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Abstract

We study the existence of solutions to the fractional elliptic equation (E1) $(-\Delta)^\alpha u + \epsilon g(|\nabla u|) = \nu$ in an open bounded regular domain Ω of $\mathbb{R}^N (N \geq 2)$, subject to the condition (E2) $u = 0$ in Ω^c, where $\epsilon = 1$ or -1, $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha \in (1/2, 1)$, ν is a Radon measure and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function. We prove the existence of weak solutions for problem (E1)-(E2) when g is subcritical. Furthermore, the asymptotic behavior and uniqueness of solutions are described when $\epsilon = 1$, ν is Dirac mass and $g(s) = s^p$ with $p \in (0, \frac{N}{N-2\alpha+1})$.

Contents

Key words: Fractional Laplacian, Radon measure, Green kernel, Dirac mass.

MSC2010: 35R11, 35J61, 35R06

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1 Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be an open bounded C^2 domain and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous function. The purpose of this paper is to study the existence of weak solutions to the semilinear fractional elliptic problem with $\alpha \in (1/2, 1)$,

$$\begin{aligned} (-\Delta)^\alpha u + \epsilon g(|\nabla u|) = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{1.1}$$

where $\epsilon = 1$ or -1 and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1)$. Here $\rho(x) = dist(x, \Omega^c)$ and $\mathfrak{M}(\Omega, \rho^\beta)$ is the space of Radon measures in Ω satisfying

$$\int_\Omega \rho^\beta d|\nu| < +\infty. \tag{1.2}$$

In particular, we denote $\mathfrak{M}^b(\Omega) = \mathfrak{M}(\Omega, \rho^0)$. The associated positive cones are respectively $\mathfrak{M}_+(\Omega, \rho^\beta)$ and $\mathfrak{M}^b_+(\Omega)$. According to the value of ϵ, we speak of an absorbing nonlinearity the case $\epsilon = 1$ and a source nonlinearity the case $\epsilon = -1$. The operator $(-\Delta)^\alpha$ is the fractional Laplacian defined as

$$(-\Delta)^\alpha u(x) = \lim_{\varepsilon \to 0^+} (-\Delta)^\alpha_\varepsilon u(x),$$

where for $\varepsilon > 0$,

$$(-\Delta)^\alpha_\varepsilon u(x) = -\int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\varepsilon(|x - z|) dz \tag{1.3}$$

and

$$\chi_\varepsilon(t) = \begin{cases} 0, & \text{if} \quad \mathrm{t} \in [0, \varepsilon], \\ 1, & \text{if} \quad \mathrm{t} > \varepsilon. \end{cases}$$

In a pioneering work, Brezis [7] (also see Bénilan and Brezis [1]) studied the existence and uniqueness of the solution to the semilinear Dirichlet elliptic problem

$$\begin{aligned} -\Delta u + h(u) = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{1.4}$$

where ν is a bounded measure in Ω and the function h is nondecreasing, positive on $(0, +\infty)$ and satisfies that

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-2\frac{N-1}{N-2}} ds < +\infty.$$

Later on, Véron [29] improved this result in replacing the Laplacian by more general uniformly elliptic second order differential operator, where $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 1]$ and h is a nondecreasing function satisfying

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-2\frac{N+\beta-1}{N+\beta-2}} ds < +\infty.$$

The general semilinear elliptic problems involving measures such as the equations involving boundary measures have been intensively studied; it was initiated by Gmira and Véron [16] and then this subject has being extended in various ways, see [4, 6, 18, 19, 20, 21] for details and [22] for a general panorama. In a recent work, Nguyen-Phuoc and Véron [24] obtained the existence of solutions to the viscous Hamilton-Jacobi equation

$$\begin{aligned} -\Delta u + h(|\nabla u|) = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{1.5}$$

when $\nu \in \mathfrak{M}^b(\Omega)$, h is a continuous nondecreasing function vanishing at 0 which satisfies

$$\int_1^{+\infty} h(s) s^{-\frac{2N-1}{N-1}} ds < +\infty.$$

During the last years there has also been a renewed and increasing interest in the study of linear and nonlinear integro-differential operators, especially, the fractional Laplacian, motivated by great applications in physics and by important links on the theory of Lévy processes, refer to [8, 12, 13, 10, 14, 26, 28, 27]. Many estimates of its Green kernel and generation formula can be found in the references [3, 11]. Recently, Chen and Véron [13] studied the semilinear fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u + h(u) = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{1.6}$$

where $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, \alpha]$. We proved the existence and uniqueness of the solution to (1.6) when the function h is nondecreasing and satisfies

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-1-k_{\alpha,\beta}} ds < +\infty,$$

where

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if} \quad \beta \in [0, \frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if} \quad \beta \in (\frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha, \alpha]. \end{cases} \tag{1.7}$$

Our interest in this article is to investigate the existence of weak solutions to fractional equations involving nonlinearity in the gradient term and with Radon measure. In order the fractional Laplacian be the dominant operator in terms of order of differentiation, it is natural to assume that $\alpha \in (1/2, 1)$.

Definition 1.1 *We say that u is a weak solution of (1.1), if $u \in L^1(\Omega)$, $|\nabla u| \in L^1_{loc}(\Omega)$, $g(|\nabla u|) \in L^1(\Omega, \rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + \epsilon g(|\nabla u|)\xi] dx = \int_\Omega \xi d\nu, \qquad \forall\ \xi \in \mathbb{X}_\alpha, \tag{1.8}$$

where $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ *is the space of functions* ξ *satisfying:*
(i) $\mathrm{supp}(\xi) \subset \bar{\Omega}$,
(ii) $(-\Delta)^\alpha \xi(x)$ *exists for all* $x \in \Omega$ *and* $|(-\Delta)^\alpha \xi(x)| \le C$ *for some* $C > 0$,
(iii) there exist $\varphi \in L^1(\Omega, \rho^\alpha dx)$ *and* $\varepsilon_0 > 0$ *such that* $|(-\Delta)^\alpha_\varepsilon \xi| \le \varphi$ *a.e. in* Ω, *for all* $\varepsilon \in (0, \varepsilon_0]$.

We denote by G_α the Green kernel of $(-\Delta)^\alpha$ in Ω and by $\mathbb{G}_\alpha[.]$ the associated Green operator defined by

$$\mathbb{G}_\alpha[\nu](x) = \int_\Omega G_\alpha(x,y)d\nu(y), \qquad \forall\ \nu \in \mathfrak{M}(\Omega, \rho^\alpha). \tag{1.9}$$

Using bounds of $\mathbb{G}_\alpha[\nu]$, we obtain in section 2 some crucial estimates which will play an important role in our construction of weak solutions. Our main result in the case $\epsilon = 1$ is the following.

Theorem 1.1 *Assume that* $\epsilon = 1$ *and* $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is a continuous function verifying* $g(0) = 0$ *and*

$$\int_1^{+\infty} g(s) s^{-1-p^*_\alpha} ds < +\infty, \tag{1.10}$$

where

$$p^*_\alpha = \frac{N}{N - 2\alpha + 1}. \tag{1.11}$$

Then for any $\nu \in \mathfrak{M}_+(\Omega, \rho^\beta)$ *with* $\beta \in [0, 2\alpha - 1)$, *problem (1.1) admits a nonnegative weak solution* u_ν *which satisfies*

$$u_\nu \le \mathbb{G}_\alpha[\nu]. \tag{1.12}$$

As in the case $\alpha = 1$, uniqueness remains an open question. We remark that the critical value p^*_α is independent of β. A similar fact was first observed when dealing with problem (1.6) where the critical value $k_{\alpha,\beta}$ defined by (1.7) does not depend on β when $\beta \in [0, \frac{N-2\alpha}{N}\alpha]$.

When $\epsilon = -1$, we have to consider the critical value $p^*_{\alpha,\beta}$ which depends truly on β and is expressed by

$$p^*_{\alpha,\beta} = \frac{N}{N - 2\alpha + 1 + \beta}. \tag{1.13}$$

We observe that $p^*_{\alpha,0} = p^*_\alpha$ and $p^*_{\alpha,\beta} < p^*_\alpha$ when $\beta > 0$. In the source case, the assumptions on g are of a different nature from in the absorption case, namely

(G) $\qquad g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function which satisfies

$$g(s) \le c_1 s^p + \sigma_0, \quad \forall s \ge 0, \tag{1.14}$$

for some $p \in (0, p^*_{\alpha,\beta})$, where $c_1 > 0$ and $\sigma_0 > 0$.

Our main result concerning the source case is the following.

Theorem 1.2 *Assume that* $\epsilon = -1$, $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ *with* $\beta \in [0, 2\alpha - 1)$ *is nonnegative,* g *satisfies* (G) *and*
(i) $p \in (0, 1)$, *or*
(ii) $p = 1$ *and* c_1 *is small enough, or*
(iii) $p \in (1, p^*_{\alpha,\beta})$, σ_0 *and* $\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}$ *are small enough.*
Then problem (1.1) admits a weak nonnegative solution u_ν *which satisfies*

$$u_\nu \ge \mathbb{G}_\alpha[\nu]. \tag{1.15}$$

We note that Bidaut-Véron, García-Huidobro and Véron in [5] obtained the existence of a renormalized solution of

$$-\Delta_p u = |\nabla u|^q + \nu \quad \text{in } \Omega,$$

when $\nu \in \mathfrak{M}^b(\Omega)$. We make use of some idea in [5] in the proof of Theorem 1.2 and extend some results in [5] to elliptic equations involving $(-\Delta)^\alpha$ with $\alpha \in (1/2, 1)$ and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1)$.

In the last section, we assume that Ω contains 0 and give pointwise estimates of the positive solutions

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p = \delta_0 \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{1.16}$$

when $0 < p < p^*_\alpha$. Combining properties of the Riesz kernel with a bootstrap argument, we prove that any weak solution of (1.16) is regular outside 0 and is actually a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p = 0 \quad &\text{in} \quad \Omega \setminus \{0\}, \\ u = 0 \quad &\text{in} \quad \Omega^c. \end{aligned} \tag{1.17}$$

These pointwise estimates are quite easy to establish in the case $\alpha = 1$, but much more delicate when the diffusion operator is non-local. We give sharp asymptotics of the behaviour of u near 0 and prove that the solution of (1.16) is unique in the class of positive solutions.

The paper is organized as follows. In Section 2, we study the Green operator and prove the key estimate

$$\|\nabla \mathbb{G}_\alpha[\nu]\|_{M^{p^*_\alpha}(\Omega, \rho^\alpha dx)} \le c_2 \|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}$$

Section 3 is devoted to prove Theorem 1.1 and Theorem 1.2. In Section 4, we consider the case where $\epsilon = 1$ in (1.1) and ν is a Dirac mass. We obtain precise asymptotic estimate and derive uniqueness.

Aknowledgements. The authors are grateful to Marie-Françoise Bidaut-Véron for useful discussions in the preparation of this work.

2 Preliminaries

2.1 Marcinkiewicz type estimates

In this subsection, we recall some definitions and properties of Marcinkiewicz spaces.

Definition 2.1 *Let $\Theta \subset \mathbb{R}^N$ be a domain and μ be a positive Borel measure in Θ. For $\kappa > 1$, $\kappa' = \kappa/(\kappa - 1)$ and $u \in L^1_{loc}(\Theta, d\mu)$, we set*

$$\|u\|_{M^\kappa(\Theta,d\mu)} = \inf\left\{ c \in [0, \infty] : \int_E |u| d\mu \le c \left(\int_E d\mu \right)^{\frac{1}{\kappa'}}, \ \forall E \subset \Theta, \ E \text{ Borel} \right\} \tag{2.1}$$

and

$$M^\kappa(\Theta, d\mu) = \{u \in L^1_{loc}(\Theta, d\mu) : \|u\|_{M^\kappa(\Theta,d\mu)} < \infty\}. \tag{2.2}$$

$M^\kappa(\Theta, d\mu)$ is called the Marcinkiewicz space of exponent κ, or weak L^κ-space and $\|.\|_{M^\kappa(\Theta,d\mu)}$ is a quasi-norm.

Proposition 2.1 *[2, 9] Assume that $1 \le q < \kappa < \infty$ and $u \in L^1_{loc}(\Theta, d\mu)$. Then there exists $c_3 > 0$ dependent of q, κ such that*

$$\int_E |u|^q d\mu \le c_3 \|u\|_{M^\kappa(\Theta,d\mu)} \left(\int_E d\mu \right)^{1-q/\kappa},$$

for any Borel set E of Θ.

The next estimate is the key-stone in the proof of Theorem 1.1.

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be a bounded C^2 domain and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1]$. Then there exists $c_2 > 0$ such that*

$$\|\nabla \mathbb{G}_\alpha[|\nu|]\|_{M^{p^*_\alpha}(\Omega,\rho^\alpha dx)} \le c_2 \|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \tag{2.3}$$

where $\nabla \mathbb{G}_\alpha[|\nu|](x) = \displaystyle\int_\Omega \nabla_x G_\alpha(x, y) d|\nu(y)|$ and p^*_α is given by (1.11).

Proof. For $\lambda > 0$ and $y \in \Omega$, we set

$$\omega_\lambda(y) = \{x \in \Omega \setminus \{y\} : |\nabla_x G_\alpha(x,y)|\rho^\alpha(x) > \lambda\}\,,\ m_\lambda(y) = \int_{\omega_\lambda(y)} dx.$$

From [11], there exists $c_4 > 0$ such that for any $(x,y) \in \Omega \times \Omega$ with $x \neq y$,

$$G_\alpha(x,y) \leq c_4 \min\left\{\frac{1}{|x-y|^{N-2\alpha}}, \frac{\rho^\alpha(x)}{|x-y|^{N-\alpha}}, \frac{\rho^\alpha(y)}{|x-y|^{N-\alpha}}\right\}, \tag{2.4}$$

$$G_\alpha(x,y) \leq c_4 \frac{\rho^\alpha(y)}{\rho^\alpha(x)|x-y|^{N-2\alpha}},$$

and by Corollary 3.3 in [3], we have

$$|\nabla_x G_\alpha(x,y)| \leq N G_\alpha(x,y) \max\left\{\frac{1}{|x-y|}, \frac{1}{\rho(x)}\right\}. \tag{2.5}$$

This implies that for any $\tau \in [0,1]$

$$G_\alpha(x,y) \leq c_4 (\frac{\rho^\alpha(y)}{|x-y|^{N-\alpha}})^\tau (\frac{\rho^\alpha(x)}{|x-y|^{N-\alpha}})^{1-\tau} = c_4 \frac{\rho^{\alpha\tau}(y)\rho^{\alpha(1-\tau)}(x)}{|x-y|^{N-\alpha}},$$

and then

$$|\nabla_x G_\alpha(x,y)| \leq c_5 \max\left\{\frac{\rho^\alpha(y)}{\rho^\alpha(x)|x-y|^{N-2\alpha+1}}, \frac{\rho^{\alpha\tau}(y)\rho^{\alpha(1-\tau)-1}(x)}{|x-y|^{N-\alpha}}\right\}. \tag{2.6}$$

Letting $\tau = \frac{2\alpha-1}{\alpha}\frac{N-\alpha}{N-2\alpha+1} \in (0,1)$, we derive

$$|\nabla_x G_\alpha(x,y)|\rho^\alpha(x) \leq c_5 \max\left\{\frac{\rho^{2\alpha-1}(y)\rho_\Omega^{1-\alpha}}{|x-y|^{N-2\alpha+1}}, \frac{\rho^{\frac{(2\alpha-1)(N-\alpha)}{N-2\alpha+1}}(y)\rho_\Omega^{\frac{(2\alpha-1)(1-\alpha)}{N-2\alpha+1}}}{|x-y|^{N-\alpha}}\right\}.$$

where $\rho_\Omega = \sup_{z\in\Omega}\rho(z)$. There exists some $c_6 > 0$ such that

$$\omega_\lambda(y) \subset \left\{x \in \Omega : |x-y| \leq c_6 \rho^{\frac{2\alpha-1}{N-2\alpha+1}}(y) \max\{\lambda^{-\frac{1}{N-2\alpha+1}}, \lambda^{-\frac{1}{N-\alpha}}\}\right\}.$$

By $N - 2\alpha + 1 > N - \alpha$, we deduce that for any $\lambda > 1$, there holds

$$\omega_\lambda(y) \subset \{x \in \Omega : |x-y| \leq c_6 \rho^{\frac{2\alpha-1}{N-2\alpha+1}}(y)\lambda^{-\frac{1}{N-2\alpha+1}}\}. \tag{2.7}$$

As a consequence,

$$m_\lambda(y) \leq c_7 \rho^{(2\alpha-1)p_\alpha^*}(y)\lambda^{-p_\alpha^*},$$

where $c_7 > 0$ independent of y and λ.

Let $E \subset \Omega$ be a Borel set and $\lambda > 1$, then

$$\int_E |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dx \leq \int_{\omega_\lambda(y)} |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dx + \lambda \int_E dx.$$

Noting that

$$\begin{aligned}\int_{\omega_\lambda(y)} |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dx &= -\int_\lambda^\infty s dm_s(y)\\ &= \lambda m_\lambda(y) + \int_\lambda^\infty m_s(y)ds\\ &\le c_8\rho^{(2\alpha-1)p^*_\alpha}(y)\lambda^{1-p^*_\alpha},\end{aligned}$$

for some $c_8 > 0$, we derive

$$\int_E |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dx \le c_8\rho^{(2\alpha-1)p^*_\alpha}(y)\lambda^{1-p^*_\alpha} + \lambda\int_E dx.$$

Choosing $\lambda = \rho^{2\alpha-1}(y)(\int_E dx)^{-\frac{1}{p^*_\alpha}}$ yields

$$\int_E |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dx \le (c_8+1)\rho^{2\alpha-1}(y)(\int_E dx)^{\frac{p^*_\alpha-1}{p^*_\alpha}}, \quad \forall y\in\Omega.$$

Therefore,

$$\begin{aligned}\int_E |\nabla\mathbb{G}_\alpha[|\nu|](x)|\rho^\alpha(x)dx &= \int_\Omega\int_E |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dxd|\nu(y)|\\ &\le \int_\Omega \rho^{2\alpha-1}(y)\left(\rho^{1-2\alpha}(y)\int_E |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dx\right)d|\nu(y)|\\ &\le (c_8+1)\int_\Omega \rho^\beta(y)\rho^{2\alpha-1-\beta}(y)d|\nu(y)|\left(\int_E dx\right)^{\frac{p^*_\alpha-1}{p^*_\alpha}}\\ &\le (c_8+1)\rho_\Omega^{2\alpha-1-\beta}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}\left(\int_E dx\right)^{\frac{p^*_\alpha-1}{p^*_\alpha}}.\end{aligned} \tag{2.8}$$

As a consequence,

$$\|\nabla\mathbb{G}_\alpha[|\nu|]\|_{M^{p^*_\alpha}(\Omega,\rho^\alpha dx)} \le c_2\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)},$$

which ends the proof. □

Proposition 2.3 *[13] Assume that $\nu\in L^1(\Omega,\rho^\beta dx)$ with $0\le\beta\le\alpha$. Then for $p\in(1,\frac{N}{N-2\alpha+\beta})$, there exists $c_9>0$ such that for any $\nu\in L^1(\Omega,\rho^\beta dx)$*

$$\|\mathbb{G}_\alpha[\nu]\|_{W^{2\alpha-\gamma,p}(\Omega)} \le c_9\|\nu\|_{L^1(\Omega,\rho^\beta dx)}, \tag{2.9}$$

where $p' = \frac{p}{p-1}$, $\gamma = \beta + \frac{N}{p'}$ if $\beta>0$ and $\gamma > \frac{N}{p'}$ if $\beta = 0$.

Proposition 2.4 *If $0\le\beta<2\alpha-1$, then the mapping $\nu\mapsto|\nabla\mathbb{G}_\alpha[\nu]|$ is compact from $L^1(\Omega,\rho^\beta dx)$ into $L^q(\Omega)$ for any $q\in[1,p^*_{\alpha,\beta})$ and there exists $c_{10}>0$ such that*

$$\left(\int_\Omega |\nabla\mathbb{G}_\alpha[\nu](x)|^q dx\right)^{\frac{1}{q}} \le c_{10}\int_\Omega |\nu(x)|\rho^\beta(x)dx, \tag{2.10}$$

where $p^*_{\alpha,\beta}$ is given by (1.13).

Proof. For $\nu \in L^1(\Omega, \rho^\beta dx)$ with $0 \le \beta < 2\alpha - 1 < \alpha$, we obtain from Proposition 2.3 that

$$\mathbb{G}_\alpha[\nu] \in W^{2\alpha-\gamma,p}(\Omega),$$

where $p \in (1, p^*_{\alpha,\beta})$ and $2\alpha - \gamma > 1$. Therefore, $|\nabla \mathbb{G}_\alpha[\nu]| \in W^{2\alpha-\gamma-1,p}(\Omega)$ and

$$\|\nabla \mathbb{G}_\alpha[\nu]\|_{W^{2\alpha-\gamma-1,p}(\Omega)} \le c_9 \|\nu\|_{L^1(\Omega,\rho^\beta dx)}. \tag{2.11}$$

By [23, Corollary 7.2], the embedding of $W^{2\alpha-\gamma-1,p}(\Omega)$ into $L^q(\Omega)$ is compact for $q \in [1, \frac{Np}{N-(2\alpha-\gamma-1)p})$. When $\beta > 0$,

$$\begin{aligned}\frac{Np}{N-(2\alpha-\gamma-1)p} &= \frac{Np}{N-(2\alpha-\beta-N\frac{p-1}{p}-1)p} \\ &= \frac{N}{N-2\alpha+1+\beta} = p^*_{\alpha,\beta}.\end{aligned}$$

When $\beta = 0$,

$$\begin{aligned}\lim_{\gamma\to(\frac{N}{p'})^+} \frac{Np}{N-(2\alpha-\gamma-1)p} &= \frac{Np}{N-(2\alpha-N\frac{p-1}{p}-1)p} \\ &= \frac{N}{N-2\alpha+1} = p^*_{\alpha,0}.\end{aligned}$$

Then the mapping $\nu \mapsto |\nabla \mathbb{G}_\alpha[\nu]|$ is compact from $L^1(\Omega, \rho^\beta dx)$ into $L^q(\Omega)$ for any $q \in [1, p^*_{\alpha,\beta})$. Inequality (2.10) follows by (2.11) and the continuity of the embedding of $W^{2\alpha-\gamma-1,p}(\Omega)$ into $L^q(\Omega)$. $\square$

Remark. If $\nu \in L^1(\Omega, \rho^\beta dx)$ with $0 \le \beta < 2\alpha - 1$ and u is the solution of

$$\begin{aligned}(-\Delta)^\alpha u = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c,\end{aligned}$$

then for any $q \in [1, p^*_{\alpha,\beta})$,

$$\left(\int_\Omega |\nabla u|^q dx\right)^{\frac{1}{q}} \le c_{10} \int_\Omega |\nu(x)| \rho^\beta(x) dx.$$

2.2 Classical solutions

In this subsection we consider the question of existence of classical solutions to problem

$$\begin{aligned}(-\Delta)^\alpha u + h(|\nabla u|) = f \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c.\end{aligned} \tag{2.12}$$

Theorem 2.1 *Assume $h \in C^\theta(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ for some $\theta \in (0,1]$ and $f \in C^\theta(\bar{\Omega})$. Then problem (2.12) admits a unique classical solution u. Moreover,*
(i) if $f - h(0) \geq 0$ in Ω, then $u \geq 0$;
(ii) the mappings $h \mapsto u$ and $f \mapsto u$ are respectively nonincreasing and nondecreasing.

Proof. We divide the proof into several steps.
Step 1. Existence. We define the operator T by

$$Tu = \mathbb{G}_\alpha\left[f - h(|\nabla u|)\right], \quad \forall u \in W_0^{1,1}(\Omega).$$

Using (2.6) with $\tau = 0$ yields

$$\begin{aligned} \|Tu\|_{W^{1,1}(\Omega)} &\leq \|\mathbb{G}_\alpha[f]\|_{W^{1,1}(\Omega)} + \|\mathbb{G}_\alpha[h(|\nabla u|)]\|_{W^{1,1}(\Omega)} \\ &\leq \left(\|f\|_{L^\infty(\Omega)} + \|h(|\nabla u|)\|_{L^\infty(\Omega)}\right) \| \int_\Omega G_\alpha(\cdot,y)dy\|_{W^{1,1}(\Omega)} \\ &= c_{11}\left(\|f\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\mathbb{R}_+)}\right), \end{aligned} \tag{2.13}$$

where $c_{11} = \| \int_\Omega G_\alpha(\cdot,y)dy\|_{W^{1,1}(\Omega)}$. Thus T maps $W_0^{1,1}(\Omega)$ into itself. Clearly, if $u_n \to u$ in $W_0^{1,1}(\Omega)$ as $n \to \infty$, then $h(|\nabla u_n|) \to h(|\nabla u|)$ in $L^1(\Omega)$, thus T is continuous. We claim that T is a compact operator. In fact, for $u \in W_0^{1,1}(\Omega)$, we see that $f - h(|\nabla u|) \in L^1(\Omega)$ and then, by Proposition 2.3, it implies that $Tu \in W_0^{2\alpha-\gamma,p}(\Omega)$ where $\gamma \in (\frac{N(p-1)}{p}, 2\alpha - 1)$ and $2\alpha - 1 > \frac{N(p-1)}{p} > 0$ for $p \in (1, \frac{N}{N-2\alpha+1})$. Since the embedding $W_0^{2\alpha-\gamma,p}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$ is compact, T is a compact operator.

Let $\mathcal{O} = \{u \in W_0^{1,1}(\Omega) : \|u\|_{W^{1,1}(\Omega)} \leq c_{10}(\|f\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\mathbb{R}_+)})\}$, which is a closed and convex set of $W_0^{1,1}(\Omega)$. Combining with (2.13), there holds

$$T(\mathcal{O}) \subset \mathcal{O}.$$

It follows by Schauder's fixed point theorem that there exists some $u \in W_0^{1,1}(\Omega)$ such that $Tu = u$.

Next we show that u is a classical solution of (2.12). Let open set O satisfy $O \subset \bar{O} \subset \Omega$. By Proposition 2.3 in [26], for any $\sigma \in (0, 2\alpha)$, there exists $c_{12} > 0$ such that

$$\|u\|_{C^\sigma(O)} \leq c_{12}\{\|h(|\nabla u|)\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\},$$

and by choosing $\sigma = \frac{2\alpha+1}{2} \in (1, 2\alpha)$, then

$$\||\nabla u|\|_{C^{\sigma-1}(O)} \leq c_{12}\{\|h(|\nabla u|)\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\},$$

and then applied [26, Corollary 2.4], u is $C^{2\alpha+\epsilon_0}$ locally in Ω for some $\epsilon_0 > 0$. Then u is a classical solution of (2.12). Moreover, from [13], we have

$$\int_\Omega [u(-\Delta)^\alpha \xi + h(|\nabla u|)\xi]dx = \int_\Omega \xi f dx, \quad \forall \xi \in \mathbb{X}_\alpha. \tag{2.14}$$

Step 2. Proof of (i). If u is not nonnegative, then there exists $x_0 \in \Omega$ such that

$$u(x_0) = \min_{x\in\Omega} u(x) < 0,$$

then $\nabla u(x_0) = 0$ and $(-\Delta)^\alpha u(x_0) < 0$. Since u is the classical solution of (2.12), $(-\Delta)^\alpha u(x_0) = f(x_0) - h(0) \geq 0$, which is a contradiction.

Step 3. Proof of (ii). We just give the proof of the first argument, the proof of the second being similar. Let h_1 and h_2 satisfy our hypotheses for h and $h_1 \leq h_2$. Denote u_1 and u_2 the solutions of (2.12) with h replaced by h_1 and h_2 respectively. If there exists $x_0 \in \Omega$ such that

$$(u_1 - u_2)(x_0) = \min_{x\in\Omega}\{(u_1 - u_2)(x)\} < 0.$$

Then

$$(-\Delta)^\alpha (u_1 - u_2)(x_0) < 0, \quad \nabla u_1(x_0) = \nabla u_2(x_0).$$

This implies

$$(-\Delta)^\alpha (u_1 - u_2)(x_0) + h_1(|\nabla u_1(x_0)|) - h_2(|\nabla u_2(x_0)|) < 0. \tag{2.15}$$

However,

$$(-\Delta)^\alpha (u_1 - u_2)(x_0) + h_1(|\nabla u_1(x_0)|) - h_2(|\nabla u_2(x_0)|) = f(x_0) - f(x_0) = 0,$$

contradiction. Then $u_1 \geq u_2$.

Uniqueness follows from Step 3. $\square$

3 Proof of Theorems 1.1 and 1.2

3.1 The absorption case

In this subsection, we prove the existence of a weak solution to (1.1) when $\epsilon = 1$. To this end, we give below an auxiliary lemma.

Lemma 3.1 *Assume that* $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is continuous and (1.10) holds with* p^*_α. *Then there is a sequence real positive numbers* $\{T_n\}$ *such that*

$$\lim_{n\to\infty} T_n = \infty \quad \text{and} \quad \lim_{n\to\infty} g(T_n)T_n^{-p^*_\alpha} = 0.$$

Proof. Let $\{s_n\}$ be a sequence of real positive numbers converging to ∞. We observe

$$\begin{aligned}\int_{s_n}^{2s_n} g(t)t^{-1-p^*_\alpha}dt &\geq \min_{t\in[s_n,2s_n]} g(t)(2s_n)^{-1-p^*_\alpha}\int_{s_n}^{2s_n} dt \\ &= 2^{-1-p^*_\alpha}s_n^{-p^*_\alpha}\min_{t\in[s_n,2s_n]} g(t)\end{aligned}$$

and by (1.10),

$$\lim_{n\to\infty}\int_{s_n}^{2s_n} g(t)t^{-1-p_\alpha^*}dt = 0.$$

Then we choose $T_n \in [s_n, 2s_n]$ such that $g(T_n) = \min_{t\in[s_n,2s_n]} g(t)$ and then the claim follows. $\square$

Proof of Theorem 1.1. Let $\beta \in [0, 2\alpha - 1)$, we define the space

$$C_\beta(\bar\Omega) = \{\zeta \in C(\bar\Omega) : \rho^{-\beta}\zeta \in C(\bar\Omega)\}$$

endowed with the norm

$$\|\zeta\|_{C_\beta(\bar\Omega)} = \|\rho^{-\beta}\zeta\|_{C(\bar\Omega)}.$$

Let $\{\nu_n\} \subset C^1(\bar\Omega)$ be a sequence of nonnegative functions such that $\nu_n \to \nu$ in sense of duality with $C_\beta(\bar\Omega)$, that is,

$$\lim_{n\to\infty}\int_{\bar\Omega}\zeta\nu_n dx = \int_{\bar\Omega}\zeta d\nu, \qquad \forall\zeta \in C_\beta(\bar\Omega). \tag{3.1}$$

By the Banach-Steinhaus Theorem, $\|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)}$ is bounded independently of n. We consider a sequence $\{g_n\}$ of C^1 nonnegative functions defined on $\mathbb{R}_+$ such that $g_n(0) = 0$ and

$$g_n \le g_{n+1} \le g, \quad \sup_{s\in\mathbb{R}_+} g_n(s) = n \quad \text{and} \quad \lim_{n\to\infty}\|g_n - g\|_{L^\infty_{loc}(\mathbb{R}_+)} = 0. \tag{3.2}$$

By Theorem 2.1, there exists a unique *nonnegative* solution u_n of (1.1) with data ν_n and g_n instead of ν and g, and there holds

$$\int_\Omega (u_n + g_n(|\nabla u_n|)\eta_1)\, dx = \int_\Omega \nu_n\eta_1 dx \le C\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \tag{3.3}$$

where $\eta_1 = \mathbb{G}_\alpha[1]$. Therefore, $\|g_n(|\nabla u_n|)\|_{\mathfrak{M}(\Omega,\rho^\alpha)}$ is bounded independently of n. For $\varepsilon > 0$ and $\xi_\varepsilon = (\eta_1+\varepsilon)^{\frac{\beta}{\alpha}} - \varepsilon^{\frac{\beta}{\alpha}} \in \mathbb{X}_\alpha$ which is concave in the interval $[0, \eta_1(\bar\omega)]$, where $\eta_1(\bar\omega) = \max_{x\in\Omega}\eta_1(x)$. By [13, Lemma 2.3 (ii)], we see that

$$\begin{aligned}(-\Delta)^\alpha\xi_\varepsilon &= \frac{\beta}{\alpha}(\eta_1+\varepsilon)^{\frac{1}{\alpha}}(-\Delta)^\alpha\eta_1 - \frac{\beta(\beta-\alpha)}{\alpha^2}(\eta_1+\varepsilon)^{\frac{\beta-2\alpha}{\alpha}}\int_\Omega\frac{(\eta_1(y)-\eta_1(x))^2}{|y-x|^{N+2\alpha}}dy \\ &\ge \frac{\beta}{\alpha}(\eta_1+\varepsilon)^{\frac{\beta-\alpha}{\alpha}},\end{aligned}$$

and $\xi_\varepsilon \in \mathbb{X}_\alpha$. Since

$$\int_\Omega (u_n(-\Delta)^\alpha\xi_\varepsilon + g_n(|\nabla u_n|)\xi_\varepsilon)\, dx = \int_\Omega \xi_\varepsilon\nu_n dx,$$

we obtain

$$\int_\Omega \left(\frac{\beta}{\alpha}u_n(\eta_1+\varepsilon)^{\frac{\beta-\alpha}{\alpha}} + g_n(|\nabla u_n|)\xi_\varepsilon\right)dx \le \int_\Omega \xi_\varepsilon \nu_n dx.$$

If we let $\varepsilon \to 0$, it yields

$$\int_\Omega \left(\frac{\beta}{\alpha}u_n\eta_1^{\frac{\beta-\alpha}{\alpha}} + g_n(|\nabla u_n|)\eta_1^{\frac{\beta}{\alpha}}\right)dx \le \int_\Omega \eta_1^{\frac{\beta}{\alpha}} \nu_n dx.$$

Using [13, Lemma 2.3], we derive the estimate

$$\int_\Omega \left(u_n\rho^{\beta-\alpha} + g_n(|\nabla u_n|)\rho^\beta\right)dx \le c_{13}\|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)} \le c_{14}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}. \quad (3.4)$$

Thus $\{g_n(|\nabla u_n|)\}$ is uniformly bounded in $L^1(\Omega,\rho^\beta dx)$. Since $u_n = \mathbb{G}[\nu_n - g_n(|\nabla u_n|)]$, there holds

$$\begin{aligned} \||\nabla u_n|\|_{M^{p^*_\alpha}(\Omega,\rho^\alpha dx)} &\le \|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)} + \|g_n(|\nabla u_n|)\|_{\mathfrak{M}(\Omega,\rho^\beta)} \\ &\le c_{15}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}. \end{aligned}$$

Since $\nu_n - g_n(|\nabla u_n|)$ is uniformly bounded in $L^1(\Omega,\rho^\beta dx)$, we use Proposition 2.4 to obtain that the sequences $\{u_n\}$, $\{|\nabla u_n|\}$ are relatively compact in $L^q(\Omega)$ for $q \in [1, \frac{N}{N-2\alpha+\beta})$ and $q \in [1, p^*_{\alpha,\beta})$, respectively. Thus, there exist a sub-sequence $\{u_{n_k}\}$ and some $u \in L^q(\Omega)$ with $q \in [1, \frac{N}{N-2\alpha+\beta})$ such that

(i) $u_{n_k} \to u$ a.e. in Ω and in $L^q(\Omega)$ with $q \in [1, \frac{N}{N-2\alpha+\beta})$;
(ii) $|\nabla u_{n_k}| \to |\nabla u|$ a.e. in Ω and in $L^q(\Omega)$ with $q \in [1, p^*_{\alpha,\beta})$.
Therefore, $g_{n_k}(|\nabla u_{n_k}|) \to g(|\nabla u|)$ a.e. in Ω. For $\lambda > 0$, we denote

$$S_\lambda = \{x \in \Omega : |\nabla u_{n_k}(x)| > \lambda\} \quad \text{and} \quad \omega(\lambda) = \int_{S_\lambda} \rho^\alpha(x)dx.$$

Then for any Borel set $E \subset \Omega$, we have that

$$\begin{aligned} \int_E g_{n_k}(|\nabla u_{n_k}|)|\rho^\alpha(x)dx &\le \int_E g(|\nabla u_{n_k}|)|\rho^\alpha(x)dx \\ &= \int_{E\cap S_\lambda^c} g(|\nabla u_{n_k}|)\rho^\alpha(x)dx + \int_{E\cap S_\lambda} g(|\nabla u_{n_k}|)\rho^\alpha(x)dx \\ &\le \tilde{g}(\lambda)\int_E \rho^\alpha(x)dx + \int_{S_\lambda} g(|\nabla u_{n_k}|)\rho^\alpha(x)dx \\ &\le \tilde{g}(\lambda)\int_E \rho^\alpha(x)dx - \int_\lambda^\infty g(s)d\omega(s), \end{aligned}$$

where $\tilde{g}(s) = \max_{t\in[0,s]}\{g(t)\}$. But

$$\int_\lambda^\infty g(s)d\omega(s) = \lim_{n\to\infty}\int_\lambda^{T_n} g(s)d\omega(s).$$

where $\{T_n\}$ is given by Lemma 3.1. Since $|\nabla u_{n_k}| \in M^{p^*_\alpha}(\Omega, \rho^\alpha dx)$, $\omega(s) \le c_{16}s^{-p^*_\alpha}$ and

$$-\int_\lambda^{T_n} g(s)d\omega(s) = -\Big[g(s)\omega(s)\Big]_{s=\lambda}^{s=T_n} + \int_\lambda^{T_n} \omega(s)dg(s)$$
$$\le g(\lambda)\omega(\lambda) - g(T_n)\omega(T_n) + c_{16}\int_\lambda^{T_n} s^{-p^*_\alpha}dg(s)$$
$$\le g(\lambda)\omega(\lambda) - g(T_n)\omega(T_n) + c_{16}\left({T_n}^{-p^*_\alpha}g(T_n) - \lambda^{-p^*_\alpha}g(\lambda)\right)$$
$$+\frac{c_{16}}{p^*_\alpha+1}\int_\lambda^{T_n} s^{-1-p^*_\alpha}g(s)ds.$$

By assumption (1.10) and Lemma 3.1, it follows

$$\lim_{n\to\infty} T_n^{-p^*_\alpha}g(T_n) = 0. \tag{3.5}$$

Along with $g(\lambda)\omega(\lambda) \le c_{16}\lambda^{-p^*_\alpha}g(\lambda)$, we have

$$-\int_\lambda^\infty g(s)d\omega(s) \le \frac{c_{16}}{p^*_\alpha+1}\int_\lambda^\infty s^{-1-p^*_\alpha}g(s)ds.$$

Notice that the above quantity on the right-hand side tends to 0 when $\lambda\to\infty$. It implies that for any $\epsilon>0$ there exists $\lambda>0$ such that

$$\frac{c_{16}}{p^*_\alpha+1}\int_\lambda^\infty s^{-1-p^*_\alpha}g(s)ds \le \frac{\epsilon}{2},$$

and $\delta>0$ such that

$$\int_E \rho^\alpha(x)dx \le \delta \Longrightarrow \tilde{g}(\lambda)\int_E dx \le \frac{\epsilon}{2}.$$

This proves that $\{g_{n_k}(|\nabla u_{n_k}|)\}$ is uniformly integrable in $L^1(\Omega,\rho^\alpha dx)$. Then $g_{n_k}(|\nabla u_{n_k}|) \to g(|\nabla u|)$ in $L^1(\Omega,\rho^\alpha dx)$ by Vitali convergence theorem. Letting $n_k\to\infty$ in the identity

$$\int_\Omega \left(u_{n_k}(-\Delta)^\alpha\xi + g_{n_k}(|\nabla u_{n_k}|)\xi\right)dx = \int_\Omega \nu_{n_k}\xi dx, \quad \forall\xi\in\mathbb{X}_\alpha,$$

it infers that u is a weak solution of (1.1). Since u_{n_k} is nonnegative, so is u. Estimate (1.12) is a consequence of positivity and

$$u_{n_k} = \mathbb{G}_\alpha[\nu_{n_k}] - \mathbb{G}_\alpha[g_{n_k}(|\nabla u_{n_k}|)] \le \mathbb{G}_\alpha[\nu_{n_k}].$$

Since $\lim_{n_k\to\infty} u_{n_k} = u$, (1.12) follows. $\square$

3.2 The source case

In this subsection we study the existence of solutions to problem (1.1) when $\epsilon = -1$.

Proof of Theorem 1.2. Let $\{\nu_n\}$ be a sequence of C^2 nonnegative functions converging to ν in the sense of (3.1), $\{g_n\}$ an increasing sequence of C^1, nonnegative bounded functions defined on $\mathbb{R}_+$ satisfying (3.2) and converging to g. We set $p_0 = \frac{p+p^*_{\alpha,\beta}}{2} \in (p, p^*_{\alpha,\beta})$, where $p^*_{\alpha,\beta}$ is given by (1.13) and $p < p^*_{\alpha,\beta}$ is the maximal growth rate of g which satisfies (1.14), and

$$M(v) = \left(\int_\Omega |\nabla v|^{p_0} dx\right)^{\frac{1}{p_0}}.$$

We may assume that $\|\nu_n\|_{L^1(\Omega,\rho^\beta dx)} \le 2\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}$ for all $n \ge 1$.
Step 1. We claim that for $n \ge 1$,

$$\begin{aligned} (-\Delta)^\alpha u_n &= g_n(|\nabla u_n|) + \nu_n \quad &&\text{in} \quad \Omega,\\ u_n &= 0 &&\text{in} \quad \Omega^c \end{aligned}$$

admits a solution u_n such that

$$M(u_n) \le \bar\lambda,$$

where $\bar\lambda > 0$ independent of n.

To this end, we define the operators $\{T_n\}$ by

$$T_n u = \mathbb{G}_\alpha\left[g_n(|\nabla u|) + \nu_n\right], \quad \forall u \in W^{1,p_0}_0(\Omega).$$

On the one hand, using (2.6) with $\tau = 0$ yields

$$\begin{aligned} \|T_n u\|_{W^{1,1}(\Omega)} &\le \|\mathbb{G}_\alpha[\nu_n]\|_{W^{1,1}(\Omega)} + \|\mathbb{G}_\alpha[g_n(|\nabla u|)]\|_{W^{1,1}(\Omega)}\\ &\le c_{11}\left(\|\nu_n\|_{L^\infty(\Omega)} + \|g_n\|_{L^\infty(\mathbb{R}_+)}\right), \end{aligned}$$

where $c_{11} = \|\int_\Omega G_\alpha(\cdot,y)dy\|_{W^{1,1}(\Omega)}$. On the other hand, by (1.14) and Proposition 2.4, we have

$$\begin{aligned} \left(\int_\Omega |\nabla (T_n u)|^{p_0} dx\right)^{\frac{1}{p_0}} &\le c_2\|g_n(|\nabla u|) + \nu_n\|_{L^1(\Omega,\rho^\beta dx)}\\ &\le c_2[\|g_n(|\nabla u|)\|_{L^1(\Omega,\rho^\beta dx)} + 2\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}] \qquad (3.6)\\ &\le c_2c_1\int_\Omega |\nabla u|^p\rho^\beta dx + c_{17}\sigma_0 + 2c_2\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \end{aligned}$$

where $c_{17} = c_2\int_\Omega \rho^\beta dx$. Then we use Hölder inequality to obtain that

$$\left(\int_\Omega |\nabla u|^p\rho^\beta dx\right)^{\frac{1}{p}} \le \left(\int_\Omega \rho^{\frac{\beta p_0}{p_0-p}} dx\right)^{\frac{1}{p}-\frac{1}{p_0}} \left(\int_\Omega |\nabla u|^{p_0} dx\right)^{\frac{1}{p_0}}, \qquad (3.7)$$

where $\int_\Omega \rho^{\frac{\beta p_0}{p_0-p}} dx$ is bounded, since $\frac{\beta p_0}{p_0-p} \geq 0$. Along with (3.6) and (3.7), we derive

$$M(T_n u) \leq c_{18} M(u)^p + c_{19} \|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)} + c_{17}\sigma_0, \tag{3.8}$$

where $c_{18} = c_2 c_1 (\int_\Omega \rho^{\frac{\beta p_0}{p_0-p}} dx)^{\frac{1}{p}-\frac{1}{p_0}} > 0$ and $c_{19} > 0$ independent of n. Therefore, if we assume that $M(u) \leq \lambda$, inequality (3.8) implies

$$M(T_n u) \leq c_{18} \lambda^p + c_{19} \|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)} + c_{17}\sigma_0. \tag{3.9}$$

Let $\bar{\lambda} > 0$ be the largest root of the equation

$$c_{18} \lambda^p + c_{19} \|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)} + c_{17}\sigma_0 = \lambda, \tag{3.10}$$

This root exists if one of the following condition holds:

(i) $p \in (0,1)$, in which case (3.10) admits only one root;

(ii) $p = 1$ and $c_{17} < 1$, and again (3.10) admits only one root;

(iii) $p \in (1, p_\alpha^*)$ and there exists $\varepsilon_0 > 0$ such that $\max \left\{ \|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \sigma_0 \right\} \leq \varepsilon_0$. In that case (3.10) admits usually two positive roots.

If we suppose that one of the above conditions holds, the definition of $\bar{\lambda} > 0$ implies that it is the largest $\lambda > 0$ such that

$$c_{18} \lambda^p + c_{19} \|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)} + c_{17}\sigma_0 \leq \lambda, \tag{3.11}$$

For $M(u) \leq \bar{\lambda}$, we obtain that

$$M(T_n u) \leq c_{18} \bar{\lambda}^p + c_{19} \|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)} + c_{17}\sigma_0 = \bar{\lambda}.$$

By the assumptions of Theorem 1.2, $\bar{\lambda}$ exists and it is larger than $M(u_n)$. Therefore,

$$\int_\Omega |\nabla (T_n u)|^{p_0} dx \leq \bar{\lambda}^{p_0}. \tag{3.12}$$

Thus T_n maps $W_0^{1,p_0}(\Omega)$ into itself. Clearly, if $u_n \to u$ in $W_0^{1,p_0}(\Omega)$ as $n \to \infty$, then $g_n(|\nabla u_n|) \to g_n(|\nabla u|)$ in $L^1(\Omega)$, thus T is continuous. We claim that T is a compact operator. In fact, for $u \in W_0^{1,p_0}(\Omega)$, we see that $\nu_n - g_n(|\nabla u|) \in L^1(\Omega)$ and then, by Proposition 2.3, it implies that $T_n u \in W_0^{2\alpha-\gamma,p}(\Omega)$ where $\gamma \in (\frac{N(p-1)}{p}, 2\alpha - 1)$ and $2\alpha - 1 > \frac{N(p-1)}{p} > 0$ for $p \in (1, \frac{N}{N-2\alpha+1})$. Since the embedding $W_0^{2\alpha-\gamma,p}(\Omega) \hookrightarrow W_0^{1,p_0}(\Omega)$ is compact, T_n is a compact operator.

Let

$$\mathcal{G} = \{u \in W_0^{1,p_0}(\Omega) : \|u\|_{W^{1,1}(\Omega)} \leq c_{11}(\|\nu_n\|_{L^\infty(\Omega)} + \|g_n\|_{L^\infty(\mathbb{R}_+)}) \quad \text{and} \quad M(u) \leq \bar{\lambda}\},$$

which is a closed and convex set of $W_0^{1,p_0}(\Omega)$. Combining with (2.13), there holds

$$T_n(\mathcal{G}) \subset \mathcal{G}.$$

It follows by Schauder's fixed point theorem that there exists some $u_n \in W_0^{1,p_0}(\Omega)$ such that $T_n u_n = u_n$ and $M(u_n) \le \bar{\lambda}$, where $\bar{\lambda} > 0$ independent of n. By the same arguments as in Theorem 2.1, u_n belongs to $C^{2\alpha+\epsilon_0}$ locally in Ω, and

$$\int_\Omega u_n(-\Delta)^\alpha \xi = \int_\Omega g_n(|\nabla u_n|)\xi dx + \int_\Omega \xi \nu_n dx, \quad \forall \xi \in \mathbb{X}_\alpha. \tag{3.13}$$

Step 2: Convergence. By (3.12) and (3.7), $g_n(|\nabla u_n|)$ is uniformly bounded in $L^1(\Omega, \rho^\beta dx)$. By Proposition 2.3, $\{u_n\}$ is bounded in $W_0^{2\alpha-\gamma,q}(\Omega)$ where $q \in (1, p^*_{\alpha,\beta})$ and $2\alpha - \gamma > 1$. By Proposition 2.4, there exist a subsequence $\{u_{n_k}\}$ and u such that $u_{n_k} \to u$ a.e. in Ω and in $L^1(\Omega)$, and $|\nabla u_{n_k}| \to |\nabla u|$ a.e. in Ω and in $L^q(\Omega)$ for any $q \in [1, p^*_{\alpha,\beta})$. By assumption (G), $g_{n_k}(|\nabla u_{n_k}|) \to g(|\nabla u|)$ in $L^1(\Omega)$. Letting $n_k \to \infty$ to have that

$$\int_\Omega u(-\Delta)^\alpha \xi = \int_\Omega g(|\nabla u|)\xi dx + \int_\Omega \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha,$$

thus u is a weak solution of (1.1) which is nonnegative as $\{u_n\}$ are nonnegative. Furthermore, (1.15) follows from the positivity of $g(|\nabla u_n])$. $\square$

4 The case of the Dirac mass

In this section we assume that Ω is an open, bounded and C^2 domain containing 0 and u a nonnegative weak solution of

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p = \delta_0 \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{4.1}$$

where $p \in (0, p^*_\alpha)$ and δ_0 is the Dirac mass at 0. We recall the following result dealing with the convolution operator $*$ in Lorentz spaces $L^{p,q}(\mathbb{R}^N)$ (see [25]).

Proposition 4.1 *Let $1 \le p_1, q_1, p_2, q_2 \le \infty$ and suppose $\frac{1}{p_1} + \frac{1}{p_2} > 1$. If $f \in L^{p_1,q_1}(\mathbb{R}^N)$ and $g \in L^{p_2,q_2}(\mathbb{R}^N)$, then $f * g \in L^{r,s}(\mathbb{R}^N)$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} - 1$, $\frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{s}$ and there holds*

$$\|f * g\|_{L^{r,s}(\mathbb{R}^N)} \le 3r \|f\|_{L^{p_1,q_1}(\mathbb{R}^N)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^N)}. \tag{4.2}$$

In the particular case of Marcinkiewicz spaces $L^{p,\infty}(\mathbb{R}^N) = M^p(\mathbb{R}^N)$, the result takes the form

$$\|f * g\|_{M^r(\mathbb{R}^N)} \le 3r \|f\|_{M^{p_1}(\mathbb{R}^N)} \|g\|_{M^{p_2}(\mathbb{R}^N)}. \tag{4.3}$$

Proposition 4.2 *Assume that $0 < p < p^*_\alpha$ and u is a nonnegative weak solution of (4.1). Then*

$$0 \le u \le \mathbb{G}_\alpha[\delta_0], \tag{4.4}$$

$|\nabla u| \in L^\infty_{loc}(\Omega \setminus \{0\})$ *and u is a classical solution of*

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p = 0 \quad &\text{in} \quad \Omega \setminus \{0\}, \\ u = 0 \quad &\text{in} \quad \Omega^c. \end{aligned} \tag{4.5}$$

Proof. Since $0 < p < p^*_\alpha$, (4.1) admits a solution. Estimate (4.4) is a particular case of (1.12). We pick a point $a \in \Omega \setminus \{0\}$ and consider a finite sequence $\{r_j\}_{j=0}^\kappa$ such that $0 < r_\kappa < r_{\kappa-1} < ... < r_0$ and $\bar{B}_{r_0}(a) \subset \Omega \setminus \{0\}$. We set $d_j = r_{j-1} - r_j$, $j = 1, ...\kappa$. By (3.4) with $\beta = 0$, it follows that

$$\int_\Omega (u + |\nabla u|^p)\, dx \le c_{20}. \tag{4.6}$$

Let $\{\eta_n\} \subset \mathbb{C}^\infty_0(\mathbb{R}^N)$ be a sequence of radially decreasing and symmetric mollifiers such that $\mathrm{supp}(\eta_n) \subset B_{\varepsilon_n}(0)$ and $\varepsilon_n \le \frac{1}{2}\min\{\rho(a) - r_0, |a| - r_0\}$ and $u_n = u * \eta_n$. Since

$$\eta_n * (-\Delta)^\alpha \xi = (-\Delta)^\alpha(\xi * \eta_n)$$

by Fourier analysis and

$$\int_{\mathbb{R}^N} (u(-\Delta)^\alpha(\xi * \eta_n) + \xi * \eta_n |\nabla u|^p) dx = \int_{\mathbb{R}^N} (u * \eta_n (-\Delta)^\alpha \xi + \eta_n * |\nabla u|^p \xi) dx$$

because η_n is radially symmetric, it follows that u_n is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u_n + |\nabla u|^p * \eta_n = \eta_n \qquad &\text{in} \ \ \Omega_n, \\ u_n = 0 \qquad &\text{in} \ \ \Omega_n^c, \end{aligned} \tag{4.7}$$

where $\Omega_n = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < \varepsilon_n\}$. We denote by $G_{\alpha,n}(x, y)$ the Green kernel of $(-\Delta)^\alpha$ in Ω_n and by $\mathbb{G}_{\alpha,n}$ the Green operator. Set $f_n = \eta_n - |\nabla u|^p * \eta_n$, then $u_n = \mathbb{G}_{\alpha,n}[f_n]$. If we set $f_{n,r_0} = f_n \chi_{B_{r_0}(a)}$, $\tilde{f}_{n,r_0} = f_n - f_{n,r_0}$, we have

$$\begin{aligned} \partial_{x_i} u_n(x) &= \int_{\Omega_n} \partial_{x_i} G_{\alpha,n}(x, y) f_n(y) dy \\ &= \int_{\Omega_n} \partial_{x_i} G_{\alpha,n}(x, y) f_{n,r_0}(y) dy + \int_{\Omega_n} \partial_{x_i} G_{\alpha,n}(x, y) \tilde{f}_{n,r_0}(y) dy \\ &= v_{n,r_0}(x) + \tilde{v}_{n,r_0}(x), \end{aligned}$$

where

$$v_{n,r_0}(x) = \int_{B_{r_0}(a)} \partial_{x_i} G_{\alpha,n}(x, y) f_n(y) dy = -\int_{B_{r_0}(a)} \partial_{x_i} G_{\alpha,n}(x, y) |\nabla u|^p * \eta_n(y) dy$$

and

$$\tilde{v}_{n,r_0}(x) = \int_{\Omega_n \setminus B_{r_0}(a)} \partial_{x_i} G_{\alpha,n}(x,y) f_n(y) dy.$$

We set $\rho_n(x) = \operatorname{dist}(x, \Omega_n^c)$, then by (2.4) and (2.5), we have

$$|\partial_{x_i} G_{\alpha,n}(x,y)| \leq c_4 N \max \left\{ \frac{1}{|x-y|^{N-2\alpha+1}}, \frac{\rho_n^{-1}(x)}{|x-y|^{N-2\alpha}} \right\}.$$

Thus, if $x \in B_{r_1}(a)$ and $y \in \Omega_n \setminus B_{r_0}(a)$, then $\rho_n(x) > d_1$ and $|x-y| > d_1$,

$$|\tilde{v}_{n,r_0}(x)| \leq c_{21} \int_{\Omega_n \setminus B_{r_0}(a)} f_n(y) dy \leq c_{20} c_{21}, \tag{4.8}$$

where $c_{21} > 0$ depends on $d_1^{-N+2\alpha-1}$, N and α. Furthermore, if $x \in B_{r_1}(a)$ and $y \in B_{r_0}(a)$,

$$|\partial_{x_i} G_{\alpha,n}(x,y)| \leq \frac{c_4 N}{|x-y|^{N-2\alpha+1}}. \tag{4.9}$$

We have already use the fact that $y \mapsto |y|^{2\alpha-N-1} \in L^{q_1}_{loc}(\mathbb{R}^N)$ with $q_1 \in (\max\{1,p\}, p^*_\alpha)$. Since f_n is uniformly bounded in $L^1(\Omega)$, there exists $c_{22} > 0$ such that

$$\|v_{n,r_0}\|_{M^{q_1}(B_{r_1}(a))} \leq c_{22}. \tag{4.10}$$

Combined with (4.8), it yields

$$\||\nabla u|^p * \eta_n\|_{M^{\frac{q_1}{p}}(B_{r_1}(a))} \leq c_{23}. \tag{4.11}$$

Next we set $f_{n,r_1} = f_n \chi_{B_{r_1}(a)}$ and $\tilde{f}_{n,r_1} = f_n - f_{n,r_1}$. Then

$$\partial_{x_i} u_n = v_{n,r_1} + \tilde{v}_{n,r_1},$$

where

$$v_{n,r_1}(x) = \int_{B_{r_1}(a)} \partial_{x_i} G_\alpha(x,y) f_n(y) dy = -\int_{B_{r_1}(a)} \partial_{x_i} G_\alpha(x,y) |\nabla u|^p * \eta_n(y) dy$$

and

$$\tilde{v}_{n,r_1}(x) = \int_{\Omega_n \setminus B_{r_1}(a)} \partial_{x_i} G_\alpha(x,y) f_n(y) dy$$

Clearly $\tilde{v}_{n,r_1}(x)$ is uniformly bounded in $B_{r_2}(a)$ by a constant c_{24} depending on the structural constants and $d_2 = r_1 - r_2$. Estimate (4.9) holds if we assume $x \in B_{r_2}(a)$ and $y \in B_{r_1}(a)$. Therefore,

$$|v_{n,r_1}(x)| \leq c_4 N \int_{B_{r_1}(a)} \frac{|\nabla u|^p * \eta_n(y)}{|x-y|^{N-2\alpha+1}} dy.$$

We derive from Proposition 4.1

$$\|v_{n,r_1}\|_{M^{q_2}(B_{r_2}(a))} \le c_{24} \||\nabla u|^p * \eta_n\|_{M^{\frac{q_1}{p}}(B_{r_1}(a))},$$

with

$$\frac{1}{q_2} = \frac{p}{q_1} + \frac{1}{q_1} - 1. \tag{4.12}$$

Notice that $q_2 > q_1$. Therefore

$$\||\nabla u|^p * \eta_n\|_{M^{\frac{q_2}{p}}(B_{r_2}(a))} \le c_{25}. \tag{4.13}$$

We iterate this construction and obtain the existence of constants c_j such that

$$\||\nabla u|^p * \eta_n\|_{M^{\frac{q_j}{p}}(B_{r_j}(a))} \le \bar{c}_j, \qquad \forall j = 1, 2, ... \tag{4.14}$$

We pick $q_1 = \frac{1}{2}(p_\alpha^* + p)$ if $p > 1$ or $q_1 = \frac{1}{2}(p_\alpha^* + 1)$ if $p \in (0, 1]$

$$\frac{1}{q_{j+1}} = \frac{p}{q_j} + \frac{1}{q_1} - 1. \tag{4.15}$$

If $p = 1$, there exists $j_0 \in \mathbb{N}$ such that $q_{j_0} > 0$ and $q_{j_0+1} \le 0$.
If $p \in (0, p_\alpha^*) \setminus \{1\}$, let $\ell = \frac{q_1 - 1}{q_1(p-1)}$, then $\ell = p\ell + \frac{1}{q_1} - 1$, thus

$$\frac{1}{q_{j+1}} = \ell + p^j \left(\frac{1}{q_1} - \ell \right) = \ell - p^j \frac{q_1 - p}{q_1(p-1)}. \tag{4.16}$$

Therefore there exists j_0 such that $q_{j_0} > 0$ and $q_{j_0+1} \le 0$. This implies

$$\||\nabla u|^p * \eta_n\|_{L^s(B_{r_{j_0+1}}(a))} \le c_{26}, \qquad \forall s < \infty \tag{4.17}$$

and

$$\||\nabla u|^p * \eta_n\|_{L^\infty(B_{r_{j_0+2}}(a))} \le c_{27}, \tag{4.18}$$

with c_{27} independent of n. Letting $n \to \infty$ infers

$$\|\nabla u\|_{L^\infty(B_{r_{j_0+2}}(a))} \le c_{27}^{\frac{1}{p}}. \tag{4.19}$$

Combining this estimate with (4.4) and using [26, Corollary 2.5] which states

$$\begin{aligned} \|u\|_{C^\beta(B_{r_{j_0+3}}(a))} \le c \Bigg(& \|u\|_{L^1(\mathbb{R}^N, \frac{dx}{1+|x|^{N+2\alpha}})} \\ & + \|u\|_{L^\infty(B_{r_{j_0+2}}(a))} + \|\nabla u\|_{L^\infty(B_{r_{j_0+2}}(a))} \Bigg), \end{aligned} \tag{4.20}$$

for any $\beta < 2\alpha$, we obtain that u remains bounded in $C^{1+\varepsilon}(K)$ for any compact set $K \subset \Omega \setminus \{0\}$ and some $\varepsilon > 0$. Using now [26, Corollary 2.4], we obtain that $C^{2\alpha+\varepsilon'}(\Omega \setminus \{0\})$ for $0 < \varepsilon' < \varepsilon$. Futhermore u is continuous up to $\partial\Omega$. As a consequence it is a strong solution in $\Omega \setminus \{0\}$. $\square$

In the next result we give a pointwise estimate of ∇u for a positive solution u of (4.1).

Proposition 4.3 *Assume that $R = \frac{1}{2}dist(0, \partial\Omega)$, $p \in (0, p^*_\alpha)$ and u is a nonnegative weak solution of (4.1). Then there exists $c_{28} > 0$ depending on R, p and α such that*

$$|\nabla u(x)| \le c_{28}|x|^{2\alpha-N-1}, \qquad \forall x \in \bar{B}_{R/4}(0) \setminus \{0\}. \tag{4.21}$$

Proof. Up to a change of variable we can assume that $R = 1$. For $0 < |x| \le 1$, there exists $b \in (0,1)$ such that $b/2 \le |x| \le b$. We set

$$u_b(y) = b^{N-2\alpha}u(by).$$

Then

$$(-\Delta)^\alpha u_b + b^{N+p(2\alpha-N-1)}|\nabla u_b|^p = 0 \qquad \text{in } \Omega_b := b^{-1}\Omega.$$

Using [26, Corollary 2.5] with $\beta < 2\alpha$, for any $a \in \Omega_b$ such that $|a| = 3/4$, there holds

$$\begin{aligned}\|u_b\|_{C^\beta(B_{\frac{3}{16}}(a))} \le c_{29} \Big(&\|u_b\|_{L^1(\mathbb{R}^N, \frac{dx}{1+|y|^{N+2\alpha}})} + \|u_b\|_{L^\infty(B_{\frac{3}{8}}(a))} \\ &+ b^{N+p(2\alpha-N-1)}\||\nabla u_b|^p\|_{L^\infty(B_{\frac{3}{8}}(a))}\Big).\end{aligned} \tag{4.22}$$

Furthermore, by the same argument as in Proposition 4.2,

$$\||\nabla u_b|^p\|_{L^\infty(B_{\frac{3}{8}}(a))} \le c_{30}\int_{\Omega_b} |\nabla u_b(y)|^p dy = c_{30}b^{p(N+1-2\alpha)-N}\int_\Omega |\nabla u(x)|^p dx, \tag{4.23}$$

and from (4.4) and (2.4)

$$u(x) \le G_\alpha(x,0) \le \frac{c_4}{|x|^{N-2\alpha}} \Longrightarrow u_b(y) \le \frac{c_4}{|y|^{N-2\alpha}}.$$

Then

$$\|u_b\|_{L^1(\mathbb{R}^N, \frac{dy}{1+|y|^{N+2\alpha}})} \le c_4 \int_{\mathbb{R}^N} \frac{dy}{|y|^{N-2\alpha}(1+|y|)^{N+2\alpha}} = c_{31}.$$

If we take $\beta = 1$, which is possible since $\alpha > 1/2$, we derive

$$|\nabla u_b(a)| \le c_{32} \Longrightarrow |\nabla u(ba)| \le c_{32}^{-1}b^{2\alpha-N-1}$$

In particular, with $|b| = 4|x|/3$ we derive (4.21) with $c_{28} = c_{32}^{-1}(\frac{4}{3})^{2\alpha-N-1}$.□

We denote

$$c_{N,\alpha} = \lim_{x\to 0} |x|^{N-2\alpha}G_\alpha(x,0). \tag{4.24}$$

It is well known that $c_{N,\alpha} > 0$ does not depend on the domain Ω and, by the maximum principle, $G_\alpha(x,0) \le c_{N,\alpha}|x|^{2\alpha-N}$ in $\Omega \setminus \{0\}$.

Theorem 4.1 *Let Ω be an open bounded C^2 domain containing 0, $\alpha \in (\frac{1}{2}, 1)$ and $0 < p < p^*_\alpha$. If u is a positive solution of problem (4.1) and $\bar{B}_R(0) \subset \Omega$, it satisfies*
(i) if $\frac{2\alpha}{N-2\alpha+1} < p < p^*_\alpha$,

$$0 < \frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) \le \frac{c_{33}}{|x|^{(N-2\alpha+1)p-2\alpha}}, \quad x \in B_{R/4}(0) \setminus \{0\};$$

(ii) if $p = \frac{2\alpha}{N-2\alpha+1}$,

$$0 < \frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) \le -c_{33} \ln(|x|), \quad x \in B_{R/4}(0) \setminus \{0\};$$

(iii) if $0 < p < \frac{2\alpha}{N-2\alpha+1}$,

$$0 < \frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) \le c_{33}, \quad x \in B_{R/4}(0) \setminus \{0\},$$

where c_{33} depends on N, p, α and R.
Furthermore, if $1 \le p < p^*_\alpha$, this solution is unique.

Proof. The existence of a nonnegative weak solution is a consequence of the subriticality assumption; the fact that this solution is a classical solution in $\Omega \setminus \{0\}$ derives from Proposition 4.2. It follows by (4.4) and (4.6) that for any $x \in \Omega \setminus \{0\}$,

$$\begin{aligned}\frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) &\le \int_\Omega G_\alpha(x,y)|\nabla u(y)|^p dy \\ &\le c_{28}^p c_4 \int_{B_{\frac{R}{4}}(0)} |x-y|^{2\alpha-N}|y|^{p(2\alpha-N-1)}dy + c_{34}\|\nabla u\|_{L^p(\Omega)} \\ &\le c_{35}\left[\int_{B_{\frac{R}{4}}(0)} |x-y|^{2\alpha-N}|y|^{p(2\alpha-N-1)}dy + 1\right]\end{aligned} \tag{4.25}$$

where $c_{34}, c_{35} > 0$ depend on N, p and α. Next we assume $0 < |x| \le \frac{R}{16}$.
Case: $\frac{2\alpha}{N-2\alpha+1} < p < p^*_\alpha$. We can write

$$\int_{B_{\frac{R}{4}}(0)} |x-y|^{2\alpha-N}|y|^{p(2\alpha-N-1)}dy = E_1 + E_2$$

with

$$E_1 = \int_{B_{\frac{R}{4}(0)} \setminus B_{\frac{R}{8}}(0)} |x-y|^{2\alpha-N}|y|^{p(2\alpha-N-1)}dy \le c_{36},$$

where $c_{36} > 0$ depends on N, α, p and R and

$$\begin{aligned}E_2 &= \int_{B_{\frac{R}{8}}(0)} |x-y|^{2\alpha-N}|y|^{p(2\alpha-N-1)}dy \\ &= |x|^{2\alpha-p(N+1-2\alpha)} \int_{B_{\frac{R}{8|x|}}(0)} |\xi-\zeta|^{2\alpha-N}|\zeta|^{p(2\alpha-N-1)}d\zeta \\ &\leq \int_{|\zeta|>2} |\xi-\zeta|^{2\alpha-N}|\zeta|^{p(2\alpha-N-1)}d\zeta\end{aligned}$$

with $\xi = x/|x|$. Since $2\alpha - N < 0$, $|\xi-\zeta|^{2\alpha-N} \leq (|\zeta|-1)^{2\alpha-N}$, then

$$E_2 \leq c_N \int_2^\infty (r-1)^{2\alpha-N} r^{p(2\alpha-N-1)+N-1}dr = c_{37}.$$

Thus (i) follows.
Case: $\frac{2\alpha}{N-2\alpha+1} = p$. We see that

$$E_2 = \int_{B_{\frac{R}{8|x|}}(0)} |\xi-\zeta|^{2\alpha-N}|\zeta|^{-2\alpha}d\zeta,$$

then clearly

$$E_2 = -\ln|x| + o(1) \quad \text{when} \quad |x| \to 0.$$

Thus (ii) follows.
Case: $0 < p < \frac{2\alpha}{N-2\alpha+1}$. We have that

$$E_2 = \int_{B_{\frac{R}{8|x|}}(0)} |\xi-\zeta|^{2\alpha-N}|\zeta|^{-2\alpha}d\zeta = c_{29}|x|^{p(N+1-2\alpha)-2\alpha} + o(1) \quad \text{when} \quad |x| \to 0.$$

Thus (iii) follows.

Uniqueness in the case $1 \leq p < p^*_\alpha$, is very standard, since if u_1 and u_2 are two positive solutions of (4.1), they satisfies

$$\lim_{x\to 0} \frac{u_1(x)}{u_2(x)} = 1.$$

Then, for any $\varepsilon > 0$, $u_{1,\varepsilon} := (1+\varepsilon)u_1$ is a supersolution which dominates u_2 near 0, it follows by the maximum principle that $w := u_2 - (1+\varepsilon)u_1$ satisfies

$$(-\Delta)^\alpha w + |\nabla u_2|^p - |\nabla u_{1,\varepsilon}|^p \leq 0$$

since w is negative near 0 and vanishes on $\partial\Omega$, if it is not always negative, there would exists $x_0 \in \Omega \setminus \{0\}$ such that $w(x_0)$ reaches a maximum and $|\nabla u_2(x_0)| = |\nabla u_{1,\varepsilon}(x_0)|$, thus $(-\Delta)^\alpha w(x_0) \leq 0$, contradiction. $\square$

Remark. If $0 < p < 1$, the nonlinearity is not convex and uniqueness does hold only if two solutions u_1 and u_2 satisfy

$$\lim_{x\to 0}(u_1(x) - u_2(x)) = 0.$$

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