}} \max_\Omega |\nabla \xi|^p \left(\int_{S_\xi} u_\delta^\ell \varphi \right)^{\frac{p-1}{\ell}};
\end{aligned}$$

and (2.4) follows as $\delta \to 0$. ■

Corollary 2.2 *Under the assumptions of Proposition 2.1, consider any ball* $B(x_0, 2\rho) \subset \Omega$, *and any* $\varepsilon \in \left(0, \frac{1}{2}\right]$. *Let* $\varphi = \xi^\lambda$ *with* ξ *such that*

$$\xi = 1 \text{ in } B(x_0, \rho), \qquad \xi = 0 \text{ in } \Omega \backslash \bar{B}(x_0, \rho(1+\varepsilon)) \qquad |\nabla \xi| \leq \frac{C_0}{\varepsilon \rho}. \tag{2.8}$$

Then for any $\ell > p-1$, *there exists* $\lambda(p, \ell) > 0$ *such that for* $\lambda \geq \lambda(p, \ell)$, *there exists* $C = C(N, p, K, \ell, \lambda) > 0$ *such that*

$$\oint_\varphi f \leq C(\varepsilon\rho)^{-p} \left(\oint_\varphi u^\ell \right)^{\frac{p-1}{\ell}}. \tag{2.9}$$

Remark 2.3 *If* $S_\xi = \cup_{i=1}^k S_\xi^i$ *where the* S_ξ^i *are* 2 *by* 2 *disjoint, then (2.4) can be replaced by*

$$\int_\Omega f\varphi \leqq C \sum_{i=1}^k \left| S_\xi^i \right| \max_{S_\xi^i} |\nabla \xi|^p \left(\oint_{S_\xi^i} u^\ell \right)^{\frac{p-1}{\ell}}. \tag{2.10}$$

2.2 Punctual estimates under strong conditions

When $\mathcal{A}_p$ is S-p-C, the estimate (2.7) of the gradient is the beginning of the proof of the well-known weak Harnack inequalities:

Theorem 2.4 ([25], [28]) *(i) Let* $\mathcal{A}_p$ *be S-p-C, and* $u \in W^{1,p}_{loc}(\Omega)$ *be nonnegative, such that*

$$-\mathcal{A}_p u \leqq 0 \qquad \text{in } \Omega;$$

then for any ball $B(x_0, 3\rho) \subset \Omega$, *and any* $\ell > p-1$,

$$\sup_{B(x_0,\rho)} u \leq C \left(\oint_{B(x_0,2\rho)} u^\ell \right)^{\frac{1}{\ell}}, \tag{2.11}$$

with $C = C(N, p, \ell, K_{1,p}, K_{2,p})$.
(ii) Let $w \in W^{1,p}_{loc}(\Omega)$ *be nonnegative, such that*

$$-\mathcal{A}_p w \geq 0 \qquad \text{in } \Omega;$$

then for any ball $B(x_0, 3\rho) \subset \Omega$, *for any* $\ell \in (0, N(p-1)/(N-p))$

$$\left(\oint_{B(x_0,2\rho)} v^\ell \right)^{\frac{1}{\ell}} \leq C \inf_{B(x_0,\rho)} v. \tag{2.12}$$

Next we give a more precise version of weak Harnack inequality (2.11). Such a kind of inequality was first established in the parabolic case in [12].

Lemma 2.5 *Let $\mathcal{A}_p$ be S-p-C, and $u \in W^{1,p}_{loc}(\Omega)$ be nonnegative, such that*

$$-\mathcal{A}_p u \leqq 0 \qquad in\ \Omega;$$

then for any $s > 0$, there exists a constant $C = C(N, p, s, K_{1,p}, K_{2,p})$, such that for any ball $B(x_0, 2\rho) \subset \Omega$ and any $\varepsilon \in \left(0, \frac{1}{2}\right]$,

$$\sup_{B(x_0,\rho)} u \leq C\varepsilon^{-\frac{Np^2}{s^2}} \left(\oint_{B(x_0,\rho(1+\varepsilon))} u^s \right)^{\frac{1}{s}}. \tag{2.13}$$

Proof. From a slight adaptation of the usual case where $\varepsilon = \frac{1}{2}$, for any $\ell > p-1$, there exists $C = C(N, \ell) > 0$ such that for any $\varepsilon \in \left(0, \frac{1}{2}\right)$,

$$\sup_{B(x_0,\rho)} u \leqq C\varepsilon^{-N} \left(\oint_{B(x_0,\rho(1+\varepsilon))} u^\ell \right)^{\frac{1}{\ell}}. \tag{2.14}$$

Thus we can assume $s \leq p-1$. We fix for example $\ell = p$, and define a sequence (ρ_n) by $\rho_0 = \rho$, and $\rho_n = \rho(1 + \frac{\varepsilon}{2} + ... + (\frac{\varepsilon}{2})^n)$ for any $n \geq 1$, and we set $M_n = \sup_{B(x_0,\rho_n)} u^p$. From (2.14) we obtain, with new constants $C = C(N, p)$,

$$M_n \leqq C(\frac{\rho_{n+1}}{\rho_n} - 1)^{-Np} \oint_{B(x_0,\rho_{n+1})} u^p \leq C(\frac{\varepsilon}{2})^{-(n+1)Np} \oint_{B(x_0,\rho_{n+1})} u^p.$$

From the Young inequality, for any $\delta \in (0,1)$, and any $r < 1$, we obtain

$$\begin{aligned} M_n &\leqq C(\frac{\varepsilon}{2})^{-(n+1)Np} M_{n+1}^{1-r} \oint_{B(x_0,\rho_{n+1})} u^{pr} \\ &\leqq \delta M_{n+1} + r\delta^{1-1/r} (C(\frac{\varepsilon}{2})^{-(n+1)Np})^{\frac{1}{r}} \left(\oint_{B(x_0,\rho_{n+1})} u^{pr} \right)^{\frac{1}{r}}. \end{aligned}$$

Defining $\kappa = r\delta^{1-1/r} C^{\frac{1}{r}}$ and $b = (\frac{\varepsilon}{2})^{-Np/r}$, we find

$$M_n \leqq \delta M_{n+1} + b^{n+1}\kappa \left(\oint_{B(x_0,\rho_{n+1})} u^{pr} \right)^{\frac{1}{r}}.$$

Taking $\delta = \frac{1}{2b}$ and iterating, we obtain

$$\begin{aligned} M_0 = \sup_{B(x_0,\rho)} u^p &\leqq \delta^{n+1} M_{n+1} + b\kappa \sum_{i=0}^{n} (\delta b)^i \left(\oint_{B(x_0,\rho_{n+1})} u^{pr} \right)^{\frac{1}{r}} \\ &\leqq \delta^{n+1} M_{n+1} + 2b\kappa \left(\oint_{B(x_0,\rho_{n+1})} u^{pr} \right)^{\frac{1}{r}}. \end{aligned}$$

Since $B(x_0, \rho_{n+1}) \subset B(x_0, \rho(1+\varepsilon))$, going to the limit as $n \to \infty$, and returning to u, we deduce

$$\sup_{B(x_0,\rho)} u \leq (2b\kappa)^{1/p} \left(\fint_{B(x_0,\rho(1+\varepsilon))} u^{pr} \right)^{\frac{1}{rp}},$$

and the conclusion follows by taking $r = s/p$. ■

It is interesting to make the link between Proposition 2.1, with the powerful estimates issued from the potential theory, involving *Wölf potentials*, proved in [20], [21] and [22]. Here we show that the lower estimates hold for any S-p-C operator.

Corollary 2.6 *Suppose that $\mathcal{A}_p$ is S-p-C. Let $f \in L^1_{loc}(\Omega)$, $f \geq 0$ and $u \in W^{1,p}_{loc}(\Omega)$ be any nonnegative such that*

$$-\mathcal{A}_p u \geqq f, \qquad in\ \Omega;$$

then for any ball $B(x_0, 2\rho) \subset \Omega$,

$$CW^f_{1,p}(B(x_0,\rho)) + \inf_{B(x_0,2\rho)} u \leq \liminf_{x \to x_0} u(x), \tag{2.15}$$

where $W^f_{1,p}$ is the Wölf potential of f defined at (1.7), and $C = C(N, p, K_{1,p}, K_{2,p})$. If u satisfies (2.3), then

$$CW^f_{1,p}(B(x_0,\rho)) + \limsup_{x \to x_0} u(x) \leq \sup_{B(x_0,2\rho)} u. \tag{2.16}$$

Proof. (i) The function $w = u - m_{2\rho}$, where $m_\rho = \inf_{B(x_0,\rho)} u$, is nonnegative in $B(x_0, 2\rho)$, and satisfies the inequality $-\mathcal{B}_p w \geq f$, where

$$w \longmapsto \mathcal{B}_p w = \operatorname{div} \mathrm{A}_p(x, w + m_{2\rho}, \nabla w)$$

is also a S-p-C operator. Then from Proposition 2.1 with ξ as in (2.8), fixing $\ell \in \left(0, \frac{N(p-1)}{N-p}\right)$ and $\varepsilon = \frac{1}{2}$, and applying Harnack inequality (2.12), there exists $C = C(N, p, K_{1,p}, K_{2,p})$ such that

$$2C \left(\rho^{1-N} \int_{B(x_0,\rho)} f \right)^{\frac{1}{p-1}} \leq \rho^{-1} \left(\fint_{B(x_0,2\rho)} (u - m_{2\rho})^\ell \right)^{\frac{1}{\ell}} \leq \rho^{-1}(m_\rho - m_{2\rho}).$$

Setting $\rho_j = 2^{1-j}\rho$, as in [20],

$$CW^f_{1,p}(B(x_0,\rho)) \leq \sum_{j=1}^{\infty} (m_{\rho_j} - m_{\rho_{j-1}}) = \lim m_{\rho_j} - \inf_{B(x_0,2\rho)} u = \liminf_{x \to x_0} u - \inf_{B(x_0,2\rho)} u.$$

(ii) The function $y = M_{2\rho} - u$ where $M_{2\rho} = \sup_{B(x_0,2\rho)} u$ satisfies the inequality $-\mathcal{C}_p w \geq f$ in $B(x_0, 2\rho)$, where

$$w \longmapsto \mathcal{C}_p w := \operatorname{div} [\mathrm{A}_p(x, M_{2\rho} - w, \nabla w)]$$

is still S-p-C. Then

$$W^f_{1,p}(B(x_0,\rho) \leqq C(\sup_{B(x_0,2\rho)} u - \limsup_{x \to x_0} u),$$

and (2.16) follows. ■

Remark 2.7 *The minorizations by Wölf potentials (2.15) and (2.16) have been proved in [20] and [22] for S-p-C operators of type* $\mathcal{A}_p u := div\left[\mathrm{A}_p(x, \nabla u)\right]$ *independent of* u*, satisfying moreover monotonicity and homogeneity properties, in particular* $\mathcal{A}_p(-u) = -\mathcal{A}_p u$*. The solutions are defined in the sense of potential theory, and may not belong to* $W^{1,p}_{loc}(\Omega)$*,* f *can be a Radon measure; majorizations by Wölf potentials are also given, with weighted operators, see [21] and [22]. In the same way Proposition 2.1 can also be extended to weighted operators, see [8, Remark 2.4] and [14], or to the case of a Radon measure when* $\mathcal{A}_p$ *is S-p-C by using the notion of local renormalized solution introduced in [3].*

2.3 A bootstrap result

Finally we give a variant of a result of [5, Lemma 2.2]:

Lemma 2.8 *Let* $d, h \in \mathbb{R}$ *with* $d \in (0,1)$ *and* y, Φ *be two positive functions on some interval* $(0, R]$*, and* y *is nondecreasing. Assume that there exist some* $K, M > 0$ *and* $\varepsilon_0 \in \left(0, \frac{1}{2}\right]$ *such that, for any* $\varepsilon \in (0, \varepsilon_0]$*,*

$$y(\rho) \leqq K\varepsilon^{-h}\Phi(\rho) y^d\left[\rho(1+\varepsilon)\right] \qquad and \ \max_{\tau\in\left[\rho, 3\frac{\rho}{2}\right]} \Phi(\tau) \leqq M\ \Phi(\rho), \qquad \forall \rho \in \left(0, \frac{R}{2}\right].$$

Then there exists $C = C(K, M, d, h, \varepsilon_0) > 0$ *such that*

$$y(\rho) \leqq C\Phi(\rho)^{\frac{1}{1-d}}, \qquad \forall \rho \in \left(0, \frac{R}{2e}\right]. \tag{2.17}$$

Proof. Let $\varepsilon_m = \varepsilon_0/2^m (m \in \mathbb{N})$, and $P_m = (1+\varepsilon_1)..(1+\varepsilon_m)$. Then (P_m) has a finite limit $P > 0$, and more precisely $P \leq e^{2\varepsilon_0} \leq e$. For any $\rho \in \left(0, \frac{R}{2e}\right]$ and any $m \geq 1$,

$$y(\rho P_{m-1}) \leq K\varepsilon_m^{-h}\Phi(\rho P_{m-1}) y^d(\rho P_m).$$

By induction, for any $m \geq 1$,

$$y(\rho) \leq K^{1+d+..+d^{m-1}}\varepsilon_1^{-h}\varepsilon_2^{-hd}..\varepsilon_m^{-hd^{m-1}}\Phi(\rho)\Phi^d(\rho P_1)..\Phi^{d^{m-1}}(\rho P_{m-1}) y^{d^m}(\rho P_m).$$

Hence from the assumption on Φ,

$$y(\rho) \leq (K\varepsilon_0^{-h})^{1+d+..+d^{m-1}} 2^{k(1+2d+..+md^{m-1})} M^{d+2d^2+..+(m-1)d^{m-1}}\Phi(\rho)^{1+d+..+d^{m-1}} y^{d^m}(\rho P_m);$$

and $y^{d^m}(\rho P_m) \leq y^{d^m}(e\rho) \leq y^{d^m}(\frac{R}{2})$, and $\lim y^{d^m}(\frac{R}{2}) = 1$, because $d < 1$. Hence (2.17) follows with $C = (K\varepsilon_0^{-h})^{1/(1-d)} 2^{h/(1-d)^2} M^{d/(1-d)^2}$. ■

3 Keller-Osserman estimates

3.1 The scalar case

First consider the solutions of inequality

$$-\mathcal{A}_p u + cu^Q \leq 0, \qquad \text{in } \Omega, \tag{3.1}$$

with $Q > p-1$ and $c > 0$. From the integral estimates of Proposition 2.1 we get easily Keller-Osserman estimates in the scalar case of the equation with absorption, without any hypothesis of monotonicity on the operator:

Proposition 3.1 *Let $Q > p-1$, $c > 0$. If $\mathcal{A}_p$ is S-p-C, and $u \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ is a nonnegative solution of (3.1), there exists a constant $C = C(N, p, K_{1,p}, K_{2,p}, Q) > 0$ such that, for any $x \in \Omega$,*

$$u(x) \leq Cc^{-1/(Q+1-p)} d(x, \partial\Omega)^{-p/(Q+1-p)}. \tag{3.2}$$

Proof. Let $B(x_0, \rho_0) \subset \Omega$, and $u \in W^{1,p}(B(x_0, \rho_0))$. From Corollary 2.2 with $\rho \leq \frac{\rho_0}{2}$, $\varepsilon = \frac{1}{2}$, and $\ell = Q$ and a function φ satisfying (2.8), we obtain for $\lambda = \lambda(p, Q)$

$$\oint_\varphi u^Q \leq c^{-1} C\rho^{-p} \left(\oint_\varphi u^Q \right)^{\frac{p-1}{Q}}, \tag{3.3}$$

where $C = C(N, p, K_{1,p}, K_{2,p}, Q)$. Then with another $C > 0$ as above,

$$\left(\oint_{B(x_0,\rho)} u^Q \right)^{\frac{1}{Q}} \leq Cc^{-\frac{1}{Q+1-p}} \rho^{-\frac{p}{Q+1-p}}.$$

Since $\mathcal{A}_p$ is S-p-C, from the weak Harnack inequality (2.11), with another constant C as above,

$$u(x_0) \leq C \left(\oint_{B(x_0,\rho)} u^Q \right)^{\frac{1}{Q}} \leq c^{-\frac{1}{Q+1-p}} \rho^{-\frac{p}{Q+1-p}},$$

and (3.2) follows by taking $\rho_0 = d(x_0, \partial\Omega)$. ■

3.2 The systems (A) and (M)

Here we prove theorems 1.1, 1.2, and Corollary 1.3. We recall that γ and ξ are defined by (1.8) under the condition (1.2) of superlinearity:

$$\gamma = \frac{p(q-1) + q\delta}{D}, \qquad \xi = \frac{q(p-1) + p\mu}{D}, \qquad D = \delta\mu - (p-1)(q-1) > 0.$$

Proof of Theorem 1.1. Consider a ball $B(x_0, \rho_0) \subset \Omega$, $\varepsilon \in \left(0, \frac{1}{2}\right]$, and a function φ satisfying (2.8) with λ large enough.

(i) Case $\mu > p-1$, $\delta > q-1$. Here C denotes different constants which only depend on N, p, q, δ, μ, and $K_{1,p}, K_{2,p}, K_{1,q}, K_{2,q}$. We take $\varepsilon = \frac{1}{2}$ and apply Corollary 2.2 with $\rho \leq \frac{\rho_0}{2}$ to the solution u with $f = v^\delta$, and with $\ell = \mu > p-1$. since $\mathcal{A}_p$ is W-p-C, from (2.9), we obtain

$$\oint_\varphi v^\delta \leq C\rho^{-p} \left(\oint_\varphi u^\mu \right)^{\frac{p-1}{\mu}}, \tag{3.4}$$

and similarly we apply it to the solution v with now $f = u^\mu$ and $\ell = \delta > q-1$: since $\mathcal{A}_q$ is W-q-C, we obtain

$$\oint_\varphi u^\mu \leq C\rho^{-q} \left(\oint_\varphi v^\delta\right)^{\frac{q-1}{\delta}}. \tag{3.5}$$

We can assume that $\oint_\varphi u^\mu > 0$. Indeed if $\oint_\varphi u^\mu = 0$, then $u = 0$ in $B(x_0, \rho_0)$. Then $\nabla u = 0$, thus $v^\delta = 0$ and then the estimates are trivially verified. Replacing (3.5) in (3.4) we deduce

$$\oint_\varphi v^\delta \leq C\rho^{-p-q\frac{p-1}{\mu}} \left(\oint_\varphi v^\delta\right)^{\frac{(q-1)(p-1)}{\mu\delta}},$$

and similarly for u, hence

$$\left(\oint_\varphi v^\delta\right)^{\frac{1}{\delta}} \leq C\rho^{-\xi}, \qquad \left(\oint_\varphi u^\mu\right)^{\frac{1}{\mu}} \leq C\rho^{-\gamma}. \tag{3.6}$$

Moreover, since $\mathcal{A}_q$ is S-q-C, then from the usual weak Harnack inequality, since $v \in L^\infty_{loc}(\Omega)$, and $\varphi(x) = 1$ in $B(x_0, \rho)$, with values in $[0,1]$,

$$\sup_{B(x_0,\frac{\rho}{2})} v \leq C \left(\oint_{B(x_0,\rho)} v^\delta\right)^{\frac{1}{\delta}} \leq \left(\oint_\varphi v^\delta\right)^{\frac{1}{\delta}} \leq C\rho^{-\xi}.$$

Similarly

$$\sup_{B(x_0,\frac{\rho}{2})} u \leq C\rho^{-\gamma},$$

because $\mathcal{A}_p$ is S-p-C.

(ii) Case $\mu > p-1$, and $\delta \leq q-1$. Here we still apply Corollary 2.2 with $\rho \leq \frac{\rho_0}{2}$, $\varepsilon \in (0, 1/4]$, and a function φ satisfying (2.8). Since $\mu > p-1$, we still obtain (3.4); and for any $k > q-1$, and λ large enough,

$$\oint_\varphi u^\mu \leq C(\varepsilon\rho)^{-q} \left(\oint_\varphi v^k\right)^{(q-1)/k}, \tag{3.7}$$

and from Lemma 2.5,

$$\left(\oint_\varphi v^k\right)^{1/k} \leq \sup_{B(x_0,\rho(1+\varepsilon))} v \leq C\varepsilon^{-\frac{Nq^2}{\delta^2}} \left(\oint_{B(x_0,\rho(1+2\varepsilon))} v^\delta\right)^{\frac{1}{\delta}}.$$

Then with new constants C, setting $m = q + \delta^{-2}Nq^2(q-1)$, and $h = (p-1)\mu^{-1}m$,

$$\oint_\varphi u^\mu \leq C\varepsilon^{-m}\rho^{-q} \left(\oint_{B(x_0,\rho(1+2\varepsilon))} v^\delta\right)^{\frac{(q-1)}{\delta}}, \tag{3.8}$$

hence from (3.4) and (3.8),

$$\oint_{B(x_0,\rho)} v^\delta \leq C \oint_\varphi v^\delta \leq C\rho^{-p} \left(\oint_\varphi u^\mu\right)^{\frac{p-1}{\mu}} \leq C\varepsilon^{-h}\rho^{-\frac{p\mu+q(p-1)}{\mu}} \left(\oint_{B(x_0,\rho(1+2\varepsilon))} v^\delta\right)^{\frac{(p-1)(q-1)}{\delta\mu}},$$

for any $\rho \leq \frac{\rho_0}{2}$. Next we apply the boostrap Lemma 2.8 with $R = \rho_0$, $y(\rho) = \oint_{B(x_0,\rho)} v^\delta$, $\Phi(r) = r^{-\frac{p\mu+q(p-1)}{\mu}}$ and 2ε. We deduce that

$$\left(\oint_{B(x_0,\rho)} v^\delta\right)^{1/\delta} \leq C\rho^{-\xi},$$

for any $\rho < \frac{\rho_0}{2}e$, and thus also

$$\sup_{B(x_0,\frac{\rho}{2})} v \leq C\left(\oint_{B(x_0,\rho)} v^\delta\right)^{\frac{1}{\delta}} \leq C\rho^{-\xi}, \qquad \sup_{B(x_0,\frac{\rho}{2})} u \leq C\left(\oint_{B(x_0,\rho)} u^\mu\right)^{1/\mu} \leq C\rho^{-\gamma}.$$

In particular

$$u(x_0) \leq C\rho_0^{-\gamma}, \qquad v(x_0) \leq C\rho_0^{-\xi}, \tag{3.9}$$

for any ball $B(x_0,\rho_0) \subset \Omega$, and the estimates (1.10) follow by taking $\rho_0 = d(x_0,\partial\Omega)$. ■

Proof of Theorem 1.2. We consider a ball $B(x_0,\rho_0)$ such that $B(x_0,2\rho_0) \subset \Omega$. From Proposition 2.1, we have the same estimates: for any $\ell > p-1, k > q-1$, $\rho \leq \rho_0$,

$$\oint_\varphi u^\mu \leq C\rho^{-q}\left(\oint_\varphi v^k\right)^{\frac{q-1}{k}}, \qquad \oint_\varphi v^\delta \leq C\rho^{-p}\left(\oint_\varphi u^\ell\right)^{\frac{p-1}{\ell}}.$$

From Lemma 2.5 (even if $\mu < p-1$), we have

$$\sup_{B(x_0,\frac{\rho}{2})} u^\mu \leq C\oint_{B(x_0,\rho)} u^\mu.$$

Taking $k < \frac{N(q-1)}{N-q}$, and using the weak Harnack inequality for v, we obtain

$$\begin{aligned}\sup_{B(x_0,\frac{\rho}{2})} u^\mu &\leq C\oint_{B(x_0,\rho)} u^\mu \leq C\oint_\varphi u^\mu \leq C\rho^{-q}\left(\oint_\varphi v^k\right)^{\frac{q-1}{k}} \\ &\leq C\rho^{-q}\left(\oint_{B(x_0,2\rho)} v^k\right)^{\frac{q-1}{k}} \leq C\rho^{-q}\inf_{B(x_0,\rho)} v^{(q-1)};\end{aligned}$$

hence (1.11) holds in $B(x_0,\frac{\rho}{2})$. Moreover if $v(x_0) = 0$, then $u = 0$ in $B(x_0,\frac{\rho}{2})$, then also $v = 0$ in $B(x_0,\frac{\rho}{2})$. Since Ω is connected, it implies that $v \equiv 0$, and then $u \equiv 0$. If $v \not\equiv 0$, then v stays positive in Ω, and we can write

$$-\mathcal{A}_q v = dv^{q-1}, \qquad \text{in } \Omega, \tag{3.10}$$

with $d(x) = u^\mu/v^{(q-1)} \leq C\rho^{-q}$ in $B(x_0,\frac{\rho}{2})$; in particular

$$d(x_0) = \frac{u^\mu(x_0)}{v^{q-1}(x_0)} \leq C\rho^{-q}, \tag{3.11}$$

thus (1.11) holds and v satisfies Harnack inequality in Ω : there exists a constant $C > 0$ such that

$$\sup_{B(x_0,\rho)} v \leq C \inf_{B(x_0,\rho)} v.$$

Therefore

$$\begin{aligned} v^\delta(x_0) &\leq \sup_{B(x_0,\rho)} v^\delta \leq C \inf_{B(x_0,\rho)} v^\delta \leq C \oint_{\varphi} v^\delta \leq C\rho^{-p} \left(\oint_{\varphi} u^\ell \right)^{\frac{p-1}{\ell}} \\ &\leq C\rho^{-p} \sup_{B(x_0,2\rho)} u^{p-1} \leq C\rho^{-p}\rho^{-q\frac{p-1}{\mu}} \inf_{B(x_0,4\rho)} v^{\frac{(q-1)(p-1)}{\mu}} \\ &\leq C\rho^{-(p+q\frac{p-1}{\mu})} v^{\frac{(q-1)(p-1)}{\mu}}(x_0); \end{aligned} \tag{3.12}$$

and (3.9) follows again from (3.12) and (3.11). ■

Remark 3.2 *Once we have proved (3.11) we can obtain the estimate on u in another way: we have the relation in the ball*

$$\mathcal{A}_p u = v^\delta \geq c u^{\frac{\delta\mu}{q-1}} \qquad in\ B(x_0, \rho_0),$$

with $c = C_1 \rho_0^{\frac{q\delta}{q-1}}$; then from Osserman-Keller estimates of Proposition 3.1 with $Q = \frac{\delta\mu}{q-1} > p-1$, we deduce that

$$u(x) \leq C_2 c^{-1/Q} \rho_0^{-\frac{p}{Q+1-p}} = C_3 \rho_0^{-\gamma}, \qquad in\ B(x_0, \frac{\rho_0}{2}).$$

The Liouville results are a direct consequence of the estimates:

Proof of Corollary 1.3. Let $x \in \mathbb{R}^N$ be arbitrary. Applying the estimates in a ball $B(x, R)$, we deduce that $u(x) \leq CR^{-\gamma}, v(x) \leq CR^{-\xi}$. Then we get $u(x) = v(x) = 0$ by making R tend to ∞. ■

Remark 3.3 *In the scalar case of inequality (3.1) it was proved in [14] that the Liouville result is also valid for a W-p-C operator. In the case of systems (A) or (M), the question is open. Indeed the method is based on the multiplication of the inequality by u^α with α large enough, and cannot be extended to the system.*

4 Behaviour near an isolated point

4.1 The systems (A_w) and (M_w).

Here we prove theorems 1.4 and 1.5. We recall that $\gamma_{a,b}$ and $\xi_{a,b}$ are defined by (1.12) under condition (1.2) :

$$\gamma_{a,b} = \frac{(p+a)(q-1)+(q+b)\delta}{D}, \ \xi_{a,b} = \frac{(q+b)(p-1)+(p+a)\mu}{D}, \ D = \delta\mu - (p-1)(q-1) > 0.$$

Proof of Theorem 1.4. It is a variant of Theorem 1.1: we consider $\Omega = B_1'$ and $x_0 \in B_{\frac{1}{2}}'$, and take $\rho_0 = \frac{|x_0|}{4}$. Here we apply Proposition 2.1 in the ball $B(x_0,\rho)$ with $\rho \leq \frac{\rho_0}{2}$ and $\varepsilon \in \left(0, \frac{1}{4}\right]$. The estimates (3.4) and (3.7) are replaced by

$$\oint_{\varphi} |x|^a v^\delta \leq C(\varepsilon\rho)^{-p}\left(\oint_{\varphi} u^\ell\right)^{\frac{p-1}{\ell}}, \qquad \oint_{\varphi} |x|^b u^\mu \leq C(\varepsilon\rho)^{-q}\left(\oint_{\varphi} v^k\right)^{\frac{q-1}{k}}, \tag{4.1}$$

for any $\ell > p-1, k > q-1$; and $2\rho_0 \leq |x| \leq 6\rho_0$ in $B(x_0, 2\rho_0)$, then in any of the cases $a \leq 0$ or $a > 0$, with a new constant C,

$$\oint_{\varphi} v^\delta \leq C\varepsilon^{-p}\rho^{-(p+a)}\left(\oint_{\varphi} u^\ell\right)^{\frac{p-1}{\ell}}, \qquad \oint_{\varphi} u^\mu \leq C\varepsilon^{-q}\rho^{-(q+b)}\left(\oint_{\varphi} v^k\right)^{\frac{q-1}{k}}. \tag{4.2}$$

Then all the proof is the same up to the change from p, q into $p+a$ and $q+b$. We deduce the same estimates with γ, ξ replaced by $\gamma_{a,b}, \xi_{a,b}$:

$$u(x_0) \leq C\,|x_0|^{-\gamma_{a,b}}, \qquad v(x_0) \leq C\,|x_0|^{-\xi_{a,b}}, \tag{4.3}$$

where C depends on $N, p, q, a, b, \delta, \mu$, and $K_{1,p}, K_{2,p}, K_{1,q}, K_{2,q}$. ■

Proof of theorem 1.5. In the same way we obtain estimate (4.3), then we only need to prove the estimate with respect to $|x|^{-\frac{N-q}{q-1}}$. We can apply to the function v the results of [2], recalled in [8, Propositions 2.2 and 2.3]: $|x|^b u^\mu \in L^1\left(B_{\frac{1}{2}}\right)$, and for any $k \in \left(0, \frac{N(q-1)}{N-q}\right)$, and $\rho > 0$ small enough,

$$\left(\oint_{B(0,\rho)} v^k\right)^{\frac{1}{k}} \leq C\rho^{-\frac{N-q}{q-1}}. \tag{4.4}$$

Moreover, arguing as in the proof of (1.11), we obtain the punctual inequality

$$u^\mu(x_0) \leq C\,|x_0|^{-(q+b)}\,v^{q-1}(x_0), \qquad \text{in } B_{\frac{1}{2}}', \tag{4.5}$$

which implies that

$$d(x_0) = |x_0|^b\,\frac{u^\mu(x_0)}{v^{q-1}(x_0)} \leq C\,|x_0|^{-q}.$$

Then v satisfies the Harnack inequality in $B_{\frac{1}{2}}'$, hence, from (4.4),

$$v(x_0) \leq \left(\oint_{B(x_0,\frac{|x_0|}{2})} v^k\right)^{\frac{1}{k}} \leq C\,|x_0|^{-\frac{N-q}{q-1}},$$

and (1.16) follows. ■

4.2 Removability results

Here we suppose that

$$(C_p)\begin{cases} \mathcal{A}_p u := div\left[\mathrm{A}_p(x,\nabla u)\right], & \mathcal{A}_p \text{ is S-}p\text{-C}, \\ (\mathrm{A}_p(x,\xi)-\mathrm{A}_p(x,\zeta)).\,(\xi-\zeta)>0, & \text{for } \xi\neq\zeta, \\ \mathrm{A}_p(x,\lambda\xi)=|\lambda|^{p-2}\,\lambda\mathrm{A}_p(x,\xi), & \text{for } \lambda\neq 0, \end{cases}$$

and similarly for $\mathcal{A}_q$. We give sufficient conditions ensuring that at least one of the functions u, v or both are bounded. We obtain the two following results, relative to systems (A_w) and (M_w):

Theorem 4.1 *Assume (1.2), $(C_p),(C_q)$. Let $u\in W^{1,p}_{loc}(B'_1)$, $v\in W^{1,q}_{loc}(B'_1)$ be nonnegative solutions of*

$$\begin{cases} -\mathcal{A}_p u+|x|^a\,v^\delta\leq 0, \\ -\mathcal{A}_q v+|x|^b u^\mu\;\leq 0, \end{cases} \qquad in\ B'_1.$$

(i) If $\gamma_{a,b}\leq\frac{N-p}{p-1}$, then u is bounded near 0; if $\xi_{a,b}\leq\frac{N-q}{q-1}$, then v is bounded.

(ii) If moreover (u,v) is a solution of (A_w) and u is bounded near 0 and $\delta>\frac{(p+a)(q-1)}{N-q}$ (or $\delta=\frac{(p+a)(q-1)}{N-q}$ if $\mathcal{A}_p=\Delta_p$) then v is also bounded. In the same way if v is bounded and $\mu>\frac{(q+b)(p-1)}{N-p}$ (or $\mu=\frac{(q+b)(p-1)}{N-p}$ if $\mathcal{A}_q=\Delta_q$) then u is also bounded.

Theorem 4.2 *Assume (1.2), $(C_p),(C_q)$. Let $u\in W^{1,p}_{loc}(B'_1)\cap C(B'_1)$, $v\in W^{1,q}_{loc}(B'_1)\cap C(B'_1)$ be nonnegative solutions of*

$$\begin{cases} -\mathcal{A}_p u+|x|^a\,v^\delta\leq 0, \\ -\mathcal{A}_q v\geq|x|^b u^\mu, \end{cases} \qquad in\ B'_1.$$

If $\gamma_{a,b}\leq\frac{N-p}{p-1}$, or if $\gamma_{a,b}>\frac{N-p}{p-1}$ and $\mu>\frac{(N+b)(p-1)}{N-p}$, then u is bounded.

The proofs require some lemmas, adapted to subsolutions of equation $\mathcal{A}_p u=0$.

Lemma 4.3 *Assume (C_p). Let $u\in W^{1,p}_{loc}(B'_1)\cap C(B'_1)$ be nonnegative, such that*

$$-\mathcal{A}_p u\leqq 0 \qquad in\ B'_1.$$

Then, either there exists $C>0$ and $r\in\left(0,\frac{1}{2}\right)$ such that

$$\sup_{|x|=\rho}u\geq C\rho^{\frac{p-N}{p-1}}, \qquad for\ any\ \rho\in(0,r), \tag{4.6}$$

or u is bounded near 0.

Proof. From our assumptions on $\mathcal{A}_p$, there exists at least a solution E of the Dirichlet problem

$$-\mathcal{A}_p E = \delta_0, \qquad \text{in } B_1,$$

where δ_0 is the Dirac mass at 0, in the renormalized sense, see [13, Theorem 3.1]. In particular it satisfies the equation in $\mathcal{D}'(B_1)$, and it is a smooth solution of equation $\mathcal{A}_p E = 0$ in B_1'. From [25], [26], there exists $C_1, C_2 > 0$ such that $C_1 |x|^{-\frac{N-p}{p-1}} \leqq E(x) \leqq C_2 |x|^{-\frac{N-p}{p-1}}$ near 0. Assume that (4.6) does not hold. Then there exists $r_n < \min(1/n, r_{n-1})$ such that

$$\sup_{|x|=r_n} u \leq \frac{1}{n} r_n^{\frac{p-N}{p-1}} \leq \frac{1}{nC_1} E(r_n).$$

Next we use the comparison theorem in the annulus $\mathcal{C}_n = \left\{x \in \mathbb{R}^N : r_n \leq |x| \leq \frac{1}{2}\right\}$ for functions in $W^{1,p}_{loc}(\mathcal{C}) \cap C(\overline{\mathcal{C}_n})$, and we find that

$$u(x) \leq \frac{1}{nC_1} E(x) + \max_{|x|=\frac{1}{2}} u, \qquad \text{in } \mathcal{C}_n.$$

Going to the limit as $n \to \infty$, we deduce that u is bounded. ∎

Our next lemma complements the results of [8, Proposition 2.2]:

Lemma 4.4 *Assume that $\mathcal{A}_p$ is W-p-C. Let $f \in L^1_{loc}(B_1'), f \geqq 0$. Let $u \in W^{1,p}_{loc}(B_1')$ be nonnegative, such that*

$$-\mathcal{A}_p u + f \leqq 0 \qquad \text{in } B_1'.$$

If $|x|^{\frac{N-p}{p-1}} u$ is bounded near 0, then $f \in L^1_{loc}(B_1)$.

Proof. Let $0 < \rho < \frac{1}{2}$. Here we apply Proposition 2.1 with $\varphi = \xi^\lambda$ given by

$$\xi = 1 \text{ for } \rho < |x| < \frac{1}{2}, \quad \xi = 0 \text{ for } |x| \leqq \frac{\rho}{2} \text{or } |x| \geqq \frac{3}{4}, \quad |\nabla \xi| \leq \frac{C_0}{\rho}.$$

From Remark 2.3, we find with for example $\ell = p$,

$$\int_{\rho \leqq |x| \leqq \frac{1}{2}} f \leqq C\rho^{N-p} \left(\fint_{\frac{\rho}{2} \leqq |x| \leqq \rho} u^\ell \right)^{\frac{p-1}{\ell}} + C \left(\fint_{\frac{1}{2} \leqq |x| \leqq \frac{3}{4}} u^\ell \right)^{\frac{p-1}{\ell}}. \tag{4.7}$$

Hence from our assumption on u, the integral is bounded, then $f \in L^1(B_{\frac{1}{2}})$. ∎

Proof of Theorem 4.1. (i) Suppose that $\gamma_{a,b} \leq \frac{N-p}{p-1}$. Then $u(x_0) \leq C |x_0|^{-\frac{N-p}{p-1}}$. Let us show that u is bounded. If $\gamma_{a,b} < \frac{N-p}{p-1}$ it is a direct consequence of Lemma 4.3. Then we can assume $\gamma_{a,b} = \frac{N-p}{p-1}$. If u is not bounded, then (4.6) holds for some $C > 0$. Let us set $f = |x|^a v^\delta$. From (4.2) with $\varepsilon = \frac{1}{4}$ then for any $r_0 \leq \frac{1}{2}$ and any x_0 such that $|x_0| = r_0$, and Lemma 2.5, taking $\rho = \frac{r_0}{4}$,

$$\begin{aligned} u^\mu(x_0) &\leq C \fint_{B(x_0,\rho)} u^\mu \leq C r_0^{-(q+b)-N\frac{q-1}{\delta}} \left(\int_{B(x_0,2\rho)} v^\delta \right)^{\frac{q-1}{\delta}} \\ &\leq C r_0^{-(q+b)-(N+a)\frac{q-1}{\delta}} \left(\int_{\frac{r_0}{2} \leqq |x| \leqq \frac{3r_0}{2}} f \right)^{\frac{q-1}{\delta}}, \end{aligned}$$

then

$$Cr_0^{-\mu\gamma_{a,b}} = Cr_0^{-(q-1)\xi_{a,b}-q-b} \leq \sup_{|x|=r_0} u^\mu \leq Cr_0^{-(q+b)-(N+a)\frac{q-1}{\delta}} \left(\int_{\frac{r_0}{2} \leqq |x| \leqq \frac{3r_0}{2}} f \right)^{\frac{q-1}{\delta}},$$

$$Cr_0^{-(q-1)\xi_{a,b}\frac{\delta}{q-1}+(N+a)} = Cr_0^0 = C \leq \int_{\frac{r_0}{2} \leqq |x| \leqq \frac{3r_0}{2}} f;$$

then for any $n \in \mathbb{N}$,

$$C \leq \int_{\frac{r_0}{2.3^n} \leqq |x| \leqq \frac{r_0}{2.3^{n-1}}} f.$$

By summation it contradicts Lemma 4.4. Similarly for v.

(ii) Suppose that (u, v) is a solution of (A_w) and u is bounded and $\delta \geq \frac{(p+a)(q-1)}{N-q}$. Here v satisfies equation $\mathcal{A}_q v = g$ with $g = |x|^b u^\mu \leqq C |x|^b$, thus $g \in L^{N/q+\varepsilon}(\Omega)$ for some $\varepsilon > 0$, then from [25], [26], if v is not bounded near 0, then there exist $C_1, C_2 > 0$ such that

$$C_1 |x|^{-\frac{N-q}{q-1}} \leqq v \leqq C_2 |x|^{-\frac{N-q}{q-1}}$$

near 0. If $\delta > \frac{(p+a)(q-1)}{N-q}$ then

$$\mathcal{A}_p u = |x|^a v^\delta \geq C_1 |x|^{a-\delta\frac{N-q}{q-1}} = C_1 |x|^{-p-\varepsilon},$$

for some $\varepsilon > 0$, then from (4.1),

$$\rho^{-p-\varepsilon} \leqq C \oint_{\varphi} |x|^{-p-\varepsilon} \leq C\rho^{-p} \left(\oint_{\varphi} u^\ell \right)^{\frac{p-1}{\ell}} \leqq C\rho^{-p},$$

which is a contradiction. If $\delta = \frac{(p+a)(q-1)}{N-q}$, then

$$C_2 |x|^{-p} \geq \mathcal{A}_p u = |x|^a v^\delta \geq C_1 |x|^{-p}.$$

Otherwise u is bounded by some M in a ball B'_r. Then the function $w = M - u$ is nonnegative and bounded and satisfies

$$-\mathcal{A}_p w \geq C_1 |x|^{-p} \qquad \text{in } B'_r.$$

But for $\mathcal{A}_p = \Delta_p$, there is no bounded solution of this inequality, from [8, Proposition 2.7], we reach a contradiction. ■

Remark 4.5 *The results obviously apply to the scalar case, finding again and improving a result of [31].*

Proof of Theorem 4.2. (i) Assume $\gamma_{a,b} \leq \frac{N-p}{p-1}$. The proof of part (i) of Theorem 4.1 is still valid and shows that u is bounded.

(ii) Assume $\gamma_{a,b} > \frac{N-p}{p-1}$ and $\mu > \frac{(N+b)(p-1)}{N-p}$. Then $\xi_{a,b} > \frac{N-q}{q-1}$, thus the estimate (1.16) for v gives $v(x_0) \leq C\,|x_0|^{-\frac{N-q}{q-1}}$, then

$$u^{\mu}(x_0) \leq C\,|x_0|^{-(q+b)}\,v^{(q-1)}(x_0) \leq C\,|x_0|^{-(N+b)}.$$

Then $\rho^{\frac{N-p}{p-1}} \sup_{|x|=\rho} u$ tends to 0, hence u is bounded from Lemma 4.3. ■

Remark 4.6 *Let us give an alternative proof of (i): the punctual inequality (4.5) implies that near 0,*

$$\mathcal{A}_p u \geq |x|^a v^{\delta} \geq C|x|^{a+\delta(q+b)/(q-1)} u^{\mu\delta/(q-1)};$$

then we are reduced to a simple scalar inequality:

$$-\mathcal{A}_p u + |x|^m u^Q \leq 0, \tag{4.8}$$

with $Q = \frac{\mu\delta}{q-1} > p-1$ *and* $m = a + \frac{\delta(q+b)}{q-1} > -p$. *And* $\gamma_{a,b} = \frac{m+p}{Q+1-p} \leq \frac{N-p}{p-1}$; *applying Theorem 4.1 to the scalar inequality (4.8), we find again that u is bounded.*

5 Sharpness of the results

In this last section we show the optimality of our results by constructing some radial solutions of systems (A_w) or (M_w) in case $\mathcal{A}_p = \Delta_p, \mathcal{A}_q = \Delta_q$. They are based on the transformation introduced in [4], valid for systems with any sign:

$$\begin{cases} -\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = \varepsilon_1\,|x|^a\,v^{\delta}, \\ -\Delta_q v = -\operatorname{div}(|\nabla v|^{q-2}\,\nabla u) = \varepsilon_2\,|x|^b\,u^{\mu}, \end{cases}$$

with $\varepsilon_1 = -1 = \varepsilon_2$ for the system with absorption, and $\varepsilon_1 = -1, \varepsilon_2 = 1$ for the mixed system: setting

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = -\varepsilon_1 r^{1+a} u^s v^{\delta}\frac{u'}{|u'|^p}, \quad W(t) = -\varepsilon_2 r^{1+b} u^{\mu} v^m \frac{v'}{|v'|^q},$$

where $t = \ln r$, and we obtain the system

$$(\Sigma)\begin{cases} X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \\ Y_t = Y\left[Y - \frac{N-q}{q-1} + \frac{W}{q-1}\right], \\ Z_t = Z\left[N + a - \delta Y - Z\right], \\ W_t = W\left[N + b - \mu X - W\right]. \end{cases}$$

And u, v are recovered from X, Y, Z, W by the relations

$$u = r^{-\gamma_{a,b}}(\,|X|^{p-1}\,Z)^{(q-1)/D}(\,|Y|^{q-1}\,W)^{\delta/D}, \qquad v = r^{-\xi_{a,b}}(\,|X|^{p-1}\,Z)^{\mu/D}(\,|Y|^{q-1}\,W)^{(p-1)/D}. \tag{5.1}$$

5.1 About Harnack inequality

Here we show that Harnack inequality can be *false* in case of system (A_w) and also for the function u of system (M_w), even in the radial case; indeed we construct nonnegative radial solutions of system (A_w) in a ball such that $u(0) = 0 < v(0)$, or by symmetry $u(0) > 0 = v(0)$ and solutions of system (M_w) such that $u(0) = 0 < v(0)$. Such solutions were constructed in [15] by using Schauder theorem, and in [7] in the case of system (A_w) for $p = q = 2$ by using system (Σ). Here we show that the construction of [7] extends to the general case. We consider the radial regular solutions, which are C^2 if $a, b \geq 0$, and C^1 if $a, b > -1$.

Proposition 5.1 *Suppose that* $\mathcal{A}_p = \Delta_p$ *and* $\mathcal{A}_q = \Delta_q$*. For any* $v_0 > 0$*, there exists a regular radial solution of* (A_w) *and* (M_w) *such that* $u(0) = 0 < v(0) = v_0$*.*

Proof. The regular solutions (u, v) with nonnegative initial data $(u_0, v_0) \neq (0, 0)$ are increasing for system (A_w), hence $X, Y < 0 < Z, W$ and u is increasing and v is decreasing for system (M_w), hence $X < 0 < Y$ and $Z, W > 0$. As shown in [4], the solutions (u, v) with $u(0) = u_0 > 0$ and $v(0) = v_0 > 0$ correspond to the trajectories of system (Σ) converging to the fixed point $N_0 = (0, 0, N + a, N + b)$ as $t \longrightarrow -\infty$, and local existence and uniqueness holds as in [4, Proposition 4.4]. As in [7] the solutions such that $u_0 = 0 < v_0$ correspond to a trajectory converging to the point $S_0 = \left(\bar{X}, 0, \bar{Z}, \bar{W}\right) = \left(-\frac{p+a}{p-1}, 0, N + a, N + b + \mu\frac{p+a}{p-1}\right)$. The linearization at S_0 gives the eigenvalues

$$\lambda_1 = \bar{X} < 0, \quad \lambda_2 = \frac{1}{q-1}(q + b + \mu\frac{p+a}{p-1}) > 0, \quad \lambda_3 = -\bar{Z} < 0, \quad \lambda_4 = -\bar{W} < 0.$$

Then the unstable manifold $\mathcal{V}_u$ has dimension 1 and $\mathcal{V}_u \cap \{Y = 0\} = \emptyset$, thus there exists a unique trajectory such that $Y < 0$ (resp. $Y > 0$) and $Z, W > 0$. There holds $\lim_{t\to-\infty} e^{-\lambda_2 t} Y = c > 0$, $\lim X = \bar{X}$, $\lim Z = \bar{Z}$, $\lim W = \bar{W}$, then from (5.1) v has a positive limit v_0, and u tends to 0. By scaling we obtain the existence and uniqueness of solutions for any $v_0 > 0$. ■

5.2 About removability

Here also we show that the results of Theorems 4.1 and 4.2 are optimal, by constructing singular solutions when the assumptions are not satisfied. We begin by system (A_w), extending [7, Proposition 3.2]. Obviously it admits a particular singular solution when $\gamma_{a,b} > \frac{N-p}{p-1}$ and $\xi_{a,b} > \frac{N-q}{q-1}$. Moreover we find other types of singular solutions:

Proposition 5.2 *Consider system* (A_w) *with* $\mathcal{A}_p = \Delta_p$ *and* $\mathcal{A}_q = \Delta_q$*.*

(i) If $\mu < \frac{(q+b)(p-1)}{N-p}$*, there exist solutions such that*

$$\lim_{\rho\to 0} \rho^{\frac{N-p}{p-1}} u = \alpha > 0, \qquad \lim_{\rho\to 0} v = \beta > 0.$$

(ii) If $\delta < \frac{(N+a)(q-1)}{N-q}$ *and* $\mu < \frac{(N+b)(p-1)}{N-p}$*, there exist solutions such that*

$$\lim_{\rho\to 0} \rho^{\frac{N-p}{p-1}} u = \alpha > 0, \qquad \lim_{\rho\to 0} \rho^{\frac{N-q}{q-1}} v = \beta > 0.$$

(iii) If $\gamma_{a,b} > \frac{N-p}{p-1}$, and either $\mu > \frac{(N+b)(p-1)}{N-p}$ or $\mu < \frac{(q+b)(p-1)}{N-p}$, there exist solutions such that

$$\lim_{\rho\to 0} \rho^{\frac{N-p}{p-1}} u = \alpha > 0, \qquad \lim_{\rho\to 0} \rho^{\frac{1}{q-1}(\frac{N-p}{p-1}\mu-(q+b))} v = \beta(\alpha) > 0.$$

The results extend by symmetry, after exchanging $u, v, a, \gamma_{a,b}$ and $v, u, b, \xi_{a,b}$.

Proof. As in [5], [7] we prove the existence of trajectories of system (Σ) and return to u, v by using (5.1).

(i) Such solutions correspond to trajectories converging to the fixed point $G_0 = (\frac{N-p}{p-1}, 0, 0, N + b - \frac{N-p}{p-1}\mu)$ of (Σ). The linearization at G_0 gives the eigenvalues

$$\lambda_1 = \frac{N-p}{p-1} > 0,\ \lambda_2 = \frac{1}{q-1}(q + b - \frac{N-p}{p-1}\mu),\ \lambda_3 = N + a > 0,\ \lambda_4 = \frac{N-p}{p-1}\mu - N - b.$$

If $\mu < \frac{(q+b)(p-1)}{N-p}$, then $\lambda_2, \lambda_4 < 0$. Then $\mathcal{V}_u$ has dimension 3, and $\mathcal{V}_u \cap \{Y = 0\}$ and $\mathcal{V}_u \cap \{Z = 0\}$ have dimension 2. This implies that $\mathcal{V}_u$ must contain trajectories such that $Y, Z < 0 < X, W$.

(ii) Such solutions correspond to the fixed point $A_0 = \left(\frac{N-p}{p-1}, \frac{N-q}{q-1}, 0, 0\right)$. All the eigenvalues are positive:

$$\lambda_1 = \frac{N-p}{p-1},\ \lambda_2 = \frac{N-q}{q-1},\ \lambda_3 = N + a - \delta\frac{N-q}{q-1},\ \lambda_4 = N + b - \mu\frac{N-p}{p-1}.$$

The unstable manifold $\mathcal{V}_u$ has dimension 4, then there exists an infinity of trajectories converging to A_0 with $X; Y, Z, W < 0$.

(iii) Such solutions correspond to the fixed point $P_0 = \left(\frac{N-p}{p-1}, Y_*, 0, W_*\right)$, with

$$Y_* = \frac{1}{q-1}(\frac{N-p}{p-1}\mu - (q + b)), \qquad W_* = N + b - \frac{N-p}{p-1}\mu.$$

The eigenvalues are given by

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = Y_*, \quad \lambda_3 = \frac{D}{q-1}(\gamma - \frac{N-p}{p-1}) > 0, \quad \lambda_4 = -W_*.$$

If $\mu > \frac{(N+b)(p-1)}{N-p}$, then $\lambda_2, \lambda_4 > 0$ and thus $\mathcal{V}_u$ has dimension 4, then there exist trajectories, with $X, Y, Z, W < 0$, converging to P_0. If $\mu < \frac{(q+b)(p-1)}{N-p}$, then $\lambda_2, \lambda_4 < 0$, $\mathcal{V}_u$ has dimension 2, and $\mathcal{V}_u \cap \{Z = 0\}$ has dimension 1, thus there also exist trajectories with $X, Z, W < 0 < Y$ converging to P_0. ■

In the same way, system (M_w) has a particular singular solution when $\gamma_{a,b} > \frac{N-p}{p-1}$ and $\xi_{a,b} < \frac{N-q}{q-1}$, and we find other singular solutions:

Proposition 5.3 *Consider system* (M_w) *with* $\mathcal{A}_p = \Delta_p, \mathcal{A}_q = \Delta_q$.

(i) If $\gamma_{a,b} > \frac{N-p}{p-1}$, *and* $\xi_{a,b} > \frac{N-q}{q-1}$, *there exist solutions such that*

$$\lim_{\rho \to 0} \rho^{\frac{N-q}{q-1}} v = \beta > 0, \qquad \lim_{\rho \to 0} \rho^{\frac{1}{p-1}(\frac{N-q}{q-1}\delta - (q+a))} u = \beta(\alpha) > 0.$$

(ii) If $\delta < \frac{(N+a)(q-1)}{N-q}$ *and* $\mu < \frac{(N+b)(p-1)}{N-p}$, *there exist solutions such that*

$$\lim_{\rho \to 0} \rho^{\frac{N-p}{p-1}} u = \alpha > 0, \qquad \lim_{\rho \to 0} \rho^{\frac{N-q}{q-1}} v = \beta > 0.$$

Proof. (i) These solutions correspond to the fixed point Q_0 deduced from P_0 by symmetry, and our assumptions imply $\delta > \frac{(N+a)(q-1)}{N-q}$, hence there exist trajectories, such that $X, Y, Z < 0 < W$ converging to Q_0.

(ii) The conclusion follows as in Proposition 5.2, (ii). ■

We refer to [5] and [6] for a description of all the (various) possible behaviours of the solutions in the case $p = q = 2$.

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References

[1] M-F. Bidaut-Véron, Local and global behaviour of solutions of quasilinear equations of Emden-Fowler type, *Arch. Rat. Mech. Anal.,* **107** (1989), 293-324.

[2] M-F. Bidaut-Véron, Singularities of solutions of a class of quasilinear equations in divergence form, Nonlinear diffusion equations and their equilibrium states, Birkauser, Boston, Basel, Berlin (1992), 129-144.

[3] M-F. Bidaut-Véron, Removable singularities and existence for a quasilinear equation, *Adv. Nonlinear Studies,* **3** (2003), 25-63.

[4] M-F. Bidaut-Véron, and H. Giacomini, A new dynamical approach of Emden-Fowler equations and systems, arXiv:1001.0562v2 [math.AP], *Adv. Diff. Eq.,* **15** (2010), 1033-1082.

[5] M-F. Bidaut-Véron and P. Grillot, Singularities in elliptic systems with absorption terms, *Ann. Scuola Norm. Sup. Pisa CL. Sci,* **28** (1999), 229-271.

[6] M-F. Bidaut-Véron and P. Grillot, Asymptotic behaviour of elliptic systems with mixed absorption and source terms, Asymtotic Anal., ***19*** (1999), 117-147.

[7] M-F. Bidaut-Véron, M. Garcia-Huidobro and C. Yarur, Large solutions of elliptic systems of second order and applications to the biharmonic equation, *Discrete and continuous dynamical. systems*, **32** (2012), 411-432.

[8] M-F. Bidaut-Véron and S. Pohozaev, Nonexistence results and estimates for some nonlinear elliptic problems, J. Anal. Mathématique, ***84*** (2001),1-49.

[9] M-F. Bidaut-Véron and L. Véron, Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations, *Invent. Math.*, 106 (1991), 489-539.

[10] L. d'Ambrosio and E. Mitidieri, A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic equations, *Advances in Math.,* **224** (2010), 967-1020.

[11] J. Davila, L. Dupaigne, 0. Goubet and S. Martinez, Boundary blow-up solutions of cooperative systems, *Ann. I.H.Poincaré-AN,* **26** (2009), 1767-1791.

[12] E. Di Benedetto, Partial Differential equations, Birkaüser (1995).

[13] G. Dal Maso, F. Murat, L.Orsina, and A. Prignet, Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa*, **28** (1999), 741-808.

[14] A. Farina and J. Serrin, Entire solutions of completely coercive quasilinear elliptic equations, *AJ. Diff. Equ.* **250** (2011), 4367-4408 and 4408-4436.

[15] J. García-Melián, R. Letelier-Albornoz and J. Sabina de Lis, The solvability of an elliptic system under a singular boundary condition, *Proc. Roy. Soc. Edinburgh,* **136** (2006), 509-546.

[16] J. García-Melian, and J. Rossi, *Boundary blow-up solutions to elliptic system of competitive type*, *J. Diff. Equ.,* **206** (2004), 156-181.

[17] J. García-Melián, Large solutions for an elliptic system of quasilinear equations, *J. Diff. Equ.,* **245** (2008), no. 12, 3735–3752.

[18] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure and Applied Math.,* **34** (1981), 525-598.

[19] J.B. Keller, On the solutions of $-\Delta u = f(u)$, *Comm. Pure Applied Math.,* **10** (1957), 503-510.

[20] T. Kilpelainen and J. Maly, Degenerate elliptic equations with measure data and non linear potentials, *Ann. Scuola Norm. Sup. Pisa,* **19** (1992), 591-613.

[21] T. Kilpelainen and J. Maly, The Wiener test and potential estimates for quasilinear elliptic equations, *Acta Matematica,* **172** (1994), 137-161.

[22] T. Kilpelainen and X. Zhong, Growth of entire $\mathcal{A}$-subharmonic functions, *Ann. Acad. Sci. Fennic Math.,* **28** (2003), 181-192.

[23] E. Mitidieri and S. Pohozaev, Non existence of positive solutions for quasilinear elliptic problems on $\mathbb{R}^N$, *Proc. Steklov Institute of Math.,* **227** (1999), 186-216.

[24] R. Osserman, 0n the inequality $-\Delta u \geq f(u)$, *Pacific J. Math.,* **7** (1957), 1641-1647.

[25] J. Serrin, Local behavior of solutions of quasilinear equations, *Acta Mathematica,* **111**, (1964), 247-302.

[26] J. Serrin, Isolated singularities of solutions of quasilinear equations, *Acta Mathematica,* **113**, (1965), 219-240.

[27] J. Serrin and H. Zou, Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, *Acta Mathematica,* **189** (2002), 79-142.

[28] N. Trudinger, On Harnack type inequalities and their application to quasilinear equations, *Comm. Pure Applied Math.,* **20** (1967), 721-747.

[29] J.L. Vazquez, *An a priori interior estimate for the solutions of a nonlinear problem representing weak diffusion*, *Nonlinear Anal.,* **5** (1981), 95-103.

[30] L. Véron, Semilinear elliptic equations with uniform blowup on the boundary, *J. Anal. Math.,* **59** (1992), 2-250.

[31] J.L. Vazquez and L. Véron, Removable singularities of some strongly nonlinear elliptic equations, *Manuscripta Math.,* **33** (1980), 129-144.

[32] M. Wu and Z. Yang, Existence of boundary blow-up solutions for a class of quasilinear elliptic systems with critical case, *Applied Math. Comput.,* **198** (2008), 574-581.