About the mass of certain second order elliptic operators

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ABSTRACT. Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ and let $f \in C^\infty(M)$, such that the operator $P_f := \Delta_g + f$ is positive. If g is flat near some point p and f vanishes around p, we can define the mass of P_f as the constant term in the expansion of the Green function of P_f at p. In this paper, we establish many results on the mass of such operators. In particular, if $f := \frac{n-2}{4(n-1)} s_g$, i.e. if P_f is the Yamabe operator, we show the following result: assume that there exists a closed simply connected non-spin manifold M such that the mass is non-negative for every metric g as above on M, then the mass is non-negative for every such metric on every closed manifold of the same dimension as M.

CONTENTS

1. Introduction

Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$, let $p \in M$ and assume that g is flat on an open neighborhood U of p. Let $f \in C^\infty(M)$ such that $f \equiv 0$ on U. Then, a *Green function* of $P_f := \Delta_g + f$ at p is a function $G_f \in L^1(M) \cap C^\infty(M \setminus \{p\})$ such that in the sense of distributions

$$P_f G_f = \delta_p \tag{1}$$

where δ_p is the Dirac distribution at p. It is well known that

Proposition 1.1. *Assume that all eigenvalues of the operator P_f are positive. Then, there exists a unique Green function G_f for P_f at p. Moreover, G_f is strictly positive on $M \setminus \{p\}$ and has the following expansion at p:*

$$G_f = \frac{1}{(n-2)\omega_{n-1} r^{n-2}} + m_f + o(1) \tag{2}$$

where $r := \mathrm{d}_g(p,\cdot)$ is the distance function to p, where ω_{n-1} is the volume of the standard $(n-1)$-sphere and where m_f is a number called the mass of P_f at the point p.

Considering the importance of this proposition for this paper, we give the proof in Section 2. These objects play a crucial role in many problems of geometric analysis in which blowing-up sequences of functions behave like Green function. The most famous one is maybe the Yamabe problem which consists in finding a metric with constant scalar curvature in a given conformal class. After Yamabe, Trudinger and Aubin had found a solution to this problem in some special cases, the remaining cases were solved by Schoen in 1984 with a test function argument in which he used the Green function of the *conformal Laplacian* or *Yamabe operator*

$$L_g := \Delta_g + \frac{n-2}{4(n-1)} s_g.$$

We give more information on the operator L_g in Paragraph 2.4. With the notation above, $L_g = P_{\frac{n-2}{4(n-1)} s_g}$. Schoen could show that the positivity of the number $m_{\frac{n-2}{4(n-1)} s_g}$ allows to solve the Yamabe problem. To prove this last step, he showed that $m_{\frac{n-2}{4(n-1)} s_g}$ can be interpreted as the ADM mass of an asymptotically flat manifold, which is regarded as the energy of an isolated system in general relativity and which can be proved to be positive in this context. Even if this interpretation is really specific to $m_{\frac{n-2}{4(n-1)} s_g}$, the number m_f for a more general f is now called *mass of the operator P_f*. For more information on the Yamabe problem, we refer the reader for instance to [20].
At a first glance, we could think from the definition that the mass m_f only depends on the local geometry around p. Unfortunately, this is not true which makes its study very difficult. In particular, the question of whether $m_{\frac{n-2}{4(n-1)} s_g} \geq 0$ with equality if and only (M,g) is conformally equivalent to the standard sphere is still open in full generality. It is proven only in some particular cases, including the context of Yamabe problem (i.e. when (M,g) is locally conformally flat, see [25]) and the case of spin manifolds, solved by Witten in [27].

The first result of this paper is Theorem 3.1 in which we show that $-m_f$ can be expressed as the minimum of a functional. Note that Hebey and Vaugon [10] have already proved a variational characterization of the mass $m_{\frac{n-2}{4(n-1)}s_g}$ but their approach is different giving rise to different applications. We then exhibit four short applications of Theorem 3.1:

- We first give an alternative proof of the positive mass theorem on spin manifolds. This proof is not simpler than the one of Ammann-Humbert [1] but has the advantage to enlighten the ingredients which make the proof work.
- We prove in a very simple way a generalization of a result of Beig and O'Murchadha who proved in [5] that near a metric of zero Yamabe constant, the mass $m_{\frac{n-2}{4(n-1)}s_g}$ is arbitrarily large.
- We prove that on every manifold, we can find many non-negative functions f for which m_f is negative.
- We prove that the positivity of $m_{\frac{n-2}{4(n-1)}s_g}$ is preserved by surgery (see Section 6.4 for a precise statement).

These facts could also be proven directly but Theorem 3.1 is nevertheless interesting for many reasons:

- The variational characterization is really easy to manipulate and helps a lot to simplify the proofs. For instance, the mass-to-infinity Theorem 6.4 becomes almost obvious with this approach.
- Theorem 3.1 makes it easy to have a good intuition without any computation of what is true or not, as can be seen for example in Section 6.4 about the preservation of the positivity of mass by surgery.
- Theorem 3.1 clarifies the situation a lot: this is particularly true for the proof of the positive mass theorem on spin manifolds (see Section 6.1).

After these applications we prove that also the negativity of $m_{\frac{n-2}{4(n-1)}s_g}$ is preserved by surgery (see Section 7 for a precise statement). The proof is more difficult than the proof of the preservation of the positivity of $m_{\frac{n-2}{4(n-1)}s_g}$ and uses Theorem 3.1 together with some techniques developed in the article [2].

As explained above, the question of whether $m_{\frac{n-2}{4(n-1)}s_g} \geq 0$ with equality if and only if (M,g) is conformally equivalent to the standard sphere is still open. It is known as the *positive mass conjecture (weak version)* and is a particular case of the standard *positive mass conjecture* which says that the ADM mass of an asymptotically flat manifold with non-negative and integrable scalar curvature must be non-negative and vanishes if and only if the manifold is $\mathbb{R}^n$ equipped with the flat metric. It turns out that both versions of the positive mass conjecture are actually equivalent: see Proposition 4.1 in [21] or Section 5 in [18] (this could also be proved using Theorem 3.1 but the proof is not really simpler and not instructive so we omit it in this paper). The positive mass conjecture is proved when $n \leq 7$ by Schoen and Yau [22] or when (M,g) is spin by Witten [27]. More recently Lohkamp has announced a complete proof in [19]. Note that the conjecture has been proved by Schoen and Yau [25] under the assumption that the manifold is conformally flat leading to the complete solution of the Yamabe problem.

Now, let M be a closed manifold. We say that PMT (for Positive Mass Theorem) is true on M if for every point $p \in M$ and for every metric g on M which is flat around p and for which L_g is a positive operator we have $m_{\frac{n-2}{4(n-1)}s_g} \geq 0$. Using that the negativity of $m_{\frac{n-2}{4(n-1)}s_g}$ is preserved by surgery we obtain the second main result of this paper which is the following:

Theorem 1.2. *Assume that PMT is true on a closed simply connected non-spin manifold of dimension $n \geq 5$, then PMT is true on all closed manifolds of dimension n.*

Note that using for instance Proposition 4.1 in [21] or Section 5 in [18] one can conclude from the assumption of this theorem that every asymptotically flat Riemannian manifold of dimension n with non-negative and integrable scalar curvature has non-negative ADM mass.

This theorem should help a lot to prove the positive mass conjecture. Indeed, it reduces the problem to finding a non-spin simply connected manifold M on which PMT is true. For instance, $\mathbb{C}P^{2m}$ or $\mathbb{C}P^{2m} \times S^k$ with $k \geq 2$ could be a good candidate to provide such an example by using its particular structure. We did not succeed until now but let us explain how some structures could help a lot to prove that PMT is true on a manifold. First, it is not difficult to find a simply connected manifold for which PMT is true: it suffices to choose a manifold which is spin (the sphere for instance). But we can also easily construct a non-spin manifold for which PMT is true (unfortunately, it is not simply connected):

Proposition 1.3. *Let $n \geq 5$, $n \equiv 1 \mod 4$. Then, the projective space $\mathbb{R}P^n$ satisfies PMT.*

The proof of this proposition is really simple and is given is Section 8.

The paper is organized as follows:

- In Section 2, we give some general preliminaries which will be used in the whole text;
- In Section 3, we give the statement of Theorem 3.1 whose goal is to establish the variational characterization of the mass;
- Sections 4 and 5 are devoted to the proof of Theorem 3.1;
- In Section 6, we give several applications of Theorem 3.1;
- In Section 7, we establish a surgery formula for the mass which will be the main ingredient in the proof of Theorem 1.2;
- In Section 8, we show how the results of Section 7 can be applied to prove Theorem 1.2.

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2. Preliminaries

In these sections, we introduce all the objects and the notation which will be needed in the paper and we give some additional information on the context of the problem.

2.1. **Notation.** All manifolds are assumed to be connected and without boundary unless otherwise stated. We denote by ξ^n the Euclidean metric on $\mathbb{R}^n$ and by σ^n the standard metric of constant sectional curvature 1 on S^n. For any Riemannian manifold (M, g) and for $p \in M$ and $r > 0$ we denote by $B(p, r)$ or by $B^g(p, r)$ the open ball of radius r centered at p. For a subset N of M we denote by $\mathrm{vol}(N)$ or $\mathrm{vol}^g(N)$ the volume of N with respect to g and by $\mathrm{d}_g(x, N)$ the distance of x to N. The scalar curvature of any Riemannian metric g will be denoted by s_g. We will use the abbreviation

$$\int_{M\setminus\{p\}} := \lim_{\varepsilon\to 0}\int_{M\setminus B(p,\varepsilon)}.$$

For any Riemannian manifold (M, g) and for any $q \in [1, \infty]$ we denote by $L^q(M)$ the space of all measurable functions on M with finite L^q-norm. The Sobolev space $H^{1,2}(M)$ is the space of all functions in $L^2(M)$ whose distributional derivative exists and is in $L^2(M)$.

2.2. **A cut-off formula.** We state a formula which is used several times in the article (see also Appendix A.3 in [2]). Let u and χ be smooth functions on a Riemannian manifold (M, g) and assume that χ has compact support. Then we have

$$\begin{aligned}\int_M |d(\chi u)|^2\, dv^g &= \int_M |u d\chi + \chi du|^2\, dv^g\\ &= \int_M (u^2|d\chi|^2 + g(\chi^2 du, du) + g(2u\chi d\chi, du))\, dv^g\\ &= \int_M (u^2|d\chi|^2 + g(\chi^2 du, du) + g(ud(\chi^2), du))\, dv^g\\ &= \int_M (u^2|d\chi|^2 + g(d(\chi^2 u), du))\, dv^g\\ &= \int_M (u^2|d\chi|^2 + \chi^2 u\Delta_g u)\, dv^g. \end{aligned} \tag{3}$$

2.3. **Properties of the Green function.** Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Let $f \in C^\infty(M)$ and assume that the operator $P_f := \Delta_g + f$ acting on $C^\infty(M)$ has only positive eigenvalues. Fix $p \in M$. A function $G_f \in L^1(M) \cap C^\infty(M \setminus \{p\})$ is called a Green function for P_f at p if for all $u \in C^\infty(M)$ we have

$$\int_{M\setminus\{p\}} G_f P_f u\, dv^g = u(p).$$

In our article we use the following properties of the Green function which are well known.

Proposition 2.1. *Assume that P_f is a positive operator. Then the following holds.*

1. *At every point $p \in M$ there exists a unique Green function G_f for P_f. Moreover G_f is strictly positive on $M \setminus \{p\}$.*
2. *Let $p \in M$ and assume that there exists an open neighborhood U of p such that g is flat on U and $f \equiv 0$ on U. Then the function G_f has the following expansion as $x \to p$*

$$G_f(x) = \frac{1}{(n-2)\omega_{n-1}r^{n-2}} + m_f + o(1), \tag{4}$$

where $r := \mathrm{d}_g(p, \cdot)$ is the distance function to p, ω_{n-1} is the volume of the standard $(n-1)$-sphere and m_f is a real number called the mass of P_f at p.

Proof. 1.: The proof is classical and we omit it here.
2.: Let η and F_η be as in Section 3. Since P_f has only positive eigenvalues, P_f is invertible on $C^\infty(M)$. Let $v := P_f^{-1}(F_\eta)$. The function $G_f := \eta r^{2-n} - v$ is smooth on $M \setminus \{p\}$, is in $L^1(M)$ and satisfies $P_f G_f = 0$ on $M \setminus \{p\}$. Moreover, near p,

$$G_f(x) = \frac{1}{(n-2)\omega_{n-1} r^{n-2}} + v(x)$$

where $P_f v = \Delta_g v = 0$. Since the manifold is flat around p and thus locally isometric to a neighborhood of 0 in $\mathbb{R}^n$ and since the Green function for the Laplacian on $\mathbb{R}^n$ at 0 is $\frac{1}{(n-2)\omega_{n-1} r^{n-2}}$, we get that $P_f v = \delta_p$ and thus G_f is a Green function for P_f. This proves the existence.

If now G and G' are Green functions for P_f then $P_f(G - G') = 0$ in the sense of distributions. By standard regularity theorems, $G - G'$ is smooth and hence, by invertibility of P_f we obtain $G = G'$. □

2.4. **The Yamabe operator.** Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. We define $f := \frac{n-2}{4(n-1)} s_g$ and denote the operator P_f by

$$L_g := \Delta_g + \frac{n-2}{4(n-1)} s_g.$$

This operator is called the *conformal Laplacian* or *Yamabe operator*. If the metric g is flat on an open neighborhood of a point $p \in M$, we will denote the mass of L_g at p by $m(M, g)$. There are several reasons why this operator is very important. First it played a crucial role in the solution of the Yamabe problem, which is a famous problem in conformal geometry. For more information on the subject, the reader may refer to [3, 9, 20]. Furthermore the mass of the operator L_g can be interpreted as the ADM mass of an asymptotically flat Riemannian manifold, which is an important quantity measuring the total energy of an isolated gravitational system in general relativity (see [24]).

In this article we will use several properties of the operator L_g. First it transforms nicely under conformal changes of the metric. Namely, if $g' = u^{\frac{4}{n-2}} g$ are two conformally related metrics, where u is a smooth positive function on M, then for all $\varphi \in C^\infty(M)$ we have

$$L_{g'}(u^{-1}\varphi) = u^{-\frac{n+2}{n-2}} L_g(\varphi) \tag{5}$$

(see e. g. [20], p. 43). Using this formula with $\varphi = u$ we obtain the equation

$$L_g(u) = \frac{n-2}{4(n-1)} s_{g'} u^{\frac{n+2}{n-2}}, \tag{6}$$

which gives a relation between the scalar curvatures of g and g'. Next we define

$$Y(M, g) := \inf\left\{ \frac{\int_M u L_g u \, dv^g}{(\int_M |u|^p \, dv^g)^{2/p}} \,\Big|\, u \in C^\infty(M),\, u \not\equiv 0 \right\},$$

where $p := \frac{2n}{n-2}$. This number is a conformal invariant called the *Yamabe constant* of (M, g). The operator L_g is positive (i.e. has only positive eigenvalues) if and only if $Y(M, g)$ is positive.

If $g' = u^{\frac{4}{n-2}} g$ and if g and g' are both flat in an open neighborhood of a point $p \in M$ and if G and G' denote the Green functions of L_g and $L_{g'}$ respectively, we have for all $x \in M \setminus \{p\}$

$$G'(x) = u(p)^{-1} u(x)^{-1} G(x)$$

(see e.g.[20], p. 63). If we write down the expansions of G and G' given by Proposition 2.1 and use that u is constant on an open neighborhood of p, it follows that $m(M, g)$ and $m(M, g')$ have the same sign (see also [25] or [9], p. 277).

3. A variational characterization of the mass

We keep the same notation as above and fix a function f such that the operator P_f is positive. Then the Green function G_f of P_f at p and the associated mass m_f are well defined. Let $\delta > 0$ such that the ball $B(p, \delta)$ around p of radius δ is contained in U and let η be a smooth function on M such that $\eta \equiv \frac{1}{(n-2)\omega_{n-1}}$ on $B(p, \delta)$ and $\mathrm{supp}(\eta) \subset U$, where ω_{n-1} denotes the volume of S^{n-1} with the standard metric. The function $F_\eta \colon M \to \mathbb{R}$ defined by

$$F_\eta(x) = \begin{cases} \Delta_g(\eta r^{2-n})(x), & x \neq p \\ 0, & x = p \end{cases}$$

is smooth on M. For every $u \in C^\infty(M)$ we define

$$I_f(u) := \int_{M\setminus\{p\}} (\eta r^{2-n} + u) P_f (\eta r^{2-n} + u) \, dv^g,$$

and

$$J_f(u) := \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^g + 2 \int_M u F_\eta \, dv^g + \int_M u P_f u \, dv^g.$$

We also define

$$\begin{aligned} \nu &:= \inf\{I_f(u) |\, u \in C^\infty(M),\, u(p) = 0\}, \\ \mu &:= \inf\{J_f(u) |\, u \in C^\infty(M)\}. \end{aligned}$$

Let us remark the following fact: if η' is another smooth function with the same properties as η, one can construct in a similar way:

$$I'_f(u) := \int_{M\setminus\{p\}} (\eta' r^{2-n} + u) P_f (\eta' r^{2-n} + u) \, dv^g.$$

Note that, for all u,

$$I_f(u) = I'_f(u - \eta' r^{2-n} + \eta r^{2-n})$$

and that $u - \eta' r^{2-n} + \eta r^{2-n}$ has a smooth extension to all of M. As a consequence, the number ν does not depend on the choice of η.

The following theorem is the main result of this article.

Theorem 3.1. *We have* $\nu = \mu = -m_f = J_f(G_f - \eta r^{2-n})$.

The proof is obtained in several steps and is done in Section 4.

4. Proof of Theorem 3.1

The proof of Theorem 3.1 proceeds in several steps. The general idea is to show that ν and μ are equal and that μ is attained by exactly one smooth function u which is such that

$$G_f = \eta r^{2-n} + u.$$

These facts will be established in the following lemmas. First we relate the functionals I_f and J_f.

Lemma 4.1. *For all $u \in C^\infty(M)$ we have*

$$I_f(u) = J_f(u) + u(p).$$

Proof. Using that $f \equiv 0$ on $\mathrm{supp}(\eta)$ we calculate

$$\begin{aligned} I_f(u) = &\int_{M\setminus\{p\}} \eta r^{2-n} \Delta_g(\eta r^{2-n})\, dv^g + \int_{M\setminus\{p\}} u \Delta_g(\eta r^{2-n})\, dv^g \\ &+ \int_{M\setminus\{p\}} \eta r^{2-n} \Delta_g u\, dv^g + \int_M u P_f u\, dv^g. \end{aligned}$$

Let $\varepsilon > 0$ and let ν be the unit normal vector field on $\partial B(p,\varepsilon)$ pointing into $M \setminus B(p,\varepsilon)$. Integrating by parts, we have

$$\begin{aligned} &\int_{M\setminus B(p,\varepsilon)} \eta r^{2-n} \Delta_g u\, dv^g - \int_{M\setminus B(p,\varepsilon)} u \Delta_g(\eta r^{2-n})\, dv^g \\ &= \int_{\partial B(p,\varepsilon)} \eta r^{2-n} \partial_\nu u\, ds^g - \int_{\partial B(p,\varepsilon)} u \partial_\nu(\eta r^{2-n})\, ds^g. \end{aligned}$$

As $\varepsilon \to 0$, the first term on the right hand side tends to 0 and the second integral on the right hand side tends to $-u(p)$. The assertion follows. □

Lemma 4.2. *We have $\mu > -\infty$ and $\nu > -\infty$. Furthermore there exists a unique function $u \in C^\infty(M)$ such that $\mu = J_f(u)$.*

Proof. Assume that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $C^\infty(M)$ such that $J_f(u_k) \to -\infty$ as $k \to \infty$. Since P_f is a positive operator, there exists $A > 0$ such that for all $k \in \mathbb{N}$ we have

$$\int_M u_k P_f u_k\, dv^g \geq A \|u_k\|^2_{L^2(M)} \geq 0.$$

From our assumption and the definition of J_f it follows that $\int_M u_k F_\eta\, dv^g \to -\infty$ as $k \to \infty$. On the other hand with Hölder's inequality we have for all $k \in \mathbb{N}$

$$\Big| \int_M u_k F_\eta\, dv^g \Big| \leq \|F_\eta\|_{L^2(M)}\, \|u_k\|_{L^2(M)}.$$

Thus we have $\|u_k\|_{L^2(M)} \to \infty$ as $k \to \infty$ and thus

$$J_f(u_k) \geq \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta\, dv^g - 2\|F_\eta\|_{L^2(M)}\, \|u_k\|_{L^2(M)} + A\|u_k\|^2_{L^2(M)} \to \infty$$

as $k \to \infty$, which is a contradiction. Thus we have $\mu > -\infty$. Next assume that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $C^\infty(M)$ such that for every $k \in \mathbb{N}$ we have $u_k(p) = 0$ and $I_f(u_k) \to -\infty$ as $k \to \infty$. By Lemma 4.1 we conclude $J_f(u_k) \to -\infty$ which is a contradiction. Thus we have $\nu > -\infty$.

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $C^\infty(M)$ such that $J_f(u_k) \to \mu$ as $k \to \infty$. As above it follows that $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^2(M)$. Since for all $k \in \mathbb{N}$ we have

$$J_f(u_k) = \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^g + 2 \int_M u_k F_\eta \, dv^g + \int_M |du_k|^2 \, dv^g + \int_M f u_k^2 \, dv^g,$$

it follows that the sequence $(|du_k|)_{k\in\mathbb{N}}$ is bounded in $L^2(M)$ and thus that $(u_k)_{k\in\mathbb{N}}$ is bounded in $H^{1,2}(M)$. Since $H^{1,2}(M)$ is reflexive there exists $u \in H^{1,2}(M)$ such that after passing to a subsequence we have $u_k \to u$ weakly in $H^{1,2}(M)$. Furthermore since the embeddings of $H^{1,2}(M)$ into $L^1(M)$ and into $L^2(M)$ are compact we obtain after passing again to sub-sequences that $u_k \to u$ strongly in $L^1(M)$ and in $L^2(M)$. For every $k \in \mathbb{N}$ we have

$$\begin{aligned} 0 &\le \int_M |du - du_k|^2 \, dv^g \\ &= \int_M |du|^2 \, dv^g + \int_M |du_k|^2 \, dv^g - 2 \int_M g(du, du_k) \, dv^g. \end{aligned}$$

By weak convergence in $H^{1,2}(M)$ the third term on the right hand side converges to $-2 \int_M |du|^2 \, dv^g$ as $k \to \infty$. It follows that

$$\int_M |du|^2 \, dv^g \le \liminf_{k\to\infty} \int_M |du_k|^2 \, dv^g.$$

Since the sequence $(u_k)_{k\in\mathbb{N}}$ converges strongly to u in $L^1(M)$ and in $L^2(M)$ we have

$$\int_M f u_k^2 \, dv^g \to \int_M f u^2 \, dv^g, \quad \int_M u_k F_\eta \, dv^g \to \int_M u F_\eta \, dv^g$$

as $k \to \infty$. It follows that

$$J_f(u) \le \liminf_{k\to\infty} J_f(u_k) = \mu$$

and therefore $J_f(u) = \mu$. For every $\varphi \in C^\infty(M)$ we have

$$0 = \frac{d}{dt} J_f(u + t\varphi)\big|_{t=0} = 2 \int_M \varphi F_\eta \, dv^g + 2 \int_M \varphi P_f u \, dv^g$$

and therefore $P_f u = -F_\eta$. Using standard results in regularity theory (see e. g. [7]) we see from this equation that u is smooth. We also see that u is the unique minimizer of J_f since P_f is invertible on $C^\infty(M)$. □

Lemma 4.3. *We have $\mu = -m_f$.*

Proof. Define $v := u + \eta r^{2-n}$. Then $G_f - v$ has a smooth extension to all of M and on $M \setminus \{p\}$ we have

$$P_f(G_f - v) = P_f(G_f - \eta r^{2-n} - u) = -F_\eta - P_f u = 0.$$

Since P_f is invertible on smooth functions, we have $v = G_f$. It follows that $u(p) = m_f$ and therefore

$$\mu = J_f(u) = I_f(u) - m_f = \int_{M\setminus\{p\}} G_f P_f G_f \, dv^g - m_f = -m_f.$$

This ends the proof. □

We are now able to prove the result:

Lemma 4.4. *We have*

$$\mu = \nu.$$

Together with Lemma 4.3 this proves Theorem 3.1

Proof. In order to show "$\mu \le \nu$" let $\varepsilon > 0$ and let $u \in C^\infty(M)$ such that $u(p) = 0$ and $I_f(u) \le \nu + \varepsilon$. Then we have $J_f(u) = I_f(u) \le \nu + \varepsilon$ and thus $\mu \le \nu + \varepsilon$.

In order to show "$\mu \ge \nu$" let $\varepsilon > 0$ and let $u \in C^\infty(M)$ such that $J_f(u) = \mu$. For $s > 0$ let $\chi_s \colon M \to [0,1]$ be a smooth function such that $\chi_s \equiv 0$ on $B(p,s)$, $\chi_s \equiv 1$ on $M \setminus B(p,2s)$ and $|d\chi_s| \le \frac{2}{s}$. We write $A_s := B(p,2s) \setminus B(p,s)$ and we obtain by (3)

$$\begin{aligned}\int_M u\chi_s P_f(u\chi_s)\, dv^g &= \int_M (u^2 |d\chi_s|^2 + \chi_s^2 u P_f u)\, dv^g \\ &\le \frac{4}{s^2}\int_{A_s} u^2\, dv^g + \int_M \chi_s^2 u P_f u \, dv^g.\end{aligned}$$

Since there exists $C > 0$ such that for all s we have $\mathrm{vol}(A_s) \le Cs^n$, the first term on the right hand side tends to 0 as $s \to 0$. We conclude that

$$\lim_{s\to 0} J_f(u\chi_s) \le J_f(u).$$

Thus we can choose s so close to 0 that we have $J_f(u\chi_s) \le \mu + \varepsilon$. Since we have $\chi_s(p) = 0$ the left hand side is equal to $I_f(u\chi_s)$. It follows that $\nu \le \mu + \varepsilon$. □

Finally we ask whether the infimum ν is attained. We immediately obtain the following answer.

Lemma 4.5. *Let $u \in C^\infty(M)$ be the unique smooth function with $J_f(u) = \mu$ given by Lemma 4.2.*

1. *If $u(p) = 0$, then there is exactly one $w \in C^\infty(M)$ with $w(p) = 0$ and $I_f(w) = \nu$, namely $w = u$.*
2. *If $u(p) \neq 0$, then there is no $w \in C^\infty(M)$ with $w(p) = 0$ and $I_f(w) = \nu$.*

Proof. If $w \in C^\infty(M)$ satisfies $w(p) = 0$ and $I_f(w) = \nu$ then by Lemma 4.1 and Lemma 4.4 we have $J_f(w) = \mu$ and thus $w = u$. Both 1. and 2. follow from this observation. □

5. Another proof of the case $P_f = L_g$

We give an alternative proof of Theorem 3.1 in the special case $f = \frac{n-2}{4(n-1)} s_g$. Let (M,g) be a closed Riemannian manifold such that g is flat on an open neighborhood U of a fixed point $p \in M$ and assume that $Y(M,g) > 0$. Then the mass $m(M,g)$ of L_g at the point p is well defined. Let $\delta > 0$ such that $B(p,\delta) \subset U$ and let η be a smooth function on M such that $\eta \equiv \frac{1}{(n-2)\omega_{n-1}}$ on $B(p,\delta)$ and $\mathrm{supp}(\eta) \subset U$, where ω_{n-1} denotes the volume of S^{n-1} with the standard metric. For every $u \in C^\infty(M)$ with $u(p) = 0$ we define

$$I_g(u) := \int_{M\setminus\{p\}} (\eta r^{2-n} + u) L_g(\eta r^{2-n} + u)\, dv^g$$

and

$$\nu := \inf\{I_g(u) |\, u \in C^\infty(M),\, u(p) = 0\}.$$

We denote by G the Green function for the conformal Laplacian L_g at the point p. It is strictly positive on $M \setminus \{p\}$ by Proposition 2.1. Thus $\widetilde{g} := G^{4/(n-2)}g$ is a Riemannian metric on $M \setminus \{p\}$. Furthermore for every $u \in C^\infty(M)$ with $u(p) = 0$ the function

$$\Phi_u := (\eta r^{2-n} + u)G^{-1}$$

has a smooth extension to all of M and in an isometric chart on U it has the expansion

$$\Phi_u(x) = 1 - Ar^{n-2} + o(r^{n-2}) \quad \text{as } x \to p \tag{7}$$

with $A := (n-2)\omega_{n-1} m(M, g)$. We prove the following theorem.

Theorem 5.1. *For every $u \in C^\infty(M)$ with $u(p) = 0$ we have*

$$I_g(u) = \int_{M\setminus\{p\}} |d\Phi_u|^2_{\widetilde{g}}\, dv^{\widetilde{g}} - m(M, g). \tag{8}$$

Proof. Let $u \in C^\infty(M)$ with $u(p) = 0$. We write $w := \eta r^{2-n} + u$. By the conformal transformation law (5) for L_g we obtain

$$L_{\widetilde{g}}\Phi_u = G^{-\frac{n+2}{n-2}} L_g w.$$

Since we have $dv^{\widetilde{g}} = G^{2n/(n-2)}dv^g$ and since by the conformal transformation law (6) for the scalar curvature we have $s_{\widetilde{g}} = 0$ it follows that

$$I_g(u) = \int_{M\setminus\{p\}} wL_g w\, dv^g = \int_{M\setminus\{p\}} \Phi_u \Delta_{\widetilde{g}} \Phi_u\, dv^{\widetilde{g}}.$$

Integrating by parts, we have for every $\varepsilon > 0$

$$\int_{M\setminus B(p,\varepsilon)} \Phi_u \Delta_{\widetilde{g}} \Phi_u\, dv^{\widetilde{g}} = \int_{M\setminus B(p,\varepsilon)} |d\Phi_u|^2_{\widetilde{g}}\, dv^{\widetilde{g}} - \int_{\partial B(p,\varepsilon)} \Phi_u \partial_{\widetilde{\nu}} \Phi_u\, ds^{\widetilde{g}},$$

where $ds^{\widetilde{g}}$ is the induced volume form on $\partial B(p, \varepsilon)$ and where

$$\widetilde{\nu} = -G^{-\frac{2}{n-2}}\partial_r$$

is the outer unit normal vector field on the boundary of $M \setminus B(p, \varepsilon)$. Using (7) we compute the following expansions as $x \to p$

$$\begin{aligned}
\widetilde{\nu} &= -(\eta^{-\frac{2}{n-2}} r^2 + o(r^2))\partial_r \\
\Phi_u \partial_{\widetilde{\nu}} \Phi_u &= A(n-2)\eta^{-\frac{2}{n-2}} r^{n-1} + o(r^{n-1}) \\
ds^{\widetilde{g}} &= G^{\frac{2(n-1)}{n-2}} ds^g = (\eta^{\frac{2(n-1)}{n-2}} r^{-2(n-1)} + o(r^{-2(n-1)}))ds^g.
\end{aligned}$$

Thus we obtain

$$\lim_{\varepsilon\to 0} \int_{\partial B(p,\varepsilon)} \Phi_u \partial_{\widetilde{\nu}} \Phi_u\, ds^{\widetilde{g}} = m(M, g)$$

and the assertion follows. □

We now obtain the following special case of Theorem 3.1.

Corollary 5.2. *We have $\nu = -m(M, g)$. The infimum is attained if and only if $m(M, g) = 0$.*

Proof. It follows from (8) that $\nu = -m(M,g)$ since one can choose $u \in C^\infty(M)$ with $u(p) = 0$ in such a way that the first term on the right hand side becomes as small as one wants. If the infimum is attained at $u \in C^\infty(M)$, then Φ_u is constant and by (7) we conclude $m(M,g) = 0$. On the other hand if $m(M,g) = 0$, then $G - \eta r^{2-n}$ has a smooth extension u to all of M satisfying $u(p) = 0$. Then with the notation from above we have $w = G$ and $\Phi_u = 1$ and therefore $I_g(u) = -m(M,g)$. □

6. Several applications

6.1. **Application 1: Positive mass theorem on spin manifolds.** Let (M,g) be a closed Riemannian manifold with positive Yamabe constant $Y(M,g)$ which means that the operator $L_g := \Delta_g + \frac{n-2}{4(n-1)} s_g$ is positive (see Paragraph 2.4). We assume that g is flat on an open neighborhood U of $p \in M$. Furthermore we assume in this section that M is a spin manifold with a fixed orientation and a fixed spin structure. We denote by G the Green function of L_g and by $m(M,g)$ the associated mass. In this section we prove the following positive mass theorem for spin manifolds.

Theorem 6.1. *Let (M,g) be a closed Riemannian spin manifold with positive Yamabe constant $Y(M,g)$ such that g is flat on an open neighborhood of a point $p \in M$. Then we have $m(M,g) \geq 0$. Furthermore we have $m(M,g) = 0$ if and only if (M,g) is conformally equivalent to (S^n, σ^n).*

This theorem solves the positive mass conjecture in the particular case of spin manifolds. This was already known by the work of Witten [27]. Let us come back on the name "mass" used for $m(M,g)$ and more generally for the numbers m_f associated to the operators P_f. Set $g' := G^{\frac{4}{n-2}} g$. This new metric is defined on $M \setminus \{p\}$. As observed by Schoen, the manifold $(M \setminus \{p\}, g')$ is asymptotically flat. We will not explain in detail what this means, but asymptotically flat manifolds are the standard models for isolated system in general relativity. To each asymptotically flat manifold with positive L^1 scalar curvature one can associate a number called the ADM-mass of the manifold which is interpreted as the energy of the isolated system. For this reason, this number should be positive but this is far to be obvious from its mathematical definition. It was proven to be true e. g. on spin manifolds by Witten [27] and in dimension $n \in \{3, ..., 7\}$ by Schoen and Yau [22] but the problem in its full generality is still open. In the particular case that the asymptotically flat manifold was obtained by blowing-up a closed manifold as above with the Green function of L_g (this procedure is sometimes called *stereographic projection* since, starting with a closed manifold (M,g) conformally equivalent to the standard sphere, then $(M \setminus \{p\}, g') = (\mathbb{R}^n, \xi)$), Schoen proved that the number $m(M,g)$ is a positive multiple of the ADM mass of $(M \setminus \{p\}, g')$. This is the reason why the number $m(M,g)$ is called the mass. In this special context, which is actually not restrictive, the positivity of the ADM mass, i.e. on M, is also open. Schoen and Yau gave a proof when the manifold is locally conformally flat in [25]. Later, inspired by Witten's proof, Ammann and Humbert gave a very simple proof for spin manifolds which are conformally flat or of dimension 3, 4 or 5 (see [1]). This last method was adapted to other situations: Jammes [14] obtained another proof of Schoen-Yau's theorem [25] for conformally flat manifolds of even dimension and Humbert, Raulot in [13] could prove a positive mass theorem for the Paneitz operator. The proof we give here is quite similar to the one of Ammann and Humbert and not simpler but it allows to understand the crucial role played by the Green function of the Dirac operator in their proof. Namely, we prove that the norm of this Green function can be used as a test function in the variational characterization of the mass given by Theorem 3.1.

Before we give the proof we recall some facts from spin geometry which we will need. Since M is spin, for every Riemannian metric g on M we can define the spinor bundle $\Sigma^g M$ over M which is a complex vector bundle of rank $2^{[n/2]}$ with a bundle metric $(.,.)$ and a connection ∇. Smooth sections of $\Sigma^g M$ are called spinors. We denote by

$$D_g : \quad \Gamma(\Sigma^g M) \to \Gamma(\Sigma^g M)$$

the Dirac operator acting on spinors. For an introduction to the concepts of spin geometry the reader may consult the books [17] or [6]. We will mainly use two important results. First, by the Schrödinger-Lichnerowicz formula we have for all $\psi \in \Gamma(\Sigma^g M)$

$$(D_g)^2\psi = \nabla^*\nabla\psi + \frac{1}{4}s_g\psi, \tag{9}$$

where $\nabla^*\nabla$ denotes the connection Laplacian on $\Sigma^g M$. Second, if $g' = w^{\frac{4}{n-1}}g$ is a metric conformal to g, where w is a smooth positive function on M, then by [12], [11] there exists an isomorphism of vector bundles

$$\beta_{g,g'} : \quad \Sigma^g M \to \Sigma^{g'} M$$

which is a fiberwise isometry such that for all $\psi \in \Gamma(\Sigma^g M)$ we have

$$D_{g'}(w^{-1}\beta_{g,g'}\psi) = w^{-\frac{n+1}{n-1}}\beta_{g,g'}D_g\psi. \tag{10}$$

Furthermore one can show that for every element ψ_0 of the fiber $\Sigma^g_p M$ over p there exists a unique Green function of D_g, i. e. a spinor ψ on $M \setminus \{p\}$ such that for every $\varphi \in \Gamma(\Sigma^g M)$ we have

$$\int_{M\setminus\{p\}} (\psi, D_g\varphi)dv^g = (\psi_0, \varphi(p)).$$

Using our assumptions one can also write down the expansion of ψ around p similarly as for the Green function of $\Delta_g + f$ in Proposition 2.1. Namely we use that g is flat on an open neighborhood U of p and we choose $\delta > 0$ such that $B(p,\delta) \subset U$. We may assume that there exists an isometric chart $B(p,\delta) \to B(0,\delta) \subset \mathbb{R}^n$ and that $\Sigma^g M$ is trivial on $B(p,\delta)$. Since L_g is positive, it is well known that D_g is invertible. Using these facts Ammann and Humbert described the expansion of ψ as follows (see [1]).

Lemma 6.2. *Let $\psi_0 \in \Sigma^g_p M$. Then there is a unique spinor ψ on $M \setminus \{p\}$ such that $D_g\psi = 0$ and such that for all $x \in B(p,\delta) \cong B(0,\delta) \subset \mathbb{R}^n$ we have in the above chart and trivialization*

$$\psi|_{B(p,\delta)}(x) = -\frac{1}{\omega_{n-1}}\frac{x}{r^n} \cdot \psi_0 + \theta(x) \tag{11}$$

where θ is a smooth spinor on $B(p,\delta)$.

From now on we assume that

$$|\psi_0| = ((n-2)\omega_{n-1})^{-\frac{n-1}{n-2}}. \tag{12}$$

Then on $B(p,\delta)$ we have by (11)

$$|\psi(x)|^{\frac{n-2}{n-1}} = \frac{1}{(n-2)\omega_{n-1} r^{n-2}} + o(1) \quad \text{as } r \to 0. \tag{13}$$

Now, let η and F_η be defined as in Section 3. By Theorem 3.1 we have

$$\begin{aligned} -m(M,g) &= \inf\{I_g(u) |\, u \in C^\infty(M),\, u(p) = 0\} \\ &= \inf\{J_g(u) |\, u \in C^\infty(M)\} \end{aligned} \tag{14}$$

where

$$\begin{aligned} I_g(u) &= \int_{M\setminus\{p\}} (\eta r^{2-n} + u) L_g(\eta r^{2-n} + u)\, dv^g, \\ J_g(u) &= \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^g + 2 \int_M u F_\eta \, dv^g + \int_M u L_g u \, dv^g. \end{aligned}$$

The function

$$u: \quad M \to \mathbb{R}, \quad u(x) := \begin{cases} |\psi(x)|^{\frac{n-2}{n-1}} - \eta(x) r(x)^{2-n}, & \text{if } x \neq p \\ 0, & \text{if } x = p \end{cases} \tag{15}$$

is smooth on the complement of the zero set of ψ.

The idea for our proof of Theorem 6.1 is to use the characterization (14) of $m(M,g)$ and to use u as a test function for our functional I_g. If ψ has non-empty zero set, then u is not smooth and we will approximate u by a sequence of smooth functions. Since the zero set of ψ has Hausdorff dimension at most $n-2$ (see [4]), the proof will also work in this case. This is the content of the following proposition.

Proposition 6.3. *There exists a sequence $(u_k)_{k\in\mathbb{N}}$ of smooth functions on M such that $u_k(p) = 0$ for all k and $\lim_{k\to\infty} J_g(u_k) = \lim_{k\to\infty} I_g(u_k) \leq 0$.*

Proof. We first write down the proof in the case that ψ is nowhere zero and consider the case of non-empty zero set afterwards. If ψ is nowhere zero then $g' := |\psi|^{\frac{4}{n-1}} g$ is a Riemannian metric on $M \setminus \{p\}$. As explained above there exists an isomorphism of vector bundles

$$\beta_{g,g'}: \quad \Sigma^g(M \setminus \{p\}) \to \Sigma^{g'}(M \setminus \{p\})$$

which is a fiberwise isometry. Furthermore with $\psi' := |\psi|^{-1} \beta_{g,g'} \psi$ we have $D_{g'}\psi' = 0$ by (10). Let $\varepsilon > 0$ be small. In what follows, the set $B(p,\varepsilon)$ is the ball of center p and radius ε for the metric g. By (9) we have

$$0 = \int_{M\setminus B(p,\varepsilon)} (D_{g'}^2 \psi', \psi')\, dv^{g'} = \int_{M\setminus B(p,\varepsilon)} \Big((\nabla^* \nabla \psi', \psi') + \frac{1}{4} s_{g'} |\psi'|^2 \Big) dv^{g'}. \tag{16}$$

Note that $|\psi'| \equiv 1$. Hence, integrating by parts:

$$\begin{aligned} \int_{M\setminus B(p,\varepsilon)} (\nabla^*\nabla\psi', \psi') dv^{g'} &= \int_{M\setminus B(p,\varepsilon)} |\nabla\psi'|^2 dv^{g'} + \int_{\partial B(p,\varepsilon)} (\nabla_\nu \psi', \psi') ds^{g'} \\ &= \int_{M\setminus B(p,\varepsilon)} |\nabla\psi'|^2 dv^{g'} + \frac{1}{2} \int_{\partial B(p,\varepsilon)} \partial_\nu |\psi'|^2 ds^{g'} \\ &= \int_{M\setminus B(p,\varepsilon)} |\nabla\psi'|^2 dv^{g'}. \end{aligned} \tag{17}$$

where ν is the outer unit normal vector field on $B(p,\varepsilon)$ and $ds^{g'}$ is the volume element induced by g' on $\partial B(p,\varepsilon)$. By Equation (6), we also have

$$s_{g'} = \frac{4(n-1)}{n-2}|\psi|^{-\frac{n+2}{n-1}} L_g(|\psi|^{\frac{n-2}{n-1}}).$$

Since $dv^{g'} = |\psi|^{\frac{2n}{n-1}} dv^g$, we obtain that

$$\int_{M\setminus B(p,\varepsilon)} s_{g'}|\psi'|^2 dv^{g'} = \frac{4(n-1)}{n-2}\int_{M\setminus B(p,\varepsilon)} |\psi|^{\frac{n-2}{n-1}} L_g(|\psi|^{\frac{n-2}{n-1}}) dv^g.$$

Taking the limit as ε tends to 0, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{M\setminus B(p,\varepsilon)} s_{g'}|\psi'|^2 dv^{g'} &= \frac{4(n-1)}{n-2}\int_{M\setminus\{p\}} |\psi|^{\frac{n-2}{n-1}} L_g(|\psi|^{\frac{n-2}{n-1}}) dv^g \\ &= \frac{4(n-1)}{n-2} I_g(u)\end{aligned}$$

where u is defined in (15). Together with (16) and (17) we obtain

$$0 = \int_{M\setminus\{p\}} |\nabla\psi'|^2 dv^{g'} + \frac{n-1}{n-2} I_g(u) \tag{18}$$

which implies $I_g(u) \leq 0$. Furthermore by (13) we have $u(p) = 0$ and by Lemma 4.1 it follows that $J_g(u) = I_g(u)$. This finishes the proof if ψ is nowhere zero.

If ψ has non-empty zero set N, then for every $s > 0$ we define

$$B_s(N) := \{x \in M \,|\, \mathrm{d}_g(x,N) < s\}$$

and for every $k \in \mathbb{N}$ we define

$$M_k := \Big\{x \in M \,\Big|\, \mathrm{d}_g(x,N) > \frac{2}{k}\Big\}.$$

Then the calculation (16) holds with M_k instead of M. If we do the calculation (17) with M_k instead of M then we obtain an extra boundary term

$$\int_{\partial M_k} (\nabla_\nu\psi', \psi') ds^{g'}$$

which vanishes since $|\psi'| \equiv 1$. Thus we conclude

$$0 = \int_{M_k\setminus\{p\}} |\nabla\psi'|^2 dv^{g'} + \frac{n-1}{n-2}\int_{M_k\setminus\{p\}} |\psi|^{\frac{n-2}{n-1}} L_g(|\psi|^{\frac{n-2}{n-1}})\, dv^g. \tag{19}$$

For every $k \in \mathbb{N}$ we choose a smooth function $\chi_k\colon M \to [0,1]$ such that $\chi_k(x) = 0$ if $\mathrm{d}_g(x,N) \leq \frac{1}{k}$, $\chi_k(x) = 1$ if $\mathrm{d}_g(x,N) \geq \frac{2}{k}$ and $|d\chi_k|_g \leq 2k$ and we define $u_k := \chi_k u$ and $A_k := \{x \in M \,|\, \frac{1}{k} < \mathrm{d}_g(x,N) < \frac{2}{k}\}$. Then we have

$$\begin{aligned}&\int_{M_k\setminus\{p\}} |\psi|^{\frac{n-2}{n-1}} L_g(|\psi|^{\frac{n-2}{n-1}})\, dv^g \\ =&I_g(u_k) - \int_{A_k} \chi_k|\psi|^{\frac{n-2}{n-1}} L_g(\chi_k|\psi|^{\frac{n-2}{n-1}})\, dv^g.\end{aligned} \tag{20}$$

Next we define ν as the outer unit normal vector field on ∂A_k and we obtain

$$\begin{aligned}&\int_{A_k}\chi_k|\psi|^{\frac{n-2}{n-1}}\Delta_g(\chi_k|\psi|^{\frac{n-2}{n-1}})\,dv^g\\&=\int_{A_k}|d(\chi_k|\psi|^{\frac{n-2}{n-1}})|^2\,dv^g-\int_{\partial A_k}\chi_k|\psi|^{\frac{n-2}{n-1}}\partial_\nu(\chi_k|\psi|^{\frac{n-2}{n-1}})\,ds^g\\&=\int_{A_k}|d(\chi_k|\psi|^{\frac{n-2}{n-1}})|^2\,dv^g-\int_{\partial B_{2/k}(N)}|\psi|^{\frac{n-2}{n-1}}\partial_\nu|\psi|^{\frac{n-2}{n-1}}\,ds^g.\end{aligned}\tag{21}$$

In order to estimate the derivatives of $|\psi|^{(n-2)/(n-1)}$ near N we note that for all $Y\in T(M\setminus\{p\})$ we have

$$\partial_Y|\psi|^{\frac{n-2}{n-1}}=\frac{n-2}{n-1}|\psi|^{-\frac{n}{n-1}}\mathrm{Re}(\nabla_Y\psi,\psi).$$

Thus there exists $C_1>0$ such that for all $k\in\mathbb{N}$ large enough, for all $x\in\overline{B_{2/k}(N)}$ and for all $Y\in T_xM$ with $|Y|=1$ we have the estimate

$$\left|\partial_Y|\psi|^{\frac{n-2}{n-1}}(x)\right|\le C_1|\psi(x)|^{-\frac{1}{n-1}}.$$

Since $|\psi|^2$ is a C^1-function there exists $C_2>0$ such that for all $k\in\mathbb{N}$ large enough and for all $x\in\overline{B_{2/k}(N)}$ we have $|\psi(x)|^2\le C_2\mathrm{d}_g(x,N)$. Thus there exists $C_3>0$ such that for all $k\in\mathbb{N}$ large enough, for all $x\in\overline{B_{2/k}(N)}$ and for all $Y\in T_xM$ with $|Y|=1$ we have

$$\left|\partial_Y|\psi|^{\frac{n-2}{n-1}}(x)\right|\le C_3k^{\frac{1}{2(n-1)}}.\tag{22}$$

Furthermore since N has Hausdorff dimension at most $n-2$ there exists $C_4>0$ such that for all $k\in\mathbb{N}$ large enough we have

$$\mathrm{vol}(A_k)\le\frac{C_4}{k^2},\quad\mathrm{vol}(\partial B_{2/k}(N))\le\frac{C_4}{k}.\tag{23}$$

Using (22), (23) and using that $|d\chi_k|_g\le 2k$ we obtain from (21) that

$$\int_{A_k}\chi_k|\psi|^{\frac{n-2}{n-1}}L_g(\chi_k|\psi|^{\frac{n-2}{n-1}})\,dv^g\to 0$$

as $k\to\infty$. Therefore we obtain from (19), (20) that $\liminf_{k\to\infty}I_g(u_k)\le 0$. Furthermore by (13) we have $u_k(p)=0$ for all k and by Lemma 4.1 it follows that $J_g(u_k)=I_g(u_k)$ for all k. This finishes the proof in the general case. □

Proof of Theorem 6.1. The first statement follows immediately from Proposition 6.3 and from Lemma 4.3.

Next let $m(M,g)=0$ and let $(u_k)_{k\in\mathbb{N}}$ be the sequence in $C^\infty(M)$ constructed in the proof of Proposition 6.3. We have $\liminf_{k\to\infty}J_g(u_k)=0$ and therefore there exists a subsequence of $(u_k)_{k\in\mathbb{N}}$ which is a minimizing sequence for the functional J_g. From the proof of Lemma 4.2 it follows that after passing again to a subsequence the sequence $(u_k)_{k\in\mathbb{N}}$ converges pointwise almost everywhere to the minimizer $G-\eta r^{2-n}$ of the functional J_g. Therefore we have

$$|\psi|^{\frac{n-2}{n-1}}=G$$

almost everywhere on $M\setminus\{p\}$ and since both functions are continuous the equality holds everywhere on $M\setminus\{p\}$. By Proposition 2.1 the function G is strictly positive on $M\setminus\{p\}$. In particular ψ is nowhere zero and $|\psi|$ and the metric g' constructed in the proof of Proposition 6.3 are independent of the choice of $\psi_0\in\Sigma_pM$ satisfying

(12). For every such spinor $\psi_0 \in \Sigma_p M$ the spinor ψ' constructed from ψ_0 as in the proof of Proposition 6.3 is a parallel spinor for the metric g' by (18). Since the choice of ψ_0 is arbitrary we obtain a trivialization of the spinor bundle $\Sigma^{g'}(M \setminus \{p\})$ by parallel spinors. As in the proof of Theorem 2.2 in [1] it follows that (M, g) is conformally equivalent to (S^n, σ^n). □

6.2. **Application 2: A mass-to-infinity theorem.** Let M be a closed Riemannian manifold of dimension $n \geq 3$. We consider a sequence g_k of metrics which converges in $C^2(M)$ to a metric g_∞. We assume that all the metrics g_k, g_∞ are flat on a fixed neighborhood U of $p \in M$. Let also $(f_k)_{k\in\mathbb{N}}$ be a sequence in $C^\infty(M)$ such that for every k we have $f_k \equiv 0$ on U and such that for every k all eigenvalues of the operator $P_k := \Delta_{g_k} + f_k$ are positive. Furthermore we assume that there exists $f_\infty \in C^\infty(M)$ such that $f_k \to f_\infty$ in $C^\infty(M)$ and we write $P_\infty := \Delta_{g_\infty} + f_\infty$. Note that we just prove the result for C^∞ for simplicity but these assumptions could easily be weakened. For every $k \in \mathbb{N}$ the Green function of the operator P_k has an expansion as in Proposition 2.1 and we will denote the mass of P_k by m_k.

Theorem 6.4. *Assume that the first eigenvalue λ_∞ of P_∞ is 0. Then we have $m_k \to \infty$ as $k \to \infty$.*

This is a generalization of a result by Beig and O'Murchadha [5] who proved it with $f_k = \frac{n-2}{4(n-1)} s_{g_k}$, i.e. P_k is the Yamabe operator of the metric g_k. The limiting metric g_∞ was assumed to have a vanishing Yamabe constant (i.e. the first eigenvalue of $P_\infty = L_{g_\infty}$ is equal to 0). With the use of Theorem 3.1, the proof is much simpler than the proof by Beig and O'Murchadha.

Proof. Let $k \in \mathbb{N}$. We choose $\delta_k > 0$ such that the ball $B(p, \delta_k)$ centered at p and of radius δ_k with respect to the metric g_k is contained in U. Then we define a smooth non-negative function η_k on M such that $\eta_k \equiv \frac{1}{(n-2)\omega_{n-1}}$ on $B(p, \delta_k)$ and such that $\mathrm{supp}(\eta_k) \subset U$. For $x \in M$ let $r_k(x)$ denote the distance of x to p with respect to the metric g_k. The function $F_{\eta_k} \colon M \to \mathbb{R}$ defined by

$$F_{\eta_k}(x) = \begin{cases} \Delta_{g_k}(\eta_k r_k^{2-n})(x), & x \neq p \\ 0, & x = p \end{cases}$$

is smooth on M. For every $u \in C^\infty(M)$ we define

$$J_k(u) := \int_{M\setminus\{p\}} \eta_k r_k^{2-n} F_{\eta_k} \, dv^{g_k} + 2 \int_M u F_{\eta_k} \, dv^{g_k} + \int_M u P_k u \, dv^{g_k}.$$

Then by Theorem 3.1 we have for all $k \in \mathbb{N}$

$$-m_k = \inf\{J_k(u) |\, u \in C^\infty(M)\}.$$

Let $u \in C^\infty(M)$ be an eigenfunction associated to λ_∞. It is a classical result that the eigenfunctions corresponding to the first eigenvalue of an operator of the form P_f are either strictly positive or strictly negative. Thus we may assume that u is strictly positive. As in the proof of Lemma 4.1 one obtains for every k:

$$\int_M u F_{\eta_k} \, dv^{g_k} = \int_{M\setminus\{p\}} u \Delta_{g_k}(\eta_k r_k^{2-n}) \, dv^{g_k} = \int_{M\setminus\{p\}} \eta_k r_k^{2-n} \Delta_{g_k} u \, dv^{g_k} - u(p).$$

Since $g_k \to g_\infty$ in $C^2(M)$ we have $\Delta_{g_k} u \to \Delta_{g_\infty} u$ in $C^0(M)$. Since $P_\infty u \equiv 0$ and since $f_\infty \equiv 0$ on U it follows that $\Delta_{g_\infty} u \equiv 0$ on U. Since $\mathrm{supp}(\eta_k) \subset U$ we conclude that

$$\int_M uF_{\eta_k}\,dv^{g_k} \to -u(p) < 0$$

as $k \to \infty$. Since $P_k u \to P_\infty u = 0$ in $C^0(M)$ it follows that

$$a_k := \int_M uP_k u\,dv^{g_k} \to 0$$

as $k \to \infty$. Now for every $k \in \mathbb{N}$ we have

$$-m_k \le J_k(a_k^{-1/3}u) = \int_{M\setminus\{p\}} \eta_k r_k^{2-n} F_{\eta_k}\,dv^{g_k} + 2a_k^{-1/3}\int_M uF_{\eta_k}\,dv^{g_k} + a_k^{-2/3}a_k$$

and the right hand side tends to $-\infty$ as $k \to \infty$. The assertion follows. □

6.3. **Application 3: Real analytic families of masses and negative mass.** In this section, we study the family of masses associated to a family of operators of the type $\Delta_g + f$. As an application, we prove that on any manifold, there exists a function f such that the operator $\Delta_g + f$ is positive but with negative mass. This shows in particular that a proof of a positive mass theorem as studied in Section 6.1 must use the conformal properties of the operator L_g. Let (M, g) be a closed Riemannian manifold such that g is flat on an open neighborhood U of a point $p \in M$. Let $\varphi, f \in C^\infty(M)$ such that $f \equiv 0$ and $\varphi \equiv 0$ on U. For every $a \in \mathbb{R}$ we define the operator $P_a := \Delta_g + f + a\varphi$. We assume that for $a = 0$ all eigenvalues of P_0 are positive. Since the operator P_0 is invertible, it follows from the Neumann series expansion of the inverse that there exists an open interval I containing 0 such that for every $a \in I$ the operator P_a is invertible (see e. g. [15, IV-1.16]). Since by a theorem of Rellich the eigenvalues of P_a are real analytic functions of a (see [15, VII-3.9]), it follows that for every $a \in I$ the operator P_a has only positive eigenvalues. Moreover we can choose I as the maximal interval with this property. For every $a \in I$ we can define the mass of P_a and we denote it by $m(a)$. Furthermore, for every $a \in \mathbb{R}$ and for every $u \in C^\infty(M)$ we define

$$\begin{aligned} I_a(u) &:= \int_{M\setminus\{p\}} (\eta r^{2-n} + u)P_a(\eta r^{2-n} + u)\,dv^g, \\ J_a(u) &:= \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta\,dv^g + 2\int_M uF_\eta\,dv^g + \int_M uP_a u\,dv^g, \end{aligned}$$

where η and F_η are as in Section 3. By Theorem 3.1, we have

$$\begin{aligned} -m(a) &= \inf\{I_a(u)\,|\, u \in C^\infty(M),\, u(p) = 0\} \\ &= \inf\{J_a(u)\,|\, u \in C^\infty(M)\}. \end{aligned}$$

The main result of this section is the following theorem.

Theorem 6.5. *1. The function $I \to \mathbb{R}$, $a \mapsto m(a)$ is real analytic.*
2. The function $I \to \mathbb{R}$, $a \mapsto m(a)$ is convex.
3. Assume that there exists a point $q \in M$ such that $\varphi(q) < 0$. Then there exists $a_\infty > 0$ such that $m(a)$ can be defined for all $a \in [0, a_\infty)$ and we have $m(a) \to \infty$ as $a \to a_\infty$.

4. If $\varphi \geq 0$, then $m(a)$ can be defined for all $a \geq 0$, the function $a \mapsto m(a)$ is non-increasing and we have

$$\begin{aligned}\lim_{a\to\infty} m(a) &= -\inf\{J_0(u)|\, u \in C^\infty(M),\, \mathrm{supp}(u) \subset M \setminus \mathrm{supp}(\varphi)\}\\ &= -\inf\{I_0(u)|\, u \in C^\infty(M),\, u(p) = 0,\, \mathrm{supp}(u) \subset M \setminus \mathrm{supp}(\varphi)\}\\ &=: m_{f,M\setminus\mathrm{supp}(\varphi)} > -\infty.\end{aligned}$$

Corollary 6.6. *There exists a function f such that P_f is positive and such that $m_f < 0$.*

Corollary 6.7. *Let $p \in S^n$. There exists a Riemannian metric g on S^n which is conformal to σ^n and flat on an open neighborhood of p such that for the operator $P_a := \Delta_g + as_g$ we have $m(a) < 0$ for all $a > \frac{n-2}{4(n-1)}$.*

6.3.1. *Proof of Theorem 6.5 Point 1.* For every $a \in I$ we denote the Green function for the operator P_a by G_a. We have

$$(P_0 + a\varphi)G_a = \delta_p, \quad P_0G_0 = \delta_p$$

and therefore

$$(P_0 + a\varphi)(G_a - G_0) = -a\varphi G_0, \tag{24}$$

where the right hand side is smooth, since φ vanishes on an open neighborhood of p. The family of bounded linear operators

$$I \ni a \mapsto P_0 + a\varphi \in B(C^2(M), C^0(M))$$

is real analytic and for every $a \in I$ the operator $P_0 + a\varphi$ is invertible. It follows that the family of bounded linear operators

$$I \ni a \mapsto (P_0 + a\varphi)^{-1} \in B(C^0(M), C^2(M))$$

is real analytic as well (see [15, VII-§1.1]). From (24) we obtain that the family of smooth functions $a \mapsto G_a - G_0$ is real analytic. The assertion follows.

6.3.2. *Proof of Theorem 6.5 Point 2.* Denote by $G'_a := \frac{d}{da}G_a$ and $G''_a := \frac{d^2}{da^2}G_a$. Differentiating twice $P_aG_a = \delta_p$, we get:

$$P_aG'_a = -\varphi G_a \text{ and } P_aG''_a = -2\varphi G'_a. \tag{25}$$

Now, observe that $G''_a(p) = m''(a)$. As a consequence, since in the sense of distributions $P_aG_a = \delta_p$ and using (25), we have

$$\begin{aligned}m''(a) &= \int_{M\setminus\{p\}} G_aP_aG''_a\, dv^g\\ &= -2\int_{M\setminus\{p\}} \varphi G'_aG_a dv^g\\ &= 2\int_{M\setminus\{p\}} G'_aP_aG'_a\, dv^g \geq 0.\end{aligned}$$

The last inequality comes from the fact that G'_a is smooth on M and that P_a is a positive operator.

6.3.3. *Proof of Theorem 6.5 Point 3.* Denote by λ_a the first eigenvalue of P_a. By assumption, $\lambda_0 > 0$. Since $\varphi(q) < 0$ there exists an open neighborhood $V \subset M$ of q such that $\varphi < 0$ on V. Let $v \neq 0$ be a non-negative function supported in V. Then, for a large enough, $\int_M vP_a v dv^g < 0$ and hence $\lambda_a < 0$. Define a_∞ as

$$a_\infty := \inf\{a > 0 |\, \lambda_a = 0\}.$$

Then, by Theorem 6.4 we have $m(a) \to \infty$ as $a \to a_\infty$.

6.3.4. *Proof of Theorem 6.5 Point 4.* Since P_0 is a positive operator and since $\varphi \geq 0$ we have for all $a \geq 0$ and for all $u \in C^\infty(M)$ with $u \not\equiv 0$

$$\int_M uP_a u\, dv^g = \int_M uP_0 u\, dv^g + a \int_M \varphi u^2\, dv^g > 0.$$

Thus for all $a \geq 0$ the operator P_a is positive and $m(a)$ can be defined.

For every $a \geq 0$ and for every $u \in C^\infty(M)$ we have

$$J_a(u) = J_0(u) + a \int_M \varphi u^2\, dv^g,$$

where the integral on the right hand side is non-negative. Let $a_1, a_2 \geq 0$ with $a_1 \leq a_2$. Then for every $u \in C^\infty(M)$ we have $J_{a_1}(u) \leq J_{a_2}(u)$. It follows that $m(a_1) \geq m(a_2)$ and thus the function $a \mapsto m(a)$ is non-increasing. Next let $u \in C^\infty(M)$ such that $\mathrm{supp}(u) \subset M \setminus \mathrm{supp}(\varphi)$. Then for all $a \geq 0$ we have $-m(a) \leq J_a(u) = J_0(u)$. Since this holds for every $u \in C^\infty(M)$ such that $\mathrm{supp}(u) \subset M \setminus \mathrm{supp}(\varphi)$, we obtain

$$-m(a) \leq \inf\{J_0(u) |\, u \in C^\infty(M),\, \mathrm{supp}(u) \subset M \setminus \mathrm{supp}(\varphi)\}.$$

Thus the function $a \mapsto m(a)$ is bounded from below and the limit $\lim_{a\to\infty} m(a)$ exists.

In the following we may assume without loss of generality that $\varphi \not\equiv 0$. We now need to obtain some properties of G_a. Let us observe that $G_0 - G_a$ is smooth. One computes that

$$P_0(G_0 - G_a) = a\varphi G_a. \tag{26}$$

Multiplying this equation by the Green function of P_0 at any point $q \in M \setminus \{p\}$ and integrating we obtain $(G_0 - G_a)(q) > 0$. It follows that $0 < G_a < G_0$ on $M \setminus \{p\}$. Therefore, since

$$1 = \int_{M\setminus\{p\}} P_a(1) G_a\, dv^g = \int_M f G_a\, dv^g + a \int_M \varphi G_a\, dv^g$$

we obtain that

$$a \int_M \varphi G_a\, dv^g \leq C \tag{27}$$

for some fixed positive constant C which is independent of a. We multiply (26) by $G_0 - G_a$ and integrate.

$$\begin{aligned} a \int_M \varphi G_a G_0\, dv^g &\geq a \int_M \varphi G_a G_0\, dv^g - a \int_M \varphi G_a^2\, dv^g \\ &= \int_M (G_0 - G_a) P_0 (G_0 - G_a)\, dv^g \\ &= \int_M |d(G_0 - G_a)|^2 dv^g + \int_M f(G_0 - G_a)^2 dv^g \end{aligned}$$

and the right hand side is positive since P_0 is a positive operator. From (27), we deduce that $a\int_M \varphi G_a G_0 dv^g$ is bounded, and hence the same holds for

$$\int_M |d(G_0 - G_a)|^2 dv^g + \int_M f(G_0 - G_a)^2 dv^g.$$

This implies that $G_0 - G_a$ is bounded in the Sobolev space $H^{1,2}(M)$. Hence, there exists a function $v_\infty \in H^{1,2}(M)$ such that after taking a subsequence the functions $G_0 - G_a$ tend to v_∞ weakly in $H^{1,2}(M)$ and strongly in $L^2(M)$. We now set $u_a := G_a - \eta r^{2-n}$. Then u_a tends to $u_\infty := -v_\infty + G_0 - \eta r^{2-n}$ weakly in $H^{1,2}(M)$ and strongly in $L^2(M)$ and pointwise almost everywhere. Observe that u_∞ is non-negative on $\mathrm{supp}(\varphi)$ since $u_a \equiv G_a$ on $\mathrm{supp}(\varphi)$. Moreover, by (27) we have

$$\int_M \varphi u_\infty \, dv^g = \lim_{a\to\infty} \int_M \varphi G_a \, dv^g = 0$$

and as a consequence, $u_\infty \equiv 0$ on $\mathrm{supp}(\varphi)$.

For all smooth functions u we have

$$J_0(u) = \int_M (|du|^2 + fu^2) \, dv^g + \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^g + 2\int_M u F_\eta \, dv^g.$$

By density of $C^\infty(M)$ in $H^{1,2}(M)$ and since u_∞ vanishes on $\mathrm{supp}(\varphi)$, we thus have

$$\begin{aligned}&\inf\{J_0(u) |\, u \in C^\infty(M),\, \mathrm{supp}(u) \subset M \setminus \mathrm{supp}(\varphi)\}\\ &= \int_M (|du_\infty|^2 + fu_\infty^2) \, dv^g + \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^g + 2\int_M u_\infty F_\eta \, dv^g.\end{aligned}$$

By weak convergence in $H^{1,2}(M)$ and strong convergence in $L^2(M)$ of u_a to u_∞, it follows that the right hand side is bounded above by (see the proof of Lemma 4.2 for details)

$$\begin{aligned}&\liminf_{a\to\infty} \int_M (|du_a|^2 + fu_a^2) \, dv^g + \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^g + 2\int_M u_a F_\eta \, dv^g\\ &= \liminf_{a\to\infty} J_0(u_a).\end{aligned}$$

This implies that

$$\inf\{J_0(u) |\, u \in C^\infty(M),\, \mathrm{supp}(u) \subset M \setminus \mathrm{supp}(\varphi)\} \leq \liminf_{a\to\infty} J_0(u_a). \tag{28}$$

From Theorem 3.1,

$$-m(a) = J_a(u_a) = J_0(u_a) + a\int_M \varphi G_a^2 \, dv^g \geq J_0(u_a)$$

which gives, together with (28) that

$$\inf\{J_0(u) |\, u \in C^\infty(M),\, \mathrm{supp}(u) \subset M \setminus \mathrm{supp}(\varphi)\} \leq -\lim_{a\to\infty} m(a).$$

This proves Point 4 of Theorem 6.5.

6.3.5. *Proof of Corollary 6.6.* Let us for a moment consider the sphere S^n. Let h be a metric on S^n which is conformal to the standard metric and which is flat on a ball $B(q,\delta)$ of radius δ for some $q \in S^n$ where $\delta > 0$ is chosen such that (M,g) is flat on $B(p,\delta)$. Let $\overline{\varphi}$ be a smooth function on S^n which is positive on $S^n \setminus \overline{B(q,\delta)}$ and which vanishes on $B(q,\delta)$. For every $a \geq 0$ let G_a be the Green function of the operator $L_h + a\overline{\varphi}$ and let $\overline{m}(a)$ be its mass. We have

$$L_h(G_0 - G_a) = a\overline{\varphi}G_a.$$

As in the lines after Equation (26) it follows that for all $a > 0$ we have $G_0 - G_a > 0$ and hence, $\overline{m}(a) = \overline{m}(a) - \overline{m}(0) = (G_a - G_0)(q) < 0$. By Point 4 of Theorem 6.5 the function $a \mapsto \overline{m}(a)$ is non-increasing. Hence, $\lim_{a\to\infty} \overline{m}(a) < 0$. Applying Point 4 of Theorem 6.5, we obtain that

$$\inf\{J(u) |\, u \in C^\infty(S^n),\, \mathrm{supp}(u) \subset B(q,\delta)\} > 0 \tag{29}$$

where J is defined as above on the sphere by

$$J(u) := \int_{S^n\setminus\{q\}} \eta r^{2-n} F_\eta \, dv^g + 2\int_{S^n} uF_\eta \, dv^g + \int_{S^n} uP_0 u \, dv^g,$$

where η is a smooth function supported in $B(q,\delta)$.

Now, let $f\colon M \to \mathbb{R}$ be a smooth function which is positive on $M \setminus \overline{B(p,\delta)}$ and 0 on $B(p,\delta)$. We consider the operator $P_a := \Delta_g + f + af$. Let $m(a)$ be the corresponding mass. For every $a \geq 0$ the operator P_a is positive. By Point 4 of Theorem 6.6, we have

$$\lim_{a+\infty} m(a) = -\inf\{J_0(u) |\, u \in C^\infty(M),\, \mathrm{supp}(u) \subset B(p,\delta)\}$$

where J_0 is constructed as above on M. Observe that since $(B(q,\delta),h) \subset (S^n,h)$ and $(B(p,\delta),g) \subset (M,g)$ are isometric, we have

$$\begin{aligned}&\inf\{J_0(u) |\, u \in C^\infty(M),\, \mathrm{supp}(u) \subset B(p,\delta)\}\\ &= \inf\{J(u) |\, u \in C^\infty(S^n),\, \mathrm{supp}(u) \subset B(q,\delta)\}.\end{aligned}$$

By (29), we obtain that $\lim_{a\to\infty} m(a) < 0$ which proves Corollary 6.6.

6.3.6. *Proof of Corollary 6.7.* It is sufficient to find a Riemannian metric g on S^n which is conformal to σ^n, flat on an open neighborhood of p and satisfies $s_g \geq 0$. Choose an open neighborhood U of p on which σ^n is conformally flat. Using stereographic projection at $-p$ we may write $\sigma^n = u^{4/(n-2)}\xi^n$ on U where with $r = |x|_{\xi^n}$ we have

$$u(r) = \left(\frac{2}{1+r^2}\right)^{\frac{n-2}{2}}.$$

Let $\varepsilon > 0$ be so small that $u''(r) < 0$ on $[0,2\varepsilon)$ and such that the preimage of $B(0,2\varepsilon) \subset \mathbb{R}^n$ under the stereographic projection is contained in U. Choose a smooth function v on $[0,\infty)$ such that v is constant on $[0,\varepsilon)$, $v = u$ on $[2\varepsilon,\infty)$ and such that on $[0,2\varepsilon)$ we have $v'(r) \leq 0$ and $v''(r) \leq 0$. We define v as a radial function on $\mathbb{R}^n$ and we obtain

$$\Delta_{\xi^n} v(x) = -v''(r) - \frac{n-1}{r}v'(r) \geq 0. \tag{30}$$

We define $g = v^{4/(n-2)}\xi^n$ on U and $g = \sigma^n$ on $S^n \setminus U$. Then g is a smooth Riemannian metric on S^n which is conformal to σ^n and flat on an open neighborhood of p. Furthermore by the conformal transformation law (6) for L_{ξ^n} and by (30) we have $s_g \geq 0$ on S^n.

6.4. **Application 4: Surgery and positivity of mass.** Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$, let $p \in M$ and assume that g is flat on an open neighborhood U of p. Let $f \in C^\infty(M)$ such that $f \equiv 0$ on U. We keep the same notation as in Section 3. Let now $\Omega \subset M$ be an open subset containing $\mathrm{supp}(\eta)$. Assume that

$$P_f|_\Omega : \quad C^\infty(\Omega) \to C^\infty(\Omega)$$

is a positive operator with respect to Dirichlet boundary condition. Then, we define

$$m_{f,\Omega} := -\inf\{I_f(u) |\, u \in C^\infty(M),\, u(p) = 0,\, \mathrm{supp}(u) \subset \Omega\}$$

Let $G_{f,\Omega}$ be the Green function of $P_f|_\Omega$ with Dirichlet boundary condition. Mimicking the proof of Theorem 3.1, one proves that $m_{f,\Omega}$ is the mass of $G_{f,\Omega}$. Clearly for any Ω the following proposition is obvious from the definitions.

Proposition 6.8. *We have*

$$m_f \geq m_{f,\Omega}.$$

This observation has nevertheless some interesting applications. A first one is the following: let (Ω, g_0) be a compact manifold with boundary and let f_0 be a function defined on Ω. Assume that (Ω, g_0) embeds isometrically in (M,g) and let f be such that P_f is positive on (M,g) and $f = f_0$ on $\Omega \subset M$. Then, the mass of P_f is bounded from below by a constant which depends only on (Ω, g_0) and f_0.

Another application seems much more interesting. Let (M,g) be a closed Riemannian manifold with positive Yamabe constant $Y(M,g)$. We assume that g is flat around a point p. Now, we perform on M a surgery of dimension $k \leq n-3$, i. e. we remove a tubular neighborhood of a sphere S^k in M and replace it by gluing the boundary with the boundary of the product $\overline{B}^{k+1} \times S^{n-k-1}$. Without loss of generality, we can assume that p does not lie in the removed part. For more information on this procedure, see for instance [2] or Section 7.2. Then, it was proven by several authors (see [8, 23, 2]) that on the new manifold N one can construct a new metric h with positive Yamabe constant which is flat around p. Moreover h can be constructed in such a way that it coincides with g on M except on an arbitrarily small open neighborhood of the removed sphere in M. Then, a natural question is: assume that the mass $m(M,g)$ of L_g is positive. Does this imply that the mass $m(N,h)$ of L_h is also positive? Observe that Proposition 6.8 gives an immediate positive answer to this question. Indeed, for $\varepsilon > 0$, define

$$\Omega_\varepsilon := \{x \in M |\, \mathrm{d}_g(x,S) > \varepsilon\}$$

where S is the surgery k-sphere. Then we prove the following theorem.

Theorem 6.9. *For every $\varepsilon > 0$ let h_ε be a Riemannian metric on N such that $Y(N,h_\varepsilon) > 0$ and $h_\varepsilon = g$ on Ω_ε. Then we have*

$$\liminf_{\varepsilon \to 0} m(N,h_\varepsilon) \geq m(M,g).$$

Proof. Let $u \in C^\infty(M)$ such that we have $-m(M,g) = J_g(u)$. Let χ_ε be a smooth function on M equal to 1 on $\Omega_{2\varepsilon}$, equal to 0 on $M \setminus \Omega_\varepsilon$ and such that $|d\chi_\varepsilon|_g \leq \frac{2}{\varepsilon}$.

We may consider the functions $\chi_\varepsilon u$ as functions on N. We write $A_\varepsilon := \Omega_\varepsilon \setminus \Omega_{2\varepsilon}$. Since on $\mathrm{supp}(\chi_\varepsilon)$ we have $h_\varepsilon = g$ we obtain by (3)

$$\begin{aligned}\int_N u\chi_\varepsilon L_{h_\varepsilon}(u\chi_\varepsilon)\,dv^{h_\varepsilon} &= \int_M (u^2|d\chi_\varepsilon|_g^2 + \chi_\varepsilon^2 u L_g u)\,dv^g \\ &\leq \frac{4}{\varepsilon^2}\int_{A_\varepsilon} u^2\,dv^g + \int_M \chi_\varepsilon^2 u L_g u\,dv^g.\end{aligned}$$

Let $k \in \{0, ..., n-3\}$ be the dimension of the surgery sphere. Since there exists $C > 0$ such that for all ε we have $\mathrm{vol}(A_\varepsilon) \leq C\varepsilon^{n-k}$, the first term on the right hand side tends to 0 as $\varepsilon \to 0$. We conclude that

$$\limsup_{\varepsilon\to 0} J_{h_\varepsilon}(u\chi_\varepsilon) \leq J_g(u).$$

Since $\mathrm{supp}(\chi_\varepsilon u) \subset \Omega_\varepsilon$ it follows that

$$\limsup_{\varepsilon\to 0}\big(-m_{\frac{n-2}{4(n-1)}s_{h_\varepsilon},\Omega_\varepsilon}\big) \leq J_g(u).$$

and thus

$$m(M,g) \leq \liminf_{\varepsilon\to 0} m_{\frac{n-2}{4(n-1)}s_{h_\varepsilon},\Omega_\varepsilon}.$$

The assertion now follows from Proposition 6.8. □

Theorem 6.9 shows that the positivity of mass is preserved by surgery of dimension $k \in \{0, ..., n-3\}$. In the next section we will obtain a much stronger result, namely that also a negative mass is preserved under such surgeries.

7. Preservation of mass by surgery

7.1. **The result.** Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ with positive Yamabe constant $Y(M, g)$. Assume that g is flat on an open neighborhood of a point $p \in M$. Then we can define the mass $m(M, g)$ at p. Let N be obtained from M by a surgery of dimension $k \in \{0, ..., n-3\}$ which does not hit the point p. Our aim is to show that the mass $m(M, g)$ at p is preserved by this procedure. More precisely we will prove the following theorem.

Theorem 7.1. *There exists a sequence of metrics (g_θ) on N such that for every θ the mass $m(N, g_\theta)$ at p can be defined and such that we have*

$$\lim_{\theta\to 0} m(N, g_\theta) = m(M, g).$$

We will study an application of this theorem to the positive mass conjecture in Section 8. But first we will prove Theorem 7.1. We will define the family of metrics g_θ in Section 7.2. The same family of metrics has been used in the article [2]. In Section 7.5 we will prove that this family of metrics has the property stated in the theorem. We will use the variational characterization of the mass according to Theorem 3.1 and we will also use some techniques from the article [2], which we briefly recall in Sections 7.3 and 7.4.

7.2. **Definition of the metrics g_θ.** We recall a construction called the connected sum along a submanifold using the notation of the article [2]. On the manifold obtained in this way we define a family of Riemannian metrics $(g_\theta)_{\theta>0}$ which is described in the same article. We will mostly be interested in surgery which is a special case of this construction. Let (M_1, g_1), (M_2, g_2) be complete Riemannian manifolds of dimension n and let W be a closed manifold of dimension $k \leq n$. Let $\bar{w}_i$: $W \times \mathbb{R}^{n-k} \to TM_i$, $i = 1, 2$, be embeddings. We assume that $\bar{w}_i$ maps $W \times \{0\}$ to the zero section of TM_i which we identify with M_i. Thus we obtain embeddings $W \to M_i$ and we will denote the images of these embeddings by $W_i' \subset M_i$. We assume that for every $x \in W$ the embeddings $\bar{w}_i$ restrict to linear isomorphisms $\{x\} \times \mathbb{R}^{n-k} \to N_{\bar{w}_i(x,0)} W_i'$, where NW_i' denotes the normal bundle of W_i' with respect to the metric g_i. For $i = 1, 2$ let r_i be the function on M_i giving the distance to W_i' and define $U^{M_i}(c) := \{x \in M_i | \, r_i(x) < c\}$ for every $c > 0$. There exists $R_{\max} > 0$ such that the maps $w_i := \exp^{g_i} \circ \bar{w}_i$ define diffeomorphisms

$$w_i : \quad W \times B^{n-k}(R_{\max}) \to U^{M_i}(R_{\max}), \quad i = 1, 2.$$

In general, the Riemannian metrics g_i do not have a corresponding product structure on $U^{M_i}(R_{\max})$. We introduce error terms T_i measuring the differences from the product metrics. Namely, if h_i denote the restrictions of g_i to W_i' and if σ^{n-k-1} is the standard metric on S^{n-k-1} we have

$$g_i = h_i + dr_i^2 + r_i^2 \sigma^{n-k-1} + T_i$$

on $U^{M_i}(R_{\max})$, $i = 1, 2$. Now, for every $\varepsilon \in (0, R_{\max})$ we define

$$N_\varepsilon := (M_1 \setminus U^{M_1}(\varepsilon)) \cup (M_2 \setminus U^{M_2}(\varepsilon)) / \sim,$$

and for every $c \in (\varepsilon, R_{\max})$

$$U_\varepsilon^N(c) := (U^{M_1}(c) \setminus U^{M_1}(\varepsilon)) \cup (U^{M_2}(c) \setminus U^{M_2}(\varepsilon)) / \sim,$$

where $\sim$ means that we identify the point $x \in \partial U^{M_1}(\varepsilon)$ with the point $w_2 \circ w_1^{-1}(x) \in \partial U^{M_2}(\varepsilon)$. Therefore we have

$$N_\varepsilon = (M_1 \setminus U^{M_1}(c)) \cup (M_2 \setminus U^{M_2}(c)) \cup U_\varepsilon^N(c).$$

We say that N_ε is obtained from M_1 and M_2 by a connected sum along W with parameter ε. Since the diffeomorphism type of the manifold N_ε is independent of the choice of ε we will often write N instead of N_ε. Our next aim is to define for a given $\theta > 0$ a Riemannian metric g_θ on N_ε for $\varepsilon > 0$ small enough. We choose numbers R_0, θ, δ_0 such that

$$R_{\max} > R_0 > \theta > \delta_0 > 0.$$

Then we choose $A_\theta \in (\theta^{-1}, (\delta_0)^{-1})$ and we put $\varepsilon := e^{-A_\theta} \delta_0$. Then we define N_ε and $U_\varepsilon^N(c)$ for $c > 0$ as above. On the set $U_\varepsilon^N(R_{\max})$ we define the coordinate function t by

$$t := \begin{cases} -\ln r_1 + \ln \varepsilon, & \text{on } U^{M_1}(R_{\max}) \setminus U^{M_1}(\varepsilon), \\ \ln r_2 - \ln \varepsilon, & \text{on } U^{M_2}(R_{\max}) \setminus U^{M_2}(\varepsilon). \end{cases}$$

We choose smooth functions F on N_ε and f on $U_\varepsilon^N(R_{\max})$ such that

$$F(x) = \begin{cases} 1, & \text{if } x \in N_\varepsilon \setminus U_\varepsilon^N(R_{\max}), \\ r_i(x)^{-1}, & \text{if } x \in U^{M_i}(R_0) \setminus U^{M_i}(\varepsilon),\ i=1,2, \end{cases}$$

$$f(x) = \begin{cases} -|t(x)| - \ln \varepsilon, & \text{if } x \in N_\varepsilon \setminus U_\varepsilon^N(\theta), \\ \ln A_\theta, & \text{if } x \in U_\varepsilon^N(\delta_0) \end{cases}$$

and such that $|df/dt| \leq 1$ for all t and $\|d^2f/dt^2\|_{L^\infty} \to 0$ as $\theta \to 0$. We choose a smooth function χ: $\mathbb{R} \to [0,1]$ such that $\chi = 0$ on $(-\infty, -1]$, $\chi = 1$ on $[1, \infty)$ and $|\chi'| \leq 1$ on $\mathbb{R}$. Then we define

$$g_\theta := \begin{cases} F^2 g_i, & \text{on } M_i \setminus U^{M_i}(\theta), \\ e^{2f(t)}(h_i + T_i) + dt^2 + \sigma^{n-k-1}, & \text{on } U^{M_i}(\theta) \setminus U^{M_i}(\delta_0), \\ \left.\begin{array}{l} A_\theta^2 \chi(t/A_\theta)(h_2 + T_2) \\ +A_\theta^2(1-\chi(t/A_\theta))(h_1+T_1) \\ +dt^2 + \sigma^{n-k-1}, \end{array}\right\} & \text{on } U_\varepsilon^N(\delta_0). \end{cases}$$

On $U_\varepsilon^N(R_0)$ we write the metric g_θ as

$$g_\theta = e^{2f(t)}\tilde{h}_t + dt^2 + \sigma^{n-k-1} + \widetilde{T}_t,$$

where $\tilde{h}_t$ is defined by

$$\tilde{h}_t := \chi(t/A_\theta)h_2 + (1-\chi(t/A_\theta))h_1,$$

for $t \in \mathbb{R}$ and where the error term $\widetilde{T}_t$ is equal to

$$\widetilde{T}_t := e^{2f(t)}(\chi(t/A_\theta)T_2 + (1-\chi(t/A_\theta))T_1).$$

On $U_\varepsilon^N(R_0)$ we also define the metric without error term

$$g'_\theta := g_\theta - \widetilde{T}_t = e^{2f(t)}\tilde{h}_t + dt^2 + \sigma^{n-k-1}. \tag{31}$$

We will need upper bounds for the error term $\widetilde{T}$ and its derivatives. As in Section 6.2 of the article [2] one can show that there exists $C > 0$ such that for all θ we have

$$|\widetilde{T}_t|_{g'_\theta} \leq Ce^{-f(t)} \tag{32}$$

$$|\nabla^{g'_\theta}\widetilde{T}_t|_{g'_\theta} \leq Ce^{-f(t)} \tag{33}$$

$$|s_{g_\theta} - s_{g'_\theta}| \leq Ce^{-f(t)}. \tag{34}$$

In the special case where $M_2 = S^n$, $W = S^k$, $k \leq n$, and $S^k \to S^n$ is the standard embedding we say that N_ε is obtained from M_1 by surgery of dimension k with parameter ε. Note that in this case $M_2 \setminus U^{M_2}(\varepsilon)$ is diffeomorphic to $\overline{B}^{k+1} \times S^{n-k-1}$.

7.3. **Limit spaces and limit solutions.** In the proof of Theorem 7.1 we will construct solutions to the equation $\Delta_g u = 0$ on certain limit spaces (V, g). We need the following lemmas which are adapted versions of Lemmas 4.1, 4.2 and 4.3 in [2].

Lemma 7.2. *Let V be a manifold of dimension n. Let $(q_\alpha)_\alpha$ be a sequence of points in V that converges to a point q as $\alpha \to 0$. Let $(\gamma_\alpha)_\alpha$ be a sequence of metrics defined on an open neighborhood O of q that converges to a metric γ_0 in the $C^2(O)$-topology as $\alpha \to 0$. Let $(b_\alpha)_\alpha$ be a sequence of positive real numbers such* *that $b_\alpha \to \infty$ as $\alpha \to 0$. Then for every $r > 0$ there exists for α small enough a diffeomorphism*

$$\Theta_\alpha : \quad B^n(r) \to B^{\gamma_\alpha}(q_\alpha, b_\alpha^{-1} r)$$

with $\Theta_\alpha(0) = q_\alpha$ such that the metric $\Theta_\alpha^*(b_\alpha^2 \gamma_\alpha)$ tends to the flat metric ξ^n in $C^2(B^n(r))$.

Proof. see the proof of Lemma 4.1 in [2]. □

Lemma 7.3. *Let V be a manifold of dimension n. Let $(g_\alpha)_\alpha$ be a sequence of metrics that converges to a metric g in C^2 on all compact sets $K \subset V$ as $\alpha \to 0$. Assume that $(U_\alpha)_\alpha$ is an increasing sequence of subdomains of V such that $\bigcup_\alpha U_\alpha = V$. Let $u_\alpha \in C^2(U_\alpha)$ be a sequence of positive functions such that $\|u_\alpha\|_{L^\infty(U_\alpha)}$ is bounded independently of α. We assume*

$$L_{g_\alpha} u_\alpha = 0$$

for all α. Then there exists a non-negative function $u \in C^2(V)$ satisfying

$$L_g u = 0$$

on V and a subsequence of u_α that tends to u in C^1 on each open set $\Omega \subset V$ with compact closure. In particular for every compact subset $K \subset V$ we have

$$\|u\|_{L^\infty(K)} = \lim_{\alpha \to 0} \| u_\alpha\|_{L^\infty(K)} \tag{35}$$

and

$$\int_K u^r \, dv^g = \lim_{\alpha \to 0} \int_K u_\alpha^r \, dv^{g_\alpha} \tag{36}$$

for every $r \geq 1$.

Proof. see the proof of Lemma 4.2 in [2]. □

Lemma 7.4. *Let ξ^n be the flat metric on $\mathbb{R}^n$ and assume that $u \in C^2(\mathbb{R}^n)$, $u \geq 0$, $u \not\equiv 0$ satisfies*

$$L_{\xi^n} u = \mu u^{p-1}$$

for some $\mu \in \mathbb{R}$ and $p := \frac{2n}{n-2}$. Assume in addition that $u \in L^p(\mathbb{R}^n)$ and that

$$\|u\|_{L^p(\mathbb{R}^n)} \leq 1.$$

Then $\mu \geq Y(S^n, \sigma^n)$.

Proof. see the proof of Lemma 4.3 in [2]. □

7.4. **L^2-estimates on WS-bundles.**

Definition 7.5. Let $n \geq 1$ and $k \in \{0, ..., n-3\}$ be integers. Let W be a closed manifold of dimension k and let I be an interval. A WS-bundle is a product $P := I \times W \times S^{n-k-1}$ equipped with a metric of the form

$$g_{\mathrm{WS}} = dt^2 + e^{2\varphi(t)} h_t + \sigma^{n-k-1} \tag{37}$$

where h_t is a smooth family of metrics on W depending on $t \in I$ and φ is a function on I.

We denote by $\pi\colon\ P \to I$ the projection onto the first factor and for every $t \in I$ we write $F_t := \pi^{-1}(t)$. Furthermore we define

$$e(h_t) := \frac{1}{2(n-1)} \mathrm{tr}_{h_t}(\partial_t h_t).$$

Definition 7.6. We say that condition (A_t) holds at $t \in I$, if the following assumptions are true:

1. $s \mapsto h_s$ is constant on an open neighborhood of t,
2. $e^{-2\varphi(t)} \inf_{x \in W} s_{h_t}(x) \geq -\frac{(n-k-2)(n-1)}{8(n-2)}$,
3. $|\varphi'(t)| \leq 1$
4. $0 \leq -2k\varphi''(t) \leq \frac{1}{2}(n-1)(n-k-2)^2$.

We say that condition (B_t) holds at $t \in I$, if the following assumptions are true:

1. $s \mapsto \varphi(s)$ is constant on an open neighborhood of t,
2. $\inf_{x \in F_t} s_{g_{\mathrm{WS}}}(x) \geq \frac{1}{2} s_{\sigma^{n-k-1}} = \frac{1}{2}(n-k-1)(n-k-2)$,
3. $\frac{(n-1)^2}{2} e(h_t)^2 + \frac{n-1}{2} \partial_t e(h_t) \geq -\frac{3}{64}(n-k-2)$.

Let P be a WS-bundle and let G be a Riemannian metric on P which is close to g_{WS} in a sense we will make precise later. Assume that u satisfies the equation

$$L_G u = 0. \tag{38}$$

Our aim is to estimate the distribution of L^2-norm of u with respect to the metric g_{WS}. If we rewrite the equation (38) in terms of the metric g_{WS} we obtain an equation of the form

$$L_{g_{\mathrm{WS}}} u = d^* A(du) + Xu + \varepsilon \partial_t u - su, \tag{39}$$

where $s, \varepsilon \in C^\infty(P)$, $A \in \Gamma(\mathrm{End}(T^*P))$ and $X \in \Gamma(TP)$ and where $dt(X) = 0$ and $A(dt) = 0$ and A is symmetric. Then the following theorem holds.

Theorem 7.7. *Assume that P is equipped with a metric g_{WS} of the form (37). Let $\alpha, \beta \in \mathbb{R}$ such that $[\alpha, \beta] \subset I$. Assume that for every $t \in I$ condition (A_t) or condition (B_t) holds. Assume that u is a positive solution of (39). Then there exists $c_0 > 0$ independent of α, β and φ such that if*

$$\|A\|_{L^\infty(P)}, \|X\|_{L^\infty(P)}, \|s\|_{L^\infty(P)}, \|\varepsilon\|_{L^\infty(P)}, \|e(h_t)\|_{L^\infty(P)} \leq c_0,$$

then

$$\int_{\pi^{-1}((\alpha+\gamma, \beta-\gamma))} u^2 \, dv^{g_{\mathrm{WS}}} \leq \frac{4(\mathrm{vol}^{g_\alpha}(F_\alpha) + \mathrm{vol}^{g_\beta}(F^\beta))}{n-k-2} \|u\|^2_{L^\infty(\pi^{-1}(\alpha,\beta))},$$

where $\gamma := \frac{\sqrt{32}}{n-k-2}$.

Note that the assertion is non-trivial only if $\beta - \alpha > 2\gamma$.

Proof. This is a special case of Theorem 5.2 in [2]. Since the proof given there is very long and technical we will not repeat it here. Note that the theorem in [2] is stated with $\|u\|_{L^\infty(P)}$ on the right hand side of the asserted estimate. However if we examine the end of the proof of Theorem 5.2 in [2] we observe that we may also put $\|u\|_{L^\infty(\pi^{-1}(\alpha,\beta))}$ as we have done. $\square$

7.5. **Proof of Theorem 7.1.** Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ with positive Yamabe constant $Y(M, g)$. Assume that g is flat on an open neighborhood U of a point $p \in M$. Then we can define the mass $m(M, g)$ at p.

Let N be obtained from M by a surgery of dimension $k \in \{0, ..., n-3\}$ which does not hit the point p. More precisely we apply the construction described in Section 7.2 with $M_1 := M$, $g_1 := g$, $M_2 := S^n$, $g_2 := \sigma^n$, $W := S^k$ such that the embedding $S^k \to S^n$ is the standard embedding and such that p is not contained in the image of the embedding $S^k \to M$. Moreover we choose the number $R_{\max} > 0$ and the open neighborhood U of p in such a way that $U \cap U^M(R_{\max}) = \emptyset$. Then for all θ which are small enough we obtain a manifold $N := N_\varepsilon$ with a Riemannian metric g_θ as described in Section 7.2. In particular g_θ coincides with g on U. By Theorem 6.1 in the article [2] and by the fact that

$$Y(M \amalg S^n, g \amalg \sigma^n) = Y(M, g)$$

(see e. g. Section 1.2 in [2]) we know that there exist positive constants $\Lambda_{n,k}$ depending only on n and k, such that

$$\min\{Y(M,g), \Lambda_{n,k}\} \le \liminf_{\theta \searrow 0} Y(N, g_\theta) \le \limsup_{\theta \searrow 0} Y(N, g_\theta) \le Y(M, g).$$

Thus if θ is small enough we have $Y(N, g_\theta) > 0$ and thus we can define the mass $m(N, g_\theta)$ at p.

We recall that by Theorem 3.1 we have

$$-m(M, g) = \inf\{J_g(u) | \, u \in C^\infty(M)\},$$

where for every $u \in C^\infty(M)$

$$J_g(u) = \int_{M \setminus \{p\}} \eta r^{2-n} F_\eta \, dv^g + 2 \int_M u F_\eta \, dv^g + \int_M u L_g u \, dv^g$$

and where η and F_η are defined as in Section 3. For $m(N, g_\theta)$ we have an analogous formula with a functional denoted by J_{g_θ}. We note that the functions η and F_η can be chosen independently of θ since we have $g = g_\theta$ for all θ on $\mathrm{supp}(\eta)$.

The proof of Theorem 7.1 is divided into several steps.

Step 1: After passing to a subsequence we have

$$\lim_{\theta \to 0} m(N, g_\theta) \ge m(M, g).$$

The proof is analogous to the proof of Theorem 6.9 and we do not repeat it here.

We choose $\delta > 0$ such that $B(p, 2\delta) \subset U$ and we choose a smooth function η on N_ε such that $\eta \equiv \frac{1}{(n-2)\omega_{n-1}}$ on $B(p, \delta)$, $\eta \equiv 0$ on $N_\varepsilon \setminus B(p, 2\delta)$ and $|d\eta|_g \le \frac{2}{\delta}$ on N_ε. For every θ we denote the Green function for L_{g_θ} at p by G_θ. Then the function u_θ: $N_\varepsilon \to \mathbb{R}$,

$$u_\theta(x) := \begin{cases} G_\theta(x) - \eta(x) r(x)^{2-n}, & x \ne p \\ m(N, g_\theta), & x = p \end{cases}$$

is smooth. For every $\alpha > 0$ which is small enough we set

$$A_\alpha := U^M(2\alpha) \setminus U^M(\alpha) \subset M.$$

Step 2: We prove that for all α, θ with $0 < \theta < \alpha < R_0$ we have

$$-m(M, g) \le -m(N, g_\theta) + 16 \int_{A_\alpha} u_\theta^2 \, dv^{g_\theta}.$$

For every α which is small enough let χ_α: $M \to [0, 1]$ be a smooth function such that $\chi_\alpha \equiv 1$ on $M \setminus U^M(2\alpha)$, $\chi_\alpha \equiv 0$ on $U^M(\alpha)$ and $|d\chi_\alpha|_g \le \frac{2}{\alpha}$. In particular for all α we have $\chi_\alpha \equiv 1$ on U. Furthermore if $\theta < \alpha$, then we have $g_\theta = F^2 g$ on $\mathrm{supp}(\chi_\alpha)$. If in addition $\alpha \in (0, R_0)$, then we obtain for all $\theta \in (0, \alpha)$

$$|d\chi_\alpha|_{g_\theta} = F^{-1} |d\chi_\alpha|_g = r |d\chi_\alpha|_g \le 2\alpha \frac{2}{\alpha} = 4. \tag{40}$$

For $0 < \theta < \alpha < R_0$ the function $v_{\alpha,\theta}\colon M \to \mathbb{R}$ defined by

$$v_{\alpha,\theta}(x) := \begin{cases} F^{\frac{n-2}{2}} \chi_\alpha u_\theta, & \text{if } x \in M \setminus W_1' \\ 0, & \text{if } x \in W_1' \end{cases}$$

is smooth. By Theorem 3.1 we have

$$\begin{aligned} -m(M,g) &\le J_g(v_{\alpha,\theta}) \\ &= \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^g + 2\int_M F^{\frac{n-2}{2}} \chi_\alpha u_\theta F_\eta \, dv^g \\ &\quad + \int_M F^{\frac{n-2}{2}} \chi_\alpha u_\theta L_g(F^{\frac{n-2}{2}} \chi_\alpha u_\theta) \, dv^g. \end{aligned}$$

Since on $\mathrm{supp}(\chi_\alpha)$ we have $g_\theta = F^2 g$ it follows from the conformal transformation property (5) of L_g that

$$L_g(F^{\frac{n-2}{2}} \chi_\alpha u_\theta) = F^{\frac{n+2}{2}} L_{g_\theta}(\chi_\alpha u_\theta).$$

Since on $\mathrm{supp}(\chi_\alpha)$ we have $dv^g = F^{-n} dv^{g_\theta}$ we obtain

$$\int_M F^{\frac{n-2}{2}} \chi_\alpha u_\theta L_g(F^{\frac{n-2}{2}} \chi_\alpha u_\theta) \, dv^g = \int_M \chi_\alpha u_\theta L_{g_\theta}(\chi_\alpha u_\theta) \, dv^{g_\theta}.$$

Now by (3) we have

$$\int_M \chi_\alpha u_\theta \Delta_{g_\theta}(\chi_\alpha u_\theta) \, dv^{g_\theta} = \int_M (u_\theta^2 |d\chi_\alpha|_{g_\theta}^2 + \chi_\alpha^2 u_\theta \Delta_{g_\theta} u_\theta) \, dv^{g_\theta}.$$

Using that on $\mathrm{supp}(F_\eta)$ we have $F \equiv 1$, $\chi_\alpha \equiv 1$ and $g_\theta = g$ we obtain

$$\begin{aligned} -m(M,g) \le & \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^{g_\theta} + 2\int_M u_\theta F_\eta \, dv^{g_\theta} \\ & + \int_M u_\theta^2 |d\chi_\alpha|_{g_\theta}^2 \, dv^{g_\theta} + \int_M \chi_\alpha^2 u_\theta L_{g_\theta} u_\theta \, dv^{g_\theta}. \end{aligned}$$

Using that $L_{g_\theta} u_\theta = -F_\eta$ and $\chi_\alpha \equiv 1$ on $\mathrm{supp}(F_\eta)$ and using that $\mathrm{supp}(F_\eta) \subset M$, we obtain

$$\begin{aligned} -m(M,g) &\le \int_{N_\varepsilon\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^{g_\theta} + 2\int_{N_\varepsilon} u_\theta F_\eta \, dv^{g_\theta} \\ &\quad + \int_M u_\theta^2 |d\chi_\alpha|_{g_\theta}^2 \, dv^{g_\theta} + \int_{N_\varepsilon} u_\theta L_{g_\theta} u_\theta \, dv^{g_\theta} \\ &= J_{g_\theta}(u_\theta) + \int_M u_\theta^2 |d\chi_\alpha|_{g_\theta}^2 \, dv^{g_\theta} \end{aligned}$$

Using (40) and that $\mathrm{supp}(d\chi_\alpha) \subset A_\alpha$ we obtain

$$-m(M,g) \le J_{g_\theta}(u_\theta) + 16\int_{A_\alpha} u_\theta^2 \, dv^{g_\theta}$$

By Theorem 3.1 we have $J_{g_\theta}(u_\theta) = -m(N, g_\theta)$ and therefore the assertion of Step 2 follows.

In the remainder of the proof we will show that the integral on the right hand side tends to 0 as α and θ tend to 0. By definition of u_θ we have $L_{g_\theta} u_\theta = -F_\eta$ for all θ, where F_η is defined as in Section 3. In particular there exists $b > 0$ such that for all θ we have

$$U_\varepsilon^N(b) \cap \mathrm{supp}(L_{g_\theta} u_\theta) = \emptyset.$$

In the following step we obtain an L^2-estimate for the functions u_θ which is independent of θ. The result is not trivial since $\mathrm{vol}^{g_\theta}(U_\varepsilon^N(b)) \to \infty$ as $\theta \to 0$.

Step 3: We prove that there exist $a \in (0, b)$ and $D > 0$ such that for every θ we have

$$\int_{U_\varepsilon^N(a)} u_\theta^2 \, dv^{g_\theta} \le D\Big(\max_{U_\varepsilon^N(b)} u_\theta\Big)^2.$$

This inequality is a special case of Lemma 6.6 in the article [2] and we follow the proof given there. Let $\tilde r \in (\varepsilon, b)$ be fixed. The manifold $P := U_\varepsilon^N(\tilde r)$ with the metric g_θ' defined in (31) is a WS-bundle, where in the notation of Section 7.4 we have $I = (\alpha, \beta)$ with $\alpha := -\ln \tilde r + \ln \varepsilon$ and $\beta := \ln \tilde r - \ln \varepsilon$. The metric g_θ' has exactly the form (37) with $\varphi = f$ and $h_t = \tilde h_t$. Let θ be small enough and let

$$t \in (-\ln \tilde r + \ln \varepsilon, -\ln \delta_0 + \ln \varepsilon) \cup (\ln \delta_0 - \ln \varepsilon, \ln \tilde r - \ln \varepsilon).$$

Then assumption (A_t) from Section 7.4 is true. Let again θ be small enough and let

$$t \in (-\ln \delta_0 + \ln \varepsilon, \ln \delta_0 - \ln \varepsilon).$$

Then we have $s_{g_\theta'} = s_{\sigma^{n-k-1}} + O(1/A_\theta)$ and the error term $e(\tilde h_t)$ from condition (B_t) satisfies

$$2(n-1)|e(\tilde h_t)| \le \Big|\mathrm{tr}_{\tilde h_t} \partial_t \tilde h_t\Big| = \Big|\mathrm{tr}_{\tilde h_t}\Big(\chi'(t/A_\theta)\frac{h_2 - h_1}{A_\theta}\Big)\Big| \le \frac{C}{A_\theta}$$

and

$$2(n-1)|\partial_t e(\tilde h_t)| = \big|\mathrm{tr}\big(\tilde h_t^{-1}(\partial_t \tilde h_t)\tilde h_t^{-1}(\partial_t \tilde h_t)\big)\big| + \big|\mathrm{tr}_{\tilde h_t} \partial_t^2 \tilde h_t\big| \le \frac{C}{A_\theta^2}.$$

Because of $1/A_\theta \le \theta$ the assumption (B_t) from Section 7.4 is true. Now on P we have $L_{g_\theta} u_\theta = 0$ and with respect to the metric $g_{\mathrm{WS}} := g_\theta'$ this equation has the form (39) as argued in Section 7.4. Using (32), (33), (34) one verifies that the error terms satisfy the pointwise estimates

$$|A(x)|_{g_{\mathrm{WS}}}, |X(x)|_{g_{\mathrm{WS}}}, |s(x)|_{g_{\mathrm{WS}}}, |\varepsilon(x)|_{g_{\mathrm{WS}}} \le Ce^{-f(t)}$$

on $U_\varepsilon^N(R_0)$, where $C > 0$ is independent of θ. In particular for every $c_0 > 0$ we obtain

$$|A(x)|_{g_{\mathrm{WS}}}, |X(x)|_{g_{\mathrm{WS}}}, |s(x)|_{g_{\mathrm{WS}}}, |\varepsilon(x)|_{g_{\mathrm{WS}}} \le c_0$$

on $U_\varepsilon^N(\theta)$ if θ is small enough. We set

$$\alpha := -\ln \tilde r + \ln \varepsilon, \quad \beta := \ln \tilde r - \ln \varepsilon.$$

If $\tilde r$ is so small that $\beta - \alpha > 2\gamma = \frac{8\sqrt{2}}{n-k-2}$, then with $P' := U_\varepsilon^N(\tilde r e^{-\gamma})$ we obtain by Theorem 7.7 that

$$\int_{P'} u_\theta^2 \, dv^{g_{\mathrm{WS}}} \le C\|u_\theta\|^2_{L^\infty(\pi^{-1}(\alpha,\beta))},$$

where

$$C = \frac{4}{n-k-2}(\mathrm{vol}^{g_\alpha}(F^\alpha) + \mathrm{vol}^{g_\beta}(F^\beta))$$

is independent of θ. Furthermore if $\tilde r$ is small enough we have

$$dv^{g_\theta} \le 2dv^{g_{\mathrm{WS}}}$$

on P' and therefore

$$\int_{P'} u_\theta^2 \, dv^{g_\theta} \le 2C\|u_\theta\|^2_{L^\infty(\pi^{-1}(\alpha,\beta))}.$$

Thus with $a := \tilde{r}e^{-\gamma}$ the assertion of Step 3 follows since the functions u_θ are positive on $U_\varepsilon^N(b)$.

Step 4: We prove that there exists $C_1 > 0$ such that for all θ we have

$$\int_N u_\theta^p \, dv^{g_\theta} \leq C_1,$$

where $p := \frac{2n}{n-2}$.

By Theorem 6.1 in the article [2] there exists a positive constant $\Lambda_{n,k}$ depending only on n and k such that we have

$$C_0 := \min\{Y(M,g), \Lambda_{n,k}\} \leq \liminf_{\theta \to 0} Y(N, g_\theta)$$

where $C_0 > 0$. Let $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By definition of $Y(N, g_\theta)$ and by Hölder's inequality we obtain for all sufficiently small θ

$$\frac{C_0}{2} \leq \frac{\int_N u_\theta L_{g_\theta} u_\theta \, dv^{g_\theta}}{(\int_N u_\theta^p \, dv^{g_\theta})^{2/p}} = -\frac{\int_N u_\theta F_\eta \, dv^{g_\theta}}{(\int_N u_\theta^p \, dv^{g_\theta})^{2/p}} \leq \frac{(\int_N F_\eta^q \, dv^{g_\theta})^{1/q}}{(\int_N u_\theta^p \, dv^{g_\theta})^{1/p}}.$$

On $\mathrm{supp}(F_\eta)$ we have $g_\theta = g$ and thus the numerator on the right hand side is independent of θ. The assertion of Step 4 follows.

Step 5: We prove that there exists $C_2 > 0$ such that for all θ we have

$$\max_{N_\varepsilon} u_\theta \leq C_2.$$

For every θ we choose $x_\theta \in N_\varepsilon$ such that

$$u_\theta(x_\theta) = \max_{N_\varepsilon} u_\theta =: m_\theta.$$

We assume that after taking a subsequence we have $m_\theta \to \infty$ as $\theta \to 0$. First we prove the following lemma.

Lemma 7.8. *Let $\alpha > 0$. Then for all sufficiently small θ there exists $x'_\theta \in U_\varepsilon^N(2\alpha)$ such that we have*

$$c_\alpha m_\theta \leq u_\theta(x'_\theta) \leq m_\theta,$$

where $c_\alpha > 0$ is independent of θ.

Proof. Let v be a solution to the Yamabe problem on (M,g), i. e. a smooth positive function on M such that the Riemannian metric $v^{4/(n-2)}g$ has constant scalar curvature 1 on M. We choose a smooth function $\chi_\alpha \colon N_\varepsilon \to [0,1]$ such that $\chi_\alpha \equiv 1$ on $N \setminus U_\varepsilon^N(2\alpha)$ and $\chi_\alpha \equiv 0$ on $U_\varepsilon^N(\alpha)$. Then for every θ the function

$$v_\theta := F^{-\frac{n-2}{2}} v \chi_\alpha + 1 - \chi_\alpha$$

on N_ε is smooth and positive and it depends on θ since F depends on θ. Now there exist constants $b_\alpha, B_\alpha > 0$ such that for every θ we have

$$b_\alpha \leq v_\theta \leq B_\alpha \tag{41}$$

on N_ε. For every θ we define the Riemannian metric

$$\tilde{g}_\theta := v_\theta^{\frac{4}{n-2}} g_\theta$$

on N_ε. Let θ be so small that on $N \setminus U_\varepsilon^N(2\alpha)$ we have $g_\theta = F^2 g$. Then on $N \setminus U_\varepsilon^N(2\alpha)$ we get $\tilde{g}_\theta = v^{4/(n-2)} g$ and thus

$$s_{\tilde{g}_\theta} \equiv 1 \text{ on } N \setminus U_\varepsilon^N(2\alpha). \tag{42}$$

For every θ we define the function

$$\tilde{u}_\theta = \frac{u_\theta}{v_\theta}$$

on N_ε and we choose $x'_\theta \in N_\varepsilon$ such that

$$\tilde{u}_\theta(x'_\theta) = \max_{N_\varepsilon} \tilde{u}_\theta.$$

Then for all θ we have by (41)

$$\tilde{u}_\theta(x'_\theta) \geq \tilde{u}_\theta(x_\theta) = \frac{m_\theta}{v_\theta(x_\theta)} \geq \frac{m_\theta}{B_\alpha} \tag{43}$$

and thus by our assumption $\tilde{u}_\theta(x'_\theta) \to \infty$ as $\theta \to 0$. By the conformal transformation law (5) for L_{g_θ} we have at x'_θ

$$(\Delta_{\tilde{g}_\theta} \tilde{u}_\theta)(x'_\theta) + \frac{n-2}{4(n-1)} s_{\tilde{g}_\theta}(x'_\theta) \tilde{u}_\theta(x'_\theta) = -F_\eta(x'_\theta) v_\theta(x'_\theta)^{-\frac{n+2}{n-2}}. \tag{44}$$

Notice that the right hand side is bounded independently of θ since on $\mathrm{supp}(F_\eta)$ the function v_θ is independent of θ. Since the first term on the left hand side is non-negative and since $\tilde{u}_\theta(x'_\theta) \to \infty$ as $\theta \to 0$ it follows that $s_{\tilde{g}_\theta}(x'_\theta) \to 0$ as $\theta \to 0$. Thus by (42) we have $x'_\theta \in U_\varepsilon^N(2\alpha)$ if θ is small enough. It remains to prove the inequalities of the assertion. First, by definition of m_θ we have $u_\theta(x'_\theta) \leq m_\theta$. Second, by (41) and (43) we have

$$u_\theta(x'_\theta) = v_\theta(x'_\theta) \tilde{u}_\theta(x'_\theta) \geq \frac{v_\theta(x'_\theta) m_\theta}{B_\alpha} \geq \frac{b_\alpha}{B_\alpha} m_\theta.$$

This finishes the proof of the lemma. □

In the remaining part of the proof of Step 5 we distinguish two cases.

Case 1: There exists $c > 0$ such that $x'_\theta \in N \setminus U_\varepsilon^N(c)$ for an infinite number of θ.

The proof is very similar to Subcase I.1 in the proof of Theorem 6.1 in [2]. After taking a subsequence we may assume that there exists $\overline{x} \in N \setminus U_\varepsilon^N(c)$ such that $\lim_{\theta \to 0} x'_\theta = \overline{x}$. For every θ we put $a_\theta := u_\theta(x'_\theta)$. In a neighborhood U of $\overline{x}$ the metric $g_\theta = F^2 g$ is independent of θ if θ is small enough. We define $\tilde{g}_\theta := a_\theta^{4/(n-2)} g_\theta$. Let $r > 0$. We apply Lemma 7.2 with $O = U$, $\alpha = \theta$, $q_\alpha = x'_\theta$, $q = \overline{x}$, $\gamma_\alpha = g_\theta = F^2 g$ and $b_\alpha = a_\theta^{2/(n-2)}$. For θ small we then obtain a diffeomorphism

$$\Theta_\theta : B^n(r) \to B^{g_\theta}(x'_\theta, a_\theta^{-\frac{2}{n-2}} r)$$

such that the sequence of metrics $(\Theta_\theta^*(\tilde{g}_\theta))$ converges to the flat metric ξ^n in $C^2(B^n(r))$. For all sufficiently small θ we have

$$B^{g_\theta}(x'_\theta, a_\theta^{-\frac{2}{n-2}} r) \cap \mathrm{supp}(F_\eta) = \emptyset$$

and thus $L_{g_\theta} u_\theta = 0$ on $B^{g_\theta}(x'_\theta, a_\theta^{-2/(n-2)} r)$. We define $\tilde{u}_\theta := a_\theta^{-1} u_\theta$. By the conformal transformation law 5 for L_{g_θ} we have

$$L_{\tilde{g}_\theta} \tilde{u}_\theta = 0$$

on $B^{g_\theta}(x'_\theta, a_\theta^{-2/(n-2)}r)$ and since $dv^{\tilde{g}_\theta} = a_\theta^p dv^{g_\theta}$ we have

$$\begin{aligned}\int_{B^{g_\theta}(x'_\theta, a_\theta^{-\frac{2}{n-2}}r)} \tilde{u}_\theta^p \, dv^{\tilde{g}_\theta} &= \int_{B^{g_\theta}(x'_\theta, a_\theta^{-\frac{2}{n-2}}r)} u_\theta^p \, dv^{g_\theta} \\ &\le \int_N u_\theta^p \, dv^{g_\theta} \\ &\le C_1\end{aligned}$$

by Step 4. Since

$$\Theta_\theta : (B^n(r), \Theta_\theta^*(\tilde{g}_\theta)) \to (B^{g_\theta}(x'_\theta, a_\theta^{-\frac{2}{n-2}}r), \tilde{g}_\theta)$$

is an isometry we can consider $\tilde{u}_\theta$ as a solution of

$$L_{\Theta_\theta^*(\tilde{g}_\theta)} \tilde{u}_\theta = 0$$

on $B^n(r)$ with

$$\int_{B^n(r)} \tilde{u}_\theta^p \, dv^{\Theta_\theta^*(\tilde{g}_\theta)} \le C_1.$$

Since $\|\tilde{u}_\theta\|_{L^\infty(B^n(r))} = |\tilde{u}_\theta(0)| = 1$ we can apply Lemma 7.3 with $V = \mathbb{R}^n$, $\alpha = \theta$, $g_\alpha = \Theta_\theta^*(\tilde{g}_\theta)$ and $u_\alpha = \tilde{u}_\theta$. We can apply this lemma since every compact subset of $\mathbb{R}^n$ is contained in some ball $B^n(r)$. We conclude that there exists a non-negative C^2-function u on $\mathbb{R}^n$ such that

$$L_{\xi^n} u = 0, \quad u(0) = 1,$$

in particular $u \not\equiv 0$. By (36) we have for every $r > 0$

$$\int_{B^n(r)} u^p \, dv^{\xi^n} = \lim_{\theta \to 0} \int_{B^{g_\theta}(x'_\theta, a_\theta^{-\frac{2}{n-2}}r)} u_\theta^p \, dv^{g_\theta} \le C_1.$$

In particular

$$\int_{\mathbb{R}^n} u^p \, dv^{\xi^n} \le C_1.$$

After dividing u by a constant we may assume that $\int_{\mathbb{R}^n} u^p dv^{\xi^n} = 1$. We have obtained a contradiction to Lemma 7.4. This finishes the proof in Case 1.

Case 2: For every $c > 0$ we have $x'_\theta \in U_\varepsilon^N(c)$ for θ sufficiently small.

The proof is very similar to Subcase I.2 in the proof of Theorem 6.1 in [2]. Again for every θ we put $a_\theta := u_\theta(x'_\theta)$. The subset $U_\varepsilon^N(c)$ is diffeomorphic to $W \times I \times S^{n-k-1}$ where I is an interval. We identify

$$x'_\theta = (y_\theta, t_\theta, z_\theta) \in W \times (-\ln R_0 + \ln \varepsilon, -\ln \varepsilon + \ln R_0) \times S^{n-k-1}.$$

By taking a subsequence we may assume that y_θ, $\frac{t_\theta}{A_\theta}$ and z_θ converge respectively to $y \in W$, $T \in [-\infty, \infty]$ and $z \in S^{n-k-1}$. First we apply Lemma 7.2 with $V = W$, $\alpha = \theta$, $q_\alpha = y_\theta$, $q = y$, $\gamma_\alpha = \tilde{h}_{t_\theta}$, $\gamma_0 = \tilde{h}_T$ and $b_\alpha = a_\theta^{2/(n-2)}$, where we define $\tilde{h}_{-\infty} = h_1$ and $\tilde{h}_\infty = h_2$. For every $r > 0$ the lemma provides diffeomorphisms

$$\Theta_\theta^y : B^k(r) \to B^{\tilde{h}_{t_\theta}}(y_\theta, a_\theta^{-\frac{2}{n-2}} e^{-f(t_\theta)} r)$$

such that $(\Theta_\theta^y)^*(a_\theta^{4/(n-2)} e^{2f(t_\theta)} \tilde{h}_{t_\theta})$ converges to the flat metric ξ^k on $B^k(r)$ as $\theta \to 0$. Second we apply Lemma 7.2 with $V = S^{n-k-1}$, $\alpha = \theta$, $q_\alpha = z_\theta$, $\gamma_\alpha = \gamma_0 =$

σ^{n-k-1} and $b_\alpha = a_\theta^{2/(n-2)}$. For every $r' > 0$ we obtain diffeomorphisms

$$\Theta_\theta^z : B^{n-k-1}(r') \to B^{\sigma^{n-k-1}}(z_\theta, a_\theta^{-\frac{2}{n-2}} r')$$

such that $(\Theta_\theta^z)^*(a_\theta^{4/(n-2)}\sigma^{n-k-1})$ converges to ξ^{n-k-1} on $B^{n-k-1}(r')$ as $\theta \to 0$. For $r, r', r'' > 0$ we define

$$U_\theta(r, r', r'') := B^{\tilde{h}_{t_\theta}}(y_\theta, a_\theta^{-\frac{2}{n-2}} e^{-f(t_\theta)} r) \times [t_\theta - a_\theta^{-\frac{2}{n-2}} r'', t_\theta + a_\theta^{-\frac{2}{n-2}} r'']$$
$$\times B^{\sigma^{n-k-1}}(z_\theta, a_\theta^{-\frac{2}{n-2}} r')$$

and

$$\Theta_\theta : B^k(r) \times [-r'', r''] \times B^{n-k-1}(r') \to U_\theta(r, r', r'')$$

by

$$\Theta_\theta(y, s, z) := (\Theta_\theta^y(y), t(s), \Theta_\theta^z(z)),$$

where $t(s) := t_\theta + a_\theta^{-2/(n-2)} s$. Then Θ_θ is a diffeomorphism and we obtain

$$\Theta_\theta^*(a_\theta^{\frac{4}{n-2}} g_\theta) = (\Theta_\theta^y)^*(a_\theta^{\frac{4}{n-2}} e^{2f(t)} \tilde{h}_t) + ds^2 + (\Theta_\theta^z)^*(a_\theta^{\frac{4}{n-2}} \sigma^{n-k-1}) + \Theta_\theta^*(a_\theta^{\frac{4}{n-2}} \widetilde{T}_t).$$

As in Subcase I.2 in the proof of Theorem 6.1 in [2] one shows that the sequence of Riemannian metrics $\Theta_\theta^*(a_\theta^{4/(n-2)} g_\theta)$ tends to the flat metric ξ^n. Then as in the proof of Case 1 above one obtains a non-negative C^2-function u satisfying

$$L_{\xi^n} u = 0, \quad u(0) = 1, \quad \int_{\mathbb{R}^n} u^p \, dv^{\xi^n} < \infty.$$

In particular $u \not\equiv 0$ and one obtains a contradiction to Lemma 7.4 as above. This finishes the proof of Step 5.

By Steps 3 and 5 we know that there exist $a > 0$ and $C > 0$ such that for every θ we have

$$\int_{U^M(a)} u_\theta^2 \, dv^{g_\theta} \le C. \tag{45}$$

We recall that for $\alpha > 0$ we have defined

$$A_\alpha := U^M(2\alpha) \setminus U^M(\alpha) \subset M.$$

Next we define

$$E := \liminf_{\alpha \to 0} \liminf_{\theta \to 0} \int_{A_\alpha} u_\theta^2 \, dv^{g_\theta}.$$

Step 6: Conclusion.

By the result of Step 2 it remains to show that $E = 0$. We proceed similarly as on p. 50 of the article [2]. Namely there exists $\delta > 0$ such that for every $\alpha \in (0, \delta)$ we have

$$\liminf_{\theta \to 0} \int_{A_\alpha} u_\theta^2 \, dv^{g_\theta} \ge \frac{E}{2}.$$

For $m \in \mathbb{N}$ we set $\alpha_m := 2^{-m}\delta$. Then we have

$$\liminf_{\theta \to 0} \int_{A_{\alpha_m}} u_\theta^2 \, dv^{g_\theta} \ge \frac{E}{2}$$

for all m. Let $N_0 \in \mathbb{N}$. The sets A_{α_m}, $m \in \mathbb{N}$, are disjoint and therefore we have

$$\int_{U^M(\delta)} u_\theta^2 \, dv^{g_\theta} \ge \int_{\bigcup_{m=1}^{N_0} A_{\alpha_m}} u_\theta^2 \, dv^{g_\theta} = \sum_{m=1}^{N_0} \int_{A_{\alpha_m}} u_\theta^2 \, dv^{g_\theta}$$

for all θ. From this we obtain

$$\begin{aligned}\liminf_{\theta\to 0}\int_{U^M(\delta)} u_\theta^2\, dv^{g_\theta} &\geq \liminf_{\theta\to 0}\sum_{m=1}^{N_0}\int_{A_{\alpha_m}} u_\theta^2\, dv^{g_\theta}\\ &\geq \sum_{m=1}^{N_0}\liminf_{\theta\to 0}\int_{A_{\alpha_m}} u_\theta^2\, dv^{g_\theta}\\ &\geq \frac{EN_0}{2}.\end{aligned}$$

Assume that $E > 0$. Since $N_0 \in \mathbb{N}$ can be chosen arbitrarily large, we obtain a contradiction to the estimate (45). Thus we have $E = 0$ and Theorem 7.1 is proved.

8. Application to the positive mass conjecture

In this section we study an application of Theorem 7.1 to the positive mass conjecture. By a simply connected manifold T we mean a connected manifold T with $\pi_1(T) = \{0\}$. If T is an oriented manifold, we denote by $-T$ the manifold T with the opposite orientation.

Lemma 8.1. *Let X_1 be a closed simply connected oriented non-spin manifold of dimension $n \geq 5$ and let X_0 be a manifold of dimension n which is oriented cobordant to X_1. Then X_1 can be obtained from X_0 by finitely many surgeries of dimension $k \in \{0, ..., n-3\}$.*

Proof. The assertion follows from the proof of Theorem C in the article [8] by Gromov and Lawson. Namely let W be an oriented cobordism from X_0 to X_1. After applying finitely many surgeries of dimension 0 or 1 to X_0 and then to W we may assume that X_0 and W are simply connected. After further applying surgeries and using that X_1 is not spin we can assume that the induced homomorphism $\pi_2(X_1) \to \pi_2(W)$ is surjective. It follows that for $i \leq 2$ we have $H_i(W, X_0) = 0$ and $H_i(W, X_1) = 0$. The assertion then follows from a result by Smale ([26], see also [16, VIII Thm. 4.1]). □

Definition 8.2. We say that a closed manifold M satisfies PMT if for every Riemannian metric g on M with $Y(M, g) > 0$ and for every point $p \in M$ such that g is flat on an open neighborhood of p we have $m(M, g) \geq 0$ at p.

Lemma 8.3. *Let M, N be two closed manifolds of dimension n such that N satisfies PMT. Assume that M is obtained from N by surgery of dimension $\ell \in \{2, ..., n-1\}$. Then M satisfies PMT.*

Proof. In general any surgery of dimension ℓ on a manifold of dimension n can be undone by a surgery of dimension $n - 1 - \ell$. Thus N can be obtained from M by surgery of dimension $k \in \{0, ..., n-3\}$ and the assertion follows from Theorem 7.1. □

Lemma 8.4. *Let M and P be two closed manifolds of the same dimension. Assume that M does not satisfy PMT and that there exists a Riemannian metric h on P with $Y(P, h) > 0$. Then the connected sum $M \# P$ does not satisfy PMT.*

Proof. Let g be a Riemannian metric on M with $Y(M,g) > 0$ such that at some point $p \in M$ we have $m(M,g) < 0$. The metric $g \amalg h$ on the disjoint union $M \amalg P$ satisfies

$$Y(M \amalg P, g \amalg h) = \min\{Y(M,g), Y(P,h)\} > 0$$

(see e. g.Section 1.2 in [2]). The Green function of $L_{g \amalg h}$ is given by

$$G_{g \amalg h} = \begin{cases} G_g & \text{on } M, \\ 0 & \text{on } P \end{cases}$$

and thus at p we have $m(M \amalg P, g \amalg h) = m(M,g) < 0$, i. e. $M \amalg P$ does not satisfy PMT. Since $M\#P$ can be obtained from $M \amalg P$ by surgery of dimension 0 Theorem 7.1 shows that $M\#P$ does not satisfy PMT. □

Theorem 8.5. *Assume that there exists a closed orientable simply-connected non-spin manifold of dimension $n \geq 5$ satisfying PMT. Then every closed manifold of dimension n satisfies PMT.*

Note that, by Proposition 4.1 in [21] or Section 5 in [18], this theorem could also be stated for the ADM-mass in the context of the standard positive mass conjecture coming from general relativity.

Proof. Let M be a closed oriented simply-connected non-spin manifold of dimension n satisfying PMT. The manifold $M\#M\#(-M)$ is oriented cobordant to M. By Lemma 8.1 the manifold M can be obtained from $M\#M\#(-M)$ by finitely many surgeries of dimension $k \in \{0, ..., n-3\}$. Therefore $M\#M\#(-M)$ can be obtained from M by finitely many surgeries of dimension $\ell \in \{2, ..., n-1\}$. Since M satisfies PMT it follows from Lemma 8.3 that $M\#M\#(-M)$ satisfies PMT. By Lemma 8.4 we conclude that $M\#(-M)$ satisfies PMT.

Let N be a closed manifold of dimension n. Assume first that N is orientable and choose an orientation on N. Assume that N does not satisfy PMT. By Lemma 8.4 it follows that $N\#(-N)$ does not satisfy PMT. Now $N\#(-N)$ is oriented cobordant to $M\#(-M)$ since both manifolds are oriented cobordant to S^n. Furthermore $M\#(-M)$ is simply connected and non-spin. By Lemma 8.1 the manifold $M\#(-M)$ can be obtained from $N\#(-N)$ by finitely many surgeries of dimension $k \in \{0, ..., n-3\}$. By Theorem 7.1 the manifold $M\#(-M)$ does not satisfy PMT which is a contradiction.

Next assume that N is not orientable. Let π: $\tilde{N} \to N$ be the two-fold orientable covering of N. Let g be a Riemannian metric on N which is flat on an open neighborhood of a point $p \in N$ and such that L_g is a positive operator. Let $\tilde{g}$ be the Riemannian metric on $\tilde{N}$ such that π is a Riemannian covering. Since the first eigenvalue $\tilde{\lambda}_0$ of $L_{\tilde{g}}$ is simple and the corresponding eigenfunctions do not change their sign, $\tilde{\lambda}_0$ is also an eigenvalue of L_g. It follows that $L_{\tilde{g}}$ is a positive operator. Now if we write $\pi^{-1}(p) = \{\tilde{p}_1, \tilde{p}_2\}$ and if $\tilde{G}_1$, $\tilde{G}_2$ denote the Green functions for $L_{\tilde{g}}$ at $\tilde{p}_1$ and $\tilde{p}_2$ respectively, then for the Green function G of L_g at p we have $G \circ \pi = \tilde{G}_1 + \tilde{G}_2$. In particular if $m^{\tilde{p}_1}(\tilde{N}, \tilde{g})$ denotes the mass of $(\tilde{N}, \tilde{g})$ at $\tilde{p}_1$, then for the mass of L_g at p we have $m(N,g) = m^{\tilde{p}_1}(\tilde{N}, \tilde{g}) + \tilde{G}_2(\tilde{p}_1) > 0$. □

It is easy to find examples of closed orientable simply-connected non-spin manifolds, e. g. $\mathbb{C}P^{2m}$ or $\mathbb{C}P^{2m} \times S^k$ with $k \geq 2$. Our hope is that among these examples one can find manifolds of dimension at least 8 satisfying PMT. However we have not yet succeeded. Among the manifolds of dimension at least 8 satisfying PMT we know examples which are simply connected and spin (by Section 6.1) and examples which are not simply-connected and non-spin: indeed, we have

Proposition 8.6. *Let $n \geq 5$, $n \equiv 1 \mod 4$. Then, the projective space $\mathbb{R}P^n$ satisfies PMT.*

Proof. Let g be a metric on $\mathbb{R}P^n$ which is flat around $p \in \mathbb{R}P^n$ such that L_g is a positive operator. Using the two-fold covering $S^n \to \mathbb{R}P^n$ one obtains as in the last part of the proof of Theorem 8.5 that the mass of L_g at p is strictly positive. □

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